Wigner function formalism in Quantum mechanics
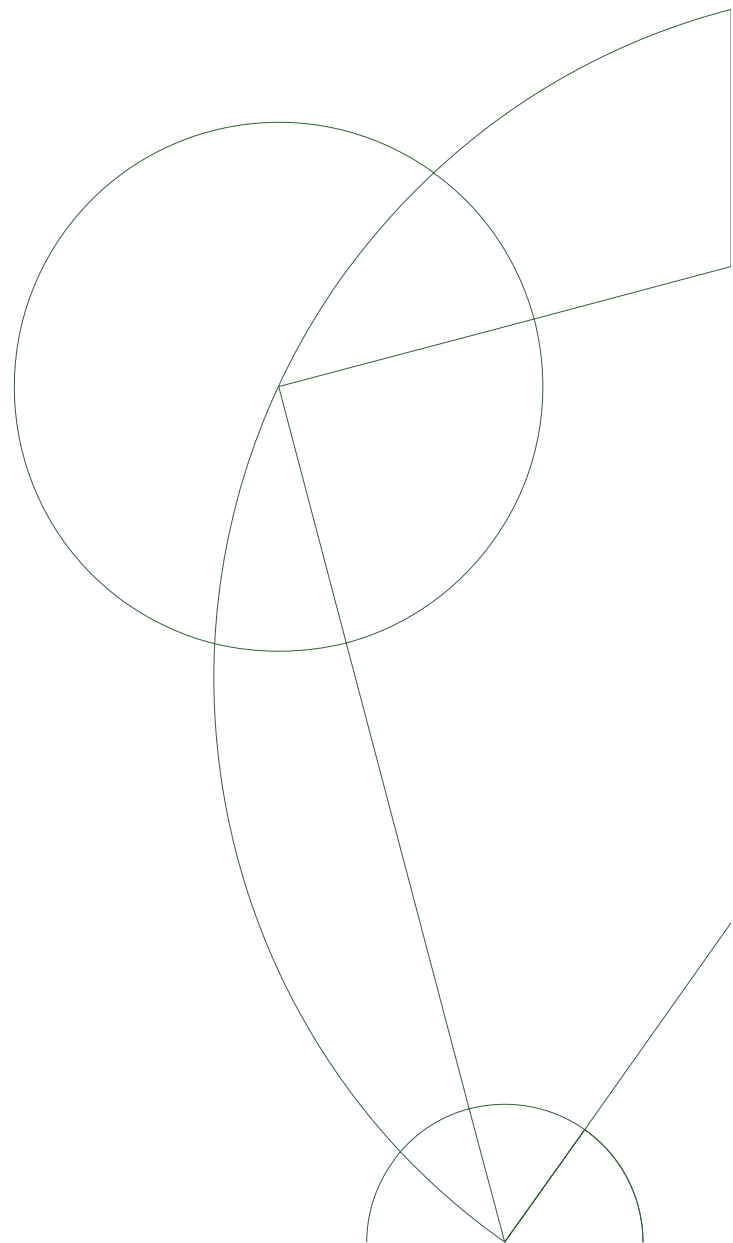
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Abstract

In this paper the Wigner function is introduced as an alternative to the Schrödinger picture to solve quantum mechanical problems. General properties of the Wigner functions are proven and discussed. The Wigner function formalism is then applied to the harmonic oscillator, the gaussian wavepacket and the driven harmonic oscillator. We see the Wigner function enables us to describe quantum mechanical systems, using only a single mathematical object. We find that the Wigner function in some cases offers an easier way to visualize the properties of quantum systems than the wavefunction does.

Contents

1 The Wigner function

We begin by introducing the Wigner function, by the motivation behind creating it in the first place. In classical Hamiltonian physics a state is described by a point in a 6N dimensional phase space for the variables position (q) and momentum (p). There is no uncertainty principle in classical physics, so it is possible to know a particle's momentum and position at the same time to an arbitrary presicion. In quantum mechanics there is an uncertainty priciple that makes it impossible to know both q or x and p at the same time. In the standard formulation of quantum mechanics one works with probability densities instead. One for the wavefunction in position-basis and one for the wave function in the momentun-basis

$$P(x) = |\psi(x)|^2, \tag{1}$$

$$P(k) = |\phi(k)|^2. \tag{2}$$

Where the two functions are connected by a fourier transform and we have used $p = \hbar k$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int dx \psi(x) e^{-ikx}. \tag{3}$$

Where the integral is from $-\infty$ to ∞. This goes for all integrals in this paper unless stated otherwise. It would be desirable to have a single function that could display the probability in both x and p. The Wigner function is a function constructed to do just that. It must also be able to give the correct expectation values for operators. What one would desire is to have a probability distribution in phase space P(x,p), that is positive everywhere and such that

$$\int \int dx dp P(x,p) A(x,p), \tag{4}$$

gives the expectation value of the operator A(x,p). Because of Heisenberg's uncertainty principle it is not possible to find such a probability distribution. The Wigner function comes close to fulfill these demands, but it will not have a direct physical interpretation as a probability distrubition we know from classical physics. For example the Wigner function can be negative in regions of phase space, which have no physical meaning if one thinks of it as a probability distrubition [1].

1.1 The Weyl-transform

In constructing the Wigner function one attempts to construct a new formalism of quantum mechanics based on a phase space formalism. In order to be succesful in creating such a formalism one needs a mapping between functions in the quantum phase space formulation and Hilbert space operators in the Schrödinger picture [1]. This mapping is given by the Weyl-transform $\tilde{A}$ of an operator $\hat{A}$ defined in the following way

$$\tilde{A}(x,p) = \int dy e^{\frac{-ipy}{\hbar}} \langle x + \frac{y}{2} | \hat{A}(\hat{x}, \hat{p}) | x - \frac{y}{2} \rangle \tag{5}$$

This transformation takes an operator and represents it with a function. We will now show a key property of the Weyl transform. This property is that the trace of the product of two operators $\hat{A}$ and $\hat{B}$ is given by

$$Tr[\hat{A}\hat{B}] = \frac{1}{2\pi\hbar} \int \int dx dp \tilde{A}(x,p) \tilde{B}(x,p). \tag{6}$$

To prove this relation we first start with the Weyl transform of the two operators.

$$\tilde{A}(x,p) = \int dy e^{\frac{-ipy}{\hbar}} \langle x + \frac{y}{2} | \hat{A}(\hat{x}, \hat{p}) | x - \frac{y}{2} \rangle, \tag{7}$$

$$\tilde{B}(x,p) = \int dy' e^{\frac{-ipy'}{\hbar}} \langle x + \frac{y'}{2}|\hat{B}(\hat{x},\hat{p})|x - \frac{y'}{2}\rangle. \tag{8}$$

We take the product of these two and integrate over all x and p to find

$$\int\int dxdp\tilde{A}(x,p)\tilde{B}(x,p) = \int\int\int\int dxdpdydy' e^{\frac{-ip(y+y')}{\hbar}} \langle x + \frac{y}{2}|\hat{A}(\hat{x},\hat{p})|x - \frac{y}{2}\rangle\times$$
$$\langle x + \frac{y'}{2}|\hat{B}(\hat{x},\hat{p})|x - \frac{y'}{2}\rangle. \tag{9}$$

To perform the p integration we use the following result

$$\int dp e^{\frac{ipy)}{\hbar}} = 2\pi\hbar\delta(y). \tag{10}$$

This gives a delta function which we will use to do the y' integration. We now have the following expression

$$\int\int dxdp\tilde{A}(x,p)\tilde{B}(x,p) = \int\int\int dxdydy'\langle x + \frac{y}{2}|\hat{A}(\hat{x},\hat{p})|x - \frac{y}{2}\rangle\times$$
$$\langle x + \frac{y'}{2}|\hat{B}(\hat{x},\hat{p})|x - \frac{y'}{2}\rangle\delta(y+y'). \tag{11}$$
$$= 2\pi\hbar\int\int dxdy\langle x + \frac{y}{2}|\hat{A}(\hat{x},\hat{p})|x - \frac{y}{2}\rangle\langle x - \frac{y}{2}|\hat{B}(\hat{x},\hat{p})|x + \frac{y}{2}\rangle.$$

With the following change of variables $u = x - \frac{y}{2}, v = x + \frac{y}{2}$ and $dudv = dxdy$ we get

$$\int\int dxdp\tilde{A}(x,p)\tilde{B}(x,p) = 2\pi\pi\hbar\int\int dudv\langle v|\hat{A}|u\rangle\langle u|\hat{B}|v\rangle = hTr[\hat{A}\hat{B}]. \tag{12}$$

Thus we have proven a key property about the Weyl transform, that we will use when defining the Wigner function.

1.2 The Wigner function

Before we define the Wigner function we introduce the density operator $\hat{\rho}$ [1]. This is for a pure state given by

$$\hat{\rho} = |\psi\rangle\langle\psi|. \tag{13}$$

We can express this in the position basis in the following way

$$\langle x|\hat{\rho}|x'\rangle = \psi(x)\psi^*(x'). \tag{14}$$

A property of the density matrix is, that it is normalized. That is $Tr[\hat{\rho}] = 1$. This we show by using the definition of the trace of an operator.

$$Tr[\hat{\rho}] = \sum_n \langle n|\hat{\rho}|n\rangle = \sum_n \langle n|\psi\rangle\langle\psi|n\rangle = \sum_n \langle\psi|n\rangle\langle n|\psi\rangle = \langle\psi|\psi\rangle = 1. \tag{15}$$

We can also get the expectation value of an operator $\hat{A}$ from $\hat{\rho}$ the following way

$$\langle A \rangle = Tr[\hat{\rho}\hat{A}] = Tr[|\psi\rangle\langle\psi|\hat{A}] = \sum_n \langle n|\psi\rangle\langle\psi|\hat{A}|n\rangle = \sum_n \langle\psi|\hat{A}|n\rangle\langle n|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle. \tag{16}$$

By using (6) we have

$$\langle A \rangle = Tr[\hat{\rho}\hat{A}] = \frac{1}{2\pi\hbar}\int\int dx dp \tilde{\rho}\tilde{A}. \tag{17}$$

We now define the Wigner function as

$$W(x,p) = \frac{\tilde{\rho}}{2\pi\hbar} = \frac{1}{2\pi\hbar}\int dy e^{-\frac{ipy}{\hbar}}\psi\left(x + \frac{y}{2}\right)\psi^*\left(x - \frac{y}{2}\right). \tag{18}$$

We can now see that we can write the expectation value on an operator $\hat{A}$ as

$$\langle A \rangle = \int\int dx dp W(x,p)\tilde{A}(x,p). \tag{19}$$

The expectation value is obtained through the average of a physical quantity represented by $\tilde{A}(x,p)$ over phase space with quasi-probability density W(x,p) characterizing the state. The expectation values of x and p are now simply given by

$$\langle x \rangle = \int\int dx dp W(x,p)x, \tag{20}$$

$$\langle p \rangle = \int\int dx dp W(x,p)p. \tag{21}$$

In general to find the expectation value of an operator from the Wigner function, one has to consider the weyl transform of said operator. Suppose we have an operator $\hat{A}(\hat{x})$ that only depends on $\hat{x}$. The Weyl transform for an operator of this form is simple. From (5) we have

$$\tilde{A} = \int dy e^{\frac{-iyp}{\hbar}}\langle x + \frac{y}{2}|\hat{A}(\hat{x})|x - \frac{y}{2}\rangle = \int dy e^{\frac{-iyp}{\hbar}}A\left(x - \frac{y}{2}\right)\delta(y) = A(x) \tag{22}$$

So the weyl transform of such an operator is simply a function A with the operator $\hat{x}$ replaced with x. This is the same for an operator $\hat{A}(\hat{p})$ since the Weyl transform can be defined in a momentun representation instead of a position representation. [1]. So for an operator $\hat{B}(\hat{p})$ that only depends on $\hat{p}$ the Weyl transform is the function $B(p)$. This extends to sums of operators that only depends on $\hat{x}$ and $\hat{p}$. Consider a Hamilton operator $\hat{H}(\hat{x},\hat{p}) = \hat{T}(\hat{p}) + \hat{U}(\hat{x})$. This operator will have the Weyl transform $H(x,p) = T(p) + U(x)$. So from this result we can determine the expectation value from this Hamilton operator.

$$\langle H \rangle = \int\int dy dp W(x,p)H \tag{23}$$

If one has an operator that is not a sum of operators only depending on $\hat{x}$ and $\hat{p}$ but have terms that depend on both $\hat{x}$ and $\hat{p}$ the Weyl transform is not easy to do. [1]

A feature of the Wigner function is that it is normalized in x,p space. This is easily seen by the following calculation.

$$\int\int dx dp W(x,p)\tilde{1} = \int\int dx dp W(x,p) = Tr[\hat{\rho}] = 1 \tag{24}$$

This follows from the weyl transform of $\hat{1}$ is 1.

$$\tilde{1} = \int dy e^{\frac{-ipy}{\hbar}} \langle x + \frac{y}{2} | \hat{1} | x - \frac{y}{2} \rangle = \int dy e^{\frac{-ipy}{\hbar}} \delta \left(x + \frac{y}{2} - \left(x - \frac{y}{2} \right) \right) = 1. \tag{25}$$

So the Wigner function is normalized in phase space. From the Wigner function it is now straightforward to obtain the probability distributions in x and p by simply integration over either x or p. If we integrate the Wigner function over p we get

$$\frac{1}{2\pi\hbar} \int dp \int dy e^{-\frac{ipy}{\hbar}} \psi \left(x + \frac{y}{2} \right) \psi^* \left(x - \frac{y}{2} \right) = \int dy \psi \left(x + \frac{y}{2} \right) \psi^* \left(x - \frac{y}{2} \right) \delta(y) = \psi^*(x)\psi(x), \tag{26}$$

where we have used that

$$\int dp e^{-\frac{ipy}{\hbar}} = h\delta(y). \tag{27}$$

The same can be done for the probability distribution in momentum, by integrating over x instead of p.

Another feature of the Wigner function is that it is always real. This can bee seen by taking the complex conjugate of (18) and changing integration variable from y to - y.

$$W(x,p)^* = \frac{1}{2\pi\hbar} \int dy e^{\frac{ipy}{\hbar}} \psi^* \left(x + \frac{y}{2} \right) \psi \left(x - \frac{y}{2} \right), \tag{28}$$

now changing variable from y to -y in the integration we see that we recover (18) again. A feature that distinguishes the Wigner probability distribution from a classical probability distribution, is the fact that the Wigner function can take on negative values. To see this we first consider two density operators $\hat{\rho}_a$ and $\hat{\rho}_b$ each with an associated state ψ_a and ψ_b. We can write the following expression

$$Tr[\hat{\rho}_a \hat{\rho}_b] = |\langle \psi_a | \psi_b \rangle|^2. \tag{29}$$

This we now Weyl transform and get

$$Tr[\hat{\rho}_a \hat{\rho}_b] = \frac{1}{2\pi\hbar} \int \int dx dp \tilde{\rho}_a \tilde{\rho}_b = \int \int dx dp W_a(x,p) W_b(x,p) = |\langle \psi_a | \psi_b \rangle|^2. \tag{30}$$

We now only consider orthogonal states that fufill $\langle \psi_a | \psi_b \rangle = 0$. We now have that

$$\int \int dx dp W_a(x,p) W_b(x,p) = 0. \tag{31}$$

This can only be true if the Wigner function is negative in regions of phase space. This is very different from the classical case and shows us, that the Wigner function does not represent a physical property. Only the integral of the Wigner function over either x or p has physical meaning. Though we cannot think of the Wigner function in the same way we think of a classical probability distribution, we can still think of it as a mathematical object, that will help us to calculate physical observables, much like the way we think of the wavefunction in the Schrödinger picture.

Another property of a Wigner function is that it must fufill $|W(x,p)| \leq \frac{1}{\pi\hbar}$ This follows from the fact that we can rewrite the definition of the Wigner function (58), as a product of two wave functions, the following way.

$$W(x,p) = \frac{1}{\pi\hbar} \int dy \psi_1(y) \psi_2^*(y) \tag{32}$$

Where we have defined the following normalised wave functions $\psi_1(y) = e^{-\frac{ipy}{\hbar}}\frac{\psi(x+\frac{y}{2})}{\sqrt{2}}$ and $\psi_2(y) = \frac{\psi(x-\frac{y}{2})}{\sqrt{2}}$ and used the following relation

$$\int dy \psi \left(x - \frac{y}{2} \right) \psi^* \left(x - \frac{y}{2} \right) = 2 \int d \left(\frac{y}{2} \right) \psi \left(x - \frac{y}{2} \right) \psi^* \left(x - \frac{y}{2} \right) = 2. \tag{33}$$

From the definition of the Wigner function (18) it is clear that an even wavefunction at 0,0 will have a Wigner function that takes on the value $+\frac{1}{\pi\hbar}$. An odd wavefunction will then have a Wigner function at 0,0 with the value $-\frac{1}{\pi\hbar}$.

1.3 Time evolution of the Wigner function

In general for a stationary state we have the solution as [1] [3]

$$\psi_n(x,t) = u_n(x)e^{-\frac{iE_n t}{\hbar}}. \tag{34}$$

Where $u_n(x)$ is a real function. By looking at the definition (18) it is clear that for a stationary state the Wigner function does not explictly depend on time. The phases containing the time evolution $e^{\frac{-iEt}{\hbar}}$ will always cancel out.

It is however still possible to derive an equation that governs the time evolution of the Wigner function. This approach uses the fact that x and p will depend on t. To describe the time evolution of a given Wigner function, we simply take the derivative with respect to t and use the Schrödinger equation to eliminate the partial derivatives of the wavefunction.

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi\hbar} \int dy e^{\frac{-ipy}{\hbar}} \left[\frac{\partial \psi^*(x-\frac{y}{2})}{\partial t} \psi \left(x + \frac{y}{2} \right) + \frac{\partial \psi(x+\frac{y}{2})}{\partial t} \psi^* \left(x - \frac{y}{2} \right) \right] \tag{35}$$

$$\frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar}{2im} \frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{1}{i\hbar} U(x)\psi(x,t) \tag{36}$$

Inserting (36) into (35) we find

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi\hbar} \int dy e^{\frac{-ipy}{\hbar}} \left[\frac{\hbar}{2im} \frac{\partial^2 \psi^*(x-\frac{y}{2})}{\partial x^2} \psi \left(x + \frac{y}{2} \right) - \frac{1}{i\hbar} U \left(x - \frac{y}{2} \right) \psi^* \left(x - \frac{y}{2} \right) \psi \left(x + \frac{y}{2} \right) \right.$$
$$\left. - \frac{\hbar}{2im} \frac{\partial^2 \psi(x+\frac{y}{2})}{\partial x^2} \psi^* \left(x - \frac{y}{2} \right) + \frac{1}{i\hbar} U \left(x + \frac{y}{2} \right) \psi \left(x + \frac{y}{2} \right) \psi^* \left(x - \frac{y}{2} \right) \right]. \tag{37}$$

Rearranging (37), we can write it as

$$\frac{\partial W}{\partial t} = \frac{1}{4\pi im} \int dy e^{\frac{-ipy}{\hbar}} \left[\frac{\partial^2 \psi^*(x-\frac{y}{2})}{\partial x^2} \psi \left(x + \frac{y}{2} \right) - \frac{\partial^2 \psi(x+\frac{y}{2})}{\partial x^2} \psi^* \left(x - \frac{y}{2} \right) \right]$$
$$+ \frac{2\pi}{i\hbar^2} \int dy e^{\frac{-ipy}{\hbar}} \left[U \left(x + \frac{y}{2} \right) - U \left(x - \frac{y}{2} \right) \right] \psi \left(x + \frac{y}{2} \right) \psi^* \left(x - \frac{y}{2} \right). \tag{38}$$

Defining

$$\frac{\partial W_T}{\partial t} = \frac{1}{4\pi im} \int dy e^{\frac{-ipy}{\hbar}} \left[\frac{\partial^2 \psi^*(x-\frac{y}{2})}{\partial x^2} \psi \left(x + \frac{y}{2} \right) - \frac{\partial^2 \psi(x+\frac{y}{2})}{\partial x^2} \psi^* \left(x - \frac{y}{2} \right) \right] \tag{39}$$

and

$$\frac{\partial W_U}{\partial t} = \frac{2\pi}{i\hbar^2} \int dy e^{\frac{-ipy}{\hbar}} \left[U\left(x + \frac{y}{2}\right) - U\left(x - \frac{y}{2}\right) \right] \psi\left(x + \frac{y}{2}\right) \psi^*\left(x - \frac{y}{2}\right) \tag{40}$$

We see that the time evolution of the Wigner function can be split into two parts. One concerning the kinetic energy and the other part concerning the potential energy. We therefore arrive at the following equation.

$$\frac{\partial W}{\partial t} = \frac{\partial W_T}{\partial t} + \frac{\partial W_U}{\partial t} \tag{41}$$

We will evaluate these integrals separately begining with the one for $\frac{\partial W_T}{\partial t}$ We notice that we can write the first part of the integral in (39) as

$$\int dy e^{\frac{-ipy}{\hbar}} \frac{\partial^2 \psi^*(x - \frac{y}{2})}{\partial x^2} \psi\left(x + \frac{y}{2}\right) = -2 \int dy e^{\frac{-ipy}{\hbar}} \frac{\partial^2 \psi^*\left(x - \frac{y}{2}\right)}{\partial y \partial x} \psi\left(x + \frac{y}{2}\right). \tag{42}$$

(42) can be integrated by parts to yield

$$\frac{-2ip}{\hbar} \int dy e^{\frac{-ipy}{\hbar}} \frac{\partial \psi^*(x - \frac{y}{2})}{\partial x} \psi\left(x + \frac{y}{2}\right) + \int dy e^{\frac{-ipy}{\hbar}} \frac{\partial \psi^*(x - \frac{y}{2})}{\partial x} \frac{\partial \psi(x + \frac{y}{2})}{\partial x}. \tag{43}$$

We now repeat the above procedure with the second part of the integral in (39).

$$-\int dy e^{\frac{-ipy}{\hbar}} \frac{\partial^2 \psi(x + \frac{y}{2})}{\partial x^2} \psi^*\left(x - \frac{y}{2}\right) = -2 \int dy e^{\frac{-ipy}{\hbar}} \frac{\partial^2 \psi(x + \frac{y}{2})}{\partial y \partial x} \psi^*\left(x - \frac{y}{2}\right). \tag{44}$$

(44) can be integrated by parts to yield

$$-\frac{2ip}{\hbar} \int dy e^{\frac{-ipy}{\hbar}} \frac{\partial \psi(x + \frac{y}{2})}{\partial x} \psi^*\left(x - \frac{y}{2}\right) - \int dy e^{\frac{-ipy}{\hbar}} \frac{\partial \psi(x + \frac{y}{2})}{\partial x} \frac{\partial^* \psi(x - \frac{y}{2})}{\partial x}. \tag{45}$$

Inserting (44) and (45) back into (39) we get the following result

$$\begin{aligned}
\frac{\partial W_T}{\partial t} &= \frac{1}{4\pi i m} \frac{-2ip}{\hbar} \int dy e^{\frac{-ipy}{\hbar}} \left[\frac{\partial \psi(x + \frac{y}{2})}{\partial x} \psi^*\left(x - \frac{y}{2}\right) + \frac{\partial \psi^*(x - \frac{y}{2})}{\partial x} \psi\left(x + \frac{y}{2}\right) \right] \\
&= -\frac{p}{m 2\pi\hbar} \frac{\partial}{\partial x} \int dy e^{\frac{-ipy}{\hbar}} \psi\left(x + \frac{y}{2}\right) \psi^*\left(x - \frac{y}{2}\right) \\
&= -\frac{p}{m} \frac{\partial W}{\partial x}.
\end{aligned} \tag{46}$$

Where the last equality follows from the definition of the Wigner function. The part concerning $\frac{\partial W_U}{\partial t}$ is handled as follows. Assume the potential $U(x)$ can be expanded in a power series in x we get

$$\begin{aligned}
U\left(x + \frac{y}{2}\right) - U\left(x - \frac{y}{2}\right) &= \sum_n \frac{1}{n!} \frac{\partial^n U(x)}{\partial x^n} \left[\left(\frac{-1}{2y}\right)^n - \left(\frac{-1}{2y}\right)^n \right] \\
&= \frac{1}{(2s + 1)!} \left(\frac{1}{2}\right)^{2s} \frac{\partial^{2s+1} U(x)}{\partial x^{2s+1}} y^{2s+1}.
\end{aligned} \tag{47}$$

If we insert (47) into (40) we get the final expression for $\frac{\partial W_U}{\partial t}$

$$\frac{\partial W_U}{\partial t} = \sum_{s=0}^{\infty} (-\hbar^2)^s \frac{1}{(2s + 1)!} \left(\frac{1}{2}\right)^{2s} \frac{\partial^{2s+1} U(x)}{\partial x^{2s+1}} \frac{\partial^{2s+1} W(x, p)}{\partial p^{2s+1}}. \tag{48}$$

So we end up with the following expressions for the time evolution of the Wigner function

$$\frac{\partial W}{\partial t} = \frac{-p}{m}\frac{\partial W(x,p)}{\partial x} + \sum_{s=0}^{\infty}(-\hbar^2)^s\frac{1}{(2s+1)!}\left(\frac{1}{2}\right)^{2s}\frac{\partial^{2s+1}U(x)}{\partial x^{2s+1}}\frac{\partial^{2s+1}W(x,p)}{\partial p^{2s+1}}. \tag{49}$$

If we in the expansion in (47) neglect derivatives of higher order than second order (as is the case for a harmonic oscillator), then we find

$$\frac{\partial W(x,p)}{\partial t} = \frac{-p}{m}\frac{\partial W(x,p)}{\partial x} + \frac{\partial U(x)}{\partial x}\frac{\partial W(x,p)}{\partial p}. \tag{50}$$

An important feature of the time evolution in this limit ($\hbar \to 0$) is that (50) is classical. There is no $\hbar$ in the equation. This equation is in fact the classical Liouville equation. This equation can be expressed by using possion brackets

$$\frac{\partial W(x,p)}{\partial t} = -\{W(x,p),H\} = -\frac{\partial W(x,p)}{\partial x}\frac{\partial H}{\partial p} + \frac{\partial W(x,p)}{\partial p}\cdot\frac{\partial H}{\partial x} \tag{51}$$

Where the possion bracket is defined as

$$\{f,g\} = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right). \tag{52}$$

Given the Hamilton $H = \frac{P^2}{2m} + U(x)$ it is clear that (51) is equal to (50). The important thing we notice at this point is, the time does not appear explictly in the Wigner functio $W(x,p)$ for a stationary state. Instead the time t and the time evolution lies in the coordinates that obey Hamilton's equations. So in this classical case one will see the Wigner function moving in phase space as a classical probability distrubition would in classical physics under the influence of the potential U(x). Every point of the Wigner function follows a trajectory given by the Liouville equation, so the Wigner function manintains it's shape as it evolves in phase space. Thus if the motion of the phase space distrubition is known for the classical analogue to the quantum problem, all one has to do is to require that every point of the Wigner function move accordingly to this. This of course is only in the limit where derivatives of higher order than second of the potential vanish. That is the limit where $\hbar \to 0$. In cases where this is not true the evolution of the Wigner Function will not equal the classical Liouville equation. Instead there will be quantum corrections that will distinguish it from the classical case.

An interesting feature of (41) is that it is equivalent to solving the Schrödinger equation. A way to see this is to show that the relation between Wigner function and the wave equation is one to one up to a constant phase. Consider the definition of the Wigner function (18). Multiply by $e^{\frac{ipx'}{\hbar}}$ and integrate over p. We then find that

$$\int dp W(x,p)e^{\frac{ipx'}{\hbar}} = \psi^*\left(x - \frac{x'}{2}\right).\psi\left(x + \frac{x'}{2}\right). \tag{53}$$

A simple change of variables to $x = \frac{x}{2}$ and $x' = x$, and we now have that

$$\psi(x) = \frac{1}{\psi^*(0)}\int dp W\left(\frac{x}{2},p\right)e^{\frac{ipx}{\hbar}}. \tag{54}$$

This means that instead of solving the Schrödinger's equation, one could in principle find the Wigner function by solving (41) and from that recover the wavefunctions. However, there are some issues with this approach. In pratice, it is not necessarily possible to obtain the Wigner function from (41), because if the given system has a potential with derivatives of higher order than 2, the equation will be a partial differential

equation of third or higher order. Anoter issue with this approach is, that not all functions of x and p, that one may find this way, are admissable Wigner functions [2].

2 Wigner functions of the Harmonic oscillator

In this section we will demonstrate how to find a Wigner function and its time evolution for a well known quantum mechanical problem, the Harmonic Oscillator. Given an ordinary harmonic oscillator, we seek to calculate the Wigner functions of its groundstate and first excited state. We begin with the well known Hamiltonian and wavefunctions for the system

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2, \tag{55}$$

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{\sqrt[4]{\pi}\sqrt{\ell}} e^{-\frac{x^2}{2\ell^2}}, \tag{56}$$

$$\psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{4\sqrt{\pi}}\sqrt{\frac{2}{\ell}} e^{-\frac{x^2}{2\ell^2}}. \tag{57}$$

Where $\ell^2 = \frac{\hbar}{m\omega}$ is the characteristic oscillator length for the system. To calculate the Wigner function, or Wigner distrubition, we simply plug in our given states in the definition and calculate it.

$$W(x,p) = \frac{1}{2\pi\hbar} \int dy e^{-\frac{ipy}{\hbar}} \psi\left(x + \frac{y}{2}\right) \psi^*\left(x - \frac{y}{2}\right), . \tag{58}$$

$$\begin{aligned} W_0(x,p) &= \frac{1}{2\pi\hbar} \int dy e^{-\frac{ipy}{\hbar}} \frac{1}{\sqrt{\pi}\ell} e^{-\frac{(x+\frac{y}{2})^2}{2\ell^2}} e^{-\frac{(x-\frac{y}{2})^2}{2\ell^2}} \\ &= \frac{1}{\sqrt{\pi}\ell h} \int dy e^{-\frac{ipy}{\hbar}} e^{-\frac{x^2}{\ell^2}-\frac{y^2}{4\ell^2}} = \frac{1}{\pi\hbar} e^{-\frac{x^2}{\ell^2}-\frac{\ell^2 p^2}{\hbar^2}}, \end{aligned} \tag{59}$$

$$W_1(x,p) = \frac{1}{\pi\hbar} \int dy e^{-\frac{ipy}{\hbar}} \frac{2x^2}{\sqrt{\pi}\ell^3} e^{-\frac{x^2}{\ell^2}-\frac{y^2}{4\ell^2}} = \frac{1}{\pi\hbar} \left(-1 + \frac{2\ell^2 p^2}{\hbar^2} + \frac{2x^2}{\ell^2}\right) e^{-\frac{x^2}{\ell^2}-\frac{\ell^2 p^2}{\hbar^2}}. \tag{60}$$

We note that our calculated Wigner functions fufill the important relation for Wigner functons which states that $\frac{1}{\hbar} \geq |W(x,p)|$. We see that the gound state reaches $+\frac{2}{\hbar}$ at the point 0,0 since the wavefunctions of the ground state is even. The first excited state has odd wavefunctions and at the point 0,0 the Wigner function reaches the value $-\frac{2}{\hbar}$.

2.1 Time evolution of the Harmonic oscillator

From the section where we derived the time evolution, we know how to calculate the time evolution of the Wigner function for a harmonic oscillator. This has a symmetric quadratic potential so the time evoluton is governed by equation (50). So given we can describe the motion classically we turn to the time evolution of a classical harmonic oscillator. This is given by the known solutions

$$x_0 = x\cos(\omega t) - \frac{p}{m\omega}\sin(\omega t), \tag{61}$$

$$p_0 = p\cos(\omega t) + m\omega x\sin(\omega t). \tag{62}$$

Now the time evolution of the Wigner function lies in the coordinates, so the Wigner function maintains its shape, but moves around in elliptical orbits in phase space. This means that

$$W(x, p, t) = W\left(x\cos(\omega t) - \frac{p}{m\omega}\sin(\omega t), p\cos(\omega t) + m\omega x\sin(\omega t), 0\right). \tag{63}$$

Going back to the Wigner function we found for the grund state, we first translate it by b in the x direction. We do this because, as we previously stated, a stationary state does not have a time evolution. By translating the function by b we will be able to observe it's motion in phase space.

$$W_0(x, p, o) = \frac{1}{\pi\hbar}e^{-\frac{(x-b)^2}{\ell^2} - \frac{\ell^2 p^2}{\hbar^2}}. \tag{64}$$

Now inserting the time evolution of the coordinates we find the time evolution of the Wigner function to be

$$\begin{aligned} W_0(x, p, t) = \frac{1}{\pi\hbar}exp[-\frac{1}{\ell^2}(x\cos(\omega t) - \frac{p}{m\omega}\sin(\omega t))^2 \\ -\frac{\ell^2}{\hbar^2}(p\cos(\omega t) + m\omega x\sin(\omega t))^2]. \end{aligned} \tag{65}$$

This we can rewrite if we use that $\ell^2 = \frac{\hbar}{m\omega}$

$$W_0(x, p, t) = \frac{1}{\pi\hbar}exp[-\frac{\ell^2}{\hbar^2}(p + \frac{b\hbar}{\ell^2}\sin(\omega t))^2 - \frac{1}{\ell^2}(x - b\cos(\omega t))^2]. \tag{66}$$

The above procedure can also be done for the first excited state (60). If we in this expression insert (61) and (62) we find the folowing time evolution of (60) by again using that $\ell^2 = \frac{\hbar}{m\omega}$

$$\begin{aligned} W_1(x, p, t) = \frac{1}{\pi\hbar}\left(-1 + \frac{2p^2\ell^2}{\hbar^2} + \frac{2x^2}{\ell^2} + \frac{2b^2}{\ell^2} - \frac{4bx}{\ell^2}\cos(\omega t) + \frac{4pb}{\hbar}\sin(\omega t)\right) \\ e^{-\frac{-x^2}{\ell^2} - \frac{p^2\ell^2}{\hbar^2} - \frac{b^2}{\ell^2} + \frac{2bx}{\ell^2}\cos(\omega t) - \frac{2pb}{\hbar}\sin(\omega t)}. \end{aligned} \tag{67}$$

Below we have plotted the Wigner function of the ground state and first excited state at different times. We see that it maintains its shape and moves around in an elliptical trajectory as we have predicted.
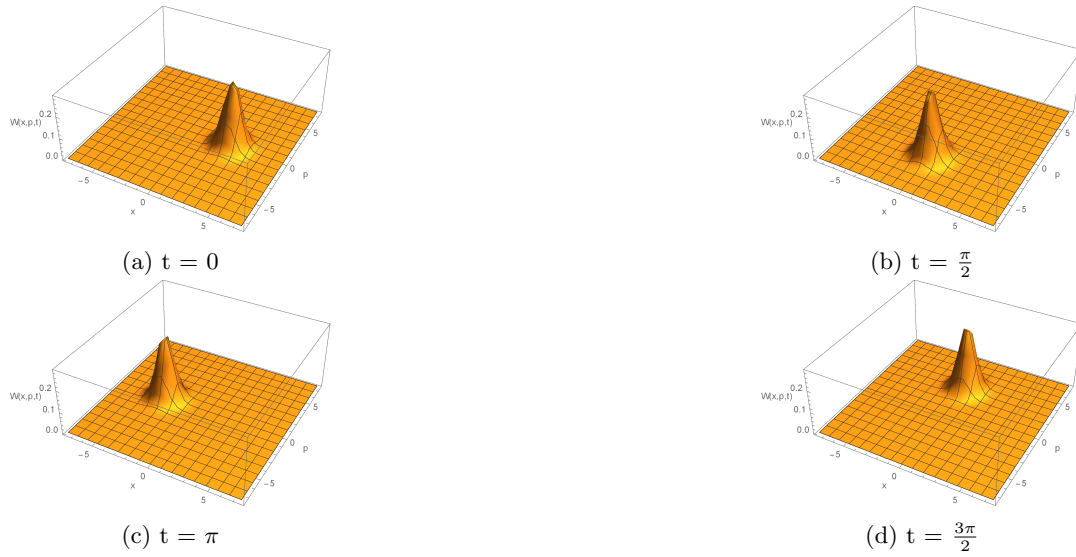
(a) t = 0

(b) t = $\frac{\pi}{2}$

(c) t = π

(d) t = $\frac{3\pi}{2}$

Figure 1: Time evolution for the ground state of the Harmonic oscillator. We have set the parameters ℓ and $\hbar$ to 1

(a) t = 0

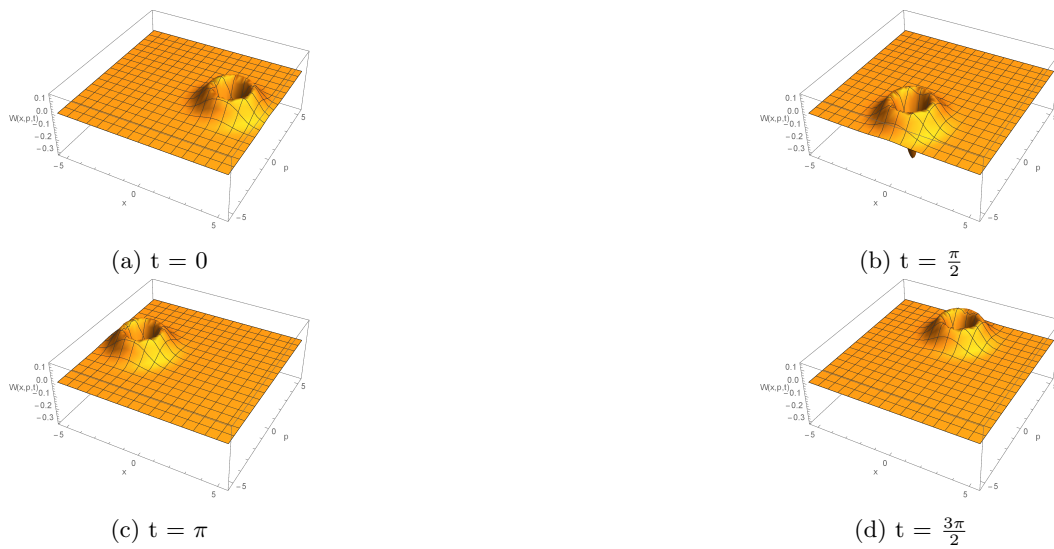(b) t = $\frac{\pi}{2}$

(c) t = π

(d) t = $\frac{3\pi}{2}$

Figure 2: Time evolution for the first excited state of the Harmonic oscillator. We have set the parameters ℓ and $\hbar$ to 1

We can now find the expectation values of x and p from these Wigner functions. For the ground state they are given as

$$
\begin{aligned}
\langle x \rangle &= \int \int dx dp W_0(x,p)x = \int \int dx dp \frac{1}{\pi \hbar} exp[-\frac{\ell^2}{\hbar^2}(p + \frac{b\hbar}{\ell^2}\sin(\omega t))^2 - \frac{1}{\ell^2}(x - b\cos(\omega t))^2]x \\
&= b\cos(\omega t),
\end{aligned}
\tag{68}
$$

and for p we have

$$
\begin{aligned}
\langle p \rangle &= \int \int dx dp W_0(x,p)p = \int \int dx dp \frac{1}{\pi \hbar} exp[-\frac{\ell^2}{\hbar^2}(p + \frac{b\hbar}{\ell^2}\sin(\omega t))^2 - \frac{1}{\ell^2}(x - b\cos(\omega t))^2]p \\
&= -\frac{b\sin(\omega t)}{\omega^2 \hbar}.
\end{aligned}
\tag{69}
$$

These expectation values is what we expect from a shifted harmonic oscillator. The expectation value for x in a ground state of a harmonic oscillator is known to be 0 because it's a gaussian function centered around 0. If we shift it so it is centered around b we expect b to be the expectation value. This is what we see here.

We can also find the expectation value for the first excited state of the energy to check our method yields the correct results. We expect this result to be $\frac{3\hbar\omega}{2}$. This is for the unshifted case. From the Wigner function (60) we find that

$$
\langle H \rangle = \int \int dy dp W_1(x,p)H = \int \int dy dp W_1(x,p)\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right) = \frac{3\hbar\omega}{2}.
\tag{70}
$$

So we see that the found Wigner function gives the correct values. Now that we know our Wigner function yields the correct results we can use it to find the expectation value of the energy in the shifted case. Again we look at the fist excited state and we find by using (67)

$$
\langle H \rangle = \int \int dy dp W_1(x,p,t)H = \int \int dy dp W_1(x,p,t)\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right) = \frac{1}{2}\omega\left(b^2 m\omega + 3\hbar\right)
\tag{71}
$$

3   The Gaussian wavepacket

Lets consider a free particle with an initial wave function that is a gaussian wavepacket $\Psi(x,0) = Ae^{-ax^2}$. It's wave function for time t is known to be [4] [1]

$$
\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{e^{\frac{-ax^2}{1+\frac{it}{\tau}}}}{\sqrt{1+\frac{it}{\tau}}},
\tag{72}
$$

where $\tau = \frac{m}{2\hbar a}$ is the characteristic time-scale for the system. We now calculate its Wigner function and note that the wave function is not a stationary state. So the time dependence will not cancel out when we perform the tranform, but will be carried over to the Wigner function.

---

[1]See the appendix for the details in how to find this wavefunction

$$W(x,p,t) = \frac{1}{2\pi\hbar} \int dy e^{\frac{-ipy}{\hbar}} \Psi^* \left(x - \frac{y}{2}, t \right) \Psi \left(x + \frac{y}{2}, t \right)$$

$$= \frac{1}{2\pi\hbar} \left(\frac{2a}{\pi} \right)^{\frac{1}{2}} \int dy e^{\frac{-ipy}{\hbar}} \frac{e^{\frac{-a\left(x - \frac{y}{2} \right)^2}{1 - \frac{it}{\tau}}}}{\sqrt{1 - \frac{it}{\tau}}} \frac{e^{\frac{-a\left(x + \frac{y}{2} \right)^2}{1 + \frac{it}{\tau}}}}{\sqrt{1 + \frac{it}{\tau}}} \qquad (73)$$

$$= \frac{1}{\pi\hbar} e^{-\frac{p^2\left(t^2 + \tau^2 \right)}{2a\tau^2\hbar^2} - 2ax^2 + \frac{2ptx}{\tau\hbar}}$$

$$= \frac{1}{\pi\hbar} e^{-\frac{2a(pt - mx)^2}{m^2} - \frac{p^2}{2a\hbar^2}}.$$

Here we have used that $\tau = \frac{m}{2\hbar a}$ to express the final result. We now see an advantage of the Wigner function. (73) appears more simple than (72) does. Here we have a relatively simple function to give us all the information about the system we need instead of a cumbersome wavefunction whose time evolution is not all that obvious.

Another way of finding this Wigner function is to used (49). Since our potential is zero we can find the Wigner function at t = 0, and then demand that the variables in the Wigner function behaves according to classical physics to obtain $W(x,p,t)$. So we begin with $\Psi(x,0) = \left(\frac{2a}{\pi} \right)^{\frac{1}{4}} e^{-ax^2}$, and from this we calculate the Wigner function

$$W(x,p,0) = \left(\frac{2a}{\pi} \right)^{\frac{1}{2}} \frac{1}{2\pi\hbar} \int dy e^{-\frac{ipy}{\hbar}} e^{-a\left(x - \frac{y}{2} \right)^2} e^{-a\left(x + \frac{y}{2} \right)^2} = \left(\frac{2a}{\pi} \right)^{\frac{1}{2}} \frac{1}{2\pi\hbar} e^{-2ax^2} \int dy e^{-\frac{ipy}{\hbar}} e^{-\frac{ay^2}{2}}$$

$$= \left(\frac{2a}{\pi} \right)^{\frac{1}{2}} \frac{1}{2\pi\hbar} e^{-2ax^2} \sqrt{\frac{2\pi}{a}} e^{-\frac{p^2}{2a\hbar^2}} = \frac{1}{\pi\hbar} e^{-\frac{2ax^2}{m^2} - \frac{p^2}{2a\hbar^2}}. \qquad (74)$$

Where we have used that the Fourier transform $\int dy e^{-\frac{ipy}{\hbar}} e^{-\frac{ay^2}{2}} = \sqrt{\frac{2\pi}{a}} e^{-\frac{p^2}{2a\hbar^2}}$. Now we have obtained the Wigner fuction at t = 0. We know from the equation of motion for the Wigner function (49), that our system's equation of motion is identical to the Louville equation. This means the variables of the Wigner function must obey classical equations of motion. So we change the variable from x to $\frac{pt}{m} - x$ or $pt - mx$ which is the equation of motion for a free particle in classical physics. And we then see that we have obtained the same result as we got with a direct calculation using the definition of the Wigner function

$$W(x,p,t) = \frac{1}{\pi\hbar} e^{-\frac{2a(pt - xm)^2}{m^2} - \frac{p^2}{2a\hbar^2}}. \qquad (75)$$

Below we have plotted this Wigner function.

(a) t = 0

(b) t = π

(c) t = 2π

(d) t = 3π
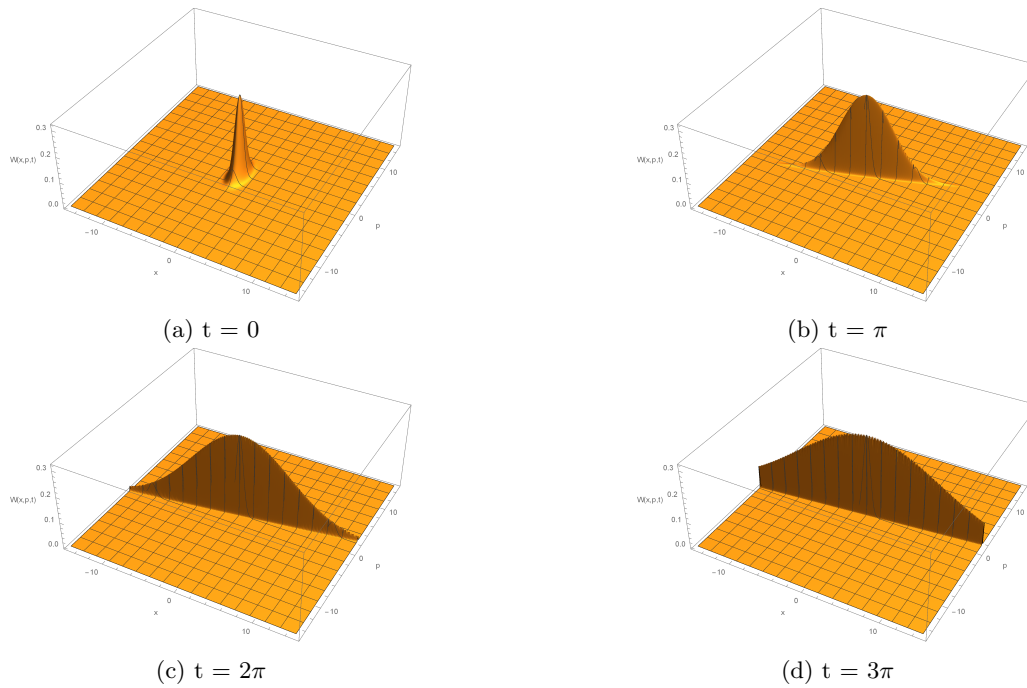
Figure 3: Time evolution of the Wigner function for a free particle. We have set a = 1, m = 1 and $\hbar = 1$

We see the Wigner function spreads out in x with time. In the Schrödinger picture we know that a wave packet spreads out in x with time. So a wavepacket that starts off to be localized spreads out and becomes delocalized. It is interesting to see this feature carried over to the Wigner function. In the Schrödinger picture when t → ∞ we have that

$$|\Psi(x,t)|^2 = \sqrt{2}m\sqrt{\frac{a}{4\pi a^2 t^2 \hbar^2 + \pi m^2}} e^{-\frac{2am^2x^2}{4a^2t^2\hbar^2+m^2}} \tag{76}$$

goes to zero. This simply states that, the probability of finding the particle at every point becomes equally, with probability zero since the wavefunction is normalized. When we look at (73) we expect that the Wigner function also goes to zero as t → ∞ which it does for a given p.

We notice that the Wigner function does not run parallel to either x or p in the xy plane. This slobe can be explained due to the classical result for a free particle motion that $x = \frac{pt}{m}$. Since that the Wigner function evolves accordingly to clssical equations of motion it must lie on this curve. Another way to explain the slope is the uncertainty principle. As already explained the more the Wigner function spreads out in x the more narrow it becomes in p.

A thing we notice about the plots of the Wigner function is that it undergoes a squeezing as t increases. As the function spreads out in x and becomes wide in this coordinate, it becomes narrow in p around 0. This is a direct consequence of the uncertainty principle. As the wavefunction spreads out over position and becomes delocalized, it gains a very well defined momentum. In this case the momentum expectation value is 0 as can be seen form the following calculation

$$\langle p \rangle = \int\int dpdx W(x,p,t)p = \int\int dpdx \frac{1}{\pi\hbar} e^{-\frac{2a(pt-xm)^2}{m^2}-\frac{p^2}{2a\hbar^2}} p = 0. \tag{77}$$

A similar calculation can be done for the expectation value for x of the system.

$$\langle x \rangle = \int \int dp dx W(x,p,t) x = \int \int dp dx \frac{1}{\pi \hbar} e^{-\frac{2a(pt-xm)^2}{m^2} - \frac{p^2}{2a\hbar^2}} x = 0. \tag{78}$$

These results are also known from the problem in the Schrödinger picture. [4]

4 The Driven Quantum Oscillator

[2] We now turn to study the behaviour of the the harmonic oscillator when exposed to a driving perturbation, $S(t)$. We will follow the method in [5] in this section and the next. The Schrödinger equation for such a system is given by

$$i\hbar \dot{\Psi}(x,t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega_0^2 x^2 - xS(t) \right] \Psi(x,t). \tag{79}$$

This can be solved by bringing it on a form that resembles the unperturbed harmonic oscillator, and hiding the perturbation behind transformations made along the way. This is done using unitary transformations. To get there, we will start by making a change of variables,

$$x \to y = x - \zeta(t), \tag{80}$$

so that we can write the Schrödinger equation in the new coordinate y,

$$i\hbar \dot{\Psi}(y,t) = \left[i\hbar \dot{\zeta}(t) \frac{\partial}{\partial y} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m\omega_0^2 (y+\zeta(t))^2 - (y+\zeta(t))S(t) \right] \Psi(y,t). \tag{81}$$

From here, it is useful to perform the following unitary transformation,

$$\Psi(y,t) = e^{im\dot{\zeta}y/\hbar} \phi(y,t), \tag{82}$$

where $\zeta(t)$ obeys the newtonian equation of motion

$$m\ddot{\zeta} + m\omega_0^2 \zeta = S(t). \tag{83}$$

Inserting this into the above and calculating the LHS and RHS seperately, we get for the LHS:

$$i\hbar \dot{\Psi}(y,t) = e^{im\dot{\zeta}y/\hbar} \left[i\hbar \dot{\phi} - my\ddot{\zeta}\phi \right]. \tag{84}$$

Using (83), we eliminate the double time derivative and get the following expression for the LHS

$$i\hbar \dot{\Psi}(y,t) = e^{im\dot{\zeta}y/\hbar} \left[i\hbar \dot{\phi} - y\phi S(t) + y\phi m\omega_0^2 \zeta \right]. \tag{85}$$

For the RHS, we get

$$\left[i\hbar \dot{\zeta}(t) \frac{\partial}{\partial y} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m\omega_0^2 (y+\zeta(t))^2 - (y+\zeta(t))S(t) \right] \Psi(y,t)$$

$$= e^{im\dot{\zeta}y/\hbar} \left[i\hbar \dot{\zeta} \frac{\partial \phi}{\partial y} - m\dot{\zeta}^2 \phi - \frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial y^2} - i\hbar \dot{\zeta} \frac{\partial \phi}{\partial y} + \frac{m}{2} \dot{\zeta}^2 \phi + \frac{m}{2} \omega_0^2 y^2 \phi + \frac{1}{2} m\omega_0^2 \zeta^2 \phi + m\omega_0^2 \zeta y\phi - yS(t)\phi - \zeta S(t)\phi \right] \tag{86}$$

Equating the RHS and LHS and reducing, we see that

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[2]This section has been done in collaboration with Mads Anders Jørgensen and hence also appears in his bachelor thesis

$$i\hbar\dot{\phi} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{m}{2}\omega_0^2 y^2 - L(\zeta, \dot{\zeta}, t)\right]\phi, \tag{87}$$

where $L = \frac{m}{2}\dot{\zeta}^2 - \frac{1}{2}m\omega_0^2\zeta^2 + \zeta S(t)$ is the lagrangian for a driven harmonic oscillator.

Introducing another unitary transformation specifically to get the lagrangian term to cancel out, we thereby end up with a form that is highly resemblant of the standard harmonic oscillator.

$$\phi(y, t) = e^{\frac{i}{\hbar}\int_0^t dt' L(\zeta, \dot{\zeta}, t')}\chi(y, t). \tag{88}$$

This transformation gives us the form of a normal, unperturbed harmonic oscillator

$$i\hbar\dot{\chi}(y, t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_0^2 y^2\right]\chi(y, t). \tag{89}$$

The energies of the harmonic oscillator are known to be $E_n = \hbar\omega_0(n + \frac{1}{2})$. Combining this with the stationary states, φ_n, of the harmonic oscillator, we get that

$$\varphi_n(y) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(y)e^{-\frac{m\omega_0}{2\hbar}y^2}, \tag{90}$$

where $H_n(x)$ are the Hermite polynomials. Given this, we have that

$$\chi_n(y, t) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(y)e^{-\frac{m\omega_0}{2\hbar}y^2 - \frac{i}{\hbar}E_n t}. \tag{91}$$

Inserting this expression into our expression for ϕ, and then that expression into the one for ψ, we get

$$\phi_n(y, t) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(y)e^{-\frac{m\omega_0}{2\hbar}y^2 - \frac{i}{\hbar}\left[E_n t - \int_0^t dt' L\right]} \tag{92}$$

$$\psi_n(y, t) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(y)e^{-\frac{m\omega_0}{2\hbar}y^2 + \frac{i}{\hbar}\left[m\dot{\zeta}(t)y - E_n t + \int_0^t dt' L\right]}. \tag{93}$$

Now, substituting our variable from y back to x

$$\psi_n(x, t) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(x - \zeta(t))e^{-\frac{m\omega_0}{2\hbar}(x - \zeta(t))^2 + \frac{i}{\hbar}\left[m\dot{\zeta}(t)(x - \zeta(t)) - E_n t + \int_0^t dt' L\right]}. \tag{94}$$

Now we have our general wavefunction for an arbitrary S(t), which appears in the lagrange function in the exponent. [5]

4.1 Periodic monochromatic driving force

Let us now try to work with a simple harmonic driving force. We set

$$S(t) = A\sin(\omega t + \theta). \tag{95}$$

Classically, this would be like placing a hamonic oscillator (with frequency ω_0) on a platform performing harmonic movement itself, with frequency ω. With this force, a solution to (83) is

$$\zeta(t) = \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)}, \tag{96}$$

where $\omega \neq \omega_0$.

In order to calculate the action, we need to first calculate the time derivative of the moving coordinate, $\dot{\zeta}$,

$$\dot{\zeta} = \frac{A\omega \cos(\omega t + \theta)}{m(\omega_0^2 - \omega^2)}. \tag{97}$$

We now calculate the action as it appears in our expression for ψ,

$$\int_0^t dt' L(\zeta, \dot{\zeta}, t') = \int_0^t dt' \left[\frac{A^2\omega^2 \cos^2(\omega t' + \theta)}{2m(\omega_0^2 - \omega^2)^2} - \frac{m\omega_0^2 \left(\frac{A\sin(\omega t' + \theta)}{m(\omega_0^2 - \omega^2)}\right)^2}{2} + \frac{A^2 \sin^2(\omega t' + \theta)}{m(\omega_0^2 - \omega^2)} \right] = \\ \frac{A^2 [\omega t(\omega_0^2 - \omega^2) + (3\omega^2 - \omega_0^2) \cos(2\theta + \omega t) \sin(\omega t)]}{4m\omega(\omega_0^2 - \omega^2)^2}. \tag{98}$$

Inserting this into (94) we find the wavefunctions

$$\psi_n(x, t) = \\ \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(x - \zeta(t)) e^{-\frac{m\omega_0}{2\hbar}(x - \zeta(t))^2 + \frac{i}{\hbar} \left[m\dot{\zeta}(t)(x - \zeta(t)) - E_n t + \frac{A^2[\omega t(\omega_0^2 - \omega^2) + (3\omega^2 - \omega_0^2) \cos(2\theta + \omega t) \sin(\omega t)]}{4m\omega(\omega_0^2 - \omega^2)^2} \right]} \tag{99}$$

[5]

4.2 Wigner functions of the Driven Quantum Oscillator

To find the Wigner function and it's time evolution of the Driven harmonic oscillator, we will take a different approach from what was done in the section about the Harmonic Oscillator. Previously the Wigner function was calculated from a stationary state and then (41) was used to determine it's time dependence. In this approach we will find the Wigner function from (99) which is not a stationary state of the Harmonic oscillator since the driven Harmonic Oscillator does not have stationary states. When we look at (99) we see that it is in fact a shifted Harmonic oscillator, with the shift depending on time. To find the Wigner function and it's time dependence, we start the definition of the Wigner function and insert our wavefunction.

$$W_n(x, p, t) = \frac{1}{2\pi\hbar} \int dy\, e^{-\frac{ipy}{\hbar}} \psi_n\left(x + \frac{y}{2}\right) \psi_n^*\left(x - \frac{y}{2}\right) = \\ \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2\pi\hbar 2^n n!} \int dy\, e^{\frac{-ipy}{\hbar}} H_n\left(\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} - \frac{y}{2}\right)/\ell\right) e^{\frac{-m\omega}{2\hbar}\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} - \frac{y}{2}\right)^2} \\ H_n\left(\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} + \frac{y}{2}\right)/\ell\right) e^{\frac{-m\omega}{2\hbar}\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} + \frac{y}{2}\right)^2} \tag{100}$$

Where again $\ell = \sqrt{\frac{\hbar}{m\omega_0}}$ is the characteristic oscillator length. This integral cannot be done analytically for arbitrary n. Only when n is specified can an expression be calculated. This is due to the Hermite polynomials in the expression. Lets first look at state of the driven oscillator with n = 0 and find it's Wigner function.

$$W_0(x,p,t) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2\pi\hbar} \int dy e^{\frac{-ipy}{\hbar}} H_0\left(\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} - \frac{y}{2}\right)/\ell\right) e^{\frac{-m\omega}{2\hbar}\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} - \frac{y}{2}\right)^2}$$

$$H_0\left(\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} + \frac{y}{2}\right)/\ell\right) e^{\frac{-m\omega}{2\hbar}\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} + \frac{y}{2}\right)^2} \tag{101}$$

$$= \frac{1}{\pi\hbar} e^{-\frac{p^2}{m\omega_0\hbar} - \frac{mx^2\omega_0}{\hbar} - \frac{A^2\omega_0\sin^2(t\omega + \theta)}{m\hbar(\omega^2 - \omega_0^2)^2} - \frac{2Ax\omega_0\sin(t\omega + \theta)}{\hbar(\omega^2 - \omega_0^2)}} \ .$$

We see that the $W_0(x,p)$ function is nearly identical to (66). The difference an extra phase factor corresponding the timedependent shift in the oscillator coming from the driving term in the potential. We note that if we in (101) turn off the driving force. That is in the limit $A \to 0$, the Wigner function we obtain are the same as in the case of the undriven oscillator. If we look at (101) near resonance when $\omega = \omega_0$, we see that the expression goes towards zero. In a classical driven oscillator resonance is an unstable point for the system and we see the same here. It is not possible to have a particle in a driven oscillator at resonance. We can also find the Wigner function for the n = 1 state,

$$W_1(x,p,t) = \frac{1}{2}\left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2\pi\hbar} \int dy e^{\frac{-ipy}{\hbar}} H_1\left(\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} - \frac{y}{2}\right)/\ell\right) e^{\frac{-m\omega}{2\hbar}\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} - \frac{y}{2}\right)^2}$$

$$H_1\left(\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} + \frac{y}{2}\right)/\ell\right) e^{\frac{-m\omega}{2\hbar}\left(x - \frac{A\sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)} + \frac{y}{2}\right)^2} \tag{102}$$

$$= \frac{1}{\pi m\omega_0\hbar^2(\omega^2 - \omega_0^2)^2} e^{-\frac{p^2}{m\omega_0\hbar} - \frac{mx^2\omega_0}{\hbar} - \frac{A^2\omega_0\sin^2(t\omega + \theta)}{m\hbar(\omega^2 - \omega_0^2)^2} - \frac{2Ax\omega_0\sin(t\omega + \theta)}{\hbar(\omega^2 - \omega_0^2)}} \times$$

$$\left(A^2\omega_0^2 + A\omega_0^2(4mx(\omega - \omega_0)(\omega + \omega_0)\sin(t\omega + \theta) - A\cos(2(t\omega + \theta))) + (\omega - \omega_0)^2(\omega + \omega_0)^2\left(m\omega_0\left(2mx^2\omega_0 - \hbar\right) + 2p^2\right)\right).$$

One can take the limit of this Wigner function as $\omega \to \omega$ as well and find that it tends to zero. Again we see that resonance is not a stable state for the system.

Since our Wigner functions of the driven oscillator resemble those of the undriven harmonic oscillator, expect the expectation value for p should be zero as it remains centered around this value, while the expectation value for x should be centered around it's shifted position. in this case that would be $\frac{A\sin(t\omega + \theta)}{m(\omega_0^2 - \omega^2)}$. Lets check our predictions by calculating the expectation values $\langle x \rangle$ and $\langle p \rangle$

$$\langle x \rangle = \int\int dx dp W_0(x,p,t) x = \frac{A\sin(t\omega + \theta)}{m(\omega_0^2 - \omega^2)}, \tag{103}$$

$$\langle p \rangle = \int\int dx dp W_0(x,p,t) p = 0. \tag{104}$$

So we see that the found Wigner function gives the correct predictions. For the n=1 state we find the expectation values to be

$$\langle x \rangle = \int\int dx dp W_1(x,p,t) x = \frac{A\omega\sin(t\omega + \theta)}{m(\omega_0^2 - \omega^2)}, \tag{105}$$

$$\langle p \rangle = \int\int dx dp W_1(x,p,t) p = 0. \tag{106}$$

From these expressions it is also clear that resonance $\omega = \omega_0$ is an unstable state of the system. $\langle x \rangle$ goes to infinity which means the particle cannot be "inside" the oscillator.

One may also be interested in the expectation value for the energy $\langle H \rangle$ for $H = \frac{p^2}{2m} + \frac{m\omega_0^2 x^2}{2} - xA\sin(\omega t + \theta)$. This is also calculated using the found Wigner functions.

$$\langle H \rangle = \int \int dx dp W_0(x,p,t) H(x,p,t) = \frac{1}{2} \left(\omega_0 \hbar + \frac{A^2 \left(2\omega^2 - \omega_0^2\right) \sin^2(t\omega + \phi)}{m \left(\omega^2 - \omega_0^2\right)^2} \right). \tag{107}$$

And for the n = 1 state one finds

$$\langle H \rangle = \int \int dx dp W_1(x,p,t) H(x,p,t) =$$
$$\frac{A^2 \left(\omega_0^2 - 2\omega^2\right) \cos(2(t\omega + \phi)) + A^2 \left(2\omega^2 - \omega_0^2\right) + 6m\omega_0 \hbar \left(\omega^2 - \omega_0^2\right)^2}{4m \left(\omega^2 - \omega_0^2\right)^2}. \tag{108}$$

Again we see why resonance is an unstable state for the system. Near resonance the energies become infinite which is unphysical.

The Wigner function for the n=0 adn n= 1 state is plotted below with the following parameters:
A $= 1$, $\hbar = 1$, m $= 1$, $\theta = 1$, $\omega = 2.05$, $\omega_0 = 2$.

(a) t $= 0$

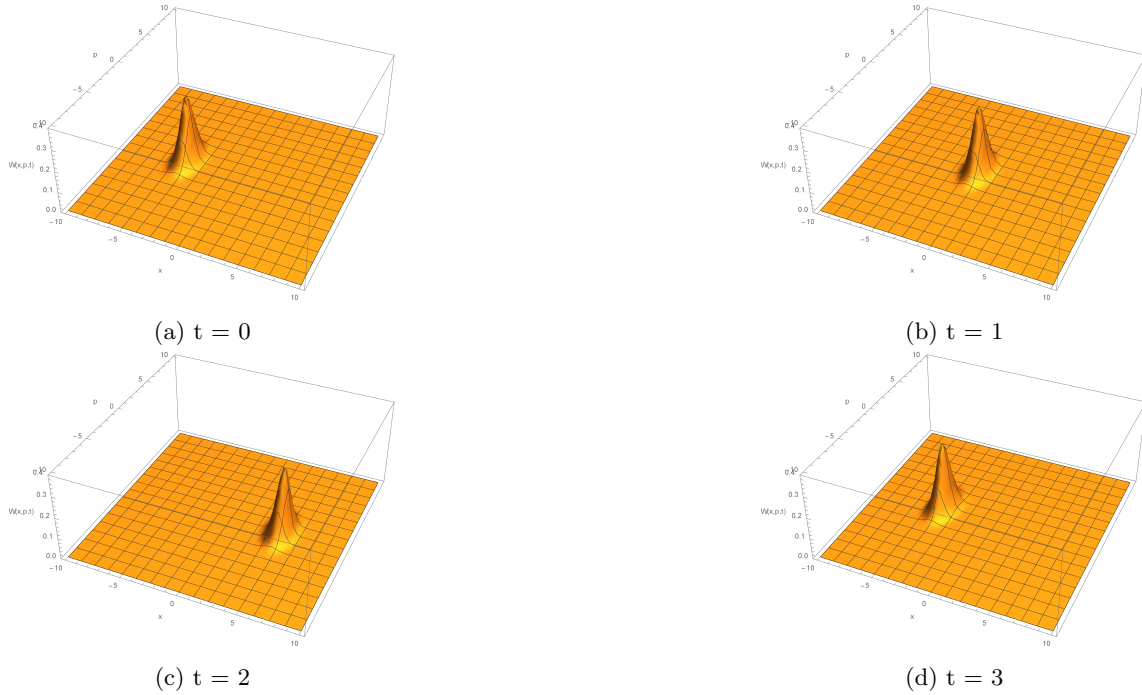(b) t $= 1$

(c) t $= 2$

(d) t $= 3$

Figure 4: The n $= 0$ Wigner function of the Driven harmonic oscillator near resonance at various times. We notice how it corresponds to the Wigner function of the ground state of the undriven harmonic oscillator now just shifted along the x axis as a function of time.

(a) t = 0

(b) t = 1

(c) t = 2

(d) t = 3

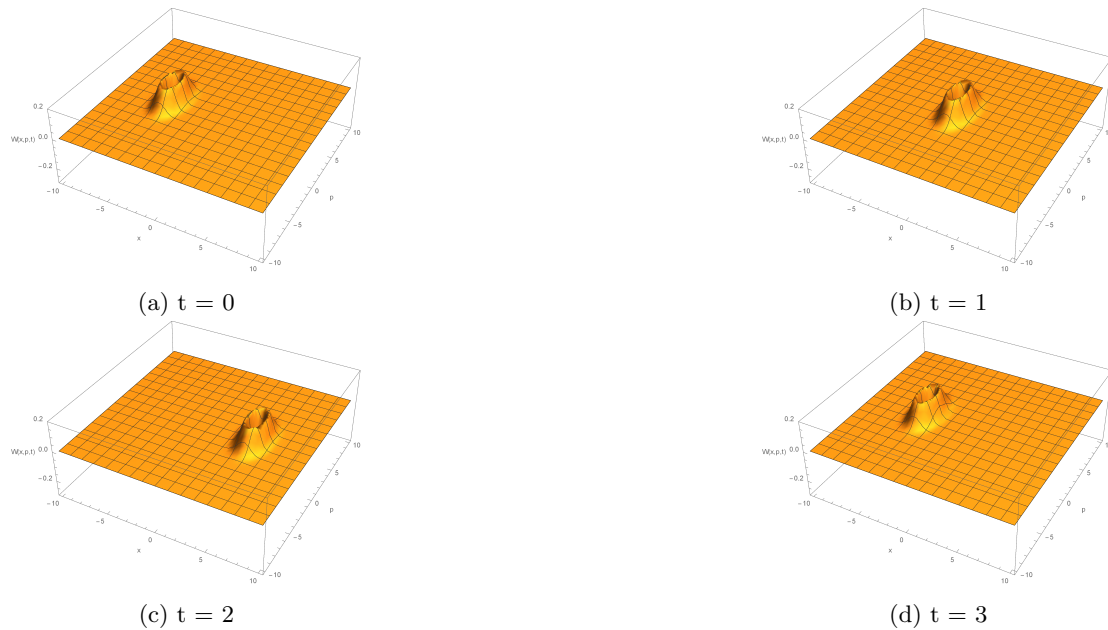Figure 5: The n = 1 Wigner function of the Driven harmonic oscillator near resonance at various times. We notice how it corresponds to the Wigner function of the first excited state of the undriven harmonic oscillator now just shifted along the x axis as a function of time.

From these plots it is clear, that our wigner functions form are the same as in the case of the undriven oscillator. Now we only have a time dependent shift in the x coordinate while the distribution along the p axis remains unchanged.

5   Conclusion

As we have shown the Wigner function offers a formalism equivalent to the formalism offered by the Schrödinger picture of Quantum mechanics. Quantum problems can be solved and their results obtained through a single mathematical object. The Wigner function has a strength in it's ability to visualize a system and it's development in time. In some cases the Wigner function has a more simple form than the wavefunction. Another strength of the Wigner formalism is that in certain limits it behaves as a probability distribution does in classical physics. This allows for a connection between the realm of classical physics and Quantum mechanics. A prospective on this paper could be to include spin in the Wigner formalism. One could ask how the formalism will look if spin is included, and if the same result concerning the time evolution can be obtained.

A   Appendix

Following [4] we will show how to obtain (72). We begin with our initial wavefunction $\Psi(x,0) = Ae^{-ax^2}$ which has to be normalized.

$$1 = |A|^2 \int dx e^{-2ax^2} = |A|^2 \sqrt{\frac{\pi}{2a}} \Rightarrow A = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}}. \tag{109}$$

We use that the wavefunctions at time t can be written as

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int dk \Phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}. \tag{110}$$

For t = 0 we know $\Psi(x,t)$, and from this can find $\Phi(k)$ via a fourier transform.

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int dx e^{-ax^2} e^{-kx} = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\pi}{a}} e^{\frac{k^2}{4a}} = \frac{1}{(2\pi a)^{\frac{1}{4}}} e^{\frac{k^2}{4a}}. \tag{111}$$

Here we used that $\int dx e^{-(ax^2+bx)} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$. $\Psi(x,t)$ is now found by inserting this result into (110) and performing the integral.

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{\frac{1}{4}}} \int dk e^{\frac{k^2}{4a}} e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} = \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{\frac{1}{4}}} \int dk e^{-\left[k^2\left(\frac{1}{4a} + \frac{i\hbar t}{2m}\right) - ikx\right]} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{\frac{1}{4}}} \frac{\sqrt{\pi}}{\sqrt{\frac{1}{4a} + \frac{i\hbar t}{2m}}} e^{\frac{-x^2}{4\left(\frac{1}{4a} + \frac{i\hbar t}{2m}\right)}} = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{e^{\frac{-ax^2}{1 + \frac{2i\hbar at}{m}}}}{\sqrt{1 + \frac{2i\hbar at}{m}}}
\end{aligned} \tag{112}$$

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