# Master's Thesis 

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## Physics of Majorana Fermions


#### Abstract

We briefly introduce the reader to the formalism of Majorana fermions and a specific realization of them: the topological nanowire. We study tunnelling through topological nanowires with large charging energy via Fermi's Golden Rule and an effective Hamiltonian approach. Tunnelling experiments probe the parity operator of the wire, $P=i \gamma_{1} \gamma_{2}$, where $\gamma_{1}, \gamma_{2}$ are the Majorana fermions hosted on the wire. We show how interference setups can be used to measure this parity. We introduce the Majorana-Cooper box which is effectively a spin- $1 / 2$ system built from Majorana fermions.

The decoherence time in conductance experiments, which yields a way to estimate the minimum time needed to perform a parity measurement, is calculated.

Next we find that adiabatically pumping a single electron through a pair of Majoranas effectively applies the operator $i \gamma_{1} \gamma_{2}$ to the state of the system.

Finally, we introduce the concept of stabilizer coding and a specific realization of a surface code that uses the Majorana-Cooper box and the above effects as its building blocks. Implementation of a complete set of logical operators is discussed.


## Acknowledgements

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A thanks also to Stephan and Morten; many of the surface code ideas were developed with them. Our discussions usually hit upon something that was poorly understood - which is a good thing! - or a new idea.

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## Chapter 1

## Introduction

### 1.1 Spukhafte Fernwirkung

In quantum physics, systems exhibit superposition. The state of the system may be described by $|1\rangle+|2\rangle$ where $|1\rangle,|2\rangle$ refer to classical configurations. This implies a range of interesting results, for example, an electron passing through a double slit may interfere with itself which changes the probability for the electron to be observed at any given point.

An endlessly more fascinating phenomenon happens when multiple particles or systems are considered: entanglement.

The full state of a two-particle system is now of the form $|12\rangle+|11\rangle$ where the first index refers to the state of the first particle and the second refers to the second particle. Certain states, called product states, can be written as

$$
|\psi\rangle|\phi\rangle,
$$

where $\psi$ is some state of the first particle and $\phi$ is some state of the second particle. It may be that $\psi$ and $\phi$ are not definite states in the classcial sense but we can still talk about the probability distributions of either particle independently of the other particle. The state mentioned above is a product state $|12\rangle+|11\rangle=|1\rangle(|2\rangle+|1\rangle)$.

When the state is not a product state it is said to be entangled. One can no longer speak of the probability of measuring the first particle in (classical) state 1. It becomes necessary to consider the state of both particles; only statements like "The probability of measuring the first particle in state 1 given that the second particle is in state 2 is $p$ " have any meaning. In some sense the two particles do not have separate existences when they are entangled. Until entanglement is broken there are not "two particles" but merely "a quantum system". This fusion is non-local in some sense. Even if the two parts of the system are entangled and then spacelike separated the two particles will always "affect" each other. It can be shown that no information can be sent via entanglement but measurements of the two particles are always correlated.

An example of an entangled state is $|12\rangle+|21\rangle$. No matter what, measurements on this system will always yield one particle in the state $|1\rangle$ and the other in the state $|2\rangle$, even if
one particle is in the Large Magellanic Cloud and the other is in Dyrehaven in Copenhagen and both are measured at 2 pm . Quite mysterious.

Entanglement in many-body systems give rise to different topological orders, where nontrivial order can be associated with long-range entanglement [1, 2]. When every subsystem/particle in the system is entangled with every other, quite interesting effects occur.

From entangled systems, particles may emerge. These emergent particles are not fundamental because they are really a particular state of many quantum particles, but for the low-energy physics they may be viewed as proper particles. The most prominent example in current research is that of Majorana fermions [3]. They are associated with self-adjoint degrees of freedom that anti-commute with each other and square to one. A regular fermionic degree of freedom, $f$, is formed from two Majoranas $\gamma_{1}, \gamma_{2}: f=\gamma_{1}+i \gamma_{2}$; Majorana fermions are sometimes called "half a fermion".

This thesis deals with some of the physics of Majorana fermions and how they may be applied to quantum computing.

Other types of particles can also emerge. From systems where the "fundamental particle" is an $n$-state system, so-called $\mathbb{Z}_{n}(=\mathbb{Z} / n \mathbb{Z})$ parafermions can emerge. Such particles exhibit exotic braiding properties and can be used for quantum computation. Refs. [4, 5] provide excellent introductions. The idea of general parafermions corresponding to finite groups, e.g. $D_{n}$, the symmetry group of an $n$-sided regular polygon; or the Platonic groups, occurred to me via another route, and I spent a few months attempting this, in the end to no avail.

Interestingly, even fermions and gauge bosons can emerge from bosonic systems [6, 7]. One starts to wonder whether there really is only one fundamental particle!

Entanglement might even be able to do even weirder stuff. There has been work, both theoretical and experimental [8], suggesting that time may be a phenomenon that emerges from quantum entanglement!

The above discussion should convince the reader that entanglement is one of the most interesting phenomena in physics (if not already so convinced!). In this thesis we deal with a specific type of system in which Majorana fermions emerge. We will explore the physics of these odd particles that live in strangely entangled systems and not talk alot about general characterizations of entanglement.

The rest of this chapter is devoted to an introduction to Majorana fermions and some general methods that we apply in the following chapters.

In the second chapter we investigate tunnelling experiments with Majorana fermions and see how the tunnelling amplitude depends on the parity of the entangled ground state of the system. Here we also get an idea of why the so-called Majorana-Cooper box might be interesting for applications.

Because the tunnelling amplitude depends on parity one can use tunnelling experiments to measure the parity. We explore the decoherence time for such experiments in chapter 3, thus showing a way to estimate the time required for a parity measurement.

The fourth chapter shows how adiabatic electron pumping across a pair of Majoranas, $\gamma_{1}, \gamma_{2}$ effectively applies the operator $i \gamma_{1} \gamma_{2}$ to the system. An interesting effect that is useful in the last chapter.

The last chapter deals with applications of Majorana physics to quantum computation. Specifically, we investigate a surface code that utilizes the Majorana-Cooper box as its fundamental building block. We discuss quantum informational aspects and the implementation of a complete set of operations.

### 1.2 Majorana Fermions

This section serves as an introduction to Majorana fermions. I have decided to start with purely formal considerations, postponing the physics until the formalism is familiar. See also refs. [3, 9] for excellent introductions that are more focused on the physics.

### 1.2.1 Majorana Formalism

Regular fermionic operators, $c_{i}, i=1,2, \ldots$, obey the anti-commutation relations

$$
\begin{align*}
& \left\{c_{i}, c_{j}\right\}=0  \tag{1.2.1}\\
& \left\{c_{i}, c_{j}^{\dagger}\right\}=\delta_{i j} \tag{1.2.2}
\end{align*}
$$

It follows from these relations that $c_{i}^{2}=0$. If we have any state $|\psi\rangle$ it will either be an eigenstate of $c_{i}$ with eigenvalue zero, or we can make one, $|\phi\rangle=c_{i}|\psi\rangle$. With such a state $|0\rangle_{i} ; c_{i}|0\rangle_{i}=0$, then $|1\rangle_{i}:=c_{i}^{\dagger}|0\rangle_{i}$ is an eigenstate of $c_{i}^{\dagger} c_{i}: c_{i}^{\dagger} c_{i} c_{i}^{\dagger}|0\rangle_{i}=c_{i}^{\dagger}|0\rangle_{i}$.

If we identify $c_{i}^{\dagger} c_{i}$ as the number operator, $c_{i}^{\dagger} c_{i} \equiv n_{i}$, then the above observations amount to "for fermionic operators, states always occur in pairs: one with no fermion and one with one fermion."

Majorana operators, $\gamma_{i}, i=1,2, \ldots$, obey

$$
\begin{equation*}
\left\{\gamma_{i}, \gamma_{j}^{\dagger}\right\}=2 \delta_{i j} \tag{1.2.3}
\end{equation*}
$$

just like the regular fermions (the factor of two can be seen as just a rescaling of the operators; the formalism will turn out to be prettier with this choice of scale). What differentiates Majorana fermions from regular ones is that

$$
\begin{equation*}
\gamma_{i}=\gamma_{i}^{\dagger} \tag{1.2.4}
\end{equation*}
$$

There is therefore no relation analogous to eq. (1.2.1). In elementary particle physics, this hermiticity condition expresses that Majorana fermions are their own antiparticles, like photons. In condensed matter physics, Majorana fermions generally arise as excitations in a mean-field Hamiltonian, schematically: $\gamma=\alpha c+\beta c^{\dagger}$. If $\alpha=\beta^{*}$, the excitation is a Majorana fermion.

We can write these relations in a different way

$$
\begin{align*}
\left\{\gamma_{i}, \gamma_{j}\right\} & =0, \quad i \neq j  \tag{1.2.5}\\
\gamma_{i}^{2} & =1 \tag{1.2.6}
\end{align*}
$$

(If we hadn't chosen the factor of 2 in eq. (1.2.3) this last expression would have equaled $\frac{1}{2}$.)
One peculiar thing about Majorana "fermions" is that no state is annihilated by them $\gamma_{i}|\psi\rangle=0 \Rightarrow 0=\gamma_{i}^{2}|\psi\rangle=|\psi\rangle$. There is no notion of the occupancy of a Majorana state. Asking whether a Majorana fermion is occupied is akin to asking whether the dog has Buddha-nature.

What would a Hamiltonian with Majorana fermions look like? Any term should of course be hermitian.

The simplest conceivable term is just a single Majorana $H_{1}=\gamma_{i}$, but we cannot say much in general. However, as we shall see, a lonely Majorana fermion is unphysical.

The next-simplest term consists of a Majorana bilinear: $i \gamma_{j} \gamma_{k}$ where $j \neq k$. A term with $j=k$ would just be proportional to the identity operator, and we have included the factor $i$ to make the expression Hermitian. Using the Majorana relations eqs. (1.2.5) and (1.2.6) one can show that

$$
\begin{equation*}
\left(i \gamma_{j} \gamma_{k}\right)^{2}=1 \tag{1.2.7}
\end{equation*}
$$

$i \gamma_{j} \gamma_{k}$ is a non-trivial operator that squares to one. This means that its eigenvalues are either +1 or -1 and that at least one is -1 . In fact, the eigenvalues always comes in pairs: for a state, $|\psi\rangle$, that is not an eigenstate of $i \gamma_{j} \gamma_{k}$ we can construct two eigenstates

This construction is simply applying the operators $1 \pm P$, for parity $P$, that projects to the $\pm$ eigenspaces of $P$.

On the other hand if we have an eigenstate $|+\rangle$ then, from (1.2.5) it has a partner


It is worth comparing these two expressions for, say, $|-\rangle$. If we apply $\gamma_{j}$ to $|+\rangle$ from eq. (1.2.8) then the result is
while for $|+\rangle$ we get
$\gamma_{j} \mp i \gamma_{k}$ and $1 \pm i \gamma_{j} \gamma_{k}$ are distinct operators but they both yield eigenstates $| \pm\rangle$. We know that the latter two are projection operators so we think of (1.2.9) and (1.2.10) as respectively telling us what the raising and lowering operators are.

With this identification of the creation and annihilation operators we have defined a regular fermion from our Majorana fermions. Let us verify this statement. We would like that the number operator satisfies

$$
\begin{equation*}
f^{\dagger} f=\frac{1}{2}\left(1-\sigma_{z}\right)=\frac{1}{2}\left(1-i \gamma_{j} \gamma_{k}\right) \tag{1.2.11}
\end{equation*}
$$

Based on eq. (1.2.10) we guess that we can write

$$
f=\gamma_{j}-i \gamma_{k}
$$

And this works!

$$
\begin{equation*}
f^{\dagger} f=\left(\gamma_{j}+i \gamma_{k}\right)\left(\gamma_{j}-i \gamma_{k}\right)=1+1-2 i \gamma_{j} \gamma_{k}=2\left(1-i \gamma_{j} \gamma_{k}\right) \tag{1.2.12}
\end{equation*}
$$

To match eq. (1.2.11) our fermion operator is defined as

$$
\begin{equation*}
f=\frac{1}{2}\left(\gamma_{j}-i \gamma_{k}\right) \tag{1.2.13}
\end{equation*}
$$

The final thing to show is the anticommutator

$$
\begin{equation*}
\left\{f, f^{\dagger}\right\}=\frac{1}{4}\left(\left\{\gamma_{j}, \gamma_{j}\right\}+\left\{\gamma_{j}, i \gamma_{k}\right\}-\left\{i \gamma_{k}, \gamma_{j}\right\}+\left\{\gamma_{k}, \gamma_{k}\right\}\right)=1, \tag{1.2.14}
\end{equation*}
$$

as required. This mapping also shows that it is probably unrealistic for a single Majorana fermion to occur as a term in the Hamiltonian because $\gamma=f+f^{\dagger}$ which means that such a term does not conserve particle number.

We now see that the simplest Majorana billinear is just equivalent to a fermion number operator. In this sense Majorana fermions occur in all condensed matter systems, but they are merely artifacts and are not physical. Interesting things can happen when we can probe the Majorana fermion itself. Thus we want to look at when Majorana fermions occur at separated locations.

The last thing we want to consider before moving on to actual physics is what happens when we have systems with many Majoranas. From the above discussion of the Majorana $\leftrightarrow$ fermion correspondence we expect that in physical systems Majoranas always occur in pairs.

Let us set many $=4$, i.e. we have Majoranas $\gamma_{j}, j=1,2,3,4$. To define the fermionic operators we can choose, say, the pairings (12) and (34) or the pairings (13) and (24). Which pairing we pick is merely convention so the two choices should be related by a basis change.

The two pairings give the following fermion operators

$$
\begin{array}{ll}
f_{1}=\frac{1}{2}\left(\gamma_{1}-i \gamma_{2}\right), \quad f_{2}=\frac{1}{2}\left(\gamma_{3}-i \gamma_{4}\right), \\
d_{1}=\frac{1}{2}\left(\gamma_{1}-i \gamma_{3}\right), \quad f_{2}=\frac{1}{2}\left(\gamma_{2}-i \gamma_{4}\right) . \tag{1.2.16}
\end{array}
$$

We of course also have their conjugates.
Given the states $|00\rangle_{f},|01\rangle_{f}:=f_{2}^{\dagger}|00\rangle_{f},|10\rangle_{f}=f_{1}^{\dagger}|00\rangle_{f},|11\rangle_{f}:=f_{1}^{\dagger} f_{2}^{\dagger}|00\rangle_{f}$, the problem is to find the corresponding states $|00\rangle_{d}$ and so on. Now, the vacuum is defined through $d_{1}|00\rangle_{d}=d_{2}|00\rangle_{d}=0$, and from this we can get the other states. If we start with an arbitrary combination of $f$-states we can find the $d$-vacuum. We start by expressing the $d$-operators in terms of the $f$-operators and then act with, say, $d_{1}$, on the state:

$$
\begin{align*}
0 & =d_{1}|00\rangle_{d}=\frac{1}{2}\left(f_{1}+f_{1}^{\dagger}-i\left(f_{2}+f_{2}^{\dagger}\right)\right)\left(a|00\rangle_{f}+b|01\rangle_{f}+c|10\rangle_{f}+d|11\rangle_{f}\right) \\
& \Rightarrow a+i d=0 \text { and } b+i c=0 \tag{1.2.17}
\end{align*}
$$

Of course we find two states corresponding to $|00\rangle_{d}$ and $|01\rangle_{d}$. Which one is the vacuum is found by considering $d_{2}$ by the same procedure:

$$
\begin{align*}
0 & =d_{2}|00\rangle_{d}=\frac{1}{2}\left(i\left(f_{1}-f_{1}^{\dagger}\right)+f_{2}-f_{2}^{\dagger}\right)\left(a|00\rangle_{f}+b|01\rangle_{f}+c|10\rangle_{f}+d|11\rangle_{f}\right) \\
& \Rightarrow i c+b=0, \text { and } i a+d=0 \tag{1.2.18}
\end{align*}
$$

So the $d$-vacuum is

$$
\begin{equation*}
|00\rangle_{d}=|01\rangle_{f}+i|10\rangle_{f} . \tag{1.2.19}
\end{equation*}
$$

From eqs. (1.2.17) and (1.2.18) we must then have

$$
\begin{align*}
|01\rangle_{d} & =|00\rangle_{f}+i|11\rangle_{f},  \tag{1.2.20}\\
|10\rangle_{d} & =|00\rangle_{f}-i|11\rangle_{f} . \tag{1.2.21}
\end{align*}
$$

The fully occupied state can be found by a variety of methods, it is

$$
\begin{equation*}
|11\rangle_{d}=|01\rangle_{f}-i|10\rangle_{f} . \tag{1.2.22}
\end{equation*}
$$

When parity is fixed, the 4 Majorana system becomes a two-level system, essentially a spin- $1 / 2$ degree of freedom. This is the case for an object we will study in great detail later in the thesis. To connect to the spin description we note that the parity operator of either pair can be thought of as $\sigma_{z}$

$$
\begin{equation*}
P=i \gamma_{1} \gamma_{2} \equiv \sigma_{z} \tag{1.2.23}
\end{equation*}
$$

$\sigma_{x}$ should flip both pairs to preserve overall parity, so by a proper choice of basis we can make the identification

$$
\begin{equation*}
i \gamma_{1} \gamma_{3} \equiv \sigma_{x} \tag{1.2.24}
\end{equation*}
$$

Note that $\left\{\sigma_{z}, \sigma_{x}\right\}=0$ as required. The idenfication of $\sigma_{y}$ is found by

$$
\begin{equation*}
i \sigma_{x} \sigma_{z}=\sigma_{y} \equiv i \gamma_{3} \gamma_{2} \tag{1.2.25}
\end{equation*}
$$

### 1.2.2 Majorana Physics

We are now in search of physical systems that harbor isolated Majorana fermions. There are several systems that have isolated Majoranas. The simplest and most relevant to this thesis is that of special type of nanowire, called a topological nanowire. This system was first considered by Kitaev [9].

A nanowire is an effectively one dimensional system and "special" refers to superconductivity and spinlessness. In a lattice model the Hamiltonian for N sites is

$$
\begin{equation*}
H=-\sum_{i=1}^{N} \mu_{i} c_{i}^{\dagger} c_{i}-\sum_{i=1}^{N-1}\left(t c_{i}^{\dagger} c_{i+1}+\Delta c_{i} c_{i+1}+\text { h.c. }\right) . \tag{1.2.26}
\end{equation*}
$$

To achieve this structure one needs strong spin-orbit coupling and a strong Zeeman effect as well as superconductivity [3].

We then introduce two Majorana operators per site:

$$
\begin{align*}
& \gamma_{i}^{+}=c_{i}+c_{i}^{\dagger}  \tag{1.2.27}\\
& \gamma_{i}^{-}=i\left(c_{i}-c_{i}^{\dagger}\right) \tag{1.2.28}
\end{align*}
$$

In terms of these the Hamiltonian is

$$
\begin{align*}
H= & -\sum_{i=1}^{N} \mu \frac{1}{2}\left(1-i \gamma_{i}^{+} \gamma_{i}^{-}\right)-\sum_{i=1}^{N-1}\left(t \frac{1}{2}\left(\gamma_{i}^{+}+i \gamma_{i}^{-}\right) \frac{1}{2}\left(\gamma_{i+1}^{+}-i \gamma_{i+1}^{-}\right)\right. \\
& \left.+\Delta \frac{1}{2}\left(\gamma_{i}^{+}-i \gamma_{i}^{-}\right) \frac{1}{2}\left(\gamma_{i+1}^{+}-i \gamma_{i+1}^{-}\right)+\text {h.c. }\right) \\
= & \frac{1}{2} \sum_{i} \mu \gamma_{i}^{+} \gamma_{i}^{-}-\sum_{i}\left(\frac{t}{4}\left(\gamma_{i}^{+} \gamma_{i+1}^{+}+\gamma_{i}^{-} \gamma_{i+1}^{-}-i \gamma_{i}^{+} \gamma_{i+1}^{-}+i \gamma_{i}^{-} \gamma_{i+1}^{+}\right)\right. \\
& \left.+\frac{\Delta}{4}\left(\gamma_{i}^{+} \gamma_{i+1}^{+}-\gamma_{i}^{-} \gamma_{i+1}^{-}-i \gamma_{i}^{+} \gamma_{i+1}^{-}-i \gamma_{i}^{-} \gamma_{i+1}^{+}\right)+\text {h.c. }\right)+ \text { const. } \\
= & \frac{1}{2} \sum_{i} \mu i \gamma_{i}^{+} \gamma_{i}^{-}-\left(\sum_{i} \frac{t+\Delta}{4}\left(\gamma_{i}^{+} \gamma_{i+1}^{+}-i \gamma_{i}^{+} \gamma_{i+1}^{-}\right)\right. \\
& \left.-\sum_{i} \frac{t-\Delta}{4}\left(\gamma_{i}^{-} \gamma_{i+1}^{-}+i \gamma_{i}^{-} \gamma_{i+1}^{+}\right)+\text {h.c. }\right) \tag{1.2.29}
\end{align*}
$$

Upon adding the hermitian conjugate the terms without an $i$ drop out as they are antihermitian. We end up with

$$
\begin{equation*}
H=\mu \sum_{i} i \gamma_{i}^{+} \gamma_{i}^{-}+\frac{t+\Delta}{2} \sum_{i} i \gamma_{i}^{+} \gamma_{i+1}^{-}+\frac{\Delta-t}{2} \sum_{i} i \gamma_{i}^{-} \gamma_{i+1}^{+} \tag{1.2.30}
\end{equation*}
$$

We consider a special point in parameter space: $\mu=0$ and $\Delta=t$. The Hamiltonian is then

$$
\begin{equation*}
H=t \sum_{i=1}^{N-1} i \gamma_{i}^{+} \gamma_{i+1}^{-} . \tag{1.2.31}
\end{equation*}
$$

$\gamma_{1}^{-}$and $\gamma_{N}^{+}$do not enter the Hamiltonian! A ground state of $H$ has negative parity for all the operators $i \gamma_{i}^{+} \gamma_{i+1}^{-}$but the parity of $i \gamma_{i}^{-} \gamma_{N}^{+}$is unconstrained. We therefore have two degenerate ground states (in fact, the whole spectrum is doubled). Because the ground states are specified by the parity of two separate Majoranas it does make sense to talk about the Majoranas as having individual (because they are separated), physical (because they are related to the parity of the ground state) existences.

The degeneracy can only be removed if a term $i \gamma_{1}^{-} \gamma_{N}^{+}$enters the Hamiltonian. In terms of the physical electrons this is

$$
\begin{equation*}
i \gamma_{1}^{-} \gamma_{N}^{+}=i\left(i\left(c_{1}-c_{1}^{\dagger}\right)\left(c_{N}+c_{N}^{\dagger}\right)=-1\left(c_{1} c_{N}+c_{1} c_{N}^{\dagger}-c_{1}^{\dagger} c_{N}-c_{1}^{\dagger} c_{N}^{\dagger}\right)\right. \tag{1.2.32}
\end{equation*}
$$

Such a term evidently couples electrons at either end of the wire. If the wire is long enough this cannot happen; real Hamiltonians are local.

The Hamiltonian (1.2.31) couples electrons and electrons and holes on different sites in such a way as to produce two ground states with long-range entanglement but where the two states are entangled differently. Far from being accidental the entanglement and degeneracy is related to the superconductivity. The entanglement comes from the p-wave superconductivty and the coupling between different sites. The two-fold degeneracy is related to the symmetry-breaking $U(1) \rightarrow \mathbb{Z}_{2}$ of superconductivity; parity but not particle number is conserved by the Hamiltonian and we get a parity degree of freedom. For a more detailed discussion start with refs. $[9,10]$.

If we tune away from the ideal parameters the above results still hold when $2|t|<|\mu|$. The ground states will no longer be exactly degenerate but the splitting is proportional to $e^{-L / \ell_{0}}$ where $L$ is the length of the nanowire and the "free" Majorana states are smeared out across the wire rather than localized at the ends[9].

The topological nanowire will be used in all the setups we consider in this thesis.

### 1.3 Low-energy Approximation Scheme

We will be interested only in the low-energy physics. This implies a number of useful approximations can be employed, and we describe these in this section. Excitations above the gap are inactive and that we can therefore usually replace $c_{1} \rightarrow \gamma_{1}^{-}$to use the notation from the previous section. The systems we consider will have a strong charging energy, which locks the charge to a specific value in the low-energy sector. However, for the processes we consider it is necessary to take fluctuations to higher or lower charge values into account.

We begin by addressing the latter from a general point of view following, more or less, ref. [11].

### 1.3.1 Effective Hamiltonians

We need a systematic way to formulate a theory for the low-energy sector that still takes the higher-energy sectors into account to some low order. Without the high-energy sectors the Hamiltonian would just be $H_{\text {Low }}$, and the idea is essentially to modify this Hamiltonian $H_{\text {Low }} \rightarrow H_{\text {Low }}+H_{\text {Fluctuations }}$ so that we have described processes that jump into the highenergy sector, hang around for a bit, and then jump back to the low-energy sector.

For this purpose it is clever to separate the state of the system $|\psi\rangle=\left|\psi_{L}\right\rangle+\left|\psi_{H}\right\rangle$, where $\left|\psi_{L}\right\rangle$ and $\left|\psi_{H}\right\rangle$ are the low-energy (active) degrees of freedom and the high-energy (passive)
degrees of freedom respectively. Introducing

$$
\begin{align*}
H_{L L} & =\left|\psi_{L}\right\rangle\left\langle\psi_{L}\right| H\left|\psi_{L}\right\rangle\left\langle\psi_{L}\right|,  \tag{1.3.1}\\
H_{H H} & =\left|\psi_{H}\right\rangle\left\langle\psi_{H}\right| H\left|\psi_{H}\right\rangle\left\langle\psi_{H}\right|,  \tag{1.3.2}\\
H_{L H} & =\left|\psi_{L}\right\rangle\left\langle\psi_{L}\right| H\left|\psi_{H}\right\rangle\left\langle\psi_{H}\right|,  \tag{1.3.3}\\
H_{H L} & =\left|\psi_{H}\right\rangle\left\langle\psi_{H}\right| H\left|\psi_{L}\right\rangle\left\langle\psi_{L}\right|=H_{L H}^{\dagger}, \tag{1.3.4}
\end{align*}
$$

we can then write the Schrödinger equation as

$$
\left(\begin{array}{cc}
H_{L L} & H_{L H}  \tag{1.3.5}\\
H_{H L} & H_{H H}
\end{array}\right)\binom{\left|\psi_{L}\right\rangle}{\left|\psi_{H}\right\rangle}=E\binom{\left|\psi_{L}\right\rangle}{\left|\psi_{H}\right\rangle} .
$$

Because $\left\langle\psi_{H} \mid \psi_{L}\right\rangle=0$ this way of writing the Schrödinger equation is still exact.
The second equation reads

$$
\begin{align*}
E\left|\psi_{H}\right\rangle & =H_{H H}\left|\psi_{H}\right\rangle+H_{H L}\left|\psi_{L}\right\rangle \\
\Rightarrow\left|\psi_{H}\right\rangle & =\left(E-H_{H H}\right)^{-1} H_{H L}\left|\psi_{L}\right\rangle . \tag{1.3.6}
\end{align*}
$$

Inserting this into the first equation yields an exact, effective Hamiltonian for $\left|\psi_{L}\right\rangle$

$$
\begin{equation*}
E\left|\psi_{L}\right\rangle=H_{L L}\left|\psi_{L}\right\rangle+H_{L H}\left|\psi_{H}\right\rangle=H_{L L}\left|\psi_{L}\right\rangle+H_{L H}\left(E-H_{H H}\right)^{-1} H_{H L}\left|\psi_{L}\right\rangle . \tag{1.3.7}
\end{equation*}
$$

The effective Hamiltonian, $H_{\text {eff }}$, deserves its own equation:

$$
\begin{equation*}
H_{\mathrm{eff}}=H_{L L}+H_{L H}\left(E-H_{H H}\right)^{-1} H_{H L} . \tag{1.3.8}
\end{equation*}
$$

This result holds for any system where you can separate your degrees of freedom into interesting and less-interesting sectors. How exactly one would express the different subHamiltonians or solve the Schrödinger equation varies from problem to problem.

In this thesis the high-energy sector will be interpreted as a charging of the system away from some energetically ideal charge number, $N_{0}$. The high-energy sector is then all states with charge numbers $N_{0} \pm 1, N_{0} \pm 2, N_{0} \pm 3, \ldots$, and $H_{H H}$ will couple these different sectors, which greatly complicates the calculation of $\left(E-H_{H H}\right)^{-1}$. However, a sector with charge $N$ has weight $\sim \frac{1}{H_{C}(N)}$ where $H_{C}$ is the charging energy. We will set the charging energy to be very large so that we can safely neglect all other sectors than $N_{0} \pm 1$. Thus $H_{H H}$ becomes a number and we write

$$
\begin{equation*}
H_{L}=H_{L}+H_{L H}^{+} \frac{-1}{U_{-}} H_{H L}^{-}+H_{L H}^{-} \frac{-1}{U_{+}} H_{H L}^{+} \tag{1.3.9}
\end{equation*}
$$

$U_{ \pm}=H_{H H ;\left(N_{0} \pm 1\right)}-E$ and $H_{H L}^{ \pm}$is the term in $H_{H L}$ that increases/lowers the charge by one. This approximation is used throughout the thesis.

### 1.3.2 Majorana Tunnelling

We now address the first point mentioned in the introductory paragraph.
We will look at systems that have a lead or a dot tunnel-coupled to one end of a topological nanowire and another lead/dot tunnel-coupled to the other end. The somethings need not be the same thing. Such a coupling will introduce a term in the Hamiltonian:

$$
\begin{equation*}
H_{T}=t_{1} f^{\dagger} c_{1}+t_{N} d^{\dagger} c_{N}+\text { h.c.. } \tag{1.3.10}
\end{equation*}
$$

$f, d$ refer to the electrons in the somethings. The states corresponding to $c_{1}, c_{N}$ are not energy eigenstates and it is convenient to express the operators in terms of energy eigenstates. Such an expansion looks like

$$
\begin{equation*}
c_{1}=a_{0}^{-} \gamma_{1}^{-} P_{-}+a_{0}^{+} \gamma_{N}^{+} P_{-}+a_{1} \lambda_{1}+a_{2} \lambda_{2}+\ldots \tag{1.3.11}
\end{equation*}
$$

and similarly for $c_{N}$. The states $\lambda_{n}$ are the (non-zero) energy eigenstates of the Hamiltonian (1.2.26), and $P_{-}$is an operator that lowers the charge on the nanowire by one. The states $k_{n}$ have energy $\sim \Delta$ and if the gap is very large these terms will contribute very little to the effective Hamiltonian (1.3.8). Furthermore since the Majoranas $\gamma_{1}^{-}, \gamma_{N}^{+}$are localized at opposite end of the wire $a_{0}^{+}$is essentially zero. A similar statement holds for $c_{N}$. Based on this we can replace (see also ref. [12])

$$
\begin{align*}
c_{1} & \rightarrow \gamma_{1}^{-} P_{-},  \tag{1.3.12}\\
c_{N} & \rightarrow \gamma_{N}^{+} P_{-} \tag{1.3.13}
\end{align*}
$$

In the rest of the thesis we will always be working with the low-energy physics and the charge raising and lowering operators are omitted, because in any amplitude they will always cancel, cf. sec. 1.3.1.

### 1.3.3 Approximations

Above we have referred to "a large charging energy" and a "large gap". Throughout the thesis we will assume that

$$
\begin{equation*}
V, T \ll E_{C}, \Delta, \tag{1.3.14}
\end{equation*}
$$

where $V$ is the voltage across the wire, $T$ is the temperature, $E_{C}$ is the energy associated with charging, and $\Delta$ is the gap induced by the superconductivity.

## Chapter 2

## Topological Nanowires with Charging Energy

We will now begin our study of physical systems.
A schematic setup for a system that is technically feasible today is shown in figure 2.1. The grey cylinder indicates the nanowire which is assume to have a strong Zeeman effect and strong spin-orbit coupling. The nanowire is placed on top of a superconductor which is indicated by the maroon box. This superconductor induces superconductivity in the nanowire via the proximity effect. The spin-orbit coupling, Zeeman effect and proximity effect is assumed to realize the model discussed in section 1.2.2, which means the nanowire hosts free Majorana fermions at the ends of the wire which is shown as red dots.

The superconductor and nanowire are floating and assumed to possess a large charging energy. Physically, the electrons in this subsystem interact via the Coulomb interaction and one can typically describe this effect by a term in the Hamiltonian $H_{C}=E_{C} N^{2}$ where $N$ is the number of particles in the subsystem and $E_{C}$ is the effective strength of the interaction. The nanowire is contacted by a gate which is shown as an inverted T. This gate can tune the local electrostatic potential and therefore adds a term in the Hamiltonian $q N V$, with $q$ the charge of the particles, $N$ the number of particles, and $V$ the potential. We can include this as a modifcation of the Coulomb charging:

$$
\begin{equation*}
H_{C}=E_{C} N^{2}+q V N=E_{C}\left(N+\frac{q V}{2 E_{C}}\right)^{2}+\text { const. } \equiv E_{C}\left(N-n_{g}\right)^{2} \tag{2.0.1}
\end{equation*}
$$

The final things in the system are the two leads. These are assumed to be metallic, i.e. to possess a continuum of states with energy $\epsilon_{k}=\frac{k^{2}}{2 m}$, where $k$ is the quantum number for the leads. These are weakly coupled to the ends of the nanowire which we describe by a tunnel-coupling, $H_{T}=\sum_{\alpha=L, R ; k} t_{\alpha ; k} c_{\alpha ; k}^{\dagger} c_{\text {wire; } \alpha}+$ h.c., where $L / R$ denotes the left/right side and we have labelled the fermions on the wire by their location. Insituting the low-energy approximations in section 1.3.2 the leads and tunnel couplings add terms

$$
\begin{equation*}
H=H_{L}+H_{R}+H_{T}=\sum_{\alpha ; k} \epsilon_{k} c_{\alpha ; k}^{\dagger} c_{\alpha ; k}+\sum_{\alpha ; k}\left(t_{\alpha ; k} c_{\alpha ; k} \gamma_{\alpha}+\text { h.c. }\right) \tag{2.0.2}
\end{equation*}
$$



Figure 2.1: A single nanowire (grey cylinder) tunnel-coupled (dashed lines) to leads (orange blobs). A gate for tuning the local electrostatic potential (black inverted T ) is shown. The nanowire sits on top of a superconductor (maroon box) which induces superconductivity in the nanowire. The nanowire is topological and therefore hosts Majorana fermions (red dots).

In fact, eqs. (2.0.1) and (2.0.2) are all there is to the Hamiltonian. Because we only consider low-energy physics the gapped nanowire effectively only contributes the two Majorana fermions to the system and because of the large charging energy the number of electrons is locked to $N_{0}=\left\lfloor n_{g}\right\rfloor$, or $N_{0}=\left\lceil n_{g}\right\rceil$ depending on which yields the lower energy.

The setup in figure 2.1 is the basic element of everything in this thesis. Sometimes there will be multiple nanowires on the superconductor and sometimes there will be multiple boxes, but understanding the physics of the single nanowire, single box system is essential to understanding the rest.

The motivation for this configuration is as follows. The fact that the superconductor is floating means that a low-energy electron cannot stay in the wire. The electron must always tunnel out again. Therefore only Majorana billinears are probed, e.g. $\gamma_{L}^{2}$ or $\gamma_{L} \gamma_{R}$. When the system has a high charging energy and two wires are placed on the superconductor it essentially becomes a spin- $1 / 2$ degree of freedom (see also the discussion in sec. 1.2.1). We need a gate because certain effects will depend on the parity, and using the gate we can control this.

In this chapter we do not need the machinery of section 1.3 . Rather we will look at the conductance through the system for low voltage over the leads from the point of view of Fermi's Golden Rule.

In this chapter we will start by investigating the setup with one nanowire (sec. 2.1). Then we will add an interference link to explore the interesting effect that the tunnelling amplitude depends on the parity of the ground state (sec. 2.2). Finally we will arrive at the main system of interest in this thesis: we replace the tunnelling link with another nanowire (sec. 2.2.2).

### 2.1 Tunnelling through a single Nanowire

We begin with the setup in figure 2.1 and calculate the current via Fermi's Golden Rule. There are two processes to consider: the rate of electrons moving from left to right and the reverse. Obviously the two are related by appropriate transformations of the problem so we need only explicitly calculate one, say the rate left to right, $\Gamma_{\rightarrow \text {. }}$. The final state will then correspond to removing an electron from the left lead and adding it to the right lead. At finite temperature the state of the system will in a mixed state, so we also need to sum over initial states weighted with the corresponding thermal probabilities, $W_{i}$.

The Hamiltonian is

$$
\begin{equation*}
H=H_{L}+H_{R}+H_{C}+H_{T} \tag{2.1.1}
\end{equation*}
$$

as discussed, and we take $H_{T}$ as the perturbation. Fermi's Golden Rule must then be applied to second order:

$$
\begin{aligned}
\Gamma_{\rightarrow}= & \left.2 \pi \sum_{q k ; i}\left|\langle f| H_{T} \frac{1}{E-H_{0}} H_{T}\right| i\right\rangle\left.\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) \\
= & \left.2 \pi \sum_{q k ; i}\left|\langle i| c_{L q}^{\dagger} c_{R k} H_{T} \frac{1}{E-H_{0}} H_{T}\right| i\right\rangle\left.\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) \\
= & 2 \pi \sum_{q k ; i} \left\lvert\,\left.\langle i| c_{L q}^{\dagger} c_{R k} \sum_{\alpha=L, R ; r}\left(t_{\alpha ; r} c_{\alpha ; r}^{\dagger} \gamma_{\alpha}+\text { h.c. }\right) \frac{1}{E-H_{0}} \sum_{\alpha=L, R ; w}\left(t_{\alpha ; w} c_{\alpha ; w}^{\dagger} \gamma_{\alpha}+\text { h.c. }\right)|i\rangle\right|^{2}\right. \\
& \times W_{i} \delta\left(E_{f}-E_{i}\right) ;
\end{aligned}
$$

by particle conservation the $c^{\dagger}$-term in the first parentheses pairs up with the $c$-term in the second and vice versa and we therefore find

$$
\begin{equation*}
\left.=2 \pi \sum_{q k ; i}\left|\langle i| c_{L q}^{\dagger} c_{R k}\left(\sum_{\alpha \beta ; r w} t_{\alpha ; r}^{*} t_{\beta ; w} \frac{\gamma_{\alpha} c_{\alpha ; r} c_{\beta ; w}^{\dagger} \gamma_{\beta}}{-U_{-}-\epsilon_{w}}+t_{\alpha ; r} t_{\beta ; w}^{*} \frac{c_{\alpha ; r}^{\dagger} \gamma_{\alpha} \gamma_{\beta} c_{\beta ; w}}{-U_{+}+\epsilon_{w}}\right)\right| i\right\rangle\left.\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) \tag{2.1.2}
\end{equation*}
$$

Here we invoke the assumption that the charging energy is large: $U_{ \pm} \mp \epsilon \approx U_{ \pm}$. Certain requirement arise for the expectation value to be non-zero. We have, for example, $r=q, w=$ $k, \alpha=L, \beta=R$ in the first term. The net effect is to cancel the inner sum, and we are left with

$$
\begin{aligned}
= & \left.2 \pi \sum_{q k ; i}\left|\langle i| c_{L q}^{\dagger} c_{R k}\left(t_{L ; q}^{*} t_{R ; k} \frac{\gamma_{L} c_{L ; q} c_{R ; k}^{\dagger} \gamma_{R}}{U_{-}}+t_{R ; k} t_{L ; q}^{*} \frac{c_{R ; k}^{\dagger} \gamma_{R} \gamma_{L} c_{L ; q}}{U_{+}}\right)\right| i\right\rangle\left.\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) \\
= & \left.2 \pi \sum_{q k ; i}\left|\langle i| \gamma_{R} \gamma_{L}\left(t_{L ; q}^{*} t_{R ; k} \frac{c_{L q}^{\dagger} c_{L ; q} c_{R ; k} c_{R ; k}^{\dagger}}{U_{-}}+t_{R ; k} k_{L ; q}^{*} \frac{c_{L ; q}^{\dagger} c_{L ; q} c_{R ; k} c_{R ; k}^{\dagger}}{U_{+}}\right)\right| i\right\rangle\left.\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) \\
= & \left.\left.2 \pi \sum_{q k ; i}\left(\left|\langle i| \gamma_{R} \gamma_{L} t_{L ; q}^{*} t_{R ; k} \frac{c_{L q}^{\dagger} c_{L ; q} c_{R ; k} c_{R ; k}^{\dagger}}{U_{-}}\right| i\right\rangle\right|^{2}+\left|\langle i| \gamma_{R} \gamma_{L} t_{R ; k} t_{L ; q}^{*} \frac{c_{L ; q}^{\dagger} c_{L ; q} c_{R ; k} c_{R ; k}^{\dagger}}{U_{+}}\right| i\right\rangle\left.\right|^{2} \\
& \left.+2 \operatorname{Re}\langle i| \gamma_{R} \gamma_{L} t_{L ; q}^{*} t_{R ; k} \frac{c_{L q}^{\dagger} c_{L ; q} c_{R ; k} c_{R ; k}^{\dagger}}{U_{-}}|i\rangle\langle i| \gamma_{R} \gamma_{L} t_{R ; k} t_{L ; q}^{*} \frac{c_{L ; q}^{\dagger} c_{L ; q} c_{R ; k} c_{R ; k}^{\dagger}}{U_{+}}|i\rangle^{*}\right) W_{i} \delta\left(E_{f}-E_{i}\right) .
\end{aligned}
$$

Because $\langle i| c_{L ; q}^{\dagger} c_{L ; q} c_{R ; k} c_{R ; k}^{\dagger}|i\rangle$ is either one or zero depending on the occupancy, $n_{q, k}(i)$, of the corresponding levels in the state $|i\rangle$, we can remove the absolute value squared and write

$$
\left.=\left.2 \pi \sum_{q k ; i}\left(\left|\langle i| \gamma_{R} \gamma_{L}\right| i\right\rangle\right|^{2}\left|t_{L ; q}\right|^{2}\left|t_{R ; k}\right|^{2}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} n_{q, k}(i)\right) W_{i} \delta\left(E_{f}-E_{i}\right)
$$

By definition $\sum_{i} n_{q, k}(i) W_{i}=n_{F}\left(\xi_{L ; k}\right)\left(1-n_{F}\left(\xi_{R ; k}\right)\right)$ where $n_{F}(\epsilon)$ is the Fermi distribution. Furthermore, remembering that the Majorana parity operator is $P=i \gamma_{R} \gamma_{L}$ we can replace $\gamma_{R} \gamma_{L}$ by $P$ under the absolute value. Hence

$$
\left.=2 \pi \sum_{q k}|\langle i| P| i\right\rangle\left.\right|^{2}\left|t_{L ; q}\right|^{2}\left|t_{R ; k}\right|^{2}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} n_{F}\left(\xi_{L ; q}\right)\left(1-n_{F}\left(\xi_{R ; k}\right)\right) \delta\left(E_{f}-E_{i}\right)
$$

The delta function fixes $q=k$. To perform the sum over $q$ we use that $n_{F}(\epsilon)\left(1-n_{F}\left(\epsilon^{\prime}\right)\right)=$ $n_{B}\left(\epsilon-\epsilon^{\prime}\right)\left(n_{F}\left(\epsilon^{\prime}\right)-n_{F}(\epsilon)\right)$ and introduce a density of states $d$ assumed to be constant, and for simplicity we also take the tunnelling amplitudes to be independent of $k, q$ :

$$
\begin{align*}
& =2 \pi|\langle i| P| i\rangle\left.\right|^{2}\left|t_{L}\right|^{2}\left|t_{R}\right|^{2}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} d^{2} n_{B}(-e V) \int \mathrm{d} \epsilon\left(n_{F}(\epsilon)-n_{F}(\epsilon-e V)\right) \\
& =2 \pi|\langle i| P| i\rangle\left.\right|^{2}\left|t_{L}\right|^{2}\left|t_{R}\right|^{2}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} d^{2} n_{B}(-e V)(-e V) . \tag{2.1.3}
\end{align*}
$$

The opposite rate $\Gamma_{\leftarrow}$ is found by $R \leftrightarrow L, V \rightarrow-V$. This is particularly simple in the $T \rightarrow 0$
limit. Assume $-e V<0$ then

$$
\begin{align*}
\Gamma_{\rightarrow} & =2 \pi|\langle i| P| i\rangle\left.\right|^{2}\left|t_{L}\right|^{2}\left|t_{R}\right|^{2}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} d^{2} e V,  \tag{2.1.4}\\
\Gamma_{\leftarrow} & =0 ;  \tag{2.1.5}\\
G & =2 \pi|\langle i| P| i\rangle\left.\right|^{2}\left|t_{L}\right|^{2}\left|t_{R}\right|^{2}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} d^{2} e^{2}, \tag{2.1.6}
\end{align*}
$$

where we have introduced the linear conductance $G=e \frac{d}{d V}\left(\Gamma_{\rightarrow}-\Gamma_{\leftarrow}\right)$.
I have deliberately kept $|\langle i| P| i\rangle\left.\right|^{2}$ even though it is always equal to one. This is to emphasize an effect which may otherwise have been lost in the algebra: the tunnelling amplitude depends on parity. Referring back to eq. (2.1.2) this may be more apparent. The extension we now suggest is to add an interference link so that the total conductance depends on the parity $P$.

### 2.2 Tunnelling with an interference link



Figure 2.2: We have modified the previous setup by adding an interference link shown as the dashed curved line connected the two leads. The gate and superconductor are not shown.

We will now explore the extension suggested at the end of the previous section. We start with describing the physics for a regular interference, which we model as a direct coupling between the leads, and then move on to the case where the interference link is another topological nanowire. We will again calculate the conductance using Fermi's Golden Rule and much will mirror our efforts in the previous section.

What we mean by an interference link setup is shown schematically in figure 2.2. The superconductor and gate from figure 2.1 are still there but they are not shown. If the interference link is another topological nanowire then both nanowires can be connected to the same superconductor and gate which will have special consequences to be investigated below.

### 2.2.1 Regular interference link

We model the interference link setup by the Hamiltonian

$$
\begin{align*}
H & =H_{L}+H_{R}+H_{C}+H_{T} \\
& =\sum_{\alpha ; k} \epsilon_{k} c_{\alpha ; k}^{\dagger} c_{\alpha ; k}++E_{C}\left(N-n_{g}\right)^{2}+\sum_{\alpha ; k}\left(t_{\alpha ; k} c_{\alpha ; k} \gamma_{\alpha}+\text { h.c. }\right)+\sum_{k, u}\left(\mathcal{T}_{k, u} c_{L ; k}^{\dagger} c_{R, u}+\text { h.c. }\right) . \tag{2.2.1}
\end{align*}
$$

This is identical to the Hamiltonian (2.1.1) except that the last term has been added to the tunnelling to describe the interference link.

The direct tunnelling term contributes to the amplitude at first order but not at second order. Therefore using Fermi's Golden Rule we get

$$
\begin{align*}
\Gamma_{\rightarrow}= & \left.2 \pi \sum_{q k ; i}\left|\langle f| H_{T} \frac{1}{E-H_{0}} H_{T}\right| i\right\rangle\left.\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) \\
= & 2 \pi \sum_{q k ; i} \left\lvert\,\langle i| c_{L q}^{\dagger} c_{R k} \sum_{\alpha=L, R ; r}\left(t_{\alpha ; r} c_{\alpha ; r}^{\dagger} \gamma_{\alpha}+\text { h.c. }\right) \frac{1}{E-H_{0}} \sum_{\alpha=L, R ; w}\left(t_{\alpha ; w} c_{\alpha ; w}^{\dagger} \gamma_{\alpha}+\text { h.c. }\right)|i\rangle\right. \\
& +\left.\langle i| c_{L q}^{\dagger} c_{R k} \sum_{k, u}\left(\mathcal{T}_{k, u} c_{L ; k}^{\dagger} c_{R, u}+\text { h.c. }\right)|i\rangle\right|^{2} \\
& \times W_{i} \delta\left(E_{f}-E_{i}\right) . \tag{2.2.2}
\end{align*}
$$

The new tunnelling term just adds another term to the rate. If we again assume that the tunnelling amplitudes, $t, \mathcal{T}$, are independent of $k, u$ we find a zero temperature conductance

$$
\begin{equation*}
G=2 \pi d^{2} e^{2}\left|\mathcal{T}^{*}-i t_{L}^{*} t_{R}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} P_{i}\right|^{2} \tag{2.2.3}
\end{equation*}
$$

where we have defined $P_{i}:=\langle i| P|i\rangle$. The conductance is boosted or suppressed depending on whether $\mathcal{T}^{*}$ and $-i t_{L}^{*} t_{R}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} P_{i}$ interfere constructively or destructively. The parity-dependent conductance can be observed by tuning from constructive to destructive interference by changing the occupancy of the nanowire via the gate.

This effect is maximized when $R \mathcal{T}^{*}=-i t_{L}^{*} t_{R}\left(\frac{1}{U_{-}}+\frac{1}{U_{+}}\right)^{2} P_{i}$ with $R \in \mathbb{R}$. Whether this is satisfied depends on the phases of $\mathcal{T}, t_{L}, t_{R}$ which are in general uncontrollable, but one can imagine tuning the overall phase by putting a magnetic flux through the sample and adjusting the field to yield maximum conductance.

### 2.2.2 Two nanowires

We now take the inteference link in figure 2.2 to be a second nanowire. This modifies the tunnelling and the charging.

The tunnelling Hamiltonian becomes

$$
\begin{equation*}
H_{T}=\sum_{j ; \alpha ; k}\left(t_{j ; \alpha ; k} c_{\alpha ; k} \gamma_{j ; \alpha}+\text { h.c. }\right), \tag{2.2.4}
\end{equation*}
$$

where $j=1,2$ labels the two nanowires.
For the charging energy there are two choices.
If we place the nanowires on the same superconductor and contact them with the same gate there are two types of contributions to the rate $\Gamma_{\rightarrow \text {. }}$. The first type is "parallel" tunnelling, i.e. electron transfer into one nanowire and out of the same one. This obviously returns us to the low-energy sector as required. The other, new, type is "crossed" tunnelling, where the electron tunnels into one nanowire and out of the other. Because the nanowires share charging energy this also returns us to the low-energy sector. For this situation the charging Hamiltonian is

$$
\begin{equation*}
H_{C}=E_{C}\left(N-n_{g}\right)^{2} \tag{2.2.5}
\end{equation*}
$$

where $N$ is the total number of electrons in the system.
We can place the nanowires on different superconductors and contact different gates to each nanowire. In that case we get just the first type of tunnelling. In this case the charging Hamiltonian becomes

$$
\begin{equation*}
H_{C}=E_{C ; 1}\left(N_{1}-n_{g ; 1}\right)^{2}+E_{C ; 2}\left(N_{2}-n_{g ; 2}\right)^{2}, \tag{2.2.6}
\end{equation*}
$$

where $N_{i}$ is the charge on the $i$ th nanowire-superconductor system and $n_{g ; i}$ is the generalization of $n_{g}$.

In the following we will refer to the first case as "One island" and the latter as "Two islands".

Two Islands The calculation of the conductance in the two islands setup is analogous the calculation is section 2.2.1, so we will not go through it in detail. There are two amplitudes that contribute:

$$
\begin{equation*}
G=2 \pi e^{2} d^{2}\left|T_{1} P_{1}+T_{2} P_{2}\right|^{2}, \tag{2.2.7}
\end{equation*}
$$

where $T_{i}$ is an effective tunnelling coupling for the $i$ th nanowire, $T_{i}=-i t_{i ; L}^{*} t_{i ; R}\left(\frac{1}{U_{i ;-}}+\frac{1}{U_{i ;+}}\right)^{2}$ and $P_{i}$ is the parity of the $i$ th nanowire.

Compare this to the situation in section 2.2.1. We see again that the conductance suppressed or enhanced depending on the relative phase between the different paths. This is of course quite standard; the interesting effect is that the phase of the effective tunnelling depends on which ground state the nanowire is in.

One Island In this case there is an interplay between the two types of tunnelling. Therefore we will go through the calculation in more detail.

The island is essentially a spin- $1 / 2$ system. The initial states can be written as one of the two

$$
\begin{align*}
& |i\rangle=(\alpha|00\rangle+\beta|11\rangle) \otimes|\psi\rangle,  \tag{2.2.8}\\
& |i\rangle=(\alpha|10\rangle+\beta|01\rangle) \otimes|\psi\rangle, \tag{2.2.9}
\end{align*}
$$

where $|\psi\rangle$ is some state of the leads. If $N_{0}$ is even, the first will be the initial state, and if $N_{0}$ is odd the second will be the initial state. For any $N_{0}>0$ the island is a two state system.

There are two types of final states corresponding to the two tunnelling types discussed above: one in which an electron has tunnelled into a given nanowire and out of the same, and one in which an electron has tunnelled into a nanowire and another electron has tunnelled out of the other. The second process changes the parity of both NWs. Thus we can describe the final states as $\left|f_{s}\right\rangle=q_{s}^{\dagger} c_{R k}^{\dagger} c_{L q}|i\rangle$ where

$$
q_{s}^{\dagger}= \begin{cases}1 & s=0  \tag{2.2.10}\\ \gamma_{1 L} \gamma_{2 L} & s=1,\end{cases}
$$

where $s=0$ labels the parallel tunnelling and $s=1$ labels the crossed tunnelling.
We then apply Fermi's Golden Rule

$$
\left.\Gamma_{\rightarrow}=2 \pi \sum_{q k, s ; i}\left|\langle i| c_{L q}^{\dagger} c_{R k} q_{s} H_{T} \frac{1}{E-H_{0}} H_{T}\right| i\right\rangle\left.\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) ;
$$

inserting the expression for the tunnelling Hamiltonian and requiring that there is an equal number of $c s$ and $c^{\dagger} \mathrm{s}$ yields

$$
\begin{align*}
= & \left.2 \pi \sum_{q k, s ; i}\left|\langle i| c_{L q}^{\dagger} c_{R k} q_{s}\left(\sum_{\substack{\alpha, \beta \\
l, j \\
l, j}} t_{l ; \alpha ; w}^{*} t_{j ; \beta ; u} \frac{c_{\alpha ; k} \gamma_{l ; \alpha} \gamma_{j ; \beta} c_{\beta u}^{\dagger}}{-U_{-}}+t_{l ; \alpha ; w} t_{j ; \beta ; u}^{*} \frac{\gamma_{j ; \beta} c_{\beta ; u}^{\dagger} c_{\alpha ; w} \gamma_{l ; \alpha}}{-U_{+}}\right)\right| i\right\rangle\left.\right|^{2} \\
& \times W_{i} \delta\left(E_{f}-E_{i}\right) \\
= & 2 \pi \sum_{q k, s ; i}\left|\sum_{\substack{\alpha, \beta \\
l, j \\
w, u}} t_{l ; \alpha ; w}^{*} t_{j ; \beta ; u} \frac{\langle i| c_{L q}^{\dagger} c_{R k} c_{\alpha ; w} c_{\beta ; u}^{\dagger} q_{s} \gamma_{j ; \beta} \gamma_{l ; \alpha}|i\rangle}{U_{-}}+t_{l ; \alpha ; w} t_{j ; \beta ; u}^{*} \frac{\langle i| c_{L q}^{\dagger} c_{R k} c_{\alpha, w}^{\dagger} c_{\beta ; u} q_{s} \gamma_{j ; \beta} \gamma_{l ; \alpha}|i\rangle}{U_{+}}\right|^{2}  \tag{2.2.11}\\
& \times W_{i} \delta\left(E_{f}-E_{i}\right) .
\end{align*}
$$

Again we enforce $\alpha=L, \beta=R, w=q, u=k$ and $\alpha=R, \beta=L, w=k, u=q$ in the first
and second terms respectively. Introducing $\frac{1}{\mathcal{U}}:=\frac{1}{U_{+}}+\frac{1}{U_{-}}$we can then write

$$
\begin{align*}
\Gamma_{\rightarrow}= & 2 \pi \sum_{q k, s ; i}\left|\sum_{l, j} t_{l ; L ; q}^{*} t_{j ; R ; k} \frac{\langle i| c_{L q}^{\dagger} c_{R k} c_{L ; q} c_{R ; k}^{\dagger} q_{s} \gamma_{j ; R} \gamma_{l ; L}|i\rangle}{U_{-}}+t_{l ; R ; k} t_{j ; L ; q}^{*} \frac{\langle i| c_{L q}^{\dagger} c_{R k} c_{R, k}^{\dagger} c_{L ; q} q_{s} \gamma_{j ; L} \gamma_{l ; R}|i\rangle}{U_{+}}\right|^{2} \\
& \times W_{i} \delta\left(E_{f}-E_{i}\right) \\
= & 2 \pi \sum_{q k, s ; i}\left|\sum_{l, j} t_{l ; L ; q}^{*} t_{j ; R ; k} \frac{\langle i| c_{L q}^{\dagger} c_{R k} c_{L ; q} c_{R ; k}^{\dagger} q_{s} \gamma_{j ; R} \gamma_{l ; L}|i\rangle}{\mathcal{U}}\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) . \tag{2.2.12}
\end{align*}
$$

The problem is now to calculate the expectation values of $q_{s} \gamma_{j ; R} \gamma_{l ; L}$. There are eight operators to consider:

$$
\begin{array}{r}
\gamma_{1 ; R} \gamma_{1 ; L}, \\
\gamma_{2 ; R} \gamma_{2 ; L}, \\
\gamma_{1 ; R} \gamma_{2 ; L}, \\
\gamma_{2 ; R} \gamma_{1 ; L}, \\
\gamma_{1 ; L} \gamma_{2 ; L} \gamma_{1 ; R} \gamma_{1 ; L}, \\
\gamma_{1 ; L} \gamma_{2 ; L} \gamma_{2 ; R} \gamma_{2 ; L}, \\
\gamma_{1 ; L} \gamma_{2 ; L} \gamma_{1 ; R} \gamma_{2 ; L}, \\
\gamma_{1 ; L} \gamma_{2 ; L} \gamma_{2 ; R} \gamma_{1 ; L} .
\end{array}
$$

Suppose we have the initial state (2.2.8) corresponding to even $N_{0}$. Then

$$
\begin{align*}
\langle i| \gamma_{1 ; R} \gamma_{1 ; L}|i\rangle & =-i\left(|\alpha|^{2}-|\beta|^{2}\right),  \tag{2.2.13}\\
\langle i| \gamma_{2 ; R} \gamma_{2 ; L}|i\rangle & =-i\left(|\alpha|^{2}-|\beta|^{2}\right),  \tag{2.2.14}\\
\langle i| \gamma_{1 ; R} \gamma_{2 ; L}|i\rangle & =-i\left(\alpha \beta^{*}-\beta \alpha^{*}\right),  \tag{2.2.15}\\
\langle i| \gamma_{2 ; R} \gamma_{1 ; L}|i\rangle & =i\left(\alpha \beta^{*}-\beta \alpha^{*}\right),  \tag{2.2.16}\\
\langle i| \gamma_{1 ; L} \gamma_{2 ; L} \gamma_{1 ; R} \gamma_{1 ; L}|i\rangle & =i\left(\alpha \beta^{*}-\beta \alpha^{*}\right),  \tag{2.2.17}\\
\langle i| \gamma_{1 ; L} \gamma_{2 ; L} \gamma_{2 ; R} \gamma_{2 ; L}|i\rangle & =i\left(\alpha \beta^{*}-\beta \alpha^{*}\right),  \tag{2.2.18}\\
\langle i| \gamma_{1 ; L} \gamma_{2 ; L} \gamma_{1 ; R} \gamma_{2 ; L}|i\rangle & =-i\left(|\alpha|^{2}-|\beta|^{2}\right),  \tag{2.2.19}\\
\langle i| \gamma_{1 ; L} \gamma_{2 ; L} \gamma_{2 ; R} \gamma_{1 ; L}|i\rangle & =i\left(|\alpha|^{2}-|\beta|^{2}\right) . \tag{2.2.20}
\end{align*}
$$

An interesting effect occurs: the crossed terms tend to cancel each other whereas the parallel terms amplify each other. When the other initial state is (2.2.9) this tendency is reversed.

With these relations the rate becomes, with $\Gamma_{\rightarrow}=\Gamma_{\|}+\Gamma_{\times}$where $\Gamma_{\|}, \Gamma_{\times}$are the terms
with $s=0$ and $s=1$ respectively.

$$
\begin{align*}
\Gamma_{\|}= & 2 \pi \sum_{q k ; i} \mid\left[\left(t_{1 ; L ; q}^{*} t_{1 ; R ; k}+t_{2 ; L ; q}^{*} t_{2 ; R ; k}\right) i\left(|\beta|^{2}-|\alpha|^{2}\right)+\left(t_{1 ; L ; q}^{*} t_{2 ; R ; k}-t_{2 ; L ; q}^{*} t_{1 ; R ; k}\right)(-i)\left(\alpha \beta^{*}-\beta \alpha^{*}\right)\right] \\
& \times\left.\frac{\langle i| c_{L q}^{\dagger} c_{R k} c_{L q} c_{R k}^{\dagger}|i\rangle}{\mathcal{U}}\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) .  \tag{2.2.21}\\
\Gamma_{\times}= & 2 \pi \sum_{q k ; i} \mid\left[\left(t_{1 ; L ; q}^{*} t_{1 ; R ; k}+t_{2 ; L ; q}^{*} t_{2 ; R ; k}\right)(-i)\left(\alpha \beta^{*}-\beta \alpha^{*}\right)+\left(t_{1 ; L ; q}^{*} t_{2 ; R ; k}-t_{2 ; L ; q}^{*} t_{1 ; R ; k}\right) i\left(|\beta|^{2}-|\alpha|^{2}\right)\right] \\
& \times\left.\frac{\langle i| c_{L q}^{\dagger} c_{R k} c_{L q} c_{R k}^{\dagger}|i\rangle}{\mathcal{U}}\right|^{2} W_{i} \delta\left(E_{f}-E_{i}\right) . \tag{2.2.22}
\end{align*}
$$

Assume now for simplicity that the tunnelling is the same for each nanowire. Then we find

$$
\begin{equation*}
G=2 \pi e^{2} d^{2} \frac{1}{\mathcal{U}^{2}}\left[4\left|t_{L}^{*} t_{R}\left(|\beta|^{2}-|\alpha|^{2}\right)\right|^{2}+4\left|t_{L}^{*} t_{R}(-i)\left(\alpha \beta^{*}-\beta \alpha^{*}\right)\right|^{2}\right] \tag{2.2.23}
\end{equation*}
$$

This result is independent of whether the system was in the initial state (2.2.8) or (2.2.9), which is remarkable. Even when the parities of the wires are different the tunnelling is not suppressed due to the additional tunnelling paths corresponding to the crossed process.

### 2.3 Low-energy Hamiltonian

We will now apply the methods of section 1.3 to get a clearer picture of the effects described above. We want to take a general approach that is applicable for the following chapters as well as here. The most interesting system is the "one-island" system described above, but this time we only connect each lead to one of the Majorana fermions. As mentioned in the introduction to this chapter this system is essentially a spin- $1 / 2$ system which, as we shall see, has some nice properties related to surface coding. This system is also typically called a Majorana-Cooper box.

The Hamiltonian for the system is, in the low-energy approximation of the tunnelling, given by

$$
\begin{equation*}
H=H_{L}+H_{R}+\left(t_{L} d_{L}^{\dagger} \gamma_{L}+t_{R} d_{R}^{\dagger} \gamma_{R}+\text { h.c. }\right)+E_{C}\left(N-n_{g}\right)^{2} . \tag{2.3.1}
\end{equation*}
$$

We now just work with general systems on the left and right and $d_{L}, d_{R}$ are the electron annihilation operators of the left and right systems respectively.

We want to write an effective Hamiltonian that does not involve the sectors where the island is charged. The Hamiltonian that connects the low-energy sector $N=N_{0}$ to the highenergy sector $N=N_{0} \pm 1$ is given by the tunnelling Hamiltonian appropriately projected:

$$
\begin{equation*}
H_{H L}=P_{N_{0}+1}\left(t_{L}^{*} \gamma_{L} d_{L}+t_{R}^{*} \gamma_{R} d_{R}\right) P_{N_{0}}+P_{N_{0}-1}\left(t_{L} d_{L}^{\dagger} \gamma_{L}+t_{R} d_{R}^{\dagger} \gamma_{R}\right) P_{N_{0}} \tag{2.3.2}
\end{equation*}
$$

$P_{N}$ projects to the sector with charge $N$ on the island. We then find

$$
\begin{align*}
H_{\mathrm{eff}}-H_{L L} & =H_{L H}\left(E-H_{H H}\right)^{-1} H_{H L} \\
& =P_{N_{0}}\left(t_{L} d_{L}^{\dagger} \gamma_{L}+t_{R} d_{R}^{\dagger} \gamma_{R}\right) P_{N_{0}+1}\left(E-E_{N_{0}+1}\right)^{-1} P_{N_{0}+1}\left(t_{L}^{*} \gamma_{L} d_{L}+t_{R}^{*} \gamma_{R} d_{R}\right) P_{N_{0}} \\
& +P_{N_{0}}\left(t_{L}^{*} \gamma_{L} d_{L}+t_{R}^{*} \gamma_{R} d_{R}\right) P_{N_{0}-1}\left(E-E_{N_{0}-1}\right)^{-1} P_{N_{0}-1}\left(t_{L} d_{L}^{\dagger} \gamma_{L}+t_{R} d_{R}^{\dagger} \gamma_{R}\right) P_{N_{0}} \tag{2.3.3}
\end{align*}
$$

We have replaced the high-energy Hamiltonian by just the energy of the $N_{0} \pm 1$ sector as explained in sec. 1.3.

In each of the two lines of eq. (2.3.3) there are two types of terms. The first type involves the product of $d_{L / R}$ and $d_{L / R}^{\dagger}$ and the other type involves the product of $d_{L / R}$ and $d_{R / L}^{\dagger}$. The first type renormalizes the dot levels, while the second term is an effective tunnel-coupling between the dots.

The renormalization terms are, with $U_{ \pm}=E_{N_{0} \pm 1}-E$,

$$
\begin{aligned}
& \frac{-1}{U_{+}}\left(\left|t_{L}\right|^{2} d_{L}^{\dagger} \gamma_{L} \gamma_{L} d_{L}+\left|t_{R}\right|^{2} d_{R}^{\dagger} \gamma_{R} \gamma_{R} d_{R}\right) \\
+ & \frac{-1}{U_{-}}\left(\left|t_{L}\right|^{2} \gamma_{L} d_{L} d_{L}^{\dagger} \gamma_{L}+\left|t_{R}\right|^{2} \gamma_{R} d_{R} d_{R}^{\dagger} \gamma_{R}\right) .
\end{aligned}
$$

We have ignored the projection operators, because they act trivially. The Majorana operators can be commuted to cancel via $\gamma^{2}=1$. In the second term we write $d d^{\dagger}=1-d^{\dagger} d$ and we find

$$
\begin{equation*}
=-\left(\frac{1}{U_{+}}-\frac{1}{U_{-}}\right)\left(\left|t_{L}\right|^{2} d_{L}^{\dagger} d_{L}+\left|t_{R}\right|^{2} d_{R}^{\dagger} d_{R}\right)+\text { const. } \tag{2.3.4}
\end{equation*}
$$

When $n_{g} \in \mathbb{Z}$ this term is zero.
Turning to the effective tunnelling we find

$$
\begin{align*}
& \frac{-1}{U_{+}}\left(t_{L} d_{L}^{\dagger} \gamma_{L} t_{R}^{*} \gamma_{R} d_{R}+t_{R} d_{R}^{\dagger} \gamma_{R} t_{L}^{*} \gamma_{L} d_{L}\right) \\
+ & \frac{-1}{U_{-}}\left(t_{L}^{*} \gamma_{L} d_{L} t_{R} d_{R}^{\dagger} \gamma_{R}+t_{R}^{*} \gamma_{R} d_{R} t_{L} d_{L}^{\dagger} \gamma_{L}\right) \\
= & \left(\frac{1}{U_{+}}+\frac{1}{U_{-}}\right)\left(t_{L} t_{R}^{*} d_{L}^{\dagger} d_{R} \gamma_{R} \gamma_{L}+\text { h.c. }\right) . \tag{2.3.5}
\end{align*}
$$

With this we can write the effective Hamiltonian as

$$
\begin{equation*}
H=H_{L}+H_{R}+\left(T \sigma_{z} d_{L}^{\dagger} d_{R}+\text { h.c. }\right) \tag{2.3.6}
\end{equation*}
$$

Here we have introduced $\sigma_{z}=i \gamma_{R} \gamma_{L}$ and introduced

$$
\begin{equation*}
T=-i\left(\frac{1}{U_{+}}+\frac{1}{U_{-}}\right) t_{L} t_{R}^{*} \tag{2.3.7}
\end{equation*}
$$

We see how, if we imagine connecting the leads in more complicated ways, the lowenergy approximation in general leads to tunnelling terms for each path through the system. Moreover, if a path "touches" a pair of Majoranas, the tunnelling is $\sim t i \gamma_{1} \gamma_{2}$. With the Hamiltonian (2.3.6) it is also considerably easier to see how the effects calculated above occur, but it is important to have a notion of the validity of this Hamiltonian; it essentially accounts for only co-tunnelling and assumes that the high-charge sectors have no dynamics.

### 2.4 Measurements

We will now discuss a general question: When can I use a current measurement to project the island to a certain state?

A general tunnelling setup consists of leads or dots that are connected to the Majorana Island in an arbitrary way and possibly have an interference link. Each path yields an effective tunnelling of $T i \gamma_{i} \gamma_{j}$, if the path starts at $\gamma_{i}$ and ends at $\gamma_{j}$, as we saw above. $T$ is a complex tunnelling amplitude which can, in principle, take any value. Therefore the general tunnelling Hamiltonian is

$$
\begin{array}{r}
H=H_{L}+H_{R}+\sum_{k, u} \lambda c_{L k}^{\dagger} c_{R u}+\text { h.c. }, \\
\lambda=A \sigma_{x}+B \sigma_{y}+C \sigma_{z}+D, \quad A, B, C, D \in \mathbb{C} . \tag{2.4.2}
\end{array}
$$

Here $k, u$ label the states in the left and right subsystems. One or both may be the trivial sum, as when one subsystem is a single level. We have assumed for simplicity that the tunnelling amplitudes do not depend on $k, u$.

As we know from the fundamental principles of quantum mechanics a measurement of the observable $\mathcal{O}$ projects to an eigenstate of $\mathcal{O}$. The current operator $J=\partial_{t} N_{L}=i\left[H, N_{L}\right]$ is

$$
\begin{align*}
J & =i \sum_{k, u} \lambda c_{L k}^{\dagger} c_{R u}-\lambda^{\dagger} c_{R u}^{\dagger} c_{L k}=i \lambda \psi_{L}^{\dagger} \psi_{R}-i \lambda^{\dagger} \psi_{R}^{\dagger} \psi_{L} \\
& =\left(\begin{array}{ll}
\psi_{L}^{\dagger} & \psi_{R}^{\dagger}
\end{array}\right) i\left(\begin{array}{cc}
0 & \lambda \\
-\lambda^{\dagger} & 0
\end{array}\right)\binom{\psi_{L}}{\psi_{R}} . \tag{2.4.3}
\end{align*}
$$

For a general operator $\lambda$ we do not have that $\left[\lambda, \lambda^{\dagger}\right]=0$. This is rather unfortunate because it means that the eigenstates of $J$ are typically entangled states.

So when $\left[\lambda, \lambda^{\dagger}\right] \neq 0$ we cannot view the current measurement as projecting the island to a definite state. In chapter 5 we will discuss how to prepare specific states of the Majorana island and we will therefore have to consider the requirement $\left[\lambda, \lambda^{\dagger}\right]=0$.

We will also see this fact mirrored in the discussion in sec. 4.2. There $\left[\lambda, \lambda^{\dagger}\right] \neq 0$ means that the adiabatic tunnelling procedure (to be described) does not correspond to a 'good' transformation on the Majorana island.

### 2.5 Concluding Remarks

In this chapter we have investigated some sample systems with Majorana fermions from the point of view of conductance (or current) measurements.

We have started with some "first principle" calculations based on Fermi's Golden Rule. It became apparent that the parity of state of the wire could have a significant influence on the conductance; a rather peculiar effect.

These calculations served as a sort of introduction to the physics of electrons tunnelling through Majorana states. We then developed a useful picture based on a low-energy effective model where each tunnelling path has an amplitude that depends on the Majorana operator of the path.

The effect of this becomes most pronounced when we consider a one-island, two-nanowires setup. This setup, which is termed a Majorana-Cooper Box in the literature, is essentially a two-level system because one can redistribute electrons between the wires and superconductor without changing the charge number. When this becomes possible the tunnelling amplitudes $\sim T \sigma_{z}$ become dynamical variables, or, in an equivalent picture, tunnelling processes affect the dynamical variables of the system. This is particularly attractive in the context of surface coding to be discussed in chapter 5 .

## Chapter 3

## Decoherence Effects

We have discussed measurements on Majorana systems, but a question of considerable interest in applications is the time-scale associated with a measurement. There are two things to consider: the physical timescale and the instrumental timescale. By physical timescale we mean the amount of time it takes for collapse to occur. By instrumental timescale we mean the amount of time it takes for the instruments to get a definite reading. In this chapter we deal with the physical timescale, but one should of course consider the instrumental timescale as well for any specific setup. We saw in chapter 2 how tunnelling through a pair of Majorana fermions on the Majorana island has an amplitude $\mathcal{T}=T \sigma_{z}$ that depends on the parity operator of the two Majoranas.

We expect that a conductance measurement in the interference setup of figure 2.2 would collapse the state to a definite parity.

In chapter 4 we will see how an adiabatic tunnelling process is equivalent to applying $\sigma_{z}$ to the state of the island. This offers a different picture of the collapse: A conductance experiment sends a macroscopic number of electrons through the sample in order to measure a current. Although this is not adiabatic at all, if we suspend disbelief and imagine that it $i s$, a conductance experiment produces a random phase flip of the state of the island:

$$
\begin{equation*}
|0\rangle+|1\rangle \rightarrow|0\rangle+(-1)^{n}|1\rangle, \tag{3.0.1}
\end{equation*}
$$



Figure 3.1: Two leads (orange blobs) are connected to the Majorana island. The island consists of a superconductor (magenta box) on top of which sits two topological nanowires (grey cylinder). The nanowires host Majorana fermions (red circles).
where $n$ is the number of electrons that passed through the island in the course of the experiment, a number which is obviously uncontrollable. At the very least we can assume that the probability that $n$ is even is equal to the probability that $n$ is odd. In that case, the probabilities $\mathcal{P}\left(\sigma_{z}=1\right)=\frac{1}{2}, \mathcal{P}\left(\sigma_{z}=-1\right)=\frac{1}{2}$ are unchanged, but $\mathcal{P}\left(\sigma_{x}=1\right)=\mathcal{P}(n$ even $)=$ $\frac{1}{2}$ and $\mathcal{P}\left(\sigma_{x}=-1\right)=\mathcal{P}(n$ odd $)=\frac{1}{2}$. In total, this heuristic argument tells us that the density matrix of the system undergoes the following transformation due to the conductance experiment

$$
\rho=\frac{1}{2}\left(\begin{array}{ll}
1 & 1  \tag{3.0.2}\\
1 & 1
\end{array}\right) \rightarrow \rho=\frac{1}{2}\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) .
$$

This is again 'collapse' but in the decoherence language.
The system we will study here is shown in figure 3.1. Two leads are connected to Majorana fermions on the Majorana-Cooper box.

We start with a brief introduction of the decoherence formalism and then turn to the decoherence effect itself. We study a simplified model where there is no interference link. This is because the calculation in the simplified system is already quite involved. One can show (it is also quite intuitive) that adding an interference link will not affect the decoherence time, as should be clear to the reader after studying the calculations.

### 3.1 Decoherence

If we have a system that can be separated into two subsystems, A and B, where perhaps only one is of interest, a natural question is "What is the state of subsystem A after a time t?" When we speak of the state of subsystem A we refer to the reduced density matrix

$$
\begin{equation*}
\rho_{A}(t)=\operatorname{Tr}_{B} \rho(t)=\operatorname{Tr}_{B} U(t) \rho(t=0) U^{\dagger}(t) . \tag{3.1.1}
\end{equation*}
$$

The results of experiments that only probe subsystem A are fully described by $\rho_{A}$, since any moment of the distribution of results can be found as

$$
\begin{equation*}
\left\langle\mathcal{O}^{n}\right\rangle=\operatorname{Tr}_{A} \mathcal{O}^{n} \rho_{A}, \tag{3.1.2}
\end{equation*}
$$

where $\mathcal{O}$ is the observable measured in the experiment. Of course, this does not hold if $\mathcal{O}$ involves operators in subsystem B.

If $\rho_{A}(t)$ is initially in some pure state and as $t \rightarrow \infty$ evolves to a mixed state then decoherence is said to occur. One can view this as a model for measurement within quantum mechanics that does not invoke the semi-mystical collapse phenomenon.

### 3.1.1 The Model

In the setup figure 3.1 the natural division is $\mathrm{A}=$ the Majorana Island and $\mathrm{B}=$ the leads. The name of the game is therefore to calculate

$$
\begin{equation*}
\rho_{I}(t)=\operatorname{Tr}_{\text {leads }} U(t) \rho(t=0) U^{\dagger}(t) \tag{3.1.3}
\end{equation*}
$$

The derivation of the low-energy Hamiltonian mirrors the derivation in section 2.3 so we will not repeat it here, except schematically. There will also be a renormalization of the energies in the leads, i.e. a term $T_{1} \psi_{L}^{\dagger} \psi_{L}+T_{1} \psi_{R}^{\dagger} \psi_{R}$. We also have tunnelling through the Majorana island which yields a term $T_{2} \sigma_{z} \psi_{R}^{\dagger} \psi_{L}+$ h.c.. The effective Hamiltonian is

$$
\begin{equation*}
H=H_{L}+H_{R}+\sigma_{z} \sum_{k, u}\left(T_{2} c_{L k}^{\dagger} c_{R u}+\text { h.c. }\right)+\sum_{k, u, a=L, R}\left(\tilde{T}_{1} c_{a k}^{\dagger} c_{a u}+\text { h.c. }\right) . \tag{3.1.4}
\end{equation*}
$$

For simplicity we have assumed that the renormalization is the same in each lead.
Since the last term only involves operators from the same lead we may write the Hamiltonian in the slightly simpler form:

$$
\begin{equation*}
H=H_{L}+H_{R}+\sigma_{z} \sum_{k, u}\left(T_{2} c_{L k}^{\dagger} c_{R u}+\text { h.c. }\right)+\sum_{k, u, a=L, R} T_{1} c_{a k}^{\dagger} c_{a u} \tag{3.1.5}
\end{equation*}
$$

where $T_{1}=2 \operatorname{Re}\left(\tilde{T}_{1}\right)$.

### 3.1.2 The Decoherence

We assume the full system is initially described by the density matrix

$$
\begin{equation*}
\rho(t=0)=\sum_{i, j=0,1} c_{i j}|i\rangle\langle j| \otimes \rho_{L}(t=0) \otimes \rho_{R}(t=0), \tag{3.1.6}
\end{equation*}
$$

where $|i\rangle$ is the eigenstate of $\sigma_{z}$ with eigenvalue $i$ and the leads are thermally distributed: $\rho_{a}(t=0)=\frac{1}{Z_{a}} e^{-\beta H_{a}}$. With a voltage applied across the leads the Hamiltonian (3.1.5) becomes

$$
H=\sum_{k, u}\left(c_{L k}^{\dagger}, c_{R k}^{\dagger}\right)\left(\begin{array}{cc}
\left(\xi_{k}+V\right) \delta_{u k}+T_{1} & \sigma_{z} T_{2}  \tag{3.1.7}\\
\sigma_{z} T_{2}^{*} & \left(\xi_{k}-V\right) \delta_{u k}+T_{1}
\end{array}\right)\binom{c_{L u}}{c_{R u}}
$$

Our approach is to take the tunnelling as the perturbation, perform the time-evolution of reduced density matrix and expand the resulting expression to second order.

We are in the interaction picture:

$$
\begin{align*}
H_{0} & =\sum_{k}\left(c_{L k}^{\dagger}, c_{R k}^{\dagger}\right)\left(\begin{array}{cc}
\xi_{k} & 0 \\
0 & \xi_{k}
\end{array}\right)\binom{c_{L k}}{c_{R k}}  \tag{3.1.8}\\
V(s) & =\sum_{\substack{k, u \\
\alpha, \beta}} c_{\alpha k}^{\dagger}\left[T_{1} \tau_{0}+\sigma_{z}\left(\operatorname{Re}\left(T_{2}\right) \tau_{x}-\operatorname{Im}\left(T_{2}\right) \tau_{y}\right)\right]_{\alpha \beta} c_{\beta u} \exp \left(i\left(\xi_{k}-\xi_{u}\right) s\right), \tag{3.1.9}
\end{align*}
$$

where $\alpha, \beta$ take values $L, R$ and we include the voltage only in the distribution functions $n_{F}\left(\xi_{k}\right)=\left\langle c_{k}^{\dagger} c_{k}\right\rangle$ to capture the out-of-equilibrium physics. The time-evolution operator
depends on $\sigma_{z}$ but since we have written the density matrix in the eigenbasis of $\sigma_{z}$ we can write

$$
\begin{align*}
\rho_{A}(t) & =\operatorname{Tr}_{L} \sum_{i, j} c_{i j} U|i\rangle\langle j| \otimes \rho_{L}(t=0) U^{\dagger}=\operatorname{Tr}_{L} \sum_{i, j} c_{i j}|i\rangle\langle j| \otimes U_{i} \rho_{L} U_{j}^{\dagger} \\
& =\sum_{i j} c_{i j}|i\rangle\langle j|\left\langle U_{j}^{\dagger}(t) U_{i}(t)\right\rangle_{0} . \tag{3.1.10}
\end{align*}
$$

Here $U_{i}$ is the time-evolution operator with $\sigma_{z}$ replaced by its eigenvalue, $\eta_{i}$.
To reiterate: We are interested in whether $\left|c_{i j}\right| \rightarrow 0$ when $t \rightarrow \infty$ as this signifies decoherence.

To calculate this expectation value we then perform a cumulant expansion, with timeordering $T_{t}$ and anti-time-ordering $\bar{T}_{t}$ :

$$
\begin{align*}
\left\langle U_{j}^{\dagger}(t) U_{i}(t)\right\rangle_{0} \approx & \left(1+i \int V_{j}(s)-\frac{1}{2} \bar{T}_{t} \iint V_{j}(s) V_{j}\left(s^{\prime}\right)\right)\left(1-i \int V_{i}(s)-\frac{1}{2} T_{t} \iint V_{i}(s) V_{i}\left(s^{\prime}\right)\right) \\
\approx & \exp \left(i \int\left\langle V_{j}\right\rangle-i \int\left\langle V_{i}\right\rangle-\frac{1}{2}\left(\iint\left\langle T_{t} V_{i}(s) V_{i}\left(s^{\prime}\right)\right\rangle-\left(\int\left\langle V_{i}\right\rangle\right)^{2}\right)\right. \\
& -\frac{1}{2}\left(\iint\left\langle\bar{T}_{t} V_{j}(s) V_{j}\left(s^{\prime}\right)\right\rangle-\left(\int\left\langle V_{j}\right\rangle\right)^{2}\right) \\
& \left.+\left(\iint\left\langle V_{j}(s) V_{i}\left(s^{\prime}\right)\right\rangle-\iint\left\langle V_{j}(s)\right\rangle\left\langle V_{i}\left(s^{\prime}\right)\right\rangle\right)\right) \tag{3.1.11}
\end{align*}
$$

We have kept only terms of second order. The first two terms in the exponent are phases and do not contribute to decoherence; we ignore them.

To simplify the notation we introduce

$$
\begin{equation*}
h_{i ; \alpha \beta}^{k u}(s) \equiv\left[T_{1} \tau_{0}+\eta_{i}\left(\operatorname{Re}\left(T_{2}\right) \tau_{x}-\operatorname{Im}\left(T_{2}\right) \tau_{y}\right)\right]_{\alpha \beta} \exp \left(i\left(\xi_{k}-\xi_{u}\right) s\right) . \tag{3.1.12}
\end{equation*}
$$

Now we dare to tackle the cumulants. Because we have always the structure $\langle V V\rangle-\langle V\rangle^{2}$ only the connected diagrams in the Wick expansion of $\langle V V\rangle$ contribute. We find

$$
\begin{align*}
\iint\left\langle V_{j}(s) V_{i}\left(s^{\prime}\right)\right\rangle_{c, 0} & =\iint \sum_{\substack{k, u, \alpha, \beta \\
k^{\prime}, u^{\prime}, \alpha^{\prime}, \beta^{\prime}}} h_{j ; \alpha \beta}^{k u}(s) h_{i ; \alpha^{\prime} \beta^{\prime}}^{k^{\prime} u^{\prime}}\left(s^{\prime}\right)\left\langle c_{\alpha k}^{\dagger} c_{\beta u} c_{\alpha^{\prime} k^{\prime}}^{\dagger} c_{\beta^{\prime} u^{\prime}}\right\rangle_{c, 0} \\
& =\iint \sum_{\substack{k^{\prime} \\
h_{j ; \alpha \beta}^{\prime}}} h_{j, h_{i}}^{k u}\left(h_{i ; \alpha^{\prime} \beta^{\prime}}\left(s^{\prime}\right) n_{F}\left(\xi_{\alpha k}\right) \delta_{k u^{\prime}} \delta_{\alpha \beta^{\prime}}\left(1-n_{F}\left(\xi_{\alpha^{\prime} k^{\prime}}\right)\right) \delta_{u k^{\prime}} \delta_{\beta \alpha^{\prime}}\right. \\
& =\iint \sum_{\substack{k, k^{\prime} \\
\alpha, \alpha^{\prime}}} n_{F}\left(\xi_{\alpha k}\right) h_{j ; \alpha \alpha^{\prime}}^{k k^{\prime}}(s)\left(1-n_{F}\left(\xi_{\alpha^{\prime} k^{\prime}}\right)\right) h_{i ; \alpha^{\prime} \alpha}^{k^{\prime} k}\left(s^{\prime}\right) \tag{3.1.13}
\end{align*}
$$

We have now introduced $\xi_{\alpha k}=\xi_{k} \pm V$ in the distribution functions to account for the
difference in chemical potential. Performing the sum over the $\alpha / \alpha^{\prime}$ indices yields

$$
\begin{align*}
& \sum_{\alpha, \alpha^{\prime}} n_{F}\left(\xi_{\alpha k}\right) h_{j ; \alpha \alpha^{\prime}}^{k k^{\prime}}(s)\left(1-n_{F}\left(\xi_{\alpha^{\prime} k^{\prime}}\right)\right) h_{i ; \alpha^{\prime} \alpha}^{k^{\prime} k}\left(s^{\prime}\right) \\
& =\left(n_{F}\left(\xi_{L k}\right)\left(1-n_{F}\left(\xi_{L k^{\prime}}\right)\right) e^{i\left(\xi_{k}-\xi_{k^{\prime}}\right)\left(s-s^{\prime}\right)}+n_{F}\left(\xi_{R k}\right)\left(1-n_{F}\left(\xi_{R k^{\prime}}\right)\right) e^{i\left(\xi_{k}-\xi_{k^{\prime}}\right)\left(s-s^{\prime}\right)}\right) T_{1}^{2} \\
& +\left(n_{F}\left(\xi_{L k}\right)\left(1-n_{F}\left(\xi_{R k^{\prime}}\right)\right) e^{i\left(\xi_{k}-\xi_{k^{\prime}}\right)\left(s-s^{\prime}\right)}+n_{F}\left(\xi_{R k}\right)\left(1-n_{F}\left(\xi_{L k^{\prime}}\right)\right) e^{i\left(\xi_{k}-\xi_{k^{\prime}}\right)\left(s-s^{\prime}\right)}\right) \eta_{j} \eta_{i}\left|T_{2}\right|^{2} \tag{3.1.14}
\end{align*}
$$

Doing the time integrals yields

$$
\begin{align*}
& \int_{0}^{t} \mathrm{~d} s \int_{0}^{t} \mathrm{~d} s^{\prime} e^{i \omega\left(s-s^{\prime}\right)}=\frac{\left[e^{i \omega t}-1\right]\left[e^{-i \omega t}-1\right]}{\omega^{2}} \\
& =2 \frac{1-\cos (\omega t)}{\omega^{2}} \tag{3.1.15}
\end{align*}
$$

With this, we have

$$
\begin{align*}
\iint & \sum_{\alpha, \alpha^{\prime}} n_{F}\left(\xi_{\alpha k}\right) h_{j ; \alpha \alpha^{\prime}}^{k k^{\prime}}(s)\left(1-n_{F}\left(\xi_{\alpha^{\prime} k^{\prime}}\right)\right) h_{i ; \alpha^{\prime} \alpha}^{k^{\prime} k}\left(s^{\prime}\right) \\
& =\left(n_{F}\left(\xi_{L k}\right)\left(1-n_{F}\left(\xi_{L k^{\prime}}\right)\right)+n_{F}\left(\xi_{R k}\right)\left(1-n_{F}\left(\xi_{R k^{\prime}}\right)\right)\right) \frac{2}{\omega_{k k^{\prime}}^{2}}\left(1-\cos \left(\omega_{k k^{\prime}} t\right) T_{1}^{2}\right. \\
& +\left(n _ { F } ( \xi _ { L k } ) ( 1 - n _ { F } ( \xi _ { R k ^ { \prime } } ) ) \frac { 2 } { \omega _ { k k ^ { \prime } } ^ { 2 } } \left(1-\cos \left(\left(\omega_{k k^{\prime}}\right) t\right)\right.\right. \\
& +n_{F}\left(\xi_{R k}\right)\left(1-n_{F}\left(\xi_{L k^{\prime}}\right)\right) \frac{2}{\omega_{k k^{\prime}}^{2}}\left(1-\cos \left(\left(\omega_{k k^{\prime}}\right) t\right)\right) \eta_{i} \eta_{j}\left|T_{2}\right|^{2} \tag{3.1.16}
\end{align*}
$$

where we introduced $\omega_{k k^{\prime}}=\xi_{k}-\xi_{k}^{\prime}$.
To evaluate the remaining sums over $k, k^{\prime}$ we again make use of the identity $\left(n_{F}(\epsilon)(1-\right.$ $\left.n_{F}\left(\epsilon^{\prime}\right)\right)=n_{B}\left(\epsilon-\epsilon^{\prime}\right)\left(n_{F}\left(\epsilon^{\prime}\right)-n_{F}(\epsilon)\right)$ and go to a continuum picture, thus:

$$
\begin{align*}
& \iint \sum_{\substack{k, k^{\prime} \\
\alpha, \alpha^{\prime}}} n_{F}\left(\xi_{\alpha k}\right) h_{j ; \alpha \alpha^{\prime}}^{k k^{\prime}}(s)\left(1-n_{F}\left(\xi_{\alpha^{\prime} k^{\prime}}\right)\right) h_{i ; \alpha^{\prime} \alpha}^{k^{\prime} k}\left(s^{\prime}\right) \\
& =d^{2} \int \mathrm{~d} \omega \int \mathrm{~d} \epsilon\left(2 n_{B}(\omega)\left(n_{F}(\epsilon)-n_{F}(\epsilon+\omega)\right)\right) \frac{2}{\omega^{2}}(1-\cos \omega t) T_{1}^{2} \\
& +d^{2} \int \mathrm{~d} \omega \int \mathrm{~d} \epsilon n_{B}(\omega+2 V)\left(n_{F}(\epsilon)-n_{F}(\epsilon+\omega+2 V)\right) \frac{2}{\omega^{2}}(1-\cos (\omega) t) \eta_{i} \eta_{j}\left|T_{2}\right|^{2} \\
& +d^{2} \int \mathrm{~d} \omega \int \mathrm{~d} \epsilon n_{B}(\omega-2 V)\left(n_{F}(\epsilon)-n_{F}(\epsilon+\omega-2 V)\right) \frac{2}{\omega^{2}}(1-\cos (\omega) t) \eta_{i} \eta_{j}\left|T_{2}\right|^{2} \tag{3.1.17}
\end{align*}
$$

The calculation of the terms with $\left\langle V_{i} V_{i}\right\rangle_{c, 0}$ and $\left\langle V_{j} V_{j}\right\rangle_{c, 0}$ is very similar. Together they yield the same as eq. (3.1.17) but without the factor of $\eta_{i} \eta_{j}$ and with an extra minus.

Integrating over $\epsilon$ in each term produces a factor of $\omega+2 V$ or $\omega-2 V$ depending on the second Fermi function. In total we get

$$
\begin{align*}
\left|c_{i j}(t)\right|= & \left|c_{i} j(0)\right| \exp \left(-\left(1-\eta_{i} \eta_{j}\right) 2 d^{2}\left|T_{2}\right|^{2} \int \mathrm{~d} \omega\left[n_{B}(\omega+2 V) \frac{\omega+2 V}{\omega^{2}}(1-\cos (\omega t))\right.\right. \\
& \left.\left.+n_{B}(\omega-2 V) \frac{\omega-2 V}{\omega^{2}}(1-\cos (\omega t))\right]\right) \tag{3.1.18}
\end{align*}
$$

To determine the timescale for decoherence we then need to calculate

$$
\begin{equation*}
\int \mathrm{d} \omega n_{B}(\omega \pm 2 V) \frac{\omega \pm 2 V}{\omega^{2}}(1-\cos (\omega t)) \tag{3.1.19}
\end{equation*}
$$

We assume that we have small temperatures so that we can set $n_{B}(\omega \pm 2 V)=-\Theta(-\omega \mp 2 V)$. With a bandwidth $D$, we may then write for the integral above

$$
\begin{equation*}
-\int_{-D}^{\mp 2 V} \mathrm{~d} \omega \frac{\omega \pm 2 V}{\omega^{2}}(1-\cos \omega t) \tag{3.1.20}
\end{equation*}
$$

If we are in the very short time limit, i.e. $\omega t \ll 1$ for all $\omega,|\omega|<D$ we can Taylor-expand the cosine-term and we find

$$
\begin{equation*}
-\int_{-D}^{\mp 2 V} \mathrm{~d} \omega(\omega \pm 2 V) t^{2}=\frac{1}{2}(D \mp 2 V)^{2} t^{2} \tag{3.1.21}
\end{equation*}
$$

In the intermediate time limit $\omega t \sim 1$ for a sizeable region of energies and $1 / t>2 V$. In this case we do a rough estimate of the integral as follows. We again assume low temperature so that the Bose function provides a cutoff for small energies and only allows negative energies. We then separate the integral into two parts: $|\omega|<1 / t$ and $|\omega|>1 / t$. For the first part we let $\omega t \rightarrow 0$ (the rough part of the estimate). The integrand is in this limit equal to $\pm 2 V \frac{t^{2}}{2}$. For the part of the integral with $|\omega|>1 / t$ we assume that the contribution from the term proportional to $\cos \omega t$ average out (another rough estimate). In total the integral becomes

$$
\begin{align*}
& -\int_{-1 / t}^{\mp 2 V} \mathrm{~d} \omega( \pm 2 V) \frac{t^{2}}{2}-\int_{-D}^{-1 / t} \mathrm{~d} \omega \frac{\omega \pm 2 V}{\omega^{2}}=( \pm 2 V) \frac{t^{2}}{2}\left(\mp 2 V+\frac{1}{t}\right)+\int_{1 / t}^{D} \mathrm{~d} \omega \frac{\omega \mp 2 V}{\omega^{2}} \\
& =2 V^{2} t^{2} \pm V t+\ln (D t) \pm 2 V\left(\frac{1}{D}-t\right)=\ln (D t) \pm V t+2 V^{2} t^{2} \pm \frac{2 V}{D} \tag{3.1.22}
\end{align*}
$$

In the long time limit $(D t \gg 1 ; 1 / t<2 V)$ we have the following expression in the exponent:

$$
\begin{align*}
& -\int_{-D}^{-2 V}(1-\cos \omega t) \frac{\omega+2 V}{\omega^{2}}-\int_{-D}^{2 V}(1-\cos \omega t) \frac{\omega-2 V}{\omega^{2}} \\
& =2 V \int_{-2 V}^{2 V} \frac{1-\cos \omega t}{\omega^{2}}-\int_{0}^{2 V} \frac{1-\cos \omega t}{\omega}+\int_{-2 V}^{0} \frac{1-\cos \omega t}{\omega}-2 \int_{-D}^{0} \frac{1-\cos \omega t}{\omega} \\
& =4 V \int_{0}^{2 V} \frac{1-\cos \omega t}{\omega^{2}}-2 \int_{0}^{2 V} \frac{1-\cos \omega t}{\omega}-2 \int_{-D}^{0} \frac{1-\cos \omega t}{\omega} \tag{3.1.23}
\end{align*}
$$

In the last two terms the integrand is zero for $|\omega|<1 / t$. For the first term the integrand is $\frac{t^{2}}{2}$ for $\omega<1 / t$. For larger $|\omega|$ we assume that the cosine averages to zero. We get

$$
\begin{align*}
& =4 V \int_{0}^{1 / t} \frac{t^{2}}{2} \mathrm{~d} \omega+4 V \int_{1 / t}^{2 V} \frac{\mathrm{~d} \omega}{\omega^{2}}-2 \int_{1 / t}^{2 V} \frac{\mathrm{~d} \omega}{\omega}-2 \int_{-D}^{-1 / t} \frac{\mathrm{~d} \omega}{\omega} \\
& =2 V t-4 V\left(\frac{1}{2 V}-t\right)-2 \ln (2 V t)+2 \ln (D t) \tag{3.1.24}
\end{align*}
$$

### 3.1.3 Summary of results

For small times the evolution of the density matrix is described by

$$
\begin{equation*}
\left|c_{i j}(t)\right|=\left|c_{i j}(0)\right| \exp \left(-\left(1-\eta_{i} \eta_{j}\right) d^{2}\left|T_{2}\right|^{2}\left[D^{2}+4 V^{2}\right] t^{2}\right) \tag{3.1.25}
\end{equation*}
$$

For intermediate times we found

$$
\begin{equation*}
\left|c_{i j}(t)\right|=\left|c_{i j}(0)\right| \exp \left(-\left(1-\eta_{i} \eta_{j}\right) 4 d^{2}\left|T_{2}\right|^{2}\left(\ln (D t)+2 V t^{2}\right)\right) . \tag{3.1.26}
\end{equation*}
$$

This result also holds for large times when $V=0$.
For large times with voltage

$$
\begin{equation*}
\left|c_{i j}(t)\right|=\left|c_{i j}(0)\right| \exp \left(-\left(1-\eta_{i} \eta_{j}\right) d^{2}\left|T_{2}\right|^{2}\left(6 V t-2+2 \ln \frac{D}{2 V}\right)\right) . \tag{3.1.27}
\end{equation*}
$$

In these expressions $d$ is the density of states, $D$ is the bandwidth, $V$ is the voltage across the system, $T_{2}$ is the effective tunnelling amplitude between the leads and $\eta_{i}$ indicates the parity corresponding to $|i\rangle$.

These results apply for both $V \neq 0$ and $V=0$ so it is easy to compare the two regimes.
For $V=0$ the decay is initially Gaussian and then becomes like a power-law. For $V \neq 0$ the decay is initially Gaussian, undergoes a modified power-law phase, and finally becomes exponential.

We see that the off-diagonal elements, which satisfy $\eta_{i} \eta_{j}=-1$, go to zero as $t \rightarrow \infty$. In agreement with the discussion in the introduction see that tunnelling through $\sigma_{z}$ effectively collapses the state onto an eigenstate of $\sigma_{z}$.

### 3.2 Concluding Remarks

From the results of this chapter the time-scale of collapse can be calculated but it depends on the density of states, the tunnelling amplitudes and the bandwidth which can vary from system to system.

This chapter is not only useful in the context of experimental procedure; it also highlights the decoherence effect itself, which will occur when we connect a continuum to a finite system. For surface code applications this effect may be harmful or completely destructive, on the other hand we might need to perform a measurement to prepare the system or manipulate it.

## Chapter 4

## Adiabatic Tunnelling

In the previous chapter we were lead somewhat naturally from considering conductance experiments for topological nanowires to considering the double nanowire setup. This setup has a great deal more interesting physics than just peculiar conductance effects.

In this chapter (and the next) we will explore the single island setup. We will find out what happens when an electron is adiabatically transferred across a Majorana pair. The system we work with is shown schematically in figure 4.1. We again have two topological nanowires (grey cylinders) hosting Majorana fermions (red circles). The nanowires sit on a superconducting island (magenta box). In this setup two of the Majoranas are connected to 'dots' (orange circles). The dots are modelled as single levels where the level energy is tunable by a gate (not shown). The island is also contacted to a gate (not shown) to control the optimal number of electrons $N_{0}$.

In ref. [13] it was shown that a single dot connected to a Majorana (which was hosted in a non-floating topological nanowire) could be used to flip the parity of the ground state by adiabatically tunnelling an electron from the dot to the nanowire. The tunnelling was described by $d^{\dagger} \gamma+$ h.c., where $d$ is the electron operator on the dot, and it turned out that the adiabatic tunnelling essentially amounted to applying the operator $\gamma d$ to the state of the system, thus flipping the parity.

Based on this idea we expect that tunnelling adiabatically from, say, left to right in the


Figure 4.1: The setup for this chapter. Two topological nanowires (grey cylinders) sit on a floating superconductor (magenta box). The nanowires host Majorana fermions (red circles). Two Majoranas are connected to single level dots (orange circles).
setup in figure 4.1, essentially amounts to applying $\gamma_{L} d_{L} d_{R}^{\dagger} \gamma_{R}$ to the state of the system. This will turn out to be right but it takes some amount of work to show it. As discussed in section 1.2 .1 the island is essentially a spin- $1 / 2$ system and the different Majorana billinears correspond to different Pauli operators. Using adiabatic tunnelling we can rotate the spin in different ways.

### 4.1 Billinear Tunnelling

We again want an effective description of the system, as in section 2.3. The calculation is very similar, so we only highlight some details. Only two Majoranas will be involved, hence the term "billinear tunnelling".

The Hamiltonian for the system is, in the low-energy approximation of the tunnelling, given by

$$
\begin{equation*}
H=\epsilon d_{L}^{\dagger} d_{L}-\epsilon d_{R}^{\dagger} d_{R}+\left(t_{L} d_{L}^{\dagger} \gamma_{L}+t_{R} d_{R}^{\dagger} \gamma_{R}+\text { h.c. }\right)+E_{C}\left(N-n_{g}\right)^{2} . \tag{4.1.1}
\end{equation*}
$$

Here $\pm \epsilon$ is the energy of the left/right dot (only the difference between the levels will turn out to matter), $d_{L}, d_{R}$ are the electron annihilation operators of the left and right dots respectively. The Hilbert space is 16 -dimensional: two states of either dot, and four states of the island. To apply the adiabatic principle we will need to find the eigenvectors of a 16 by 16 matrix, which, while in principle doable, leads to very unwieldy expressions. A better approach is to use the method outlined in section 1.3 to find an effective 4 by 4 Hamiltonian. It will turn out that we have two 2-dimensional sectors that do not couple! That is a real improvement.

After performing the low-energy projection the effective Hilbert space has basis vectors

$$
\begin{aligned}
& |00+\rangle, \\
& |00-\rangle, \\
& |01+\rangle, \\
& |01-\rangle, \\
& |10+\rangle, \\
& |10-\rangle, \\
& |11+\rangle, \\
& |11-\rangle .
\end{aligned}
$$

However, if we notice that the Hamiltonian (4.1.1) preserves particle number, we see that the first pair and the last pair do not couple to each other or to the middle four states. Thus if we start with one electron in the left dot the only active states are $|10 \pm\rangle$ and $|01 \pm\rangle$. Since we want to tunnel an electron across the system we would like to operate in this subspace. The Hilbert space is effectively two spin- $1 / 2$ systems: $\mathcal{H}_{1 / 2} \otimes \mathcal{H}_{1 / 2}=$ Island $\otimes$ Dots. Furthermore, if we choose a good basis, i.e. if $\pm$ in the states above refer to the eigenvalues of $i \gamma_{R} \gamma_{L}$, there is no coupling between the different parities.

With

$$
\begin{align*}
i \gamma_{R} \gamma_{L} & =\sigma_{z}  \tag{4.1.2}\\
T & =-i\left(\frac{1}{U_{+}}+\frac{1}{U_{-}}\right) t_{L} t_{R}^{*}  \tag{4.1.3}\\
\delta \epsilon_{L / R} & =-\left(\frac{1}{U_{+}}-\frac{1}{U_{-}}\right)\left|t_{L / R}\right|^{2} \tag{4.1.4}
\end{align*}
$$

the effective Hamiltonian becomes

$$
H_{\mathrm{eff}}=\left(\begin{array}{cc}
\epsilon+\delta \epsilon_{L} & T \sigma_{z}  \tag{4.1.5}\\
T^{*} \sigma_{z} & -\epsilon+\delta \epsilon_{R}
\end{array}\right) .
$$

Written out

$$
H_{\mathrm{eff}}=\left(\begin{array}{cccc}
\epsilon+\delta \epsilon_{L} & T & 0 & 0  \tag{4.1.6}\\
T^{*} & -\epsilon+\delta \epsilon_{R} & 0 & 0 \\
0 & 0 & \epsilon+\delta \epsilon_{L} & -T \\
0 & 0 & -T^{*} & -\epsilon+\delta \epsilon_{R}
\end{array}\right)
$$

Thus we have shown the assertion in the beginning of the section: we have two decoupled 2 by 2 systems. As we also claimed we see that only $\left(\epsilon+\delta \epsilon_{L}\right)-\left(-\epsilon+\delta \epsilon_{R}\right)$ has an effect on the dynamics, so we can absorb $\delta \epsilon_{L}-\delta \epsilon_{R}$ into the definition of $\epsilon$. The final result is worth writing out

$$
H_{\mathrm{eff}}=\left(\begin{array}{cccc}
\epsilon & T & 0 & 0  \tag{4.1.7}\\
T^{*} & -\epsilon & 0 & 0 \\
0 & 0 & \epsilon & -T \\
0 & 0 & -T^{*} & -\epsilon
\end{array}\right)
$$

A remarkably simple description of a seemingly complicated system.
The idea is to adiabatically tune $\epsilon$ from $\epsilon=-\infty$ (an electron on the left dot) to $\epsilon=\infty$ (an electron on the right dot), and to determine if this induces a transformation on the state of the island. To find out, we invoke the adiabatic principle, and we therefore need to solve the Schrödinger problem for the Hamiltonian (4.1.7).

It is easy; thanks to our work on finding a proper low-energy description we have a system that is equivalent to an electron in a magnetic field. The field is $B=( \pm \operatorname{Re} T, \mp \operatorname{Im} T, \epsilon)$, where the upper sign is chosen for the $\sigma_{z}=1$ sector and the lower sign is chosen for the $\sigma_{z}=-1$ sector. The energies are

$$
\begin{equation*}
E_{ \pm}= \pm|B|= \pm \sqrt{\epsilon^{2}+\left|T \sigma_{z}\right|^{2}} \tag{4.1.8}
\end{equation*}
$$

It is essential that the energies do not depend on the parity; we can apply the adiabatic theorem to each sector separately. This also tells us that the process should happen on a timescale $t$ that satisfies $2 E_{+}(\epsilon=0) \gg \frac{1}{t}$, i.e. the energy splitting should be very large compared to the adiabatic timescale.

The ground state is given by $\mathbf{B} \cdot \mathbf{S}|\psi\rangle=-\frac{1}{2} B|\psi\rangle$. This a standard problem in quantum mechanics and they are (in spin- $1 / 2$ notation)

$$
\begin{equation*}
|\psi\rangle=\binom{\sin (\theta / 2)}{-e^{i \phi} \cos (\theta / 2)} \tag{4.1.9}
\end{equation*}
$$

where $\theta, \phi$ are the spherical angles of the magnetic field

$$
\begin{align*}
& \theta=\arctan \frac{|T|}{\epsilon}  \tag{4.1.10}\\
& \phi=\arctan \frac{-\operatorname{Im} T}{\operatorname{Re} T}=: \phi_{0}, \quad \sigma_{z}=1  \tag{4.1.11}\\
& \phi=\arctan \frac{-\operatorname{Im} T}{\operatorname{Re} T}+\pi, \quad \sigma_{z}=-1 \tag{4.1.12}
\end{align*}
$$

As $\epsilon$ is tuned from $-\infty$ to $\infty, \theta$ runs from $\pi$ to 0 . If the initial state of the system is $(\alpha|+\rangle+\beta|-\rangle) \otimes|10\rangle$, then the state at a given $\theta$ is

$$
|\psi\rangle=e^{-i \int^{t(\theta)} E(t) d t}\left[\alpha\left(\begin{array}{c}
\sin (\theta / 2)  \tag{4.1.13}\\
-e^{i \phi_{0}} \cos (\theta / 2) \\
0 \\
0
\end{array}\right)+\beta\left(\begin{array}{c}
0 \\
0 \\
\sin (\theta / 2) \\
e^{i \phi_{0}} \cos (\theta / 2)
\end{array}\right)\right] .
$$

Let now $\epsilon \rightarrow \infty$. Then

$$
|\psi\rangle=e^{i \varphi}\left[\alpha\left(\begin{array}{c}
0  \tag{4.1.14}\\
-1 \\
0 \\
0
\end{array}\right)+\beta\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right)\right]
$$

or

$$
\begin{equation*}
|\psi\rangle=e^{i \varphi+i \pi}(\alpha|+\rangle-\beta|-\rangle) \otimes|01\rangle . \tag{4.1.15}
\end{equation*}
$$

Here $\varphi$ is the (generally uncontrollable) dynamical phase plus $\phi_{0}$.
We see that as far as the island goes, the adiabatic tunnelling process is equivalent to the operation

$$
\begin{equation*}
|\psi\rangle \rightarrow \sigma_{z}|\psi\rangle \tag{4.1.16}
\end{equation*}
$$

up to a random phase.

### 4.2 Other Kinds of Tunnelling

It should be clear that if we had coupled the dots to a different pair of Majoranas in figure 4.1, a result similar to the one above would hold. That is, in general, tunnelling through a pair of Majoranas, $\gamma_{j}, \gamma_{k}$ is equivalent to the operation

$$
\begin{equation*}
|\psi\rangle \rightarrow i \gamma_{j} \gamma_{k}|\psi\rangle, \tag{4.2.1}
\end{equation*}
$$

up to a phase. We can therefore perform the operations $\sigma_{x}, \sigma_{y}, \sigma_{z}$ on the state.
Can we perform an arbitrary operation? We can imagine connecting the dots to multiple Majoranas and perhaps adding a direct coupling. This leads to a low-energy Hamiltonian that contains a tunnelling term

$$
\begin{equation*}
\left(A \sigma_{x}+B \sigma_{y}+C \sigma_{z}+D\right) c_{L}^{\dagger} c_{R}+\text { h.c. } \tag{4.2.2}
\end{equation*}
$$

In general

$$
\begin{align*}
\lambda & =\left(A \sigma_{x}+B \sigma_{y}+C \sigma_{z}+D\right),  \tag{4.2.3}\\
\lambda^{\dagger} & =\left(A^{*} \sigma_{x}+B^{*} \sigma_{y}+C^{*} \sigma_{z}+D^{*}\right), \tag{4.2.4}
\end{align*}
$$

can not be diagonlized simultaneously. Only when $A B^{*}, A C^{*}, B C^{*}$ are real can both $\lambda$ and $\lambda^{\dagger}$ be diagonalized, but this requires careful tuning of the tunnelling parameters and is unrealistic. Therefore the problem no longer separates to two independent sectors. Instead we need to consider the full Hamiltonian

$$
H=\left(\begin{array}{cc}
\epsilon \sigma_{0} & \lambda  \tag{4.2.5}\\
\lambda^{\dagger} & -\epsilon \sigma_{0}
\end{array}\right) .
$$

For general $A, B, C, D$ the eigenvalues and vectors are uninformative. It is not worthwhile to consider the general expression. We can, however, say that the clean result found in the previous section only holds when a single tunnelling path is active.

One case that does not require tuning is when two of $A, B, C$ are zero. Without loss of generality assume $A=B=0$ and $C, D$ non-zero. The eigenvalues of $\lambda$ and $\lambda^{\dagger}$ are $D \pm C$ and $D^{*} \pm C^{*}$ respectively. The problem now is that this leads to effective magnetic fields of different magnitude for the two sectors. Therefore the dynamical phases of the sectors are different, and the effect of such an operation is to randomly scramble the state of the island, which is rarely useful.

Although it is probably difficult to fine-tune, let us consider what happens when we do get a nice operation. When $D=0$ and $A, B, C$ are 'parallel', i.e. have the same phase, we get two equal-energy sectors and we can write

$$
\begin{equation*}
\lambda=T\left(\sigma_{x}+\alpha \sigma_{y}+\beta \sigma_{z}\right), \quad T \in \mathbb{C}, \alpha, \beta \in \mathbb{R} \tag{4.2.6}
\end{equation*}
$$

The adiabatic transfer is equivalent to the operation $\sigma_{x}+\alpha \sigma_{y}+\beta \sigma_{z}$.

### 4.3 Concluding Remarks

We have discussed the remarkable effect that pumping an electron adiabatically across a Majorana pair, $\gamma_{1}, \gamma_{2}$ amounts to applying the operator $i \gamma_{1} \gamma_{2}$ to the system.

We saw that other, more complicated operations were in general possible only if we had some fine-tuning in the system. However, in spite of these limitations we will make use of this effect time and again in chapter 5 ; even having just $\sigma_{x}, \sigma_{y}, \sigma_{z}$ and the results of chapters 2 and 3 at our disposal gives us a nice set of tools.

## Chapter 5

## Majorana Surface Code

The Holy Grail in condensed matter physics is at present applications towards quantum computing. The formalism of quantum computing is quite developed and we will not attempt to cover it in depth here; there are many good references to be found for instance ref. [14].

Suffice it to say, a keystone of quantum computing is encoding information into a quantum system. This information might be encoded as simply the coefficients of the state in some basis, e.g. one could encode the 'information' $(\alpha, \beta)$ into a spin- $1 / 2$ system by forming a state $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$. In typical systems, such a spin- $1 / 2$ system, this kind of state is not very stable and this type of encoding would be quite bad. One could improve this by using more states to encode the information $|\psi\rangle=\sum_{n} c_{n}(\alpha, \beta)|n\rangle$ and thus distribute the information in a clever way that reduces corruption of the information. One could for example form the state $\alpha|\rightarrow\rangle+\beta|\leftarrow\rangle$ where $|\rightarrow\rangle,|\leftarrow\rangle$ are degenerate states that have a different value of some non-local observable; the two ground states of a Majorana-Cooper box come to mind.

One approach is called stabilizer coding. In stabilizer coding the Hilbert space is reduced to a certain subspace by measuring or otherwise fixing a number of observables, $\mathcal{O}_{n}$, to some value, say $\mathcal{O}_{n}=1$. Information can be encoded into the system by using the non-fixed degrees of freedom and stabilizer coding has the nice property that some errors are detectable by measuring $\mathcal{O}_{n} \neq 1$.

In this chapter we will first introduce the stabilizer coding formalism, then discuss how logical operators that act on the encoded information should be defined in terms of physical operators and how they should be updated when operations (measurements, electronpumping, etc.) are performed on the system. We then turn to a specific application namely a stabilizer code that utilizes the Majorana-Cooper box as its fundamental building block. This type of system is termed a Majorana surface code. To build a quantum computer one needs a certain set of generating operations to be able to perform an arbitrary computation and we will discuss how to implement these generating operations.

### 5.1 Stabilizer Coding

In this section we give a brief discussion of the concept of stabilizer coding.

We wish to encode $k$ logical qubits into $n$ physical qubits. The physical Hilbert space is $\mathcal{H}=\bigotimes^{n} \mathcal{H}_{1 / 2}$ and a typical error process is $\mathbb{1} \otimes \mathbb{1} \otimes \ldots \otimes \sigma_{i} \otimes \mathbb{1} \otimes \ldots$, a single-bit error. Naturally, multiple-bit errors can also occur, but as the number of bits involved in the error increases, the probability of the error goes down.

Loosely speaking, we would like to be able to measure when errors happen without measuring the state of the system. So we cannot simply encode the bits in a state

$$
\binom{\alpha_{1}}{\beta_{1}} \otimes \ldots \otimes\binom{\alpha_{k}}{\beta_{k}} \otimes \operatorname{arbitrary} \otimes \ldots
$$

because, e.g. the error $\sigma_{x} \otimes \mathbb{1} \otimes \ldots$ is only visible if we measure the state of the first physical qubit.

This problem can be remedied if the error can be associated with a change of an operator $\mathcal{O}$ that acts trivially on all the encoded bit-states. More precisely, all code states should be eigenstates with the same eigenvalue of the observable $\mathcal{O}$, and when an error occurs this eigenvalue should change. For such an operator we can measure $\mathcal{O}$ without affecting the encoded information and thus the error can be detected.

This is the central idea of stabilizer coding: we want to project to a code subspace, $\mathcal{C}$, defined by

$$
\begin{equation*}
\left.\mathcal{C}=\left\{|\psi\rangle\left|\mathcal{O}_{j}\right| \psi\right\rangle=|\psi\rangle, j=1,2, \ldots\right\}, \tag{5.1.1}
\end{equation*}
$$

where $\mathcal{O}_{j}$ are some operators usually termed stabilizers. What errors can be detected, how many errors that can be detected and how the information is encoded physically depends on the $\mathcal{O}_{j}$.

Let us take a specific example of a code subspace: Shor's code [15]. We will just state what the code subspace is. There are two states

$$
\begin{align*}
|\overline{0}\rangle & =(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle),  \tag{5.1.2}\\
|\overline{1}\rangle & =(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle) . \tag{5.1.3}
\end{align*}
$$

We have encoded 1 qubit in 9 qubits. If a bit-flip error, $\sigma_{x}^{1}$, happens on the first physical qubit this will be observable by measuring $\sigma_{z}^{1} \sigma_{z}^{2}$. Notice that the code states both have eigenvalue 1 for this operator, but the bit-flipped have eigenvalue -1 . This does not in itself give enough information to correct the error because we can only conclude that either physical bit 1 or 2 has flipped. We also need to measure e.g. $\sigma_{z}^{1} \sigma_{z}^{3}$, which will tell us that it was bit 1 that flipped, unless, of course, a multiple-bit error happened. Phase-flips, $\sigma_{z}$, are treated similarly. To detect a phase-error we have to measure $\sigma_{x}^{1} \sigma_{x}^{2} \sigma_{x}^{3} \sigma_{x}^{4} \sigma_{x}^{5} \sigma_{x}^{6}$ and $\sigma_{x}^{1} \sigma_{x}^{2} \sigma_{x}^{3} \sigma_{x}^{7} \sigma_{x}^{8} \sigma_{x}^{9}$.

The mathematical properties of stabilizer codes, how they correct errors, the structure of the code space $\mathcal{C}$ and how to construct the stabilizer operators is an interesting story in its own right. I refer the reader to refs. $[15,16,17]$ for further inquiries into the stabilizer code formalism.

### 5.2 Logical Operators

In the surface code setup to be discussed below we will have a set of stabilizers and once these are fixed to $\mathcal{O}_{n}=1$, we encode one or more logical qubits into the physical qubits. To perform computations we need a set of logical operators to manipulate or measure the logical bits. In this section we deal first with the definition of these.

To perform certain operations it will be necessary to dynamically, but step-wise, change the set of stabilizers. This process should not destroy the encoded information but it changes the code subspace and consequently how the information is encoded. The logical operators therefore have to be updated correspondingly.

A logical qubit is an ordered pair of numbers $(\alpha, \beta),|\alpha|^{2}+|\beta|^{2}=1, \alpha, \beta \in \mathbb{C}$ and to encode a qubit means to prepare a physical state

$$
\begin{equation*}
(\alpha, \beta) \rightarrow|\psi(\alpha, \beta)\rangle \tag{5.2.1}
\end{equation*}
$$

in such a way that $|\psi(\alpha, \beta)\rangle$ is in one-to-one correspondence with $(\alpha, \beta)$. For two logical qubits an encoding is

$$
\begin{equation*}
\left(\alpha_{1}, \beta_{1}, \alpha_{2}, \beta_{2}\right) \rightarrow\left|\psi\left(\alpha_{1}, \beta_{1}, \alpha_{2}, \beta_{2}\right)\right\rangle, \tag{5.2.2}
\end{equation*}
$$

and so on for multiple qubits. We will use linear encodings that satisfy

$$
\begin{equation*}
|\psi(\alpha, \beta)\rangle=\alpha|\psi(1,0)\rangle+\beta|\psi(0,1)\rangle, \tag{5.2.3}
\end{equation*}
$$

with the obvious generalization to more logical qubits.
We also assume that

$$
\begin{equation*}
\langle\psi(1,0) \mid \psi(0,1)\rangle=0 . \tag{5.2.4}
\end{equation*}
$$

A logical operation is a linear operator $L$ acting on the logical state $(\alpha, \beta)$

$$
\begin{equation*}
L(\alpha, \beta)=\left(\alpha^{\prime}, \beta^{\prime}\right), \tag{5.2.5}
\end{equation*}
$$

and the question is how to represent a given logical operator $L$ in terms of physical operators. If we denote the representation of $L$ by $\mathcal{L}_{L}$ we should have

$$
\begin{equation*}
\mathcal{L}_{L}|\psi(\alpha, \beta)\rangle=\left|\psi\left(\alpha^{\prime}, \beta^{\prime}\right)\right\rangle \tag{5.2.6}
\end{equation*}
$$

for all possible values of $\alpha$ and $\beta$. This tells us that $\mathcal{L}_{L}$ should have matrix elements

$$
\langle\psi(1,0)| \mathcal{L}_{L}|\psi(1,0)\rangle=\alpha
$$

when

$$
\begin{equation*}
L(1,0)=(\alpha, 0) \tag{5.2.7}
\end{equation*}
$$

and similarly for the other matrix elements.
We will call this a good representation of $L$.
Such a representation is not unique; because we are in the subspace $\mathcal{C}$ the stabilizer operators act trivially, $\mathcal{O}_{n}=1$, and if we have a representation $\mathcal{L}_{L}$, then we have an equivalence class of representations with equivalence given by

$$
\begin{equation*}
\mathcal{L}_{L}^{\prime} \sim \mathcal{L}_{L}, \text { if } \mathcal{L}_{L}^{\prime}=\mathcal{O} \mathcal{L}_{L}, \tag{5.2.8}
\end{equation*}
$$

where $\mathcal{O}$ is some product of stabilizers and other trivial operators if such exist. It should be noted that it is not necessarily true that all good representations of $L$ are in the same equivalence class.

When we have multiple encoded qubits the definitions (5.2.7) and (5.2.8) are extended in the obvious way.

A fact we will often use is that for logical operators $L, M, N$

$$
\begin{equation*}
[L, M]_{ \pm}=N \Rightarrow\left[\mathcal{L}_{L}, \mathcal{L}_{M}\right]_{ \pm}=\mathcal{L}_{N} \tag{5.2.9}
\end{equation*}
$$

which can be seen by comparing matrix elements on either side.
We now deal with operations on the physical Hilbert space that do not leave the code subspace invariant. In practice, we will deal with ceasing a stabilizer fixation; measuring an operator; and adding a new operator to the stabilizer set.

If a stabilizer is no longer fixed the only thing that is affected is the equivalence classes of representations, eq. (5.2.8). Barring error processes that entangle the encoded information with the new degree of freedom the encoding and thus all operators are unchanged.

When measuring any operator, $A$, we randomly project to an eigenstate of $A$. This procedure is described by applying a projection operator $P_{j}$ to the state. This, at the very least, changes the encoding and could possibly completely destroy the information. A nondestructive measurement is quite special. A non-destructive measurement should satisfy

$$
\begin{equation*}
P_{j}|\psi(\alpha, \beta)\rangle=\left|\psi^{\prime}(\alpha, \beta)\right\rangle, \quad \forall \alpha, \beta \in \mathbb{C}, \forall j \tag{5.2.10}
\end{equation*}
$$

where $\psi^{\prime}$ is another encoding. Then, since the projector is linear

$$
\begin{align*}
\alpha P_{j}|\psi(1,0)\rangle+\beta P_{j}|\psi(0,1)\rangle & =\alpha\left|\psi^{\prime}(1,0)\right\rangle+\beta\left|\psi^{\prime}(0,1)\right\rangle, \quad \forall \alpha, \beta \in \mathbb{C},  \tag{5.2.11}\\
& \Rightarrow \\
P_{j}|\psi(1,0)\rangle & =\left|\psi^{\prime}(1,0)\right\rangle  \tag{5.2.12}\\
P_{j}|\psi(0,1)\rangle & =\left|\psi^{\prime}(0,1)\right\rangle . \tag{5.2.13}
\end{align*}
$$

Suppose now we have a good representation, $\mathcal{L}_{L}$, of the logical operator $L$, with $\left[\mathcal{L}_{L}, P_{j}\right]=0$. When $L(1,0)=(\alpha, \beta)$

$$
\begin{align*}
\left\langle\psi^{\prime}(1,0)\right| \mathcal{L}_{L}\left|\psi^{\prime}(1,0)\right\rangle & =\left\langle\psi^{\prime}(1,0)\right| P_{j} \mathcal{L}_{L}|\psi(1,0)\rangle \\
& =\left\langle\psi^{\prime}(1,0)\right| P_{j}(\alpha|\psi(1,0)\rangle+\beta|\psi(0,1)\rangle)=\alpha . \tag{5.2.14}
\end{align*}
$$

Hence the representation remains good after the projection. Below we will find a good representation with $\left[\mathcal{L}_{L}, P_{j}\right]=0$ to find a class of good representations after measurement of $A$.

Finally, we deal with adding a new stabilizer. Any kind of fixation of an operator is essentially applying an operator $P_{j}$ to the state of the physical system. Therefore the above discussion about measurements apply equally well here. In fact, we will stabilize exactly by measuring the stabilizers.

Finally, we will introduce some notation. We will need the logical operators $\sigma_{z}, \sigma_{x}$. To distinguish logical operators from physical operators we write the logical Pauli operators as $Z, X, Y$ and keep the usual notation for the physical operators.

### 5.3 Surface Code Setup

We now discuss a specific setup proposed in ref. [18]. The surface code is a large number of Majorana-Cooper boxes connected by tunnelling link in the manner shown in figure 5.1. To describe the physics in this system it is instructive to consider the low-energy Hamiltonian. By the methods of sec. 1.3 we can find the Hamiltonian to lowest order in $E_{C}$ as follows. Any electron that tunnels away from a starting island should always return there, in order that the system returns to the low-energy sector. For each tunnelling path there is a corresponding term in the Hamiltonian. Imagine, for example, an electron tunnelling along some path and then back along the same path. Such a process involves every Majorana fermion along the path twice, and therefore it is trivial. We should therefore look for the shortest noncontractible closed loop. This is given by tunnelling once around one of the 'holes' between the Majorana-Cooper boxes in figure 5.1. The corresponding term in the Hamiltonian is

$$
\begin{equation*}
H_{\text {loop }}=\frac{5 \operatorname{Re}\left(t^{4}\right)}{16 E_{C}^{3}} \gamma_{1} \gamma_{2} \gamma_{3} \gamma_{4} \gamma_{5} \gamma_{6} \gamma_{7} \gamma_{8} \tag{5.3.1}
\end{equation*}
$$

Here $\gamma_{1}$ through $\gamma_{8}$ are the operators along the loop. The point is that low-energy degrees of freedom are given by plaquette operators

$$
\begin{equation*}
\mathcal{O}=\prod_{k=1}^{8} \gamma_{k} \tag{5.3.2}
\end{equation*}
$$

Plaquette operators square to one

$$
\begin{equation*}
\mathcal{O}^{2}=1 \tag{5.3.3}
\end{equation*}
$$

and different plaquettes commute with each because they always share two or zero Majorana fermions. We note that a number of different boundary conditions are possible depending on how the network is terminated.

For each Majorana-Cooper box we associate a Majorana billinear with $\sigma_{z}$ and another billinear with $\sigma_{x}$ and this leads to $x$-type plaquettes and $z$-type plaquettes. For example in


Figure 5.1: The surface code setup. A number of Majorana-Cooper boxes (magenta, with grey cylinders, with red dots) are tunnel-coupled in a 'zig-zag' pattern.
the inset of figure 5.1 we define $\sigma_{z}$ and $\sigma_{x}$ in terms of Majorana billinears on box 1. With this definition the plaquette bounded by boxes $1,4,5,6$ is a $z$-type plaquette. The plaquette bounded by boxes 1,3 and 4 is of $x$-type, and so on. This choice on box 1 fixes the bases on all the other boxes in such a way that a $z$-type plaquette is written as a product of $\sigma_{z}$, $\mathcal{O}^{z}=\sigma_{z}^{1} \sigma_{z}^{2} \sigma_{z}^{3} \sigma_{z}^{4}$ and similarly for $x$-type plaquettes. This leads to the staggered pattern shown in the figure.

The plaquette operators will act as both stabilizers and as a way to encode information. Plaquette stabilizers are fixed by measuring them.

The simplest qubit is a non-stabilized plaquette. Compound qubits can be form by joining together more plaquettes.

To perform computations we need a set of logical operators. A complete set of logical, single-qubit operators are $X, Y, Z[19]$. To perform general computations we also need a few other single-qubit operations and a so-called CNOT-operation which entangles two qubits. In this section we stick to the single-bit Pauli-operators.

In figure 5.2 we have encoded a qubit into the plaquette operator shown in grey; this is called a single-cut qubit because we have 'cut away' one stabilizer. The state of the system is (ignoring all the stabilized plaquettes)

$$
\alpha|0\rangle+\beta|1\rangle,
$$

where $|0\rangle,|1\rangle$ refer to the two eigenstates of the plaquette operator. The plaquette itself acts as the $Z$ operator. To construct a representation of $X$ we need an operator that anticommutes with the plaquette and commutes with all stabilizers and other logical operators


Figure 5.2: Pauli-operators for plaquette qubits. The grey area indicates the plaquette, $\sim Z$. The operator corresponding to the black line anticommutes with the plaquette and is a representation of $X$. A different representation is shown as the blue line.
present in the system. Therefore the $X$ should share an odd number of Majoranas with the logical qubit and an even number with all others. A choice is the 'string' operator shown as a black line in figure 5.2. This string should go to the edge of the sample so that it ends on a Majorana that is not stabilized.

By multiplying this specific choice with plaquette operators we move around in the equivalence class. For example, by multiplying by the stabilizer to the right of the qubit plaquette we get the string operator shown as a blue line, using the property of Majorana fermions that $\gamma^{2}=1$. With both of $X, Z$ we also have $Y=i X Z$.

This setup has the disadvantage that to access the $X$-operator we have to involve a very large number of Majorana fermions. To remedy this problem we can form double cut qubits. These are qubits formed by an entangled state of two non-stabilized plaquettes. A double cut qubit is shown schematically in figure 5.3. The $X$-operator must now be a simultaneous flip of both plaquettes and is shown as the black string in the figure. For the $Z$-operator we can choose either of the grey plaquettes. We will mainly be dealing with double-cut qubits as these are the most attractive in a practical setting.

To actually access or apply any of these operators we can use the results from chapter 4. To apply the $X$ operator we connect a dot to the ends of the string, see figure 5.4. A large number of paths contribute to the tunnelling operator, for example there is the direct path and the path that goes around the turquoise plaquettes. The effective tunnelling term for a given path scale as $t^{N} E_{C}^{N-1}$ where $N$ is the number of Majorana-Cooper boxes involved in the path. Insofar as this allows to approximate the tunnelling Hamiltonian as the direct


Figure 5.3: A schematic double-cut qubit. The grey squares symbolize the plaquettes, compare figure 5.2. The black line between the plaquettes is string of Majorana operators that act as $X$.
path we get

$$
\begin{equation*}
H_{T}=T \sigma_{x} d_{1}^{\dagger} d_{2}+\text { h.c. } \tag{5.3.4}
\end{equation*}
$$

Here $\sigma_{x}$ is the string operator joining the two plaquettes, which is a representation of $X$. Using the results of chapter 4 the effect of pumping a single electron is to apply the $X$ operator ${ }^{1}$.

To apply the $Z$ operator consider connecting dots as in figure 5.5. There are three main paths to consider: the direct path, the red path, and the blue path. When $n_{g} \in \mathbb{Z}$ for both Majorana-Cooper boxes, one can show that the direct tunnelling is zero [18]. The red and blue paths lead to a tunnelling term

$$
\begin{equation*}
H_{T}=\left(T_{1} \sigma_{z}+T_{2} \mathcal{O}_{2}\right) d_{1}^{\dagger} d_{2}+\text { h.c. } \tag{5.3.5}
\end{equation*}
$$

where $\mathcal{O}_{2}$ is the stabilizer encircled by the blue path and $\sigma_{z}$ is the plaquette operator. This is effectively 1 . Generically this fact does not allow us to apply $Z$ using adiabatic transport as we saw in chapter 4 . However, if we suppress a tunnelling link (which does not destroy or alter the state of the system) anywhere along the blue path, $T_{2} \rightarrow 0$ and the adiabatic operation becomes well-defined.

There is an alternative to this 'hardware-based' approach. When considering a general algorithm in the surface code, all applications of say $Z$ can be commuted through the other operators until it hits another $Z$ at which point we use $Z^{2}=1$. If there is no other $Z$ to 'annihilate' with we apply the $Z$ to the eventual measurement needed to do readout, i.e. we flip the sign of a $X$ measurement, but do nothing to a $Z$-measurement. See also ref. [19].

To measure or fix these operators the discussion above applies but instead of adiabatically tunnelling we measure a current. By measuring the current from lead to lead in figure 5.4 we will project to an eigenstate of $X$. We need an interference link to know which eigenstate $X$ we project to.

Similarly, the setup in figure 5.5 can be used to measure $Z$. We again need an interference link, this time both for readout but also so the measurement does not simply project to a definite relative parity. The tunnelling with intereference is $\sim T+T_{1} \mathcal{O}_{1}+T_{2} \mathcal{O}_{2}$.

With these basic ingredients we are ready to discuss the more complicated operations needed for completeness.

[^0]

Figure 5.4: To apply the $\sigma_{x}$ operator we connect dots to the ends of the string. The effective tunnelling is $\sim T \sigma_{x}$.


Figure 5.5: To apply the $\sigma_{x}$ operator we connect dots to the ends of the string. The effective tunnelling is $\sim T \sigma_{x}$.

### 5.4 Surface Code Operation

To perform universal quantum computation we need the operators: Hadamard; S-gate; Tgate; CNOT [20]. The first three are single-bit operations while the last is a two-bit operation.

The Hadamard operation, $H$, is

$$
H=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}
1 & 1  \tag{5.4.1}\\
1 & -1
\end{array}\right),
$$

which satisfies

$$
\begin{align*}
& H \sigma_{z} H=\sigma_{x},  \tag{5.4.2}\\
& H \sigma_{x} H=\sigma_{z} . \tag{5.4.3}
\end{align*}
$$

It is a basis change.
The S- and T-gates are

$$
\begin{array}{r}
S=\left(\begin{array}{ll}
1 & 0 \\
0 & i
\end{array}\right), \\
T=\left(\begin{array}{cc}
1 & 0 \\
0 & e^{i \pi / 4}
\end{array}\right) . \tag{5.4.5}
\end{array}
$$

The CNOT or Controlled-not is

$$
C=\left(\begin{array}{llll}
1 & 0 & 0 & 0  \tag{5.4.6}\\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

When applying the CNOT gate to a two-qubit system we define the control bit and the target bit. In the above representation of CNOT the first tensor factor is the control while the second factor is the target.

As we will see below, if we can perform a "qubit move" we can implement CNOT. With CNOT we can do Hadamard. With special ancilla states, CNOT and Hadamard, we can apply the $S$ - and $T$-gates [19].

### 5.4.1 Qubit move

A qubit move is moving an encoded qubit to an adjacent plaquette. This is done by first ceasing stabilization of the target plaquette, then measuring a Majorana billinear joining the two plaquettes and finally measuring the original plaquette. In figure 5.6 the information is originally encoded in the right plaquette, which is in the state

$$
\begin{equation*}
|\psi\rangle=\alpha|0\rangle+\beta|1\rangle . \tag{5.4.7}
\end{equation*}
$$

The left plaquette is stabilized to begin with and the first step is to stop stabilization.
Next we measure the Majorana billinear indicated by the black line joining two Majorana fermions. This operator is equal to the $X$ string of the left box times the $X$ string of the right box

$$
\begin{equation*}
i \gamma_{1} \gamma_{2}=X^{L} X^{R} \tag{5.4.8}
\end{equation*}
$$

Measuring this operator entangles the two plaquettes. There are two things worth considering: what happens to the state of the system and how the representations of the logical operators are updated. The update of the logical operators can be handled by the discussion of section 5.2 and the evolution of the state will provide a less abstract counterpoint. A measurement of, say, $i \gamma_{1} \gamma_{2}=1$, puts the plaquettes in the state

$$
\begin{equation*}
|0\rangle \otimes(\alpha|0\rangle+\beta|1\rangle) \rightarrow(\alpha+\beta)|++\rangle+(\alpha-\beta)|--\rangle, \tag{5.4.9}
\end{equation*}
$$

where $\pm$ refer to the eigenstates of $X$ and the first tensor factor is the left plaquette and the second factor is the right plaquette.

Notice that $X^{R}$ commutes with the measurement operator and hence this is still a good representation of $X . Z$ is no longer represented by just the right plaquette but now has to be represented by the right plaquette times the left plaquette $Z=\sigma_{z}^{L} \sigma_{z}^{R}$, which was also a good representation before the measurement since the left plaquette was stabilized before initiating the move.


Figure 5.6: Moving a qubit from the right to the left plaqeutte is done via measurement of the Majorana billinear indicated with a black line in the figure. To complete the move the right plaquette is measurement.

That these operators work as advertised can also be verified by direct inspection of the state.

The final step is measuring the right plaquette. For example, if we measure $\sigma_{z}^{R}=1$, then

$$
\begin{equation*}
(\alpha+\beta)|++\rangle+(\alpha-\beta)|--\rangle \rightarrow(\alpha|0\rangle+\beta|1\rangle) \otimes|0\rangle \tag{5.4.10}
\end{equation*}
$$

$\sigma_{z}^{L} \sigma_{z}^{R}$ commutes with the measurement so this is still a good representation of $Z$. Because effectively $\sigma_{z}^{R}=1$ after the measurement, we can use $\sigma_{z}^{L}$ as $Z$. However, to find a good representation of $X$ we notice that before the $\sigma_{z}^{R}$ measurement we could have used either $\sigma_{x}^{R}$ or $\sigma_{x}^{L}=i \gamma_{1} \gamma_{2} \sigma_{x}^{R}$ since $i \gamma_{1} \gamma_{2}$ is trivial on the projected subspace. $\sigma_{x}^{L}$ commutes with the measurement and is therefore a good representation.

Again we can verify this result by directly inspecting the state.
It is worth noting that if the results of measurements were not $(1,1)$ but $(1,-1)$ or any other combination, the final state would have not be $\alpha|0\rangle+\beta|1\rangle$ but rather $\left(\sigma_{x}\right)^{n_{x}}\left(\sigma_{z}\right)^{n_{z}}(\alpha|0\rangle+$ $\beta|1\rangle$ ) where $n_{z}=0,1$ when the first measurement yields $1,-1$ respectively and similarly for $n_{x}$ with respect to the second measurement. However, this does not affect the logical operators.

Thus, when moving a qubit from plaquette 1 to adjacent plaquette 2 the representations of $X, Z$ moves from

$$
\begin{equation*}
\left(\sigma_{x}^{1}, \sigma_{z}^{1}\right) \rightarrow\left(\sigma_{x}^{2}, \sigma_{z}^{2}\right) \tag{5.4.11}
\end{equation*}
$$

up to equivalence.
In a more pictorial way of speaking, as we move the qubit, it drags its string operator along with it.

### 5.4.2 CNOT

Now that the qubit move can be completed we can perform a CNOT on a two-qubit system consisting of a $z$-type plaquette and an $x$-type plaquette. This is done by "braiding" the $z$-type plaquette around the $x$-type plaquette.

The first part of the braiding is shown in figure 5.7. We start with a $z$-type and an $x$ type plaquette shown as red and blue boxes with solid outlines respectively. The blue string


Figure 5.7: The first part of the CNOT operation is shown. A $z$-plaquette (red box, solid outline) is moved stepwise until it reaches the end of the first part of the braid (red box, dashed outline). The string operator (blue line) is updated by dragging it along the plaquette (dashed, blue line). Also shown are the $x$-type plaquette (blue box, solid outline) and its string (red line).
attached to the $z$-plaquette is the $x$-string operator for this bit and the red string attached to the $x$-plaquette is the $z$-string operator for the $x$-bit.

The $z$-type is moved stepwise along the plaquettes with dashed outlines until it reaches the red plaquette with a dashed outline. As we saw in section 5.6 the string operator merely follows along, here shown as a blue dashed line.

For the last part of the CNOT-operation we want to move the $z$-plaquette across the $z$-string of the $x$-plaquette. We cannot keep the representations of the logical operators as they are because the operator shown as a red string in figure 5.7 will not commute with the operator shown as a blue string once they cross each other; they will share one Majorana and thus will anti-commute where-as logical operators belonging to different qubits should commute.

This indicates that we must choose a different representation, namely one where the $z$ string will crosses the $x$-string an even number of times. Before we move across the red string consider the representation $Z_{2}^{\prime}=Z_{2} \mathcal{O}$, where $Z_{2}^{\prime}$ is the new representation of logical $\sigma_{z}, Z_{2}$ is the old and $\mathcal{O}$ is the stabilized plaquette directly above the red string. This operator commutes with the qubit move and does not cross either the old or updated $x$-string and will therefore remain a good representation after the move. Continuing this process until the braid is completed we have the following.

$$
\begin{align*}
Z_{1} & =\mathcal{O}_{1}  \tag{5.4.12}\\
X_{1} & =S_{1} \mathcal{O}_{2} \times \text { stabilized plaquettes },  \tag{5.4.13}\\
Z_{2} & =S_{2} \mathcal{O}_{1} \times \text { stabilized plaquettes }  \tag{5.4.14}\\
X_{2} & =\mathcal{O}_{2} \tag{5.4.15}
\end{align*}
$$

Here $\mathcal{O}_{i}$ is the plaquette-operator where the $i$ th bit is encoded (red and blue boxes in figure 5.7 ) and $S_{i}$ is the corresponding string-operator (blue and red lines in figure 5.7). The red string is updated to the red, dashed string, and the blue string and the plaquettes are updated as before.

The result is that the string operators $S_{1}$ and $S_{2}$ are no longer representations of logical $\sigma_{x}, \sigma_{z}$ but rather

$$
\begin{align*}
& S_{1}=X_{1} X_{2},  \tag{5.4.16}\\
& S_{2}=Z_{1} Z_{2} \tag{5.4.17}
\end{align*}
$$

and nothing happens to the operators that represent $Z_{1}, X_{2}$.
These statements show that the qubit braid is a CNOT operation as follows. The braid is equivalent to an operation $\mathcal{L}_{C}$ on the state of the system $|\psi\rangle \rightarrow \mathcal{L}_{C}|\psi\rangle$. This physical operation is a representation of a logical operation, and we can find $C$ by finding the transformations of the representations of a basis set of logical operators. In fact, $\mathcal{L}_{C}$ is the trivial operator because the qubit move does not alter the state of the system, it merely moves it from one tensor factor to another. Thus,

$$
\begin{equation*}
\mathcal{L}_{C}^{\dagger} S_{1} \mathcal{L}_{C}=S_{1}, \tag{5.4.18}
\end{equation*}
$$

and so on. However, we saw that $S_{1}$ is no longer a representation of $X_{1}$ so we must have $C^{\dagger} X_{1} C=X_{1} X_{2}$. Using the same line of reasoning for all the above operators we can write

$$
\begin{align*}
C^{\dagger} Z_{1} C & =Z_{1}  \tag{5.4.19}\\
C^{\dagger} X_{1} C & =X_{1} X_{2}  \tag{5.4.20}\\
C^{\dagger} Z_{2} C & =Z_{1} Z_{2}  \tag{5.4.21}\\
C^{\dagger} X_{2} C & =X_{2} \tag{5.4.22}
\end{align*}
$$

This is enough to determine $C$ because all other operators can be written as products of these, e.g. $C^{\dagger} X_{1} Z_{2} C=C^{\dagger} X_{1} C C^{\dagger} Z_{2} C$.

These relations are exactly the action of CNOT (5.4.6) on the basis operators. Therefore the braiding is equivalent to CNOT. See also ref. [19].

### 5.4.3 Hadamard

In ref. [19] there is an algorithm for implementing the Hadamard transform, but in this section we present an alternative based on ref. [21]. The Hadamard is essentially an operation where we move stored information from a $z$-type qubit to an $x$-type qubit.

The setup for the Hadamard protocol is shown in figure 5.9. The information is initially stored in the green $z$-type bit and is to be moved to the blue $x$-type bit. For the $z$-type bit we identify $Z_{1}$ with the leftmost green plaquette, $\mathcal{O}_{1}$, and $X_{1}$ with the Majorana billinear shown as a blue string, $b_{1}$. The $x$-type is initially stabilized, then stabilization is ceased and


Figure 5.8: The final part of the braid is completed. The $z$-string has to be updated to the red dashed line so that it does not intersect the $x$-string to preserve the commutator.
it becomes a bit with $X_{2}$ identified as the leftmost blue plaquette, $\mathcal{O}_{2}$, and $Z_{2}$ identified as the Majorana billinear shown as a red string, $b_{2}$.

Two leads are connected to a pair of Majoranas as shown. The leads also have an interference link between them (not shown) so that the conductance between them is

$$
\begin{equation*}
G \sim\left|T_{d}+T b_{1} b_{2}\right|^{2} \tag{5.4.23}
\end{equation*}
$$

thus allowing us to measure $b_{1} b_{2}$.
Since $X_{2}$ was initially stabilized we are still in the subspace where it acts trivially. Hence a good representation of $Z_{1}$ is $\mathcal{O}_{1} \mathcal{O}_{2}$. With $X_{1}$ identified as the blue string both representations commute with the measurement of $b_{1}$, which we now perform.

Since $b_{1} b_{2}=1$ we can identify $X_{1}=b_{1} b_{1} b_{2}=b_{2}=Z_{2}$. We then measure both green plaquettes. The final step is to identify $Z_{1}$ with $\mathcal{O}_{1} \mathcal{O}_{2} \mathcal{O}_{1}=\mathcal{O}_{2}=X_{2}$. Thus

$$
\begin{align*}
& Z_{1} \rightarrow X_{2},  \tag{5.4.24}\\
& X_{1} \rightarrow Z_{2} \tag{5.4.25}
\end{align*}
$$

which is exactly the Hadamard-gate (5.4.1).

### 5.4.4 S- and T-gates

Ref. [19] has an algorithm for the S- and T-gates but it relies on the preparation of the ancilla states

$$
\begin{align*}
\left|A_{S}\right\rangle & =\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)  \tag{5.4.26}\\
\left|A_{T}\right\rangle & =\frac{1}{\sqrt{2}}\left(|0\rangle+\frac{1+i}{\sqrt{2}}|1\rangle\right) \tag{5.4.27}
\end{align*}
$$



Figure 5.9: The setup for the Hadamard gate. The information is initially stored in the green $z$ type qubit and is to be moved to the blue $x$-type qubit. Leads are connected to two Majoranas (red circles) as shown, so we can measure $X_{1} Z_{2}$ where $X_{1}$ is the (representation of) the logical $X$ on the $z$-bit and $Z_{2}$ is (the representation of) the logical $Z$ on the $x$-bit. The orange blobs are leads tunnel-coupled (dashed lines) to Majoranas.

Here we discuss only how to prepare these states.
The basic idea is to measure the operators

$$
\begin{align*}
& \left|A_{S}\right\rangle\left\langle A_{S}\right|,  \tag{5.4.28}\\
& \left|A_{T}\right\rangle\left\langle A_{T}\right| . \tag{5.4.29}
\end{align*}
$$

Obviously they have eigenstates $\left|A_{S / T}\right\rangle$ and if we were lucky enough to project to this state the preparation is complete. The final check we then need to perform is whether the other eigenstates can be transformed to $\left|A_{S / T}\right\rangle$ or if we have to do repeated measurements to get the right state.

S-gate ancilla We deal with the $S$-gate first.

$$
\left|A_{S}\right\rangle\left\langle A_{S}\right|=\frac{1}{2}\left(\begin{array}{cc}
1 & -i  \tag{5.4.30}\\
i & 1
\end{array}\right)=\frac{\mathbb{1}+\sigma_{y}}{2} .
$$

We see that by measuring $\sigma_{y}$ we can prepare the ancilla state we want. Of course, this is obvious from eq. (5.4.28), but this way of constructing the required measurement is helpful when we consider the T-gate ancilla.

It is easiest to the preparation at the edge of the code, see figure 5.10. A number of Majorana-Cooper boxes are shown. On the active boxes Majorana fermions are shown as red circles.

Operating on the edge allows us to make measurements and transformations on the boxes at the edge withouth affecting more than one plaquette.

We assume the active plaquette is a $z$-type. The active plaquette is stabilized before preparation begins so the state is

$$
\begin{equation*}
\alpha_{1}|0000\rangle+\alpha_{2}|0011\rangle+\alpha_{3}|1001\rangle+\ldots \tag{5.4.31}
\end{equation*}
$$



Figure 5.10: We prepare the ancilla states at the boundary by measuring Majorana billinears (Majoranas indicated by red dots, billinears by the solid and dotted lines) in the manner described in the text.

We cease stabilization and measure the $\sigma_{z}$ operator, i.e. the billinear indicated by the solid line in the figure. Assuming we get $\sigma_{z}=1$ the state of the plaquette becomes

$$
\begin{equation*}
\alpha_{1}|0000\rangle+\alpha_{2}|0110\rangle+\alpha_{3}|1010\rangle+\ldots=|0\rangle_{\text {Rest }}|0\rangle_{\text {Box }} \tag{5.4.32}
\end{equation*}
$$

Here we have separated the degrees of freedom into "box" degrees of freedom and "the rest", where the box refers to the box on which we are doing measurements.

After $\sigma_{z}$ measurement we measure the $\sigma_{y}$-operator, i.e. the billinear indicated by the dashed line in the figure. This projects the state to

$$
\begin{equation*}
|0\rangle_{\text {Rest }} \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)=|0\rangle_{\text {Plaquette }} \pm i|1\rangle_{\text {Plaquette }} . \tag{5.4.33}
\end{equation*}
$$

If the upper sign is found we are done. If the lower sign is found we can perform $\sigma_{z}$ on the box and we are done.

If the intermediate measurement was $\sigma_{z}=-1$ we can e.g. correct by performing $\sigma_{x}$ on the box.

T-gate ancilla We need to measure the operator

$$
\left|A_{T}\right\rangle\left\langle A_{T}\right|=\frac{1}{2}\left(\begin{array}{cc}
1 & e^{-i \pi / 4}  \tag{5.4.34}\\
e^{i \pi / 4} & 1
\end{array}\right)=\frac{1}{2}\left(\mathbb{1}+\frac{\sigma_{x}+\sigma_{y}}{\sqrt{2}}\right)
$$

Essentially we have to measure $\sigma_{x}+\sigma_{y}$. As we know from chapter 2 this can be done if we can construct a tunnelling Hamiltonian $H_{T}=T\left(\sigma_{x}+\sigma_{y}\right)$, so it will require some fine-tuning.

We will go into somewhat more detail than previously because there are some technical points that we need to consider.

Consider the setup in figure 5.11. Two leads are tunnel-coupled the Majorana-Cooper box. One of the leads is only coupled to one Majorana fermion while the other is connected
to two Majorana fermions. This creates two tunnelling paths: one in the $\sigma_{x^{-}}$'channel' and one in the $\sigma_{y}$-channel. The Hamiltonian for this system is

$$
\begin{equation*}
H=H_{l e a d_{1}}+H_{\text {lead }_{2}}+E_{C}\left(N-N_{g}\right)^{2}+\left[t_{1} \psi_{1} \gamma_{1}+t_{2} \psi_{2} \gamma_{2}+t_{3} \psi_{2} \gamma_{3}+\text { h.c. }\right] . \tag{5.4.35}
\end{equation*}
$$

Referring again to figure 5.11 the $\sigma_{x}$-operator for the MCB is $\sigma_{x}=i \gamma_{1} \gamma_{2}$ while $\sigma_{y}=i \gamma_{3} \gamma_{1}$.
With the methods of section 1.3 we can then go to the effective Hamiltonian

$$
\begin{align*}
H= & H_{\text {lead }_{1}}+H_{\text {lead }_{2}}+\left[\left(T_{x} \sigma_{x}+T_{y} \sigma_{y}\right) \psi_{1}^{\dagger} \psi_{2}+\text { h.c. }\right] \\
& -\mathcal{T}_{1} \psi_{1}^{\dagger} \psi_{1}-\mathcal{T}_{2} \psi_{2}^{\dagger} \psi_{2}-\mathcal{T}_{3} \psi_{2}^{\dagger} \psi_{2}+\mathcal{T}_{\perp} \sigma_{z} \psi_{2}^{\dagger} \psi_{2} . \tag{5.4.36}
\end{align*}
$$

We have introduced

$$
\begin{array}{r}
T_{x}=-i t_{1} t_{2}^{*}\left(\frac{1}{U_{+}}+\frac{1}{U_{-}}\right), \\
T_{y}=-i t_{1} t_{3}^{*}\left(\frac{1}{U_{+}}+\frac{1}{U_{-}}\right), \\
\mathcal{T}_{i}=\left|t_{i}\right|^{2}\left(\frac{1}{U_{+}}-\frac{1}{U_{-}}\right), \\
\mathcal{T}_{\perp}=2 \operatorname{Re}\left[-i\left(t_{2} t_{3}^{*}\left(\frac{1}{U_{+}}+\frac{1}{U_{-}}\right)\right]\right. \tag{5.4.40}
\end{array}
$$

For the conductance measurement to yield a product state we need $t_{2}=t_{3}$, whence $T_{x}=$ $T_{y}=: T$, so $G \sim|T|^{2}\left|\sigma_{x}+\sigma_{y}\right|^{2}$. A measurement of the conductance between the leads with an interference link will then project to an eigenstate of $\sigma_{x}+\sigma_{y}$.

One might worry about whether the $\mathcal{T}_{\perp}$-term produces decoherence in the $\sigma_{z}$-basis, but if $t_{2}=t_{3}$ then $\mathcal{T}_{\perp}=0$ so we need not worry about this.

If the measurement of $\sigma_{x}+\sigma_{y}$ yields +1 we have the desired ancilla state. If the measurement yields -1 we can perform the following operation on the box to get a better state:

$$
\begin{equation*}
|\downarrow\rangle \rightarrow \sigma_{y} \sigma_{x}|\downarrow\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle+e^{-i \pi / 4}|1\rangle\right) . \tag{5.4.41}
\end{equation*}
$$

While this is not the desired ancilla state, the algorithm of ref. [19] still works.
Again we start by measuring $\sigma_{z}$ on the box. Assume we get +1 . Then the state of the plaquette is $|0\rangle_{\text {Rest }}|0\rangle_{\text {Box }}$. Measuring $\sigma_{x}+\sigma_{y}$ as above projects to a "good" state regardless of the measurement outcome, so after measurement

$$
\begin{equation*}
\left.|0\rangle_{\text {Rest }}|0\rangle_{\text {Box }} \rightarrow|0\rangle_{\text {Rest }} \mid \text { Good }\right\rangle_{\text {Box }} \tag{5.4.42}
\end{equation*}
$$

This ends our discussion of ancilla preparation and surface code operation in general.


Figure 5.11

### 5.5 Concluding Remarks

In this section we have introduced the concept of stabilizer coding and surface codes [14, 15, $16,17,18,19]$.

I have presented what I think is the cleanest way (at least in this context) to think about the idea of encoding information into physical states and how physical operations correspond to informational operations.

A certain set of operations are required for universal quantum computation to be achievable [20], and we have discussed how to implement these operations partly based on ref. [19] and partly based on some simple suggestions inspired by the discussions of the preceeding chapters.

Needless to say we have merely scratched the surface. There are many more things that may be worth investigating when considering a specific problem and the setup presented here is by no means the only way to implement a surface code. For a general introduction see ref. [19].

## Chapter 6

## Conclusion

We have covered alot of ground and it is useful to take a moment to look back at the way we came. Majorana fermions emerge in certain systems and they have peculiar properties. We explored how Majorana fermions behave in tunnelling experiments and how the MajoranaCooper box is affected by adiabatic electron pumping.

It turned out that the tunnelling across a pair of Majorana fermions has amplitude $\operatorname{Ti} \gamma_{1} \gamma_{2}$, so flipping the parity of the state flips the sign of the amplitude, an effect which among other things allows us to project to definite parity states via current measurements (chapter 2.

Pumping an electron across a pair of Majoranas yields a rotation $i \gamma_{1} \gamma_{2}$ of the state of the system. This is probably what one would expect, but we also saw how this expectation fails in more general setups.

These studies yielded a set of tools that allowed us to implement a surface code setup in the final chapter. Here some basic quantum informational aspects were discussed as well as implementation and operation of the surface code.

A lot of work may still be put into the surface code. Some of the gates we have suggested may not be optimal and perhaps a different fundamental building block will turn out to be preferable to the Majorana-Cooper box. For example, one can consider modified MajoranaCooper boxes where a larger number of topological nanowires are on the box. The simplest extension is similar to spin- 1 for three nanowires and $n_{g}=1$; you can go as high-dimensional as you please by simply adding more wires. Parafermionic systems, if we can think of a clever physical realization, may also turn out to be useful.

One should also not forget that implementing the setup experimentally will be a huge challenge and perhaps this effort will reveal questions we have not yet thought to address.

From a more general point of view, I hope this thesis has provided some idea of the interesting things that topological order can do. A lot of interesting questions came to mind while doing this work and I doubt that the fascination of entanglement will be extinguished any time soon.

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[^0]:    ${ }^{1}$ It is only the algebra of the operators $Z, X$ that matter for the results of chapter 4 to hold. Therefore we can apply those results even though that was done for $\sigma_{x}=i \gamma_{1} \gamma_{2}$ and not the long strings considered here.

