Effects of Non-Hermiticity on Symmetry and Topology in Condensed Matter Systems

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Abstract

This thesis provides a review of the effects of relaxing the Hermiticity condition usually imposed on Hamiltonian operators in quantum systems and explore the consequences of the non-Hermiticity on the symmetry and the topology. The complex-valued spectrum of the non-Hermitian Hamiltonians provide additional degrees of freedom when it comes to the definition of energy gaps and this leads to the definition of so-called point- and line-gaps. In non-Hermitian systems, the typical particle-hole, time-reversal and chiral symmetries are no longer sufficient in characterizing non-Hermitian Hamiltonians. This has therefore lead to the definition of the Bernard-LeClair (BL) symmetries, which provide a generalization of the Hermitian symmetry operations to the non-Hermitian case. This generalization of the symmetry operations has profound effects on the number of possible symmetry classes and it is shown how the relaxation of the Hermiticity requirement results in a generalization of the 10 Altland-Zirnbauer classes described for Hermitian systems. The topological classification of the BL symmetry classes is obtained by using topological K-theory, and it is shown that this results in a total of 54 distinct symmetry classes for line-gapped systems and a total of 38 distinct symmetry classes for point-gapped systems defined by the BL symmetries. The thesis also provides a generalization of the dimensional raising formalism originally defined by Teo and Kane for Hermitian systems and it is shown that similar K-group isomorphisms exist for the new non-Hermitian symmetry classes.

Contents

1 Introduction **3**

2 Non-Hermitian Physics **6**
2.1 Biorthogonal Basis . . . 6
2.2 Complex Spectrum . . . 8
2.3 Spectral Singularities . . . 11
2.4 Exceptional Points . . . 14

3 Bernard-LeClair Symmetry **17**
3.1 Systems with Multiple Symmetries . . . 19
3.2 Eigenvalue Relations . . . 22
3.3 Extended Hamiltonian . . . 24
3.4 Hermitianization of Line-Gapped Hamiltonians . . . 27

4 Topological Classification of Symmetry Classes **31**
4.1 Classification in Terms of K-Groups . . . 31
4.2 K-Theory . . . 33
4.3 Classification for Line Gaps . . . 37
4.4 Classification for Point Gaps . . . 42

5 Dimensional Raising **48**
5.1 Spatially Modulated Hamiltonians . . . 48
5.2 Hamiltonian Mappings . . . 49
5.3 Invertibility . . . 51
5.4 Representative Hamiltonians . . . 54
5.5 Non-Hermitian Bott Clocks . . . 56

6 Novel Properties intrinsic to Non-Hermitian Topology **60**
6.1 Forgetful Functor from Line-Gap to Point-Gap Topology . . . 60
6.2 Non-Hermitian Skin Effect . . . 61
6.3 Duality of Non-Hermitian Skin Effect and Hermitian Exact Zero Modes . . . 64
6.4 Different Kinds of Perturbations . . . 65

7 Conclusion **68**

1 Introduction

While the assumption of Hermiticity of the Hamiltonians is effective in describing the physics of isolated quantum systems, the interest in non-Hermitian systems has seen a growing interest recently as non-Hermiticity naturally arise in the description of open quantum systems. The non-Hermiticity is a consequence of the openness of the systems, this is because an open system typically will be in a non-equilibrium state with its surroundings, thereby allowing for mechanisms by which the energy of the system can be attenuated or amplified.

A general fact about Hermitian matrices is that the eigenvalues are real-valued and that the eigenvectors corresponding to different eigenvalues are orthogonal. These conditions are not generally satisfied in non-Hermitian systems, where the eigenvalues of the Hamiltonian will generally be complex-valued and it is possible for eigenvectors to become linearly dependent. This property of the eigenvectors not being orthogonal has multiple implications for novel intrinsic non-Hermitian physics.

One of the most interesting features arising from the non-orthogonality of the eigenvectors is that of exceptional points, which describes a point in parameter space, at which two or more eigenvalues become degenerate and their corresponding eigenvectors become equal and thereby coalescing into a single eigenvector. This has the implication that the geometric multiplicity becomes lower than that of the algebraic multiplicity, which is prohibited in Hermitian physics. These defects in the spectrum leads to novel features that have no Hermitian analogue.

Another important consequence of the intrinsic non-Hermiticity is the non-Hermitian skin effect [11], which leads to a breakdown of the typical bulk-boundary correspondence, which establishes a relation between topology of the bulk of a system and the existence of edge modes characterized by the topology of the bulk. An example of a system in which this breakdown occurs is the case of non-reciprocal transport on a 1D chain, which is can be described by a non-Hermitian Su-Schrieffer-Heeger model with imbalanced left/right hopping terms. This asymmetry in the hopping terms causes the otherwise delocalized bulk states of a system to instead favor one side of a system and this causes the eigenstates of the system to pile up at the boundary of the finite system. This has had the consequence that the standard Bloch theory is rendered invalid for systems with certain non-Hermitian features, this has lead to the construction of a new theories that can adequately describe non-Hermitian systems, one of the more useful is that of the non-Bloch theory[11][18].

The notion of symmetry in physical systems has played a major role in the development of topological phases in condensed matter.

Specifically the internal symmetries of a system, that is, the symmetries that do not relate to the spatiality of the system. In the Hermitian realm, the typical internal symmetries that are considered are the particle-hole symmetry, time-reversal symmetry and chiral symmetry. Together, the different possible combinations of these three symmetries constitute the 10 Altland-Zirnbauer symmetry classes[1], which are used to classify symmetry-protected topological phases of topological insulators/superconductors. It was shown by Bernard and LeClair [2], that the three symmetry transformations used in the AZ classification did not suffice in the classification of non-Hermitian random matrices. They proposed a generalization of the symmetry transformations and this lead them to obtain 38 distinct symmetry classes, contrasting the 10 symmetry classes of Altland and Zirnbauer.

The effect of relaxing the Hermiticity constraint on Hamiltonians, means that for non-Hermitian systems there exist additional degrees of freedom, that is, additional parameters which can be tuned. This implies that the underlying manifold, that is the spectrum of the Hamiltonian, must feel the effect of the non-Hermiticity, and it is therefore expected that the non-Hermiticity changes the topological classification of such systems. This therefore means that it is a very relevant to investigate the effects of relaxing the Hermiticity constraint of Hamiltonians and to explore the implications this has for the topology of non-Hermitian systems.

In section 2, some of the properties that arise due to the relaxation of the Hermiticity constraint will be derived and contrasted to the Hermitian results. It will shown how the complex-valued eigenvalues of non-Hermitian Hamiltonians leads to different types of energy gaps in the spectrum of the Hamiltonians, which will be denoted as point-gaps and line-gaps. This distinction between energy gaps has important implications for the classification of the topological phases of non-Hermitian systems, which will be derived in section 4.

In section 3, the symmetry transformations originally defined by Bernard and LeClair [2], which considered only the case corresponding to point-gaps, will be generalized to also encompass line-gapped systems. This section will also provide some tools that will be necessary in the classification of the topological phases of the different symmetry classes and it will be shown how a line-gapped non-Hermitian Hamiltonian can be flattened to either a Hermitian or anti-Hermitian Hamiltonian possessing the same symmetries and while retaining the line-gap. These results will be used in section 4, where the topological classification of the new symmetry classes will be done by considering matrix representations of the symmetry transformations as generators of a Clifford algebra. This allows for the symmetry classification to be expressed in terms of topological K-theory, where each symmetry classification is assigned a K-group [9], that characterizes the topological phases. This classification scheme leads to a total of 38 symmetry classes for point-gapped systems and 54 symmetry classes for line-gapped systems.

In section 5, a review will be given of the dimensional raising procedure for Hermitian systems originally derived by Teo and Kane [12], where they proved the existence of isomorphisms between the K-group of a given symmetry class in d-dimensions with the K-group of a different symmetry class in $d \pm 1$-dimensions. Following the review, it will be shown that the result of Teo and Kane can easily be generalized in such a way that it is applicable for all the non-Hermitian symmetry classes for both point- and line-gapped systems.

Section 6 finishes the thesis with a couple of examples of novel features that are intrinsic to non-Hermitian systems, such as the emergence of the non-Hermitian skin effect and how it arise due to non-Hermitian point-gap topology. Additionally, it will be shown how exceptional points and exceptional rings can be produced by non-Hermitian perturbations of Hermitian systems.

2 Non-Hermitian Physics

This section will provide a review of a selection of properties unique to non-Hermitian systems which will be useful in the discussion of topological phases of matter in the following sections.

2.1 Biorthogonal Basis

This section is based on [15].

In the usual Hermitian quantum mechanical theory the Hamiltonian operators are assumed to be Hermitian and the eigenvalues are therefore guaranteed to be real and there exist a complete orthonormal set of basis vectors that span the eigenspaces. These features are no longer guaranteed when the Hermiticity constraint is relaxed, instead the eigenvectors form a biorthogonal system.

To see this, construct a non-Hermitian Hamiltonian $H = h + i\Gamma$, where $h^\dagger = h$ and $\Gamma^\dagger = \Gamma$, and define the right eigenvectors of H to be $\{\,|\phi_n\rangle\}$ with eigenvalues $\{\kappa_n\}$, such that

$$H\,|\phi_n\rangle = \kappa_n\,|\phi_n\rangle \quad \langle\phi_n|\,H^\dagger = \kappa_n^*\,\langle\phi_n|\,. \tag{1}$$

Now assume that the eigenvalues of H are non-degenerate. Now since the Hamiltonian is non-Hermitian it is necessary to define the action of $H^\dagger$ as well. To do this, define a new set of eigenvectors $\{\,|\chi_n\rangle\}$, which are eigenvectors of $H^\dagger$, and has eigenvalues $\{\nu_n\}$, then the action of $H^\dagger$ is

$$H^\dagger\,|\chi_n\rangle = \nu_n\,|\chi_n\rangle \quad \langle\chi_n|\,H = \nu_n^*\,\langle\chi_n|\,. \tag{2}$$

The set of eigenvectors $\{\,|\chi_n\rangle\}$ correspond to the set of left eigenvectors of H. Now to determine the orthogonality of the left and right eigenvectors, notice that $2i\Gamma = H^\dagger - H$ and that $2h = H^\dagger + H$, this allows for the inner product between two of the right eigenvectors to be written as

$$\langle\phi_m|\phi_n\rangle = 2i\frac{\langle\phi_m|\,\Gamma\,|\phi_n\rangle}{\kappa_m^* - \kappa_n} = 2\frac{\langle\phi_m|\,h\,|\phi_n\rangle}{\kappa_m^* + \kappa_n} \tag{3}$$

which holds for $m \neq n$. similarly, for the inner product of two left eigenvectors

$$\langle\chi_m|\chi_n\rangle = 2i\frac{\langle\chi_m|\,\Gamma\,|\chi_n\rangle}{\nu_m^* - \nu_n} = 2\frac{\langle\chi_m|\,H\,|\chi_n\rangle}{\nu_m^* + \nu_n}. \tag{4}$$

In the case where H is non-Hermitian, the eigenvectors are generally not orthogonal. This means that the typical projection operators defined in Hermitian quantum mechanics are no longer usable.

From the definitions 1 and 2, it is seen that

$$\langle \chi_m | H | \phi_n \rangle = \nu_m^* \langle \chi_m | \phi_n \rangle = \kappa_n \langle \chi_m | \phi_n \rangle . \tag{5}$$

This implies that $\langle \chi_m | \phi_n \rangle = 0$ if $\kappa_n \neq \nu_m^*$, and $\langle \chi_m | \phi_n \rangle \neq 0$ if $\kappa_n = \nu_m^*$. Due to the fact that $\langle \chi_m | \phi_n \rangle = 0$ can not hold for all $|\chi_m\rangle$, there must exist a ν_m such that $\nu_m^* = \kappa_n$. But since the eigenvalues are assumed to be non-degenerate, there can be at most one ν_m such that $\nu_m^* = \kappa_n$. It can therefore be assumed that for each right eigenvalue κ_n there exist a corresponding left eigenvalue $|\chi_n\rangle$ such that $\kappa_n = \nu_n^*$. Hence it follows that $\langle \chi_m | \phi_n \rangle \neq 0$ for $n = m$ and $\langle \chi_m | \phi_n \rangle = 0$ for $n \neq m$. A pair consisting of a left and a right eigenvector, that satisfy

$$\langle \chi_m | \psi_n \rangle = \delta_{mn}, \tag{6}$$

where the indexing is determined by the eigenvalues, is said to be a biorthogonal system. This provides a notion of orthogonality between left and right eigenvectors and that for a given right eigenvalue of H there exist a corresponding complex conjugate left eigenvalue. In the Hermitian limit, where the left and right eigenvectors are the same, the biorthogonal system simply reduce to the typical orthogonal system.

Since the eigenvectors in the non-Hermitian case are generally not orthogonal, it is convenient to introduce the notion of an *exact* set of basis vectors instead. A set of basis vectors is called exact if it is minimal and complete. To show the exactness of the set of basis vectors, it is sufficient to show that the basis vectors are linearly independent, since the spectrum of H is non-degenerate. The linear independence of the eigenvectors can be proven by contradiction.

Assume that the eigenvectors $\{ |\phi_n\rangle \}$ are linearly dependent, this implies the existence of a set of complex coefficients $\{c_n\}$ such that $\sum_n |c_n|^2 \neq 0$ and such that

$$\sum_n c_n |\phi_n\rangle = 0. \tag{7}$$

Multiplying this by $\langle \chi_m |$ from the left yields $c_m \langle \chi_m | \phi_m \rangle = 0$. Since the $\langle \chi_m | \phi_m \rangle \neq 0$ when K is non-degenerate, it follows that the coefficients c_m in 7 must all be zero, which is a contradiction. This therefore shows that for a non-degenerate H, the right eigenvectors are linearly independent, and they therefore constitute a minimal and complete basis for the Hilbert space. Exactness of the right eigenvectors $\{\phi_n\}$ implies exactness of the left eigenvectors $\{\chi_m\}$ when the Hilbert space is finite dimensional. This can be shown by carrying out the same procedure as above for the left eigenvector and the result is analogous.

2.2 Complex Spectrum

This section is based on [3], [5].

In Hermitian physics the Hamiltonian operators are always Hermitian and due to the Hermiticity the eigenvalues are guaranteed the be real, this is no longer the case in non-Hermitian physics where the eigenvalues are generally complex-valued. An implication of the complex eigenvalues is that the time-evolution in non-Hermitian quantum mechanics is no longer necessarily unitary. To see this feature consider a system where the time-evolution is described by the Schroedinger equation,

$$i\frac{d}{dt}\Psi = H\Psi,$$

then taking the time-derivative of the inner product of a generic state Ψ and using the distributivity of the derivative it is seen that

$$\begin{aligned}\frac{d\langle\Psi,\Psi\rangle}{dt} &= \langle\frac{d\Psi}{dt},\Psi\rangle + \langle\Psi,\frac{d\Psi}{dt}\rangle \\ &= \langle -iH\Psi,\Psi\rangle + \langle\Psi,-iH\Psi\rangle \\ &= \langle\Psi, i(iH^* - H)\Psi\rangle\end{aligned}$$

This has the implication that the time-evolution of the system is no longer necessarily unitary for non-Hermitian systems, which is ensured by the real-valuedness of the eigenvalues in Hermitian systems. A physical interpretation of this non-unitary time-evolution is that the imaginary part of the energy eigenvalues correspond to either a gain or loss of the system energy, which as an example could be an open system that radiates heat to the surroundings and thereby dissipate energy.

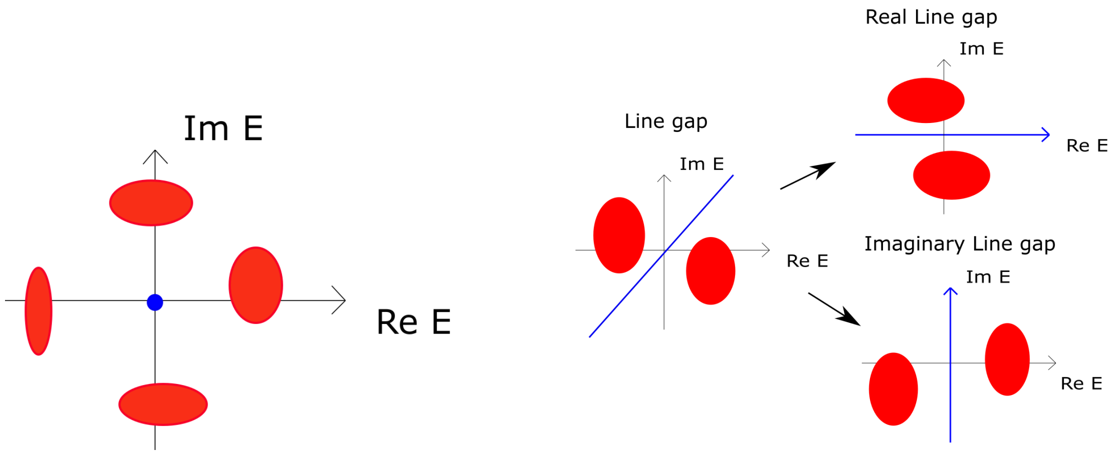

Figure 1: (left) Spectrum of non-Hermitian Hamiltonian with the red circle indicating the set of eigenvalues for all values of the the wave vector k, at the origin there is a point gap represented by a blue circle. (right) Spectrum of non-Hermitian Hamiltonian with the red circles indicating the set of eigenvalues for all values of the the wave vector k, separating the red ellipses is the line gap represented by a blue line.

The real-valued spectrum in the Hermitian case can be consider as a one-dimensional line, whereas the complex-valued spectrum in the non-Hermitian case can be considered as a two-dimensional plane. This idea of the spectrum being a surface rather than a line, has the consequence that it is possible to define different types of energy gaps, since the co-dimension of an energy gap now can take the values 0 and 1. This leads to the definition of two types of energy gaps, the first is the point gap, which is a zero dimensional gap in the complex plane and the second is the line gap, which is a non-contractible continuous line that gaps the spectrum.

In order to distinguish these types of gaps, a formal definition is required to ensure that the two types of gaps are topological distinct.

A non-Hermitian Hamiltonian $H(k)$ has a point gap at $E_p \in \mathbb{C}$ if and only if it is invertible for all values of k in its domain, except for at E_p. Similarly, a non-Hermitian Hamiltonian $H(k)$ has a line gap, where the line gapping the spectrum is a smooth curve $\gamma \in \mathbb{C}$, if and only if it is invertible for all k in its domain, except for the points on the curve γ.

Since it is the topological features that arise due to the gaps that are of interest, a few simplifications can be made to the definitions of the gaps above. Due to the topology of the spectrum being invariant under continuous deformation and translation, it is therefore always possible to translate the location of the point gap to the origin, while preserving the topology of the Hamiltonian. Similarly, for the line gap, the spectrum can be continuously deformed such that the curve γ lines up with either the real or the imaginary axis, which leads to two different types of line gaps, a real and an imaginary line gap, which are related to each other by Wick rotation i.e. $H \to iH$. This motivates the following, simplified, definitions:

Definition: A non-Hermitian Hamiltonian $H(k)$ has a point gap if and only if it is non-singular for all values of k in its domain, and all the eigenvalues are nonzero.

Definition: A non-Hermitian Hamiltonian $H(k)$ has a real (imaginary) line gap if and only if it is non-singular for all k in its domain and the real (imaginary) part of the eigenvalues is nonzero.

The figures 1 shows the spectrum of a generic Hamiltonian with a point-gap (left) and a line-gap (right).

In order to demonstrate the nature of point-gaps it is instructive to consider an example. The minimal example of a system demonstrating a point-gap is the Hatano-Nelson model, which describes a chain with nearest neighbor hopping and is given by

$$H = \sum_n \left(t_L c_n^\dagger c_{n+1} + t_R c_{n+1}^\dagger c_n \right), \tag{8}$$

where $t_L, t_R \in \mathbb{R}$ and $c_n^\dagger$ is the creation operator for a state at a point n. The Hamiltonian is Hermitian in the case of $t_L = t_R$. The eigenvalues of the Hamiltonian are given by $E(k) = (t_L + t_R)\cos(k) + i(t_L - t_R)\sin(k)$ for a given wave vector k.

To show this, a method of defining a topological invariant for a point-gapped system must be given. The winding number for a system with a point-gap can be defined as

$$w = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dk \partial_k \ln E(k). \tag{9}$$

The winding number effectively returns a quantity that can be used to distinguish topologically inequivalent paths, e.g. whether a solution wraps clockwise or counter-clockwise around the point-gap, or in the topologically trivial case $w = 0$, the spectrum is point-gapless. When the hopping terms are chosen to be equal $t_L = t_R$, the Hamiltonian is Hermitian and the winding number is zero and it is therefore considered topologically trivial. If, however, the hopping terms are unequal, then the Hamiltonian is non-Hermitian and as it turns out, it also becomes topologically non-trivial. In the case where $|t_L| < |t_R|$, the eigenvalues wind around the point-gap in the clockwise direction, which is characterized by the winding number $w = +1$, and in the case of $|t_L| > |t_R|$ it winds counter-clockwise, characterized by the winding number $w = -1$.

2.3 Spectral Singularities

This section is based on [8]

This section will introduce the idea of spectral singularities, which arise in the context of complex scattering potentials in non-Hermitian systems. These spectral singularities lead to singularities in the scattering matrix of a system and additional they also lead to the definition of so-called exceptional points. Exceptional points are points in the spectrum of a non-Hermitian Hamiltonian, for which the eigenvalues become degenerate, but in contrast to Hermitian degeneracies, the eigenvectors coalesce and become the same.

Consider the typical Schrodinger operator in the form

$$H = -\frac{d^2}{dx^2} + v(x), \tag{10}$$

together with the Schrodinger equation

$$-\psi''(x) + v(x)\psi(x) = k^2\psi(x), \quad x \in \mathbb{R}. \tag{11}$$

Now let $v(x)$ be a real/complex-valued scattering potential defined on $\mathbb{R}$, which satisfy $|v(x)| \to 0$ as $x \to \pm\infty$ such that

$$\int_{-\infty}^{\infty} (1 + |x|\,)|v(x)|\, dx < \infty. \tag{12}$$

It then follows that for each $k \in \mathbb{R}$, 11 admits a pair of solutions for the asymptotic boundary conditions

$$\lim_{x\to\pm\infty} \mathrm{e}^{\mp ikx}\psi_{k\pm}(x) = 1, \quad \lim_{x\to\pm\infty} \mathrm{e}^{\mp ikx}\psi'_{k\pm}(x) = \pm ik,$$

these solutions are the called the Jost solutions of 11. Formally, this leads to the definition:

Definition 1. Let $v : \mathbb{R} \to \mathbb{C}$ be a function satisfying (12) and H be the Schroedinger operator (10) that is defined by $v \in \mathbb{R}$. A real and positive number $k_\star^2$ is called a spectral singularity of H or v, if the Jost solutions $\psi_{k_\star,\pm}$ of (11) are linearly dependent.

The Jost solutions correspond to the scattering states of the potential $v(x)$ and for real scattering potentials they are always linearly independent.

Let the right- and left-incident scattering solutions of the Schrodinger equation be denote as $\psi_k^r(x)$ and $\psi_k^r(x)$, respectively. These scattering solutions satisfy the boundary conditions

$$\psi_k^l(x) \to \begin{cases} A^l(k)(\mathrm{e}^{ikx} + R^l(k)\mathrm{e}^{-ikx}) & \text{for} \quad x \to -\infty \\ A^l(k)T^l(k)\mathrm{e}^{ikx} & \text{for} \quad x \to \infty \end{cases} \tag{13}$$

$$\psi_k^r(x) \to \begin{cases} A^l(k)T^l(k)\mathrm{e}^{-ikx} & \text{for} \quad x \to -\infty \\ A^r(k)(\mathrm{e}^{-ikx} + R^r(k)\mathrm{e}^{-ikx}) & \text{for} \quad x \to \infty \end{cases} \tag{14}$$

where $A^{l/r}$, $R^{l/r}$, and $T^{l/r}$ are complex-valued functions. Due to the linearity of the Schrodinger equation, the choice of $A^{l/r}$ does not affect the physically measurable quantities. This is however not the case for the reflection and the transmission amplitudes $R^{l/r}$ and $T^{l/r}$. This is because the experimentally observable left and right reflection and transmission coefficients are determined by the squared modulus $|R^{l/r}|^2$ and $|T^{l/r}|^2$, respectively. Another consequence of the linearity of the Schroedinger equation is that $T^l = T^r$ and they can therefore simply be denoted as T. These equations for $\psi_k^{l/r}$ coincide with the Jost solutions $\psi_{k\pm}$, which for $A^{l/r}(k)T(k) = 1$ can be written as

$$\psi_{k+}(x) \to \begin{cases} T^{-1}(k)(\mathrm{e}^{ikx} + R^l(k)\mathrm{e}^{-ikx}) & \text{for} \quad x \to -\infty \\ \mathrm{e}^{ikx} & \text{for} \quad x \to \infty \end{cases} \tag{15}$$

$$\psi_{k-}(x) \to \begin{cases} \mathrm{e}^{-ikx} & \text{for} \quad x \to -\infty \\ T^{-1}(k)(\mathrm{e}^{-ikx} + R^r(k)\mathrm{e}^{ikx}) & \text{for} \quad x \to \infty \end{cases} \tag{16}$$

It can be seen that by the existence of the Jost solutions it follows that $T(k) \neq 0$, and therefore that perfectly absorbing potentials do not exist. It turns out that the coefficients of the $\mathrm{e}^{\pm ikx}$ terms that appear on the RHS of 15 and 16 coincide with the entries of a 2×2 complex matrix known as the transfer matrix.

Since $v(x) \to 0$ as x goes to $\pm\infty$ it follows that every solution $\psi(x)$ of the Schroedinger equation satisfies the boundary conditions

$$\psi(x) \to A_\pm(k)\mathrm{e}^{ikx} + B_\pm(k)\mathrm{e}^{-ikx} \quad \text{for} \quad x \to \pm\infty \tag{17}$$

where $A_\pm(k)$ and $B_\pm(k)$ are complex-valued coefficients. The transfer matrix $M(k)$ is defined by the relation

$$A = \begin{bmatrix} A_+(k) \\ B_+(k) \end{bmatrix} = M(k) \begin{bmatrix} A_-(k) \\ B_-(k) \end{bmatrix}.$$

The entries M_{ij} of $M(k)$ can now be related to the reflection and transmission amplitudes, this is done by comparing the equations 13, 14 and 17, which yields

$$M_{11} = T - \frac{R^l R^r}{T}, \quad M_{12} = \frac{R^r}{T}, \quad M_{21} = -\frac{R^l}{T}, \quad M_{22} = \frac{1}{T}, \tag{18}$$

from which it can be seen that $\det(M(k)) = 1$. The Jost solutions (13) and (14) can now be expressed in terms of the transfer matrix entries as

$$\psi_{k+}(x) \to \begin{cases} M_{22}(k)\mathrm{e}^{ikx} - M_{21}(k)\mathrm{e}^{-ikx} & \text{for} \quad x \to -\infty \\ \mathrm{e}^{ikx} & \text{for} \quad x \to \infty \end{cases} \tag{19}$$

$$\psi_{k-}(x) \to \begin{cases} \mathrm{e}^{-ikx} & \text{for} \quad x \to -\infty \\ M_{22}(k)\mathrm{e}^{-ikx} + M_{21}(k)\mathrm{e}^{ikx} & \text{for} \quad x \to \infty \end{cases}. \tag{20}$$

These equations leads to a formal characterization of spectral singularities, which can be stated as the following theorem[8]:

Theorem Let $v : \mathbb{R} \to \mathbb{C}$ and H be as defined above, $M(k)$ be the transfer matrix of v, $M_{ij}(k)$ be the entries of $M(k)$, and $k_\star$ be a positive real number. Then $k_\star^2$ is a spectral singularity of H (or v) if and only if $M_{22}(k_\star) = 0$.

Proof. $k_\star^2$ is a spectral singularity of H whenever $\psi_{k_\star-}$ and $\psi_{k_\star+}$ are linearly dependent, and according to (19) and (20) and the fact that these equations determine $\psi_{k\pm}$ uniquely, this is only the case if and only if $M_{22}(k_\star) = 0$.

A physical meaning of spectral singularities can be obtained by combining 18 with this theorem. The spectral singularities correspond to the energies $k_\star^2$, for which the reflection and the transmission amplitudes are divergent.

The latter is a characteristic property of resonances, for they satisfy the outgoing boundary conditions. As seen from (36) and (37), for the cases that $k_\star$ corresponds to a spectral singularity and $\psi_{k_\star\pm}$ become linearly dependent, they also satisfy the outgoing boundary conditions.

The main difference between the wave function of a resonance and the Jost solutions $\psi_{k_\star\pm}$ at a spectral singularity $k_\star^2$ is that the former satisfies the Schroedinger equation for a non-real value of k^2, whereas the latter does not. The imaginary part of k^2 for a resonance determines its width and therefore the spectral singularities can be identified with the energies of certain zero-width resonances. The spectral singularities are distinguished from the bound states in the continuum, since they determine non-decaying scattering states with positive and real energies. Although bound states in the continuum are also associated with zero-width resonances, their wave functions are square-integrable solutions of the Schroedinger equation.

The fact that the reflection and transmission amplitudes and consequently the reflection and transmission coefficients $\left|R^{l/r}(k)\right|^2$ and $|T(k)|^2$ diverge for a resonance does not conflict with the well-known unitary condition

$$\left|R^{l/r}(k)\right|^2 + |T(k)|^2 = 1, \tag{21}$$

because the k-value for a resonance is not real. For a spectral singularity, k is real and 21 is therefore violated, which implies that real potentials cannot support a spectral singularity. This leads to the conclusion that in the case of Hermitian Hamiltonians, for which the potential functions are necessarily real-valued, it is not possible for a system to possess spectral singularities.

The scattering matrix can expressed in terms of the elements of the transfer matrix as

$$S = \begin{bmatrix} -\frac{M_{21}}{M_{22}} & \frac{1}{M_{22}} \\ M_{11} - \frac{M_{12}M_{21}}{M_{22}} & \frac{M_{12}}{M_{22}}, \end{bmatrix}$$

and it is seen that at the spectral singularity, where $M_{22}(k_\star) = 0$, the scattering matrix entries are divergent.

A special case of spectral singularities are that of exceptional points, which are points in a parameter space, for which the two or more eigenvalues become degenerate, and their corresponding eigenvectors not only become linearly dependent, but actually coalesce into the same eigenvectors. This causes the geometric multiplicity of the eigenvalues to become smaller than the algebraic multiplicity. This leads to the formal definition of an exceptional point[8]:

Definition 2. Let V be a vector space, m be a positive integer, $H(x) : V \to V$ be a linear operator depending on m real parameters $x_1, x_2, \ldots, x_m$ and identify these with local coordinates of a point x of a parameter space M. Now suppose that for each $x \in M$ the eigenvalues of $H(x)$ have finite geometric multiplicity and form a countable set of isolated points in $\mathbb{C}$ that is denoted by $E_n(x)$, where $n \in \mathcal{N}$ is a spectral label taking values in a discrete set. Let $\mu_n(x)$ be the geometric multiplicity of $E_n(x)$, i.e. the dimension of the span of the eigenvectors of $H(x)$ that are associated with the eigenvalue of $E_n(x)$. A point $x_0 \in M$ is called an exceptional point of $H(x)$ if there exists $n \in \mathcal{N}$, $\epsilon \in \mathbb{R}^+$, and a parameterized curve in M, i.e. a continuous function, $\gamma : (-\epsilon, \epsilon) \to M$, such that $\gamma(0) = x_0$ and for all $t \neq 0$, $\mu_n(\gamma(t)) \neq \mu_n(x)$.

If for all $x \in M$, $H(x)$ is a Hermitian operator, the geometric multiplicity of the eigenvalues $E_n(x)$ do not undergo discontinuous changes and an exceptional point cannot exist and non-Hermiticity is therefore a necessary condition for the emergence of exceptional points.

2.4 Exceptional Points

This section is based on [16].

This section will expand on the concept of exceptional points defined in the previous section and explore how exceptional points arise in non-Hermitian physical systems. The minimal example of a system with an exceptional point, is the system described by the Hamiltonian

$$H(\alpha) = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix} \tag{22}$$

where $\alpha \in \mathbb{C}$ and the energy eigenvalues are $E_{\pm}(\alpha) = \pm\sqrt{\alpha}$. Notice that the eigenvalues are not analytic in α since $|\partial_\alpha E(\alpha)| \to \infty$ as $\alpha \to 0$. The biorthogonality of non-Hermitian systems has the implication that the right and left eigenvectors of $H(\alpha)$ are generally not the same, and it is therefore necessary to define both the right and left eigenvectors independently. The right eigenvectors of $H(\alpha)$ are of the form

$$\psi_{R,\pm} = \begin{pmatrix} \pm\sqrt{\alpha} \\ 1 \end{pmatrix} \tag{23}$$

and the left eigenvectors are of the form

$$\psi_{L,\pm} = \begin{pmatrix} 1, & \pm\sqrt{\alpha} \end{pmatrix}. \tag{24}$$

The eigenvalues are the same at $\alpha = 0$, but in contrast to the typical degeneracies of Hermitian systems, the eigenvectors are not orthogonal for $\alpha \neq 1$. It is seen that in the case of $\alpha = 0$, there only exist a single right eigenvector and a single left eigenvector. This means that the $H(\alpha)$ now has become defective, meaning that the geometric multiplicity is now smaller than the algebraic multiplicity for the case where $E = 0$, and it therefore follows that there exists an exceptional point in the spectrum of $H(\alpha)$ located at $\alpha = 0$. The real and imaginary spectrum is showing in figure 2, and it can be seen that the square root singularity behavior near the exceptional point results in the spectrum corresponding to two intersection Riemann sheets.

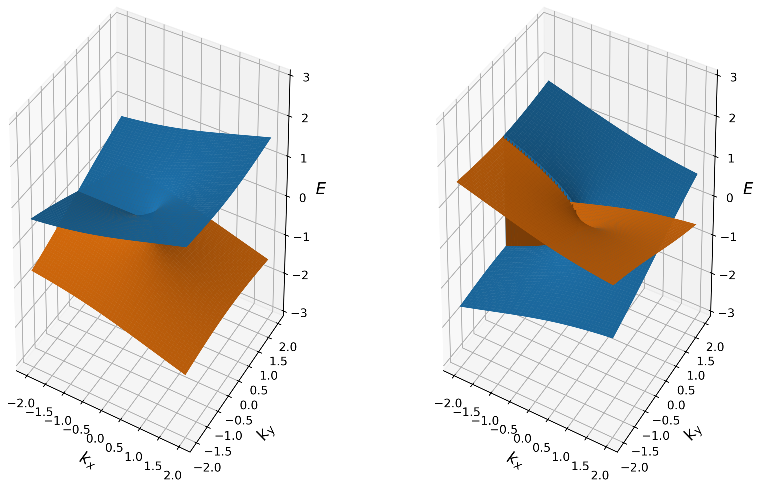

Figure 2: Real(left) and imaginary(right) part of the eigenvalues of 23 as functions of k_x and k_y.

An interesting feature of exceptional points can be observed by considering the effects of moving along a loop enclosing the exceptional point. To see this, write $\alpha = |\alpha|\, \mathrm{e}^{i \arg \alpha}$, then the eigenvalues can be written as $E_\pm = \pm|\alpha|^{1/2}\mathrm{e}^{i \arg \alpha/2}$, where on the principal domain $\arg \alpha$ takes values $-\pi < \arg \alpha \leq \pi$. Now in order to ensure that the loop can be chosen unambiguously, notice that there exist an energy gap $\Delta E = 2|\alpha|^{1/2}\mathrm{e}^{i \arg \alpha/2}$, which closes at $\alpha = 0$, but is open for $\alpha \neq 0$. By traversing the loop from $\arg \alpha$ to $\arg \alpha + 2\pi$ that is enclosing the exceptional point, it is then seen that the eigenvectors transform as

$$\psi_{R/L,\pm} \to \psi_{R/L,\mp} \tag{25}$$

and the eigenvalues transform similarly as $E_\pm \to E_\mp$. It is therefore seen that the effect of traversing a loop around an exceptional point effectively correspond to interchanging the eigenvalues between two intersection Riemann sheets. The topological nature of exceptional points will be explored through some examples in section 6.4.

3 Bernard-LeClair Symmetry

This section is based on [2],[4],[3],[5],[13],[6].

In Hermitian systems topological insulators/superconductors are typically defined as having topological states that are protected by certain symmetries, this symmetries are the particle-hole $\mathcal{C}$ symmetry, time-reversal symmetry $\mathcal{T}$ and chiral symmetry $\mathcal{P}$ defined such that

$$\{H, \mathcal{C}\} = [H, \mathcal{T}] = \{H, \mathcal{P}\} = 0, \tag{26}$$

where H is the Hamiltonian of the system. The chiral symmetry $\mathcal{P}$ is a unitary symmetry, and it is possible to multiply it by a phase such that $\mathcal{P}^2 = I$. The symmetries $\mathcal{C}$ and $\mathcal{T}$ are anti-unitary symmetries, which satisfy $\mathcal{C}^2, \mathcal{T}^2 = \pm I$. There is a total of 10 combinations of these symmetries and their squares, these 10 classes constitute the famous Altland-Zirnbauer classes for topological insulators/superconductors. These symmetry operations are however only sufficient for the classification of Hermitian systems, and in order to provide a classification for non-Hermitian systems, a proper generalization must be derived. The first steps towards this generalization was by Bernard and LeClair who proposed a generalization of the symmetry operations for complex random matrices in [2], this generalization lead to a total of 38 symmetry classes for generic complex matrices. Their classification is, however, only valid in the case of point gaps, as in their derivation Hamiltonians that can be mapped to each other under Wick rotations were treated as being equivalent, but, as was shown above, this is only valid for point gaps. It is therefore necessary to provide a further generalization in order to account for line gaps, as the action $H \to iH$, maps a real line gap to an imaginary line gap and vice versa.

The symmetry classification for non-Hermitian Hamiltonians, where the symmetry constraint can be expressed generically as

$$H = \epsilon_X U_X \mathcal{O}(H) U_X^\dagger$$

where $\epsilon_X = \pm 1$, U_X is a unitary operator and $\mathcal{O}(H)$ is an operation on the Hamiltonian. The possible operations on the Hamiltonian are identity, Hermitian conjugation, transposition or complex conjugation, and they will be defined to the symmetry transformations $X = P, Q, C, K$, respectively. Due to the fact that transposition correspond to spatial inversion and complex conjugation correspond time-reversal, the C, K-symmetries are to be considered as generalized versions of the Hermitian particle-hole and time-reversal symmetries, respectively, and they are therefore also chosen such that they change the sign of the momentum variables of the Hamiltonian.

The action of the symmetry operators should preserve the inner product of the eigenvectors and they should therefore be implemented as unitary operators satisfying the relations

$$qq^\dagger = 1, \quad pp^\dagger = 1, \quad cc^\dagger = 1, kk^\dagger = 1.$$

In addition to the unitarity of the operators, they are also required to be involutions. In the case of $X = C, K$ application of the symmetry operation once flips the sign of the momentum of the Hamiltonian together with a factor ϵ_X, acting with the symmetry operation again should flip the momentum back to the original sign and since $\epsilon_X = \pm 1$, it naturally squares to one. Hence it is instructive to determine the action of the symmetry operators under two applications. To see this, apply the C-symmetry operation twice:

$$h(k) = c(ch^T(k)c^{-1})^T c^{-1} = cc^* h(k) c^T c^\dagger,$$

the case of K-symmetry is completely analogous. Hence it is seen that $[XX^*, H] = 0$ for $X = C, K$, which implies that X^2 or XX^* should be proportional to the identity matrix. Similarly, it can be shown that $[X^2, H] = 0$ for $X = P, Q$. An appropriate choice of a $U(1)$ phase allows for the choice of $X^2 = I$ for $X = P, Q$. Similarly, when $X = C, K$ $XX^* = \pm I = \eta_X I$, where $\eta_X = \pm 1$ for $X = C, K$. The generalized Bernard-LeClair symmetry transformations can therefore be written as

$$h(k) = -ph(k)p^{-1}, \quad p^2 = I \qquad \text{P symmetry} \tag{27}$$

$$h(k) = \epsilon_q q h^\dagger(k) q^{-1}, \quad q^2 = I \qquad \text{Q symmetry} \tag{28}$$

$$h(-k) = \epsilon_c c h^T(k) c^{-1}, \quad cc^* = \eta_c I \qquad \text{C symmetry} \tag{29}$$

$$h(-k) = \epsilon_k k h^*(k) k^{-1}, \quad kk^* = \eta_k I \qquad \text{K symmetry.} \tag{30}$$

Notice that under the action of Wick rotation, the sign of ϵ_k and ϵ_q can be changed. This leads to a unification of certain symmetries for point-gapped systems and is the reason for the original 38 symmetry classes found by Bernard and LeClair. The Wick rotation can also be used to map a system with an imaginary line gap to one with a real line gap, it does however not change the underlying symmetries of the system, except for the sign change of ϵ_k and ϵ_q.

The AZ classes of Hermitian Hamiltonians were originally defined in terms of particle-hole, time-reversal and chiral symmetries. This can be translated to the language of BL symmetries by considering the P-symmetry as sublattice symmetry, the Q-symmetry as chiral symmetry and pseudo-Hermiticity, the C-symmetry with $\epsilon_c = -1$ as PHS, and the K-symmetry with $\epsilon_k = +1$ as TRS. Similarly, define a symmetry PHS† as a C-symmetry with $\epsilon_c = +1$ and TRS† as a K-symmetry with $\epsilon_k = -1$.

The AZ class can now be defined as having $\epsilon_q = +1$, PHS given by the C-symmetry with $\epsilon_c = -1$ and TRS with $\epsilon_k = +1$. The positivity of ϵ_q correspond to Hermiticity of the system, this can be seen since $h(k) = \epsilon_q q h^\dagger(k) q^{-1}$ is trivially satisfied for a Hermitian system. The chiral symmetry of the AZ class correspond to the Q-symmetry with $\epsilon_q = -1$, but due to the inherent Hermiticity, the system also possess Q-symmetry with $\epsilon_q = +1$, and the existence of simultaneous Q-symmetries with both $\epsilon_q = +1$ and $\epsilon_q = -1$ implies the existence of a P-symmetry, which aligns with the typical definition of the chiral symmetry $\mathcal{P}$ in Hermitian systems that was seen in 26. However, due to the fact that generally $H \neq H^\dagger$, the chiral and sublattice symmetries are typically distinct in non-Hermitian systems.

Similarly to the definition of the AZ class, two other classes can be defined for the new non-Hermitian systems with $\epsilon_q = -1$, these classes are called the NH AZ class and the AZ† class.

The NH AZ class is defined by the coefficients of the BL symmetry operators being fixed to be $\epsilon_q = -1$ $\epsilon_c = -1$ and $\epsilon_k = +1$. The NH AZ class is similar in nature to that of the Hermitian AZ class, where the C-symmetry correspond to PHS and the K-symmetry correspond to TRS.

Define the AZ† as the class where the coefficients of the BL symmetry operators are fixed to be $\epsilon_q = -1$ and $\epsilon_c = +1$ and $\epsilon_k = -1$. The NH AZ and the AZ† classes are distinct due to the fact that the lack of Hermiticity implies that generally $H^T \neq H^*$. An important note is that due to the definition of the NH AZ and AZ† classes, they do not include any configuration with P-symmetry. This therefore allows for the definition of two additional symmetry classes with Q-symmetry, which correspond to the NH AZ and the AZ† classes with the inclusion of a pseudo-hermiticity given by a Q-symmetry with $\epsilon_q = +1$, these classes with be denoted as the NH pAZ and pAZ† classes, respectively. Similarly, define another two classes NH sAZ and sAZ†, which correspond to adding a P-symmetry.

3.1 Systems with Multiple Symmetries

This section is based on [4].

In order to provide a unique classification for a given system, it is important to determine how symmetries might interact with one another in systems with multiple symmetries. This involves determining the commutation relations between different symmetries and finding whether combinations of symmetries might imply additional symmetries. Beginning with deriving the commutation relations for the different symmetry operations, starting with a system possessing simultaneous P- and Q-symmetry. The presence of both P- and Q-symmetry implies that

$$-php^{-1} = \epsilon_q q h^\dagger q^{-1},$$

which by substituting $-php^{-1}$ for h on the right-hand-side and substituting $\epsilon_q q h^\dagger q^{-1}$ for h left-hand side, yields

$$-p(\epsilon_q q h^\dagger q^{-1})p^{-1} = \epsilon_q q(-php^{-1})^\dagger q^{-1}.$$

This equality implies that

$$-pqh^\dagger q^\dagger p^\dagger = -qph^\dagger p^\dagger q^\dagger,$$

and using the fact that the transformations are unitary, i.e. $p^\dagger = p^{-1}$ and $q^\dagger = q^{-1}$, then by multiplication of pq on both sides of the equality, it is seen that $p^\dagger q^\dagger pq = \lambda I$, or $\lambda p = q^\dagger pq$. Squaring both sides of this equation yields $\lambda^2 = 1$ and thus that

$$p = \epsilon_{pq} q^\dagger pq, \quad \epsilon_{pq} = \pm 1. \tag{31}$$

For the other symmetry combinations the commutation relations can, by the same procedure, be shown to be

$$c = \epsilon_{pc} pcp^T, \quad k = \epsilon_{pk} pkp^T, \quad c = \epsilon_{qc} qcq^T \tag{32}$$

where $\epsilon_{\mu\nu} = \pm 1$ for $\mu, \nu = p, q, c, k$.

To fully classify a symmetry class uniquely requires knowledge of the types of symmetry considered, the sign of the commutation relations between the symmetry operators and the sign of C-symmetry coefficient ϵ_c, and the absolute values of the C- and K-symmetry operators η_c and η_k.

In general the ϵ_Xs are not independent of one another, for instance, a system having both Q, C-symmetry, the K-symmetry can be written as $k = qc$ with $\epsilon_k = \epsilon_q \epsilon_c$ and $\eta_k = \epsilon_{qc}\eta_c$. Another example is the case of simultaneous P- and C-symmetry, here $\tilde{c} = pc$ is also a C-symmetry, but with $\epsilon_{\tilde{c}} = -\epsilon_c$ and $\eta_{\tilde{c}} = \epsilon_{pc}\eta_c$, which leads to the equivalence between $(\epsilon_c, \eta_{pc}, \eta_c)$ and $(-\epsilon_c, \eta_{pc}, \eta_{pc}\eta_c)$.

Even though knowledge of the items listed above determines the symmetry class uniquely, knowing the symmetry class does not uniquely determine all the coefficients of a given class. There are some symmetry classes that possess the same symmetry operators, but with different combinations of the signs of the coefficients.

For a Hamiltonian with P- and C-symmetry, there exist an additional C-symmetry $\tilde{c} = pc$. Then using the definitions of the symmetry operators 27, the coefficients of $\tilde{c}$ are

$$\begin{aligned} h = -php^{-1} = -\epsilon_c \tilde{c} h^T \tilde{c}^{-1} &\Rightarrow \quad \epsilon_{\tilde{c}} = -\epsilon_c \\ \tilde{c}\tilde{c}^* = pcp^* c^* = pcp^T c^T = \epsilon_{cp} cc^* = \epsilon_{cp}\eta_c &\Rightarrow \quad \eta_{\tilde{c}} = \epsilon_{cp}\eta_c \\ p\tilde{c}p^T = cp^T = \epsilon_{cp} pc = \epsilon_{cp}\tilde{c} &\Rightarrow \quad \epsilon_{\tilde{c}p} = \epsilon_{cp} \\ \tilde{c}^\dagger q^\dagger \tilde{c} = c^\dagger p^\dagger q^\dagger pc = \epsilon_{pq} c^\dagger q^\dagger c = \epsilon_{pq} q^\dagger &\Rightarrow \quad \epsilon_{q\tilde{c}} = \epsilon_{qp}\epsilon_{qc} \end{aligned}$$

For the case of simultaneous P- and K-symmetry, there exists an additional K-symmetry $\tilde{k} = pk$, which has coefficients $\eta_{\tilde{k}} = \epsilon_{pk}\eta_k$, $\epsilon_{p\tilde{k}} = \epsilon_{pk}$.

For the combination of P- and Q-symmetries, an additional Q-symmetry exist given by $\tilde{q} = qp$, which in order to satisfy the condition that $\tilde{q}^2 = 1$ it requires an additional phase factor which, using that $qpqp = \epsilon_{pq}$, can be defined to be

$$\tilde{q} = \sqrt{\epsilon_{pq}}qp.$$

This additional Q-symmetry alters the commutation relations in the case where a C-symmetry is present as can be seen by considering the product

$$\begin{aligned} c\tilde{q}^* &= (\sqrt{\epsilon_{pq}})^* cq^* p^* \\ &= (\sqrt{\epsilon_{pq}})^* \epsilon_{qc}\epsilon_{pc} qpc = \epsilon_{qc}\epsilon_{pc}\epsilon_{pq}\tilde{q}c \end{aligned}$$

hence the commutator relation for the C-symmetry and the new Q-symmetry is given by $\epsilon_{\tilde{q}c} = \epsilon_{qc}\epsilon_{pc}\epsilon_{pq}$.

3.2 Eigenvalue Relations

This section is based on [4],[6].

In this section relations between the existence of certain symmetries and eigenvalue pairs will be derived. This includes a non-Hermitian version of the Kramer's theorem and a new additional theorems.

K-symmetry: Consider a Hamiltonian H with K-symmetry and let v_k be a right eigenvector of H with eigenvalue λ_k, then

$$\begin{aligned} h_k v_k = \lambda_k v_k &= \epsilon_k k h^*_{-k} k^\dagger v_k \\ \Rightarrow \epsilon_k \lambda_k (k^\dagger v_k) &= h^*_{-k}(k^\dagger v_k) \\ \Rightarrow \epsilon_k \lambda_k^* (k^T v_k^*) &= h_{-k}(k^T v_k^*) \end{aligned}$$

Hence K symmetry implies the existence of a right eigenvector $k^T v_k^*$ for h_{-k} with eigenvalue $\lambda_{-k} = \epsilon_k \lambda_k^*$. At the time-reversal invariant momenta this guarantees that there exist a eigenvalue pair $(\lambda_k, \epsilon_k \lambda_k^*)$. The case where $\epsilon_k = -1$ is reminiscent of the notion of time-reversal symmetry in Hermitian systems.

C-symmetry: Now to show the existence of energy eigenvalue pairs for a system with C-symmetry. To this end consider a system described by a non-Hermitian Hamiltonian h_k with C-symmetry. The C-symmetry includes the action of transposition of the Hamiltonian, it is therefore necessary to consider the biorthogonality of the eigenvectors of a non-Hermitian Hamiltonian. It was shown in section 2.1 that the eigenvectors of non-Hermitian systems typically form biorthogonal systems, which means that the set of left eigenvectors $\{u_n\}$ and the set of right eigenvectors $\{v_n\}$ generally are distinct and satisfy the biorthogonality relation $u_n v_m = \delta_{nm}$ given by 6.

Now, let v_k be a right eigenvector of h_k with eigenvalue λ_k,

$$\begin{aligned} h_k v_k = \lambda_k v_k &= \epsilon_c c h^T_{-k} c^\dagger v_k \\ \Rightarrow \epsilon_c \lambda_k (c^\dagger v_k) &= h^T_{-k}(c^\dagger v_k) \\ \Rightarrow (v_k^T c^*) h_{-k} &= \epsilon_c \lambda_k (v_k^T c^*) \end{aligned}$$

which implies the existence of a left eigenvector $v_k^T c^*$ of h_{-k} with eigenvalue $\lambda_{-k} = \epsilon_c \lambda_k$. When $\epsilon_c = -1$, this guarantees the existence of a pair of eigenvalues $(\lambda_k, -\lambda_k)$ at the time-reversal invariant momenta.

In the case where $\epsilon_c = +1$ the two eigenvalues are the same. Therefore in order to ensure the existence of a degeneracy at the time-reversal invariant momenta, the eigenvectors v_k and $v_k^T c^*$ must remain different at $k = -k$ and are therefore required to not form a biorthogonal pair. To satisfy this criteria it is sufficient to show that the right and left eigenvectors v_k and $v_k^T c^*$ are orthogonal, as this will prove the existence of a pair of independent eigenvectors at the time-reversal invariant momenta, and since v_k and $v_k^T c^*$ have the same eigenvalues they form a degeneracy at $k = -k$. This can be shown for the case where $\eta_c = -1$. Define $\lambda = v_k^T c^* v_k$, and using that $cc^* = \eta_c = -1$ which implies $c^\dagger = -c^*$, it follows that

$$\lambda = v_k^T c^* v_k = v_k^T c^\dagger v_k = v_k^T (-c^*) v_k = -\lambda$$

where the second equality was done by transposition. It is seen that $\lambda = 0$, which imply that $v_k^T c^*$ and v_k are orthogonal eigenvectors. Hence a symmetry protected degeneracy is guaranteed for systems having C-symmetry with $\epsilon_c = -1$ and $\eta_c = -1$, and this provides a biorthogonal generalization of the Kramer's degeneracy.

Q-symmetry: For the case of Q symmetry, consider a Hamiltonian h_k with Q-symmetry and let v_k be an eigenvector of h_k with eigenvalue λ_k, then it follows that

$$\begin{aligned} h_k v_k = \lambda_k v_k = \epsilon_q q h_k^\dagger q^\dagger v_k \\ \Rightarrow \epsilon_q \lambda_k (q^\dagger v_k) = h_k^\dagger (q^\dagger v_k) \\ \Rightarrow \epsilon_q \lambda^* (v_k^\dagger q) = (v_k^\dagger q) h_k \end{aligned}$$

it is thereby seen that the presence of Q-symmetry results in a relation between the left and right eigenvectors, implying that for every k-point, there exists an eigenvalue pair $(\lambda_k, \epsilon_q \lambda_k^*)$.

3.3 Extended Hamiltonian

This section is based on [4],[5].

Now is order to determine the topological nature of point-gapped non-Hermitian Hamiltonians it is useful to define the so-called extended Hamiltonian, which for a generic non-Hermitian Hamiltonian h is defined as

$$H_E := \begin{bmatrix} 0 & h(k) \\ h^\dagger(k) & 0 \end{bmatrix}.$$

This Hamiltonian is naturally Hermitian and in the case of a point-gapped system it can be shown that this construction preserves the symmetries of the original non-Hermitian systems, but due to the Hermiticity of the extended Hamiltonian it allows the usage of the well-developed topological classification methods that are used in Hermitian physics.

For the case of a point-gap, it is possible to define a unique polar decomposition as long as the Hamiltonian is invertible. So without loss of generality, define the point-gap to be positioned at the origin $E = 0$, hence the matrix is invertible, and the unique polar decomposition is given by

$$h(k) = U(k)P(k)$$

where $U(k) = H(k)P(k)$ is a unitary operator and $P(k) = \sqrt{h^\dagger(k)h(k)}$ is positive-definite and Hermitian operator, both operators are continuous in k.

By the uniqueness of the polar decomposition, it can be seen that the unitary operator $U(k)$ preserve the BL symmetries 27. For the case where $X = P, K$, the action on the Hamiltonian is the identity or complex conjugation, respectively, and it follows that

$$\begin{aligned} h(k) &= U(k)P(k) && (33) \\ &= \epsilon_X U_X \mathcal{O}(H(s_X k))U_X^\dagger && (34) \\ &= \epsilon_X U_X \mathcal{O}(U(s_X k))U_X^\dagger U_X \mathcal{O}(P(s_X k))U_X^\dagger && (35) \end{aligned}$$

and for the case where $X = C, Q$, the action on the Hamiltonian is transposition or Hermitian conjugation, respectively, and it follows that

$$\begin{aligned} h(k) &= U(k)P(k) && (36) \\ &= \epsilon_X U_X \mathcal{O}(U(s_X k))U_X \mathcal{O}(U(s_X k)P(s_x k)U(s_X k)^\dagger)U_X^\dagger && (37) \end{aligned}$$

where it is seen that in both cases 33 and 36, the unitary matrix $U(k)$ is given by

$$U(k) = \epsilon_X U_X \mathcal{O}(U(s_X k))U_X^\dagger,$$

it can thereby be seen that $U(k)$ has the same symmetries as the original Hamiltonian.

Now since $h(k)$ and $U(k)$ are continuous in the parameter k, it is possible to define a continuous symmetry-preserving path that preserves the point-gap. This path can be given as the interpolation $h_\lambda(k) = (1-\lambda)h(k) + \lambda U(k)$ where $\lambda \in [0,1]$. To see that the point-gap is indeed preserved, rewrite the interpolation as $h_\lambda(k) = \lambda U(k)((1-\lambda)P(k) + \lambda I)$, which is the product of unitary and positive-definite operators, this implies that the point-gapped at $E = 0$ is preserved for all k.

Now, it is possible to define the extended Hamiltonian for a point-gapped system as

$$H_U(k) = \begin{bmatrix} 0 & U(k) \\ U^\dagger(k) & 0 \end{bmatrix}.$$

It is therefore seen that a non-Hermitian Hamiltonian possessing a point gap, can be written as a extended Hermitian Hamiltonian, with entries that are unitary matrices. This has the implication that the point-gapped spectrum of the Hamiltonian can be mapped onto a unit circle surrounding the point-gap at the origin as depicted in figure 3.

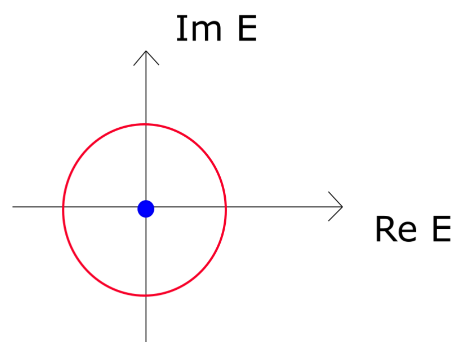

Figure 3: Spectrum of the unitary flattened Hamiltonian, where the blue dot represents the point gap and the red circle represent the eigenvalues of the flattened unitary Hamiltonian.

The extended Hamiltonian is Hermitian and has a chiral symmetry given by $\Sigma = \sigma_z \otimes I$. Additionally, it is also spectrally flattened, i.e. $H_U^2 = \sigma_0 \otimes I$. The extended Hamiltonian carries the symmetries of the original non-Hermitian Hamiltonian, however, in a slightly different form. The new symmetry transformations are given by

$$P = \sigma_0 \otimes p \tag{38}$$

$$Q = \sigma_x \otimes q, \tag{39}$$

$$C = \sigma_x \otimes c, \tag{40}$$

$$K = \sigma_0 \otimes k \tag{41}$$

and satisfy the commutation relations

$$\{P, H_U\} = 0 \tag{42}$$

$$H_U(k) = \epsilon_q Q H Q^\dagger, \tag{43}$$

$$H_U(k) = \epsilon_c C H_U^*(-k) C^\dagger, \tag{44}$$

$$H_U(k) = \epsilon_k K H_U^*(-k) K^\dagger \tag{45}$$

where $P^2 = Q^2 = \sigma_0 \otimes I$, $CC^* = \eta_C \sigma_0 \otimes I$ and $KK^* = \eta_K \sigma_0 \otimes I$. Additionally,

$$\{Q, \Sigma\} = \{C, \Sigma\} = 0 \tag{46}$$

$$[P, \Sigma] = [K, \Sigma] = 0. \tag{47}$$

The commutation relations between the corresponding symmetry operators are unaffected by this procedure and are still given by the relations 31 and 32.

3.4 Hermitianization of Line-Gapped Hamiltonians

This section is based on [5].

In this section it will be shown that a generic non-Hermitian Bloch Hamiltonian possessing a real line gap, can be continuously deformed to a Hermitian Hamiltonian while preserving the line gap and all of the BL symmetries. This will allow for the usage of the results regarding the unitary flattening of Hermitian Hamiltonians and thereby grant access to the K-group tools that have already been thoroughly studied. It is sufficient to only consider the case of a real line gap, since it is always possible to map an imaginary line gapped Hamiltonian to a real line gapped Hamiltonian by Wick rotation.

Firstly, in order to define a generic NH Bloch Hamiltonian that can be continuous deformed into a Hermitian Hamiltonian, it is convenient to define a projection operator, that projects onto the eigenbasis corresponding to the eigenvalues with either positive or negative real parts. Such a projector can be defined using the Riesz projector which is defined as

$$P(k) := \oint_{C_\lambda} \frac{dz}{2\pi i} \frac{1}{zI - H(k)}$$

where C_λ is some simple closed curve in the complex plane that separates an eigenvalue $\lambda \in \mathbb{C}$ of $H(k)$ from the rest of the spectrum of $H(k)$.

The NH Hamiltonian $H(k)$ can be written in terms of projectors, and in the case of a real line gap, it is convenient to choose the paths such that the projectors project onto the eigenbasis corresponding to eigenvalues that have positive/negative real parts. Such paths can be defined as $C_+ = \{z \ : |z| = r, \mathrm{Re}\, z > 0\} \cup \{z \ : \mathrm{Re}\, z = 0, |\mathrm{Im}\, z| \leq r\}$ and $C_- = -C_+$, where $r > \max_k ||H(k)||$. This definition ensures that the paths $C_\pm$ do not intersect the spectrum of $H(k)$, which is necessary since $H(k)$ has a real line gap. The NH Hamiltonian $H(k)$ can now be written as a linear combination of itself and itself in terms of its $C_\pm$ projectors, with a parameter $\lambda \in [0, 1]$ that can be varied, as to map between the two limits continuously as

$$H_\lambda(k) = (1 - \lambda)H(k) + \lambda \left(\oint_{C_+} - \oint_{C_-} \right) \frac{dz}{2\pi i} \frac{1}{zI - H(k)},$$

where the form of the second term is chosen such that it allows for the definition of the projectors of the Hamiltonian that project to the eigenspaces with positive/negative real parts, which correspond to projecting onto the eigenbasis on either side of the real line gap.

In the limit where $\lambda = 0$, the Hamiltonian is obviously just $H(k)$, but in the other limit where

$\lambda = 1$, the Hamiltonian takes the form

$$H_{\lambda=1}(k) = P_+(k) - P_-(k), \tag{48}$$

where $P_\pm(k)$ is a projector onto the eigenspaces corresponding to the eigenvalues with positive/negative real parts. The eigenvalues of $H_\lambda(k)$ are of the form

$$\epsilon_\lambda(k) = (1-\lambda)\epsilon(k) + \lambda \mathrm{sgn}\,\mathrm{Re}\,\epsilon_k,$$

where ϵ_k is an eigenvalue of $H(k)$. It can be seen that the sign of the real part of the eigenvalues remains the same under the continuous interpolation of H_λ and the real-line gap is thereby never crossed.

Now to determine whether the paths $C_\pm$ preserve the BL symmetries of $H(k)$, consider the Hamiltonian under one of the BL symmetry transformation 27,

$$H(k) = \epsilon_X U_X \mathcal{O}(H(s_X k)) U_X^\dagger.$$

The paths $C_\pm$ does not cross the spectrum of $H(k)$ if and only if it does not cross the spectrum of $X(H(k))$, this is due to $||H(k)|| = ||X(H(k))||$ and the lower bound for r is unchanged. Hence for any path $C_\pm$, the following relation holds

$$\frac{1}{zI - H(k)} = \epsilon_X U_X \frac{1}{\epsilon_X zI - \mathcal{O}(H(s_x k))} U_X^\dagger. \tag{49}$$

It can be shown that for any of the BL symmetry operations, the path integral transforms as

$$\mathcal{O}\left[\oint_{C_\pm} \frac{dz}{2\pi i} \frac{1}{zI - H(k)}\right] = \oint_{C_\pm} \frac{dz}{2\pi i} \frac{1}{zI - X(H(k))}. \tag{50}$$

This is easily seen for the cases where X represent the identity or the transpose, however, the result is less obvious in the case of complex conjugation or Hermitian conjugation. Now consider the path integral under complex conjugation

$$\begin{aligned} \mathcal{O}\left[\oint_{C_\pm} \frac{dz}{2\pi i} \frac{1}{zI - H(k)}\right]^* &= -\oint_{C_\pm^*} \frac{dz^*}{2\pi i} \frac{1}{z^* I - H^*(k)} \\ &= \oint_{C_\pm} \frac{dz}{2\pi i} \frac{1}{zI - H^*(k)}, \end{aligned}$$

where it was used that complex conjugation merely changes the direction of the path $C_\pm$, but not the path itself. Now by combining 49 and 50 yields

$$
\begin{aligned}
P_+(k) &= \oint_{C_+} \frac{dz}{2\pi i} \frac{1}{zI - H(k)} && (51)\\
&= \oint_{C_+} \frac{d(\epsilon_X z)}{2\pi i} U_X \frac{1}{\epsilon_X z I - X\left[H(s_X k)\right]} U_X^\dagger && (52)\\
&= U_X \mathcal{O}\left[\oint_{C_+} \frac{d(\epsilon_X z)}{2\pi i} \frac{1}{\epsilon_X z I - H(s_X k)}\right] U_X^\dagger && (53)\\
&= U_X \mathcal{O}\left[\oint_{C_{\epsilon_X}} \frac{dz}{2\pi i} \frac{1}{zI - H(s_X k)}\right] U_X^\dagger && (54)\\
&= U_X \mathcal{O}\left[P_{\epsilon_X}(s_X k)\right] U_X^\dagger. && (55)
\end{aligned}
$$

Now due to the fact that $P_+(k) + P_-(k) = I$, it follows by 48 that $P_\pm(k) = \frac{1}{2}(I \pm H_{\lambda=1}(k))$. Using this property of the projectors in the last equality of 51 it is seen that

$$
H_1(k) = \epsilon_X U_X \mathcal{O}\left[H_1(s_X k)\right] U_X^\dagger,
$$

which shows that the interpolation Hamiltonian $H_\lambda(k)$, being a real linear combination of $H(k)$ and $H_1(k)$, respects the BL symmetries.

Now what remains is to show that $H_{\lambda=1}(k)$ can be continuously Hermitianized in a way that preserves the BL symmetries and the line gap. For a given NH Bloch Hamiltonian $H_{\lambda=1}(k)$, there always exist a decomposition of the form $H_{\lambda=1}(k) = h_+(k) + ih_-(k)$ where

$$
\begin{aligned}
h_+(k) &= \frac{1}{2}\left[H_{\lambda=1}(k) + H_{\lambda=1}^\dagger(k)\right], && (56)\\
h_-(k) &= \frac{1}{2i}\left[H_{\lambda=1}(k) - H_{\lambda=1}^\dagger(k)\right]. && (57)
\end{aligned}
$$

Both of these terms are continuous in k and $h_+(k)$ satisfy all of the symmetries of $H_{\lambda=1}(k)$, since it is the sum of $H_{\lambda=1}(k)$ and $H_{\lambda=1}^\dagger(k)$. Due to $H_{\lambda=1}(k)$ being involutory, it follows that

$$
\begin{aligned}
\left[h_+(k) + ih_-(k)\right]^2 &= h_+^2(k) - h_-^2(k) + i\left\{h_+(k), h_-(k)\right\}\\
&= I,
\end{aligned}
$$

from which it can be deduced that

$$
\begin{aligned}
h_+^2(k) &= I + h_-^2(k), && (58)\\
\{h_+(k), h_-(k)\} &= 0. && (59)
\end{aligned}
$$

Now consider the interpolation Hamiltonian, that under the interpolation preserves the symmetries,

$$H_\lambda(k) = (1-\lambda)H_{\lambda=1}(k) + \lambda h_+(k), \quad \lambda \in [0,1]\,.$$

Then by 58, it follows that

$$H_\lambda^2(k) = I + (1-(1-\lambda)^2)h_-^2(k) \geq I,$$

this implies that all the eigenvalues of $H_\lambda^2(k)$ are real and greater than 1, from which it follows that the eigenvalues of $H_\lambda(k)$ are real and has absolute values greater or equal to 1. It has thereby been shown that for any NH Hamiltonian with a real line gap, it is possible to continuously deform it to a topologically equivalent Hermitian Hamiltonian. If the NH Hamiltonian H possessed an imaginary line gap instead, then it can be mapped to a Hamiltonian with a real line gap by the map $H \to iH$, and the NH Hamiltonian can be continuously deformed into an anti-Hermitian.

4 Topological Classification of Symmetry Classes

This section is based on [9],[10],[5],[3].

In this section the topological classification of the BL symmetry classes will be expressed in terms of Clifford generators, which then, through the Atiyah-Bott-Shapiro isomorphism, allows for the classification to be done in terms of topological K-groups. These K-groups can be shown to be equal to different sets of integers which provides a classification of the topological phases and these K-groups effective describe the inequivalent ways a mass term can be included in the set of Clifford algebra generators.

4.1 Classification in Terms of K-Groups

This section is based on [9],[10].

The classification for the periodic lattice is more involved than that of the continuum model, since the topology that can be classified by the K-theory methods of this chapter on T^d will generally be different than the classification on S^d and will typically in the form direct sums of the K-groups found in S^d, in this case the K-groups on T^d determine the topological classes of weak topological insulators. The lattice models will therefore be omitted and the focus will be on continuum models for which the K-groups on S^d determine the topological classes of strong topological insulators. The topological classification is done by considering the set of generators of a Clifford algebra is constituted by the symmetry operations on a family of Hamiltonians that possess the same BL symmetries.

This set of Clifford generators consists of the BL symmetry operations of the Hamiltonians in zero dimensions. In higher dimensions the set of Clifford generators must include representations of the momentum terms since they adds restrictions on the symmetries of the system. The topological classification is then determined by considering the inequivalent ways of including a mass term, that anti-commutes with the momentum terms, to the set of Clifford algebra generators. This effectively correspond to determining the inequivalent ways of gapping the system.

It is at this point instructive to formally give a definition of Clifford algebras and show some of its properties.

A Clifford algebra is an algebra generated by a vector space V with the quadratic form

$$xy + yx = -2q(x, y) \tag{60}$$

for $x, y \in V$. In the case where V is a real vector space $V = \mathbb{R}^{p+q}$, together with the quadratic form $Q(x) = q(x, x)$, given by

$$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \tag{61}$$

then the Clifford algebra is a real Clifford algebra and it will be denoted as $Cl_{p,q} := Cl(\mathbb{R}^{p+q}, Q)$. The real Clifford algebra is generated by p negative generators which squared to -1 and q positive generators that square to $+1$. If these generators form an orthonormal basis of $\mathbb{R}^{p+q}$, then the generators $e_1, \ldots, e_{p+q}$ generates $Cl_{p,q}$, and they satisfy the relation

$$\{e_i, e_j\} = \begin{cases} -2\delta_{ij} & \text{, for } i \leq p \\ +2\delta_{ij} & \text{, for } i > p \end{cases} \tag{62}$$

If the vector space is chosen to be complex, that is if $V = \mathbb{C}^{p+q}$, then the complex Clifford algebra can be defined as the complexification of the real Clifford algebra with complex quadratic form as $\mathbb{C}l_{p,q} \equiv Cl_{p,q} \otimes_{\mathbb{R}} \mathbb{C} \cong Cl(\mathbb{C}^{p+q}, Q \otimes \mathbb{C})$, where $\otimes_{\mathbb{F}}$ is the tensor product over the field $\mathbb{F}$, which in this section will be either be $\mathbb{R}$ or $\mathbb{C}$. In the complex Clifford algebra, there is no distinction between positive and negative generators, since the complexification effectively correspond to the inclusion of an imaginary number i, which can be multiplied onto the generators and change the sign of the square e.g. $e_j^2 = \pm 1 = \mp (ie_j)^2$, and it can therefore simply be written as $\mathbb{C}l_n$ where $n = p + q$.

The negative generators of the Clifford algebra can be represented by skew-symmetric and the positive generators can be represented by real symmetric matrices.

Now in order to classify free-fermion Hamiltonians, the main idea is to look for ways of including an additional Clifford algebra generator, which represents a mass term. For this, one considers a representation of the form $Cl_{p+1,0}$, where the action of $e_1, \ldots, e_p$ is fixed, such a problem is called a "Clifford algebra extension problem with p negative generators".

It is convenient to frame the problem in terms of positive generators, as the object we seek is an index of a classifying space for the given family of Hamiltonians. It is a simple task to reformulate the Clifford algebra problem in terms of positive generators, as there exist an isomorphism $Cl_{0,p+2} \cong Cl_{p,0} \otimes \mathbb{R}(2)$, where $\mathbb{R}(2)$ is a 2×2 real matrix. This isomorphism maps the generators as

$$e_j \to e'_j \otimes (i\sigma_y) \quad \text{for } j = 1, \ldots, p, \tag{63}$$

$$e_{p+1} \to I \otimes \sigma_z, \tag{64}$$

$$e_{p+2} \to I \otimes \sigma_x. \tag{65}$$

It turns out that it is possible to distinguish between the algebras that have different representations theories by means of the so-called Morita equivalence theorem. It follows from the Morita theorem that the representation theory of algebras A and $A \otimes B$, are Morita equivalent if the algebra B is simple, and it is a fact that matrix algebras are simple, which means they have no nontrivial ideals. The set of algebras that are Morita equivalent constitute a Morita equivalence class. It therefore follows that the Clifford algebras $Cl_{0,p+2}$ and $Cl_{p,0} \otimes \mathbb{R}(2)$ are Morita equivalent. Additionally, a property of Clifford algebras is the Bott periodicity identities given by

$$Cl_{r+1,s+1} \cong Cl_{r,s} \otimes_{\mathbb{R}} Cl_{1,1}, \tag{66}$$

$$Cl_{r+8,0} \cong Cl_{r,0} \otimes_{\mathbb{R}} Cl_{8,0}, \tag{67}$$

$$Cl_{0,s+8} \cong Cl_{0,s} \otimes_{\mathbb{R}} Cl_{0,8}, \tag{68}$$

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2, \tag{69}$$

and due to the fact that the algebras $Cl_{1,1}, Cl_{8,0}, Cl_{0,8}, \mathbb{C}l_2$ are all matrix algebras and therefore simple, it turns out that the Morita classes of the real Clifford algebras $Cl_{r,s}$ depends only on the difference $r - s \bmod 8$, and the Morita classes of the complex Clifford algebras $\mathbb{C}l_n$ depends only on $n \bmod 2$. This means that the Clifford algebra generated for given family of Hamiltonians depends only on the difference between the number of positive and negative generators of the algebra. This property is what leads to the 8-periodic structure of the real symmetry classes in the Altland-Zirnbauer classification.

The Hamiltonian of a translational invariant systems can be written in momentum space as

$$H = \frac{i}{4} \sum_{\boldsymbol{p}} \sum_{j,k} A_{jk}(\boldsymbol{p}) \hat{c}_{-\boldsymbol{p},j} \hat{c}_{\boldsymbol{p},k}, \tag{70}$$

where the indices j, k signify the particle flavors. The matrix $A(\boldsymbol{p})$ with entries $A_{jk}(\boldsymbol{p})$ is skew-Hermitian and the entries satisfy $A^*_{jk}(\boldsymbol{p}) = A_{jk}(-\boldsymbol{p})$. The symmetry operators of $A(\boldsymbol{p})$ are given by Clifford generators which can be represented as real matrices that are independent of $\boldsymbol{p}$. The Clifford algebra extension problem can now be formulated by transforming the matrix $A(\boldsymbol{p})$ to be the Clifford generator e_{q+1} and by transforming the remaining q negative generators to positive generators. The problem now consists of a set of real symmetric matrices $e_1, \ldots, e_q$ and a Hermitian matrix e_{q+1}, which satisfy the condition $e^*_{q+1}(\boldsymbol{p}) = e_{q+1}(-\boldsymbol{p})$. It will be shown in the next section that this is sufficient to determine the topological classification of the BL symmetry classes.

4.2 K-Theory

This section is based on [9].

The topological classification of a Clifford algebra extension problem can be determined by using topological K-theory, which essentially is used to determine the "difference" between vector bundles. The generators of the Clifford algebra can be considered to be Clifford modules acting Dirac bundles, which are vector bundle where the fibers are the Dirac operators.

k	0	1	2	3	4	5	6	7
Cl_k	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$
N_k	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$

Table 1: The table shows the real Clifford algebras Cl_k and the corresponding Grothendieck group, defined by the index $k = q - p$, where q and p are the numbers of positive and negative generators, respectively.

The Clifford algebras are $\mathbb{Z}_2$-graded, which means that they admit a decomposition $Cl_{p,q} = Cl^0_{p,q} \oplus Cl^1_{p,q}$, where $Cl^0_{p,q}$ is the even part and $Cl^1_{p,q}$ is the odd part. It can then be shown that there exist isomorphisms between the Clifford algebras and their corresponding even part, given by

$$Cl_{p,q} \cong Cl^0_{p+1,q}, \tag{71}$$

$$Cl_n \cong Cl^0_{n+1}. \tag{72}$$

Now define the Grothendieck group N_k, which is the group of equivalence classes of irreducible real representations of Cl_k, where $k = q - p$ is the index of the Morita class. The table 1 shows the Grothendieck groups corresponding to different Clifford algebras Cl_k. This group is a free abelian group that is generated by the distinct irreducible representations over $\mathbb{R}$. Now let M_k be the Grothendieck group of the graded representations of Cl_k and let N_k be the Grothendieck group of the ungraded representations of Cl_k. It has been shown by Atiyah, Bott and Shapiro that there exist isomorphisms[9]

$$\frac{M_{r,s}}{i^* M_{r,s+1}} \cong KO^{r-s}(\star), \qquad \frac{M^{\mathbb{C}}_n}{i^* M^{\mathbb{C}}_{n+1}} \cong K^n(\star) \tag{73}$$

for the real and complex Clifford algebras, respectively, where $M_{r,s}$ denotes the graded Clifford algebra with r negative and s positive generators and i^* is a functor that forgets about the action of the $k+1$th element of the Clifford algebra.

It can then be seen from 71 that $M_k = N_{k-1}$. This has the implication that

$$M_k / i^* M_{k+1} \cong N_{k-1} / i^* N_k \tag{74}$$

then from the table 1, it can be seen that the quotient yields $\mathbb{Z}$ when M_{k-1} correspond to $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}_2$ when the dimension of the representation Cl_k is twice the dimension of the representation of Cl_{k-1}, and it follows that

$$M_k/i^*M_{k+1} \cong \begin{cases} \mathbb{Z}, & \text{for} k = 0, 4 \pmod 8 \\ \mathbb{Z}_2, & \text{for} k = 1, 2 \pmod 8 \\ 0, & \text{otherwise} \end{cases} . \tag{75}$$

There exist a one-to-one correspondence between the K-groups at a point $\star$ and the connected components of the Cartan symmetric spaces. This isomorphism is given by

$$KO^{-k}(\star) \cong \pi_0(R_k), \quad K^{-k}(\star) \cong \pi_0(C_k) \tag{76}$$

which therefore implies that

$$M_k/i^*M_{k+1} \cong \pi_0(R_k), \quad M_k^{\mathbb{C}}/i^*M_{k+1}^{\mathbb{C}} \cong \pi_0(R_k). \tag{77}$$

This means that there exist a relation between the connected components of the symmetric spaces, the K-groups and the Grothendieck groups. This result means that the topological classification of a system only requires knowledge of the Clifford algebra generators.

In summation, through the isomorphisms proposed above, it is possible to obtain a topological classification of a family of Hamiltonians, for which the symmetries are described by a Clifford algebra Cl_k. It follows these isomorphisms that the classification of the Clifford algebra extension problems for the different symmetry classes can be characterized in terms of K-groups, which can be denoted as $K_{\mathbb{F}}(s;d)$, where $s = q - p$ is the index of the Clifford algebra $Cl_{p,q}$ and d is the dimension of the system. For $d = 0$, the K-group classifications correspond to disjoint sections of the classifying space, which is described by the zeroth homotopy group as

$$K_{\mathbb{R}}(s;0) \cong \pi_0(R_s), \quad K_{\mathbb{C}}(s;0) \cong \pi_0(C_s).$$

for the real and complex K-group, respectively. The index of the classifying space corresponding to a given Clifford algebra extension problem is determined by the number of positive generators and the classifying space for the Clifford algebra extension problem $Cl_{0,s} \rightarrow Cl_{0,s+1}$ is R_s. The tables 2 and 3 summarize the K-groups for different values of the index k for both the real and the complex K-groups and how they relate to the 10 AZ classes.

AZ Class	$K_{\mathbb{R}}(s-n,0)$	$n=0$	1	2	3	4	5	6	7
AI	$s=0$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0 $\mathbb{Z}_2$	$\mathbb{Z}_2$	
BDI	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	4	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	6	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 2: The table shows the real AZ classes and their corresponding K-groups for different values of the index $s-n$, where n and s are the numbers of positive and negative generators, respectively.

AZ Class	$K_{\mathbb{C}}(s-n,0)$	$n=0$	1
A	$s=0$	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$

Table 3: The table shows the complex AZ classes and their corresponding K-groups for different values of the index $s-n$, where n and s are the numbers of positive and negative generators, respectively.

4.3 Classification for Line Gaps

This section is based on [5],[3],[9].

In this section the K-group classifications of systems with line gaps will be derived. It is sufficient to only consider systems with real line gaps, since systems with an imaginary line gap can be mapped to a system with a real line gap by Wick rotation $H \to iH$. The Wick rotation preserves the BL symmetries 27, except for the fact that the sign of ϵ_q and ϵ_k is flipped. It was shown in section 3.4 that any NH Hamiltonian possessing a real line gap can be continuously deformed into a Hermitian Hamiltonian while preserving the BL symmetries and the line gap. This has the implication that for a system with a real line gap, the P and Q symmetries are unified when $\epsilon_q = +1$ and the C-symmetry with $\epsilon_c = \pm 1$ is unified with the K-symmetry $\epsilon_k = \pm 1$. The K-group classifications can now be derived.

No-Symmetry: Starting with the case of a system with possessing no BL symmetries, the classification coincides with the AZ class A, which is $K_{\mathbb{C}}(0; d)$.

P-Symmetry: For the case of P-symmetry, the classification is the same as AIII, which is $K_{\mathbb{C}}(1; d)$.

Q-Symmetry: The P and Q-symmetry is unified in the presence of line gaps, therefore the classification for systems with $\epsilon_q = -1$ is the same as that for P-symmetry. In the case of $\epsilon_q = 1$, the operator q commutes with H and the classification is therefore doubled and given by $K_{\mathbb{C}}(0; d) \oplus K_{\mathbb{C}}(0; d)$.

C or K-Symmetry: The classifications of the C, K-symmetries are the same and it is therefore enough to just consider the case of C-symmetry. The property that any NH Hamiltonian can be continuously deformed into a Hermitian/anti-Hermitian Hamiltonian, while preserving the line gap, the action of C-symmetry with $\epsilon_c = 1$ correspond to time-reversal symmetry and $\epsilon_c = -1$ corresponds to particle-hole symmetry in the AZ classification. Therefore the corresponding AZ classes for $\epsilon_c = 1$ is AI for $\eta_c = 1$ and AII for $\eta = -1$, which is $K_{\mathbb{R}}(0; d)$ and $K_{\mathbb{R}}(4; d)$. The corresponding AZ classes for $\epsilon_c = -1$ is D for $\eta_c = 1$ and C for $\eta = -1$, which is $K_{\mathbb{R}}(2; d)$ and $(K_{\mathbb{R}}(6; d))$.

Now for systems with two symmetries. Starting off with systems that have a P-symmetry and an additional symmetry, this can be done by considering the classification of the system with only P-symmetry and then determining the effect of including an additional symmetry.

P,Q-Symmetry: First consider a system with P- and Q-symmetry, the classification is only dependent on ϵ_{PQ}. When $\epsilon_{PQ} = -1$ the operator Q anticommutes with both P and $H(k)$, which implies that the K-group classification becomes $K_{\mathbb{C}}(2;d)$. When $\epsilon_{PQ} = +1$, the operator PQ anticommutes with P and $H(k)$ and the classification is therefore doubled $K_{\mathbb{C}}(1;d) \oplus K_{\mathbb{C}}(1;d)$.

P,C-Symmetry: Next, consider the system with P- and C-symmetry. Now, fix $\epsilon_C = -1$, then the operator $C\mathcal{K} := \mathcal{C}$ is a particle-hole symmetry and P is a chiral symmetry. This implies that, when $\epsilon_{PC} = 1$, that there exist a chiral symmetry $\Gamma = \mathcal{T}\mathcal{C}$ for which $\mathcal{T}^2 = \mathcal{C}^2 = \eta_C$, which coincide with the AZ class BDI if $\eta_C = +1$ and AZ class CII if $\eta_C = -1$, which has the classification $K_{\mathbb{R}}(1;d)$ and $K_{\mathbb{R}}(5;d)$, respectively. Similarly, if $\epsilon_C = -1$, then the chiral symmetry is $\Gamma = i\mathcal{T}\mathcal{C}$ for which $\mathcal{T}^2 = -\mathcal{C}^2 = -\eta_C$, which coincide with the AZ class DIII if $\eta_C = +1$ and CI if $\eta_C = -1$, which has the classification $K_{\mathbb{R}}(3;d)$ and $K_{\mathbb{R}}(7;d)$, respectively.

P,K-Symmetry: The classifications of systems with simultaneous P- and K-symmetry with $\epsilon_K = +1$ is fixed are almost the same as the classifications for systems with simultaneous P- and C-symmetry with $\epsilon_C = -1$ fixed. The only difference is that the case when $\epsilon_{PK} = -1$, here the classifications coincide with AZ class CI for $\eta_K = +1$ and AZ class DIII $\eta_K = -1$ and are therefore given by $K_{\mathbb{R}}(7;d)$ and $K_{\mathbb{R}}(3;d)$, respectively.

For systems with multiple BL symmetries, neither of which are P-symmetries, the classification can simply be done by considering systems with Q- and C-symmetries, since the existence of more than one of the symmetries Q, C, K implies the existence of the others. Therefore, the classification can be determined by considering a system with a C-symmetry and observing the effect of including a Q-symmetry.

Q,C,K-Symmetry: When $\epsilon_Q = +1$ and $\epsilon_{QC} = +1$, the operator Q commutes with $C\mathcal{K}$, $H(k)$ and i, which implies the doubling of the K-groups corresponding to the systems with just the C-symmetry. The classifications are therefore $K_{\mathbb{R}}(0;d) \oplus K_{\mathbb{R}}(0;d)$ for $\epsilon_c = +1$, $\eta_C = +1$, $K_{\mathbb{R}}(4;d) \oplus K_{\mathbb{R}}(4;d)$ for $\epsilon_c = +1$, $\eta_C = -1$, $K_{\mathbb{R}}(2;d) \oplus K_{\mathbb{R}}(2;d)$ for $\epsilon_c = -1$, $\eta_C = +1$, and $K_{\mathbb{R}}(6;d) \oplus K_{\mathbb{R}}(6;d)$ for $\epsilon_c = -1$, $\eta_C = -1$. When $\epsilon_Q = +1$ and $\epsilon_{QC} = -1$, then the operator iQ commutes with $C\mathcal{K}$, $H(k)$ and i, which implies the complexification of the K-groups corresponding to the systems with just the C-symmetry. The classifications of all the combinations of $\epsilon_C = \pm 1$ and $\eta_C = \pm 1$ are the same $K_{\mathbb{C}}(0;d)$. When $\epsilon_Q = -1$ the classifications become that of the AZ classes with chiral symmetry.

P,Q,C,K-Symmetry: It just remains to show the classifications of systems with all the BL symmetries, this can be done by considering the classification of the systems without P-symmetry and then determining the effect of including it. When $\epsilon_{PQ} = 1$, there exist an operator $\sqrt{\epsilon_{PC}\epsilon_{QC}}PQ$, which commutes with $C\mathcal{K}, Q, H(k)$ and i. This implies that that the K-group is doubled if $\epsilon_{PC}\epsilon_{QC} = +1$ and complexified if $\epsilon_{PC}\epsilon_{QC} = -1$. When $\epsilon_{PQ} = -1$ and $\epsilon_{PC} = +1$, then the operator P commutes with $\mathcal{C} := C\mathcal{K}$ and anticommutes with $\mathcal{T} = \eta_C\sqrt{\epsilon_{QC}}QC\mathcal{K}$. Similarly, when $\epsilon_{PQ} = -1$ and $\epsilon_{PC} = -1$, then the operator P anticommutes with $\mathcal{C} := C\mathcal{K}$ and commutes with $\mathcal{T} = \eta_C\sqrt{\epsilon_{QC}}QC\mathcal{K}$. This implies that the inclusion of a P-symmetry with $\epsilon_{PC} = +1$ in the form iP, corresponds to adding a negative generator and thereby shifting the K-group classification from $K_{\mathbb{R}}(s; d)$ to $K_{\mathbb{R}}(s-1; d)$. Similarly, the inclusion of a P-symmetry with $\epsilon_{PC} = -1$ in the form P, corresponds to adding a positive generator and thereby shifting the K-group classification from $K_{\mathbb{R}}(s; d)$ to $K_{\mathbb{R}}(s+1; d)$.

Counting up the total number of distinct symmetry classifications for line gaps, there is one with no symmetry, one with P-symmetry, two with Q-symmetry, four with C, four with K-symmetry, two with PQ, four with PC, four with PK, 16 with QC and 16 with PQC-symmetry. Hence the total number of distinct classes for line-gapped systems is 54 and the results are summarized in the tables 4 and 5.

#	Symmetry	Generator Relations	K-Group	NH Class
1			$K_{\mathbb{C}}(0;d)$	NH A & A†
2	P		$K_{\mathbb{C}}(1;d)$	NH AIII & AIII†
3	Q	$\epsilon_Q = 1$	$K_{\mathbb{C}}(0;d) \oplus K_{\mathbb{C}}(0;d)$	NH pA & pA†
4	Q	$\epsilon_Q = -1$	$K_{\mathbb{C}}(1;d)$	
5	C	$\epsilon_C = 1, \eta_C = 1$	$K_{\mathbb{R}}(0;d)$	AI†
6	C	$\epsilon_C = 1, \eta_C = -1$	$K_{\mathbb{R}}(4;d)$	AII†
7	C	$\epsilon_C = -1, \eta_C = 1$	$K_{\mathbb{R}}(2;d)$	NH D
8	C	$\epsilon_C = -1, \eta_C = -1$	$K_{\mathbb{R}}(6;d)$	NH C
9	K	$\epsilon_K = 1, \eta_K = 1$	$K_{\mathbb{R}}(0;d)$	NH AI
10	K	$\epsilon_K = 1, \eta_K = -1$	$K_{\mathbb{R}}(4;d)$	NH AII
11	K	$\epsilon_K = -1, \eta_K = 1$	$K_{\mathbb{R}}(2;d)$	D†
12	K	$\epsilon_K = -1, \eta_K = -1$	$K_{\mathbb{R}}(6;d)$	C†
13	PQ	$\epsilon_{PQ} = 1$	$K_{\mathbb{C}}(1;d) \oplus K_{\mathbb{C}}(1;d)$	NH pAIII & pAIII†
14	PQ	$\epsilon_{PQ} = -1$	$K_{\mathbb{C}}(0;d)$	
15	PC	$\eta_C = 1, \epsilon_{PC} = 1$	$K_{\mathbb{R}}(1;d)$	sAI†
16	PC	$\eta_C = -1, \epsilon_{PC} = 1$	$K_{\mathbb{R}}(5;d)$	sAII†
17	PC	$\eta_C = 1, \epsilon_{PC} = -1$	$K_{\mathbb{R}}(3;d)$	NH sD
18	PC	$\eta_C = -1, \epsilon_{PC} = -1$	$K_{\mathbb{R}}(7;d)$	NH sC
19	PK	$\eta_K = 1, \epsilon_{PK} = 1$	$K_{\mathbb{R}}(1;d)$	NH sAI
20	PK	$\eta_K = -1, \epsilon_{PK} = 1$	$K_{\mathbb{R}}(5;d)$	NH sAII
21	PK	$\eta_K = 1, \epsilon_{PK} = -1$	$K_{\mathbb{R}}(7;d)$	sD†
22	PK	$\eta_K = -1, \epsilon_{PK} = -1$	$K_{\mathbb{R}}(3;d)$	sC†
23	QC	$\epsilon_Q = 1, \epsilon_C = 1, \eta_C = 1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(0;d) \oplus K_{\mathbb{R}}(0;d)$	pAI†
24	QC	$\epsilon_Q = 1, \epsilon_C = 1, \eta_C = -1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(4;d) \oplus K_{\mathbb{R}}(4;d)$	pDAII†
25	QC	$\epsilon_Q = 1, \epsilon_C = -1, \eta_C = 1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(2;d) \oplus K_{\mathbb{R}}(2;d)$	NH pD
26	QC	$\epsilon_Q = 1, \epsilon_C = -1, \eta_C = -1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(6;d) \oplus K_{\mathbb{R}}(6;d)$	NH pC
27	QC	$\epsilon_Q = 1, \epsilon_C = 1, \eta_C = 1, \epsilon_{QC} = -1$	$K_{\mathbb{C}}(0;d)$	pC†
28	QC	$\epsilon_Q = 1, \epsilon_C = 1, \eta_C = -1, \epsilon_{QC} = -1$	$K_{\mathbb{C}}(0;d)$	pD†
29	QC	$\epsilon_Q = 1, \epsilon_C = -1, \eta_C = 1, \epsilon_{QC} = -1$	$K_{\mathbb{C}}(0;d)$	NH pAII
30	QC	$\epsilon_Q = 1, \epsilon_C = -1, \eta_C = -1, \epsilon_{QC} = -1$	$K_{\mathbb{C}}(0;d)$	NH pAI
31	QC	$\epsilon_Q = -1, \epsilon_C = 1, \eta_C = 1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(1;d)$	BDI†
32	QC	$\epsilon_Q = -1, \epsilon_C = 1, \eta_C = -1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(5;d)$	CII†
33	QC	$\epsilon_Q = -1, \epsilon_C = -1, \eta_C = 1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(1;d)$	NH BDI
34	QC	$\epsilon_Q = -1, \epsilon_C = -1, \eta_C = -1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(5;d)$	NH CII
35	QC	$\epsilon_Q = -1, \epsilon_C = 1, \eta_C = 1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(7;d)$	CI†
36	QC	$\epsilon_Q = -1, \epsilon_C = 1, \eta_C = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(3;d)$	DIII†
37	QC	$\epsilon_Q = -1, \epsilon_C = -1, \eta_C = 1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(3;d)$	NH DIII
38	QC	$\epsilon_Q = -1, \epsilon_C = -1, \eta_C = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(7;d)$	NH CI

Table 4: Symmetry classifications of the BL classes defined by 27 in terms of K-groups for systems with line gaps.

#	Symmetry	Generator Relations	K-Group	NH Class
39	PQC	$\eta_C = 1, \epsilon_{PQ} = 1, \epsilon_{PC} = 1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(1;d) \oplus K_{\mathbb{R}}(1;d)$	NH pBDI & pBDI†
40	PQC	$\eta_C = -1, \epsilon_{PQ} = 1, \epsilon_{PC} = 1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(5;d) \oplus K_{\mathbb{R}}(5;d)$	NH pCII & pCII†
41	PQC	$\eta_C = 1, \epsilon_{PQ} = 1, \epsilon_{PC} = 1, \epsilon_{QC} = -1$	$K_{\mathbb{C}}(1;d)$	NH pDIII & pCI†
42	PQC	$\eta_C = -1, \epsilon_{PQ} = 1, \epsilon_{PC} = 1, \epsilon_{QC} = -1$	$K_{\mathbb{C}}(1;d)$	NH pCI & pDIII†
43	PQC	$\eta_C = 1, \epsilon_{PQ} = 1, \epsilon_{PC} = -1, \epsilon_{QC} = 1$	$K_{\mathbb{C}}(1;d)$	NH pBDI & pBDI†
44	PQC	$\eta_C = -1, \epsilon_{PQ} = 1, \epsilon_{PC} = -1, \epsilon_{QC} = 1$	$K_{\mathbb{C}}(1;d)$	NH pCII & pCII†
45	PQC	$\eta_C = 1, \epsilon_{PQ} = 1, \epsilon_{PC} = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(3;d) \oplus K_{\mathbb{R}}(3;d)$	NH p$^+$DIII & p$^+$DIII†
46	PQC	$\eta_C = -1, \epsilon_{PQ} = 1, \epsilon_{PC} = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(7;d) \oplus K_{\mathbb{R}}(7;d)$	NH p$^+$CI & p$^+$CI†
47	PQC	$\eta_C = 1, \epsilon_{PQ} = -1, \epsilon_{PC} = 1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(0;d)$	sBDI†
48	PQC	$\eta_C = -1, \epsilon_{PQ} = -1, \epsilon_{PC} = 1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(4;d)$	sCII†
49	PQC	$\eta_C = 1, \epsilon_{PQ} = -1, \epsilon_{PC} = 1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(2;d)$	sCI†
50	PQC	$\eta_C = -1, \epsilon_{PQ} = -1, \epsilon_{PC} = 1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(6;d)$	sDIII†
51	PQC	$\eta_C = 1, \epsilon_{PQ} = -1, \epsilon_{PC} = -1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(2;d)$	NH sBDI
52	PQC	$\eta_C = -1, \epsilon_{PQ} = -1, \epsilon_{PC} = -1, \epsilon_{QC} = 1$	$K_{\mathbb{R}}(6;d)$	NH sCII
53	PQC	$\eta_C = 1, \epsilon_{PQ} = -1, \epsilon_{PC} = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(4;d)$	NH sDIII
54	PQC	$\eta_C = -1, \epsilon_{PQ} = -1, \epsilon_{PC} = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(0;d)$	NH sCI

Table 5: Symmetry classifications of the BL classes defined by 27 in terms of K-groups for systems with line gaps.

4.4 Classification for Point Gaps

This section is based on [5],[3],[9].

In this section the topological classification in terms of K-groups will be determined for point-gapped systems. The systems will be described in terms of the extended Hamiltonian formalism described in section 3.3, in which all the Hamiltonians are Hermitian and possess the chiral symmetry $\Sigma = \sigma_0 \otimes I$.

No-Symmetry: For a system with no symmetry, the chiral symmetry Σ arising from extending the Hamiltonian results in the classification being that of AZ class AIII, which is $K_{\mathbb{C}}(1; d)$

Now for the cases of systems with just one BL symmetry.

P-Symmetry: If the Hamiltonian has P symmetry, then due to $[P, \Sigma] = 0$, it follows that $P\Sigma$ commutes with both Σ and $H_U(k)$, and the classification is therefore correspond to a doubled AZ class AIII $K_{\mathbb{C}}(1; d) \oplus K_{\mathbb{C}}(1; d)$ When the Hamiltonian has Q-symmetry, it is possible to fix the value of $\epsilon_q = -1$, since the sign can be flipped under Wick rotation, which in the case of point gaps preserve the topological classification. The symmetry operator Q commutes with Σ and H_U, and it therefore correspond to the AZ class A which has the classification $K_{\mathbb{C}}(2; d) = K_{\mathbb{C}}(0; d)$.

C-Symmetry: When the Hamiltonian has C-symmetry, $C\mathcal{K}$ commutes with Σ and by 42 it follows that $CH_UC^\dagger = \epsilon_c H_U(-k)$ and $CC^* = \eta_c \sigma_0 \otimes I$. If $\epsilon_c \eta_c = -1$, then the classification corresponds to that of AZ class DIII, which is $K_{\mathbb{R}}(3; d)$. If $\epsilon_c \eta_c = 1$, then the classification corresponds to that of AZ class, which is $K_{\mathbb{R}}(7; d)$.

K-Symmetry: When the Hamiltonian has K-symmetry then due to the classification being unchanged under Wick rotation, $\epsilon_k = 1$ can be fixed without loss of generality. It follows from 46 that $K\mathcal{K}$ commutes with Σ. This implies that when $\eta_k = +1$, the classification is that of AZ class BDI, which is $K_{\mathbb{R}}(1; d)$. When $\eta_k = -1$, the classification is that of AZ class CII, which is $K_{\mathbb{R}}(5; d)$.

Now consider the symmetry classes having P-symmetry together with an additional symmetry.

P, Q-Symmetry: Consider a system that has simultaneous P- and Q-symmetry, then if $\epsilon_{PQ} = 1$ a complex Clifford algebra extension problem can be constructed as $Cl_3(\mathbb{C}) = \{\Sigma, Q, i\Sigma QP\} \rightarrow Cl_4(\mathbb{C}) = \{\Sigma, Q, i\Sigma QP, H_U\}$ for the extended Hamiltonian, the classifying space of this extension problem is C_3, hence the K-group classification is $K_{\mathbb{C}}(3; d) = K_{\mathbb{C}}(1; d)$. When $\epsilon_{PQ} = -1$, there exist an element ΣP that commutes with Σ, Q, H_U, hence the classification is the doubled classification of AZ class A, which is $K_{\mathbb{C}}(2; d) \oplus K_{\mathbb{C}}(2; d) = K_{\mathbb{C}}(0; d) \oplus K_{\mathbb{C}}(0; d)$.

P,C-Symmetry: Now consider a system that has simultaneous P- and C-symmetry, then there is a total of four different classifications, corresponding to the different combinations of $\eta_C = \pm 1$ and $\epsilon_{PC} = \pm 1$. When $\eta_C = +1$ and $\epsilon_{PC} = +1$, then $C\mathcal{K}$ commutes with Σ, this implies the existence of an operator $iP\Gamma$, which is anti-involutory and commutes with all the elements of the Clifford algebra. The K-group is therefore complexified, and since the classification with Σ is that of AZ class BDI, the classification becomes $K_{\mathbb{C}}(5;d) = K_{\mathbb{C}}(1;d)$. When $\epsilon_{PC} = -1$ then there exist an element $P\Sigma$, which for the case of $\eta_C = +1$ correspond to the doubling of the AZ class DIII, which yields $K_{\mathbb{C}}(3;d) \otimes K_{\mathbb{C}}(3;d)$. Similarly, when $\eta_C = -1$, the classification corresponds to the doubling of the AZ class CI, which yields $K_{\mathbb{C}}(7;d) \otimes K_{\mathbb{C}}(7;d)$.

P,K-Symmetry: Now consider a system that has simultaneous P- and K-symmetry, when $\epsilon_{PK} = +1$ there are two classifications, corresponding to the two cases $\epsilon_{PK} = \pm 1$. When $\epsilon_{PK} = +1$, there exist an operator $P\Sigma$ that commutes with all other elements of the Clifford algebra, this implies a doubling of the K-group classification for the system with K-symmetry, which in the case of $\eta_K = +1$ becomes $K_{\mathbb{R}}(1;d) \otimes K_{\mathbb{R}}(1;d)$ and in the case of $\eta_K = -1$ it becomes $K_{\mathbb{R}}(5;d) \otimes K_{\mathbb{R}}(5;d)$. When $\epsilon_{PK} = +1$, then there exist an element $iP\Sigma$ which is anti-involutory and commutes with all other elements of the Clifford algebra. This implies that the K-group classification of the system with just the K-symmetry becomes complexified, which yields $K_{\mathbb{C}}(7;d) = K_{\mathbb{C}}(1;d)$.

Q,C,K-Symmetry: For systems with multiple BL symmetries, neither of which are P-symmetries, the classification can simply be done by considering systems with Q- and C-symmetries, since the existence of more than one of the symmetries Q, C, K implies the existence of the others. Consider therefore a system with Q- and C-symmetry, then Σ anti-commutes with Q, $C\mathcal{K}$ and H, which implies that the K-group classification is changed from $K_{\mathbb{R}}(s;d)$ to $K_{\mathbb{R}}(s+1;d)$ when $\epsilon_C = +1$. Similarly, in the case of $\epsilon_C = -1$, the operator $i\Sigma$ anti-commutes with Q, $C\mathcal{K}$ and H, which implies that the K-group classification is changed from $K_{\mathbb{R}}(s;d)$ to $K_{\mathbb{R}}(s-1;d)$.

P,Q,C,K-Symmetry: Now for the case where the Hamiltonian has all the symmetries. The classifications can be determined by considering the the Clifford algebra extension without the P-symmetry and then seeing the effect of including it.

Let $\mathcal{C} := C\mathcal{K}$ and $\mathcal{T} := \sqrt{\epsilon_{qc}}QC\mathcal{K}$, then the Clifford algebra extension of H_U can be constructed as $\{\mathcal{C}, i\mathcal{C}, i\mathcal{T}\mathcal{C}, \Sigma\} \to \{\mathcal{C}, i\mathcal{C}, i\mathcal{T}\mathcal{C}, \Sigma, H_U\}$, and the corresponding K-group is $K_{\mathbb{R}}(1 + 2\eta_C - \epsilon_{qc}; d)$, which is valid for the fixed choice of $\epsilon_C = -1$.

The classes with $\epsilon_C = +1$, can be mapped to $\epsilon_C = -1$, by the composition $C\Sigma$ which also a C-symmetry. Due to the fact that $\{C, \Sigma\} = \{Q, \Sigma\} = 0$, the mapping $C \to C\Sigma$ carries with it the mapping $(\epsilon_C, \eta_C, \epsilon_{QC}) \to (-\epsilon_C, -\eta_C, -\epsilon_{QC})$. When $\epsilon_{PQ} = -1$, there exist an element $\sqrt{-\epsilon_{PC}}P\Sigma$ that commutes with all other elements of the Clifford algebra, this has the implication that the K-group classification $K_{\mathbb{R}}(1 + 2\eta_C - \epsilon_{qc}; d)$, is doubled if $\epsilon_{pc} = -1$ and complexified if $\epsilon_{PQ} = 1$. When

$\epsilon_{PQ} = 1$, there exist an element $\sqrt{\epsilon_{PC}\epsilon_{QC}}PQ\Sigma$ that anti-commute with all other elements in the Clifford algebra, this implies a change from $K_{\mathbb{R}}(s; d)$ to $K_{\mathbb{R}}(s-\epsilon_{PC}\epsilon_{QC}; d)$, where $s = 1+2\eta_C - \epsilon_{QC}$. Which is invariant under the gauge transformation $\epsilon_{qc} \to \epsilon_{pq}\epsilon_{pc}\epsilon_{qc}$.

Counting up the total number of distinct symmetry classifications for point gaps, there is one with no symmetry, one with P-symmetry, one with Q-symmetry, four with C, two with K-symmetry, two with PQ, four with PC, three with PK, eight with QC and 12 with PQC-symmetry. Hence the total number of distinct symmetry classes for point-gapped systems is 38 and the symmetry classes are summarized in table 6. In the tables 7, 8 and 9, the classification for the AZ and the AZ† classes are summarized for both point and line gaps.

#	Symmetry	Generator Relations	K-Group	NH Class
1			$K_{\mathbb{C}}(1;d)$	NH A & A†
2	P		$K_{\mathbb{C}}(1;d) \oplus K_{\mathbb{C}}(1;d)$	NH sA
3	Q		$K_{\mathbb{C}}(0;d)$	NH AIII & AIII†
4	C	$\epsilon_C = 1, \eta_C = +1$	$K_{\mathbb{R}}(7;d)$	AI†
5	C	$\epsilon_C = 1, \eta_C = -1$	$K_{\mathbb{R}}(3;d)$	AII†
6	C	$\epsilon_C = -1, \eta_C = +1$	$K_{\mathbb{R}}(3;d)$	NH D
7	C	$\epsilon_C = -1, \eta_C = -1$	$K_{\mathbb{R}}(7;d)$	NH C
8	K	$\eta_K = +1$	$K_{\mathbb{R}}(1;d)$	NH AI & D†
9	K	$\eta_K = -1$	$K_{\mathbb{R}}(5;d)$	NH AII & C†
10	PQ	$\epsilon_{PQ} = +1$	$K_{\mathbb{C}}(1;d)$	
11	PQ	$\epsilon_{PQ} = -1$	$K_{\mathbb{C}}(0;d) \oplus K_{\mathbb{C}}(0;d)$	
12	PC	$\eta_C = +1, \epsilon_{PC} = +1$	$K_{\mathbb{C}}(1;d)$	
13	PC	$\eta_C = -1, \epsilon_{PC} = +1$	$K_{\mathbb{C}}(1;d)$	
14	PC	$\eta_C = +1, \epsilon_{PC} = +1$	$K_{\mathbb{R}}(3;d) \oplus K_{\mathbb{R}}(3;d)$	NH sD
15	PC	$\eta_C = -1, \epsilon_{PC} = -1$	$K_{\mathbb{R}}(7;d) \oplus K_{\mathbb{R}}(7;d)$	NH sC
16	PK	$\eta_K = +1, \epsilon_{PK} = +1$	$K_{\mathbb{R}}(1;d) \oplus K_{\mathbb{R}}(1;d)$	NH sAI
17	PK	$\eta_K = -1, \epsilon_{PK} = +1$	$K_{\mathbb{R}}(5;d) \oplus K_{\mathbb{R}}(5;d)$	NH sAII
18	PK	$\eta_K = \pm 1, \epsilon_{PK} = -1$	$K_{\mathbb{C}}(1;d)$	
19	QC	$\epsilon_C = +1, \eta_C = +1, \epsilon_{QC} = +1$	$K_{\mathbb{R}}(0;d)$	BDI†
20	QC	$\epsilon_C = +1, \eta_C = -1, \epsilon_{QC} = +1$	$K_{\mathbb{R}}(4;d)$	CII†
21	QC	$\epsilon_C = -1, \eta_C = +1, \epsilon_{QC} = +1$	$K_{\mathbb{R}}(2;d)$	NH BDI
22	QC	$\epsilon_C = -1, \eta_C = -1, \epsilon_{QC} = +1$	$K_{\mathbb{R}}(6;d)$	NH CII
23	QC	$\epsilon_C = +1, \eta_C = +1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(6;d)$	CI†
24	QC	$\epsilon_C = +1, \eta_C = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(2;d)$	DIII†
25	QC	$\epsilon_C = -1, \eta_C = +1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(4;d)$	NH DIII
26	QC	$\epsilon_C = -1, \eta_C = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(0;d)$	NH CI
27	PQC	$\eta_C = +1, \epsilon_{PQ} = +1, \epsilon_{PC} = +1, \epsilon_{QC} = +1$	$K_{\mathbb{R}}(1;d)$	pBDI
28	PQC	$\eta_C = -1, \epsilon_{PQ} = +1, \epsilon_{PC} = +1, \epsilon_{QC} = +1$	$K_{\mathbb{R}}(5;d)$	pCII
29	PQC	$\eta_C = +1, \epsilon_{PQ} = +1, \epsilon_{PC} = +1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(5;d)$	NH pDIII
30	PQC	$\eta_C = -1, \epsilon_{PQ} = +1, \epsilon_{PC} = +1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(1;d)$	NH pCI
31	PQC	$\eta_C = +1, \epsilon_{PQ} = +1, \epsilon_{PC} = -1, \epsilon_{QC} = \pm 1$	$K_{\mathbb{R}}(3;d)$	NH pBDI & pDIII
32	PQC	$\eta_C = -1, \epsilon_{PQ} = +1, \epsilon_{PC} = -1, \epsilon_{QC} = \pm 1$	$K_{\mathbb{R}}(7;d)$	NH pCII & pCI
33	PQC	$\eta_C = +1, \epsilon_{PQ} = -1, \epsilon_{PC} = +1, \epsilon_{QC} = \pm 1$	$K_{\mathbb{C}}(0;d)$	
34	PQC	$\eta_C = -1, \epsilon_{PQ} = -1, \epsilon_{PC} = +1, \epsilon_{QC} = \pm 1$	$K_{\mathbb{C}}(0;d)$	
35	PQC	$\eta_C = +1, \epsilon_{PQ} = -1, \epsilon_{PC} = -1, \epsilon_{QC} = +1$	$K_{\mathbb{R}}(2;d) \oplus K_{\mathbb{R}}(2;d)$	NH sBDI
36	PQC	$\eta_C = -1, \epsilon_{PQ} = -1, \epsilon_{PC} = -1, \epsilon_{QC} = +1$	$K_{\mathbb{R}}(6;d) \oplus K_{\mathbb{R}}(6;d)$	NH sCII
37	PQC	$\eta_C = +1, \epsilon_{PQ} = -1, \epsilon_{PC} = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(4;d) \oplus K_{\mathbb{R}}(4;d)$	NH sDIII
38	PQC	$\eta_C = -1, \epsilon_{PQ} = -1, \epsilon_{PC} = -1, \epsilon_{QC} = -1$	$K_{\mathbb{R}}(0;d) \oplus K_{\mathbb{R}}(0;d)$	NH sCI

Table 6: Symmetry classifications of the BL classes defined by 27 in terms of K-groups for systems with point gaps.

At this point it is convenient to write tables summing up the topological classification for the

AZ and AZ† classes for point gaps and real/imaginary line gaps for different dimensions.

AZ Class	Gap	Classifying Space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	L_r	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L_i	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0

Table 7: Topological classification for the complex AZ symmetry classes, in d-dimensions for point gaps denoted P and real and imaginary line gaps denoted L_r and L_i, respectively.

AZ Class	Gap	Classifying Space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
AI	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	L_r	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
	L_i	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
BDI	P	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	L_r	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
	L_i	$\mathcal{R}_2 \times \mathcal{R}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$	0	0	0
D	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
DIII	P	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	L_r	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L_i	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AII	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L_r	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	L_i	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	P	$\mathcal{R}_6$	0	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	L_r	$\mathcal{R}_5$	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L_i	$\mathcal{R}_6 \times \mathcal{R}_6$	0	0	$2\mathbb{Z}\oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
C	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$
	L	$\mathcal{R}_6$	0	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CI	P	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
	L_r	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$
	L_i	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Table 8: Topological classification for the real AZ symmetry classes, in d-dimensions for point gaps denoted P and real and imaginary line gaps denoted L_r and L_i, respectively.

AZ Class	Gap	Classifying Space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
AI†	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$
	L	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
BDI†	P	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
	L_r	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	L_i	$\mathcal{R}_0 \times \mathcal{R}_0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0
D†	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	L_r	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	L_i	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII†	P	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	L_r	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L_i	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
AII†	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	L	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CII†	P	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	L_r	$\mathcal{R}_5$	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L_i	$\mathcal{R}_4 \times \mathcal{R}_4$	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0
C†	P	$\mathcal{R}_5$	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L_r	$\mathcal{R}_6$	0	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	L_i	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI†	P	$\mathcal{R}_6$	0	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	L_r	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_2$
	L_i	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Table 9: Topological classification for the real AZ† symmetry classes, in d-dimensions for point gaps denoted P and real and imaginary line gaps denoted L_r and L_i, respectively.

5 Dimensional Raising

This section is based on [12],[6].

This section provides a review of the dimensional raising isomorphism derived by Teo and Kane in [12] for the AZ classes. Following the review, it will be shown that the dimensional raising maps can be generalized to hold for the non-Hermitian symmetry classes.

5.1 Spatially Modulated Hamiltonians

In this section spatially modulated Hamiltonians of the form $H(k, r)$ are considered. The base space of these Hamiltonians are parameterized by momentum vectors k, defined on the d-dimensional surface of the d-torus T^d, and the real-space coordinates r are defined on the D-dimensional surface of S^D surrounding some defect of the Hamiltonian. The base space of such a Hamiltonian will therefore be $T^d \times S^D$, but as discussed in section 4.1, the K-group classification only correspond to the strong topological invariants when the base space is a sphere, and that for toroidal base spaces, the effects of the weak topological invariants must be included. The base space will therefore be restricted to be of the form S^{d+D}.

For the case where $D = 0$, the real space coordinates are chosen from the surface of S^0, which can be considered as two points $\{-1, 1\}$, in the case where there is a non-trivial distinction between the Hamiltonians $H(k, +1)$ and $H(k, -1)$ defined the interface between two topologically distinct phases.

For the case where $D = 1$ the one-parameter families of Hamiltonians describe point defects in $d = 2$ and line defects in $d = 3$. In the case where $d = 1$ this could be considered as describing some adiabatic cycle. When $D = 2$ the two-parameter family of Hamiltonians describe a point gap in $d = 3$ and an adiabatic cycle in $d = 2$, it is however undefined in $d = 1$ as the dimension of the defect can not exceed the dimension of the space in which it is embedded.

In the classification of the D-parameter families of Hamiltonians in d-dimensions, subjected to some symmetry constraints, leads to a natural generalization of the periodic table of symmetry classes for topological insulators/superconductors. The table typically describes the case of $D = 0$. It can be shown that for $D > 0$ the topological classifications depend only on the co-dimension of the defect $\delta = d - D$. This means that all line defects with $\delta = 2$ will have the same topological classification regardless of the choice of d, similarly for points defects with $\delta = 1$ and adiabatic cycles with $\delta = 0$.

5.2 Hamiltonian Mappings

In this section the dimensional raising maps will be defined, there are two kinds of mappings that do this. The first kind removes an anti-unitary symmetry from a chiral Hamiltonian, making it non-chiral with just one anti-unitary symmetry. The second kind adds an anti-unitary symmetry to a Hamiltonian that already possess a single anti-unitary symmetry, thereby making it chiral. The dimensional raising map can now be expressed explicitly as a mapping from a Hamiltonian on a $(d+D)$-dimensional sphere parameterized by $(k,r) \in S^{d+D}$ to a Hamiltonian on a $(d+D+1)$-dimensional sphere S^{d+D+1}. Let Π be a chiral symmetry operator which satisfy $\{H_c(k,r),\Pi\} = 0$, and define the dimensional raising map that maps a chiral Hamiltonian to a non-chiral Hamiltonian as

$$H_{nc}(k,r,\Theta) = \cos\theta H_c(k,r) + \sin\theta\Pi \tag{78}$$

for $-\pi/2 \leq \theta \leq \pi/2$. This choice of Hamiltonian has the property that for $\theta = \pm\pi/2$ the Hamiltonian is simply equal to $\pm\Pi$ and therefore independent of k and r. This in turn allows for the base space $T^d \times S^D$ defined by k and r to be contracted to a point. This new Hamiltonian will then be defined on the suspension $\Sigma(T^d \times S^D)$ of the base space. If the original base space is a $d+D$-dimensional sphere, then the base space $(k,r,\theta) \in S^{d+D} \times [0,\pi]$ of the original Hamiltonian can be seen as a $(d+D+1)$-dimensional sphere S^{d+D+1} by reducing S^{d+D} to points at $\theta = \pi/2$ and $\theta = -\pi/2$, respectively.

Now assume that the chiral Hamiltonian H_c is flattened i.e. $H_c^2 = 1$. It follows that $H_{nc}^2 = 1$ since $\{H_c,\Pi\} = 0$.

For the real Clifford algebra classes, the second term on the right-hand side of 78 either violates the time-reversal symmetry Θ or the particle-hole symmetry Ξ, depending on if θ represent a position or momentum variable. This therefore results in a new non-chiral symmetry class, which can either be a clockwise or counter-clockwise rotation of the symmetry clock.

In order to determine which direction the rotation corresponds to, it is first required that $[\Theta,\Xi] = 0$ then $(\Theta\Xi)^2 = \Theta^2\Xi^2 = (-1)^{(s-1)/2}$. This allows for the unitary chiral symmetry operator to be written in the form

$$\Pi = i^{(s-1)/2}\Theta\Xi. \tag{79}$$

It can then be seen that in the case where θ represents a momentum variable, the time-reversal symmetry will be violated for $s = 1 \bmod 4$, and particle-hole symmetry will be violated for $s = 3 \bmod 4$. Notice that this implies the mapping $s \to s+1$ when θ is a momentum variable, corresponding to a clockwise rotation. Similarly, when θ is a position variable, the mapping is $s \to s-1$, corresponding to a counter-clockwise rotation.

Now the goal is to construct a chiral Hamiltonian from a non-chiral Hamiltonian by the inclusion of a symmetry. This can be done by a similar procedure to the Bogoliubov de-Gennes description of superconductors, where the number of bands are doubled. This allows for the chiral Hamiltonian to be written as

$$H_c(k, r, \theta) = \cos\theta H_{nc}(k, r) \otimes \tau_z + \sin\theta 1 \otimes \tau_a \tag{80}$$

where $a = x$ or y. The Hamiltonian 80 is defined on a base space that is the suspension of the original base space. This new Hamiltonian has a chiral symmetry since it anticommutes with the unitary chiral operator $\Pi = i\tau_z\tau_a$. If the non-chiral Hamiltonian is flattened such that $H_{nc}^2 = 1$, then it follows that $H_c^2 = 1$ and the energy gap is therefore preserved.

For the real Clifford algebra classes, the choice of $a = x$ or $a = y$ must be made such that the second term in 80 preserves the existing anti-unitary symmetry of H_{nc}. This is dependent on whether θ represents a position variable or a momentum variable and on the original anti-unitary symmetry of the Hamiltonian.

Consider the case where H_{nc} has TRS with the operator Θ, and where θ is a momentum (position) variable, then it is required that $a = y$ ($a = x$).

Similarly, consider the case where H_c has time-reversal symmetry, Θ, and θ represents a momentum (position) variable, then it is necessary that $a = y$ ($a = x$). Now assume that H_c also has a particle-hole symmetry defined by $\Xi = \tau_x\Theta$ ($\Xi = i\tau_y\Theta$) which satisfy $\Xi^2 = \Theta^2$ ($\Xi^2 = -\Theta^2$).

By similar arguments, it can be shown that when H_{nc} has particle-hole symmetry, the resulting Hamiltonian H_c is given by a clockwise (counter-clockwise) rotation $s \to s + 1$ ($s \to s - 1$) when θ represents a momentum (position) variable.

The maps defined by 78 and 80 are seen to be mappings between Hamiltonians in different dimensions and different symmetry classes. Naturally, Hamiltonians which are topologically equivalent will be mapped to topologically equivalent images, this can be seen by noticing that the mappings can be made continuously on a smooth interpolation between the original Hamiltonians. The dimensional mappings can therefore be considered to be maps between equivalence classes of Hamiltonians. The dimensional maps are in fact homomorphisms, this is due to the mapping of a direct sum of two Hamiltonians is mapped to the direct sum of the two new Hamiltonians, this action thereby preserves the group property of the equivalence classes. The maps 78 and 80 are therefore K-group homomorphisms

$$K_{\mathbb{F}}(s; D, d) \to K_{\mathbb{F}}(s + 1; D, d + 1) \tag{81}$$

$$K_{\mathbb{F}}(s; D, d) \to K_{\mathbb{F}}(s - 1; D + 1, d) \tag{82}$$

where D is the co-dimension of the defect and d is the dimension of the system and $\mathbb{F} = \mathbb{R}, \mathbb{C}$ is the field on which the Clifford algebra is based.

5.3 Invertibility

In this section it will be shown that K-group homomorphisms 81 and 82 are in fact isomorphisms, namely, that the mappings are invertible.

In general a Hamiltonian can not be constructed from a lower dimensional Hamiltonian by using 78 and 80, however, it turns out that it is possible to deform a generic Hamiltonian into the forms described in 78 or 80. This provides an inverse map and the K-group homomorphisms are in fact isomorphisms.

Let $H(k, r, \theta)$ be a Hamiltonian defined on a $d + D + 1$-dimensional sphere parameterized by $(k, r, \theta) \in S^{D+d+1}$. Let θ be the azimuthal angle of S^{D+d+1}, such that it points towards the north (south) pole for $\theta = 0$ ($\theta = \pi$), and let the pair (r, k) parameterize the $d+D$-dimensional latitudinal circle of the sphere, such that the Hamiltonian satisfy the conditions

$$H(k, r, \theta = 0) = \text{const.}, \quad H(k, r, \theta = \pi) = \text{const.}.$$

The Hamiltonian can be spectrally flattened by continuously deformations

$$H^2(k, r, \theta) = 1.$$

The parameterization of (k, r, θ) above provides a dimensional reduction from $S^{d+D+1} \to S^{d+D}$ by fixing the azimuthal angle θ to i.e. $\theta = \pi/2$. This dimensional reduction is however not ensured to be invertible, since the flattened form of the Hamiltonian generally do not resemble the form of the dimensional raising maps. This can be mended by the introducing an artificial action

$$S[H] = \int d\boldsymbol{k} d\theta \, \mathrm{Tr} \left(\partial_\theta H \partial_\theta H \right)$$

In the space of gapped symmetry preserving Hamiltonians the action S can be interpreted as a "height" function. Given a generic Hamiltonian, it is always possible to find a "downhill" direction and these "downhill" vectors can be integrated into a deformation trajectory. Due to the action being positive definite, it is bounded below, and therefore any deformation trajectory must end at a Hamiltonian that minimizes the action.

Due to the flatness condition $H^2 = 1$, any minimized Hamiltonian must satisfy the Euler-Lagrange equation

$$\partial_\theta^2 + H = 0.$$

The solutions are linear combinations of $\cos\theta$ and $\sin\theta$. Since the base space is compactified to points at $\Theta = \pm\pi/2$ the coefficients of $\sin\Theta$ must be constant. The minimal Hamiltonian must therefore be of the form

$$H(k, r, \Theta) = \sin\theta H_1(k, r) + \cos\theta H_2 \tag{83}$$

which due to the flatness condition imply

$$H_1^2(k, r) = 1, \quad H_2^2 = 1 \quad \{H_1(k, r), H_2\} = 0. \tag{84}$$

By fixing $\theta = \pi/2$, a dimensional reduction from $H(k, r, \theta)$ to $H_1(k, r)$ has been obtained.

To conclude the proof it remains to be shown that 81 and 82 are invertible for both odd and even values of s.

Starting with the proof for the case of odd s. Let $H(k, r, \theta)$ be non-chiral and let $\Pi = H_0$ and $H_c(k, r) = H_1(k, r)$, then 83 takes the form of 78. Then due to 84, it is seen that H_1 must have chiral symmetry. This proves that 81 and 82 are invertible for odd s.

Ξ^2

Now to prove it for the case of even s, let $H(k, r, \theta)$ be chiral, then H_0 and $H_1(k, r)$ both anti-commute with the chiral symmetry operator. Now let $H_0 = \tau_a$ and $\Pi = i\tau_z\tau_a$ where $\tau_a = \tau_x$ is chosen when Θ is a position and $\tau_a = \tau_y$ when θ is a momentum variable. It then follows that $\{H_1, \tau_x\} = \{H_1, \tau_y\} = 0$, which imply that

$$H_1(k, r) = h(k, r) \otimes \tau_z.$$

Notice that if $H_{nc} = h$, then 83 is of the same form as 80. Because τ_z anti-commute with either Θ or Ξ, it follows that $h(k, r)$ has exactly one anti-unitary symmetry and hence it is not chiral. This proves the invertibility of the K-group homomorphisms for even s.

The K-group homomorphisms are therefore isomorphisms, this implies the existence an inverse to the dimensional raising map, that lowers the dimension and change the classification accordingly. This means that the K-group classification of a lower dimensional system can be derived from the classification of a higher dimensional system and vice versa.

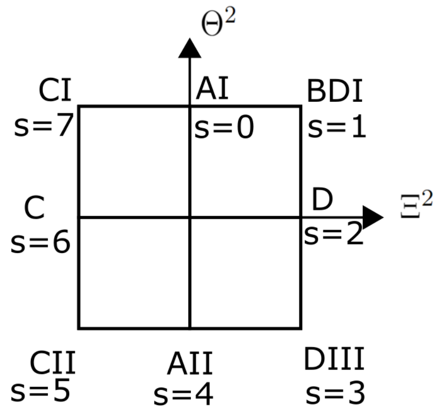

Figure 4: Bott clock for the AZ class where Θ is the time-reversal symmetry and Ξ is the particle-hole symmetry.

Due to the Bott periodicity of the Clifford algebras, namely, that the periodicity of the real Clifford algebra is 8 and the periodicity of the complex Clifford algebras is 2. This lead to a periodic table describing topological classifications of topological insulators and topological insulators. Then in the development of the dimensional raising methods, it was found that the procedure of raising the dimension of a given system naturally had this same periodicity, 2-periodic for the complex classes and 8-periodic for the real classes. This lead to the notion of the *Bott clock*, which is a two-dimensional plane, where the x, y-axis are the squared modulus of the time-reversal symmetry (Θ) and the particle-hole symmetry (Ξ) operators describing the given Altland-Zirnbauer class, and the entries on the clock are given by the variable s defined in 79.

5.4 Representative Hamiltonians

To describe Hamiltonians for the real symmetry classes, define two types of Dirac matrices, the position-like Dirac matrices Γ_μ and the momentum-like Dirac matrices γ_i. These Dirac matrices satisfy the anti-commutation relations $\{\Gamma_\mu, \Gamma_n\} = 2\delta_{\mu\nu}$, $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, $\{\Gamma_m, \gamma_j\} = 0$, and they are distinguished by their symmetries under anti-unitary symmetry operations. For time-reversal symmetry

$$[\Gamma_\mu, \Theta] = \{\gamma_i, \Theta\} = 0$$

and for particle-hole symmetry

$$\{\Gamma_\mu, \Xi\} = [\gamma_i, \Xi] = 0.$$

Now define a generic Hamiltonian that can be written as a combination of p momentum-like Dirac matrices $\gamma_{1,\ldots,p}$ and $q+1$ position-like matrices $\Gamma_{0,\ldots,q}$

$$H(k,r) = R(k,r)\cdot\Gamma + K(k,r)\cdot\gamma$$

which due to the symmetry constraints the coefficients must satisfy the involution

$$\begin{aligned} R(-k,r) &= R(k,r) \\ K(-k,r) &= -K(k,r). \end{aligned}$$

This effect can be characterized by a unit vector defined on a $(p+q)$-sphere as

$$d(k,r) = \frac{(K,R)}{\sqrt{|K|^2 + |R|^2}} \in S^{p+q}.$$

For a Hamiltonian $H(k,r)$ in symmetry class s, it can be shown that the value of s is related to the number of momentum and position like Dirac matrices as

$$p - q = s \bmod 8. \tag{85}$$

To show this, consider a Hamiltonian with $H_0 = R_0(k,r)\Gamma_0$, where $\Gamma_0 = 1$, and hence $(p,q) = (0,0)$. This Hamiltonian has time-reversal symmetry with $\Theta = K$, and hence correspond to the symmetry class AI with $s = 0$. This Hamiltonian can now be mapped using the dimensional raising maps to a Hamiltonian H_s in class s. The dimensional raising maps define a new Clifford algebra with an additional generator that is either position-like or momentum-like. The mappings that introduce an additional position-like generator $p \to p+1$ correspond to a clockwise rotation on the symmetry clocks $s \to s+1$, and the addition of a momentum-like generator $q \to q+1$ corresponds to a counter-clockwise rotation $s \to s-1$. This thereby proves the validity of the 85, as this mapping can be done for all symmetry classes and therefore 85 is valid for all choices of the indices (p, q). This is in agreement with the fact that for the real Clifford algebras it was found in section 4.1 that the Morita equivalence classes were dependent only on the difference between the number of positive and negative generators.

5.5 Non-Hermitian Bott Clocks

In this section, the remaining Bott clocks for the new non-Hermitian symmetry classes will be presented. The general procedure of dimensional raising will be presented and an example of the dimensional raising of the NH AZ classes will be shown.

The dimensional raising maps shown above was derived originally by Teo and Kane in [12] for the Hermitian AZ classes. The dimensional raising procedure can be generalized such that it works for the non-Hermitian classes as well. This is done by replacing the chiral symmetry Π by the Q-symmetry with $\epsilon_q = -1$ and replacing the anti-unitary operators Θ (TRS) and Ξ (PHS) with the anti-unitary operators of the BL symmetries C and K defined in 27. The action of the C-symmetry was defined to correspond to PHS (TRS†) for $\epsilon_c = -1$ ($\epsilon_c = +1$) and the K-symmetry was defined to correspond to TRS (PHS†) for $\epsilon_k = +1$ ($\epsilon_k = -1$). Now because the x- and y-axis of the Bott clock for the AZ class was defined to be given by squared modulus of the particle-hole symmetry Ξ^2 and the time-reversal symmetry Θ^2, respectively. This naturally leads to the convention of choosing the x- and y-axis of the Bott clock for the NH AZ class to be given by η_k and η_c, respectively. Similarly, for the AZ† class, the x- and y-axis of the Bott clock are given by η_k and η_c, respectively.

The chiral symmetry Π in 78 with the Q-symmetry with $\epsilon_q = -1$ and the dimensional raising map for the chiral system becomes

$$H_{nc}(k, r, \theta) = \cos\theta H_c(k, r) + \sin\theta q. \tag{86}$$

In order to see the dimensional raising procedure in action, consider a system in the NH AZ symmetry which is defined by having $\epsilon_q = -1$, $\epsilon_c = -1$ and $\epsilon_k = +1$. Therefore the first system on the clock is the system with a K-symmetry as its only symmetry, this correspond to the NH AI class which for a point-gap has the K-group classification $K_{\mathbb{R}}(1; d)$. Therefore the dimensional raising problem is a matter of finding the right τ_a, such that the non-chiral Hamiltonian H_{nc}, possessing only a K-symmetry with $\epsilon_k = +1$ is mapped to the chiral Hamiltonian H_c, which has both C- and K-symmetry, and therefore also a Q-symmetry.

These conditions are satisfied by choosing $\tau_a = \sigma_y$ in 80, since the new Hamiltonian has a K-symmetry in the form of $\tilde{k} = k \otimes I$ with $\eta_k = +1$, and the Hamiltonian is

$$H_c(k, r, \theta) = \cos\theta H_{nc}(k, r,) \otimes \tau_z + \sin\theta I \otimes \sigma_y. \tag{87}$$

Then in order for the new C-symmetry to have the right sign, it must be defined as $\tilde{c} = k \otimes \sigma_x$, for which $\eta_{\tilde{c}} = +1$ and the new entry correspond to $s = 1$ on the Bott clock and the symmetry class is that of the NH BDI class, which for a point-gap has the K-group classification $K_{\mathbb{R}}(2; d)$.

Similarly, the next step is to dimensional raise the chiral Hamiltonian to a non-chiral Hamiltonian which has C-symmetry with $\epsilon_c = +1$ and $\eta_k = +1$.

$$H_{nc}(k, r, \theta) = \cos\theta H_c(k, r) + \sin\theta q. \tag{88}$$

In order to find the correct Hamiltonian, the commutator ϵ_{qc} must be found, ϵ_{qc} can be found by noticing the relations $\eta_k = \eta_c\epsilon_{qc} = \eta_c\epsilon_{qk}$. Now since $\eta_k = +1$ it follows that $\epsilon_{qk} = +1$. This has the implication that the operator c commutes with the chiral operator q, which leads to the correct sign for the C-symmetry in the second term of 88, but it results in the wrong sign for the K-symmetry, and therefore the K-symmetry is removed by the dimensional raising and the Hamiltonian is therefore no longer chiral. At this stage, the Hamiltonian possess only a C-symmetry with $\epsilon_c = +1$ and $\eta_c = +1$, which correspond to the NH D class which for a point-gap has the K-group classification $K_{\mathbb{R}}(3; d)$.

The remaining six steps that are required to return to the clock entry $s = 0$ follows this exact prescription of adding an anti-unitary symmetry and obtaining a chiral symmetry, then removing an anti-unitary symmetry and so on.

The clocks for the NH AZ and the AZ† classes are shown in figure 5 and the coefficients $\eta_c, \eta_k, \epsilon_{qc}$ corresponding to the symmetry class at the different clock entries s are given in tables 11 and 12, respectively. The clocks can be reproduced for the NH PAZ and PAZ† classes as well, by simply performing the dimensional raising without the inclusion of the pseudo-hermiticity and then including it afterwards, similarly to how it was treated during the classification of the symmetry classes in terms of K-groups cf. section 4.3.

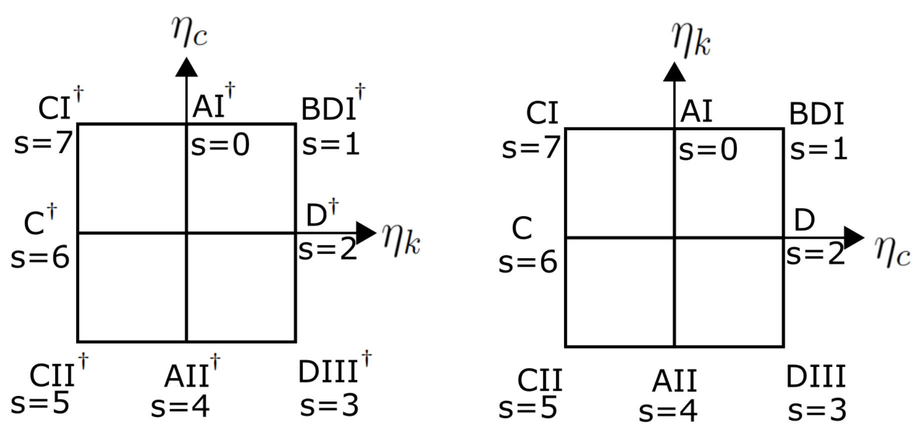

Figure 5: Bott clocks for the AZ†(left) and the NH AZ(right) classes, where $\eta_c = cc^*$ and $\eta_k = kk^*$.

The table 10 shows the K-group corresponding to the AZ, AZ† and NH AZ classes in the case of a point gap. It can be seen that the classifications, relative to the AZ class are simply shifted from $s \to s-1$ for the AZ† class and $s \to s+1$ for the NH AZ class. This fact can easily be seen in the extended Hamiltonian formalism, simply by noticing that the set of generators for the two classes contain an additional chiral symmetry Σ ($i\Sigma$), with $\Sigma^2 = +1$, for AZ† (NH AZ).

K-Group	AZ Class	AZ† Class	NH AZ Class
$K_{\mathbb{C}}(0;d)$	A	AIII	AIII
$K_{\mathbb{C}}(1;d)$	AIII	A	A
$K_{\mathbb{R}}(0;d)$	AI	BDI	CI
$K_{\mathbb{R}}(1;d)$	BDI	D	AI
$K_{\mathbb{R}}(2;d)$	D	DIII	BDI
$K_{\mathbb{R}}(3;d)$	DIII	AII	D
$K_{\mathbb{R}}(4;d)$	AII	CII	DIII
$K_{\mathbb{R}}(5;d)$	CII	C	AII
$K_{\mathbb{R}}(6;d)$	C	CI	CII
$K_{\mathbb{R}}(7;d)$	CI	AI	C

Table 10: K-Group classification for a point-gapped system in d-dimensions for the AZ, AZ† and non-Hermitian AZ classes.

Clock Entry	NH AZ Class	η_c, η_k	ϵ_{qc}
$s=0$	NH AI	$\eta_k = +1$	
$s=1$	NH BDI	$\eta_c = +1, \eta_k = +1$	$\epsilon_{qc} = +1$
$s=2$	NH D	$\eta_c = +1$	
$s=3$	NH DIII	$\eta_c = +1, \eta_k = -1$	$\epsilon_{qc} = -1$
$s=4$	NH AII	$\eta_k = -1$	
$s=5$	NH CII	$\eta_c = -1, \eta_k = -1$	$\epsilon_{qc} = +1$
$s=6$	NH C	$\eta_c = -1$	
$s=7$	NH CI	$\eta_c = -1, \eta_k = +1$	$\epsilon_{qc} = -1$

Table 11: Table showing the clock entries s for the NH AZ classes and the η_c, η_k, where x-axis is $x = \eta_c$ and the y-axis is $y = \eta_k$, and the commutator ϵ_{qc} that determines the symmetry of the chiral Hamiltonians under the dimensions raising.

Clock Entry	AZ† Class	η_c, η_k	ϵ_{qc}
$s = 0$	AI†	$\eta_c = +1$	
$s = 1$	BDI†	$\eta_c = +1, \eta_k = +1$	$\epsilon_{qc} = +1$
$s = 2$	D†	$\eta_k = +1$	
$s = 3$	DIII†	$\eta_c = -1, \eta_k = +1$	$\epsilon_{qc} = -1$
$s = 4$	AII†	$\eta_c = -1$	
$s = 5$	CII†	$\eta_c = -1, \eta_k = -1$	$\epsilon_{qc} = +1$
$s = 6$	C†	$\eta_k = -1$	
$s = 7$	CI†	$\eta_c = +1, \eta_k = -1$	$\epsilon_{qc} = -1$

Table 12: Table showing the clock entries s for the AZ† classes and the η_c, η_k, where x-axis is $x = \eta_c$ and the y-axis is $y = \eta_k$, and the commutator ϵ_{qc} that determines the symmetry of the chiral Hamiltonians under the dimensions raising.

6 Novel Properties intrinsic to Non-Hermitian Topology

This section will provide some examples of some of the different features that arise in non-Hermitian systems due to an intrinsic non-Hermitian topology.

6.1 Forgetful Functor from Line-Gap to Point-Gap Topology

This section is based on [13].

The map from a set of point or line gapped Hamiltonians to their corresponding K-group on a fixed base manifold is a functor. And due to the fact that a line gapped Hamiltonian must also be point gapped, the point gap topological phase must therefore be included in the line gap topological phase, this inclusion produce a homomorphism between the K-groups. Hence it is possible to classify the intrinsic point-gap topological phases of a system, but removing the overlap between the point-gapped topological phase and the line gapped topological phase, i.e. the point-gapped topological phases can be determined by looking at the complement of the image of such group homomorphism. To derive these homomorphisms, consider a generic non-Hermitian Hamiltonian $h(k)$ in d dimensions, h can be extended to a Hermitian Hamiltonian in the form

$$H(k) = \begin{bmatrix} 0 & h(k) \\ h^{\dagger}(k) & 0 \end{bmatrix}_{\sigma}$$

where the σ subscript denotes the basis of the Hamiltonian. This $H(k)$ satisfy the chiral symmetry $\Gamma H(k)\Gamma^{-1} = -H(k)$ where $\Gamma = \sigma_z$.

If h is point-gapped, then H is also point gap. The classification of the topology of h is simply that of the classification of the Hermitian Hamiltonian H. Denote the topological phase with respect to the point-gap as K_P. In the presence of a real (an imaginary) line gap, it has been shown in section 3.4 that the Hamiltonian h can be flattened to a Hermitian (anti-Hermitian) Hamiltonian that preserves the symmetry operations and the line gap. This Hermiticity (anti-Hermiticity) property constraints the form of H to have an additional chiral symmetry $\Gamma_r = \sigma_y$ ($\Gamma_i = \sigma_x$). Hence in the presence of a real (an imaginary) line gap, the topological phases K_r (K_i) with respect to the point gap are distinct. Due to the presence of these different chiral symmetry for the real and imaginary line gaps, it is possible to define a forgetful functor, which forgets the existence of these chiral symmetries. To this end, define a homomorphism $f_r : K_r \to K_P$ ($f_i : K_i \to K_P$) from the topological phase of a real (an imaginary) line gap to the topological phase of the corresponding point gap. This forgetful functor is valid in all dimensions by the property of the dimensional isomorphisms defined in 5.2, and it is therefore only necessary to compute the topological phases in zero dimensions.

A consequence of this forgetful functor is the following:

Let h be a point-gapped non-Hermitian Hamiltonian that lies in the image of either f_r or f_i, then h can be flattened to a Hermitian or anti-Hermitian. Hence the topology of h is simply given by that of the Hermitian extended Hamiltonian H.

Conversely, if h is point-gapped but is not in the image of just one of the homomorphisms, it hints at the existence of some property that is inherent to point-gap topology. Hence the quotient group $K_P/(\mathrm{Im}\, f_r \cup \mathrm{Im}\, f_i)$ must therefore contain some information about some properties that are intrinsic to the point-gap topology.

As an example, consider the case of DIII† in $d = 2$. Here the homomorphisms are given by $f_r : \mathbb{Z}_2 \to \mathbb{Z}$, for which there is no nontrivial homomorphism, and $f_i : \mathbb{Z} \to \mathbb{Z}$, where an element $n \in \mathbb{Z}$ is mapped as $f_i(n) = 2n$. Then, it is seen that

$$\frac{K_P}{f_r \cup f_i} = \frac{Z}{0 \cup 2\mathbb{Z}} = \mathbb{Z}_2.$$

Hence the topological invariant of the intrinsic point-gap topology is an element of $\mathbb{Z}_2$. This is interesting since the topological invariant of the point gap, determined by K-theory, is an element of $\mathbb{Z}$. This discrepancy indicates the existence of an intrinsic point-gap topology.

6.2 Non-Hermitian Skin Effect

This section is based on [11]

In this section the motivation for the non-Bloch theory in which the Brillouin zone must be replaced with a generalized Brillouin zone is provided. The invalidation fo typical Bloch theory was motivated by the emergence of the non-Hermitian skin effect which occurs in some non-Hermitian systems and originate as a product of systems with non-trivial point-gap topology under periodic boundary conditions. The skin effect manifests itself as a bunching up of eigenstates at the edges of a system instead of being delocalized in the bulk. The minimal example of this is the 1D Hatano-Nelson model with asymmetric hopping as shown in section 2.2. In this system, there is a preference for the hopping to be in a certain direction, in a finite system this leads to a bunching up of the eigenstates as they get to the edge of the system.

The non-Hermitian skin effect has the implication that it breaks the typical bulk-boundary conditions, that exist in the typical Bloch theory periodic systems and the systems must instead be described using the so-called non-Bloch theory.

This can be demonstrated by starting from a generic one-band tight-binding Hamiltonian in $d = 1$ with hopping indices given by $-m \leq i - j \leq n$,

$$H = \sum_{i,j} t_{i-j} \left|i\right\rangle \left\langle j\right| = \sum_{k \in BZ} H(k) \left|k\right\rangle \left\langle k\right|,$$

where $H(k) = \sum_{r=-m,\dots,n} t_r \mathrm{e}^{ikr}$ is the Fourier transform of the hopping term (t_0 is understood to be the on-site potential). For periodic boundary conditions, the wave vector takes values within the Brillouin zone with $0 \le k \le 2\pi$, and $\mathrm{e}^{ik} \in U(1)$. Define $z := \mathrm{e}^{ik}$ and consider z as a general point in the complex plane. Each Hamiltonian $H(k)$ can then be expressed as a holomorphic function in the form

$$H(z) = t_{-m} z^{-m} + \cdots + t_n z^n = \frac{P_{m+n}(z)}{z^m},$$

where $P_{m+n}(z)$ is a polynomial of degree $m+n$. Geometrically, the characteristic equation defines a 2D Riemann surface in the 4D space parameterized by $(\beta, E) \in \mathbb{C}^2$, defining the function $f(\beta, E) = \prod_\mu^n [E - E_\mu(\beta)] = 0$, each root $E = E_\mu(\beta)$ corresponds to a branch of the multi-valued function. For fixed boundary conditions the corresponding Bloch band $\{E_\mu(\beta_{BZ,\mu}), \mu = 1, \dots, n\}$ or GBZ band $\{E_\mu(\beta_{GBZ,\mu}, \mu = 1, \dots, n)\}$ become a set of closed loops on the Riemann surface.

The Hamiltonian $H(k)$ has a pole of order m at $z = 0$ and it has $m + n$ zeroes. Along any oriented loop $\mathcal{C}$ and for any reference point $E_b \in \mathcal{C}$, it is possible to define a winding number of $H(z)$ as

$$w_{\mathcal{C},E_b} := \frac{1}{2\pi} \oint_{\mathcal{C}} \frac{d}{dz} \arg(H(z) - E_b) dz.$$

The integral on the right-hand side can be evaluated using the Cauchy's argument principle, which is defined in the following lemma.

Lemma: If a function $f(z)$ is meromorphic in a region R enclosed by a contour γ, let N be the number of complex roots of $f(z)$ in γ, and P be the number of poles in γ, with each zero and pole counted as many times as its multiplicity and order, respectively, then

$$N - P = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z) dz}{f(z)}. \tag{89}$$

The Cauchy's argument principle gives a relation between the winding number of a complex function $f(z)$ and the total number of zeroes and poles enclosed by the contour $\mathcal{C}$ given by

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f'(z)}{f(z)} dz = Z - P$$

which is valid as long as f is a meromorphic function. The number of poles is always m and the winding number is therefore determined by the number of zeros of $P_{m+n}(z) - z^m E_b$ that lie within the unit circle. Now, instead of evaluating $H(z)$ along the standard BZ, consider evaluating $H(z)$

along another closed loop, namely, the so-called generalized BZ, which is defined as

$$GBZ := \{z||H_m^{-1}(H(z))| = |H_{m+1}^{-1}H(z)|\},$$

where the $H_i^{-1}(E)$'s satisfying $|H_i^{-1}(E)| = |H_{i+1}^{-1}(E)|$ are the $m+n$ branches of the inverse function of $H(z)$. This therefore lead us to the following theorem.

Theorem: Let $H(z)$ denote a bulk Hamiltonian in 1D with $z := \mathrm{e}^{ik}$ with $k \in \mathbb{C}$ and let $\{z_i|\text{for } i = 1, 2, \ldots, 2M; |z_1| \leq |z_2| \leq \cdots \leq |z_{2M}|\}$ be the solutions to the characteristic equation $\det(H(z) - E) = 0$ for some $E \in \mathbb{C}$. In the Hermitian limit $|z_i| = 1$ for all z_i, however, this is not necessarily the case for complex k.

The generalized BZ is the closed curve in $\mathbb{C}$ that enclose the mth order pole and exactly m zeros of the polynomial $P_{m+n}(z) - Ez^m$ for any $E \in \mathbb{C}$. This has the implication that if $w_{BZ,E_b} \neq 0$ for arbitrary E_b, then it must follow that the image of the GBZ can not enclose any area, and therefore it must be a 1D arc, and it necessarily have $w_{GBZ,E_b} = 0$.

6.3 Duality of Non-Hermitian Skin Effect and Hermitian Exact Zero Modes

This section is based on [17].

There exist a duality between the existence of the non-Hermitian skin effect of a non-Hermitian Hamiltonian and the existence of exact zero modes of the corresponding extended Hamiltonian.

The suppression of the non-Hermitian skin effect in the presence of open boundary conditions can be related to the lack of exact zero modes of the extended Hermitian Hamiltonian.

To see this duality, let $H - E$ be a non-Hermitian Hamiltonian in the non-Hermitian class A, where E is a complex energy, then consider the corresponding extended Hamiltonian, which is in class AIII,

$$H_E := \begin{bmatrix} 0 & H - E \\ H^\dagger - E^* & 0 \end{bmatrix}.$$

When $H - E$ has a nontrivial topology, there must exist boundary exact zero modes, which are composed of the skin mode $|E_R\rangle$ and the corresponding left-eigenstate $|E_L\rangle$, hence

$$H_E \begin{pmatrix} 0 \\ |E_R\rangle \end{pmatrix} = 0 \quad H_E \begin{pmatrix} |E_L\rangle \\ 0 \end{pmatrix} = 0.$$

The zero modes are localized at opposite edges and they therefore correspond to boundary zero modes of the class AIII topological insulator.

Similarly, a non-Hermitian Hamiltonian in class AII† has a corresponding extended Hamiltonian in class DIII, which in one-dimension hosts Majorana doublets as topological zero modes at both ends of the system. Just as in the previous case, the modes are given by

$$(0, |E_R\rangle)^T, (|E_R\rangle^*, 0)^T, (|E_L\rangle), (0, T\,|E_L\rangle^*) \tag{90}$$

Due to the fact that the energy spectrum can be chosen non-uniquely, it is important in the topological classification that the energy term E is not just chosen arbitrarily to be zero, as it might might break certain symmetries, e.g. the K-symmetry in the AZ† classes. Since the non-Hermitian skin effect relies on the existence of exact zero modes, and hence on the existence of time-reversal pairs which share the same complex eigenenergies, which only occur in the presence of the K-symmetry in class AZ†.

6.4 Different Kinds of Perturbations

This section is based on [7].

In this section the non-Hermitian topology that arise from perturbing a Hermitian Hamiltonian by some different non-Hermitian terms. In order to determine the topology of the systems, it is convenient to use a result from section 5.1, where it was shown that the topology of a system is dependent only on the co-dimension of the gap embedded in the system. Therefore let $\delta = d - d_{gap}$ be the co-dimension of a d-dimensional system with a d_{gap}-dimensional gap, where the values $d_{gap} = 0, 1, 2$ correspond to an exceptional point, line, surface. The topological invariant can then be defined on the surface of a sphere $S^{\delta-1}$ enclosing the d_{gap}-dimensional exceptional manifold.

To show the effects of different kinds of perturbations, consider a 2D model possessing a Weyl point described by the Hermitian Hamiltonian

$$H(k) = k_x\sigma_x + k_y\sigma_y. \tag{91}$$

A Hermitian perturbation will simply create an energy gap, such that the energy eigenvalues no longer touch at a Weyl point. However, if the perturbations are non-Hermitian, this is no longer the case, hence the perturbations that will be considered in this section are non-Hermitian perturbations of the form $i\gamma\sigma_z$ and $i\gamma\sigma_x$ for some $\gamma \in \mathbb{R}$.

Pair of Exceptional Points

If the non-Hermitian perturbation is given by $i\gamma\sigma_y$, such that the Hamiltonian becomes

$$H(k) = k_x\sigma_x + (k_y + i\gamma)\sigma_y, \tag{92}$$

then $H(k)$ becomes non-Hermitian and without symmetries. When the non-Hermitian term is zero, that is $\gamma = 0$, the system is simultaneously point- and line-gapped, since there is no distinction between the point- and line-gaps in Hermitian physics. When $\gamma \neq 0$, the Weyl-point bifurcate into two exceptional points, that are located at $k_y = \pm\gamma$ and are connected by a Fermi arc. The energy eigenvalues are real-valued as long as $|k_y| > \gamma$, but when $|k_y| < \gamma$ they become purely imaginary. At the exceptional points, the eigenvalues result in a square root singularity, that acts as a branch cut of a Riemann surface in the complex plane, that is self-intersection as shown in figure 6. This has the implications that the eigenvalues E_+ and E_-, can be swapped by traversing around a loop S^1 enclosing one of the exceptional points in momentum space. The system is in the symmetry classes AZ AI and it therefore possess an open point-gap at the energy $E(k_y = \gamma)$, but the line-gap is closed on S^1. This means that the Hamiltonian is invertible on S^1 and it is therefore possible to define a winding number for the point gap as

$$W := \oint_{S^1} \frac{dk}{2\pi i} \nabla_k \log\det(H(k) - E(k_y = \gamma)). \tag{93}$$

When the integration is carried out over a loop S^1 enclosing one of the exceptional points at $k_y = \pm\gamma$, the winding number is $W = \pm 1$, which implies that the exceptional points are topologically stable.

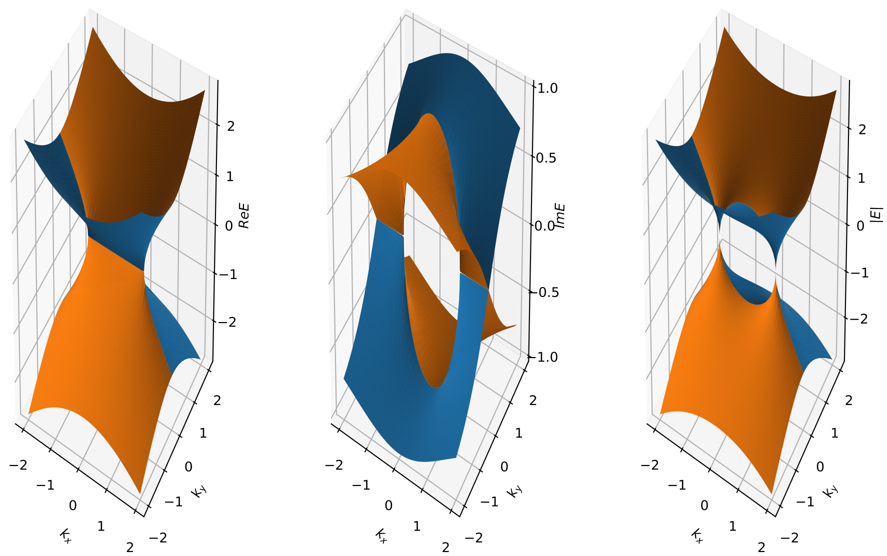

Figure 6: The figures shows the real part (left), the imaginary part (middle) and the absolute value (right) of the spectrum of the Hamiltonian 92 in the z-axis as a function of the k_x and k_y.

Exceptional Ring

Now to show a system possessing a ring of exceptional points, consider the Hamiltonian

$$H(k) = k_x\sigma_x + k_y\sigma_y + (k_z + i\gamma)\sigma_z \tag{94}$$

which is in AZ class AI and has energy eigenvalues $E_\pm = \pm\sqrt{k_x^2 + k_y^2 - (k_z\gamma)^2}$. The non-Hermitian perturbation creates a ring of exceptional points, located at $k_x^2 + k_y^2 = \gamma^2$ for $k_z = 0$ as shown in figure 7.

Now for the system defined by 94 in $d = 3$, it can be seen from the table 7, that there is an open line-gap on the surface of a sphere S^2 that enclose the exceptional ring, this results in the Chern number remaining well-defined and it correspond to the $\mathbb{Z}$ topological invariant for a line gapped system with co-dimension $p = 3$. The point gap is open on S^1 that goes across the execeptional ring, this implies that the winding number is also well-defined and takes values in $\mathbb{Z}$. The fact that the topological invariants exist for both the point and the line gap, and that the indices are independent of one another, means that the they both provide the stability of the exceptional ring. If the system were 2-dimensional instead of 3-dimensional, then it would not be possible to embed the sphere S^2, which means that the line-gap would not be open in 2-dimensions. This results in the exceptional ring being unstable in 2-dimensions.

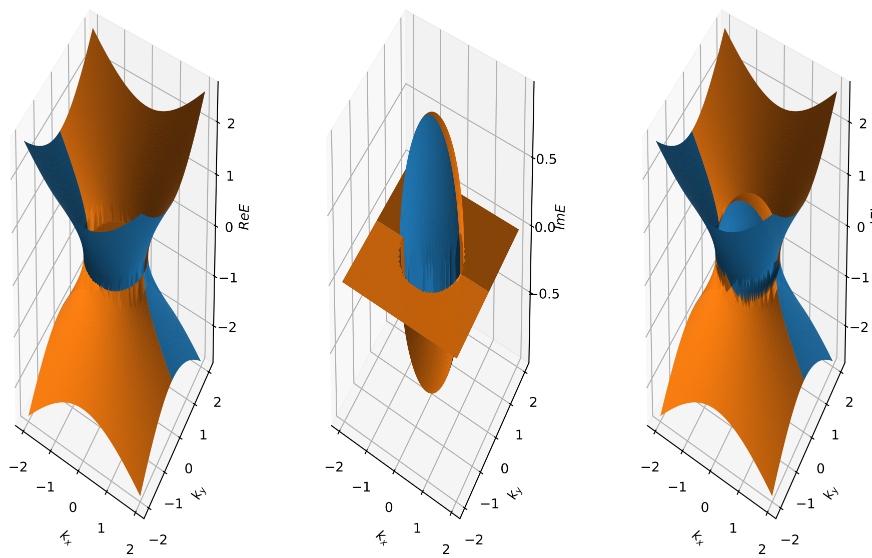

Figure 7: The figures shows the real part (left), the imaginary part (middle) and the absolute value (right) of the spectrum of the Hamiltonian 94 in the z-axis as a function of the k_x and k_y.

7 Conclusion

To start off, an overview of some properties that arise in physical systems due to the effects of non-Hermiticity were provided. It was shown that an implication of the non-Hermiticity was that the typical orthogonality of the eigenvectors of a Hermitian Hamiltonian, no longer necessarily would be satisfied in the non-Hermitian case. It was found that the non-Hermiticity of a Hamiltonian lead to the set of left and right eigenvectors forming a biorthogonal system, instead of the typical orthogonal system, which is obtained in the Hermitian limit of this construction. It was found that the biorthogonality of the eigenvectors of a non-Hermitian Hamiltonian lead to the possibility of exceptional points forming the energy spectrum. It was then shown that the complex spectrum of the non-Hermitian Hamiltonians allowed for the definition of two types of energy gaps, namely, point- and line-gaps, which turned out to have a major impact on the classification of the topological phases.

The Bernard-LeClair symmetries, which generalize the particle-hole symmetry, the time-reversal symmetry and the chiral symmetry of Hermitian systems to the more general case of non-Hermitian systems were derived in section 2 and it was shown how this lead to the definition of multiple new non-Hermitian symmetry classes analogous to the Hermitian AZ class. It was then shown that the new symmetries lead to the existence of certain types of eigenvalues pairs, including a generalized non-Hermitian version of the Kramer's theorem. Following this, it was shown that Hamiltonians corresponding to systems with point-gaps or real/imaginary line-gaps could be flattened to a unitary or Hermitian/anti-Hermitian Hamiltonian, respectively. These two flattening procedures would later be useful in the classification of the topological phases for the new non-Hermitian symmetry classes. Section 4 provided an outline of how the topological classification of the Bernard-LeClair symmetry classes could be derived by considering representations of the symmetry operators as generators of Clifford algebras. It was then shown how the generated Clifford algebras could be related to topological K-groups, which provided a classification of the topological phases for a given symmetry class. Next the topological classification of all the BL symmetry classes was derived in terms of the topological K-groups, and it was found that the total number of distinct symmetry classes were 54 and 38 for line- and point-gaps, respectively.

In section 5, a review of the dimensional raising procedure derived by Teo and Kane was provided. A generalization of the dimensional raising was then derived for the BL symmetry classes, for both point- and line-gaps, and it was shown that the Bott clocks, that were originally defined for the AZ class, could also be constructed for the new non-Hermitian symmetry classes.

To finish of the thesis a few examples were given of physical mechanisms that arise due to the non-Hermitian topology. This included the non-Hermitian skin effect and how it arise due to the point-gap topology of a non-Hermitian system. It was also shown that certain non-Hermitian perturbations of a Hermitian Hamiltonian could produce exceptional points in different ways depending on the nature of the perturbation. It was shown that it is possible to create a pair of exceptional points that provide a symmetry-protected non-trivial topological phase, where the distance between the exceptional points scaled with the magnitude of the perturbation. Similarly, it was shown that if the perturbation was defined differently, it would produce an ring of exceptional points.

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