Master's thesis

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Computation of Amplitudes using New Techniques

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Abstract

This thesis is concerned with computation of amplitudes at tree-level. More particularly it is a goal to apply already established computational techniques like Britto-Cachazo-Feng-Witten recursion relations that connect n-point amplitudes to a product of less than n-point amplitudes and the Kawai-Lewellen-Tye relations that establishes a connection between amplitudes of gravity to those of gauge theory at tree-level. These mentioned techniques are good examples of tools that effectively compute amplitudes which previously relied on heavy calculations using Feynman rules. An aim of the thesis will also be to implement these techniques numerically using Mathematica in order to realize and interpret gluonic amplitudes expressed in terms of purely fermionic amplitudes at tree-level.

Contents

Introduction

It is interesting to examine computations of 3- and 4-jet processes due to their crucial importance in experiments. Especially the techniques that for instance simplify computations of gluon amplitudes with n external legs. New computational techniques can revolutionize the field of computable amplitudes, for instance in cases where it is not enough to use numerical implementation of traditional Feynman diagrams. Detecting new physics demands a good understanding of background Standard Model processes [1]. Hence getting better at computing amplitudes of the background will certainly prove helpful when trying to reveal new physics.

There has already been a great amount of work done in the field of amplitude computations and many tree-level amplitudes have been worked out including amplitudes with matter and some proved as well. The new prospect that this thesis will address is computations of gluonic amplitudes using techniques like Kawai-Lewellen-Tye relations with matter, relating gravity amplitudes to amplitudes of gauge theory. The relations create the opportunity to relate gauge theory amplitudes of certain particles to gauge theory amplitudes with different particles. Hence one can establish relations between gluonic amplitudes and amplitudes with matter. Furthermore using the Bern-Carrasco-Johansson relations a reduction can be made in the amount of independent amplitudes to compute. More specifically, it is expected that some nice result will be uncovered relating gluonic amplitudes to matter amplitudes of gauge theory. But first, the journey will go through understanding many other computational techniques that each has some efficiency measured by their reduction in number of independent amplitudes.

In a way the subject of amplitude computations is concerned with clever representations that for instance factorizes the vast amount of amplitudes in such a way that many of the zero contributions are subtracted before computation is applied and before numerical methods are implemented.

Acknowledgments

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Chapter 1

S-matrix

Quantum field theory is used to understand the physics of high energy experiments. Mathematically the theory can be formulated using the path integral or S-matrix formalism when quantizing quantum theories, this then leads to a parallel diagrammatic description well-known as Feynman diagrams. Basically when one masters the Feynman rules, established upon quantization using the path integral, one is able to compute scattering amplitudes. Unfortunately, when setting out to compute amplitudes of scattering experiments, it may in some cases prove impossible when the amount of terms to compute exceeds the resources for numerical implementation or demands greater computational techniques than those available. This observation motivates and forms the ground for utilizing alternative approaches when doing computations of which some will be introduced in this thesis. First, a short introduction to the scattering amplitude that will be used widely throughout this thesis.

1.1 Scattering Amplitude

In high energy experiments the physics of particles can be extracted through the knowledge of scattering amplitudes [2]. Amplitudes constitute objects that are used in computing the probability for a certain process to occur in collision experiments. In such experiments the set-up is to collide incoming particles and detect the outcome. The particles are each described in terms of quantum states, which then in an interacting theory will have non-trivial overlap with other particles. Defining the *out* states to describe particles after the collision and *in* states to denote incoming particles one is able to construct an object proportional to the probability of a physical process. The overlap can then be defined as,

$$_{\text{out}}\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots | \boldsymbol{k}_A \boldsymbol{k}_B \rangle_{\text{in}} \equiv \langle \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots | S | \boldsymbol{k}_A \boldsymbol{k}_B \rangle \tag{1.1}$$

The *in* and *out* states are asymptotically defined, hence separated by an infinite amount of time. The S-matrix has an interesting part, which is responsible for the interactions of the colliding particles. This part is defined as the T-matrix in

$$S = \mathbf{1} + iT. \tag{1.2}$$

The scattering amplitude, $\mathcal{M}$, can now be constructed through the T-matrix as,

$$\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots | iT | \boldsymbol{k}_A \boldsymbol{k}_B \rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f) \cdot i\mathcal{M}(k_A, k_B \to p_f), \quad (1.3)$$

where an overall momentum conservation has been extracted, since four-momentum must be conserved in the experiment. In this thesis, $\mathcal{A}_n \equiv \mathcal{M}_n$ of the above formula, will denote the color-dressed amplitude. The label n denotes the number of external particles. Furthermore all particles of the amplitude will be outgoing, which corresponds to the convention used in most articles. Finally, a flat Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(+, -, -, -)$, will be used for summing Lorentz indices.

Chapter 2

Color Ordering of External Particles

Quantum Chromodynamics (QCD) is a non-abelian gauge theory with color dynamics. The group of gauge transformations correspond to the generators of the group $SU(3)$, which can be generalized to $SU(N_c)$ [3]. The latter theory then has N_c number of colors. Using the tools at hand one is actually able to go as far as to strip off color factors of several Feynman diagrams that belongs to a class of color-ordered amplitudes. This incredible observation is formulated in [3, 4].

2.1 Color Stripping

Gluons carry an adjoint color index $a = 1, 2, \ldots, N_c^2 - 1$, where the adjoint information tells that the indices are associated with the generators of the adjoint representation with the form f^{abc}. The quarks and anti-quarks carry an N_c and $\overline{N}_c$ index respectively in the fundamental representation where $i, \bar{j} = 1, 2, \ldots, N_c$. The generators of $SU(N_c)$ in the fundamental representation are $N_c \times N_c$ traceless hermitian matrices, $(T^a)_i^{\bar{j}}$, and their normalization can be chosen to be $\text{Tr}(T^a T^b) = \delta^{ab}$.

The color factors of the Feynman diagrams are, a $(T^a)_i^{\bar{j}}$ for each gluon-quark-quark vertex, a structure constant f^{abc}, defined through $[T^a, T^b] = i\sqrt{2}\,f^{abc}\,T^c$, for each pure gluon 3-vertex, and a contracted pair of structure constants, $f^{abe}f^{cde}$, for each pure gluon 4-vertex.

The framework for computing amplitudes using color Feynman rules amounts to very laborious work that can be efficiently simplified by expressing the color structure more explicitly through $(T^a)_i^{\bar{j}}$ such that,

$$
\begin{aligned}
\text{Tr}(T^a T^b T^c) &= \text{Tr}(T^c T^a T^b) = \text{Tr}(T^c(T^b T^a + [T^a, T^b])) \\
&= \text{Tr}(T^c T^b T^a) + i\sqrt{2}\,\text{Tr}(T^c f^{abd} T^d) \\
&= \text{Tr}(T^c T^b T^a) + i\sqrt{2}\,f^{abc}\text{Tr}(T^c T^d) \\
&= \text{Tr}(T^c T^b T^a) + i\sqrt{2}\,f^{abd}\delta^{cd} \\
&= \text{Tr}(T^c T^b T^a) + i\sqrt{2}\,f^{abc} \\
\Rightarrow f^{abc} &= -\frac{i}{\sqrt{2}}\Big(\text{Tr}(T^a T^b T^c) - \text{Tr}(T^a T^c T^b)\Big).
\end{aligned}
\tag{2.1}
$$

The above equation implies that the color part of the amplitude can be expressed as products of traces. So the exposed symmetry of color in this moment consist partially of cyclicity, one can say locally among a product of 3 generators (T^as). But actually the symmetry can be even further refined to a explicit structure of global traces for each color ordered amplitude, where global is used for an overall color factor of the color-ordered amplitude. This is realized using the definition of the trace and Fierz rearrangement,

$$(T^a)^{\bar{j}_1}_{i_1}(T^a)^{\bar{j}_2}_{i_2} = \delta^{\bar{j}_2}_{i_1}\delta^{\bar{j}_1}_{i_2} - \frac{1}{N_c}\delta^{\bar{j}_1}_{i_1}\delta^{\bar{j}_2}_{i_2}. \tag{2.2}$$

As an example (2.2) is applied to the color Feynman rule for contracting two gluon three vertices or equivalently one gluon four-vertex. The contraction is,

$$
\begin{aligned}
f^{abx}f^{xcd} =& \frac{-i}{\sqrt{2}}\Big(\mathrm{Tr}\big(T^aT^bT^x\big) - \mathrm{Tr}\big(T^xT^bT^a\big)\Big) \\
& \times \frac{-i}{\sqrt{2}}\Big(\mathrm{Tr}\big(T^xT^cT^d\big) - \mathrm{Tr}\big(T^dT^cT^x\big)\Big) \\
=& -\frac{1}{2}[\mathrm{Tr}(T^aT^bT^x)\mathrm{Tr}(T^xT^cT^d) - \mathrm{Tr}(T^aT^bT^x)\mathrm{Tr}(T^dT^cT^x) \\
& - \mathrm{Tr}(T^xT^bT^a)\mathrm{Tr}(T^xT^cT^d) + \mathrm{Tr}(T^xT^bT^a)\mathrm{Tr}(T^dT^cT^x)]. \tag{2.3}
\end{aligned}
$$

Examine the first term

$$
\begin{aligned}
\mathrm{Tr}(T^aT^bT^x)\mathrm{Tr}(T^xT^cT^d) =& T^a_{ij}T^b_{jk}T^x_{ki}T^x_{lm}T^c_{mn}T^d_{nl} \\
=& T^a_{ij}T^b_{jk}(\delta_{km}\delta_{il} - \frac{1}{N_c}\delta_{ki}\delta_{lm})T^c_{mn}T^d_{nl} \\
=& T^a_{ij}T^b_{jk}T^c_{kn}T^c_{ni} - \frac{1}{N_c}T^a_{ij}T^b_{ji}T^c_{ln}T^d_{nl} \\
=& \mathrm{Tr}(T^aT^bT^cT^d) - \frac{1}{N_c}\mathrm{Tr}(T^aT^b)\mathrm{Tr}(T^cT^d) \\
=& \mathrm{Tr}(T^aT^bT^cT^d) - \frac{1}{N_c}\delta^{ab}\delta^{cd}. \tag{2.4}
\end{aligned}
$$

Working out the other trace contractions will imply that the terms of order $\frac{1}{N_c}$ drop out due to the alternating sign.

$$
\begin{aligned}
\Rightarrow f^{abx}f^{xcd} =& -\frac{1}{2}[\mathrm{Tr}(T^aT^bT^cT^d) - \mathrm{Tr}(T^aT^bT^dT^c) - \mathrm{Tr}(T^bT^aT^cT^d) + \mathrm{Tr}(T^bT^aT^dT^c)] \\
=& -\frac{1}{2}\mathrm{Tr}\left([T^a, T^b][T^c, T^d]\right).
\end{aligned}
$$
$$\tag{2.5}$$

The invariance under the group of non-Abelian gauge transformations of the amplitude, originating from the Yang Mills Lagrangian, can be applied to ensure that the gluon amplitude is invariant under cyclic permutations of the external legs. Going one step further this amounts to a color decomposition of the n-gluon tree amplitude, where one thinks of the color factor as stripped off the kinematic part of the amplitude. The more technical explanation is to combine (2.1) and (2.2) such that the n-gluon amplitude is

reduced to a sum of single traces times respective kinematic parts. The kinematic part is called a color-ordered amplitudes, and these exhibit a color order of the external legs. One should think of the class of color-ordered amplitudes as sums of Feynman diagrams with a particular color-ordering. The result is,

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1^{\lambda_1}), \dots, \sigma(n^{\lambda_n})), \quad (2.6)$$

where g is the gauge coupling $(\frac{g^2}{4\pi} = \alpha_s)$ and k_i, λ_i are the external gluon momenta and corresponding helicity. The last part, $A_n^{\text{tree}}(1^{\lambda_1}, \dots, n^{\lambda_n})$ is the sub-amplitude that contains all the kinematic information. Henceforth, the word sub-amplitude will occasionally just be rephrased as color-ordered amplitude. Furthermore S_n is the set of all permutations of n objects and Z_n is the subset of cyclic permutations, which preserves the ordering of the generators in the trace. The combination of the two is such that the sum in (2.6) runs over all distinct cyclic orderings in the trace.

The sub-amplitudes in (2.6) have been separated from their individual color part, hence they are color-ordered amplitudes, a word that will be used widely in this thesis. Important attributes of the sub-amplitudes, that will reduce the number of independent amplitudes, are,

$$A_n(1, 2, \dots, n) = (-1)^n A_n(n, n-1, \dots, 2, 1), \quad (2.7)$$

which is a reflection symmetry and,

$$A_n(1, 2, \dots, n) = A_n(\sigma_l(1), \sigma_l(2), \dots, \sigma_l(n)), \quad (2.8)$$

where $\sigma_l(i) = i + l$ with $\sigma_1(n) = n + 1 = 1$ corresponding to a period of n labels, thus a map that cyclically permutes the labels on the external legs. Hence the sub-amplitudes has cyclic symmetry.

A closer look at the sub-amplitudes in (2.6) reveals gauge invariance. It has its origin in that the amplitude squared is gauge invariant order by order in N_c. Squaring the full amplitude implies products of color factors having the form [4],

$$\sum_{a_i=1}^{N_c^2-1} \text{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) \left[\text{Tr}(T^{b_1} T^{b_2} \cdots T^{b_n}) \right]^* = N_c^{n-2}(N_c^2 - 1)(\delta_{\{\alpha\}\{\beta\}} + \mathcal{O}(N_c^{-2})), \quad (2.9)$$

where $\delta_{\{\alpha\}\{\beta\}}$ is equal to one if and only if the permutations are the same or cyclically related. Notice that (2.9) means that the individual sub-amplitudes of (2.6) squared are gauge invariant, since the color is manifestly invariant due to its form in (2.9). The invariance will allow one to pick different gauges for every sub-amplitude computations, a property that will reduce the amount of Feynman diagrams. In the following section, concerning Helicity decomposition, this freedom allows for different suitable choices of reference momenta in the gluon polarization vectors.

2.2 Photon-Decoupling Relation

Amplitudes with different permutations of the external legs are related through the Dual Ward Identities in string theory. The analogue relations in quantum electrodynamics

(QED) are the Ward identities, where one external leg corresponding to a polarization of the amplitude is substituted for a momentum vector and thereby shown to give zero. This result was motivated by the current being conserved $\partial_\mu j^\mu(x) = 0$ for the classical equations of motion, thus it was expected to hold in quantum theory [2]. A similar result can be obtained for the color-ordered gluon tree amplitudes, nicely agreeing with the result of the colorless QED amplitude. Not relying on string theory, the observation is that gluons do not couple to photons, hence substituting a gluon for a photon in the amplitude should result in a vanishing relation. Now the substitution is manifestly performed in the color trace where a $T^a \to T^{a_{U(1)}}$. What this means is that the color decomposition of the full amplitude takes the form,

$$
\begin{aligned}
0 = \mathcal{A}_n^{\text{tree}}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} &\text{Tr}(T^{a_{\sigma(U(1))}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}) \\
&\times A_n^{\text{tree}}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n})),
\end{aligned} \tag{2.10}
$$

where one should interpret σ working on $U(1)$ as the permutation of the photon leg. The photon generator is $(T^{a_{U(1)}})_i^j = \frac{1}{\sqrt{N_c}} \delta_i^j$ and inserting it into (2.10) implies,

$$
0 = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n})). \tag{2.11}
$$

This means that the color-ordered sub-amplitudes have to vanish for each color factor, due to the latter being independent. Phrased differently one can dynamically understand the relation as: If leg one is assigned to the photon, then the cyclically color-ordered amplitudes sharing the same color coefficient are all the partially ordered permutations of the merged sets $\{\alpha\} = \{1\}$ and $\{\beta\} = \{2, \ldots, n\}$, meaning permutations that preserve the individual ordering of the two sets. A clever concept of permutation also used later with the BCJ relations, still to be explained. The corresponding relation for photon-decoupling is,

$$
\begin{aligned}
0 = &A_n^{\text{tree}}(1, 2, 3, \ldots, n) + A_n^{\text{tree}}(2, 1, 3, \ldots, n) + A_n^{\text{tree}}(2, 3, 1, \ldots, n) \\
&+ \cdots + A_n^{\text{tree}}(2, 3, \ldots, 1, n),
\end{aligned} \tag{2.12}
$$

for which the shared color factor would be $\text{Tr}(T^{a_2} T^{a_3} \cdots T^{a_n})$ of the amplitudes in the above photon-decoupling relation. The relation (2.12) was for a given permutation of $\{\beta\}$, but generally the result is valid for arbitrary color-ordered permutations of the set in accordance to $\sigma \in S_{n-1}/Z_{n-1}$ if the photon leg is fixed to some label. Of course including this label in the permutations is similar to a relabeling of the photon leg as in (2.11) indicated by $\sigma(1^{\lambda_1})$.

2.3 Amplitudes with Fermion Pairs

It is interesting to also examine amplitudes with matter, since gluons can decay into quark-antiquark pairs. These pairs can then be measured in a particle detector [5, 2]. These amplitudes will prove helpful later on, when questions are addressed on how to

take advantage of relations with amplitudes expressed in terms of amplitudes with different external particles. For instance gluonic amplitudes expressed in terms of matter amplitudes through the Kawai-Lewellen-Tye relations [6] and using the relations of supersymmetry (both explained later). In order to compute any full amplitude one needs the corresponding color factors. The color factors for quark pairs or gluino pairs are found in the following subsections.

Quark Pairs

In the case of tree amplitudes with quark-antiquark pairs the color factors can as well be stripped off the actual amplitude like in the gluon case [4]. The color factor is obtained by repeated use of (2.2) in color factors consisting of contractions between color factors of the fundamental representation $((T^{a_i})_i^{\bar{j}})$ and color factors of the adjoint representation (f^{abc}). The result is valid for diagrams with more than one quark pair,

$$\Lambda(\{n_i\}, \{\alpha\}) = \frac{(-1)^p}{N_c^p} (T^{a_1} \ldots T^{a_{n_1}})_{i_1 \alpha_1} (T^{a_{n_1+1}} \ldots T^{a_{n_2}})_{i_2 \alpha_2} \ldots (T^{a_{n_{m-1}+1}} \ldots T^{a_n})_{i_m \alpha_m} \tag{2.13}$$

with m being the number of $q\bar{q}$ pairs and n the number of external gluons. Here $i_1, \ldots, i_n$ are the color indices of the fermions and $\alpha_1, \ldots, \alpha_n$ are the color indices of the anti-fermions. The $n_1, \ldots, n_{m-1}$ is an arbitrary partition of an arbitrary permutation of n gluons legs where $1 \leq n_i \leq n$. To indicate that a $q\bar{q}$ is connected by a fermion line the corresponding α of the anti-fermion is assigned to $\bar{i}$ such that $\{\alpha\} = (\alpha_1, \ldots, \alpha_m)$ is a generic permutation of $\{\bar{i}\} = (\bar{i}_1, \ldots, \bar{i}_m)$. The power p is determined by the the number of times that α_i coincides with $\bar{i}_j$. For example in the case $\{\alpha\} \equiv \{\bar{i}\}$ one gets $p = m - 1$. A quick check can be done using (2.2) on the case of one $q\bar{q}$ pair where $p = 0$ for $(\bar{i}_1, \bar{i}_2) = (\alpha_2, \alpha_1)$ which is exactly the first term. And if $(\bar{i}_1, \bar{i}_2) = (\alpha_1, \alpha_2)$ then $p = 2 - 1 = 1$ which is exactly the order of the second term in (2.2). In the pure fermionic amplitude, i.e. one with only fermion pairs, the color structure simplifies to,

$$D(\{\alpha\}) = \frac{(-1)^p}{N^p} \delta_{i_1 \alpha_1} \delta_{i_2 \alpha_2} \ldots \delta_{i_m \alpha_m}. \tag{2.14}$$

And the colordressed amplitude with m quark pairs and n gluons then becomes,

$$\mathcal{M}_{m,n} = \sum \Lambda_p(\{n_i\}, \{\alpha\}) \widetilde{m}^p_{\{n_i\}, \{\alpha\}}(q, h). \tag{2.15}$$

where the sum is over all permutations of external particles just as in (2.6) and $\widetilde{m}$ is a sub-amplitude depending on kinematic and helicity parameters h of the particles involved. Also note that p identifies the color factor with its color-ordered amplitude in (2.15).

Gluino Pairs

The color factors are almost the same, but gluino pairs do not have to be adjacent pairs like for the quark case with equal flavor and in a form of a particle-anti-particle pair, instead they are allowed to mix flavor and come in any pairs they want. Therefore they
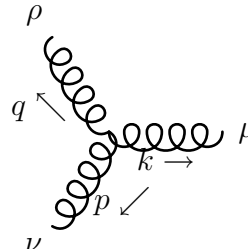
do not carry any indices of the fundamental representation as seen in their color trace factor,

$$\mathcal{M}_{n+2}(g_1, \ldots, g_n, \Lambda_{n+1}, \Lambda_{n+2}) = \sum_{\{1,\ldots,n+2\}'} \text{Tr}(\lambda^{a_1} \lambda^{a_2} \cdots \lambda^{a_{n+2}})$$
$$\times m_\Lambda(p_1, \ldots, p_n, q_{n+1}, q_{n+2}), \qquad (2.16)$$
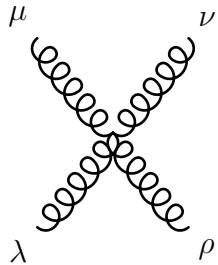
where momenta labeled with q are fermionic and $\{1, \ldots, n+2\}'$ denotes a set of non-cyclic permutations. A more elaborated treatment is still to come in a later chapter on supersymmetry with gluino amplitudes and their Super Ward identities that mixes fermions with bosons.

2.4 Color Feynman Rules

The color-ordered Feynman rules in the Lorentz-Feynman gauge are for tree-level diagrams stated below [3],

$$= \frac{i}{\sqrt{2}}(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho),$$

$$= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho}),$$

$$= \frac{i}{\sqrt{2}}\gamma_\mu,$$

$$= -\frac{i}{\sqrt{2}}\gamma_\mu,$$

μ ⚬⚬⚬⚬⚬⚬⚬ ν $\qquad = -i\frac{\eta_{\mu\nu}}{p^2},$

⟶ $\qquad = \frac{i}{\not{p}}.$

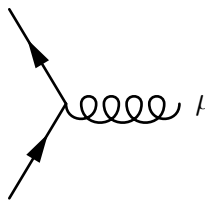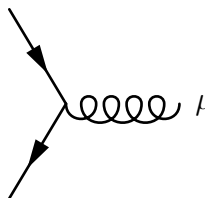In these diagrams straight lines represent fermions, wavy lines gluons and all momenta are outgoing. These have to be combined with their corresponding color part in the color decomposition (2.6) for gluons or in (2.15) for mixed gluon-fermion amplitudes in order to obtain the full color dressed amplitude. To use the color-ordered Feynman rules one should draw all color-ordered graphs, i.e. all planar diagrams (tree-level diagrams have no crossing lines in the plane). It can be shown that the class of planar diagrams will dominate in the large N_c limit [7], which is an encouraging fact for calculating tree-level diagrams along with applications to experiments. It should be mentioned that this class also holds loop diagrams in which no lines cross.

The three gluon vertex [3] exhibits the attribute of being anti-symmetric under permutation of two indices. Let $\mu \leftrightarrow \nu$ and $k \leftrightarrow p$,

$$
\begin{aligned}
(\text{3-vertex})_{\mu\nu\rho} &= \frac{i}{\sqrt{2}}(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho) \\
\Rightarrow (\text{3-vertex})_{\nu\mu\rho} &= \frac{i}{\sqrt{2}}(\eta_{\mu\rho}(k-q)_\nu + \eta_{\rho\nu}(q-p)_\mu + \eta_{\nu\mu}(p-k)_\rho) \\
&= -(\text{3-vertex})_{\mu\nu\rho}.
\end{aligned}
\tag{2.17}
$$

which is an attribute used later in the chapter on Bern-Carrasco-Johansson relations, when expanding 4-point amplitudes in terms of numerators. The above stated color Feynman rules offers a method to "compute" amplitudes, though this method quickly turns out to be incomprehensible in the way that the amount of color Feynman diagrams to compute builds up unconvincingly fast already in the 5-point tree-level gluon amplitude. Feynman diagrams are nice in, but they are not gauge invariant and when it comes to evaluation they demand heavy computations even for low n-point amplitudes. The part of the problem concerning computation suddenly needs a whole other kind of thinking. To quickly analyze the 5-point case one gets tree diagrams with one or two propagators and these are then evaluated with the color-ordered Feynman rules and additionally using gluon polarization (introduced in next chapter) for each external gluon leg. In the latter case one is forced to evaluate 15 Feynman diagrams or 5 color-ordered diagrams [5].

Chapter 3

Helicity Decomposition

The spinor helicity formalism is a great tool in eliminating many Feynman diagrams. Efficiency is measured in how the non-zero contributions are extracted, for instance for particular helicity structures an amiable compact formula exists. These are the maximally-helicity-violating amplitudes that were found by Parke-Taylor [8] and later used on an extensive scale in many tree-level computations and also used as basic ingredients in Cachazo-Svrcek-Witten recursion introduced in next chapter. But first, the subject of this chapter will be to introduce the spinor helicity formalism [3].

3.1 Spinor Helicity Formalism

Spinors are known as solutions to the Dirac equation, a relativistic equation motivated by the non-relativistic Shrödinger equation. The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0, \tag{3.1}$$

and the formalism to be explained will be concerned with the massless case.

The positive and negative energy solutions of definite helicity in the massless case are respectively $u_\pm(k) = \frac{1}{2}(1 \pm \gamma_5)u(k)$ and $v_\mp(k) = \frac{1}{2}(1 \pm \gamma_5)v(k)$ each with a subindex $\pm$ indicating the helicity. Similar relations are obtained for the conjugate spinors $\overline{u_\pm(k)} = \overline{u(k)}\frac{1}{2}(1 \mp \gamma_5)$ and $\overline{v_\mp(k)} = \overline{v(k)}\frac{1}{2}(1 \mp \gamma_5)$. The spinors and spinor product are defined for later use.

$$|i^\pm\rangle \equiv |k_i^\pm\rangle \equiv u_\pm(k_i) = v_\mp(k_i), \quad \langle i^\pm| \equiv \langle k_i^\pm| \equiv \overline{u_\pm(k_i)} = \overline{v_\mp(k_i)}, \tag{3.2}$$

$$\langle i\,j\rangle \equiv \langle i^-|j^+\rangle = \overline{u_-(k_i)}u_+(k_j), \quad [i\,j] \equiv \langle i^+|j^-\rangle = \overline{u_+(k_i)}u_-(k_j), \tag{3.3}$$

where $\bar{u} = u^\dagger\gamma^0$. In order for the above formalism to make sense a choice has to be made regarding the representation of the γ-matrices. Using the Dirac representation the matrices take the following form,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.4}$$

and the massless spinors can be chosen as,

$$u_+(k) = v_-(k) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{k^+} \\ \sqrt{k^-}e^{i\varphi_k} \\ \sqrt{k^+} \\ \sqrt{k^-}e^{i\varphi_k} \end{bmatrix} , \quad u_-(k) = v_+(k) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{k^-}e^{-i\varphi_k} \\ -\sqrt{k^+} \\ -\sqrt{k^-}e^{-i\varphi_k} \\ \sqrt{k^+} \end{bmatrix} , \quad (3.5)$$

where the spinor indices have been suppressed and

$$e^{\pm i\varphi_k} \equiv \frac{k^1 \pm ik^2}{\sqrt{(k^1)^2 + (k^2)^2}} = \frac{k^1 \pm ik^2}{\sqrt{k^+k^-}} , \qquad k^\pm = k^0 \pm k^3. \quad (3.6)$$

With the details so far of this section explicit expressions for the spinor products can be obtained,

$$\langle i\,j\rangle = \sqrt{|s_{ij}|}e^{i\phi_{ij}} \quad (3.7)$$

$$[i\,j] = \sqrt{|s_{ij}|}e^{-i(\phi_{ij}+\pi)}. \quad (3.8)$$

See appendix A for further details.

The following identities will prove very useful when computing amplitudes.

$$\langle i\,j\rangle [j\,i] = \langle i^-|j^+\rangle\langle j^+|i^-\rangle = \text{Tr}\big(\frac{1}{2}(1-\gamma_5)\,\not{k}_i\,\not{k}_j\big) = 2k_i \cdot k_j = s_{ij}, \quad (3.9)$$

where $s_{ij} = (k_i+k_j)^2 = 2k_i \cdot k_j$ (some papers use instead $\langle i\,j\rangle [i\,j] = s_{ij}$). The Gordon identity and projection operator,

$$\langle i^\pm|\gamma^\mu|i^\pm\rangle = 2k_i^\mu, \qquad |i^\pm\rangle\langle i^\pm| = \frac{1}{2}(1\pm\gamma_5)\,\not{k}_i. \quad (3.10)$$

Antisymmetry of the spinor products,

$$\langle j\,i\rangle = -\langle i\,j\rangle, \qquad [j\,i] = -[i\,j], \qquad \langle i\,i\rangle = [i\,i] = 0. \quad (3.11)$$

Fierz rearrangement,

$$\langle i^+|\gamma^\mu|j^+\rangle\langle k^+|\gamma_\mu|l^+\rangle = 2\,[i\,k]\,\langle l\,j\rangle. \quad (3.12)$$

Charge conjugation of current,

$$\langle i^+|\gamma^\mu|j^+\rangle = \langle j^-|\gamma^\mu|i^-\rangle. \quad (3.13)$$

Schouten identity,

$$\langle i\,j\rangle\,\langle k\,l\rangle = \langle i\,k\rangle\,\langle j\,l\rangle + \langle i\,l\rangle\,\langle k\,j\rangle. \quad (3.14)$$

Furthermore for the n-point amplitude, momentum conservation, $\sum_{i=1}^n k_i^\mu = 0$, provides an identity when combined with (3.9),

$$\sum_{\substack{i=1 \\ i\neq j,k}}^n [j\,i]\,\langle i\,k\rangle = 0. \quad (3.15)$$

The polarization vector is for a massless gauge boson of helicity ± 1,

$$\varepsilon_\mu^\pm(k, q) \;=\; \pm \frac{\langle q^\mp | \gamma_\mu | k^\mp \rangle}{\sqrt{2} \langle q^\mp | k^\pm \rangle} \;, \tag{3.16}$$

where q denotes the reference momenta a redundant degree of freedom in the polarization that can be fixed by a choice of gauge [9]. Remember that the sub-amplitudes each are gauge invariant, hence it leaves a vast amount of possibilities to reduce computations in a clever way.

As a transverse polarization this should have the property of being orthogonal to the momentum itself and it is later confirmed to be the case for the above polarization. This is seen by $\not{k}|k^\pm\rangle = 0$ in $\varepsilon^\pm(k, q)$. Thus $\varepsilon^\pm(k, q)$ is transverse to k for any q. The relations can be stated as,

$$\varepsilon^\pm(k, q) \cdot k = 0. \tag{3.17}$$

In classical optical sense this corresponds to a polarization that lies in a plane perpendicular to the direction of the momentum. Thus it can be stated that the polarization is not longitudinal but maybe is transverse or circularly polarized. Since the case concerns helicity the circular polarization is what one should have in mind. And it is confirmed using complex conjugation, since the helicity is reversed by complex conjugation,

$$(\varepsilon_\mu^+)^* = \varepsilon_\mu^-. \tag{3.18}$$

For clockwise and anti-clockwise circular polarizations the dot product is expected to be $\eta_{11} \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}) \cdot \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}) = -(1+1)\frac{1}{2} = -1$ and the dot product of equal circular polarizations should equal zero. The calculated standard normalization using spinor helicity formalism is,

$$\varepsilon^+ \cdot (\varepsilon^+)^* \;=\; \varepsilon^+ \cdot \varepsilon^- \;=\; -\frac{1}{2} \frac{\langle q^- | \gamma_\mu | k^- \rangle \langle q^+ | \gamma^\mu | k^+ \rangle}{\langle q\,k \rangle [q\,k]} \;=\; -1,$$

$$\varepsilon^+ \cdot (\varepsilon^-)^* \;=\; \varepsilon^+ \cdot \varepsilon^+ \;=\; \frac{1}{2} \frac{\langle q^- | \gamma_\mu | k^- \rangle \langle q^- | \gamma^\mu | k^- \rangle}{\langle q\,k \rangle^2} \;=\; 0. \tag{3.19}$$

Henceforth the polarization in the spinor helicity formalism agrees with the conditions for a polarization in a classical optical sense. Although the real understanding of helicity is that the quantum states are separated into a direct sum of contrasting helicity parts expressed in terms of chiral projections $\frac{1}{\sqrt{2}}(1 \pm \gamma^5)$, which is seen in the Weyl decomposition. The helicity appropriate for a polarization vector is ± 1, this is easily seen by doing a rotation around the k axis for which the overall phase doubles, thus the difference in phase for the vector helicity states is enlarged which is exhibited in $1 = \frac{1}{2} - (-\frac{1}{2}) \rightarrow 1 - (-1) = 2$ in terms of the assigned helicities for respectively a spinor and a vector.

The choice of reference momenta can be shown to correspond to a choice of gauge. And changing the reference momentum just corresponds to an on-shell gauge transformation, since the polarization vector shifts with an object proportional to the momentum and thereby is unchanged. However, separate reference momenta can be chosen for each

external gluon momenta, but one should be careful when calculating gauge invariant quantities and not change the reference momenta within the calculation. Such gauge invariant pieces are for instance the sub-amplitudes, such that one choice of gauge is allowed each piece in the color decomposition. As mentioned in the beginning of this section the choice of reference momenta is the key in substantially simplifying helicity computations. The choice almost seems arbitrary, but one should avoid the singularity that occurs when choosing $q_i = k_i$ for $\varepsilon_i^\pm$ in the denominator of (3.16). The following identities are widely used for doing simplifications where $\varepsilon_i^\pm(q) \equiv \varepsilon^\pm(k_i, q_i = q)$,

$$\varepsilon_i^\pm(q) \cdot q = 0, \tag{3.20}$$

$$\varepsilon_i^+(q) \cdot \varepsilon_j^+(q) = \varepsilon_i^-(q) \cdot \varepsilon_j^-(q) = 0, \tag{3.21}$$

$$\varepsilon_i^+(k_j) \cdot \varepsilon_j^-(q) = \varepsilon_i^+(q) \cdot \varepsilon_j^-(k_i) = 0, \tag{3.22}$$

$$\not{\varepsilon}_i^+(k_j)|j^+\rangle = \not{\varepsilon}_i^-(k_j)|j^-\rangle = 0, \tag{3.23}$$

$$\langle j^+| \not{\varepsilon}_i^-(k_j) = \langle j^-| \not{\varepsilon}_i^+(k_j) = 0. \tag{3.24}$$

It will prove useful to notice the advantage of choosing reference momenta of like helicity gluons to be identical, and to equal the external momenta of an opposite helicity set of gluons. Furthermore one can use symmetries such as parity to reverse all helicities in an amplitude and charge conjugation to exchange quarks and anti-quarks or equivalently flip the helicity on a quark line (see appendix B).

A choice of convention regarding helicities is made in which one assigns helicity labels to the particles when they are considered outgoing with positive energy, and if they are ingoing the helicity is reversed.

3.2 MHV Amplitude Formula

As mentioned earlier a famous result for gluonic tree-level color-ordered amplitudes was found. Maximally-helicity-violating amplitudes (MHV) was conjectured by Park-Taylor [8] in the 1980s and later on proved by Berends-Giele in [10]. The formula states,

$$A_n^{\text{tree,MHV},jk} \equiv A_n^{\text{tree}}(1^+, \ldots, j^-, \ldots, k^-, \ldots, n^+) = i\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle}, \tag{3.25}$$

where all external momenta are outgoing. An MHV amplitude is color stripped and defined to have two external legs of negative helicity and the rest of positive helicity, which means that the dots in the above formula are filled out with gluonic legs of positive helicity. Another result very useful result is [4],

$$A_n^{\text{tree}}(1^\pm, 2^+, \ldots, n^+) = 0, \quad n > 3. \tag{3.26}$$

That the MHV amplitude is invariant under cyclic permutations and reflection symmetry of the external legs is best examined in an easy case. First, the cyclicity is illustrated to hold in the denominator and numerator of the 3-point MHV tree amplitude,

$$\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle \quad \rightarrow \quad \langle 3\,1\rangle\,\langle 1\,2\rangle\,\langle 2\,3\rangle = \langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle\,. \tag{3.27}$$

The numerator, $\langle j\,k \rangle^4$, is unchanged. Hence generalizing the above computation to the n-point case only requires the observation that the spinor products $\langle i\,j \rangle$ move one step to the right for a cyclic permutation $i \to i+1$. Formally the product of spinor products changes as,

$$\langle 1\,2 \rangle \langle 2\,3 \rangle \cdots \langle (n-1)\,n \rangle \langle n\,1 \rangle \to \langle n\,1 \rangle \langle 1\,2 \rangle \langle 2\,3 \rangle \cdots \langle (n-2)\,(n-1) \rangle \langle (n-1)\,n \rangle . \quad (3.28)$$

Second, the reflection symmetry is confirmed to hold in the following analysis. Consider again the 3-point case as above of which the denominator upon reflection changes as,

$$\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle \;\to\; \langle 3\,2 \rangle \langle 2\,1 \rangle \langle 1\,3 \rangle = (-)^3 \langle 2\,3 \rangle \langle 1\,2 \rangle \langle 3\,1 \rangle = -\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle , \quad (3.29)$$

confirming the reflection symmetry using anti-symmetry of the spinor product. The numerator is still unchanged now due to the even power equal to four. This easily generalizes to n-point pointserving that the operation of reflection is to switch the order of the indices inside every spinor product and of course also reverse the order of factors in the product of spinor products. Finally, the MHV formula (3.25) is widely used in computations of amplitudes, and in the next chapter on CSW recursion it is illustrated how to efficiently combine it with recursion techniques. In CSW recursion the diagrammatic part of the computation consist of scalar diagrams (different Feynman diagrams) with vertices interpreted as MHV amplitudes connected by scalar propagators, a funny competitor to the Feynman rules.

Before ending this section some analysis and examples should be included. The result (3.26) is best understood by examining the color-ordered Feynman rules. Looking at the constituents of the amplitude the 3-vertices for gluons on page 9 implies that contractions occur for polarization vectors. From (3.20)-(3.24) the observation is that $\varepsilon_i \cdot \varepsilon_j$ will comply with the choice of reference momenta, thus implying a vanishing possibility. In more detail an n-point amplitude can at most have $n-2$ vertices, case with all vertices equal to 3-vertices, that each can at most be contracted with $n-2$ momenta vectors. So $n-(n-2)=2$ is the number of non-contracted polarization vectors, implying that the amplitude will be linear in $\varepsilon_i \cdot \varepsilon_j$. Exchanging any of the 3-vertices for 4-vertices will not change the outcome of the argument, since this just amounts to a higher number of polarization vectors to be contracted. The idea is to find a choice of reference momenta forcing all contracted polarizations to be zero, implying a total vanishing of the amplitude. This will minimize the number of amplitudes to compute. The vanishing of $A_n^{tree}(1^+, 2^+, \dots, n^+)$ is satisfied using (3.21) by taking all reference momenta to be the same, for instance equal to a null momentum [3]. If one helicity is opposite to the rest, e.g. $A_n^{tree}(1^-, 2^+, \dots, n^+)$, then this is seen to vanish by choosing the reference momenta to be $q_1 = k_n, q_2 = q_3 = \cdots = q_n = k_1$. One new contraction appears,

$$\varepsilon_j^+(k_1) \cdot \varepsilon_1^-(k_n) = 0, \quad \text{for } j = 2, \dots, n \quad , \quad (3.30)$$

which is seen to vanish by (3.22). However non of the analyzed case were of MHV type, thus an example should be given in order to demonstrate the labour that is avoided using (3.25) instead of the Feynman color-ordered rules.

MHV 4-Point Gluonic Amplitudes

A 4-point tree-level MHV amplitude is computed using the color-ordered Feynman rules to draw all possible distinguishable Feynman diagrams one gets for a specific color-order $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$,

$$(3.31)$$

where only the first diagram has a non-zero contribution. This observation is supported when choosing the reference momenta to be $q_1 = q_2 = k_4$ and $q_3 = q_4 = k_1$, since then the only non-vanishing contraction of polarization vectors is $\varepsilon_2^- \cdot \varepsilon_3^+$. However, it does not hurt to work out the details. The first diagram in (3.31) is,

$$
\begin{aligned}
(1) = {}& \varepsilon_1^{-\,\nu}\varepsilon_2^{-\,\rho}\frac{i}{\sqrt{2}}(\eta_{\nu\rho}(k_1 - k_2)_\alpha + \eta_{\rho\alpha}(k_2 - k)_\nu + \eta_{\alpha\nu}(k - k_1)_\rho)(-i\frac{\eta^{\alpha\beta}}{s_{12}}) \\
& \times \varepsilon_3^{+\,\lambda}\varepsilon_4^{+\,\mu}\frac{i}{\sqrt{2}}(\eta_{\lambda\mu}(k_3 - k_4)_\beta + \eta_{\mu\beta}(k_4 + k)_\lambda + \eta_{\beta\lambda}(-k - k_3)_\mu) \\
= {}& -\left(\frac{i}{\sqrt{2}}\right)^2\left(\frac{-i}{s_{12}}\right)4(\varepsilon_1^- \cdot k_2)(\varepsilon_2^- \cdot \varepsilon_3^+)(\varepsilon_4^+ \cdot k_3) \\
= {}& -4\left(\frac{i}{\sqrt{2}}\right)^2\left(\frac{-i}{s_{12}}\right)\left(-\frac{\langle 4^+|\,\slashed{k}_2\,|1^+\rangle}{\sqrt{2}\,[4\,1]}\right)\left(-\frac{\langle 4^+|\,\gamma_\mu\,|2^+\rangle}{\sqrt{2}\,[4\,2]}\right) \\
& \times \left(\frac{\langle 1^-|\,\gamma^\mu\,|3^-\rangle}{\sqrt{2}\,\langle 1\,3\rangle}\right)\left(\frac{\langle 1^-|\,\slashed{k}_3\,|4^-\rangle}{\sqrt{2}\,\langle 1\,4\rangle}\right) \\
= {}& i\frac{\langle 1\,2\rangle\,[4\,3]^2}{[1\,2]\,\langle 1\,4\rangle\,[4\,1]},
\end{aligned}
\tag{3.32}
$$

where $k = -(k_1 + k_2) = k_3 + k_4$ is the momentum of the propagator. The second diagram in (3.31) is,

$$
\begin{aligned}
(2) = {}& \varepsilon_4^{+\,\nu}\varepsilon_1^{-\,\rho}\frac{i}{\sqrt{2}}(\eta_{\nu\rho}(k_4 - k_1)_\mu + \eta_{\rho\mu}(k_1 - k)_\nu + \eta_{\mu\nu}(k - k_4)_\rho) \\
& \times \left(-i\frac{\eta^{\mu\alpha}}{s_{14}}\right)\varepsilon_3^{+\,\lambda}\varepsilon_2^{-\,\sigma}\frac{i}{\sqrt{2}}(\eta_{\lambda\sigma}(k_3 - k_2)_\alpha + \eta_{\sigma\alpha}(k_2 + k)_\lambda + \eta_{\alpha\lambda}(-k - k_3)_\sigma) \\
= {}& \left(\frac{i}{\sqrt{2}}\right)^2(\varepsilon_1^-{}_\mu\varepsilon_4^+ \cdot (2k_1 + k_4) + \varepsilon_4^+{}_\mu\varepsilon_1^- \cdot (-k_1 - 2k_4)) \times \dots \\
= {}& 0,
\end{aligned}
\tag{3.33}
$$

due to the polarization being transverse to its own momenta and due to (3.20). The third diagram in (3.31) is,

$$
\begin{aligned}
(3) &= \varepsilon_1^{-\lambda}\varepsilon_2^{-\mu}\varepsilon_3^{+\nu}\varepsilon_4^{+\rho}\left(i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})\right) \\
&= i\left(\varepsilon_2^- \cdot \varepsilon_4^+ \varepsilon_1^- \cdot \varepsilon_3^+ - \frac{1}{2}(\varepsilon_2^- \cdot \varepsilon_3^+ \varepsilon_1^- \cdot \varepsilon_4^+ + \varepsilon_2^- \cdot \varepsilon_1^- \varepsilon_3^+ \cdot \varepsilon_4^+)\right) \\
&= 0,
\end{aligned}
\tag{3.34}
$$

since only $\varepsilon_2^- \cdot \varepsilon_3^+ \neq 0$ and all other contractions are zero. Summing the three terms, of which two are zero, one gets,

$$
\begin{aligned}
A_4^{tree}(1^-, 2^-, 3^+, 4^+) &= i\frac{\langle 1\,2\rangle\,[4\,3]^2}{[1\,2]\,\langle 1\,4\rangle\,[4\,1]} \\
&= i\frac{\langle 1\,2\rangle^2}{\langle 1\,2\rangle\,[1\,2]\,\langle 1\,4\rangle\,[4\,1]}\left(\frac{\langle 1\,2\rangle\,[2\,1]}{\langle 3\,4\rangle}\right)^2 \\
&= -i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 4\,1\rangle\,\langle 3\,4\rangle}\frac{[2\,1]}{[4\,1]\,\langle 3\,4\rangle} \\
&= -i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 4\,1\rangle\,\langle 3\,4\rangle\,\langle 2\,3\rangle}\frac{[1\,2]\,\langle 1\,4\rangle}{[3\,2]\,\langle 3\,4\rangle} \\
&= -i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 4\,1\rangle\,\langle 3\,4\rangle\,\langle 2\,3\rangle}\frac{-\langle 4\,3\rangle\,[3\,2]}{\langle 4\,3\rangle\,[3\,2]} \\
&= i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle},
\end{aligned}
\tag{3.35}
$$

where momentum conservation was used by applying $s_{12} = s_{43}$ and $s_{23} = s_{14}$ and (3.15). The result in (3.35) nicely reproduces the MHV result in (3.25).

3.3 MHV Gravity Amplitudes

As already stated the MHV result for gluons has a nice and compact expression. The result also exist in a gravity version, hence for amplitudes with gravitons as external legs [11].

Chapter 4

CSW Relation

The Cachazo-Svrcek-Witten (CSW) relation is a clever representation of color-ordered tree amplitudes in terms of tree-level Feynman scalar diagrams with fictitious MHV vertices corresponding to MHV amplitudes continued off-shell [12, 13]. This clever method splits the color-ordered amplitude partly into products of MHV amplitudes and scalar propagators. The discovery was motivated by a string theory on twistor space introduced in [9].

The CSW diagrams are tree diagrams constructed in such a way that the number of MHV vertices depends on the number of negative helicity legs in the color-ordered amplitude. Thus the MHV vertices may have several legs going out. And of course the MHV vertices have exactly two negative helicity legs. In the following these kind of scalar diagrams will instead be called MHV diagrams. Finally, a field theory proof exists of CSW recursion relation for tree-level gluon amplitudes in [14]. And in [15] the class of MHV vertices is extended to also include MHV vertices with one and two quark-antiquark lines.

4.1 MHV Vertex

First a short detour, returning to the Dirac equation (3.1). The Dirac representation $(1/2, 1/2)$ is a reducible representation due to its block diagonal form. It can thus be decomposed into a Weyl representation like $(1/2, 0)$ and $(0, 1/2)$ with 2-dimensional spinors. For the massless case this implies Weyl equations of the form [2],

$$i\bar{\sigma} \cdot \partial \psi_L = 0 \qquad i\sigma \cdot \partial \psi_R = 0, \tag{4.1}$$

where ψ_R, ψ_L are the right and left-handed Weyl two component spinors, with respectively positive and negative helicity and $\sigma^\mu \equiv (1, \boldsymbol{\sigma}), \bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$. Applying a Fourier transform to momentum space the helicities of the Weyl spinors are correct, when helicity is defined as,

$$h \equiv \frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{4.2}$$

such that a right-handed particle has $h = +\frac{1}{2}$ from $\frac{1}{2}\sigma_i \hat{p}_i \psi_R = \frac{1}{2}\psi_R$ by (4.1). Now define the two component Weyl spinors to be for an n-particle process [9],

$$(\lambda_i)_a \equiv [u_+(k_i)]_a, \qquad (\widetilde{\lambda}_i)_{\dot{a}} \equiv [u_-(k_i)]_{\dot{a}}, \qquad i = 1, 2, \ldots, n \quad , \tag{4.3}$$

in respectively $(1/2, 0)$ and $(0, 1/2)$ where $a, \dot{a} = 1, 2$. The spinor products are needed in the following and are related to the two component spinors through,

$$\langle j\, l \rangle = \varepsilon^{\alpha\beta}(\lambda_j)_\alpha(\lambda_l)_\beta = \bar{u}_-(k_j)u_+(k_l) \tag{4.4}$$

$$[j\, l] = \varepsilon^{\dot{\alpha}\dot{\beta}}(\widetilde{\lambda}_j)_{\dot{\alpha}}(\widetilde{\lambda}_l)_{\dot{\beta}} = \bar{u}_+(k_j)u_-(k_l) \tag{4.5}$$

where $u_\pm$ corresponds to the four component spinors of chapter 3.

As already found in the previous chapter the color-ordered tree-level MHV amplitude can crudely be stated to have a structure like,

$$A_n = \frac{\langle \lambda_x\, \lambda_y \rangle^4}{\prod_{i=1}^n \langle \lambda_i\, \lambda_{i+1} \rangle} \tag{4.6}$$

where color factor, delta function for energy-momentum conservation and the Yang-Mills coupling constant are omitted for the amplitude. The goal is to use MHV amplitudes to represent the vertex in the MHV diagram. The notation used explicitly indicates the appearance of the two component spinors. In order to be able to insert MHV amplitudes for vertices, the MHV amplitudes needs to be continued off-shell, since it will be connected to intermediate propagators and since MHV amplitudes by construction are on-shell. The underlying framework is only shortly motivated in this section, since similar tools regarding off-shell shifts are needed in the next section on BCFW recursion. Quickly explained, a form of the bi-spinor momentum (elaborated in chapter 5.2) is,

$$p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}, \tag{4.7}$$

which is valid up to a scaling for the on-shell case. In order to continue the momentum off-shell a shift is needed of its spinor. If one then picks an off-shell spinor $\lambda_a = q_{a\dot{a}}\eta^{\dot{a}}$, then the on-shell momentum can be shifted in the case of and intermediate propagator with momentum $q = p_1 + p_2$. Thus an i'th leg with on-shell momentum p_i goes off-shell like,

$$\begin{aligned} p_{i,a\dot{a}} = \lambda_{i,a}\widetilde{\lambda}_{i,\dot{a}} &\rightarrow q_{a\dot{b}}\eta^{\dot{b}}\widetilde{\lambda}_{i,\dot{a}} \\ &= (\lambda_{1\,a}\widetilde{\lambda}_{1\,\dot{b}} + \lambda_{2\,a}\widetilde{\lambda}_{2\,\dot{b}})\eta^{\dot{b}}\widetilde{\lambda}_{i,\dot{a}}, \end{aligned} \tag{4.8}$$

hence notice the bi-spinor momentum suddenly increases in rank from one to two by the continuation. λ_i is the spinor of the i'th particle. The interpretation of unshifted λ for other external legs that are on-shell is unchanged. Continuing the MHV amplitude off-shell is done by shifting a λ_j in (4.6).

4.2 MHV Feynman Rules

The point has come to sample the puzzle using off-shell MHV amplitudes for MHV vertices. The amplitudes used for the MHV vertices are the mostly plus amplitudes

remembering that mostly minus amplitudes can be computed from the mostly plus ones (see appendix B). Sewing together the MHV diagrams a helicity is assigned to each gluon and the off-shell intermediate propagators are $\frac{1}{p^2}$. A simple analysis tells that in counting the number of vertices, v, of a n-leg tree-level gluon amplitude with q for the number of negative helicity gluons, there will be $v-1$ number of propagators connecting the vertices. The total color-ordered amplitude is then the sum of all possible MHV diagrams.

Furthermore, the total number of negative gluons appearing in the MHV diagram, counting external as well as internal gluons, is $2v$ since each vertex has two gluons of negative helicity. The propagators connect at one end with a negative helicity to each of the vertices, due to the flip in helicity of the gluon based on incoming gluons of one helicity is equivalent to outgoing gluons of opposite helicity. Indeed, the number of external gluon legs with negative helicity is $q = 2v - (v - 1) = v + 1$. Alternatively, when setting out to draw all MHV diagrams the preferred form is,

$$v = q - 1. \tag{4.9}$$

The MHV Feynman rules are: (1) draw all possible tree graphs of v vertices and $v - 1$ links assigning opposite helicity to the two ends of internal lines, (2) distribute the external gluons among the vertices in all possible ways while preserving cyclic ordering of the color-ordered amplitude.

Checking that the CSW method fits in the case of a color-ordered MHV amplitude ($q = 2$) one gets that $v = 2 - 1 = 1$. Applying the MHV Feynman rules, draw one MHV vertex, which exactly just returns the MHV amplitude. The vanishing result for $q < 2$ holds and is implied by a count of zero number of vertices to draw when $v < 1$. A very nice example is found in [12]. Lastly, further work on constructing MHV vertices for tree-level gluon amplitudes can be found in [16, 17, 18, 19]. And Generalizations to other particles are found for fermions and scalars in [15, 20, 21], for Higgses [22]. Attempts on generalizing to gravity are found in [23, 24].

Chapter 5

BCFW Recursion Relation

The Britto-Cachazo-Feng-Witten (BCFW) recursion relation is amazing in the way that it simplifies computation of n-point amplitudes into a problem where already known less than n-point amplitudes can be recycled. The BCF relation [25], as it was in the initial state before the proof provided by Witten [26], was inspired by tree level amplitudes being related to one-loop amplitudes. The inspiration was the unitarity based method, which will not be elaborated in this thesis in order to avoid too many unnecessary technicalities concerning one-loop amplitudes. The focus of the present chapter will instead be partly on the proof of the BCF relation for tree-level amplitudes that ensures the ground for application to MHV amplitudes. The BCFW recursion relations has been proved for the general gluonic case [26, 1]. An example is worked out for a 6-point pure gluonic tree-level amplitude. Finally, BCFW was first concerned with gluonic amplitude computations, but has later been extended to include massless fermions [15, 20], Higgs boson [22], massive gauge bosons [27], photons [28] and massive fermions [29].

5.1   Introducing On-Shell Factorization

Early attempts to establish efficient recursion relations are seen in [3], where an auxiliary quantity with initially only one external off-shell leg is constructed using standard Feynman rules of Yang-Mills theory to compute tree level amplitudes by taking the path of the off-shell momentum back into the diagram and in the end amputating the off-shell propagator in the computation of an $(n + 1)$-amplitude and instead contracting with an appropriate on-shell gluon polarization vector ε_{n+1}^μ plus taking the limit $P_{1,n}^2 = k_{n+1}^2 \to 0$.

On-shell factorization can be understood in terms of tree level color stripped diagrams. If one tries pictorially to tear apart a tree diagram with all external legs divided in each hand, the right and left part will show out to be connected by a new propagator. This is a "shared leg" among what will later turn out to be separated left and right amplitudes. Mathematically the diagram splits into products of amplitudes that are cut off each other such that their former propagator becomes a shifted off-shell external leg, which has to be contracted by a polarization vector with the off-shell momentum. The new propagator parting the amplitude turns out to be mathematically the outcome of a residual. As the last word hints the full machinery is performed using complex variable

theory and the residue is basically implied by the pole that appears in a propagator when the momenta goes on-shell. This is seen when integrating in the complex plane over all values of complex momenta. Furthermore, the shift of momentum goes through the whole diagram, i.e. occurs in all the propagators connecting the right amplitude to the left amplitude, due to momentum conservation. Hence the sum of residues corresponds to individual propagator momenta going on-shell. Thereby one encounters that the number of poles corresponds to the number of propagator momenta going on-shell. The performed splitting is then interpreted as two off-shell amplitudes, where the off-shellness refers to one leg with corresponding momentum being off-shell.

For color-ordered amplitudes the intermediate momentum will always be a sum of cyclically adjacent momenta, $P_{i,j}^\mu \equiv (k_i + k_{i+1} + \cdots + k_j)^\mu$. Thus a true pole or singularity will appear for $P_{i,j}^\mu \to 0$ if external leg momenta go to zero. Channels with more than three adjacent momenta will be referred to as multi-particle channels. Two particle channels will be called collinear channels. The latter will exactly be the channels connected with non-vanishing amplitudes appearing in MHV amplitude factorization as seen in [1]. A clever factorization is seen in the later presented BCFW machinery where one chooses the momentum deformation, i.e. the off-shell continuation, in a proper way. But first, one should introduce a pair of tools that will support the correctness of amplitude computations by offering a check in some limits.

The Soft and Collinear Limits

The soft limit of an amplitude can be understood as when all components of a momentum vector k_u goes to zero, which is an infrared limit. The soft limit can be obtained for massless particles. For instance in the case of a tree level color-ordered amplitude the limit is,

$$A_n^{\text{tree}}(\dots, a, s^+, b, \dots) \to \text{Soft}(a, s^+, b) A_{n-1}^{\text{tree}}(\dots, a, b, \dots), \quad k_s \to 0, \tag{5.1}$$

where the Soft factor is,

$$\text{Soft}(a, s^+, b) = \frac{\langle a\, b \rangle}{\langle a\, s \rangle \langle s\, b \rangle}. \tag{5.2}$$

Another tool is the collinear limit. The collinear singularities arise when two outgoing momenta of external particles are proportional such that their sum is lightlike. In this special case the amplitude factorizes in a way that the intermediate propagator momentum between two amplitude factors is described by two external legs corresponding to a three point amplitude connected through a collinear channel to some other amplitude. Due to momentum conservation in vertices the propagator connecting the two amplitudes (right and left parts) will cause a singularity in the massless case. The limit is performed similarly to the Soft limit and z describes the longitudinal momentum sharing among two particles $k_a = z k_P$, $k_b = (1 - z) k_P$. The collinear limit can be stated as,

$$A_n^{\text{tree}}(\dots, a^{\lambda_a}, b^{\lambda_b}, \dots) \xrightarrow{a||b} \sum_{\lambda=\pm} \text{Split}_{-\lambda}^{\text{tree}}(z, a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{tree}}(\dots, P^\lambda, \dots), \tag{5.3}$$

with $k_P = k_a + k_b$ for the intermediate state P^λ and λ denotes the helicity of the intermediate state. The Split function must reflect a form with a collinear channel and

a three amplitude, since it is exactly these parts of the amplitude that are affected by the momenta becoming collinear.

An expression for the Split function can be extracted in the MHV case as done in [3]. Examine the 5-point MHV tree-level gluon amplitude. Choosing two momenta to be collinear, $k_4 \parallel k_5$, is done by setting $k_4 = zk_p$ and $k_5 = (1-z)k_p$. This is similar to a scaling of the momenta, hence it implies for the individual spinor bracket a factor of $\sqrt{\text{scaling}}$, which is seen in (3.5). Thus a scaling of k_4 and k_5 implies respectively a factor $\sqrt{z}$ and $\sqrt{1-z}$. Implementing it into the MHV amplitude one gets,

$$
\begin{aligned}
A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) &= i \frac{\langle 1\,2 \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,5 \rangle \langle 5\,1 \rangle} \\
&\xrightarrow{k_4 \parallel k_5} \frac{1}{\sqrt{z(1-z)}\,\langle 4\,5 \rangle} \times i \frac{\langle 1\,2 \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,P \rangle \langle P\,1 \rangle} \\
&= \text{Split}_-^{\text{tree}}(4^+, 5^+) \times A_4^{\text{tree}}(1^-, 2^-, 3^+, P^+).
\end{aligned}
\tag{5.4}
$$

Using similar methods other Split functions are obtained [3],

$$
\begin{aligned}
\text{Split}_-^{\text{tree}}(a^-, b^-) &= 0, \\
\text{Split}_-^{\text{tree}}(a^+, b^+) &= \frac{1}{\sqrt{z(1-z)}\,\langle a\,b \rangle} \\
\text{Split}_+^{\text{tree}}(a^+, b^-) &= \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle a\,b \rangle} \\
\text{Split}_-^{\text{tree}}(a^+, b^-) &= - \frac{z^2}{\sqrt{z(1-z)}\,[a\,b]}.
\end{aligned}
\tag{5.5}
$$

Limits can also be obtained for $g \to \bar{q}q$ and $q \to qg$. Finally the the two limits explained above offers a check for future computations in the way that amplitudes should reduce to a form reflecting either (5.1) or (5.3) depending on the limit taken.

5.2 Review of the Proof

In proving the BCF relation a contour integral is considered in [1] with an analytically continued amplitude,

$$
A(z) = A(k_1, \ldots, k_j(z), k_{j+1}, \ldots, k_l(z), \ldots, k_n),
\tag{5.6}
$$

that remains on-shell and has a complex parameter z, i.e. all external gluon momenta are on-shell. The complex parameter is introduced through a shift in the spinors as [9],

$$
\tilde{\lambda}_j \to \tilde{\lambda}_j - z\tilde{\lambda}_l, \quad \lambda_l \to \lambda_l + z\lambda_j,
\tag{5.7}
$$

where the corresponding shift in the momenta is best understood by recalling that momentum has 2×2 representation that is a result of the Lorentz group $SO(3,1)$ being locally isomorphic to $SL(2) \times SL(2)$ [9]. The Weyl representation of the Lorentz group

implies that momentum, p_μ, can be expressed as a bi-spinor through the Pauli sigma matrices,

$$p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu, \tag{5.8}$$

where a and $\dot{a}$ are spinor indices of the holomorphic and anti-holomorphic spinors that transform as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ in the Weyl representation respectively. Any 2×2 matrix $p_{a\dot{a}}$ has rank at most 2 and can be written as $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \mu_a \tilde{\mu}_{\dot{a}}$ for some spinors λ, μ, $\tilde{\lambda}$ and $\tilde{\mu}$. The rank of a 2×2 matrix is less than two if and only if its determinant vanishes. Using this together with $p_\mu p^\mu = \det(p_{a\dot{a}})$ on-shell momentum is up to a scaling given as,

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}. \tag{5.9}$$

The result is momentum expressed in terms of spinors, i.e. an expression where it is explicit what a complex shift implies for the corresponding momenta. Namely something like:

$$p_{l,a\dot{a}} \rightarrow (\lambda_{l,a} + z\lambda_{j,a})\widetilde{\lambda}_{l,\dot{a}} = \lambda_{l,a}\tilde{\lambda}_{l,\dot{a}} + z\lambda_{j,a}\widetilde{\lambda}_{l,\dot{a}}, \tag{5.10}$$

where the index l denotes a label for an external leg.

The previously mentioned contour integral that is needed in order to prove the BCF relation has the form,

$$\frac{1}{2\pi i} \oint \frac{dz}{z} A(z). \tag{5.11}$$

If $A(z)$ is a rational function, meaning that $A(z) \rightarrow 0$ as $z \rightarrow \infty$, then the contour integral vanishes and the obtained expression is,

$$A(0) = -\sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{A(z)}{z}, \tag{5.12}$$

where the sum runs over poles z_α different from zero. Poles in the complex z-plane is only possible for factorizations into product of amplitudes, where the j'th and the l'th leg are located in separate camps, which will be denoted as the right or left amplitude parts. This is seen by the z-independent sum of the two corresponding shifted momenta. Determining the poles is done examining the on-shell condition, where only one of j or l is in the set $\{r, r+1, \ldots, s\}$. Using the notation of [1] the momentum shift in the Lorentz representation takes the form,

$$k_j^\mu \rightarrow k_j^\mu(z) = k_j^\mu - \frac{z}{2} \langle j^- | \gamma^\mu | l^- \rangle \tag{5.13}$$

$$k_l^\mu \rightarrow k_l^\mu(z) = k_l^\mu + \frac{z}{2} \langle j^- | \gamma^\mu | l^- \rangle \tag{5.14}$$

$$\tag{5.15}$$

The shift in momenta preserves the on-shellness of the gluons, $k_j^2(z) = 0 = k_l^2(z)$, and the overall momentum conservation. Now the on-shell condition is,

$$
\begin{aligned}
0 = [K_{r\dots s}(z)]^2 &= (k_r + k_{r+1} + \cdots + k_j(z) + \cdots + k_s)^2 \\
&= \sum_{m,i=r}^{s} 2k_i \cdot k_m - \frac{z}{2} 2\langle j^-|\gamma^\mu|l^-\rangle \sum_{i=r}^{s} k_{i,\mu} \\
&= K_{r\dots s}^2 - z \sum_{i=r}^{s} \langle j\, i\rangle [i\, l] \\
&= K_{r\dots s}^2 - z\langle j^-|K_{r\dots s}|l^-\rangle.
\end{aligned}
\tag{5.16}
$$

Accordingly poles occur at,

$$
z_{rs} = \frac{K_{r\dots s}^2}{\langle j^-|K_{r\dots s}|l^-\rangle},
\tag{5.17}
$$

where

$$
\langle a^-|K_{r\dots s}|b^-\rangle = \overline{u_-(k_a)}\gamma_\mu u_-(k_b) K_{r\dots s}^\mu = \sum_{i=r}^{s} \langle a\, i\rangle [i\, b].
\tag{5.18}
$$

Evaluating the residue one gets,

$$
- \operatorname*{Res}_{z=z_{rs}} \left(\frac{1}{z} \frac{i}{K_{r\dots s}^2(z)} \right) = \frac{-i}{z} \frac{(z-z_{rs})}{(z_{rs}-z)\langle j^-|K_{r\dots s}|l^-\rangle}\bigg|_{z=z_{rs}} = \frac{i}{K_{r\dots s}^2}.
\tag{5.19}
$$

The final expression is then for tree-level BCFW recursion formed as,

$$
A(0) = \sum_{r,s,h} A_L^h(z=z_{rs}) \frac{i}{K_{r\dots s}^2} A_R^{-h}(z=z_{rs}),
\tag{5.20}
$$

where each amplitude on the right-hand side is responsible for a part in the tree-level diagram separated by an intermediate momentum, which is of course either a function of the external momenta from the left amplitude or the right amplitude, due to momentum conservation. Generically the sum in (5.20) is a double sum labeled r, s over all recursive diagrams for which the legs j and l always appear on opposite sites; h labels the helicity sum. Thus the poles gives a natural interpretation of the amplitude being splitted into right and left amplitude parts separated by a propagator.

Choice of Shift

The choice of the complex shift in momentum is not obvious. It is interesting to know how it affects the required rationality of the amplitude. A discussion of the shifts can be found in [26, 1]. The following shifts are possible : $[+,+\rangle$, $[+,-\rangle$, $[-,-\rangle$, $[-,+\rangle$. The notation is defined as $[h_j, h_l\rangle$ denoting the j-th and l-th external shifted legs of (5.7) from before and additionally specifying their helicities. Notice that this is for the case of pure gluonic amplitudes in which the rationality of the amplitude depends on

the helicity. As a start the n-point MHV amplitude is examined for convergence when applying shifts. When color stripped and simplified into a more compact form the MHV amplitude is for a particular color-ordering given as,

$$A(z) = \frac{\langle k\,,m\rangle^4}{\prod_{i=1}^n \langle i\,,i+1\rangle}, \tag{5.21}$$

where k and m are defined to be the two gluons of negative helicity and where the color factor $\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})$, a delta function $(2\pi)^4\delta^{(4)}(\sum_i \lambda_i^a\widetilde{\lambda}_i^{\dot a})$ for the four-momentum in terms of spinors and a coupling constant g^{n-2} for Yang-Mills theory have been omitted. The shifted legs are implicit in the formula such that the form can be referred to in the examination of any shift. A notation different from earlier is used with a comma in order to emphasize separation in the spinor product and to easier refer to specific labelled Weyl spinors. The shifted spinors are still those of (5.7). But a little bit more notation needs to be introduced in order to explicitly see what the shift does to the spinor product. The new notation is not different from (3.3) in terms of physical meaning. The spinor products are defined in the two-dimensional Weyl representation as:

$$\langle j\,l\rangle = \varepsilon^{\alpha\beta}(\lambda_j)_\alpha(\lambda_l)_\beta = \overline{u_-(k_j)}u_+(k_l), \tag{5.22}$$

$$[j\,l] = \varepsilon^{\dot\alpha\dot\beta}(\widetilde{\lambda}_j)_{\dot\alpha}(\widetilde{\lambda}_l)_{\dot\beta} = \overline{u_+(k_j)}u_-(k_l), \tag{5.23}$$

where $\varepsilon^{\alpha\beta}$ is an anti-symmetric tensor of rank two. Examining the $[+,+\rangle$-shift it is easy to see for the case of a MHV amplitude Yang-Mills tree amplitude (5.21) that the shift affects the denominator of the expression, namely the factor of $\langle j-1\,,j\rangle\langle j\,,j+1\rangle$. For this shift the amplitude will go as $\mathcal{O}(1/z^2)$ for large z and is therefore rational as defined below (5.11). In the $[+,-\rangle$ case the numerator contributes with a dependence of $\mathcal{O}(z^4)$ for large z, thus the amplitude is not ensured to be rational in this case. Next case is the $[-,+\rangle$ case in which only the denominator is affected such that the overall expression goes as $\mathcal{O}(1/z^2)$ for large z, due to the shift of the positive helicity gluon appearing in the denominator. The last case is the $[-,-\rangle$ shift for which the numerator will be proportional to a polynomial of $\mathcal{O}(z^4)$ for large z and the denominator will be proportional to a polynomial of $\mathcal{O}(z^2)$ for large z, thus the overall amplitude will most likely not be rational due to a major dependence by power counting of $\mathcal{O}(z^2)$ for large z. Wether the shifted gluons are adjacent or not does not change the above arguments, since no assumption was made regarding this issue.

Having finished the case of MHV amplitudes, one should further go on and exploit the general case of any gluonic tree level amplitude. The arguments will be of Feynman character using the rules for Yang-Mills tree amplitudes. Now, assume that a $[-,+\rangle$-shift is performed. The triggering point is to examine the worst-case scenario, which is the case of tree diagrams with only three vertices connected by propagators and with external legs contributing each with a gluon polarization vector. The gluonic 3-vertex is proportional to the momentum, which is not the case for four vertices. As a reminder, the shift of momentum is performed as,

$$j'th, \quad p_{a\dot a} = (\lambda_a\widetilde{\lambda}_{\dot a})_j \to \lambda_j(\widetilde{\lambda}_j - z\widetilde{\lambda}_l) = \lambda_j\widetilde{\lambda}_j - z\lambda_j\widetilde{\lambda}_l \tag{5.24}$$

$$l'th, \quad p_{a\dot a} = (\lambda_a\widetilde{\lambda}_{\dot a})_l \to (\lambda_l + z\lambda_j)\widetilde{\lambda}_l = \lambda_l\widetilde{\lambda}_l + z\lambda_j\widetilde{\lambda}_l, \tag{5.25}$$

where the work of the shift only differs in sign. The journey of z-dependence in a Feynman diagram starts at the shifted j'th momentum in a left part of the amplitude and goes through propagators of intermediate momentum separating the amplitude into a right and left amplitude i.e. it ends up in the l'th momentum in the right amplitude by taking a unique path where it appears in propagators and vertices encountered on the path taken. That it ends up in l is a result of momentum conservation, when expressing the intermediate momentum in terms of momenta appearing either in the left or right amplitude parts. Hence counting in a CSW framework, one gets that the j'th momenta appears at most in $r + 1$ vertices countered by r that is the amount of times the j'th momentum appears in the propagators. In all, the counting leads to a factor linear in z. The actual argument for eliminating the linear z-dependence is realized by examining contractions with external gluon polarizations in (3.16), which in the notation introduced of this section, takes the form,

$$\varepsilon_{a\dot{a}}^{(-)} = \frac{\lambda_a \widetilde{\mu}_{\dot{a}}}{[\widetilde{\lambda}, \widetilde{\mu}]}, \quad \varepsilon_{a\dot{a}}^{(+)} = \frac{\mu_a \widetilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle} \tag{5.26}$$

For the case of the j'th gluon only $\widetilde{\lambda}_j(z)$ is linear in z, thus for a $[-,+\rangle$-shift $\varepsilon_j^{(-)} \sim \frac{1}{z}$. An analogue reasoning for the l'th gluon leads to $\varepsilon_l^{(+)} \sim \frac{1}{z}$. Hence the conclusion is that the overall z-dependence goes as $A(z) \sim \frac{1}{z}$, which vanishes for $z \to \infty$. Thus for the $[-,+\rangle$-shift it has been argued that the BCFW recursion relations hold in the case of MHV and general gluonic Feynman tree diagrams.

5.3 Exploiting the Tool

Choosing the labels of the shifted legs to be $(n-1)$-th and n-th corresponding to a shift $[-,+\rangle$ the BCFW recursion relation can be stated as in [25, 26],

$$A_n(1, 2, \ldots, (n-1)^-, n^+) =$$

$$\sum_{i=1}^{n-3} \sum_{h=\pm} \left(A_{i+2}(\hat{n}, 1, 2, \ldots, i, -\hat{P}_{n,i}^h) \frac{1}{P_{n,i}^2} A_{n-i}(+\hat{P}_{n,i}^{-h}, i+1, \ldots, n-2, \widehat{n-1}^-) \right), \tag{5.27}$$

where a hat on a label denotes a shift of the corresponding spinor and where momentum conservation dictates,

$$P_{n,i} = p_n + p_1 + \cdots + p_i, \tag{5.28}$$

$$\hat{P}_{n,i} = P_{n,i} + \frac{P_{n,i}^2}{\langle n-1| P_{n,i} |n]} \lambda_{n-1} \widetilde{\lambda}_n, \tag{5.29}$$

$$\hat{p}_{n-1} = p_{n-1} - \frac{P_{n,i}^2}{\langle n-1| P_{n,i} |n]} \lambda_{n-1} \widetilde{\lambda}_n, \tag{5.30}$$

$$\hat{p}_n = p_n + \frac{P_{n,i}^2}{\langle n-1| P_{n,i} |n]} \lambda_{n-1} \widetilde{\lambda}_n \tag{5.31}$$

$$t_i^{[r]} \equiv (p_i + p_{i+1} + \cdots + p_{i+r-1})^2 \tag{5.32}$$

$$\langle i| \sum_r p_r |j] \equiv \sum_r \langle i\,r\rangle [r\,j]. \tag{5.33}$$

The corresponding shifts can be read off,

$$\widetilde{\lambda}_{\widehat{n-1}} = \widetilde{\lambda}_{n-1} - \frac{P_{n,i}^2}{\langle n-1|\, P_{n,i}\, |n]}\widetilde{\lambda}_n \tag{5.34}$$

$$\lambda_{\widehat{n}} = \lambda_n + \frac{P_{n,i}^2}{\langle n-1|\, P_{n,i}\, |n]}\lambda_{n-1}. \tag{5.35}$$

The following identities will be handy when working out spinor products with $\hat{P}_{n,i}$.

$$\langle \bullet\, \hat{P}_{n,i}\rangle = -\, \langle\bullet|\, P_{n,i}\, |n]\, \frac{1}{\omega} \tag{5.36}$$

$$[\hat{P}_{n,i}\, \bullet] = -\, \langle n-1|\, P_{n,i}\, |\bullet]\, \frac{1}{\bar{\omega}}, \tag{5.37}$$

where $\omega = [\hat{P}_{n,i}|n]$ and $\omega = \langle n-1|\hat{P}_{n,i}\rangle$ and the bullet is for an arbitrary spinor label.

The following is nice to work out,

$$w\bar{w} = [\hat{P}_{n,i}\, n]\, \langle n-1\, \hat{P}_{n,i}\rangle = (-\langle n-1|\, P_{n,i}\, |n]\, \frac{1}{\bar{w}})(-\langle n-1|\, P_{n,i}\, |n]\, \frac{1}{w})$$

$$= \frac{1}{w\bar{w}}(\langle n-1|\, P_{n,i}\, |n])^2$$

$$\Rightarrow w\bar{w} = \pm\, \langle n-1|\, P_{n,i}\, |n]. \tag{5.38}$$

To see the BCFW in action an example is worked out. A nice start is to examine the case of a 6-point pure gluon amplitude. One could compute the Next-to-MHV (NMHV) color-ordered amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ using a $[3, 4\rangle = [-, +\rangle$-shift. Applying carefully BCFW for the first time requests a relabeling of the external gluon legs in order to obtain a form like (5.27). Hence, first using cyclic symmetry and then renaming the labels, one gets the helicity structure $(5^+, 6^+, 1^-, 2^-, 3^-, 4^+) = (1', 2', 3', 4', 5'^-, 6'^+)$. For example in this renaming $P_{4,3} = p_4 + p_5 + p_6 + p_1 = -(p_2 + p_3)$. Instead of the relabeling a $|$ will indicate how the amplitude splits into right and left parts. There are two terms

contributing due to vanishing helicity structures. The terms are,

$$A_1 = A_6(2^-, 3^- | 4^+, 5^+, 6^+, 1^-) = A_5(\hat{4}^+, 5^+, 6^+, 1^-, -\hat{P}_{4,3}^-) \frac{1}{P_{4,3}^2} A_3(\hat{P}_{4,3}^+, 2^-, \hat{3}^-)$$

$$= i \frac{-\langle 1\,\hat{P}\rangle^3}{\langle \hat{4}\,5\rangle \langle 5\,6\rangle \langle 6\,1\rangle \langle \hat{P}\,\hat{4}\rangle} \frac{1}{P_{4,3}^2} i \frac{\langle 2\,\hat{3}\rangle^3}{\langle \hat{P}\,2\rangle \langle \hat{3}\,\hat{P}\rangle}$$

$$\langle \hat{4}\,a\rangle = \frac{\langle a|\,4+3\,|2]}{[2\,4]} \tag{5.39}$$

$$\langle \hat{P}\,\hat{4}\rangle = -\langle \hat{4}|\,2+3\,|4]\frac{1}{\omega} = -\frac{1}{\omega}(\langle \hat{4}\,2\rangle\,[2\,4] + \langle \hat{4}\,3\rangle\,[3\,4])$$

$$= -\frac{1}{\omega} t_2^{[3]} \tag{5.40}$$

$$\frac{\langle 2\,\hat{3}\rangle^3}{\langle \hat{P}\,2\rangle \langle \hat{3}\,\hat{P}\rangle} = -\frac{\omega^2\,\langle 2\,3\rangle^3}{\langle 2|\,2+3\,|4]\,\langle 3|\,2+3\,|4]}$$

$$= \frac{\omega^2\,\langle 2\,3\rangle}{[3\,4]\,[2\,4]} \tag{5.41}$$

$$\Rightarrow A_1 = \frac{(\langle 1|\,2+3\,|4])^3}{\langle 5|\,4+3\,|2]\,\langle 5\,6\rangle \langle 6\,1\rangle\,[2\,3]\,[3\,4]\,t_2^{[3]}}. \tag{5.42}$$

Applying the map of labels that takes $i \to i+3$ and conjugation on A_1 (see appendix B) leads to,

$$A_2 = A_6(6^+, 1^-, 2^-, 3^- | 4^+, 5^+) = \frac{(\langle 3|\,4+5\,|6])^3}{\langle 3\,4\rangle \langle 4\,5\rangle\,[6\,1]\,[1\,2]\,\langle 5|\,3+4\,|2]\,t_3^{[3]}}. \tag{5.43}$$

The sum is,

$$A_1 + A_2 = \frac{1}{\langle 5|\,4+3\,|2]} \left(\frac{(\langle 1|\,2+3\,|4])^3}{\langle 5\,6\rangle \langle 6\,1\rangle\,[2\,3]\,[3\,4]\,t_2^{[3]}} + \frac{(\langle 3|\,4+5\,|6])^3}{\langle 3\,4\rangle \langle 4\,5\rangle\,[6\,1]\,[1\,2]\,t_3^{[3]}} \right). \tag{5.44}$$

$A_3 = A_6(1^-, 2^-, 3^- | 4^+, 5^+, 6^+)$ does not contribute.

One may also have keen interest in seeing BCFW recursion applied to the case of an MHV tree amplitude, i.e. to see that it actually agrees with the already known MHV result of [8]. This can be shown using induction and a comparison is done in [1]. Finally, as for any other recursion technique it is nice to measure the efficiency relative to previously found computational methods. The BCFW recursion does not at first sight express a reduction factor in the factorial set of independent amplitudes, as will be seen to be the case for KK relations and as well for BCJ relations (upcoming chapters), which each have $(n-2)!$ and $(n-3)!$ respectively. Instead the reduction takes place in the way that already known lower point amplitude can be recycled. For instance one can measure the efficiency of computation for the closed set of gluon amplitudes found in [25], where amplitudes appear with the form,

$$A_{p,q} = A(1^-, 2^-, \ldots, p^-, (p+1)^+, \ldots, (p+q)^+). \tag{5.45}$$

This set exhibits the nice property that only two terms are non-zero after being processed by BCFW recursion. Hence two terms appear, each with separate parts in either

of the sets $\{A_{p+q-1}\}$ and $\{A_3\}$ of amplitudes. The information obtained allows a vague judgment for the efficiency that is bounded by its worst-case scenario equal to the maximum size of the union of the sets $\{A_{p+q-1}\}$ and $\{A_3\}$. The two sets each consists of independent tree-level amplitudes. A naive estimate that assumes $A_{p+q} = A_n$, implies, utilizing BCJ relations, $(n - 1 - 3)! + (3 - 3)! = (n - 4)! + 1$ number of independent amplitudes. But then again this underestimates the efficiency, since BCFW is amazing in the way that it allows for recursion, i.e. the mentioned measure of efficiency can be questionized for its missing link of the BCFW recycling property, since BCFW could as well have been applied to the A_{n-1} part factorizing it into lower point amplitudes. The latter iterative step would probably change the estimate of the independent amplitudes using BCJ.

Measuring the efficiency gets more complex with the inclusion of helicity structures. Instead it is common to measure it by the number of independent amplitudes. A little scheme of how to extend the difficulty of computation may include the following stages,

$$\begin{pmatrix} \text{Compute the number of independent basis elements} \\ \text{Compute MHV independent helicity structures} \\ \text{Do NMHV} \\ \text{Do N}^n\text{MHV} \\ \text{Do 1-loop order} \\ \text{Do higher loop orders} \end{pmatrix}$$

5.4   3-Point Amplitude

A momentum configuration $(p_1, \ldots, p_n)$ can be deformed into several complex planes, one for each momenta. The only thing of concern in this context is the preservation of on-shellness for the momenta. Hence choosing proper shifts is allowed as long as on-shellness is not messed up. In the case of the three point amplitude amplitude this proves handy. Examining the following shifts of momenta,

$$p_i(z) = p_i + zq, \quad p_j(z) = p_j - zq, \tag{5.46}$$

the on-shell property requires that,

$$\begin{aligned} p_i(z)^2 = p_i^2 + 2zq \cdot p_i + z^2 q^2 = 0 \quad p_j(z)^2 = 0 \\ \Rightarrow q \cdot p_i = q \cdot p_j = q^2 = 0 \end{aligned} \tag{5.47}$$

Computing the 3-point amplitude for all external momenta outgoing and on-shell exploits some nice features that are needed when performing BCFW recursion. The momentum conservation implies,

$$\begin{aligned} p_1 + p_2 + p_3 = 0 \quad \Rightarrow \quad p_1 = -(p_2 + p_3) \\ \Rightarrow 0 = p_1^2 = (p_2 + p_3)^2 = 2p_2 \cdot p_3 = \langle 2\,3 \rangle \, [3\,2] \end{aligned} \tag{5.48}$$

similar calculations lead to,

$$2p_i \cdot p_j = \langle i\,j \rangle \, [j\,i] = 0, \quad \forall i, j = 1, 2, 3 \tag{5.49}$$

For this to hold one needs either that $\langle i\,j \rangle = 0$ or $[j\,i] = 0$ i.e. that all the left-handed spinors or all the right-handed spinors are proportional. This can be formulated as the following two equations,

$$\lambda_1 \sim \lambda_2 \sim \lambda_3, \tag{5.50}$$

$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3 \tag{5.51}$$

which will imply zeros by anti-symmetry of the spinor products. This comes in handy when performing BCFW recursion since a 3-point MHV tree amplitude factor usually occurs as part of the recursion factorization. The first contribution states,

$$A_3^{\text{tree}}(1^-, 2^-, 3^+) = i\frac{\langle 1\,2 \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle}, \tag{5.52}$$

which relies on a choice of complex momenta being related as in (5.51) implying $[1\,2] = [2\,3] = [3\,1] = 0$. Conversely one could instead choose (5.50) setting (5.52) to zero by $\langle 1\,2 \rangle = \langle 2\,3 \rangle = \langle 3\,1 \rangle = 0$ and giving a non-zero contribution for,

$$A_3^{\text{tree}}(1^+, 2^+, 3^-) = -i\frac{[1\,2]^4}{[1\,2][2\,3][3\,1]}. \tag{5.53}$$

Furthermore the all plus helicity case gives,

$$A_3^{\text{tree}}(1^+, 2^+, 3^+) = 0, \tag{5.54}$$

due to choosing reference momenta as $(q_1 = q_2 = k_3, q_3 = k_1)$ implying zeros for the $\varepsilon_i^+ \cdot \varepsilon_j^+$ for almost all i, j, while the total zero of the amplitude is ensured by momentum conservation and transverse property combined with $\varepsilon_i^+(q) \cdot q = 0$. The argument exhausts the possible contractions between the 3-vertex in (2.17) and gluon polarizations.

Chapter 6

KK Relations

The Kleiss-Kuijf (KK) relation was first formulated in 1988 [30]. The relations reduce the number of independent color-ordered gluon amplitudes by a factor of $(n-1)$. The idea is to relate different b-functions that are connected to scattering amplitudes (explained later) in terms of each other to build a recursion relation that expresses any amplitude in a $(1, n)$-basis of amplitudes. The KK relations that will be motivated in the following are,

$$A_n^{tree}(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i \in OP(\{\alpha\}, \{\beta^T\})} A_n^{tree}(1, \{\sigma\}_i, n) \tag{6.1}$$

where $OP(\{\alpha\}, \{\beta^T\})$ restricts the sum to ordered permutations, by which is meant all the permutations of $\{\alpha\} \cup \{\beta^T\}$ that preserves the individual order of the elements in the sets $\{\alpha\}$ and $\{\beta^T\}$. With $\{\beta^T\}$ being equal to $\{\beta\}$ but with reversed order of elements. Here n_β is the number of elements in the set $\{\beta\}$. This method reduces the number of independent amplitudes from $(n-1)!$ of the cyclically independent amplitudes to $(n-2)!$ identified with the number of basis amplitudes in a $(1, n)$-basis.

6.1 Reflection Symmetry

The reflection symmetry is an attribute of color-ordered amplitudes and a working tool in reducing the amount of independent amplitudes. It will be nicely illuminated how this reduction takes place when exploiting the structure of this symmetry. An example in the case of 4-point amplitudes is worked out assuming only cyclic symmetry. In this case the concerned amplitudes are,

$$A_4(1, \mathcal{P}(2, 3, 4)) = \{A_4^{\text{tree}}(1, 2, 3, 4), A_4^{\text{tree}}(1, 2, 4, 3), A_4^{\text{tree}}(1, 3, 2, 4) \tag{6.2}$$

$$, A_4^{\text{tree}}(1, 3, 4, 2), A_4^{\text{tree}}(1, 4, 2, 3), A_4^{\text{tree}}(1, 4, 3, 2)\}, \tag{6.3}$$

where $\mathcal{P}$ is for all permutations. Reflection now implies that,

$$A_4^{\mathrm{tree}}(1,2,3,4) = A_4^{\mathrm{tree}}(1,4,3,2) \tag{6.4}$$
$$A_4^{\mathrm{tree}}(1,2,4,3) = A_4^{\mathrm{tree}}(1,3,4,2) \tag{6.5}$$
$$A_4^{\mathrm{tree}}(1,3,2,4) = A_4^{\mathrm{tree}}(1,4,2,3) \tag{6.6}$$
$$A_4^{\mathrm{tree}}(1,3,4,2) = A_4^{\mathrm{tree}}(1,2,4,3) \tag{6.7}$$
$$A_4^{\mathrm{tree}}(1,4,3,2) = A_4^{\mathrm{tree}}(1,2,3,4) \tag{6.8}$$
$$A_4^{\mathrm{tree}}(1,4,2,3) = A_4^{\mathrm{tree}}(1,3,2,4). \tag{6.9}$$

Enumerating the left-hand side of (6.4) by assigning the symbol P_1 to $A_4^{\mathrm{tree}}(1,2,3,4)$ in row one and so forth and letting R denote the reflection operator, one gets that the relations pair as,

$$RP_1 = P_5, \ RP_5 = P_1, \ RP_2 = P_4, \ RP_4 = P_2, \ RP_3 = P_6, \ RP_6 = P_3, \tag{6.10}$$

which exactly corresponds to reducing the number of independent terms by a factor of two. This holds more generally for any set independent amplitudes by extrapolating the argument. The statement can be rephrased as to show that two independent permutations are never mapped to the same permutation. To prove this by contradiction assume the contrary, i.e. that two independent permutations, (P_i, P_j), map into the same permutation,

$$RP_i = P_l \ \wedge \ RP_j = P_l \tag{6.11}$$
$$\Rightarrow R^2 P_i = RP_l \ \wedge \ R^2 P_j = R \tag{6.12}$$
$$\Rightarrow P_i = RP_l \ \wedge \ P_j = RP_l \tag{6.13}$$
$$\Rightarrow P_i = P_j \tag{6.14}$$

which is a contradiction. It was used that $R^2 = 1$ which is a way of saying that reflection ensures a closed form. It is precisely this closed form structure that reduces the number of terms by a factor two.

6.2   KK Origin Revisited

In [30] the KK relations are established on the ground of the Berends-Giele recursion relations. The b-functions are exploited for their mutual relationship through properties like, cyclicity, reflection and the sub-cyclic property. The last is also called the photon-decoupling identity. Dicking a little further into the Berends-Giele relations the b-functions are expressed using the quantities as momentum, polarization and that only three or four gluon vertices appear. The following $H[j,m]^\mu$, where the gluons are denoted $i_j, i_{j+1}, \ldots, i_m$, almost corresponds to the color-ordered amplitude already mentioned in chapter 2 on color decomposition, i.e. being a function of momenta and polarizations only, but it misses one external gluon leg contraction to draw the equality. For a single gluon it is defined as,

$$H[j,j]^\mu = \varepsilon(j)^\mu, \tag{6.15}$$

and for several gluons as,

$$H[j,p]^\mu = \sum_{l=j}^{p-1} \frac{A[j,l,p]^\mu}{D[j,l]D[l+1,p]} \tag{6.16}$$

$$+ \sum_{l=j}^{p-2} \sum_{m=l+1}^{p-1} \frac{B[j,l,m,p]^\mu}{D[j,l]D[l+1,m]D[m+1,p]}, \tag{6.17}$$

where $(j \leq l < m < p)$

$$A[j,l,p]^\mu = H[j,l]^\nu H[l+1,p]_\nu (K[j,l]^\mu - K[l+1,p]^\mu) \tag{6.18}$$
$$- 2K[j,l]^\nu H[l+1,p]_\nu H[j,l]^\mu \tag{6.19}$$
$$+ 2K[l+1,p]^\nu H[j,l]_\nu H[l+1,p]^\mu \tag{6.20}$$
$$B[j,l,m,p]^\mu = 2H[j,l]^\nu H[m+1,p]_\nu H[l+1,m]^\mu \tag{6.21}$$
$$- H[j,l]^\nu H[l+1,m]_\nu H[m+1,p]^\mu \tag{6.22}$$
$$- H[l+1,m]^\nu H[m+1,p]_\nu H[j,l]^\mu. \tag{6.23}$$

The propagator functions D are defined as,

$$D[j,p] = \begin{cases} K[j,p]^\nu K[j,p]_\nu & \text{if } j < p, \\ 1 & \text{if } j = p \end{cases} \tag{6.24}$$

Notice the structure in (6.16) where A and B reflects the branching into either a three-gluon vertex or a four-gluon vertex each connected by "blobs" containing all possible Feynman diagrams. The reasoning can be further iterated. Finally, the full b-function is formed by connecting the n'th gluon leg to these vertices and is given by,

$$b(1, i_2, \ldots, i_n) = H[1,1]^\nu H[2,n]_\nu. \tag{6.25}$$

which exactly identifies with the color-ordered amplitude of chapter 2.

   To show how the explicit reduction takes place using (6.1), it is convenient to work out at simple 4-point gluon example. Assume that the amplitudes are cyclically independent, then one has a set of $(n-1)! = 6$ amplitudes equal to (6.2). To show that the set shrinks, apply the KK relations (6.1) with independent basis amplitudes $A_4(1, \mathcal{P}(2,3), 4) = \{A_4^{\text{tree}}(1,2,3,4), A_4^{\text{tree}}(1,3,2,4)\}$,

$$A_4^{\text{tree}}(1,2,3,4) = A_4^{\text{tree}}(1,2,3,4) \tag{6.26}$$
$$A_4^{\text{tree}}(1,2,4,3) = -A_4^{\text{tree}}(1,2,3,4) - A_4^{\text{tree}}(1,3,2,4) \tag{6.27}$$
$$A_4^{\text{tree}}(1,3,2,4) = A_4^{\text{tree}}(1,3,2,4) \tag{6.28}$$
$$A_4^{\text{tree}}(1,3,4,2) = -A_4^{\text{tree}}(1,3,2,4) - A_4^{\text{tree}}(1,2,3,4) \tag{6.29}$$
$$A_4^{\text{tree}}(1,4,2,3) = A_4^{\text{tree}}(1,3,2,4) \tag{6.30}$$
$$A_4^{\text{tree}}(1,4,3,2) = A_4^{\text{tree}}(1,2,3,4), \tag{6.31}$$

from which it is seen that four amplitudes can be eliminated from the equation system. An encouraging observation in (6.26-6.31) is that (6.27) for instance corresponds to the

photon-decoupling relation,

$$A_4^{\text{tree}}(1,2,4,3) = -\left(A_4^{\text{tree}}(1,2,3,4) + A_4^{\text{tree}}(1,3,2,4)\right) = -\left(A_4^{\text{tree}}(4,3,2,1) + A_4^{\text{tree}}(1,3,2,4)\right)$$

$$= -\left(A_4^{\text{tree}}(1,4,3,2) + A_4^{\text{tree}}(1,3,2,4)\right) \tag{6.32}$$

$$\Rightarrow \sum_{\mathcal{P}_i \in \mathcal{Z}_3} A_4(1, \mathcal{P}_i(2,4,3)) = 0 \tag{6.33}$$

and that (6.29) just satisfies it's equality implied by (6.27) combined with reflective and cyclic symmetry. The other equality signs (6.30),(6.31) are seen to hold using the just mentioned symmetries. Moreover, (6.27) can be shown to hold in the MHV case utilizing instead the Schouten identity (3.14). Indeed, the KK relations have been proved using repeatedly the Schouten identity [31]. Hence the properties that were assumed in constructing the KK relations are nicely rediscovered. Now the KK relations constitute an amazing result in history of amplitudes. But today there exist other alternatives offering an additional reduction to $(n-3)!$ number of basis amplitudes, such as the Bern-Carrasco-Johansson relations, which are the subject of the next chapter.

Chapter 7

BCJ Relations

The Bern-Carrasco-Johansson relations (BCJ) was first formulated in a paper from 2008 [5] and conjectured a possible rearrangement of contributing Feynman diagrams into a convenient representation with kinematic factors satisfying Jacobi identities. The inspiration originates from a paper showing that the 4-point gluon case obeys the kinematic Jacobi identity [32]. And of course the color factors already satisfy such an identity. Furthermore, this reorganization shrinks the number of color-ordered basis amplitudes by a factor of $(n-2)$ from $(n-2)!$ of the KK relations to $(n-3)!$ amplitudes.

7.1   Kinematic Jacobi Identities

The idea is best illustrated in the 4-point gluon case. Now there are three independent, independent as understood in terms of cyclic and reflective symmetries, color-ordered amplitudes $A_4(1,2,3,4)$, $A_4(1,3,4,2)$ and $A_4(1,4,2,3)$. In a Feynman interpretation each of these will show out to have different pole dependencies in terms of Mandelstam variables. The BCJ relations have been proved in [33, 34] and using only quantum field theory in [35] utilizing on-shell recursion techniques [25, 26].

First, one should apply the photon-decoupling relation as mentioned in (2.12), which implies a vanishing relation between the three amplitudes that only can be the result of proportionality with the vanishing sum of Mandelstam variables $s + t + u = 0$. The basis for this conclusion relies on the fact that color-ordered tree amplitudes appearing in the photon-decoupling relation are rational functions of polarization vectors, spinors, momenta and Mandelstam variables, $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$ and $u = (k_1 + k_3)^2$ for which all momenta k_i are outgoing. Momentum conservation takes the form,

$$k_1 + k_2 + \cdots + k_{n-1} + k_n = 0. \tag{7.1}$$

Examining the photon-decoupling relation it is clear that neither helicity nor the spacetime dimension influences the relation. Since the dimension appears implicitly in the number of momentum components and components of the polarization vectors, which is unspecified and the same is true for helicity. Hence the list of candidates reduces to momentum dependence through the Mandelstam variables and thereby implies,

$$A_4(1,2,3,4) + A_4(1,3,4,2) + A_4(1,4,2,3) = (s + t + u)\chi = 0 \tag{7.2}$$

Now each independent amplitude exhibits pole dependence corresponding to the color-ordering of the external particles. For instance $A_4(1,2,3,4)$ can not have an overall proportionality factor equal to the pole factor in the amplitude, thus the overall factor is different from s, t which leaves out only one possible factor i.e. u. Similarly for $A_4(1,3,4,2)$ one gets t as proportionality factor and for $A_4(1,4,2,3)$ a factor of s. This leads to the relations,

$$A_4(1,2,3,4) = u\chi, \qquad A_4(1,3,4,2) = t\chi, \qquad A_4(1,4,2,3) = s\chi. \qquad (7.3)$$

Eliminating χ the following three relations are obtained,

$$tA_4(1,2,3,4) = uA_4(1,3,4,2), \qquad sA_4(1,2,3,4) = uA_4(1,4,2,3)$$
$$sA_4(1,3,4,2) = tA_4(1,4,2,3), \qquad (7.4)$$

where each amplitude can be expressed in a kinematic representation like,

$$A_4(1,2,3,4) = \frac{n_s}{s} + \frac{n_t}{t}$$
$$A_4(1,3,4,2) = -\frac{n_u}{u} - \frac{n_s}{s} \qquad (7.5)$$
$$A_4(1,4,2,3) = -\frac{n_t}{t} + \frac{n_u}{u}$$

where signs appear due to the anti-symmetry of three vertices in the color-ordered Feynman rules and where quartic terms (gluon four-vertices) have been absorbed into the kinematic numerators, n_i, of the cubic diagrams through multiplication by a pole factor of the absorbing term. Due to the KK relation only two of the three amplitudes are independent. Inserting the expressions of (7.5) into (7.4) the obtained result is exactly,

$$sA_(1,2,3,4) = uA_4(1,4,2,3)$$
$$\Rightarrow s\left(\frac{n_s}{s} + \frac{n_t}{t}\right) = u\left(-\frac{n_t}{t} + \frac{n_u}{u}\right)$$
$$n_s + \frac{sn_t}{t} = -u\frac{n_t}{t} + n_u \Rightarrow n_u = n_s + \frac{n_t}{t}(s+u) = n_s - n_t \qquad (7.6)$$
$$\Rightarrow n_u = n_s - n_t,$$

i.e. Jacobi relations for kinematic numerators. A similar derivation can be done for the other two relations in (7.4) returning identical Jacobi expressions. It is important to get a good feeling for how the Jacobi identity really works. The Jacobi identity is permutation invariant, which is seen by,

$$[[A,B],C] = [A,[B,C]] - [B,[A,C]],$$
$$[[A,C],B] = [A,[C,B]] - [C,[A,B]] = -[A,[B,C]] + [[A,B],C],$$
$$[[B,A],C] = [B,[A,C]] - [A,[B,C]],$$
$$[[B,C],A] = [B,[C,A]] - [C,[B,A]] = -[B,[A,C]] - [[A,B],C], \qquad (7.7)$$
$$[[C,A],B] = [C,[A,B]] - [A,[C,B]],$$
$$[[C,B],A] = [C,[B,A]] - [B,[C,A]] = [[A,B],C] + [B,[A,C]].$$

This implies for at set of n elements that the maximal number of independent Jacobi relations is $\frac{n(n-1)(n-2)}{3!} = \frac{n!}{(n-3)!3!}$.

Color dressing each of the four point color-ordered tree-level gluon amplitudes using the method of section 2 by summing over all colors one gets,

$$
\begin{aligned}
c_1 A(1234) &= [\mathrm{Tr}(T^{1234}) + \mathrm{Tr}(T^{4321})] A(1234), \\
c_2 A(1342) &= [\mathrm{Tr}(T^{1342}) + \mathrm{Tr}(T^{2431})] A(1342), \\
c_3 A(1423) &= [\mathrm{Tr}(T^{1423}) + \mathrm{Tr}(T^{3241})] A(1423),
\end{aligned}
\tag{7.8}
$$

where $T^{1234} \equiv T^{a_1} T^{a_2} T^{a_3} T^{a_4}$ and c_i is identified with the color factor of the color-ordered amplitude. Computing the color dressed amplitudes one uses (2.6),

$$
\begin{aligned}
\mathcal{A}_4(1234) &= c_1 A(1234) + c_2 A(1342) + c_3 A(1423) \\
&= c_1\left(\frac{n_s}{s} + \frac{n_t}{t}\right) + c_2\left(-\frac{n_u}{u} - \frac{n_s}{s}\right) + c_3\left(-\frac{n_t}{t} + \frac{n_u}{u}\right) \\
&= (c_1 - c_2)\frac{n_s}{s} + (c_1 - c_3)\frac{n_t}{t} + (c_3 - c_2)\frac{n_u}{u}.
\end{aligned}
\tag{7.9}
$$

The first color factor of n_s will show out to have direct connection with the expected colorfactor of the three gluon vertex,

$$
c_1 - c_2 = \mathrm{Tr}(T^{1234}) + \mathrm{Tr}(T^{4321}) - (\mathrm{Tr}(T^{1342}) + \mathrm{Tr}(T^{2431}))
\tag{7.10}
$$

Computing the color factor for a four point diagram with a s-channel will have a form similar to (2.5),

$$
\begin{aligned}
f^{a_1 a_2 b} f^{a_4 a_3 b} &= -\frac{1}{2}[\mathrm{Tr}(T^{1243}) - \mathrm{Tr}(T^{1234}) - \mathrm{Tr}(T^{2143}) + \mathrm{Tr}(T^{2134})] \\
&= \frac{1}{2}(c_1 - c_2) \\
\Rightarrow (i\sqrt{2})^2 f^{a_1 a_2 b} f^{a_3 a_4 b} &= c_1 - c_2.
\end{aligned}
\tag{7.11}
$$

Thus by setting $\widetilde{f}^{abc} \equiv i\sqrt{2} f^{abc} = \mathrm{Tr}([T^a, T^b] T^c)$ one gets the natural identification,

$$
c_s \equiv \widetilde{f}^{a_1 a_2 b} \widetilde{f}^{b a_3 a_4} = c_1 - c_2
\tag{7.12}
$$

Finally, doing similar derivations for the other color factors in (7.9) gives the result for the full color dressed amplitude,

$$
\mathcal{A}_4^{tree} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right),
\tag{7.13}
$$

where the color factors are exactly products of structure factors in the adjoint representation that are partially contracted,

$$
c_u \equiv \widetilde{f}^{a_4 a_2 b} \widetilde{f}^{b a_3 a_1}, \quad c_s \equiv \widetilde{f}^{a_1 a_2 b} \widetilde{f}^{b a_3 a_4}, \quad c_t \equiv \widetilde{f}^{a_2 a_3 b} \widetilde{f}^{b a_4 a_1}.
\tag{7.14}
$$

These color factors obey the Jacobi identity implying $c_u = c_s - c_t$.

Gauge Invariance of 4-Point Jacobi Identity

The kinematic numerators are not unique gauge-invariant quantities. And they can be assumed to be local in the external polarization and momenta. One can say that Feynman diagrams are inside these functions s.t. contributions with are particular pole dependence are gathered together in a specific n_i. For instance Feynman diagrams with s-channel dependence will be inside n_s. But it is known that Feynman diagrams are not gauge invariant. More specifically in a theory with redundant degrees of freedom there exist a choice of gauge in order to fix these, and gauge transformations are responsible for linking different choices of gauge. This changes the field variables which affects the picture of the Feynman diagrams. Thus the gathering of diagrams is not unique and the numerators are therefore gauge dependent. Since it is possible to come up with a gauge transformation that gathers Feynman diagrams differently inside the numerators. One such manipulation is illustrated in the following,

$$n'_s = n_s + \alpha(k_i, \varepsilon_i)s, \quad n'_t = n_t - \alpha(k_i, \varepsilon_i)t, \quad n'_u = n_u - \alpha(k_i, \varepsilon_i)u, \tag{7.15}$$

where $\alpha(k_i, \varepsilon_i)$ is local. The Jacobi numerator identity is unchanged under the local gauge transformation,

$$-n'_s + n'_t + n'_u = -n_s + n_t + n_u - \alpha(k_i, \varepsilon_i)(s + t + u) = 0 \tag{7.16}$$

Though this is a special property to four points and will not hold for higher n, where only specific choices of numerators will ensure the Jacobi identity. The movement of Feynman diagrams between the numerators is best seen choosing $\alpha(k_i, \varepsilon_i) = n_u/u$ which implies $n'_u = 0$ and that n'_s and n'_t will have a u pole dependence.

Now the step from KK relations to BCJ consist of applying the new Jacobi identity for the numerators thereby eliminating one amplitude in the set of independent basis amplitudes. Implying a reduction from $(n-2)! = 2$ to $(n-3)! = 1$ number of basis amplitudes in the 4-point case.

7.2 Fundamental BCJ Relation

A compact representation of the BCJ relation [5], proved up to eight external particles and extrapolated to n particles, that incorporates the new kinematic Jacobi Identity is,

$$A_n^{tree}(1, 2, \{\alpha\}, 3, \{\beta\}) = \sum_{\{\sigma\}_j \in POP(\{\alpha\}, \{\beta\})} A_n^{tree}(1, 2, 3, \{\sigma\}_j) \prod_{k=4}^{m} \frac{F(3, \{\sigma\}_j, 1|k)}{s_{2,4,\dots,k}} \tag{7.17}$$

where POP stands for partially ordered permutations of the merged $\{\alpha\}$ and $\{\beta\}$ sets. Similar to OP in the KK relations it preserves an ordering of elements, though for this case it is only partially such that only the order of the elements in $\{\beta\}$ is maintained. The relation holds also for the case of the sets being empty. This also implies that if $\{\alpha\}$ is empty then the relation is trivial, due to just one occurring permutation. The

function F is defined as,

$$
F(3, \sigma_1, \sigma_2, \ldots, \sigma_{n-3}, 1|k) \equiv F(\{\rho\}|k) = \left\{ \begin{array}{ll} \sum_{l=t_k}^{n-1} \mathcal{G}(k, \rho_l) & \text{if } t_{k-1} < t_k \\ -\sum_{l=1}^{t_k} \mathcal{G}(k, \rho_l) & \text{if } t_{k-1} > t_k \end{array} \right\}
$$
$$
+ \left\{ \begin{array}{ll} s_{2,4,\ldots,k} & \text{if } t_{k-1} < t_k < t_{k+1} \\ -s_{2,4,\ldots,k} & \text{if } t_{k-1} > t_k > t_{k+1} \\ 0 & \text{else} \end{array} \right\}, \tag{7.18}
$$

where t_k denotes the position of the k'th leg in the set $\{\rho\}$ except for t_3 and t_{m+1} that are defined as,

$$
t_3 \equiv t_5 \qquad t_{m+1} \equiv 0, \tag{7.19}
$$

which in the case of $m = 4$ implies $t_3 = t_{m+1} = 0$. Furthermore the function $\mathcal{G}$ is defined,

$$
\mathcal{G}(i, j) = \left\{ \begin{array}{ll} s_{i,j} & \text{if } i < j \text{ or } j = 1, 3 \\ 0 & \text{else} \end{array} \right\}, \tag{7.20}
$$

and

$$
s_{i,j} = (k_i + k_j)^2, \qquad s_{2,4,\ldots,i} = (k_2 + k_4 + \cdots + k_i)^2, \tag{7.21}
$$

with all momenta massless and outgoing.

The fundamental BCJ relation is a linear representation of BCJ relations that has a field theory proof using only BCFW on-shell recursion in [35]. It can be stated as,

$$
\begin{aligned}
0 = {} & s_{12} A_n(1, 2, \ldots, n) + (s_{12} + s_{23}) A_n(1, 3, 2, 4, \ldots, n) \\
& + (s_{12} + s_{23} + s_{24}) A_n(1, 3, 4, 2, 5, \ldots, n) + \ldots \\
& + (s_{12} + s_{23} + s_{24} + s_{25} + \cdots + s_{2(n-1)}) A_n(1, 3, 4, \ldots, n-1, 2, n),
\end{aligned} \tag{7.22}
$$

from which similar equations are derived by permutation of the labels. This representation exhibits a structure where leg 2 is moved one step to the right each time adding a new factor of $s_{2,i}$ to the previous factors with i being the label of the leg passed by 2. Applying the formula to the 4-point case one gets,

$$
0 = s_{12} A(1, 2, 3, 4) + (s_{12} + s_{23}) A(1, 3, 2, 4)
$$
$$
0 = s_{13} A(1, 3, 2, 4) + (s_{13} + s_{32}) A(1, 2, 3, 4)
$$
$$
0 = s_{13} A(1, 3, 4, 2) + (s_{13} + s_{34}) A(1, 4, 3, 2)
$$
$$
\Rightarrow \begin{pmatrix} s & -u & 0 \\ -s & u & 0 \\ -t & 0 & u \end{pmatrix} \begin{pmatrix} A(1, 2, 3, 4) \\ A(1, 3, 2, 4) \\ A(1, 3, 4, 2) \end{pmatrix} = \bar{0} \tag{7.23}
$$
$$
\begin{pmatrix} s & -u & 0 \\ -s & u & 0 \\ -t & 0 & u \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -s & u & 0 \\ -t & 0 & u \end{pmatrix}
$$
$$
\Rightarrow A(1, 3, 2, 4) = \frac{s}{u} A(1, 2, 3, 4), \quad A(1, 3, 4, 2) = \frac{t}{u} A(1, 2, 3, 4), \tag{7.24}
$$

hence showing that two of the three amplitudes can be eliminated confirming $(n-3)! = (4-3)! = 1$ basis amplitude equal to $A(1,2,3,4)$. That this fits with (7.4) is of no surprise. Actually noticing the dependent equation identified by the row of zeros in the matrix one could instead have started out with just two independent basis amplitudes as implied by the KK relation. Adopting this starting point the KK relations implies 6 basis amplitudes for instance chosen as $A(1, \mathcal{P}(2,3,4), 5)$. The linear relations implied by (7.22) are,

$$0 = s_{21}A_5^{\text{tree}}(1,2,3,4,5) + (s_{21}+s_{23})A_5^{\text{tree}}(1,3,2,4,5) + (s_{21}+s_{23}+s_{24})A_5^{\text{tree}}(1,3,4,2,5)$$
$$0 = s_{21}A_5^{\text{tree}}(1,2,4,3,5) + (s_{21}+s_{24})A_5^{\text{tree}}(1,4,2,3,5) + (s_{21}+s_{24}+s_{23})A_5^{\text{tree}}(1,4,3,2,5)$$
$$0 = s_{23}A_5^{\text{tree}}(3,2,4,5,1) + (s_{23}+s_{24})A_5^{\text{tree}}(3,4,2,5,1) + (s_{23}+s_{24}+s_{25})A_5^{\text{tree}}(3,4,5,2,1)$$
$$0 = s_{24}A_5^{\text{tree}}(4,2,5,1,3) + (s_{24}+s_{25})A_5^{\text{tree}}(4,5,2,1,3) + (s_{24}+s_{25}+s_{21})A_5^{\text{tree}}(4,5,1,2,3)$$
$$0 = s_{24}A_5^{\text{tree}}(4,2,3,5,1) + (s_{24}+s_{23})A_5^{\text{tree}}(4,3,2,5,1) + (s_{24}+s_{23}+s_{25})A_5^{\text{tree}}(4,3,5,2,1)$$
$$0 = s_{23}A_5^{\text{tree}}(3,2,5,1,4) + (s_{23}+s_{25})A_5^{\text{tree}}(3,5,2,1,4) + (s_{23}+s_{25}+s_{21})A_5^{\text{tree}}(3,5,1,2,4)$$

$$1^{st}\text{column} \qquad\qquad 2^{nd}\text{column} \qquad\qquad 3^{rd}\text{column.}$$

The first column consists of basis amplitudes that in some cases have been cyclically permuted in order to keep coefficients simple in the form s_{2i}. Amplitudes occur that differ from the basis amplitudes in the last four rows. But notice that those appearing in row three and four are equal by cyclicity and the same for the ones in row five and six. On these one should apply the KK relations through which they are expressed in terms of the basis. Afterwards, the equation system is solved by putting it into a reduced echelon form using the method of Gaussian elimination with Mathematica,

$$\begin{pmatrix} 1 & 0 & -\frac{s_{24}+s_{25}}{s_{21}} & -\frac{s_{25}}{s_{21}} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{s_{23}+s_{25}}{s_{21}} & -\frac{s_{25}}{s_{21}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (7.25)$$

showing that the matrix has rank two. The only relation of s_{ij} needed in this case was $s_{21}+s_{23}+s_{24}+s_{25}=0$. Hence some dependencies have polluted the equation system and may have been introduced through the wish of coefficient only with the form s_{2i}. The fundamental BCJ relation is expected to give four linear independent equations in order to arrive at two independent basis amplitudes as predicted by $(n-3)!$ for $n=5$. If one allows for different coefficients, s_{ij}, with no restrictions on (i,j) a different looking equation system is obtained,

$$0 = s_{12}A_5^{\text{tree}}(1,2,3,4,5) + (s_{12}+s_{23})A_5^{\text{tree}}(1,3,2,4,5) + (s_{12}+s_{23}+s_{24})A_5^{\text{tree}}(1,3,4,2,5)$$
$$0 = s_{12}A_5^{\text{tree}}(1,2,4,3,5) + (s_{12}+s_{24})A_5^{\text{tree}}(1,4,2,3,5) + (s_{12}+s_{24}+s_{23})A_5^{\text{tree}}(1,4,3,2,5)$$
$$0 = s_{13}A_5^{\text{tree}}(1,3,2,4,5) + (s_{13}+s_{32})A_5^{\text{tree}}(1,2,3,4,5) + (s_{13}+s_{32}+s_{34})A_5^{\text{tree}}(1,2,4,3,5)$$
$$0 = s_{13}A_5^{\text{tree}}(1,3,4,2,5) + (s_{13}+s_{34})A_5^{\text{tree}}(1,4,3,2,5) + (s_{13}+s_{34}+s_{32})A_5^{\text{tree}}(1,4,2,3,5)$$
$$0 = s_{14}A_5^{\text{tree}}(1,4,2,3,5) + (s_{14}+s_{42})A_5^{\text{tree}}(1,2,4,3,5) + (s_{14}+s_{42}+s_{43})A_5^{\text{tree}}(1,2,3,4,5)$$
$$0 = s_{14}A_5^{\text{tree}}(1,4,3,2,5) + (s_{14}+s_{43})A_5^{\text{tree}}(1,3,4,2,5) + (s_{14}+s_{43}+s_{42})A_5^{\text{tree}}(1,3,2,4,5).$$
$$(7.26)$$

For these relations the KK relations are not needed instead one needs the full set of independent relations among the s_{ij}s,

$$s_{23} = - (s_{24} + s_{25} + s_{34} + s_{35} + s_{45})$$
$$s_{15} = - (s_{25} + s_{35} + s_{45})$$
$$s_{14} = - (s_{24} + s_{34} + s_{45}) \tag{7.27}$$
$$s_{13} = - (s_{23} + s_{34} + s_{35})$$
$$s_{12} = - (s_{23} + s_{24} + s_{25}),$$

obtained from $p_i \cdot \sum_{j=1}^n p_j = 0 \Rightarrow \sum_{j=1}^n s_{ij} = 0$ and $(\sum_{i=1}^n p_i)^2 = 0 \Rightarrow \sum_{i<j}^n s_{ij} = 0$. The total set of s_{ij} is given by,

$$(ij) = \{(12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\}, \tag{7.28}$$

of which five elements can be eliminated. The formula for the number of independent s_{ij} is $\frac{n(n-3)}{2}$ in the 4-point and 5-point case. Hence in the above case the Basis$(ij) = \{(24), (25), (34), (35), (45)\}$. Solving (7.26) in terms of $A_5^{\text{tree}}(1,2,3,4,5), A_5^{\text{tree}}(1,2,4,3,5)$ one gets,

$$A_5^{\text{tree}}(1,3,2,4,5) = \frac{1}{s_{13}} \left[(s_{34} + s_{35}) A_5^{\text{tree}}(1,2,3,4,5) + s_{35} A_5^{\text{tree}}(1,2,4,3,5) \right] \tag{7.29}$$

$$A_5^{\text{tree}}(1,3,4,2,5) = \frac{s_{34} - s_{25}}{s_{13}^2 s_{14}(2s_{43} + s_{42} + s_{45})} \left[s_{34} s_{32} s_{42} A_5^{\text{tree}}(1,2,3,4,5) \right.$$
$$\left. + s_{35}(s_{45}(s_{32} + s_{35}) + s_{43} + s_{45}) A_5^{\text{tree}}(1,2,4,3,5) \right] \tag{7.30}$$

$$A_5^{\text{tree}}(1,4,2,3,5) = \frac{1}{s_{14}} \left[(s_{43} + s_{45}) A_5^{\text{tree}}(1,2,4,3,5) + s_{45} A_5^{\text{tree}}(1,2,3,4,5) \right] \tag{7.31}$$

$$A_5^{\text{tree}}(1,4,3,2,5) = \frac{-1}{s_{25} s_{14}} \left[s_{24} s_{35} A_5^{\text{tree}}(1,2,4,3,5) + s_{45}(s_{23} + s_{25}) A_5^{\text{tree}}(1,2,3,4,5) \right]. \tag{7.32}$$

Notice that the s_{ij}s inside the square bracket are expressed only in terms of the corresponding independent set. This is important in order to clarify that no division of zero has taken place while overall factors are easily seen to be non-zero. The solutions can be checked by insertion of MHV amplitudes. Rounding off, the result agrees nicely with the expected number of independent amplitudes for the 5-point case, which equals 2.

7.3 BCJ and Gravity

The BCJ relation can also be applied to gravity. But one subtle question arises whether how to interpret a color factor in the same way as done above or differently, though gravity does not share the same symmetry under the $SU(3)$ group. In [5] it is conjectured that gravity amplitudes can be constructed as a square of gauge theory, i.e. that gravity $\sim$ (gauge theory) $\times$ (gauge theory). Where individual Jacobi like identities have been imposed for each color stripped gauge theory amplitude s.t. $n_i + n_j + n_k = 0$ and

$\widetilde{n}_i + \widetilde{n}_j + \widetilde{n}_k = 0$. The KLT relation (see chapter 8) supports the conjecture by reproducing the same relations [36, 37, 38]. The squaring relation itself has been checked up to eight points. Hence the conjectured outcome for the four point case is,

$$-iM_4^{tree}(1,2,3,4) = \frac{n_s \widetilde{n}_s}{s} + \frac{n_t \widetilde{n}_t}{t} + \frac{n_u \widetilde{n}_u}{u}. \tag{7.33}$$

And the more general extrapolated expression is,

$$\frac{-i}{(\kappa/2)^{n-2}} \mathcal{M}_n^{tree}(1,2,3,\ldots,n) = \sum_{diags.i} \frac{n_i \widetilde{n}_i}{\prod_j s_{\alpha_i}}, \tag{7.34}$$

where the sum is over the same set of diagrams appearing in the amplitudes from which it is derived. The color dressed gauge theory amplitudes have the form,

$$\frac{1}{g^{n-2}} \mathcal{A}_n^{tree}(1,2,3,\ldots,n) = \sum_{diags.i} \frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}}$$

$$\frac{1}{\widetilde{g}^{n-2}} \widetilde{\mathcal{A}}_n^{tree}(1,2,3,\ldots,n) = \sum_{diags.i} \frac{\widetilde{n}_i \widetilde{c}_i}{\prod_{\alpha_i} s_{\alpha_i}}. \tag{7.35}$$

Notice that the extrapolated expression for gravity in (7.34) has a form similar to a product of the two gauge theory amplitudes when the latter are color stripped.

Chapter 8

KLT Relation

The Kawai-Lewellen-Tye relation (KLT) is yet another amazing result [6]. In the field theory limit at tree-level it reduces to a relation with gravity amplitudes expressed as a product of gauge theory amplitudes. The relation has been proved to hold for gravity expressed in terms of Yang-Mills amplitudes in [39, 40]. And the coupling of gravitons to matter is elaborated in [41]. Furthermore, the relation has given birth to a number of vanishing relations, as the ones in [42, 43, 31], where different helicity sectors are chosen for the amplitudes on the right-hand side of the KLT relation. The KLT relation is connected to the BCJ relation, which is elaborated in [36, 40, 38, 37]. Finally, very nice reviews are found in [44, 45].

8.1 Gravity Feynman Rules

The KLT relation was found in string theory relating closed strings to a product of open strings, which offers an interpretation of gravity expressed as the square of gauge theory in the low energy limit well below the Planck scale of 10^{19}GeV. In comparison the Large Hadron Collider is aimed at an energy level of 14 TeV. The relation creates a possibility for constructing perturbative quantum gravity. But there is a problem, since quantum gravity is a non-renormalizable theory in contrast to quantum chromodynamics that is renormalizable [44, 46]. Thus the quantum gravity part should be treated as an effective field theory [47]. In the cases of 4- and 5-point amplitudes the KLT relation in the field theory limit [6] reduces to,

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)\widetilde{A}_4^{\text{tree}}(1,2,4,3), \qquad (8.1)$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)\widetilde{A}_5^{\text{tree}}(2,1,4,3,5)$$
$$+ is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\widetilde{A}_5^{\text{tree}}(3,1,4,2,5), \qquad (8.2)$$

where tilde in $\widetilde{A}$ indicates a separate basis choice of the amplitudes, which will soon be explained utilizing a BCJ interpretation. The gauge theory amplitudes of the right-hand side are color-ordered amplitudes. First, one should notice the possibility of applying color-ordered Feynman rules of section 2.4 to express gravity amplitudes. The rules including gravity coupled to matter can be found in [44], where one also finds expressions

for graviton polarization tensors. These are

$$\varepsilon_{\mu\nu}^+ = \varepsilon_\mu^+ \varepsilon_\nu^+, \qquad \varepsilon_{\mu\nu}^- = \varepsilon_\mu^- \varepsilon_\nu^-, \tag{8.3}$$

where $\varepsilon_\mu^\pm$ are found in (3.16). Further details on deriving Feynman rules for perturbative quantum gravity are found in [46]. Hence the rules can be seen as a new set of color-ordered Feynman rules for gravity theories at tree-level.

8.2 Compact KLT Expression

The KLT relations in the field theory limit is rephrased in a compact form using $\mathcal{S}$-functions in [39]. First, the $\mathcal{S}$-function is

$$\mathcal{S}[i_1, \ldots, i_k | j_1, \ldots, j_k] = \prod_{t=1}^{k} \left(s_{i_t 1} + \sum_{q>t}^{k} \theta(i_t, i_q) s_{i_t i_q} \right), \tag{8.4}$$

where $\theta(i_t, i_q)$ is zero if the pair (i_t, i_q) has same ordering at both sets $\mathcal{I} \equiv \{i_1, \ldots, i_k\}$ and $\mathcal{J} \equiv \{j_1, \ldots, j_k\}$ and otherwise unity. For example

$$\mathcal{S}[2, 3, 4 | 2, 4, 3] = s_{21}(s_{31} + s_{34})s_{41}. \tag{8.5}$$

A dual $\widetilde{\mathcal{S}}$-function will also be needed and is

$$\widetilde{\mathcal{S}}[i_1, \ldots, i_k | j_1, \ldots, j_k] = \prod_{t=1}^{k} \left(s_{j_t n} + \sum_{q<t} \theta(j_q, j_t) s_{j_t j_q} \right), \tag{8.6}$$

where again $\theta(j_a, j_b)$ is zero if j_a sequentially comes before j_b in $\{i_1, \ldots, i_k\}$ and otherwise unity. Notice that the sum is over $q < t$ and n is just a number not to confuse with external legs. For example

$$\widetilde{\mathcal{S}}[2, 3, 4 | 4, 3, 2]_{p_n} = s_{4n}(s_{3n} + s_{34})(s_{2n} + s_{23} + s_{24}). \tag{8.7}$$

The functions $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ have several properties. A property to be used later is the annihilation of amplitudes, which is understood as

$$\sum_{\alpha \in S_{n-2}} \mathcal{S}[\alpha_{2,n-1} | j_2, \ldots, j_{n-1}]_{p_1} A_n(n, \alpha_{2,n-1}, 1) = 0. \tag{8.8}$$

Here $\alpha_{2,n-1}$ denotes an ordering of the legs $2, 3, \ldots, n-1$.

In terms of the $\mathcal{S}$-functions the KLT relation can be expressed in the field theory limit as,

$$M_n = (-1)^{n+1} \sum_{\sigma \in S_{n-3}} \sum_{\alpha \in S_{j-1}} \sum_{\beta \in S_{n-2-j}} A_n(1, \sigma_{2,j}, \sigma_{j+1,n-2}, n-1, n) \mathcal{S}[\alpha_{\sigma(2),\sigma(j)} | \sigma_{2,j}]_{p_1}$$

$$\times \widetilde{\mathcal{S}}[\sigma_{j+1,n-2} | \beta_{\sigma(j+1),\sigma(n-2)}]_{p_{n-1}} \widetilde{\mathcal{A}}_n(\alpha_{\sigma(2),\sigma(j)}, 1, n-1, \beta_{\sigma(j+1),\sigma(n-2)}, n), \tag{8.9}$$

where $j = [n/2]$ is a fixed number ($[x]$ denotes integer of x). Another form of the n-point KLT relations is found in appendix A of [48].

Before working out examples one should take notice of the KLT basis dependence. The form of the KLT relation relies on a choice of basis amplitudes. A_n and $\widetilde{A}_n$ appearing in the relation each demands a choice of basis amplitudes. In a BCJ basis the number of independent amplitudes is $(n-3)!$ as found in chapter 7. In (8.1) the basis chosen is A_n^{tree} in terms of

$$A_n^{\text{tree}}(1, \mathcal{P}\{2, \ldots, n-2\}, n-1, n), \tag{8.10}$$

and $\widetilde{A}_n^{\text{tree}}$ in terms of

$$\widetilde{A}_n^{\text{tree}}(\mathcal{P}\{i_1, \ldots, i_j\}, 1, n-1, \mathcal{P}\{l_1, \ldots, l_{j'}\}, n). \tag{8.11}$$

The following example shows a 4-point MHV tree-level graviton amplitude expressed as a product of MHV gluon amplitudes. The number of gluonic basis amplitudes in each representation is one, i.e. $A_n^{\text{tree}}(1, 2, 3, 4)$ and $\widetilde{A}_n^{\text{tree}}(2, 1, 3, 4) = \widetilde{A}_n^{\text{tree}}(4, 3, 1, 2) = \widetilde{A}_n^{\text{tree}}(1, 2, 4, 3)$. The KLT relation is then

$$
\begin{aligned}
M_4^{\text{tree}}(1_h^-, 2_h^-, 3_h^+, 4_h^+) &= -is_{12} A_4^{\text{tree}}(1_g^-, 2_g^-, 3_g^+, 4_g^+) \widetilde{A}_4^{\text{tree}}(1_g^-, 2_g^-, 4_g^+, 3_g^+) \\
&= -s_{12} ii \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle} i \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,4\rangle \langle 4\,3\rangle \langle 3\,1\rangle} \\
&= -s_{12} i \frac{\langle 1\,2\rangle^6}{\langle 2\,3\rangle \langle 3\,4\rangle^2 \langle 4\,1\rangle \langle 2\,4\rangle \langle 3\,1\rangle} \\
&= i \frac{\langle 1\,2\rangle^7 [1\,2]}{\langle 2\,3\rangle \langle 3\,4\rangle^2 \langle 4\,1\rangle \langle 2\,4\rangle \langle 3\,1\rangle} \\
&= i \langle 1\,2\rangle^8 \frac{[1\,2]}{\langle 3\,4\rangle \langle 1\,2\rangle \langle 1\,3\rangle \langle 1\,4\rangle \langle 2\,3\rangle \langle 2\,4\rangle \langle 3\,4\rangle} \\
&= i \langle 1\,2\rangle^8 \frac{[1\,2]}{\langle 3\,4\rangle N(4)}
\end{aligned}
\tag{8.12}
$$

where $N(n) \equiv \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \langle i\,j\rangle$. Comparing the right-hand side of (8.12) with a similar expression for a 4-point gravity tree amplitude in [44, 11], the equality is seen to hold. A general formula for MHV gravity amplitudes exists for $n > 4$ in [11] and has been numerically verified for $n \leq 11$.

The idea of trying different helicity sectors for respectively A and $\widetilde{A}$ nicely results in vanishing identities for (8.9). It basically consist of flipping a helicity in one sector [42]. The result is non-linear vanishing relations in A and $\widetilde{A}$. This has been further exploited to include multiple flips of helicities as seen in [43, 31].

Total Crossing Symmetry

The gravity amplitude is totally crossing symmetric [44], i.e. it is invariant under arbitrary permutations of external legs. This attribute of the graviton amplitude is well illustrated in the 4-point case. The following four graviton amplitudes, M_4, are related through KLT to a product of gauge theory amplitudes in this case chosen to be pure gluonic,

$$M_4(1, 2, 3, 4) = s_{12} A_4(1, 2, 3, 4) \widetilde{A}_4(1, 2, 4, 3) \tag{8.13}$$

$$M_4(1, 3, 2, 4) = s_{13} A_4(1, 3, 2, 4) \widetilde{A}_4(1, 3, 4, 2). \tag{8.14}$$

The two relations differ by an interchange of legs $(2 \leftrightarrow 3)$. By cyclic permutation and reflection symmetry $\widetilde{A}(1,3,4,2) = \widetilde{A}(1,2,4,3)$ and using a BCJ interpretation (7.4) the gluon amplitude appearing on the right-hand side of (8.14) is related through,

$$s_{12}A(1,2,3,4) = s_{13}A(1,3,2,4) \tag{8.15}$$

which implies (8.13) = (8.14).

8.3 KLT Connection to BCJ

BCJ relations can be used to express gravity amplitudes as previously noted in section 7.3. However, exploring attributes of the S-kernel (has S-function as field theory limit), a connection is obtained to the Fundamental BCJ relation of section 7.2. The $\mathcal{S}$-function annihilation in (8.8) of the amplitude can be shown to equal the Fundamental BCJ relation and thereby vanish [39].

The Fundamental BCJ relations imply also j-independence in (8.9) as argued in [39]. Hence a whole family of KLT relations is established, where the family member consists of different choices of j. And all the j choices are related through the BCJ relations. Note that the j-independence limits the application of (8.9) to particles allowed in the BCJ relations.

Chapter 9

Supersymmetry Applied to Amplitudes

Supersymmetry finds its way to the field of amplitudes through Nair's formula [49, 15] that in the case of extended supersymmetry $\mathcal{N} = 4$ spans the range of amplitudes with four generators of the Poincare group. With these it is possible to establish amplitudes with particles such as scalars (spin=0), gluinos (spin=1/2), gluons (spin=1), gravitinos (spin=3/2) and gravitons (spin=2) and their respective superpartners.

This thesis will address further question on amplitudes with gluinos or just amplitudes with multiple quark pairs. Since the quarks might as well be taken as gluinos in tree-level color stripped amplitudes of QCD [4]. Another very useful result of supersymmetry is obtained by the vacuum annihilation by the supercharge. Hereby giving birth to vanishing relations with different particle content.

Although this helping tool sounds very promising and up to date with what counts as valuable reduction techniques, supersymmetry has not yet been seen in experiments, hence one has to be careful concluding anything when biased by the nice results. The relations can instead serve as the ground for good guesses when proving amplitude recursion relations using induction or they can serve as firm testing tools when doing computations.

Supersymmetric amplitudes may not be confirmed experimentally, however, they surely are helpful in tasks concerning amplitudes with matter. And actually many of the supersymmetric relations have been proved to hold using induction that only requires a simple Feynman computation of some low point amplitudes and a n assumption to show that it holds in the $(n + 1)$-point amplitude case.

9.1 Supersymmetric Ward Identity

A quite useful contribution by supersymmetry are for example the supersymmetric Ward identities. The derivation thereof relies on the fact that the supercharge Q annihilates the vacuum [3],

$$0 = \langle 0|[Q, \Phi_1 \Phi_2 \cdots \Phi_n]|0\rangle = \sum_{i=1}^{n} \langle 0|\Phi_1 \cdots [Q, \Phi_i] \cdots \Phi_n|0\rangle, \tag{9.1}$$

where the fields Φ_i are fields with helicity eigenstates that can take the form of gluons, $g^{\pm}(k)$, and their superpartners, gluinos, $\Lambda^{\pm}(k)$ creating respective particles with momenta k. The commutators are,

$$
\begin{aligned}
[Q(\eta), g^{\pm}(k)] &= \mp \, \Gamma^{\pm}(k, \eta) \Lambda^{\pm}(k), \\
[Q(\eta), \Lambda^{\pm}(k)] &= \mp \, \Gamma^{\mp}(k, \eta) g^{\pm}(k),
\end{aligned}
\tag{9.2}
$$

with $\Gamma(k, \eta)$ being linear in the Grassmann spinor parameter η and satisfying the Jacobi identity of the supersymmetry algebra,

$$
0 = [[Q(\eta), Q(\zeta)], \Phi(k)] + [[Q(\zeta), \Phi(k)], Q(\eta)] + [[\Phi(k), Q(\eta)], Q(\zeta)].
\tag{9.3}
$$

Now [3] goes all the way to find the Γs and the result is just stated here as,

$$
\Gamma^{+}(k, q) = \theta \, [q \, k], \quad \Gamma^{-}(k, q) = \theta \, \langle q \, k \rangle .
\tag{9.4}
$$

where θ is a grassman parameter. With two negative helicities the vacuum annihilation takes the form,

$$
\begin{aligned}
0 =& \langle 0 | [Q(\eta(q)), g_1^{-} g_2^{-} \Lambda_3^{+} g_4^{+} \cdots g_n^{+}] | 0 \rangle \\
=& \Gamma^{-}(k_1, q) A_n(\Lambda_1^{-}, g_2^{-}, \Lambda_3^{+}, \ldots, g_n^{+}) + \Gamma^{-}(k_2, q) A_n(g_1, \Lambda_2^{-}, \Lambda_3^{+}, \ldots, g_n^{+}) \\
&+ - \Gamma^{-}(k_3, q) A_n(g_1^{-}, g_2^{-}, g_3^{+}, \ldots, g_n^{+}),
\end{aligned}
\tag{9.5}
$$

where q is an arbitrary massless vector which can be chosen to simplify the relations. Setting $q = k_1$ implies a relation between a gluino pair amplitude and a pure gluon amplitude,

$$
\begin{aligned}
A_n(g_1^{-}, g_2^{-}, g_3^{+}, \ldots, g_n^{+}) =& \frac{\langle 1 \, 2 \rangle}{\langle 1 \, 3 \rangle} A_n(g_1^{-}, \Lambda_2^{-}, \Lambda_3^{+}, g_4^{+}, \ldots, g_n^{+}) \\
A_n(g_1^{-}, \Lambda_2^{-}, \Lambda_3^{+}, g_4^{+}, \ldots, g_n^{+}) =& \frac{\langle 1 \, 3 \rangle}{\langle 1 \, 2 \rangle} \frac{\langle 1 \, 2 \rangle^4}{\prod_{m=1}^{n} \langle m \, m + 1 \rangle} \\
=& \frac{\langle 1 \, 3 \rangle \langle 1 \, 2 \rangle^3}{\prod_{m=1}^{n} \langle m \, m + 1 \rangle}.
\end{aligned}
\tag{9.6}
$$

However, the same result could have been derived using BCFW recursion as in [50]. The amplitude structure is assumed to be $(g_1^{-}, g_2^{+}, \ldots, \Lambda_i^{-}, \ldots, \Lambda_j^{+}, \ldots, g_n^{+})$ for $(i > 3)$ the reasoning of the article allows one to choose a shift of $[g_1^{-}, g_2^{+}\rangle$. The first non-zero contribution of the BCFW recursion is,

$$
A_3^{L}(\hat{2}^{+}, 3^{+}, -\hat{P}_{2,3}^{-}) \frac{1}{t_2^{[2]}} A_{n-1}^{R}(\hat{P}_{2,3}^{+}, 4^{+}, \ldots, \Lambda_i^{-}, \ldots, \Lambda_j^{+}, \ldots, n^{+}, \hat{1}^{-}),
\tag{9.7}
$$

corresponding to $(i = 1)$ in the BCFW recursion sum. The computation for,

$$
A_n^{\text{tree}}(4^{+}, 5^{+}, \ldots, \Lambda_i^{-}, \ldots, \Lambda_j^{+}, \ldots, n^{+}, 1^{-}|2^{+}, 3^{+}),
\tag{9.8}
$$

is,

$$\frac{\langle \hat{1}\, i\rangle^3 \langle \hat{1}\, j\rangle}{\langle \hat{1}\, \hat{p}\rangle \langle \hat{p}\, 4\rangle \langle n\, \hat{1}\rangle \prod_{m=4}^{n-1} \langle m\, m+1\rangle} = \frac{\langle 1\, i\rangle^3 \langle 1\, j\rangle}{\langle 1\, \hat{p}\rangle \langle \hat{p}\, 4\rangle \langle n\, 1\rangle \prod_{m=4}^{n-1} \langle m\, m+1\rangle}$$

$$= \frac{\langle 1\, i\rangle^3 \langle 1\, j\rangle}{-\frac{1}{w} \langle 1|\, P_{n,i}\, |2]\, \frac{1}{w} \langle 4|\, P_{n,i}\, |2]\, \langle n\, 1\rangle \prod_{m=4}^{n-1} \langle m\, m+1\rangle}$$

$$= \frac{\langle 1\, i\rangle^3 \langle 1\, j\rangle}{\frac{-1}{w^2} \langle 1\, 3\rangle [3\, 2] \langle 4\, 3\rangle [3\, 2] \langle n\, 1\rangle \prod_{m=4}^{n-1} \langle m\, m+1\rangle}$$

$$= \frac{\langle 1\, i\rangle^3 \langle 1\, j\rangle}{\frac{-1}{w^2} \langle 1\, 3\rangle \langle 4\, 3\rangle [3\, 2]^2 \langle n\, 1\rangle \prod_{m=4}^{n-1} \langle m\, m+1\rangle} \qquad (9.9)$$

$$\frac{1}{t_2^{[2]}} \cdot \frac{[\hat{2}\, 3]^3}{[3\, \hat{p}]\, [\hat{p}\, \hat{2}]} = \frac{1}{\langle 2\, 3\rangle [3\, 2]\, \frac{1}{\bar{w}} \langle 1|\, P_{n,i}\, |3]\, \frac{-1}{\bar{w}} \langle 1|\, P_{n,i}\, |2]}\, \frac{[2\, 3]^3}{}$$

$$= \frac{1}{\langle 2\, 3\rangle [3\, 2]\, \frac{1}{\bar{w}^2} \langle 1\, 2\rangle [2\, 3]^2 \langle 1\, 3\rangle}\, [2\, 3]^3 \qquad (9.10)$$

$$\Rightarrow \quad \frac{\langle 1\, i\rangle^3 \langle 1\, j\rangle}{\frac{-1}{w^2} \langle 1\, 3\rangle \langle 4\, 3\rangle [3\, 2]^2 \langle n\, 1\rangle \prod_{m=4}^{n-1} \langle m\, m+1\rangle} \cdot \frac{1}{\langle 2\, 3\rangle [3\, 2]\, \frac{1}{\bar{w}^2} \langle 1\, 2\rangle [2\, 3]^2 \langle 1\, 3\rangle}\, [2\, 3]^3$$

$$= \frac{(w\bar{w})^2 \langle 1\, i\rangle^3 \langle 1\, j\rangle [2\, 3]^3}{\langle 1\, 3\rangle^2 \langle 4\, 3\rangle [2\, 3]^5 \langle n\, 1\rangle \prod_{m=4}^{n-1} \langle 2\, 3\rangle \langle 1\, 2\rangle}$$

$$= \frac{\langle 1|\, 2+3\, |2]^2 \langle 1\, i\rangle^3 \langle 1\, j\rangle}{\langle 1\, 3\rangle^2 \langle 4\, 3\rangle [2\, 3]^2 \langle n\, 1\rangle \prod_{m=4}^{n-1} \langle 2\, 3\rangle \langle 1\, 2\rangle}$$

$$= \frac{\langle 1\, i\rangle^3 \langle 1\, j\rangle}{\langle 4\, 3\rangle \langle n\, 1\rangle \prod_{m=4}^{n-1} \langle 2\, 3\rangle \langle 1\, 2\rangle}$$

$$= \frac{-\langle 1\, i\rangle^3 \langle 1\, j\rangle}{\prod_{m=1}^{n} \langle m\, m+1\rangle}, \qquad (9.11)$$

which is correct up to a sign. This calculation is basically using a proof by induction, assuming a less than n-point generalized [8] result for the MHV amplitude with one fermion pair. Of course an inductional step does not work without a computation of the amplitudes that is independent of the assumption. Such a computation is usually done for a low n-point case and can be found in [3]. One could imagine that other terms in the recursion sum would be non-zero, but in this case many are zero by (3.26). The question of whether the term $(i = n - 3)$,

$$A_{n-1}^L(\hat{2}^+, 3^+, \ldots, \Lambda_i^-, \ldots, \Lambda_j^+, \ldots, (n-1)^+, -\hat{P}_{2,n-1}^-)\frac{1}{P_{2,n-1}^2} A_3^R(\hat{P}_{2,n-1}^+, n^+, \hat{1}^-), \quad (9.12)$$

contributes in the BCFW recursion sum is conducted by the computation,

$$A_3^R(\hat{P}_{2,n-1}^+, n^+, \hat{1}^-) \sim \left[\hat{P}_{2,n-1}\, n\right] = -\langle 1|\, P_{2,n-1}\, |n]\, \frac{1}{\bar{w}}$$

$$= -\langle 1|\, (p_2 + p_3 + \cdots + p_{n-1})\, |n]\, \frac{1}{\bar{w}} = \langle 1|\, (p_1 + p_n)\, |n]\, \frac{1}{\bar{w}} = 0. \qquad (9.13)$$

Now that almost all possible cases have been exhausted another subtlety can be addressed by the term for which the left and right amplitude are connected by a fermion propagator, meaning the case where the fermion pair splits. Thereby implying that both amplitudes have a fermion pair. However, in this case a violation of helicity occurs for the fermion pairs in each of the right and left amplitudes, since all legs are outgoing and two fermions of same helicity appear in the right amplitude as well in the left amplitude violating helicity conservation along the fermion line.

Additionally, R-parity sets a requirement on what superpartners and other particles can decay into, since R-parity is a conserved value in scattering experiments. Assigning a value of minus one to superpartner and plus to others and defining the overall R-parity value of ingoing particles to be computed as the product of R-parity values for each of the particles. This value should then match the overall R-parity value for the outgoing particles. In connection to amplitudes remember that a convention was made to have all external legs as outgoing, hence some momenta has to be reversed using parity transformation and the particles reversed should then somehow play after the rules of R-parity in order to be non-zero.

9.2   The 3 Quark Pair Computation

The formulation of the problem in this thesis is to explore KLT relations for factorization into pure multi quark pair amplitudes. First a BCJ interpretation will be applied in order to establish results for amplitudes solely with quark pairs and then insert these into the KLT relations matched with a left-hand side consisting of gluon amplitudes multiplied with scalar amplitudes. The question whether color is to be included or not is still to be answered. The color issue is tackled by interpreting only one of the factors to have color. Reviewing results in QED one could get equipped with the results for 6-point scalar amplitudes, but instead this thesis will take an unconventional approach as done in [41]. The scalar amplitude is dressed with color in a BCJ kind of way just like $A(scalar) \sim \sum \frac{c_i}{s_{ij}}$, where the sum is over all possible poles and vaguely proposed for the 4-point case. Generalizing this approach to 6-point one expects products of s_{ij}s in the denominator to account for possible poles. Combining this with the results of [51], new results are obtained for color factor of 6-point quark pair amplitudes that may be classified into leading order, sub-leading order and sub-sub-leading order which is determined by the value of $p = 0, 1, 2$ respectively in $\mathcal{O}(\frac{1}{N_c^p})$.

Opening the problem, one encounters that helicity constrains the set of non-zero contributions. The explanation is that quark pairs must have opposite helicity in case all external legs are outgoing, since no interaction vertex can change the helicity. Hence the amplitudes with three quark pairs must be NMHV amplitudes. Thus firstly the order in the color N_c is chosen as either $\mathcal{O}(1/N_c^0), \mathcal{O}(1/N_c), \mathcal{O}(1/N_c^2)$. Already, at this point exploiting the KLT relations for sub and sub-sub-leading orders rises the question of how to manage factors of N_c on the left-hand side, where only products of pure scalar amplitudes with pure gluonic amplitudes occur. Both of which not even the gluonic amplitude can contribute through the normalization factor of its color-factor, since it is properly defined to be a Kronecker-delta function. In using the KLT diagram rules one gets color factors, $c_i \sim f^{abc} f^{dec}$, from the scalar amplitude of the left-hand side, but the

expression still misses the order factors of N_c. The interpretation regarding this problem is elaborated later on in this section. Instead for now, an example is worked out.

A good starting point seems to be the simplest pure fermionic KLT construction with two fermion pairs $(q, \bar{q})$,

$$\mathcal{M}_4(1_g^{-a_1}, 2_g^{+a_2}, 3_g^{-a_3}, 4_g^{+a_4}) = -i(\frac{\kappa}{2})^2 s_{12} A_4(1_{\bar{q}}^{-a_1}, 2_q^{+a_2}, 3_{\bar{q}}^{-a_3}, 4_q^{+a_4}) \widetilde{A}_4(1_{\bar{q}}^-, 2_q^+, 4_q^+, 3_{\bar{q}}^-)$$

$$(9.14)$$

$$\mathcal{M}_4(1_g^{-a_1}, 2_g^{+a_2}, 3_g^{-a_3}, 4_g^{+a_4}) = -i(\frac{\kappa}{2})^2 s_{12} A_4(1_g^-, 2_g^+, 4_g^+, 3_g^-) \widetilde{A}_4(1_s^{-a_1}, 2_s^{+a_2}, 4_s^{+a_4}, 3_s^{-a_3}).$$

$$(9.15)$$

Working out the right-hand side of (9.14) one gets,

$$
\begin{aligned}
A_4(1_{\bar{q}}^{-a_1}, 2_q^{+a_2}, 3_{\bar{q}}^{-a_3}, 4_q^{+a_4}) = \quad & \delta_{a_1 a_4} \delta_{a_3 a_2} A^0(1_{\bar{q}}^-, 4_q^+, 3_{\bar{q}}^-, 2_q^+) \\
& - \frac{1}{N_c} \delta_{a_1 a_2} \delta_{a_3 a_4} A^0(1_{\bar{q}}^-, 2_q^+, 3_{\bar{q}}^-, 4_q^+) \\
= \quad & \delta_{a_1 a_4} \delta_{a_3 a_2} \frac{-\langle 1\,3 \rangle^2}{\langle 1\,4 \rangle \langle 3\,2 \rangle} - \frac{1}{N_c} \delta_{a_1 a_2} \delta_{a_3 a_4} \frac{-\langle 1\,3 \rangle^2}{\langle 1\,2 \rangle \langle 3\,4 \rangle} \\
= \quad & \frac{\langle 1\,3 \rangle^2 [2\,1]}{\langle 3\,4 \rangle} \left(\frac{\delta_{a_1 a_4} \delta_{a_3 a_2}}{t} + \frac{1}{N_c} \frac{\delta_{a_1 a_2} \delta_{a_3 a_4}}{s} \right) \quad (9.16)
\end{aligned}
$$

$$s_{12} \frac{[2\,1]}{\langle 3\,4 \rangle} = \quad [4\,3][2\,1] \quad (9.17)$$

$$
\begin{aligned}
[4\,3][2\,1] \frac{\langle 1\,3 \rangle^3 \langle 2\,4 \rangle}{\langle 1\,2 \rangle \langle 2\,4 \rangle \langle 4\,3 \rangle \langle 3\,1 \rangle} = \quad & [4\,3][2\,1] \frac{-\langle 1\,3 \rangle^2}{\langle 1\,2 \rangle \langle 4\,3 \rangle} = \frac{[2\,1]\langle 1\,3 \rangle \langle 1\,2 \rangle [2\,4]}{\langle 1\,2 \rangle \langle 4\,3 \rangle} \\
= \quad & \frac{\langle 1\,3 \rangle [1\,2][2\,4]}{\langle 4\,3 \rangle} = [4\,2]^2 \quad (9.18)
\end{aligned}
$$

$$\mathcal{M}_4(1_g^{-a_1}, 2_g^{+a_2}, 3_g^{-a_3}, 4_g^{+a_4}) = -i(\frac{\kappa}{2})^2 [2\,4]^2 \langle 1\,3 \rangle^2 \left(\frac{\delta_{a_1 a_4} \delta_{a_3 a_2}}{t} + \frac{1}{N_c} \frac{\delta_{a_1 a_2} \delta_{a_3 a_4}}{s} \right) \quad (9.19)$$

where the labels of the δ-functions are of the fundamental representation. Working on the right-hand side of (9.15) one gets,

$$\mathcal{M}_4(1_g^{-a_1}, 2_g^{+a_2}, 3_g^{-a_3}, 4_g^{+a_4}) = -i(\frac{\kappa}{2})^2 [2\,4]^2 \langle 1\,3 \rangle^2 \frac{-1}{u} \left(\frac{c_s}{s} + \frac{c_t}{t} \right), \quad (9.20)$$

where c_t, c_s are defined as in (2.5). Hence a pattern identification with the quark pair side is for example that $\frac{-c_s}{u} \rightarrow \delta_{a_1 a_2} \delta_{a_3 a_4}$. This is not the most convincing kind of pattern, due to the factor of $\frac{-1}{u}$, but one may go on with further calculations for higher point cases and try to guess the results implied by this identification. The result may turn out to be correct for higher point cases if the assumption holds. This is the kind of approach that will be taken in the following computation of the 6-point case. Of course there is still the missing factor $1/N_c$ for the sub-leading order, which will be taken care of later.

In the 6-point case the amplitudes with permutations corresponding to the leading order, $\mathcal{O}(1/N_c^0)$, are [51],

$$A_1 = A_6^0(1_q^+, 2_{\bar{q}}^-, 3_p^+, 4_{\bar{p}}^-, 5_r^+, 6_{\bar{r}}^-)$$
$$= \frac{\langle 2\,4\rangle^2\,[1\,5]^2\,\langle 4|\,3+4\,|5]}{s_{234}\,\langle 3\,4\rangle\,[5\,6]\,\langle 4|\,3+2\,|1]\,\langle 2|\,3+4\,|5]}$$
$$+ \frac{\langle 2\,6\rangle^2\,[3\,5]^2\,\langle 2|\,5+4\,|3]}{s_{345}\,\langle 1\,2\rangle\,[3\,4]\,\langle 6|\,1+2\,|3]\,\langle 2|\,3+4\,|5]}$$
$$- \frac{\langle 4\,6\rangle^2\,[1\,3]^2\,\langle 6|\,3+2\,|1]}{s_{123}\,[1\,2]\,\langle 5\,6\rangle\,\langle 4|\,3+2\,|1]\,\langle 6|\,1+2\,|3]}$$

$$(9.21)$$

$$A_2 = A_6^0(1_q^+, 2_{\bar{q}}^-, 3_p^+, 4_{\bar{p}}^-, 5_r^-, 6_{\bar{r}}^+)$$
$$= \frac{\langle 2\,4\rangle^2\,[6\,1]^2\,\langle 4|\,3+4\,|5]}{s_{234}\,\langle 3\,4\rangle\,[5\,6]\,\langle 4|\,3+2\,|1]\,\langle 2|\,3+4\,|5]}$$
$$- \frac{\langle 2|\,5+4\,|3]^3}{s_{345}\,\langle 1\,2\rangle\,[3\,4]\,\langle 6|\,5+4\,|3]\,\langle 2|\,3+4\,|5]}$$
$$+ \frac{\langle 4\,5\rangle^2\,[1\,3]^2\,\langle 6|\,3+2\,|1]}{s_{123}\,[1\,2]\,\langle 5\,6\rangle\,\langle 4|\,3+2\,|1]\,\langle 6|\,5+4\,|3]}$$

$$(9.22)$$

A nice coincidence is that the sub-sub-leading amplitudes are related to above just by cyclic permutations. The notation of the above amplitudes is to always keep q on position 1 and gather the quark pairs into groups of two element starting from position 2 and going to the right. This means that the quark pair groupings in (9.21) are $\{(2_{\bar{q}}^-, 3_p^+), (4_{\bar{p}}^-, 5_r^+), (6_{\bar{r}}^-, 1_q^+)\}$ where A_2 has $\{(2_{\bar{q}}^-, 3_p^+), (4_{\bar{p}}^-, 5_r^-), (6_{\bar{r}}^+, 1_q^+)\}$ instead. This corresponds to the case $p = 0$. If all quark flavors are taken to be the same, then A_2 will vanish due to helicity conservation.

The 6-point KLT relation nicely takes it form through the momentum S-kernel formulation. In the field theory limit the 6-point KLT relation is,

$$M_6(1,2,3,4,5,6) = -s_{12}s_{13}A_6^{\text{tree}}(1,2,3,4,5,6)\big[s_{14}\widetilde{A}_6^{\text{tree}}(5,6,2,3,4,1)$$
$$+(s_{14}+s_{34})\widetilde{A}_6^{\text{tree}}(5,6,2,4,3,1) + (s_{14}+s_{34}+s_{24})\widetilde{A}_6^{\text{tree}}(5,6,4,2,3,1)\big]$$
$$-s_{12}(s_{13}+s_{23})A_6^{\text{tree}}(1,2,3,4,5,6)\big[s_{14}\widetilde{A}_6^{\text{tree}}(5,6,3,2,4,1)$$
$$+(s_{14}+s_{24})\widetilde{A}_6^{\text{tree}}(5,6,3,4,2,1) + (s_{14}+s_{24}+s_{34})\widetilde{A}_6^{\text{tree}}(5,6,4,3,2,1)\big]$$
$$+\mathcal{P}(2,3,4)$$

$$(9.23)$$

The color is assigned to the amplitudes without tilde. This implies for the permutations,

$$\mathcal{P}(2,3,4) = \{(234),(243),(324),(342),(423),(432)\}, \tag{9.24}$$

that a color order can be assigned to $A_6^{\text{tree}}(1,2,3,4,5,6)$, since it is responsible for the color factor. Writing out the permutations,

(123456)	(124356)	(132456)	(134256)	(142356)	(143256)	
0	x	0	1	x	1	(9.25)

where 0,1 and x indicate respectively leading order, sub-leading order and vanishing factor. Further analysis can be worked out for all permutations of the amplitude with tilde, where zeros will occur due to vanishing pairings like quark-quark pairs and anti-quark-anti-quark pair that both violate helicity conservation in the case of equal flavor and are not allowed in the theory. However, the tilde amplitude is not dressed with color in the present case. Notice in (9.25) that two permutations are possible for the leading order case. Working out a nice scheme for permutations of $\widetilde{A}$ one is able to identify one column for each permutation case in (9.23),

$$
\begin{array}{cccccc}
(562341) & (562431) & (563241) & (563421) & (564231) & (564321) \\
 & \text{x} & & & \text{x} & \\
(562431) & (562341) & (563421) & (563241) & (564321) & (564231) \\
\text{x} & & & & & \text{x} \\
(564231) & (563241) & (564321) & (562341) & (563421) & (562431) \\
\text{x} & & & & & \text{x} \\
(563241) & (564231) & (562341) & (564321) & (562431) & (563421) \\
 & \text{x} & & & \text{x} & \\
(563421) & (564321) & (562431) & (564231) & (562341) & (563241) \\
 & & \text{x} & \text{x} & & \\
(564321) & (563421) & (564231) & (562431) & (563241) & (562341) \\
 & & \text{x} & \text{x} & & \\
\end{array}
\tag{9.26}
$$

here $(562341) = A_6^{\text{tree}}(5, 6, 2, 3, 4, 1) = A_6^{\text{tree}}(1, 5, 6, 2, 3, 4)$, since the first leg is always held fixed when determining quark pairs. Unfortunately the notation is inconsistent in how the pairing is understood for a color-ordered amplitude. This is observed when reverting the order of the indices for instance $A_6^{\text{tree}}(1, 2, 3, 4, 5, 6)$ has $\{(23), (45), (61)\}$ and $A_6^{\text{tree}}(6, 5, 4, 3, 2, 1) = A_6^{\text{tree}}(1, 6, 5, 4, 3, 2)$ has $\{(65), (43), (21)\}$. Extracting the zeros

from the KLT relation one gets,

$$
\begin{aligned}
M6 = {}& -s_{12}s_{13}s_{14}A_6^{\text{tree}}(1,2,3,4,5,6)A_6^{\text{tree}}(5,6,2,3,4,1) \\
& -s_{12}s_{13}s_{14}A_6^{\text{tree}}(1,4,3,2,5,6)A_6^{\text{tree}}(5,6,4,3,2,1) \\
& -s_{13}s_{14}(s_{12}+s_{23})A_6^{\text{tree}}(1,3,2,4,5,6)A_6^{\text{tree}}(5,6,2,3,4,1) \\
& -s_{13}s_{14}A_6^{\text{tree}}(1,3,4,2,5,6)(s_{12}A_6^{\text{tree}}(5,6,3,4,2,1)+(s_{12}+s_{24})A_6^{\text{tree}}(5,6,3,2,4,1) \\
& +(s_{12}+s_{23}+s_{24})A_6^{\text{tree}}(5,6,2,3,4,1)) \\
& -s_{14}(s_{13}+s_{34})A_6^{\text{tree}}(1,4,3,2,5,6)(A_6^{\text{tree}}(5,6,3,4,2,1)s_{12}+A_6^{\text{tree}}(5,6,3,2,4,1)(s_{12}+s_{24}) \\
& +(s_{12}+s_{23}+s_{24})A_6^{\text{tree}}(5,6,2,3,4,1)) \\
& -s_{12}s_{13}(s_{14}+s_{34})A_6^{\text{tree}}(1,3,4,2,5,6)A_6^{\text{tree}}(5,6,4,3,2,1) \\
& -s_{12}s_{14}A_6^{\text{tree}}(1,2,4,3,5,6)((s_{13}+s_{34})A_6^{\text{tree}}(5,6,2,3,4,1) \\
& +(s_{13}+s_{23}+s_{34})A_6^{\text{tree}}(5,6,3,2,4,1)) \\
& -s_{14}(s_{12}+s_{24})A_6^{\text{tree}}(1,4,2,3,5,6)((s_{13}+s_{34})A_6^{\text{tree}}(5,6,2,3,4,1) \\
& +(s_{13}+s_{23}+s_{34})A_6^{\text{tree}}(5,6,3,2,4,1)) \\
& -s_{12}s_{14}A_6^{\text{tree}}(1,4,2,3,5,6)((s_{13}+s_{23})A_6^{\text{tree}}(5,6,4,3,2,1) \\
& +(s_{13}+s_{23}+s_{34})A_6^{\text{tree}}(5,6,3,4,2,1)) \\
& -s_{12}(s_{14}+s_{24})A_6^{\text{tree}}(1,2,4,3,5,6)((s_{13}+s_{23})A_6^{\text{tree}}(5,6,4,3,2,1) \\
& +(s_{13}+s_{23}+s_{34})A_6^{\text{tree}}(5,6,3,4,2,1)) \\
& -s_{12}s_{13}A_6^{\text{tree}}(1,3,2,4,5,6)(s_{14}A_6^{\text{tree}}(5,6,3,2,4,1)+(s_{14}+s_{24})A_6^{\text{tree}}(5,6,3,4,2,1) \\
& +(s_{14}+s_{24}+s_{34})A_6^{\text{tree}}(5,6,4,3,2,1)) \\
& -s_{12}(s_{13}+s_{23})A_6^{\text{tree}}(1,2,3,4,5,6)(s_{14}A_6^{\text{tree}}(5,6,3,2,4,1) \\
& +(s_{14}+s_{24})A_6^{\text{tree}}(5,6,3,4,2,1)+(s_{14}+s_{24}+s_{34})A_6^{\text{tree}}(5,6,4,3,2,1)).
\end{aligned}
$$

$$(9.27)$$

In this relations 6 of the 15 s_{ij}s may be eliminated. Furthermore one should check that the amplitudes are independent. And notice that this relation does not distinguish between any orders, hence a more careful treatment is needed where corresponding orders for the non tilde amplitude are inserted. Refining the computation to leading order the result is using Mathematica,

$$
\begin{aligned}
M6 = {}& (-s_{24}-s_{34}-s_{45}-s_{46})(-s_{34}-s_{35}-s_{36}-s_{45}-s_{46}-s_{56}) \\
& \times (s_{24}+s_{25}+s_{26}+s_{45}+s_{46}+s_{56})A_6^{\text{tree}}(1,2,3,4,5,6)A_6^{\text{tree}}(5,6,2,3,4,1) \\
& -(-s_{34}-s_{35}-s_{36})(s_{34}+s_{35}+s_{36}+s_{45}+s_{46}+s_{56})A_6^{\text{tree}}(1,2,3,4,5,6) \\
& \times (A_6^{\text{tree}}(5,6,3,2,4,1)(-s_{24}-s_{34}-s_{45}-s_{46})+A_6^{\text{tree}}(5,6,3,4,2,1) \\
& (-s_{34}-s_{45}-s_{46})+A_6^{\text{tree}}(5,6,4,3,2,1)(-s_{45}-s_{46})) \\
& -(-s_{24}-s_{25}-s_{26})(-s_{24}-s_{34}-s_{45}-s_{46})A_6^{\text{tree}}(1,3,2,4,5,6)A_6^{\text{tree}}(5,6,2,3,4,1) \\
& \times (s_{24}+s_{25}+s_{26}+s_{45}+s_{46}+s_{56}) \\
& -(s_{34}+s_{35}+s_{36}+s_{45}+s_{46}+s_{56})A_6^{\text{tree}}(1,3,2,4,5,6) \\
& \times (s_{24}+s_{25}+s_{26}+s_{45}+s_{46}+s_{56}) \\
& \times (A_6^{\text{tree}}(5,6,3,2,4,1)(-s_{24}-s_{34}-s_{45}-s_{46})+A_6^{\text{tree}}(5,6,3,4,2,1) \\
& \times (-s_{34}-s_{45}-s_{46})+A_6^{\text{tree}}(5,6,4,3,2,1)(-s_{45}-s_{46}))
\end{aligned}
$$

$$(9.28)$$

The involved amplitudes should be related by symmetry to A_1, A_2 of (9.21). And more carefulness should be added to the treatment, since the above results lacks the appropriate delta functions of (2.14) which are by no means equal. Next step is then to plug in the NMHV results for three quark pairs. When the day is the computation of the right-hand side will tell something about the left-hand side of the KLT relation, which has a form of products of pure scalar amplitudes multiplied with pure gluonic amplitudes of gauge theory. In the latter amplitudes of the left-hand side the color part is assigned to the scalar amplitude. It is still unclear how the orders of N_c will appear in order to match those of the right-hand side.

9.3   The 3 Gluino Pair Computation

Choosing instead to examine the supersymmetric case, gluinos suddenly take the focus in the picture. Wondering if the disease of the unmatched $1/N_c$ factors still appear in the KLT relation one first examines the color factor of the gluino pairs,

$$\mathcal{M}_4(1_g^{-a_1}, 2_g^{+a_2}, 3_g^{-a_3}, 4_g^{+a_4}) = -i(\frac{\kappa}{2})^2 s_{12} A_4(1_{\widetilde{\Lambda}}^{-a_1}, 2_{\Lambda}^{+a_2}, 3_{\widetilde{\Lambda}}^{-a_3}, 4_{\Lambda}^{+a_4}) \widetilde{A}_4(1_{\widetilde{\Lambda}}^{-}, 2_{\Lambda}^{+}, 4_{\Lambda}^{+}, 3_{\widetilde{\Lambda}}^{-})$$

$$(9.29)$$

$$\mathcal{M}_4(1_g^{-a_1}, 2_g^{+a_2}, 3_g^{-a_3}, 4_g^{+a_4}) = -i(\frac{\kappa}{2})^2 s_{12} A_4(1_g^{-}, 2_g^{+}, 4_g^{+}, 3_g^{-}) \widetilde{A}_4(1_s^{-a_1}, 2_s^{+a_2}, 4_s^{+a_4}, 3_s^{-a_3}).$$

$$(9.30)$$

A quick look in [4] cures the problem. Traces of T^a now appear in a matching pattern in both cases, since the gluinos have color factors similar to gluons without order factors $1/N_c$. Remembering that gluinos are allowed to mix flavor in vertices the computation may seem more difficult. Returning to previously suggested scalar amplitude the structure is identified to be similarly to the structure of a recursion relation in [3]. The recursion in mind uses a KK kind of basis and puts amplitudes into the form $A = \sum_{(n-2)!} A(1, \dots, n) \mathrm{Tr}(f^{abc} f^{des} \dots)$. The important observation is that the just mentioned form actually exactly in a simple case corresponds to the BCJ kind of color assignment for the scalar amplitude with $f^{abx} f^{xcd} \to \mathrm{Tr}(f^{abc} f^{des})$. This aims the target on studying KLT relation in the color stripped case, since the colors now match eachother. One question to be answered is what role the scalar amplitude will play, since KLT must split into products of amplitudes fulfilling the total spin conservation of the particles involved, hence zero spin particles have to be assigned to one of the sectors. Assuming this to be trivial a relation is obtained for gluon amplitudes through the KLT relation in terms of pure gluino pair amplitudes. Finally, this new result will be very interesting since these kind of relations usually are obtained through supersymmetric Ward identities.

Appendices

Appendix A

Helicity Identities

Spinor products in terms of Lorentz products:

$$\langle i\,j\rangle \;=\; \overline{u_-(k_i)}u_+(k_j) = u_-^\dagger(k_i)\gamma^0 u_+(k_j) \tag{A.1}$$

$$= \;\frac{1}{\sqrt{2}}\left(\sqrt{k^-}e^{i\varphi_{k_i}}\,,\; -\sqrt{k_i^+}\,,\; -\sqrt{k_i^-}e^{i\varphi_{k_i}}\,,\; \sqrt{k_i^+}\right)\gamma^0\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{k_j^+}\\ \sqrt{k_j^-}e^{i\varphi_{k_j}}\\ \sqrt{k_j^+}\\ \sqrt{k_j^-}e^{i\varphi_{k_j}}\end{pmatrix} \tag{A.2}$$

$$= \;\sqrt{k_i^- k_j^+}\,e^{i\varphi_{k_i}} - \sqrt{k_i^+ k_j^-}\,e^{i\varphi_{k_j}} \tag{A.3}$$

$$= \;\sqrt{k_i^- k_j^+}\frac{k_i^1 + ik_i^2}{\sqrt{k_i^+ k_i^-}} - \sqrt{k_i^+ k_j^-}\frac{k_j^1 + ik_j^2}{\sqrt{k_j^+ k_j^-}} \tag{A.4}$$

$$= \;\frac{1}{\sqrt{k_i^+ k_j^+}}\left[k_j^+ k_i^1 - k_i^+ k_j^1 + i(k_j^+ k_i^2 - k_i^+ k_j^2)\right] \tag{A.5}$$

$$= \;\sqrt{|s_{ij}|}\,e^{i\phi_{ij}} \tag{A.6}$$

Antisymmetry of the spinors:

$$\langle i\,j\rangle = \langle i^-|j^+\rangle = \overline{u_(k_i)}u_+(k_j) = \sqrt{|s_{ij}|}e^{i\phi_{ij}} = -\sqrt{|s_{ji}|}e^{i\phi_{ji}} = -\langle j\,i\rangle \tag{A.7}$$

because of the antisymmetry in (i,j) in the cos and sin expressions below:

$$\cos\phi_{ij} \;=\; \frac{k_i^1 k_j^+ - k_j^1 k_i^+}{\sqrt{|s_{ij}|k_i^+ k_j^+}}\,,\qquad \sin\phi_{ij} \;=\; \frac{k_i^2 k_j^+ - k_j^2 k_i^+}{\sqrt{|s_{ij}|k_i^+ k_j^+}}\,. \tag{A.8}$$

The momentum assignments used above are,

$$k^0 = \frac{k^+ + k^-}{2} \tag{A.9}$$

$$k^1 = \frac{\sqrt{k^+ k^-}}{2} \left(e^{i\phi_k} + e^{-i\phi_k} \right) \tag{A.10}$$

$$k^2 = \frac{\sqrt{k^+ k^-}}{2i} \left(e^{i\phi_k} - e^{-i\phi_k} \right) \tag{A.11}$$

$$k^3 = \frac{k^+ - k^-}{2}. \tag{A.12}$$

Appendix B

Parity Reversing All Helicities

One has to show that a parity operation on an amplitude reverses all helicities and apply this particularly to a MHV-amplitude. The parity operator has the following properties, the operator reverses the 3-momentum, it is unitary and invoking the operator twice returns an unchanged momentum state. Formulated mathematically this says $P^{-1} = P^\dagger = P$.

Following the notation in [3]:

$$
u_+(\widetilde{k}) = \frac{1}{\sqrt{2}}
\begin{bmatrix}
\sqrt{\widetilde{k}^+} \\
\sqrt{\widetilde{k}^-}e^{i\phi_{\widetilde{k}}} \\
\sqrt{\widetilde{k}^+} \\
\sqrt{\widetilde{k}^-}e^{i\phi_{\widetilde{k}}}
\end{bmatrix}
= \frac{1}{\sqrt{2}}
\begin{bmatrix}
\sqrt{k^-} \\
-\sqrt{k^+}e^{i\phi_k} \\
\sqrt{k^-} \\
-\sqrt{k^+}e^{i\phi_k}
\end{bmatrix}
= \gamma^0 \frac{1}{\sqrt{2}}
\begin{bmatrix}
\sqrt{k^-} \\
-\sqrt{k^+}e^{i\phi_k} \\
-\sqrt{k^-} \\
\sqrt{k^+}e^{i\phi_k}
\end{bmatrix}
$$

$$
= \gamma^0 e^{i\phi_k} \frac{1}{\sqrt{2}}
\begin{bmatrix}
\sqrt{k^-}e^{-i\phi_k} \\
-\sqrt{k^+} \\
-\sqrt{k^-}e^{-i\phi_k} \\
\sqrt{k^+}
\end{bmatrix}
= \gamma^0 e^{i\phi_k} u_-(k). \tag{B.1}
$$

Hence the spinor product satisfies the relation,

$$
Pu_+(k)P = \gamma^0 e^{i\phi_k} u_-(k) \Rightarrow u_+(k) = \gamma^0 P e^{i\phi_k} u_-(k)P
$$
$$
\Rightarrow \gamma^0 u_+(k) = -e^{i\phi_k} P u_-(k)P \Rightarrow u_-(\widetilde{k}) = -\gamma^0 e^{-i\phi_k} u_+(k) \tag{B.2}
$$
$$
P \langle i\,j \rangle P = P \overline{u_-(k_i)} u_+(k_j) P = P u_-^\dagger(k_i) \gamma^0 P P u_+(k_j) P
$$
$$
= (Pu_-(k_i)P)^\dagger \gamma^0 P u_+(k_j)P = (-\gamma^0 e^{-i\phi_{k_i}} u_+(k_i))^\dagger \gamma^0 \gamma^0 e^{i\phi_{k_i}} u_-(k_j)
$$
$$
= -u_+^\dagger(k_i) \gamma^0 e^{i\phi_{k_i}} e^{i\phi_{k_j}} u_-(k_j) = -e^{i(\phi_{k_i}+\phi_{k_j})} \overline{u_+(k_i)} u_-(k_j)
$$
$$
= -e^{i(\phi_{k_i}+\phi_{k_j})} [i\,j] \tag{B.3}
$$

The phase can be dropped, since it will just contribute to an overall phase in the end. The conclusion is that parity allows one to flip all helicities by exchanging $\langle i\,j \rangle \leftrightarrow [i\,j]$ and multiplying with -1 if an odd number of gluons appear in the amplitude.

Ex. Knowing that $A(1^+, 2^-, 3^-) = i\frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}$ one can compute $A(1^-, 2^+, 3^+)$,

$$
\begin{aligned}
A(1^-, 2^+, 3^+) &= P(1^+, 2^-, 3^-)P = iP\frac{1}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle}\langle 2\,3\rangle^4\,P \\
&= i(P\,\langle 1\,2\rangle\,PP\,\langle 2\,3\rangle\,PP\,\langle 3\,1\rangle\,P)^{-1}(P\,\langle 2\,3\rangle\,P)^4 \\
&= i(-1)^3 e^{-2i(\phi_{k_1}+\phi_{k_2}+\phi_{k_3})}e^{4i(\phi_{k_2}+\phi_{k_3})}\frac{[2\,3]^4}{[1\,2]\,[2\,3]\,[3\,1]} \\
&= -i\frac{[2\,3]^4}{[1\,2]\,[2\,3]\,[3\,1]}\cdot(\text{phase})
\end{aligned}
\tag{B.4}
$$

Next step is to attack the case of a MHV-amplitude:

$$
\begin{aligned}
A_n^{\text{tree},MHV,jk} &\equiv A_n^{tree}(1^+, \ldots, j^-, \ldots, k^-, \ldots, n^+) = i\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle} \\
&= (-1)^n i\frac{(P\,[j\,k]\,P)^4}{P\,[1\,2]\,PP\,[2\,3]\,P\cdots P\,[n\,1]\,P} \\
&= (-1)^n i\frac{P\,[j\,k]^4\,P}{P\,[1\,2]\,[2\,3]\cdots[n\,1]\,P} \\
&= (-1)^n iP\,[j\,k]^4\,PP^{-1}\frac{1}{[1\,2]\,[2\,3]\cdots[n\,1]}P^{-1} \\
&= P(-1)^n i\frac{[j\,k]^4}{[1\,2]\,[2\,3]\cdots[n\,1]}P \\
&= PA_n^{tree}(1^-, \ldots, j^+, \ldots, k^+, \ldots, n^-)P,
\end{aligned}
\tag{B.5}
$$

which shows the basic idea to be used in a proof by induction. Charge conjugation flips the helicity on a quark line, that is seen through the transformation of the helicity part of the spinor-helicity state. A representation for charge conjugation is $C = i\gamma^2$ and furthermore the operator has the properties that it exchanges quarks for anti-quarks and it is a unitary operator. Thus $C^{-1} = C^\dagger = C$.

$$
C^{-1}|i^+\rangle C = Cu_+(k_i)C = C\frac{1}{2}(1+\gamma^5)u(k_i)C \tag{B.6}
$$

$$
C\gamma^5 C = i\gamma^2\gamma^5 i\gamma^2 = -1\gamma^2\gamma^5\gamma^2 = (\gamma^2)^2\gamma^5 = -\gamma^5 \tag{B.7}
$$

$$
\Rightarrow C|i^+\rangle C = \frac{1}{2}(1-\gamma^5)u(k_i) = u_-(k_i) = v_+(k_i), \tag{B.8}
$$

here it was used that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_{4\times 4}$ and that $(\gamma^i)^2 = -\mathbf{1}_{4\times 4}$ for $i = 1, 2, 3$.

Appendix C

Playing with Mandelstam Variables

The Mandelstam variables have a nice vanishing relation that will be established in the derivation below using $k_i^2 = 0$ and $\sum_i k_i = 0 \Rightarrow (\sum_i k_i)^2 = 0$ and all momenta outgoing,

$$(\sum_i k_i)^2 = 2\left[(k_1 \cdot k_2 + k_1 \cdot k_3 + k_1 \cdot k_4 + k_2 \cdot k_3 + k_2 \cdot k_4 + k_3 \cdot k_4\right]$$

$$k_1^2 = 0 \quad \Rightarrow \quad (k_2 + k_3 + k_4)^2 = 0 \quad \Rightarrow \quad k_2 \cdot k_3 + k_2 \cdot k_4 + k_3 \cdot k_4 = 0$$

$$\Rightarrow k_1 \cdot k_2 + k_1 \cdot k_3 + k_1 \cdot k_4 = 0 \tag{C.1}$$

$$s + t + u = (k_1 + k_2)^2 + (k_1 + k_4) + (k_1 + k_3)^2 = 2\left[k_1 \cdot k_2 + k_1 \cdot k_4 + k_1 \cdot k_3\right]$$

$$\Rightarrow s + t + u = 0$$

Appendix D

Mathematica Sheets

```
ClearAll["Global'*"]

SetDirectory[NotebookDirectory[]];

<< def.math

(*s[i_,j_]:=spa[i,j]spb[j,i];*)

M6 = -s[1, 2] s[1,
    3] A_ 123456*(s[1,
        4] A_ 562341 + (s[1, 4] + s[4, 3]) A_ 562431 + (s[1, 4] + s[4, 3] +
        s[4, 2]) A_ 564231) -
  s[1, 2] (s[1, 3] +
    s[2, 3]) A_ 123456*(s[1,
        4] A_ 563241 + (s[1, 4] + s[4, 2]) A_ 563421 + (s[1, 4] + s[2, 4] +
        s[4, 3]) A_ 564321)

-s[1, 2] s[1,
   4] A_ 124356*(s[1,
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        s[3, 2]) A_ 563241)
-s[1, 2] (s[1, 4] +
   s[2, 4]) A_ 124356*(s[1,
        3] A_ 564231 + (s[1, 3] + s[3, 2]) A_ 564321 + (s[1, 3] + s[2, 3] +
        s[4, 3]) A_ 563421)

-s[1, 3] s[1,
   2] A_ 132456*(s[1,
        4] A_ 563241 + (s[1, 4] + s[4, 2]) A_ 563421 + (s[1, 4] + s[4, 2] +
        s[4, 3]) A_ 564321)
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   s[2, 3]) A_ 132456*(s[1,
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```

```
      s[4, 2]) A_ 564231)

-s[1, 3] s[1,
   4] A_ 134256*(s[1,
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       s[2, 3]) A_ 562341)
-s[1, 3] (s[1, 4] +
   s[4, 3]) A_ 134256*(s[1,
      2] A_ 564321 + (s[1, 2] + s[2, 3]) A_ 564231 + (s[1, 2] + s[3, 2] +
      s[4, 2]) A_ 562431)

-s[1, 4] s[1,
   2] A_ 142356*(s[1,
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       s[4, 3]) A_ 563421)
-s[1, 4] (s[1, 2] +
   s[2, 4]) A_ 142356*(s[1,
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      s[3, 2]) A_ 563241)

-s[1, 4] s[1,
   3] A_ 143256*(s[1,
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       s[4, 2]) A_ 562431)
-s[1, 4] (s[1, 3] +
   s[3, 4]) A_ 143256*(s[1,
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      s[3, 2]) A_ 562341)


-123456 A_ s[1, 2] s[1,
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    564231 A_ (s[1, 4] + s[2, 4] + s[3, 4])) -
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-124356 A_ s[1, 2] s[1,
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   563241 A_ (s[1, 3] + s[2, 3] + s[3, 4]))

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   563421 A_ (s[1, 3] + s[2, 3] + s[3, 4]))

-132456 A_ s[1, 2] s[1,
```

```
3] (563241 A_ s[1, 4] + 563421 A_ (s[1, 4] + s[2, 4]) +
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(*redo*)

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      s[3, 2]) A_ 563241] -
  s[1, 2] (s[1, 4] + s[2, 4]) A_ 124356*[
  s[1, 3] A_ 564231 + (s[1, 3] + s[3, 2]) A_ 564321 + (s[1, 3] + s[2, 3] +
      s[4, 3]) A_ 563421] -
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       s[4, 3]) A_ 564321] -
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       s[4, 2]) A_ 562431] -

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       s[4, 2]) A_ 562431] -
  s[1, 4] (s[1, 3] + s[3, 4]) A_ 143256*[
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       s[3, 2]) A_ 562341]


(*redo2*)

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       s[4, 2]) A564231) -
  s[1, 2] (s[1, 3] +
     s[2, 3]) A123456*(s[1,
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       s[4, 3]) A564321) -

  s[1, 2] s[1,
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       s[3, 2]) A563241) -
  s[1, 2] (s[1, 4] +
```

```
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    s[4, 3]) A564321) -
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   s[2, 3]) A132456*(s[1,
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    s[4, 2]) A564231) -

 s[1, 3] s[1,
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    s[2, 3]) A562341) -
 s[1, 3] (s[1, 4] +
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    s[4, 3]) A563421) -
 s[1, 4] (s[1, 2] +
   s[2, 4]) A142356*(s[1,
    3] A562431 + (s[1, 3] + s[4, 3]) A562341 + (s[1, 3] + s[3, 4] +
    s[3, 2]) A563241) -

 s[1, 4] s[1,
   3] A143256*(s[1,
    2] A564321 + (s[1, 2] + s[3, 2]) A564231 + (s[1, 2] + s[3, 2] +
    s[4, 2]) A562431) -
 s[1, 4] (s[1, 3] +
   s[3, 4]) A143256*(s[1,
    2] A563421 + (s[1, 2] + s[4, 2]) A563241 + (s[1, 2] + s[2, 4] +
    s[3, 2]) A562341)


-A134256 s[1, 3] s[1,
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   A562341 (s[1, 2] + s[2, 3] + s[2, 4])) -
 A143256 s[1, 3] s[1,
  4] (A564321 s[1, 2] + A564231 (s[1, 2] + s[2, 3]) +
```

```
     A562431 (s[1, 2] + s[2, 3] + s[2, 4])) -
 A143256 s[1,
    4] (A563421 s[1, 2] + A563241 (s[1, 2] + s[2, 4]) +
    A562341 (s[1, 2] + s[2, 3] + s[2, 4])) (s[1, 3] + s[3, 4]) -
 A134256 s[1,
    3] (A564321 s[1, 2] + A564231 (s[1, 2] + s[2, 3]) +
    A562431 (s[1, 2] + s[2, 3] + s[2, 4])) (s[1, 4] + s[3, 4]) -
 A124356 s[1, 2] s[1,
    4] (A562431 s[1, 3] + A562341 (s[1, 3] + s[3, 4]) +
    A563241 (s[1, 3] + s[2, 3] + s[3, 4])) -
 A142356 s[1,
    4] (s[1, 2] + s[2, 4]) (A562431 s[1, 3] + A562341 (s[1, 3] + s[3, 4]) +
    A563241 (s[1, 3] + s[2, 3] + s[3, 4])) -
 A142356 s[1, 2] s[1,
    4] (A564231 s[1, 3] + A564321 (s[1, 3] + s[2, 3]) +
    A563421 (s[1, 3] + s[2, 3] + s[3, 4])) -
 A124356 s[1,
    2] (s[1, 4] + s[2, 4]) (A564231 s[1, 3] + A564321 (s[1, 3] + s[2, 3]) +
    A563421 (s[1, 3] + s[2, 3] + s[3, 4])) -
 A123456 s[1, 2] s[1,
    3] (A562341 s[1, 4] + A562431 (s[1, 4] + s[3, 4]) +
    A564231 (s[1, 4] + s[2, 4] + s[3, 4])) -
 A132456 s[1,
    3] (s[1, 2] + s[2, 3]) (A562341 s[1, 4] + A562431 (s[1, 4] + s[3, 4]) +
    A564231 (s[1, 4] + s[2, 4] + s[3, 4])) -
 A132456 s[1, 2] s[1,
    3] (A563241 s[1, 4] + A563421 (s[1, 4] + s[2, 4]) +
    A564321 (s[1, 4] + s[2, 4] + s[3, 4])) -
 A123456 s[1,
    2] (s[1, 3] + s[2, 3]) (A563241 s[1, 4] + A563421 (s[1, 4] + s[2, 4]) +
    A564321 (s[1, 4] + s[2, 4] + s[3, 4]))

A1 = (spa[2, 4]^2 spb[1, 5]^2 (spa[4, 3] spb[3, 5] + spa[4, 2] spb[2, 5]))/
    ((s[2, 3] + s[2, 4] + s[3, 4]) spa[3, 4] spb[5,
       6] (spa[4, 3] spb[3, 1] + spa[4, 2] spb[2, 1]) (spa[2, 3] spb[3, 5] +
       spa[2, 4] spb[4, 5])) +

  (spa[2, 6]^2 spb[3, 5]^2 (spa[2, 5] spb[5, 3] + spa[2, 4] spb[4, 3]))/
    ((s[3, 4] + s[3, 5] + s[4, 5]) spa[1, 2] spb[3,
       4] (spa[6, 1] spb[1, 3] + spa[6, 2] spb[2, 3]) (spa[2, 3] spb[3, 5] +
       spa[2, 4] spb[4, 5])) -

  (spa[4, 6]^2 spb[1, 3]^2 (spa[6, 3] spb[3, 1] + spa[6, 2] spb[2, 1]))/
    ((s[1, 2] + s[1, 3] + s[2, 3]) spb[1, 2] spa[5,
       6] (spa[4, 3] spb[3, 1] + spa[4, 2] spb[2, 1]) (spa[6, 1] spb[1, 3] +
       spa[6, 2] spb[2, 3]))
```

```
-(spa[4, 6]^2 spb[1,
    3]^2 (spa[2, 6] spb[1, 2] + spa[3, 6] spb[1, 3]))/((s[1, 2] + s[1, 3] +
      s[2, 3]) spa[5, 6] spb[1,
    2] (spa[2, 4] spb[1, 2] + spa[3, 4] spb[1, 3]) (-spa[1, 6] spb[1, 3] -
      spa[2, 6] spb[2, 3])) + (spa[2, 6]^2 spb[3,
    5]^2 (-spa[2, 4] spb[3, 4] - spa[2, 5] spb[3, 5]))/((s[3, 4] + s[3, 5] +
    s[4, 5]) spa[1, 2] (-spa[1, 6] spb[1, 3] - spa[2, 6] spb[2, 3]) spb[3,
    4] (spa[2, 3] spb[3, 5] + spa[2, 4] spb[4, 5])) + (spa[2, 4]^2 spb[1,
    5]^2 (-spa[2, 4] spb[2, 5] - spa[3, 4] spb[3, 5]))/((s[2, 3] + s[2, 4] +
    s[3, 4]) spa[3,
    4] (spa[2, 4] spb[1, 2] + spa[3, 4] spb[1, 3]) (spa[2, 3] spb[3, 5] +
      spa[2, 4] spb[4, 5]) spb[5, 6])

(*go on with A2 *)

A2 = (spa[2, 4]^2 spb[6, 1]^2 (spa[4, 3] spb[3, 5] + spa[4, 2] spb[2, 5]))/
   ((s[2, 3] + s[2, 4] + s[3, 4]) spa[3, 4] spb[5,
     6] (spa[4, 3] spb[3, 1] + spa[4, 2] spb[2, 1]) (spa[2, 3] spb[3, 5] +
       spa[2, 4] spb[4, 5])) -

  ((spa[2, 5] spb[5, 3] + spa[2, 4] spb[4, 3])^3)/
   ((s[3, 4] + s[3, 5] + s[4, 5]) spa[1, 2] spb[3,
     4] (spa[6, 4] spb[4, 3] + spa[6, 5] spb[5, 3]) (spa[2, 3] spb[3, 5] +
       spa[2, 4] spb[4, 5])) +

  (spa[4, 5]^2 spb[1, 3]^2 (spa[6, 3] spb[3, 1] + spa[6, 2] spb[2, 1]))/
   ((s[1, 2] + s[1, 3] + s[2, 3]) spb[1, 2] spa[5,
     6] (spa[4, 3] spb[3, 1] + spa[4, 2] spb[2, 1]) (spa[6, 5] spb[5, 3] +
       spa[6, 4] spb[4, 3]))

(spa[4, 5]^2 spb[1,
    3]^2 (spa[2, 6] spb[1, 2] + spa[3, 6] spb[1, 3]))/((s[1, 2] + s[1, 3] +
      s[2, 3]) spa[5, 6] spb[1,
    2] (spa[2, 4] spb[1, 2] + spa[3, 4] spb[1, 3]) (spa[4, 6] spb[3, 4] +
      spa[5, 6] spb[3, 5])) - (-spa[2, 4] spb[3, 4] -
    spa[2, 5] spb[3, 5])^3/((s[3, 4] + s[3, 5] + s[4, 5]) spa[1, 2] spb[3,
    4] (spa[4, 6] spb[3, 4] + spa[5, 6] spb[3, 5]) (spa[2, 3] spb[3, 5] +
      spa[2, 4] spb[4, 5])) + (spa[2, 4]^2 spb[1,
    6]^2 (-spa[2, 4] spb[2, 5] - spa[3, 4] spb[3, 5]))/((s[2, 3] + s[2, 4] +
      s[3, 4]) spa[3,
    4] (spa[2, 4] spb[1, 2] + spa[3, 4] spb[1, 3]) (spa[2, 3] spb[3, 5] +
      spa[2, 4] spb[4, 5]) spb[5, 6])

A562431 = 0;
A564231 = 0;
```

```
A562431 = 0;
A564231 = 0;
A564231 = 0;
A562431 = 0;
A564231 = 0;
A562431 = 0;
A562431 = 0;
A564231 = 0;
A564231 = 0;
A562431 = 0;

s[2, 3] = -(s[2, 4] + s[2, 5] + s[2, 6] + s[3, 4] + s[3, 5] + s[3, 6] +
    s[4, 5] + s[4, 6] + s[5, 6]);
s[2, 1] = -(s[2, 3] + s[2, 4] + s[2, 5] + s[2, 6]);
s[3, 1] = -(s[3, 2] + s[3, 4] + s[3, 5] + s[3, 6]);
s[4, 1] = -(s[4, 2] + s[4, 3] + s[4, 5] + s[4, 6]);
s[5, 1] = -(s[5, 2] + s[5, 3] + s[5, 4] + s[5, 6]);
s[6, 1] = -(s[6, 2] + s[6, 3] + s[6, 4] + s[6, 5]);

M6

-A132456 A562341 (-s[2, 4] - s[2, 5] - s[2, 6]) (-s[2, 4] - s[3, 4] -
   s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] + s[2, 6] + s[4, 5] + s[4, 6] +
   s[5, 6]) -
 A123456 (-s[3, 4] - s[3, 5] - s[3, 6]) (A564321 (-s[4, 5] - s[4, 6]) +
   A563421 (-s[3, 4] - s[4, 5] - s[4, 6]) +
   A563241 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6])) (s[3, 4] + s[3, 5] +
   s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) -
 A124356 (A563421 (-s[3, 5] - s[3, 6]) +
   A564321 (-s[3, 4] - s[3, 5] - s[3, 6])) (-s[3, 4] - s[4, 5] -
   s[4, 6]) (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) -
 A142356 (A563421 (-s[3, 5] - s[3, 6]) +
   A564321 (-s[3, 4] - s[3, 5] - s[3, 6])) (-s[2, 4] - s[3, 4] - s[4, 5] -
   s[4, 6]) (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) -
 A132456 (A564321 (-s[4, 5] - s[4, 6]) +
   A563421 (-s[3, 4] - s[4, 5] - s[4, 6]) +
   A563241 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6])) (s[2, 4] + s[2, 5] +
   s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] +
   s[4, 5] + s[4, 6] + s[5, 6]) -
 A134256 A564321 (-s[2, 4] - s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] + s[2, 6] +
   s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] +
   s[4, 6] + s[5, 6]) -
 A123456 A562341 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] +
   s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] +
   s[4, 5] + s[4, 6] + s[5, 6]) -
 A143256 A564321 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] +
```

```
       s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] +
       s[4, 5] + s[4, 6] + s[5, 6]) -
  A124356 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (s[3, 4] + s[3, 5] +
       s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (A563241 (-s[3, 5] - s[3, 6]) +
       A562341 (s[2, 4] + s[2, 5] + s[2, 6] + s[3, 4] + s[4, 5] + s[4, 6] +
          s[5, 6])) -
  A142356 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (s[2, 4] + s[3, 4] +
       s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] +
       s[5, 6]) (A563241 (-s[3, 5] - s[3, 6]) +
       A562341 (s[2, 4] + s[2, 5] + s[2, 6] + s[3, 4] + s[4, 5] + s[4, 6] +
          s[5, 6])) -
  A134256 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] +
       s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (A562341 (-s[2, 5] - s[2, 6]) +
       A563421 (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) +
       A563241 (s[2, 4] + s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] +
          s[5, 6])) -
  A143256 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] +
       s[2, 6] + s[3, 4] + s[4, 5] + s[4, 6] +
       s[5, 6]) (A562341 (-s[2, 5] - s[2, 6]) +
       A563421 (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) +
       A563241 (s[2, 4] + s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] +
          s[5, 6]))

Simplify[%]

-A132456 A562341 (s[2, 4] + s[2, 5] + s[2, 6]) (s[2, 4] + s[3, 4] + s[4, 5] +
     s[4, 6]) (s[2, 4] + s[2, 5] + s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) -
  A124356 (A563421 (s[3, 5] + s[3, 6]) +
       A564321 (s[3, 4] + s[3, 5] + s[3, 6])) (s[3, 4] + s[4, 5] +
       s[4, 6]) (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) -
  A142356 (A563421 (s[3, 5] + s[3, 6]) +
       A564321 (s[3, 4] + s[3, 5] + s[3, 6])) (s[2, 4] + s[3, 4] + s[4, 5] +
       s[4, 6]) (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) -
  A123456 (s[3, 4] + s[3, 5] + s[3, 6]) (A564321 (s[4, 5] + s[4, 6]) +
       A563421 (s[3, 4] + s[4, 5] + s[4, 6]) +
       A563241 (s[2, 4] + s[3, 4] + s[4, 5] + s[4, 6])) (s[3, 4] + s[3, 5] +
       s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) +
  A134256 A564321 (s[2, 4] + s[4, 5] + s[4, 6]) (s[2, 4] + s[2, 5] + s[2, 6] +
       s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] +
       s[4, 6] + s[5, 6]) +
  A123456 A562341 (s[2, 4] + s[3, 4] + s[4, 5] + s[4, 6]) (s[2, 4] + s[2, 5] +
       s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] +
       s[4, 5] + s[4, 6] + s[5, 6]) +
  A143256 A564321 (s[2, 4] + s[3, 4] + s[4, 5] + s[4, 6]) (s[2, 4] + s[2, 5] +
       s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] +
       s[4, 5] + s[4, 6] + s[5, 6]) +
```

```
   A132456 (A564321 (s[4, 5] + s[4, 6]) +
      A563421 (s[3, 4] + s[4, 5] + s[4, 6]) +
      A563241 (s[2, 4] + s[3, 4] + s[4, 5] + s[4, 6])) (s[2, 4] + s[2, 5] +
      s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] +
      s[4, 5] + s[4, 6] + s[5, 6]) +
   A124356 (s[2, 4] + s[3, 4] + s[4, 5] + s[4, 6]) (s[3, 4] + s[3, 5] +
      s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (-A563241 (s[3, 5] + s[3, 6]) +
      A562341 (s[2, 4] + s[2, 5] + s[2, 6] + s[3, 4] + s[4, 5] + s[4, 6] +
        s[5, 6])) +
   A142356 (s[2, 4] + s[3, 4] + s[4, 5] + s[4, 6]) (s[2, 4] + s[3, 4] +
      s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] +
      s[5, 6]) (-A563241 (s[3, 5] + s[3, 6]) +
      A562341 (s[2, 4] + s[2, 5] + s[2, 6] + s[3, 4] + s[4, 5] + s[4, 6] +
        s[5, 6])) -
   A134256 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] +
      s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (-A562341 (s[2, 5] + s[2, 6]) +
      A563421 (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) +
      A563241 (s[2, 4] + s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] +
        s[5, 6])) -
   A143256 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] +
      s[2, 6] + s[3, 4] + s[4, 5] + s[4, 6] +
      s[5, 6]) (-A562341 (s[2, 5] + s[2, 6]) +
      A563421 (s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) +
      A563241 (s[2, 4] + s[3, 4] + s[3, 5] + s[3, 6] + s[4, 5] + s[4, 6] +
        s[5, 6]))

(*s[i_,j_]:=sus [i,j]*)

(*The leading order contribution*)

M6 = -s[1, 2] s[1,
     3] A123456*(s[1,
       4] A562341 + (s[1, 4] + s[4, 3]) A562431 + (s[1, 4] + s[4, 3] +
        s[4, 2]) A564231) -
  s[1, 2] (s[1, 3] +
     s[2, 3]) A123456*(s[1,
       4] A563241 + (s[1, 4] + s[4, 2]) A563421 + (s[1, 4] + s[2, 4] +
        s[4, 3]) A564321) -

  s[1, 3] s[1,
     2] A132456*(s[1,
       4] A563241 + (s[1, 4] + s[4, 2]) A563421 + (s[1, 4] + s[4, 2] +
        s[4, 3]) A564321) -
  s[1, 3] (s[1, 2] +
     s[2, 3]) A132456*(s[1,
       4] A562341 + (s[1, 4] + s[4, 3]) A562431 + (s[1, 4] + s[3, 4] +
```

```
      s[4, 2]) A564231)

-A132456 A562341 (-s[2, 4] - s[2, 5] - s[2, 6]) (-s[2, 4] - s[3, 4] -
    s[4, 5] - s[4, 6]) (s[2, 4] + s[2, 5] + s[2, 6] + s[4, 5] + s[4, 6] +
    s[5, 6]) +
 A123456 A562341 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6]) (-s[3, 4] -
    s[3, 5] - s[3, 6] - s[4, 5] - s[4, 6] - s[5, 6]) (s[2, 4] + s[2, 5] +
    s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) -
 A123456 (-s[3, 4] - s[3, 5] - s[3, 6]) (A564321 (-s[4, 5] - s[4, 6]) +
    A563421 (-s[3, 4] - s[4, 5] - s[4, 6]) +
    A563241 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6])) (s[3, 4] + s[3, 5] +
    s[3, 6] + s[4, 5] + s[4, 6] + s[5, 6]) -
 A132456 (A564321 (-s[4, 5] - s[4, 6]) +
    A563421 (-s[3, 4] - s[4, 5] - s[4, 6]) +
    A563241 (-s[2, 4] - s[3, 4] - s[4, 5] - s[4, 6])) (s[2, 4] + s[2, 5] +
    s[2, 6] + s[4, 5] + s[4, 6] + s[5, 6]) (s[3, 4] + s[3, 5] + s[3, 6] +
    s[4, 5] + s[4, 6] + s[5, 6])
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