GENERAL RELATIVITY

AND

COSMOLOGY

Lecture notes

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Preface

The following lecture notes on general relativity and cosmology grew out of a one semester course on these topics and classical gauge theory by Jan Ambjørn and the present author. Subsequently semesters were abandoned and replaced by "Blocks", which have an extension of approximately only two months. Therefore the classical field theory part (which anyhow strongly needed a revision) was dropped, and the general relativity and cosmology chapters were revised.

These lecture notes are introductory, and do not in any way pretend to be comprehensive. Several important topics have been left out. For example, gravitational radiation is not discussed at all. There are two reasons for the brevity of the notes: the alloted time was short (a couple of months, four hours a week), and it was hoped that by making the notes equally short, there is a bigger chance of getting through that general relativity and cosmology are exciting subjects. Sometimes the trouble with exposing the beauty of physics is that one has to walk a very long way so people start to feel that they rather walk in a desert than in a beautiful garden. Students hungry for more comprehensive studies are referred to the enormous literature.

Poul Olesen

Denne bog / forelæsningsnote udvalgt af Martin Skogstad-von Qualen,
og er en del af et projekt der har til formål at gøre tekster, forfattet
af danske fysikere, tilgængelige for offentligheden. Teksterne, der
hovedsageligt skrevet i 90erne, blev anskaffet i så fine udgaver som muligt,
hvorefter de blev skannet på en fotokopimaskine i en opløsning
på 600 dpi (S/H) i tiff format. De skannede billeder dannede
så udgangspunkt for en pdf-fil, hvorpå der blev udført elektronisk
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Bookmarks og links blev herefter tilføjet for at lette navigationen
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Projektet blev startet i november 2018 og omkring juni 2019
var alle bøger / noter klar til at blive lagt op på www.nbi.ku.dk.

Skulle nogen have spørgsmål til projektet kan Martin kontaktes
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En stor tak til professor emeritus Henrik Smith som også er forfatter
til flere af bøgerne, og til lektor i fysik ved NBI Jens Paaske for
samarbejde, opmuntring, valg af materiale og lån af fine eksemplarer
af værkerne.

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Chapter 1

General relativity

1.1 The principle of equivalence

Einstein's general theory of relativity is a beautiful piece of art which connects gravitational fields with geometry of space and time and thus provides a scheme in which our universe can be discussed.

Einstein's starting point was the principle of equivalence, which can be understood in the context of Newton's mechanics. We have the general equation of motion

$$\vec{F} = m_i \ddot{\vec{x}} \tag{1.1}$$

where $\ddot{\vec{x}}$ is the acceleration and m_i is the inertial mass – for a given force the acceleration is smaller the larger the mass is, i.e., the body is more inert the larger m_i is.

In a constant gravitational field the force is given by

$$\vec{F}_g = m_g \; \vec{g} \tag{1.2}$$

where $\vec{g}$ and m_g are constants. It is clear that a priori the parameter m_g is not related to the inertial mass. Newton made experiments where the period of oscillation of a pendulum made up from different materials were studied, and he found no variation with m_i/m_g. Later on many very precise experiments were made which showed that $m_i = m_g$ to a high accuracy, and this was accepted to such an extent that most text books today (and at Einstein's time) do not bother to put any indices on the masses.

Let us consider a constant gravitational field. With $m_i = m_g = m$ one has the equation of motion

$$\ddot{\vec{x}} = \vec{g} \tag{1.3}$$

Thus, if we introduce the coordinate

$$\vec{y} = \vec{x} - \frac{1}{2} \; \vec{g} t^2 \tag{1.4}$$

we get

$$\ddot{\vec{y}} = 0 \tag{1.5}$$

Therefore we conclude that an observer living in the y-system sees no effect of the gravitational field, because eq. (1.5) shows that particles move in straight lines as if there was no force. On the other hand, eq. (1.4) shows that the observer is freely falling ($\frac{1}{2}\vec{g}t^2$ is just the displacement pertinent to a free fall). All this is true irrespective of any mass because $m_i = m_g$. If $m_i \neq m_g$, the coordinate transformation (1.4) would have to be replaced by

$$\vec{y} = \vec{x} - \frac{1}{2}\frac{m_g}{m_i}\,\vec{g}\,t^2 \tag{1.6}$$

and hence the $\vec{y}$ system would depend on which material we consider through the ratio m_g/m_i.

When $m_g = m_i$ the transformation (1.4) is universal, and is easily seen to eliminate the gravitational field also if other forces (e.g. electrostatic forces) are at work. If the gravitational field varies in space we can apply the transformation (1.4) in a sufficiently small domain.

In Newtonian mechanics we therefore know that an observer in a sufficiently small freely falling elevator is unable to detect a gravitational field. Einstein's principle of equivalence generalizes this to *any* physical phenomena: **In any arbitrary gravitational field it is possible at each space-time point to select locally inertial systems (freely falling small elevators) such that the laws of physics in these are the same as in special relativity**.

One can use this statement to obtain some insight into the way in which gravity influences other physical phenomena by writing down in each of the small elevators some law of physics and then transform it to a general coordinate system. In the next section we shall consider the simplest example, namely a particle which is freely falling in an arbitrary gravitational field.

Some remarks on the history of the Einsteinian version of the equivalence principle[1]*:*
After having finished the special theory of relativity Einstein thought about the problem of how Newton gravity should be modified in order to fit in with special relativity. At this point Einstein experienced what he called the "happiest thought of my life", namely that an observer falling from the roof of a house experiences no gravitational field!

1.2 Gravitation and geometry

Let us consider a particle which moves under the influence of a gravitational field only. Thus in each of the (infinitely many) freely falling systems of inertia we can apply special relativity with no forces acting on the particle.

In special relativity an event is described by a four vector $y^\alpha = (y^0, \vec{y})$, where y^0 is the time. Since the elevators are a priori small, we need, however, to consider an infinitesimal four vector $dy^\alpha = (dy^0, d\vec{y})$. The proper time[2]

$$d\tau^2 = (dy^0)^2 - (d\vec{y})^2 \tag{1.7}$$

is an invariant, i.e., if we make a Lorentz transformation from y^α to y'^α then

$$d\tau^2 = (dy^0)^2 - (d\vec{y})^2 = (dy^0)'^2 - (d\vec{y}\,')^2 \tag{1.8}$$

[1]Most of the historical remarks in these notes are taken from MacTutor History of Mathematics (http://www-history.mcs.st -andrews.ac.uk/HistTopics/General_relativity.html), where much more information can be found.

[2]Here $d\tau^2$ means $(d\tau)^2$. This convention of leaving out the bracket in the square of infinitesimal quantities will be used in the following, unless it leads to confusion.

For light $d\tau = 0$, and eq. (1.8) says that the speed of light $|d\vec{y}/dt|$ is equal to one in all systems (Michelson-Morley's experiment). The proper time has the following physical interpretation: Let us consider a clock (or any physical system which specifies a time, e.g. a particle which decays with a certain life time), which by definition marks time by small intervals dt when the clock is at rest. In the rest system the velocity $\vec{v} = d\vec{y}/dy^0$ vanishes. Thus,

$$d\tau^2 = (dy^0)^2(1 - \vec{v}^2) = (dy^0)^2 \tag{1.9}$$

in the rest system. Thus $d\tau = dy^0$ = the interval between two ticks on the clock (at rest). In a moving system

$$d\tau^2 = (dy^0)'^2 \; (1 - \vec{v}'^2)$$

which leads to the formula for the time dilatation $dy^{0'} = d\tau/\sqrt{1 - v'^2}$.

In four vector notation we write the proper time as

$$d\tau^2 = -\eta_{\alpha\beta} dy^\alpha \; dy^\beta \tag{1.10}$$

where $\eta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\eta_{11} = \eta_{22} = \eta_{33} = 1, \eta_{00} = -1$. This convention is called *mostly positive.* In eq. (1.10) we use the summation convention: Whenever an index occurs two times in a product it is to be summed. Thus, eq. (1.10) means

$$d\tau^2 = -\sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \eta_{\alpha\beta} dy^\alpha dy^\beta \tag{1.11}$$

To obtain the effect of gravity, we should notice that eq. (1.10) is valid in any of the freely falling elevators. However, in general the elevators are different in different space-time points. If we denote the space-time coordinates in an arbitrary coordinate system by x^μ, then the y's are functions of the x's,

$$y^\alpha = y^\alpha(x). \tag{1.12}$$

For an example, see the Newtonian case (1.4). Inserting this in eq. (1.10) we get

$$d\tau^2 = -\eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \, dx^\mu dx^\nu \equiv -g_{\mu\nu}(x) dx^\mu dx^\nu \tag{1.13}$$

where

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} = g_{\nu\mu}(x) \tag{1.14}$$

is called the metric tensor.

Eq. (1.10) has an almost obvious geometric interpretation: In a curved space (e.g. the surface of a sphere) one can introduce local coordinate systems where Euclidian geometry is valid, in spite of the fact that this geometry is not valid in general in curved space. The local Euclidean geometry corresponds to eq. (1.10) if we replace $\eta_{\alpha\beta}$ by $-\delta_{\alpha\beta}$ where $\delta_{\alpha\beta}$ is the Kronecker symbol ($\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta, \delta_{\alpha\beta} = +1$ for $\alpha = \beta$). $d\tau^2$ then means the distance between two points computed by the law of Pythagoras. Then eq. (1.13) is the same distance written in arbitrary coordinates. Similarly, in Einstein's gravity locally one has pseudo-Euclidian=Minkowsky geometry, due to the principle of equivalence, but the geometry of space is, in general, non-pseudo-Euclidean (=non-Minkowskian), and the deviations from "flat space" (=Minkowski space) represent the effects of the gravitational field.

Some remarks on the history of Einstein's geometrical gravity:

According to historians of physics it is not known how Einstein got the idea of relating gravity with geometry. One hypothesis is that he was inspired by a rotating disk. Here the measuring rods will become Lorentz-contracted, and hence the length of the periphery of a circle will be different from $2\pi\times$(radius). This means a deviation from Euclidean geometry. In any case, in papers on gravitation published in 1912, he realized that the Lorentz transformations will not always be applicable in a gravity theory based on the equivalence principle. Space-time were dynamically influenced by gravity. According to the philospher Kant, Euclidean geometry should be considered as an a priori description of space. Therefore the notion of space and time as dynamical quantities represented a strong break with philosophical traditions: Kant had not been right!

Einstein now remembered that as a student he had studied Gauss's theory of surfaces. To proceed, Einstein got help with the mathematical formulation of the theory of general relativity from his former classmate Marcel Grossmann, and the latter (being a professor of descriptive geometry in Zurich) pointed out the relevance of differential geometry that had previously been investigated by a number of mathematicians (Riemann, Ricci, Levi-Civita, ...). Einstein says about this period: "...in all my life I have not laboured so hard, and I have become imbued with great respect for mathematics, the subtler part of which I had in my simple-mindedness regarded as pure luxury until now."

1.3 Motion in an arbitrary gravitational field

We shall now apply the principle of equivalence to see how gravity influences space in the simple case where there are no other forces than gravity. So let us consider a particle which moves under the influence of an arbitrary gravitational field. In the freely falling system special relativity applies, and we have the equation of motion

$$\frac{d^2y^\alpha(x)}{d\tau^2} = 0 \tag{1.15}$$

The solution to this equation is that y^α is a linear function of τ inside a small elevator, where there are no forces. This simply means straight line motion inside the elevator, in accordance with the findings of Gallilei.

Using that the y^α's depend on x^μ we have

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\frac{dy^\alpha}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\partial y^\alpha}{\partial x^\mu}\,\frac{dx^\mu(\tau)}{d\tau}\right) \\ &= \frac{\partial y^\alpha}{\partial x^\mu}\,\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau} \end{aligned} \tag{1.16}$$

This looks somewhat like an equation of motion (because of $d^2x^\mu/d\tau^2$) with a force. We can remove the factor multiplying the second derivative of x^μ by the following trick: By the rules of differentiation we have ($\delta^\lambda_\mu = +1$ for $\lambda = \mu$, $\delta^\lambda_\mu = 0$ for $\lambda \neq \mu$)

$$\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\mu} = \delta^\lambda_\mu \tag{1.17}$$

Thus, multiplying eq. (1.16) by $\partial x^\lambda/\partial y^\alpha$ and summing over α we get

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1.18}$$

where

$$\Gamma^\lambda{}_{\mu\nu} = \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial y^\alpha} \tag{1.19}$$

is called the Christoffel symbol or the affine connection (sometimes denoted $\{{}^{\lambda}_{\mu\nu}\}$). We see that the Christoffel symbol is proportional to the "gravitational force".

So far the metric and $\Gamma^\lambda_{\mu\nu}$ have been expressed in terms of the functional relation between the local freely falling elevators and the arbitrary system x^μ. We shall now show that the Christoffel symbol can be expressed in terms of the metric tensor.

From eq. (1.19) we see that $\Gamma^\lambda_{\mu\nu}$ depends on the second derivatives of y^α, whereas from eq. (1.14) $g_{\mu\nu}$ depends only on the first derivative. Therefore let us differentiate $g_{\mu\nu}$ by use of the definition (1.14),

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \eta_{\alpha\beta}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda}\frac{\partial y^\beta}{\partial x^\nu} + \eta_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial^2 y^\beta}{\partial x^\nu \partial x^\lambda} \tag{1.20}$$

From (1.19) we have

$$\Gamma^\lambda_{\mu\nu}\frac{\partial y^\beta}{\partial x^\lambda} = \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\left(\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^\lambda}\right) = \frac{\partial^2 y^\beta}{\partial x^\mu \partial x^\nu} \tag{1.21}$$

where we used the chain rule for differentiation

$$\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^\lambda} = \delta^\beta_\alpha \tag{1.22}$$

Thus we can express the second derivative of y in terms of Γ by means of eq. (1.21). Using this in eq. (1.20) we get

$$\begin{aligned}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= \eta_{\alpha\beta}\Gamma^\rho_{\mu\lambda}\frac{\partial y^\alpha}{\partial x^\rho}\frac{\partial y^\beta}{\partial x^\nu} + \eta_{\alpha\beta}\Gamma^\rho_{\nu\lambda}\frac{\partial y^\beta}{\partial x^\rho}\frac{\partial y^\alpha}{\partial x^\mu} \\ &= g_{\rho\nu}\Gamma^\rho_{\mu\lambda} + g_{\rho\mu}\Gamma^\rho_{\lambda\nu}\end{aligned} \tag{1.23}$$

where we used the expression (1.14) for the metric. Using eq. (1.23) we get

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2g_{\sigma\nu}\Gamma^\sigma_{\lambda\mu} \tag{1.24}$$

If one wishes $g_{\mu\nu}$ can be thought of as a 4×4 matrix. One can then consider the inverse, which we denote $g^{\mu\nu}$,

$$g_{\mu\nu}(x)g^{\nu\sigma}(x) = \delta^\sigma_\mu \tag{1.25}$$

The inverse exists:

$$g^{\nu\sigma} = \eta^{\alpha\beta}\frac{\partial x^\nu}{\partial y^\alpha}\frac{\partial x^\sigma}{\partial y^\beta} \tag{1.26}$$

because the transformations $y \to x$ and $x \to y$ are non-singular coordinate transformations. Eq. (1.24) now gives the following relation between Γ and g,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\,g^{\lambda\sigma}\left[\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right] \tag{1.27}$$

1.4 The Newton limit

Eqs. (1.18) and (1.27) determine the motion of a particle in a gravitational field provided we know how $g_{\mu\nu}(x)$ depend on the gravitational field. Later we shall see that the second derivatives of $g_{\mu\nu}$ are determined through Einstein's field equations in terms of matter distributions.

At present we shall study a much more modest problem, namely the Newton limit where all velocities are small relative to the velocity of light, $|d\vec{x}/d\tau| << 1$, and where the problem is static, i.e. $g_{\mu\nu}$ is time-independent. To the lowest non-trivial approximation eq. (1.18) then gives ($t = x^0$)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 \simeq 0 \tag{1.28}$$

Using eq. (1.27) and the fact that all time derivatives vanish we have

$$\Gamma^\mu_{00} \simeq -\frac{1}{2}\, g^{\mu\sigma}\, \frac{\partial g_{00}}{\partial x^\sigma} \tag{1.29}$$

Since we are interested in small effects of gravity we write

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad , \quad |h_{\mu\nu}| \ll 1 \tag{1.30}$$

where $h_{\mu\nu}$ is the correction to the constant metric tensor $\eta_{\mu\nu}$. Then eq. (1.29) gives

$$\Gamma^\mu_{00} \simeq -\frac{1}{2}\, \eta^{\mu\sigma} \frac{\partial h_{00}}{\partial x^\sigma} \tag{1.31}$$

so that $\Gamma^0_{00} \simeq 0$ since h_{00} does not depend on time. Hence the $\mu = 0$ component of eq. (1.28) becomes simply

$$\frac{d^2t}{d\tau^2} \simeq 0$$

i.e.

$$\frac{dt}{d\tau} = \text{constant}$$

For $\mu = i, \ \ i = 1, 2, 3$, eq. (1.28) becomes by use of eq. (1.31)

$$\frac{d^2\vec{x}}{d\tau^2} - \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \vec{\nabla} h_{00}(\vec{x}) \simeq 0 \tag{1.32}$$

and because dt is proportional to $d\tau$

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\, \vec{\nabla} h_{00}(\vec{x}) \tag{1.33}$$

This equation can immediately be compared to Newton's equation

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi(\vec{x}) \tag{1.34}$$

where $\phi(\vec{x})$ is the gravitational potential. The Newtonian potential is determined by Poisson's equation

$$\nabla^2\phi(\vec{x}) = 4\pi G\ \rho(\vec{x}) \tag{1.35}$$

where G i Newton's constant ($G = G/c^2 = 7.41 \times 10^{-29}$ cm/g). For a point mass one has the well known result

$$\phi = -\frac{GM}{r} \tag{1.36}$$

Comparing eqs. (1.33) and (1.34) we get $h_{00} =$ constant -2ϕ, and requiring that at very large distances from the point mass, space should be flat we get

$$g_{00}(\vec{x}) = -(1 + 2\phi(\vec{x})) \tag{1.37}$$

Thus we see that sufficiently close to a point mass space-time must indeed be slightly "curved"!

The "curvature" indicated by eq. (1.37) can be observed. Recalling that the proper time is the time observed on a freely falling watch, we have

$$d\tau^2 = -\eta_{00}\ dt^2_{falling} = -g_{\mu\nu}(x)dx^\mu dx^\nu = -g_{00}(\vec{x})dt^2 \tag{1.38}$$

where the last expression is valid in a gravitational field where the clock is (approximately) at rest. Thus the time measured in this system in the point x is

$$dt = \frac{d\tau}{\sqrt{-g_{00}(x)}} \tag{1.39}$$

This is not in itself an observable effect since all clocks and physical processes in the point will suffer the same effect. However, we can compare two different points. Here it is important to note that if e.g. an atom emits light in one point, this light will travel to another point in a constant time if the metric is time independent. This follows because $d\tau = 0$ for light, and the line element $0 = g_{\mu\nu}(\vec{x})dx^\mu dx^\nu$ can be solved for $dt = dx^0$ and subsequently integrated over the distance between the two points ($\int dt$), and the resulting integrated travel time for light is time-independent. Therefore it follows that if a wave length is emitted in time interval dt in one point, this wave length will be observed in the other point in the same time interval dt.

If the clock is a physical system with a frequency ν, we therefore get ($\nu = 2\pi/dt$)

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} \tag{1.40}$$

In the weak field approximation (1.37) we then have

$$\frac{\Delta\nu}{\nu_1} = \frac{\nu_2 - \nu_1}{\nu_1} = \phi(x_2) - \phi(x_1) \tag{1.41}$$

which is an observable effect. Here ν_2 is the frequency of light emitted in the point 2, but in accordance with what was said above this is also the frequency when this light is observed in point 1, provided the metric is time-independent. Now two spectra emitted from the same type of atom can be identified, even if the spectral lines are displaced by the amount $\Delta\nu$, so we can compare an atomic spectrum emitted from the sun and observed on earth with the same atomic spectrum emitted on earth.

If we consider light which passes from the sun to the earth we have for the sun's potential

$$\phi_\odot = -\frac{GM_\odot}{R_\odot} = -\ 2.2 \times 10^{-6} \tag{1.42}$$

whereas the earth's potential can be ignored relative to $\phi_\odot$. In eq. (1.42) G should be replaced by G/c^2 in ordinary units. The frequency of light from the sun is thus shifted by 2.2 parts per million relative to light from earthbound sources. Taking into account various other effects the best experimental result is 1.05 ± 0.05 times the predicted value.

Another experiment made 1960 consists in emitting light from a tower of height 22.6m. The falling light is then observed on the ground. From (1.41) the frequency shift should be

$$\begin{aligned} \Delta\phi &= \phi_{\text{top}} - \phi_{\text{bottom}} = \frac{(980 \text{ cm/sec}^2)(2260 \text{ cm})}{(3 \times 10^{10} \text{ cm/sec})^2} \\ &= 2.46 \times 10^{-15} \end{aligned} \tag{1.43}$$

The experimental value is

$$(2.57 \pm 0.26) \times 10^{-15} \tag{1.44}$$

in excellent agreement with the prediction.

1.5 The principle of general covariance

So far we have studied the effects of gravity by use of the principle of equivalence according to which the physics of special relativity is valid in freely falling local systems of inertia, and the effects of gravity can then be obtained by transforming to an arbitrary system. Such a procedure is in general rather complicated.

Einstein introduced (1916) a new principle which leads to a much more systematic way of obtaining the physics of gravity from the physics without gravity, namely the principle of general covariance. Using his own words (well, translated to English) this principle states:

The general laws of nature are to be expressed by equations which hold good for all systems of coordinates, that is, are covariant (i.e. preserve their form) with respect to any substitutions whatever ("generally covariant").

In this connection the "laws of nature" are to be understood as those which are valid in special relativity. It then follows that if general covariance is satisfied then the equivalence principle is also satisfied: in each point there are freely falling local elevators in which the laws of nature are those of special relativity, and from general covariance they are thus valid laws in all coordinate systems.

We need to specify precisely what we mean by "local elevators". Clearly they should not be too large, because then from experience with tidal forces we know that gravitational effects can be observed on a sufficiently large scale. The equation of motion in a gravitational field is given by equation (1.18). At each point $x = x_0$ we can select the elevator such that $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$ (= flat Minkowski space), and in order to have local straight-line motion we can select the elevator such that

$$\left.\frac{\partial g_{\mu\nu}(x)}{\partial x^\sigma}\right|_{x=x_0} = 0 \tag{1.45}$$

Because of the connection between the Cristoffel symbol $\Gamma^\lambda_{\mu\nu}$ and $g_{\mu\nu}$ given by eq. (1.27) this ensures that to lowest order near each point of space-time we have no effect from gravitational forces. However, it should be emphasized that second derivatives of $g_{\mu\nu}(x)$

are not in general assumed to vanish. This amounts to saying that we only assume that effects of gravity can be transformed away on a scale, which is small relative to the scale of the gravitational field.

1.6 Contravariant and covariant tensors

We shall now present a systematic construction of certain quantities, tensors, which are suitable for applying the principle of general covariance. For tensors there exist transformations when the coordinates are transformed ($x \to x'$). The tensor transformations are linear and homogeneous for the components of the tensors. Hence all components in the x'-system vanish if they vanish in the x-system. A law of nature requiring that all components of a tensor vanish is thus valid in all systems, if it is valid in one system. Tensor laws thus follow the principle of general covariance if we ensure that they are valid in special relativity. The simplest quantity is a scalar quantity, which is invariant under $x \to x'$. Numbers like 17, π, etc. are examples. Also, the proper time is an example of an invariant. In general one can also have a scalar field $\phi(x)$ defined in each space-time point, transforming like $\phi'(x') = \phi(x)$.

Another quantity is the coordinate differential dx^μ which transforms like

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \tag{1.46}$$

Any quantity which transform like (1.46) is called a contravariant vector, i.e., U^μ is a contravariant vector if under a transformation $x \to x'$ one has $U^\mu \to U'^\mu$ with

$$U'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\, U^\nu(x) \tag{1.47}$$

It should be noticed that since $d\tau$ is invariant, it follows from (1.46) that the four-velocity $dx^\mu/d\tau = (dx^0/d\tau, d\vec{x}/d\tau)$ is a contravariant vector.

In special relativity x^μ is a vector. It is important to realize that this is not the case in general relativity, where the transformation $x^\mu \to x'^\mu, \quad x'^\mu \to x^\mu$ is completely arbitrary. In general relativity we can only afford that the coordinate differentials dx^μ are vectors. The physical reason for this difference is, as mentioned before, that in the equivalence principle we must restrict ourselves to infinitesimal elevators.

A covariant vector is defined by

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\, A_\nu(x) \tag{1.48}$$

From (1.47) and (1.48) we can form a scalar

$$A'_\mu(x')U'^\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\sigma} A_\nu(x)U^\sigma(x) = A_\nu(x)U^\nu(x) \tag{1.49}$$

We say that by **contracting** the indices of two vectors we obtain an invariant.

From a scalar $\phi(x)$ we can form a covariant vector by differentiation,

$$\frac{\partial \phi'(x')}{\partial x'^\mu} = \frac{\partial \phi(x)}{\partial x^\nu}\,\frac{\partial x^\nu}{\partial x'^\mu} \tag{1.50}$$

A general tensor can have arbitrarily many indices, e.g. $T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}$. Its transformation is given by

$$T'^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\sigma_1}}\cdots\frac{\partial x'^{\mu_n}}{\partial x^{\sigma_n}}\frac{\partial x^{\rho_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\rho_m}}{\partial x'^{\nu_m}}T^{\sigma_1\ldots\sigma_n}{}_{\rho_1\ldots\rho_m}(x) \qquad (1.51)$$

A tensor with upstairs as well as downstairs indices is called a mixed tensor. The transformation law (1.51) can easily be remembered by noticing that T transforms the same way as if it had been a product of n contravariant and m covariant vectors. In other words, a product of vectors is a tensor, e.g. $A^\mu B^\nu C_\rho$ transforms as a tensor $D^{\mu\nu}{}_\rho$, etc. etc.

The reader should be warned that the summation convention requires some care. Suppose for example we have the two relations $A = B^\mu C_\mu$ and $D = E^\mu F_\mu$. What is then AD? Well, multiplying the two relations together, we apparently get $AD = B^\mu C_\mu E^\mu F_\mu$. This expression is, however, meaningless and hence *wrong*. The summation indices occur four instead of two times. Before multiplying A and D together, we must ensure that the summation indices in A and D are different. Thus, keeping the indices in A, we should use a different name for the indices in D, e.g. $D = E^\sigma F_\sigma$. We then obtain the correct expression $AD = B^\mu C_\mu E^\sigma F_\sigma$, where each summation index occurs only twice.

The metric tensor $g_{\mu\nu}$ is a covariant tensor, as is easily seen from the definition (1.14),

$$\begin{aligned} g'_{\mu\nu}(x') &= \eta_{\alpha\beta}\,\frac{\partial y^\alpha}{\partial x'^\mu}\,\frac{\partial y^\beta}{\partial x'^\nu} \\ &= \eta_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\rho}\,\frac{\partial y^\beta}{\partial x^\sigma}\,\frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x^\sigma}{\partial x'^\nu} \\ &= \frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x^\sigma}{\partial x'^\nu}\,g_{\rho\sigma}(x) \end{aligned} \qquad (1.52)$$

Also, $g^{\mu\nu}$ is a contravariant tensor. The index of a tensor can be "raised" or "lowered" by means of $g^{\mu\nu}$ or $g_{\mu\nu}$. E.g. $T_{\mu\nu} = g_{\mu\sigma}g_{\nu\rho}\overline{T}^{\sigma\rho}$. By going to a freely falling system we see that the tensors T and $\overline{T}$ are the same physical object, and $\overline{T}$ is denoted T. The symbol δ^ν_ρ is easily seen to be a mixed tensor.

Often the determinant of $g_{\mu\nu}$ occurs,

$$g \equiv -\det g_{\mu\nu}$$

From the transformation law (1.52), which in matrix form reads $(g') = (\frac{\partial x}{\partial x'})(g)^T(\frac{\partial x}{\partial x'})$, where (g) etc. denotes the matrix $g_{\rho\sigma}$, and where the superscript T denotes the transpose of the matrix, one obtains

$$g'(x') = \left|\frac{\partial x}{\partial x'}\right|^2 g(x) \qquad (1.53)$$

where $|\partial x/\partial x'|$ is the Jacobi determinant, $|\partial x/\partial x'| = |\det(\partial x^\mu/\partial x'^\nu)|$. Eq. (1.53) has the important consequence that ($d^4x = dx^0dx^1dx^2dx^3$)

$$\sqrt{g'(x')}\;d^4x' = \sqrt{g'(x')}\;\left|\frac{\partial x'}{\partial x}\right|d^4x = \sqrt{g(x)}\;d^4x = \text{invariant} \qquad (1.54)$$

where in the first step we used the usual Jacobi transformation. Thus the measure of integration d^4x is **not** invariant, but should be multiplied by $\sqrt{g}$.

Finally let us mention a few simple rules for tensors which follow from the transformation (1.51): a sum of two tensors is a tensor, a product of two tensors is a new tensor (with more indices), e.g.

$$A^{\mu\nu}{}_{\lambda}\, B_{\sigma}{}^{\rho\alpha} = T^{\mu\nu}{}_{\lambda\sigma}{}^{\rho\alpha} \tag{1.55}$$

and a contraction in a tensor is a new tensor with fewer indices, e.g.

$$T^{\mu\nu\rho\sigma}{}_{\mu} = \tilde{T}^{\nu\rho\sigma} \tag{1.56}$$

1.7 Differentiation

We saw in the last section that differentiation of a scalar leads to a vector (see eq. (1.50)). However, in general it is not true that differentiation of a tensor leads to a new tensor. This is related to the fact that differentiation is defined by comparing the tensor in two different points (and then taking the limit where the points approach one another). However, a tensor transforms differently in the two points, as can be seen e.g. from the transformation law (1.51).

In the following we shall discuss the concept of differentiation versus covariance following Einstein's original 1916 paper. To see what happens let us consider some trajectory $x^\nu(\tau)$. Starting from a scalar $\phi(x)$ we have

$$\text{invariant} = \frac{d\phi(x)}{d\tau} = \frac{\partial\phi(x)}{\partial x^\mu}\frac{dx^\mu}{d\tau} \tag{1.57}$$

where the invariance follows from the fact that $d\tau$ and $d\phi$ are individually invariant. However, $dx^\mu/d\tau$ is a vector, and consequently it follows that

$$V_\mu(x) = \frac{\partial\phi(x)}{\partial x^\mu} \tag{1.58}$$

is a covariant vector. This statement has already been checked directly in eq. (1.50) . Now let us differentiate (1.57) once more,

$$\text{invariant} = \frac{d^2\phi}{d\tau^2} = \frac{\partial^2\phi}{\partial x^\mu \partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + \frac{\partial\phi}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} \tag{1.59}$$

Next let us take the trajectory $x_\mu(\tau)$ to be a path for a particle which falls freely in an arbitrary gravitational field. Then we have from the equation of motion (1.18)

$$\text{invariant} = \frac{d^2\phi}{d\tau^2} = \left(\frac{\partial^2\phi}{\partial x^\mu \partial x^\nu} - \Gamma^\sigma_{\mu\nu}\frac{\partial\phi}{\partial x^\sigma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{1.60}$$

Since

$$\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

is a contravariant tensor, and since the left hand side of (1.60) is an invariant, it follows that the quantity in the bracket in eq. (1.60) is a covariant tensor. Inserting (1.58) we have that

$$V_{\nu;\mu} = \frac{\partial V_\nu}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu}V_\sigma \tag{1.61}$$

is a covariant tensor with indices μ and ν. $V_{\nu;\mu}$ is called the covariant derivative of V_ν.

The result (1.61) was derived for a trajectory of a particle falling freely in some gravitational field. By choosing all sorts of such fields we can, however, manage that the curve is completely arbitrary. Hence eq. (1.61) should be valid for x^μ being an arbitrary point in the space-time continuum. Alternatively one can use the connection (1.27) between $\Gamma^\sigma_{\mu\nu}$ and the metric, as well as the fact that the metric $g_{\mu\nu}$ is a tensor, to show directly that (1.61) is a tensor.

If we differentiate a contravariant tensor V^ν the following quantity

$$V^\nu{}_{;\mu} = \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\sigma} V^\sigma \tag{1.62}$$

is a mixed tensor (T^ν_μ). The concept of covariant differentiation can be generalized to an arbitrary tensor. One has e.g.

$$C^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial C^{\mu\sigma}{}_\lambda}{\partial x^\rho} + \Gamma^\mu{}_{\rho\nu} C^{\nu\sigma}{}_\lambda + \Gamma^\sigma{}_{\rho\nu} C^{\mu\nu}{}_\lambda - \Gamma^\kappa{}_{\lambda\rho} C^{\mu\sigma}{}_\kappa \tag{1.63}$$

We leave it to the diligent reader to verify this statement.

An important property of the covariant derivative is that it reduces to the ordinary derivative in a freely falling local elevator. This follows from the definition of locality given in (1.45) according to which the first derivatives of the metric tensor can be required to vanish in a local system. Since $\Gamma^\lambda_{\mu\nu}$ is directly related to these derivatives through eq. (1.27) it follows that

$$\Gamma^\lambda_{\mu\nu}(x)\,|_{x=x_0} = 0 \tag{1.64}$$

From this we have the very important conclusion that in spite of the notation $\Gamma^\lambda_{\mu\nu}$ (mixed tensor notation) it is **not true** that $\Gamma^\lambda_{\mu\nu}$ is a tensor, because if it was a tensor then it follows from (1.64) that in any point there exists a freely falling elevator where $\Gamma^\lambda_{\mu\nu}$ vanishes, and consequently it should vanish in all coordinate systems. Since $\Gamma^\lambda_{\mu\nu}$ does not vanish in general, it follows that it is not a tensor. Using the transformation law for the tensor $g_{\mu\nu}$ and the relation (1.27) one can easily see that $\Gamma^\lambda_{\mu\nu}$ transforms as

$$\Gamma'^\lambda_{\mu\nu}(x') = \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \Gamma^\alpha_{\beta\gamma}(x) + \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} \tag{1.65}$$

The first term on the right hand side is what one would get if $\Gamma^\lambda_{\mu\nu}$ was a tensor, but the second term shows that $\Gamma^\lambda_{\mu\nu}$ transforms in a more complicated way.

Because of (1.64) it follows that in the small elevators the covariant derivative is just the ordinary derivative. This is rather satisfactory since we are supposed to use the laws of special relativity in each elevator, and these laws, of course, contain only the ordinary derivatives.

From the definition (1.45) of a local elevator, i.e. from the fact that the first derivatives of $g_{\mu\nu}$ can be taken to vanish in a local elevator, and from the fact (see eq. (1.52)) that $g_{\mu\nu}$ **is** a tensor, it follows that the covariant derivative of $g_{\mu\nu}$ vanishes,

$$g_{\mu\nu;\sigma} = 0 \tag{1.66}$$

Finally let us consider motion along a trajectory $x^\mu(\tau)$. In this case one can only talk about differentiation along the curve $x^\mu(\tau)$. We can then project the covariant derivative to the tangent $dx^\mu/d\tau$. Eq. (1.62) is then replaced by

$$\frac{DV^\nu}{D\tau} \equiv \frac{dV^\nu}{d\tau} + \Gamma^\nu_{\mu\sigma} V^\sigma \frac{dx^\mu}{d\tau} \tag{1.67}$$

which is easily shown to be a contravariant vector with index ν. In a local elevator it reduces to the ordinary derivative $dV^\nu/d\tau$.

1.8 A property of the determinant of $g_{\mu\nu}$

We shall show an interesting property related to the determinant of $g_{\mu\nu}$. To this end we need to differentiate a determinant. Let us therefore start by considering an $n \times n$ determinant with elements a_{rs}. It is defined by

$$\det a = \sum (-1)^{P(\alpha_1,\alpha_2,\ldots,\alpha_n)} a_{1\alpha_1} a_{2\alpha_2} \ldots a_{n\alpha_n} \tag{1.68}$$

where P is even (odd) for even (odd) permutations of the reference sequence $1, 2, \ldots, n$. In the sum (1.68) we can take an element a_{rs} outside a bracket, and all the terms in (1.68) containing the particular factor a_{rs} can then be written $a_{rs}A_{rs}$ (no summation over r and s), where A_{rs} is a sum of terms which consist of $n-1$ factors. A_{rs} is called the complement. Because of the construction (1.68) of the determinant, A_{rs} is independent of the elements in the r'th row and the s'th column (we have already picked the element a_{rs}, and each term in the sum (1.68) has only one element from the r'th row and the s'th column). Now each of the $n!$ products in $\det a$ contains one and only one element in the r'th row and we have (no sum over r)

$$a_{r1}A_{r1} + a_{r2}A_{r2} + \cdots + a_{rn}A_{rn} = \det a \tag{1.69}$$

The n complements $A_{r1}, \ldots, A_{rn}$ are all independent of the elements $a_{r1}, \ldots, a_{rn}$ in the r'th row. Thus the quantities $A_{r1}, \ldots, A_{rn}$ are unchanged if in $\det a$ we interchange the elements in the r'th row with other elements. In particular we can keep $A_{r1}, \ldots, A_{rn}$ unchanged if we replace the elements in the r'th row with the corresponding elements $a_{s1}, a_{s2}, \ldots, a_{sn}$ in the s'th row ($r \neq s$). However, the new determinant has the value 0, since two rows have the same elements, i.e.

$$a_{s1}A_{r1} + a_{s2}A_{r2} + \cdots + a_{sn}A_{rn} = 0 \qquad (s \neq r) \tag{1.70}$$

Eqs. (1.69) and (1.70) can be used to construct the inverse determinant or matrix. Thus the inverse of

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n_1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \tag{1.71}$$

is

$$\begin{pmatrix} \frac{A_{11}}{\det a} & \frac{A_{21}}{\det a} & \ldots & \frac{A_{n1}}{\det a} \\ \frac{A_{12}}{\det a} & \frac{A_{22}}{\det a} & \ldots & \frac{A_{n2}}{\det a} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{A_{1n}}{\det a} & \frac{A_{2n}}{\det a} & \ldots & \frac{A_{nn}}{\det a} \end{pmatrix} \tag{1.72}$$

The correctness of (1.72) is seen by multiplying (1.71) and (1.72) together by use of (1.69) and (1.70). The result is the unit matrix.

Now let us differentiate $\det a$, i.e. consider the variation $\delta \det a$ arising from (1.68). Each term in the sum (1.68) has n factors, and the variation is a product of $n-1$ factors multiplied by δa_{rs}, according to the usual rule for differentiation of a product. The $n-1$ factors do not contain a_{rs}. Thus, collecting all terms which are multiplied by δa_{rs} amounts to collecting the terms in the complement A_{rs}. According to (1.72) this means that

$$\delta \det a = \sum A_{rs}\delta a_{rs} = \sum (a^{-1})_{sr} \det a \; \delta a_{rs} \tag{1.73}$$

where $(a^{-1})_{sr}$ stands for the elements in the inverse matrix a^{-1}.

From this general result let us return go $g_{\mu\nu}$. With $g = -\det g_{\mu\nu}$ we have ($g_{\mu\nu}$ is symmetric)

$$dg = g^{\mu\nu} g \; dg_{\mu\nu} \tag{1.74}$$

since $g^{\mu\nu}$ was defined as the inverse of $g_{\mu\nu}$. Using (1.25) we have $g_{\mu\nu}g^{\mu\nu} = 4$, i.e.

$$g_{\mu\nu} \; dg^{\mu\nu} + g^{\mu\nu} \; dg_{\mu\nu} = 0 \tag{1.75}$$

so (1.74) has the alternative form

$$dg = -g \; g_{\mu\nu} \; dg^{\mu\nu} \tag{1.76}$$

From (1.74) we have

$$g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{1}{g} \frac{\partial g}{\partial x^\lambda} = \frac{\partial \ln g}{\partial x^\lambda} \tag{1.77}$$

This property turns out to be rather useful.

1.9 Some special derivatives

In mathematical physics the curl of a vector is defined by

$$\text{curl } \vec{V} = \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3} , \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1} , \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2} \right) \tag{1.78}$$

For the covariant derivative of a vector V_μ we can similarly define the curl to be $V_{\mu;\nu} - V_{\nu;\mu}$. Using the expression (1.61) we see that the terms involving the Christoffel symbols drop out and we just have

$$V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu} \tag{1.79}$$

i.e. the curl is given in terms of just the ordinary derivatives.

The divergence of a vector is given by

$$V^\mu{}_{;\mu} = \frac{\partial V^\mu}{\partial x^\mu} + \Gamma^\mu{}_{\mu\nu} V^\nu . \tag{1.80}$$

Using the expression (1.27) we have

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2} \, g^{\mu\sigma} \left[\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right] = \frac{1}{2} \, g^{\mu\sigma} \, \frac{\partial g_{\mu\sigma}}{\partial x^\nu} , \tag{1.81}$$

since the first and the third terms are seen to be identical by interchange of the summation indices μ and σ. Now we can use (1.77) to simplify this to

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2} \frac{\partial}{\partial x^\nu} \ln g = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\nu} \sqrt{g} \tag{1.82}$$

Eq. (1.80) can then be written in the elegant form

$$V^{\mu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{g}\, V^{\mu}) \tag{1.83}$$

In eq. (1.54) we mentioned that $\sqrt{g} d^4x$ is invariant. Gauss' theorem in four dimensions can then be written

$$\int d^4x \sqrt{g}\, V^{\mu}{}_{;\mu} = \int d^4x \frac{\partial}{\partial x^{\mu}} (\sqrt{g}\, V^{\mu}) = \int d\Sigma_{\mu} \sqrt{g}\, V^{\mu} \tag{1.84}$$

where $\sum_{\mu}$ are 3-"surfaces" (e.g. for $\mu = 0,\ d\Sigma_o = dx^1 dx^2 dx^3$). If these surfaces are taken at infinity and if $V^{\mu} \to 0$ at infinity, one has

$$\int d^4x \sqrt{g}\, V^{\mu}{}_{;\mu} = 0 \tag{1.85}$$

For tensors similar simplifications occur. The covariant derivative of a tensor $T^{\mu\nu}$ is given by

$$T^{\mu\nu}{}_{;\sigma} = \frac{\partial T^{\mu\nu}}{\partial x^{\sigma}} + \Gamma^{\mu}{}_{\sigma\lambda} T^{\lambda\nu} + \Gamma^{\nu}{}_{\sigma\lambda} T^{\mu\lambda} \tag{1.86}$$

Thus the divergence is given by

$$T^{\mu\nu}{}_{;\mu} = \frac{\partial T^{\mu\nu}}{\partial x^{\mu}} + \Gamma^{\mu}{}_{\mu\lambda} T^{\lambda\nu} + \Gamma^{\nu}{}_{\mu\lambda} T^{\mu\lambda} \tag{1.87}$$

Using (1.82) we can combine the first two terms just like we did in eq. (1.83),

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{g}\, T^{\mu\nu}) + \Gamma^{\nu}{}_{\mu\lambda} T^{\mu\lambda} \tag{1.88}$$

If the tensor is antisymmetric,

$$F^{\mu\nu} = -F^{\nu\mu} \tag{1.89}$$

then the last term in (1.88) drops out (because $\Gamma^{\nu}_{\mu\lambda}$ is symmetric in μ and λ), and we have the simple result

$$F^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{g}\, F^{\mu\nu}) \tag{1.90}$$

The usual Laplace operator acting on a scalar equals divgrad. In our case there is a similar result. The gradient of the scalar S is just $\partial S/\partial x^{\mu}$, which when multiplied by the metric can be contravariant. We can then apply (1.83) to obtain

$$\Box_g\, S \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{g}\, g^{\mu\sigma} \frac{\partial S}{\partial x^{\sigma}} \right) \tag{1.91}$$

where $\Box_g$ generalizes the usual d'Alembert operator $\partial^2/(\partial x^{\mu})^2$. In a local elevator $\Box_g$ reduces to the d'Alembertian.

1.10 Some applications to physics

Having developed the apparatus of tensors and generalized differentiation let us enjoy ourselves with some applications to physics.

First let us consider again the problem of a particle which falls in a gravitational field $g_{\mu\nu}(x)$, but this time we apply the principle of general covariance. Thus, first we must know what happens in special relativity. Here the particle is completely free. Its four-velocity is given by

$$U^\alpha = \frac{dy^\alpha}{d\tau} = \left(\frac{dy^0}{d\tau}\,,\,\frac{d\vec{y}}{d\tau}\right) \tag{1.92}$$

where y^α are the coordinates in special relativity. The equation of motion is simply

$$\frac{dU^\alpha}{d\tau} = 0 \tag{1.93}$$

Now we must make this equation generally covariant. This was done in eq. (1.67), where we showed that $d/d\tau$ should be replaced by $D/D\tau$, i.e. we have

$$\frac{DU^\mu}{D\tau} = \frac{dU^\mu}{d\tau} + \Gamma^\mu{}_{\lambda\nu}U^\nu U^\lambda = 0 \tag{1.94}$$

This equation is correct because

1) it is valid in any freely falling local elevator ($\Gamma^\mu{}_{\lambda\nu} = 0$ locally),

2) it is generally covariant (a vector $DU^\mu/D\tau$ vanishes in one system, and hence it vanishes in all systems).

From this example we see that once we have been through the somewhat tiresome procedure of setting up tensor calculus, the result (1.94) can be derived in a much simpler way than by applying the equivalence principle directly.

To present another example let us consider a small gyroscope which moves in the gravitational field $g_{\mu\nu}(x)$. In special relativity it is thus not influenced by any forces. The gyroscope taken to be at rest has an angular momentum. In the limit where it can be considered to be a point we say that it has a spin $\vec{S}$. The spin is conserved if no external force acts. We can then introduce the four vector $S_\alpha = (0, \vec{S})$, and hence

$$\frac{dS_\alpha}{d\tau} = 0 \tag{1.95}$$

Also, since the gyroscope is at rest (the spin is defined in the rest frame) we have

$$U^\alpha S_\alpha = 0 \tag{1.96}$$

These equations are valid in special relativity. Under the influence of the gravitational field they become

$$\frac{DS_\mu}{D\tau} \equiv \frac{dS_\mu}{d\tau} - \Gamma^\lambda{}_{\mu\nu}\, U^\nu S_\lambda = 0 \tag{1.97}$$

$$S_\mu U^\mu = 0 \tag{1.98}$$

Eq. (1.97), for example, describes the precession of a gyroscope carried in a satellite moving in the gravitational field from the earth. To actually utilize eq. (1.97) we need, however, more accurate result for the metric than the one obtained in the Newtonian limit (see eq. (1.37)).

Newton's equation (1.1) can be generalized to special relativity,

$$m\,\frac{dU^\alpha}{d\tau} = f^\alpha \tag{1.99}$$

where f^α is called the four-force. In general relativity this becomes

$$m\,\frac{DU^\mu}{D\tau} = F^\mu \tag{1.100}$$

where F^μ is a generalized four-force.

We now turn to electrodynamics. The Maxwell equations can be expressed in terms of an antisymmetric tensor $F^{\alpha\beta}$, with $F^{12} = B_3, F^{23} = B_1, F^{31} = B_2$ ($\vec{B}$ is the magnetic field), $F^{01} = E_1, F^{02} = E_2, F^{03} = E_3$ ($\vec{E}$ is the electric field). Furthermore, the current is a four-vector, $J^\alpha = (\rho, \vec{J})$, where ρ is the charge density and $\vec{J}$ is the current. The equations

$$\text{div}\ \vec{E} = \vec{\nabla}\vec{E} = \rho \quad , \quad \vec{\nabla}\times\vec{B} = \text{curl}\ \vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{J} \tag{1.101}$$

become

$$\frac{\partial}{\partial x^\alpha}\ F^{\alpha\beta} = -J^\beta \tag{1.102}$$

The other Maxwell equations

$$\text{div}\ \vec{B} = \vec{\nabla}\vec{B} = 0 \quad , \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{1.103}$$

can be expressed as

$$\frac{\partial}{\partial x^\alpha}\ F_{\beta\gamma} + \frac{\partial}{\partial x^\beta}\ F_{\gamma\alpha} + \frac{\partial}{\partial x^\gamma}\ F_{\alpha\beta} = 0 \tag{1.104}$$

where the indices are permuted cyclically. Eqs. (1.102) and (1.104) are of course valid in special relativity; as a matter of fact, they [or rather, the equivalent eqs. (1.101) and (1.103)] are the *raison d'etre* for special relativity.

Now we can apply our tensor apparatus to obtain the effects of gravitation on electromagnetism. Eq. (1.102) is simple to write in a generally covariant form,

$$F^{\mu\nu}{}_{;\mu} = -J^\nu \tag{1.105}$$

Now $F^{\mu\nu}$ is antisymmetric, so we can simplify as in eq. (1.89),

$$\frac{\partial}{\partial x^\mu}(\sqrt{g}\ F^{\mu\nu}) = -\sqrt{g}\ J^\nu \tag{1.106}$$

To make (1.104) generally covariant we lower the indices,

$$F_{\mu\nu} = g_{\mu\rho}\ g_{\nu\sigma}F^{\rho\sigma} \tag{1.107}$$

which is still antisymmetric. Eq. (1.104) then becomes

$$F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = 0 \tag{1.108}$$

where e.g.

$$F_{\mu\nu;\lambda} = \frac{\partial F_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho{}_{\lambda\mu} F_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} F_{\mu\rho} \tag{1.109}$$

Adding the three terms in (1.108) it is seen that the six terms containing the Γ's cancel, and we thus get the simple result

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} = 0 \tag{1.110}$$

In special relativity the four-current is conserved. It is easy to see that the same must be true in general relativity. From eq. (1.106) we have

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} \left(\sqrt{g}\ F^{\mu\nu}\right) = 0 = -\frac{\partial}{\partial x^\nu} \left(\sqrt{g}\ J^\nu\right) \tag{1.111}$$

because $F^{\mu\nu}$ is antisymmetric. The conservation law

$$\frac{\partial}{\partial x^\nu}(\sqrt{g}\ J^\nu) = 0 \tag{1.112}$$

precisely corresponds to the vanishing of the four-divergence of J^ν [compare with eq. (1.85)] in generally covariant form.

Finally we mention that in special relativity the Lorentz force

$$\frac{d\vec{p}}{dt} = e[\vec{E} + \vec{v} \times \vec{B}] \tag{1.113}$$

can be written [compare with eq. (1.99)]

$$f^\alpha = eF^\alpha{}_\gamma \frac{dx^\gamma}{d\tau} \tag{1.114}$$

In general relativity the generalized four force (1.100) therefore becomes,

$$F^\mu = e\ F^\mu{}_\nu \frac{dx^\nu}{d\tau} \tag{1.115}$$

where F^μ is a vector since $F^\mu{}_\nu$ is a tensor and $dx^\nu/d\tau$ is a vector.

1.11 Curvature

The principle of equivalence (and the principle of general covariance) shows that a gravitational field can be represented by a metric tensor $g_{\mu\nu}(x)$ which is not everywhere constant. The opposite is, however, not true. Suppose space is just flat Minkowski space, and $g_{\mu\nu}(x) = \eta_{\mu\nu}$ everywhere. Thus

$$d\tau^2 = -\eta_{\mu\nu} dx^\mu\ dx^\nu \tag{1.116}$$

We can, however, always transform the x-coordinate to a different system, $x \to x'$, and then [3].

$$
\begin{aligned}
d\tau^2 &= -g'_{\mu\nu}(x')dx'^\mu\, dx'^\nu \\
g'_{\mu\nu}(x') &= \eta_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\,\frac{\partial x^\sigma}{\partial x'^\nu}
\end{aligned}
\tag{1.117}
$$

In the new system it appears that the metric does not represent flat space. However, it is clear from the derivation that the apparent non-flatness is an illusion.

The question then arises whether we can somehow avoid being fooled by somebody who claims to have an interesting gravitational field $g_{\mu\nu}$, but who has obtained this field from flat space by applying some complicated mathematical transformation $x \to x'$. To avoid such a joke we can notice that in flat space, if we differentiate a vector two times, the derivatives commute. In curved space derivatives are replaced by covariant derivatives, which in general do not commute. To see this let us notice that

$$T_{\mu\nu;\kappa} = \frac{\partial T_{\mu\nu}}{\partial x^\kappa} - \Gamma^\lambda{}_{\nu\kappa}T_{\mu\lambda} - \Gamma^\lambda{}_{\mu\kappa}T_{\lambda\nu} \tag{1.118}$$

In this equation $T_{\mu\nu}$ is an arbitrary tensor, so we can take $T_{\mu\nu} = V_{\mu;\nu}$, since the covariant derivative of a vector V_μ is a tensor. Thus

$$V_{\mu;\nu;\kappa} = \frac{\partial V_{\mu;\nu}}{\partial x^\kappa} - \Gamma^\lambda{}_{\nu\kappa}V_{\mu;\lambda} - \Gamma^\lambda{}_{\mu\kappa}\, V_{\lambda;\nu} \tag{1.119}$$

From this we get ($\Gamma^\lambda_{\nu\kappa}$ is symmetric in ν and κ!)

$$V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = \frac{\partial V_{\mu;\nu}}{\partial x^\kappa} - \frac{\partial V_{\mu;\kappa}}{\partial x^\nu} - \Gamma^\lambda{}_{\mu\kappa}V_{\lambda;\nu} + \Gamma^\lambda{}_{\mu\nu}\, V_{\lambda;\kappa} \tag{1.120}$$

Inserting now

$$V_{\mu;\nu} = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda{}_{\mu\nu}V_\lambda \tag{1.121}$$

we obtain (the ordinary derivatives of V_μ commute and thus drop out)

$$V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = -R^\sigma{}_{\mu\nu\kappa}V_\sigma \tag{1.122}$$

where

$$R^\sigma{}_{\mu\nu\kappa} = \frac{\partial \Gamma^\sigma{}_{\mu\nu}}{\partial x^\kappa} - \frac{\partial \Gamma^\sigma{}_{\mu\kappa}}{\partial x^\nu} + \Gamma^\lambda{}_{\mu\nu}\Gamma^\sigma{}_{\kappa\lambda} - \Gamma^\lambda{}_{\mu\kappa}\Gamma^\sigma{}_{\nu\lambda} \tag{1.123}$$

The quantity $R^\sigma_{\mu\nu\kappa}$ is a mixed tensor because the left-hand side of eq. (1.122) is a tensor and because V_σ on the right-hand side of (1.122) is a vector. $R^\sigma_{\mu\nu\kappa}$ is called the Riemann-Christoffel curvature tensor (or just the curvature tensor).

Now it is clear that nobody can fool us any longer, since if space is flat, it follows that the curvature tensor vanishes in the flat system. However, if a tensor vanishes in one system, it is zero in any other system.

[3]Such a transformation is very often used e.g. in quantum mechanics, where the hydrogen atom is most easily solved in spherical coordinates, where $-d\tau^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2$. The spherical coordinates $x^1 = r, x^2 = \theta, x^3 = \varphi$ thus look "non-flat" with $g_{11} = 1, g_{22} = (x^1)^2, g_{33} = (x^1)^2(\sin(x^2))^2$ ($g_{ij} = 0$ for $i \neq j$). It is clear that this "non-flat" metric does not represent any gravitational field, since we also have $-d\tau^2 = (dx)^2 + (dy)^2 + (dz)^2$, where x, y, z are the Cartesian coordinates

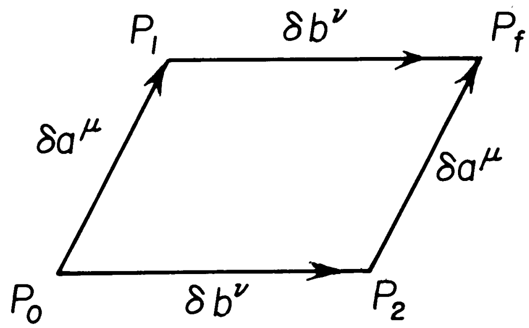

Figure 1.1: Parallel transport of a vector from P_0 to P_f through two different paths $P_0P_1P_f$ and $P_0P_2P_f$

From the curvature tensor (1.123) we can construct a covariant tensor. We leave it to the hard-working reader to show that it can be written in the form

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= g_{\lambda\sigma}\ R^{\sigma}{}_{\mu\nu\kappa} \\ &= \tfrac{1}{2}\ \left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right] \\ &\quad + g_{\eta\sigma}[\Gamma^{\eta}{}_{\nu\lambda}\Gamma^{\sigma}{}_{\mu\kappa} - \Gamma^{\eta}{}_{\kappa\lambda}\ \Gamma^{\sigma}{}_{\mu\nu}] \end{aligned} \tag{1.124}$$

If we go to a local elevator, the $\Gamma\Gamma$-terms can be brought to vanish, and the curvature tensor is thus locally given by the second derivatives of the metric.

1.12 Parallel transport and curvature

In section 10 we discussed the "transport" of vectors which fall freely in a gravitational field and which are not influenced by any forces. Thus, in any local elevator the vector, S_α say, satisfies the simple equation $dS^\alpha/d\tau = 0$. Here S_α can be the spin of a small gyroscope or the velocity vector. The equation satisfied in an arbitrary system is thus

$$\frac{dS^\alpha}{d\tau} = -\ \Gamma^{\alpha}{}_{\eta\rho}\ S^\rho \frac{dx^\eta}{d\tau} \tag{1.125}$$

The vector S^α is said to be parallel transported if it satisfies (1.125).

Let us now consider the transport from a point P_0 to a point P_f through two different paths, namely $P_0P_1P_f$ and $P_0P_2P_f$, as shown in fig. 1.1.

The parallelogram has sides δa^μ and δb^ν, which are considered to be infinitesimal. In special relativity the vector would be the same in P_f whether we used the path $P_0P_1P_f$ or $P_0P_2P_f$. In general relativity we do not expect to obtain the same result, since the vector S^α can experience different gravitational fields on the two paths.

Let us consider the path $P_0P_1P_f$. The change in S^α can be computed from (1.125)

$$S^\alpha(P_1) = S^\alpha(P_0) - \Gamma^{\alpha}{}_{\eta\rho}\big|_0\ S^\rho(P_0)\delta\ a^\eta \tag{1.126}$$

where the index 0 on Γ means that Γ is evaluated in P_0. Next we want to go from P_1 to P_f. In order to apply (1.125) we need Γ in the point P_1, which is given by

$$\Gamma^{\alpha}{}_{\gamma\beta}|_{P_1} = \Gamma^{\alpha}{}_{\gamma\beta}|_0 + \frac{\partial \Gamma^{\alpha}{}_{\gamma\beta}}{\partial x^\eta}|_0\ \delta a^\eta \tag{1.127}$$

From (1.125) we have

$$S^\alpha(P_0, P_1, P_f) = S^\alpha(P_1) - \Gamma^\alpha{}_{\gamma\beta}|_{P_1}\ S^\beta(P_1)\delta b^\gamma \tag{1.128}$$

Inserting (1.126) and (1.127) in (1.128) we get to second order

$$\begin{aligned} S^\alpha(P_0, P_1, P_f) &= S^\alpha(P_1) - \left\{\Gamma^\alpha{}_{\gamma\beta}|_0 + \frac{\partial \Gamma^\alpha{}_{\gamma\beta}}{\partial x^\eta}|_0\ \delta a^\eta\right\}\left\{S^\beta(P_0) - \Gamma^\beta{}_{\varepsilon\rho}|_0\ S^\rho(P_0)\delta a^\varepsilon\right\}\delta b^\gamma \\ &= S^\alpha(P_1) - \Gamma^\alpha{}_{\gamma\beta}|_0\ S^\beta(P_0)\delta b^\gamma - \frac{\partial \Gamma^\alpha{}_{\gamma\beta}}{\partial x^\eta}|_0\ S^\beta(P_0)\delta a^\eta \delta b^\gamma \\ &\quad +\Gamma^\alpha{}_{\gamma\beta}|_0\ \Gamma^\beta{}_{\varepsilon\rho}|_0\ S^\rho(P_0)\delta a^\varepsilon\ \delta b^\gamma \end{aligned} \tag{1.129}$$

Furthermore, $S^\alpha(P_1)$ can be expressed in terms of $S^\alpha(P_0)$ through (1.126), so we get

$$\begin{aligned} S^\alpha(P_0, P_1, P_f) &= S^\alpha(P_0) - \Gamma^\alpha{}_{\eta\rho}|_0\ S^\rho(P_0)\delta a^\eta - \Gamma^\alpha{}_{\gamma\beta}|_0\ S^\beta(P_0)\delta b^\gamma \\ &\quad - \frac{\partial \Gamma^\alpha{}_{\gamma\beta}}{\partial x^\eta}|_0\ S^\beta(P_0)\delta a^\eta\ \delta b^\gamma + \Gamma^\alpha{}_{\gamma\beta}|_0\ \Gamma^\beta{}_{\eta\rho}|_0\ S^\rho(P_0)\delta a^\eta\ \delta b^\gamma \end{aligned} \tag{1.130}$$

For the path $P_0P_2P_f$ we can obtain $S^\alpha(P_0, P_2, P_f)$ from the above expression just by interchange of δa and δb. Subtracting we find

$$\Delta S^\alpha \equiv S^\alpha(P_0, P_1, P_f) - S^\alpha(P_0, P_2, P_f) = -\ R^\alpha{}_{\beta\gamma\eta}\ S^\beta(P_0)\delta a^\eta\ \delta b^\gamma \tag{1.131}$$

where $R^\alpha_{\gamma\beta\eta}$ is the curvature tensor defined in eq. (1.123).

From eq. (1.131) we see that as expected, if there exists a gravitational field then in general a vector, which is parallel transported through two different paths to the same point, does not acquire the same value.

If, on the other hand, we assume that the Riemann tensor vanishes, then it follows that S^α becomes independent of the path, and only depends on the space-time point. In other words, S^α is a field $S^\alpha(x)$, which satisfies the differential equation (1.125). This equation simply means that the covariant derivative of the field $S^\alpha(x)$ must vanish. At one point we can prescribe a value of S^α, and the value in any other point (in a domain where $R^\alpha_{\gamma\beta\eta}$ vanishes) is then obtained by integrating (1.125). If the curvature tensor vanishes we can always define everywhere inertial coordinates ("flat coordinates") by parallel transport. Thus, it is necessary and sufficient for a metric $g_{\mu\nu}(x)$ to be equivalent to a flat metric $\eta_{\mu\nu}$ that $R^\alpha_{\beta\gamma\delta} = 0$ everywhere.

1.13 Properties of the curvature tensor

The covariant form of the curvature tensor (1.124) has a number of symmetries which can be read off from (1.124),

$$R_{\lambda\mu\nu\kappa} = +\ R_{\nu\kappa\lambda\mu} \quad \left(\boxed{\lambda\mu}\,\boxed{\nu\kappa} \to \boxed{\nu\kappa}\,\boxed{\lambda\mu}\right) \tag{1.132}$$

$$R_{\lambda\mu\nu\kappa} = -\ R_{\mu\lambda\nu\kappa} \quad (\lambda \leftrightarrow \mu) \tag{1.133}$$

$$R_{\mu\lambda\nu\kappa} = +\ R_{\lambda\mu\kappa\nu} \quad (\lambda \leftrightarrow \mu\ ,\ \kappa \leftrightarrow \nu) \tag{1.134}$$

$$R_{\lambda\mu\nu\kappa} = -\ R_{\lambda\mu\kappa\nu} \quad (\kappa \leftrightarrow \nu) \tag{1.135}$$

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0 \quad (\mu, \nu, \kappa \text{ permuted cyclically}) \tag{1.136}$$

Eq. (1.135) is a simple consequence of the commutation rule (1.122) for covariant derivatives (the left hand side of (1.122) is clearly antisymmetric in ν and κ).

We can form the contracted curvature tensor

$$R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa} = R^{\lambda}{}_{\mu\lambda\kappa} \tag{1.137}$$

which is called the Ricci tensor. Because of (1.132) it is symmetric

$$R_{\mu\kappa} = R_{\kappa\mu} \tag{1.138}$$

The contraction (1.137) is essentially unique: if we contracted instead the first two indices or the last two indices we get zero because of (1.133) and (1.135). Contraction of the other indices leads again to $R_{\mu\kappa}$ because of (1.132), (1.133), (1.134) and (1.135).

We can make a further contraction in order to get a scalar

$$R = g^{\mu\kappa}\ R_{\mu\kappa} = g^{\lambda\nu} g^{\mu\kappa} R_{\lambda\mu\nu\kappa} \tag{1.139}$$

Contraction of λ and μ and ν and κ gives zero because of antisymmetry in these indices.

Apart from the algebraic properties mentioned above the curvature tensor also satisfies important differential identities. These are most easily derived from (1.124) by going to a local elevator where Γ vanishes. Differentiating $R_{\lambda\mu\nu\kappa}$ with respect to x^η one obtains third derivatives on the metric as well as $\Gamma\ \partial\Gamma/\partial x^\eta$-terms. However, these terms vanish in the local system. Permuting the indices ν, κ, and η cyclically and adding the three expressions one finds that this sum vanishes. In generally covariant language this means

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \tag{1.140}$$

This equation is valid in the local elevators, where the covariant derivatives reduce to ordinary derivatives, and it is generally covariant and hence it holds in all systems.

Remembering that the covariant derivative of the metric tensor vanishes we find by contracting λ and ν in (1.140)

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}{}_{\mu\kappa\eta;\nu} = 0 \tag{1.141}$$

Contracting μ and κ we get

$$R_{;\eta} - R^{\mu}{}_{\eta;\mu} - R^{\nu}{}_{\eta;\nu} = 0$$

and

$$\left(R^{\mu}{}_{\eta} - \frac{1}{2}\ \delta^{\mu}{}_{\eta} R\right)_{;\mu} = 0 \tag{1.142}$$

Again, since the covariant derivative of the metric vanishes, we have from (1.142)

$$\left(R^{\mu\nu} - \frac{1}{2}\ g^{\mu\nu} R\right)_{;\mu} = 0 \tag{1.143}$$

It turns out that eq. (1.143) plays a fundamental role in the theory of general relativity, as we shall soon see.

1.14 The energy-momentum tensor

We now turn to the essential problem facing us: Suppose we are given some distribution of matter e.g. our universe, how do we determine the gravitational field $g_{\mu\nu}(x)$? So far, we only know how to proceed in the Newton limit (sect. 4), where we found (eqs. (1.35) and (1.37))

$$\nabla^2 \; g_{00}(\vec{x}) = - \; 8\pi \; G\rho(\vec{x}) \tag{1.144}$$

where $\rho(\vec{x})$ is the mass density in the limit where all velocities are small relative to the light velocity.

Eq. (1.144) suggests that we should attempt to understand how $\rho(\vec{x})$ can be generalized to arbitrary relativistic velocities. If we can find this, we know how to represent some matter distribution (e.g. in our universe) in terms of special relativity.

Let us consider a simple situation where we have a flow of matter. In special relativity the relevant quantity is not the mass density but the energy density (remember the famous formula $E = mc^2$). The proper density $\rho(x)$ is then defined as the density measured by an observer moving with the flow. The left-hand side of eq. (1.144) suggests that in more general situations it should be represented by a tensor with two indices (this tensor should somehow at least contain second derivatives of the metric tensor). Thus, we should attempt to generalize the right-hand side to a tensor with two indices, which for small velocities reduce to the right-hand side of eq. (1.144). In special relativity the flow of matter is characterized by its density and the four-velocity $U^\mu(x)$ of matter at the point x^μ. From this we can construct a tensor with two indices

$$T^{\mu\nu}(x) = \rho(x) \; U^\mu(x) \; U^\nu(x), \quad U^\nu = dx^\nu/d\tau \tag{1.145}$$

Let us consider T^{00},

$$T^{00} = \rho \left(\frac{dx^0}{d\tau}\right)^2 = \rho\frac{1}{1-v^2} \tag{1.146}$$

where v is the usual velocity $\vec{v} = d\vec{x}/dx^0$, and where we used the special relativistic relation between $d\tau, dx^0$, and $\vec{v}$,

$$d\tau^2 = (dx^0)^2 - (d\vec{x})^2 = (dx^0)^2 \; (1 - \vec{v}^2) \tag{1.147}$$

From eq. (1.146) we see that in the Newton limit $v << 1$ we get $T^{00} \approx \rho$, so eq. (1.144) becomes approximately

$$\nabla^2 \; g_{00}(x) \approx - \; 8\pi \; G \; T_{00}(x) \tag{1.148}$$

Recalling that the mass of a small volume of moving material increases by a factor $1/\sqrt{1-v^2}$ relative to the rest mass, and that the volume decreases by a factor $\sqrt{1-v^2}$, we see that T^{00} in eq. (1.146) is the density measured by a fixed observer who sees the matter passing by with a velocity $\vec{v}$. Thus T^{00} is simply the **relativistic energy density**. The other components are just

$$\begin{aligned} T^{ij} &= T^{ji} = \rho \; \frac{v_i v_j}{1-v^2} \\ T^{i0} &= T^{0i} = \frac{\rho \; v_i}{1-v^2} \end{aligned} \tag{1.149}$$

The quantity $T^{\mu\nu}$ is called the **energy-momentum** tensor (of special relativity). T^{0i} is the density of momentum (as seen by a fixed observer) and T^{ij} is the current of momentum.

$T^{\mu\nu}$ has the property that for closed systems the energy and momentum are conserved. In differential form we just have

$$\frac{\partial T^{0\nu}}{\partial x^\nu} = \frac{\partial}{\partial t}\left(\frac{\rho}{1-v^2}\right) + \vec{\nabla}\left(\frac{\rho\vec{v}}{1-v^2}\right) = 0 \tag{1.150}$$

which expresses the conservation of a quantity of material with density $\rho/(1-v^2)$ moving with a velocity $\vec{v}(x)$.

We also have

$$\begin{aligned} \frac{\partial T^{1\nu}}{\partial x^\nu} &= \frac{\partial}{\partial t}\left(\frac{\rho v_x}{1-v^2}\right) + \frac{\partial}{\partial x}\left(\frac{\rho {v_x}^2}{1-v^2}\right) + \frac{\partial}{\partial y}\left(\frac{\rho v_x v_y}{1-v^2}\right) + \frac{\partial}{\partial z}\left(\frac{\rho v_x v_z}{1-v^2}\right) \\ &= \frac{\rho}{1-v^2}\left(\frac{\partial v_x}{\partial t} + \vec{v}\,\vec{\nabla} v_x\right) \end{aligned} \tag{1.151}$$

where we used eq. (1.150). Thus

$$\frac{\partial T^{i\nu}}{\partial x^\nu} = \frac{\rho}{1-v^2}\left(\frac{\partial v^i}{\partial t} + \vec{v}\,\vec{\nabla} v^i\right) = 0 \tag{1.152}$$

since the quantity in the bracket in eq. (1.152) is just the change in the velocity for an observer following the stream of matter, and in the absence of any forces this velocity cannot change. Thus we have by combining (1.150) and (1.152)

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \tag{1.153}$$

and we say that the energy-momentum tensor is conserved. From (1.153) it follows that the total energy-momentum P^α is conserved, because (1.149) implies that

$$P^\alpha = \int d^3x\; T^{0\alpha}(\vec{x}, t) \tag{1.154}$$

and hence from (1.153)

$$\frac{dP^\alpha}{dt} = \int d^3x\; \frac{\partial T^{0\alpha}}{\partial t} = \int d^3x\; \frac{\partial T^{\mu\alpha}}{\partial x^\mu} = 0 \tag{1.155}$$

where we used that

$$\int d^3x\; \frac{\partial T^{i\alpha}}{\partial x^i} = 0$$

if $T^{i\alpha}$ vanishes at infinity. From eqs. (1.146) and (1.149) it is easily verified that P^α defined by eq. (1.154) is indeed a four-vector (remember that volume d^3x decreases by a factor $\sqrt{1-v^2}$)

The energy-momentum tensor (1.145) represents an extremely simple physical system. A somewhat more useful case can be constructed by considering a **perfect fluid**, which is defined as a fluid characterized by a velocity field $\vec{v}(x)$ such that an observer moving along with the fluid sees it as being isotropic around each point. Ideal fluids are often used to approximately describe our universe at large scales (much larger than the size of galaxies), and hence the energy-momentum distribution of such fluids is of particular relevance in the theory of gravity.

To find the energy-momentum tensor we simply use the rest frame where

$$\begin{aligned} (T^{00})_{\rm rest} &= \rho \\ (T^{i0})_{\rm rest} &= (T^{oi})_{\rm rest} = 0 \\ (T^{ij})_{\rm rest} &= (T^{ji})_{\rm rest} = p\ \delta_{ij} \end{aligned} \tag{1.156}$$

The (00)-component is the same as in (1.146) since $\vec{v} = 0$, and the result for the (ij)-component expresses the meaning of isotropy. Similarly the $(i0)$-components must vanish due to isotropy. In (1.146) ρ is again the proper energy density, whereas p is called the **pressure**.

We can now express $T^{\mu\nu}$ in a frame where the fluid moves with velocity $\vec{v}(\vec{x}, t)$,

$$T^{\mu\nu} = p\ \eta^{\mu\nu} + (p + \rho)U^\mu U^\nu \tag{1.157}$$

To see that $T^{\mu\nu}$ is correct notice that it is a tensor (in special relativity), and that it reduces to (1.156) in the rest frame. Conservation of energy and momentum in the differential form (for a closed system)

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \tag{1.158}$$

leads to the special relativistic equations for an ideal fluid,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1 - v^2}{p + \rho}\left[\nabla p + \mathbf{v}\frac{\partial p}{\partial t}\right]. \tag{1.159}$$

We recommend the reader to verify this result.

To make the above result generally covariant we proceed as done many times before. The conservation equation (1.158) becomes replaced by

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{1.160}$$

Using eq. (1.88) we get

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}(\sqrt{g}\ T^{\mu\nu}) = -\Gamma^\nu_{\mu\lambda}\ T^{\mu\lambda} \tag{1.161}$$

where we can say that the right-hand side represents a gravitational force density. The total energy-momentum can (tentatively) be written down in analogy with (1.154),

$$P^\mu = \int d^3x\ \sqrt{g}\ T^{\mu 0} \tag{1.162}$$

which is, however, not conserved due to the non-vanishing of the right-hand side of eq. (1.161). Physically non-conservation of P^μ is due to the possibility of exchanging energy and momentum between matter and gravitation. Also, P^μ is **not** a contravariant vector.

The explicit form of the energy-momentum tensor for an ideal fluid interacting with gravity is given by

$$T^{\mu\nu} = p\ g^{\mu\nu} + (p + \rho)U^\mu U^\nu \tag{1.163}$$

since $T^{\mu\nu}$ is a tensor, and since it reduces to (1.157) in special relativity.

The energy-momentum tensors constructed above are symmetric, $T^{\mu\nu} = T^{\nu\mu}$. This is a general property, assumed to be valid in the following.

1.15 Einstein's field equations for gravitation

We are now able to proceed to present Einstein's field equations, which were "derived" by plausibility arguments in 1916 ("Die Grundlage der allgemeinen Relativitätstheorie", Annalen der Physik (Leipzig) **49** (1916) 769). We return to the Newtonian limit, which can be written as in eq. (1.148), i.e.

$$\nabla^2 \; g_{00}(x) = - \; 8\pi G \; T_{00}(x) \tag{1.164}$$

The left-hand side contains the second derivative of the metric tensor, whereas the right-hand side is the (00)-component of the energy-momentum tensor. Thus it is natural to generalize this equation to

$$E_{\mu\nu}(x) = - \; 8\pi G \; T_{\mu\nu}(x) \tag{1.165}$$

in arbitrary coordinates, where the tensor $E_{\mu\nu}$ should depend on the metric and its first and second derivatives. From (1.164) it is clear that we need to include second derivatives, and the reason we do not include higher derivatives is primarily because of simplicity. However, such terms would have to be multiplied by a constant with dimension of a power of length ($\partial^3 g/\partial x^3$-terms would have to be multiplied by some constant with dimension of length, $\partial^4 g/\partial x^4$ by (length)^2 etc.). Looking around among the fundamental constants of nature there is only one constant with dimension of length, the so-called *Planck length*,

$$L_{Planck}{}^2 = \frac{G\hbar}{c^3} \tag{1.166}$$

Using $G/c^2 \simeq 7.41 \times 10^{-29}$ cm/g and $\hbar \simeq 10^{-27}$ erg $\times s$ one obtains

$$L_{Planck} \simeq 10^{-33} cm \tag{1.167}$$

It therefore follows that if the constants of dimension length to a power are present, they are incredibly small and of no relevance for large scale phenomena in our universe. They could be relevant in the very early universe, which is supposed to be very small (big bang). Due to the occurrence of $\hbar$ it is clear that quantum phenomena must be involved, and hence such terms could be of relevance in quantum gravity. [4] However, here we shall proceed to consider only scales which are large relative to 10^{-33} cm.

The Einstein tensor $E_{\mu\nu}(x)$ in eq. (1.165) thus depends on the geometry of space-time, and should be linear in the second derivatives of $g_{\mu\nu}$. Also, it should be a tensor which represents the genuine physical contents of gravitational fields. For this we have only one candidate[5], namely the curvature tensor and quantities derived from it. Since $E_{\mu\nu}$ has two indices, it should depend on the contracted curvature tensor. Because of the symmetries of $R_{\mu\nu\rho\sigma}$, we saw in sect. 13 that there is essentially only one such tensor, namely $R_{\mu\nu}$. Could we take

$$E_{\mu\nu} = R_{\mu\nu} \quad ? \tag{1.168}$$

The answer is no. In sect. 14 it was shown that the energy-momentum tensor satisfies differential conservation

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{1.169}$$

[4] Such terms occur in superstring theories (one of which could be the "right" theory of quantum gravity).

[5] Mathematicians have shown that the curvature tensor is the only tensor that can be constructed from the first and second derivatives of the metric tensor, and which is linear in the second derivatives of the metric tensor.

Going back go eq. (1.165) we therefore also have

$$E^{\mu\nu}{}_{;\nu} = 0 \qquad (1.170)$$

From eq. (1.143) it follows, however, that $R^{\mu\nu}{}_{;\nu} \neq 0$ in general. Thus eq. (1.168) is wrong[6].

However, eq. (1.143) shows that the right answer is

$$E_{\mu\nu} = c(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R) \qquad (1.171)$$

which is covariantly conserved. Here c is a dimensionless constant, which we shall fix by requiring that we get the correct Newton limit. In a non-relativistic system the pressure terms are much smaller than the energy-density, so $|T_{ij}| << |T_{00}|$. Eqs. (1.165) and (1.171) reduce to

$$c(R_{00} - \frac{1}{2}\, g_{00} R) = -\, 8\pi G\, T_{00} \qquad (1.172)$$

Since $|T_{ij}| << |T_{00}|$, we also have that E_{ij} is very small, so from (1.171)

$$R_{ij} \approx \frac{1}{2}\, g_{ij}\; R \qquad (1.173)$$

In the weak field limit, to the first approximation $g_{ij} \simeq \eta_{ij}$. Thus

$$R \cong R_{ii} - R_{00} \simeq \frac{3}{2}\, R - R_{00}$$

i.e.

$$\frac{1}{2}\, R \approx R_{00} \qquad (1.174)$$

Inserting this in (1.171) we get

$$E_{00} \approx c(R_{00} + \frac{1}{2}\, R) \approx 2c\; R_{00} \qquad (1.175)$$

Now R_{00} can be obtained from (1.124), where the $\Gamma\Gamma$-terms are of higher order and hence can be left out in the Newton limit. Thus

$$R_{\lambda\mu\nu\kappa} \approx \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right] \qquad (1.176)$$

Since all time derivatives vanish we get

$$R_{0000} \approx 0 \quad , \quad R_{i0j0} \approx \frac{1}{2}\, \frac{\partial^2 g_{00}}{\partial x^i \partial x^j} \qquad (1.177)$$

so from (1.175) we get

$$E_{00} \approx 2c\; R_{00} \approx 2c(R_{i0i0} - R_{0000}) \approx c\nabla^2\; g_{00} \qquad (1.178)$$

[6]It is interesting that the first field equation that Einstein published actually *was* precisely (1.168). He was aware that it could not be generally valid.

This shows that we obtain eq. (1.164) provided $c = 1$. Consequently **Einstein's field equations** read

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = -\ 8\pi G\ T_{\mu\nu} \tag{1.179}$$

This equation can be written in a different form. By contraction we get

$$R^{\mu}{}_{\mu} - \frac{1}{2}\, \delta^{\mu}{}_{\mu}\ R = R - \frac{1}{2}\ \cdot 4 \cdot R = -R = -\ 8\pi G\ T^{\mu}{}_{\mu}$$

or

$$R = 8\pi\ G\ T^{\mu}{}_{\mu} \tag{1.180}$$

This can be inserted in eq. (1.179), and we then get

$$R_{\mu\nu} = -\ 8\pi\ G(T_{\mu\nu} - \ \frac{1}{2}\ g_{\mu\nu} T^{\alpha}{}_{\alpha}) \tag{1.181}$$

Eq. (1.181) is, of course, fully equivalent to eq. (1.179).

In empty space $T_{\mu\nu} = 0$, and (1.181) then gives the vacuum Einstein equation

$$R_{\mu\nu} = 0 \tag{1.182}$$

Flat Minkowski-space satisfies this equation globally and locally, but in the local case there are other non-trivial solutions (due to the non-linearity of eq. (1.182)).

Eq. (1.179) can be generalized by adding the "cosmological term",

$$R_{\mu\nu} - \ \frac{1}{2}\ g_{\mu\nu} R - \Lambda g_{\mu\nu} = -\ 8\pi\ G\ T_{\mu\nu} \tag{1.183}$$

This is allowed from the point of view of covariant conservation of $T_{\mu\nu}$, since the covariant derivative of $g_{\mu\nu}$ vanishes. However, eq. (1.183) does not satisfy the Newton limit. The cosmological constant Λ must therefore be rather small, and in this chapter we shall ignore it. However, a small Λ, which seems to exist, has profound cosmological effects, as we shall see later. It should be noticed that consistency of the Einstein equation (1.183) requires that $T_{\mu\nu}$ is symmetric, $T^{\mu\nu} = T^{\nu\mu}$.

To conclude this section we now have a non-linear set of second order differential equations (1.179), which allows one to compute the gravitational field from a given energy-momentum distribution. This is precisely the contents of Einstein's general relativity.

Remarks on the history of Einstein's field equations.

In 1913 Einstein and Grossmann published a paper where gravity was for the first time described by the metric tensor, and the Ricci tensor was introduced as an important tool. The theory was, however, not right. Also, a later paper from 1914 by Einstein contains a treatise of differential geometry. The contents of this paper was corrected for technical errors by Levi-Civita. The flaw in Einstein's 1914 paper (mainly that the covariant conservation of the energy-momentum tensor on the right-hand side was not correctly implemented by the geometrical left-hand side, as explained above in connection with (1.168)) was corrected in November 1915 by Einstein and by the matematician Hilbert, who derived the consistent field equation from a variational principle. On November 25 Einstein submitted a paper which for the first time gives the correct field equation. Five days before, Hilbert had actually submitted a paper which also contained the right field equation.

1.16 The time-dependent spherically symmetric metric

In order to gain insight in Einstein's equations we shall start by considering a time-dependent spherically symmetric gravitational field $g_{\mu\nu}(x)$. By this we mean that the metric should be the same when the rectangular coordinates x^1, x^2, x^2 are subjected to a rotation. Intuitively this means that the proper time interval can only depend on the following quantities (which are rotationally invariant),

$$t, r, dt, \vec{x}d\vec{x} = r\ dr \quad , \quad (d\vec{x})^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1.184}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Thus

$$d\tau^2 = A(r,t)dt^2 - B(r,t)dr^2 - C(r,t)drdt - D(r,t)r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1.185}$$

In general relativity it is important to realize that the coordinates are arbitrary. Thus, we are free to make transformations $x^\mu \to x'^\mu$ in (1.185). In particular, we can always try to select such transformations in a way which simplifies the expression (1.185). First, the function D can easily be removed by introducing a new radial variable, [7]

$$r' = r\sqrt{D(r,t)} \tag{1.186}$$

$d\tau^2$ then becomes dependent on new functions A', B', and C', which are functions of t and r'. Dropping the primes, we can write

$$d\tau^2 = A(r,t)dt^2 - B(r,t)dr^2 - C(r,t)drdt - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1.187}$$

The "mixed" $drdt$-term can also be removed by resetting our clocks by use of a new time coordinate t', which is defined by

$$dt'(r,t) = \eta(r,t)[A(r,t)dt - \frac{1}{2}\ C(r,t)dr] \tag{1.188}$$

Here $\eta(r,t)$ is an integrating factor defined such that (1.188) becomes a perfect differential with $\eta A = \partial t'/\partial t, -\frac{1}{2}\eta C = \partial t'/\partial r$. This requires the integrability condition

$$\frac{\partial}{\partial r}\left(\eta(r,t)A(r,t)\right) = -\ \frac{1}{2}\ \frac{\partial}{\partial t}\left(\eta(r,t)C(r,t)\right) \tag{1.189}$$

This is a first order partial differential equation for η, since A and C are given. Once we are given η at a certain time for all r, we can compute it at any other time. Using that

$$\frac{1}{A\eta^2}dt'^2 = A\ dt^2 - C\ dtdr + \frac{C^2}{4A}dr^2 \tag{1.190}$$

the proper time (1.187) becomes

$$d\tau^2 = \frac{1}{\eta^2 A}dt'^2 - \left(B + \frac{C^2}{4A}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

[7]It could happen that this transformation produces a constant r'. In the following, for simplicity we only consider non-trivial transformations (r'= const turns out to be inconsistent with the Einstein equation).

or, renaming these functions,

$$d\tau^2 = E(r,t)dt^2 - \ F(r,t)dr^2 - r^2(d\theta^2 + \sin^2\theta\ d\varphi^2) \tag{1.191}$$

This form of the metric is called the "standard" form of the metric, first derived by Weyl. It should be noted that the transformations used above do not involve the angles, and therefore the spherical symmetry is still manifest.

The metric tensor thus has the form

$$\begin{array}{llll} g_{rr} = F & , \ g_{\theta\theta} = r^2 & , \ g_{\varphi\varphi} = r^2\sin^2\theta & , \ g_{tt} = -E, \\ g^{rr} = \dfrac{1}{F} & , \ g^{\theta\theta} = \dfrac{1}{r^2} & , \ g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta} & , \ g^{tt} = -\dfrac{1}{E} \end{array} \tag{1.192}$$

whereas the determinant g is

$$g = r^4 EF\sin^2\theta \tag{1.193}$$

The invariant volume element is thus

$$r^2\sin\theta\sqrt{E(r,t)F(r,t)}\ dr\ d\theta\ d\varphi\ dt \tag{1.194}$$

If E and F were constants, this would just be the usual volume element in spherical coordinates.

1.17 A digression: A simpler method for computing $\Gamma^\lambda_{\mu\nu}$

In order to write down the Einstein field equations we need first to know the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ for the metric (1.192). One method is to use eq. (1.27) directly, i.e. to use

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\ g^{\lambda\sigma}\left[\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right] \tag{1.195}$$

This is a straightforward but tedious method. A somewhat simpler way consists in obtaining the Γ's from a variational principle.

First let us consider some functional $F = F(x^\mu, \dot{x}^\nu)$, where $\dot{x}^\mu = dx^\mu(\tau)/d\tau$, and demand that the trajectory $x^\mu = x^\mu(\tau)$ is obtained by requiring the minimum variation of the integral

$$\int F(x^\mu, \dot{x}^\mu)d\tau$$

i.e.

$$\begin{aligned} \delta\int F(x^\mu,\dot{x}^\mu)d\tau &= \int \delta F(x^\mu,\dot{x}^\mu)d\tau \\ &= \int\left[\frac{\partial F}{\partial x^\mu}\ \delta x^\mu + \frac{\partial F}{\partial \dot{x}^\mu}\ \delta\dot{x}^\mu\right]d\tau = 0 \end{aligned} \tag{1.196}$$

Now $\delta\dot{x}^\mu = d\delta x^\mu/d\tau$. Requiring that the variations vanish at the ends of the interval of integration, we obtain by a partial integration,

$$\delta\int F d\tau = \int\left[\frac{\partial F}{\partial x^\mu} - \frac{d}{d\tau}\ \frac{\partial F}{\partial\dot{x}^\mu}\right]\delta x^\mu d\tau = 0$$

or the Euler-Lagrange equation

$$\frac{d}{d\tau}\frac{\partial F}{\partial \dot{x}^{\mu}} = \frac{\partial F}{\partial x^{\mu}} \tag{1.197}$$

Now we can easily check that if we take

$$F = g_{\mu\nu}(x)\,\frac{dx^{\mu}}{d\tau}\,\frac{dx^{\nu}}{d\tau} \tag{1.198}$$

then the Euler-Lagrange equation becomes

$$2\frac{d}{d\tau}\left[g_{\mu\sigma}(x)\,\frac{dx^{\mu}}{d\tau}\right] = \frac{\partial g_{\mu\nu}(x)}{\partial x^{\sigma}}\,\frac{dx^{\mu}}{d\tau}\,\frac{dx^{\nu}}{d\tau} \tag{1.199}$$

Doing the differentiation on the left-hand side we get

$$\frac{\partial g_{\mu\sigma}}{\partial x^{\rho}}\,\frac{dx^{\rho}}{d\tau}\,\frac{dx^{\mu}}{d\tau} + g_{\mu\sigma}\frac{d^2x^{\mu}}{d\tau^2} = \frac{1}{2}\,\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\,\frac{dx^{\mu}}{d\tau}\,\frac{dx^{\nu}}{d\tau}$$

so after multiplication with $g^{\alpha\sigma}$ we get

$$\begin{aligned} &\frac{d^2x^{\alpha}}{d\tau^2} + \frac{1}{2}\,g^{\alpha\sigma}\left(2\,\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)\frac{dx^{\mu}}{d\tau}\,\frac{dx^{\nu}}{d\tau} \\ = &\frac{d^2x^{\alpha}}{d\tau^2} + \frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0, \end{aligned} \tag{1.200}$$

where in the last step we used that the names of the summation indices are arbitrary and $dx^{\mu}dx^{\nu}$ is symmetric in μ and ν. From (1.195) and (1.18) we see that this is precisely the equation of motion for a freely falling particle in a gravitational field specified by the metric $g_{\mu\nu}(x)$.

As a side–remark we mention that use of the Euler-Lagrange equation (1.196) also shows that (1.199) can be obtained from the variational principle

$$\delta\int\sqrt{g_{\mu\nu}(x)\frac{dx^{\mu}}{d\tau}\,\frac{dx^{\nu}}{d\tau}}\;d\tau = 0 \tag{1.201}$$

We leave it to the reader to verify this statement. Eq. (1.201) shows that freely falling particles follow a curve with the shortest "distance", since the square root times $d\tau$ in (1.201) is the distance, i.e. (1.201) can be written $\delta\int d\tau = 0$. The curve followed by a freely falling particle is therefore called a geodesic.

To return to the main issue of obtaining the Christoffel symbols we see that eq. (1.200), or

$$\frac{d^2x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\,\frac{dx^{\mu}}{d\tau}\,\frac{dx^{\nu}}{d\tau} = 0 \tag{1.202}$$

allows one to obtain the Γ's from a variational principle using the Euler-Lagrange equations (1.197) with F given by (1.198). In our case the line element (1.191) therefore leads to the variational principle

$$\delta\int\left[E\dot{t}^2 - F\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\;\dot{\varphi}^2\right]d\tau = 0 \tag{1.203}$$

where $\dot{t} = dt/d\tau,\quad \dot{r} = dr/d\tau,\quad \dot{\theta} = d\theta/d\tau\quad$ and $\quad\dot{\varphi} = d\varphi/d\tau.$

1.18 The Christoffel symbols for the time-dependent spherically symmetric metric

Let us start by applying eq. (1.203) to the case where x^μ in the Euler-Lagrange equation (1.198) is the time. In that case (1.203) gives from (1.197)

$$\frac{d}{d\tau}(2E\dot{t}) = \dot{t}^2 \frac{\partial E}{\partial t} - \frac{\partial F}{\partial t}\dot{r}^2 \tag{1.204}$$

Performing the differentiation on the left-hand side (remember that E depends on r and t, which in turn depends on τ)

$$2E\ddot{t} + 2\dot{t}^2 \frac{\partial E}{\partial t} + 2\dot{t}\dot{r}\frac{\partial E}{\partial r} = \dot{t}^2 \frac{\partial E}{\partial t} - \frac{\partial F}{\partial t}\dot{r}^2$$

or

$$\ddot{t} + \frac{1}{2E}\frac{\partial E}{\partial t}\dot{t}^2 + \frac{1}{E}\frac{\partial E}{\partial r}\dot{t}\dot{r} + \frac{1}{2E}\frac{\partial F}{\partial t}\dot{r}^2 = 0 \tag{1.205}$$

From this we get by comparison with (1.202) the following Γ's (remember that mixed terms like $\dot{t}\dot{r}$ occur twice in (1.202))

$$\begin{aligned}
\Gamma^t_{tt} &= \frac{1}{2E}\frac{\partial E}{\partial t} \\
\Gamma^t_{tr} &= \frac{1}{2E}\frac{\partial E}{\partial r} = \Gamma^t_{rt} \\
\Gamma^t_{rr} &= \frac{1}{2E}\frac{\partial F}{\partial t}
\end{aligned} \tag{1.206}$$

Next let us use the Euler-Lagrange equation (1.198) for the problem (1.203) when we vary $x^\mu = r$,

$$-\frac{d}{d\tau}(2F\dot{r}) = \frac{\partial E}{\partial r}\dot{t}^2 - \frac{\partial F}{\partial r}\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\varphi}^2 \tag{1.207}$$

Proceeding as before we perform the $d/d\tau$ on the left-hand side and obtain

$$\ddot{r} + \frac{1}{F}\frac{\partial F}{\partial t}\dot{t}\dot{r} + \frac{1}{2F}\frac{\partial F}{\partial r}\dot{r}^2 + \frac{1}{2F}\frac{\partial E}{\partial r}\dot{t}^2 - \frac{r}{F}\dot{\theta}^2 - \frac{r}{F}\sin^2\theta\dot{\varphi}^2 = 0 \tag{1.208}$$

Comparing with (1.202) we see that

$$\begin{aligned}
&\Gamma^r_{tr} = \Gamma^r_{rt} = \tfrac{1}{2F}\tfrac{\partial F}{\partial t}\ , \quad \Gamma^r_{rr} = \frac{1}{2F}\frac{\partial F}{\partial r} \\
&\Gamma^r_{tt} = \tfrac{1}{2F}\tfrac{\partial E}{\partial r}\ , \quad \Gamma^r_{\theta\theta} = -\tfrac{r}{F}\ , \quad \Gamma^r_{\varphi\varphi} = \tfrac{-r\sin^2\theta}{F}
\end{aligned} \tag{1.209}$$

Proceeding now with the variable θ we get (since E and F are independent of θ)

$$-\frac{d}{d\tau}(2r^2\dot{\theta}) = -2r^2\sin\theta\cos\theta\dot{\varphi}^2 \tag{1.210}$$

which gives

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0 \tag{1.211}$$

i.e.

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \quad , \quad \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta \ \cos\theta \tag{1.212}$$

Finally we need only to vary φ,

$$-\frac{d}{d\tau}(2r^2 \sin^2\theta\dot{\varphi}) = 0 \tag{1.213}$$

or

$$\ddot{\varphi} + \frac{2}{r}\ \dot{r}\ \dot{\varphi} + 2\ \frac{\cos\theta}{\sin\theta}\ \dot{\theta}\dot{\varphi} = 0 \tag{1.214}$$

Hence the only remaining non-vanishing Christoffel symbols become

$$\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r} \quad , \quad \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \frac{\cos\theta}{\sin\theta} \tag{1.215}$$

The reader can now make a psychological experiment where he/she compares the time it takes to obtain the Γ's by means of the variational principle and by the direct application of the expression (1.195). This would also provide a check of the correctness of the results (1.206), (1.209), (1.212), and (1.215).

1.19 The Ricci tensor

In order to study the Einstein equation (1.179) or (1.181) we need to know the Ricci tensor $R_{\mu\nu}$. From eqs. (1.82), (1.123), (1.137) we have

$$R_{\mu\kappa} = \frac{1}{2}\ \frac{\partial^2 \ln g}{\partial x^\mu \partial x^\kappa} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^\lambda} + \Gamma^{\eta}_{\mu\lambda}\ \Gamma^{\lambda}_{\kappa\eta} - \frac{1}{2}\ \Gamma^{\eta}_{\mu\kappa}\frac{\partial \ln g}{\partial x^\eta} \tag{1.216}$$

Inserting the Γ's found in the last section as well as g given by eq. (1.193), we get the following non-vanishing components of the Ricci tensor,

$$\begin{aligned}
R_{rr} &= \frac{1}{2E}\ \frac{\partial^2 E}{\partial r^2} - \frac{1}{4E^2}\left(\frac{\partial E}{\partial r}\right)^2 - \frac{1}{4EF}\ \frac{\partial E}{\partial r}\frac{\partial F}{\partial r} \\
&\quad -\frac{1}{rF}\ \frac{\partial F}{\partial r} - \frac{1}{2E}\ \frac{\partial^2 F}{\partial t^2} + \frac{1}{4E^2}\ \frac{\partial E}{\partial t}\ \frac{\partial F}{\partial t} + \frac{1}{4EF}\ \left(\frac{\partial F}{\partial t}\right)^2 \\
R_{\theta\theta} &= -1 + \frac{1}{F} - \frac{r}{2F^2}\ \frac{\partial F}{\partial r} + \frac{r}{2EF}\ \frac{\partial E}{\partial r} \\
R_{tt} &= \frac{-1}{2F}\ \frac{\partial^2 E}{\partial r^2} + \frac{1}{4F^2}\ \frac{\partial E}{\partial r}\ \frac{\partial F}{\partial r} - \frac{1}{rF}\ \frac{\partial E}{\partial r} \\
&\quad + \frac{1}{4EF}\left(\frac{\partial E}{\partial r}\right)^2 + \frac{1}{2F}\ \frac{\partial^2 F}{\partial t^2} - \frac{1}{4F^2}\left(\frac{\partial F}{\partial t}\right)^2 - \frac{1}{4EF}\ \frac{\partial E}{\partial t}\ \frac{\partial F}{\partial t} \\
R_{tr} &= \frac{-1}{rF}\ \frac{\partial F}{\partial t} \\
R_{\varphi\varphi} &= \sin^2\theta\ R_{\theta\theta}
\end{aligned} \tag{1.217}$$

Thus, given some energy-momentum tensor $T_{\mu\nu}$ we can in principle obtain second order differential equation for the metric functions E and F by use of e.g. eq. (1.181),

$$R_{\mu\nu} = -\ 8\pi G\ (T_{\mu\nu} - \frac{1}{2}\ g_{\mu\nu} T^{\alpha}{}_{\alpha}) \tag{1.218}$$

For example,

$$\frac{1}{rF}\ \frac{\partial F}{\partial t} = 8\pi G\ T_{tr} \tag{1.219}$$

etc. etc.

1.20 The Schwarzschild solution

We shall now consider the case where space is empty, except for a mass M situated at $r = 0$. Thus, except for $r = 0$ we must satisfy the vacuum Einstein equations $R_{\mu\nu} = 0$. Using (1.217) we see that this implies that the time-dependence drops out: From $R_{tr} = 0$ we get that F is independent of time, and it is then easily checked that all the time derivatives drop out of the other R's: From the last eq. (1.217) it follows that we only need to consider $R_{rr} = R_{\theta\theta} = R_{tt} = 0$. It is seen that R_{rr} and R_{tt} contain rather similar terms. Therefore we form

$$\frac{R_{rr}}{F} + \frac{R_{tt}}{E} = \frac{-1}{rF}\left(\frac{1}{F}\ \frac{\partial F}{\partial r} + \frac{1}{E}\ \frac{\partial E}{\partial r}\right) = 0 \tag{1.220}$$

This gives

$$\frac{\partial \ln F}{\partial r} = -\frac{\partial \ln E}{\partial r} \tag{1.221}$$

or

$$E(r,t)\ F(r) = f(t). \tag{1.222}$$

We now impose the boundary condition that the metric (1.191) should become flat Minkowski at infinity, so the function $f(t)$ in (1.222) is fixed to be 1 for $r \to \infty$, and hence also for all other r's. Thus

$$E(r) = \frac{1}{F(r)} \tag{1.223}$$

Now we can take $R_{\theta\theta}$ and R_{rr} to vanish (because of (1.220) R_{tt} then vanishes automatically). Using the result (1.223) we get

$$R_{\theta\theta} = -1 + E(r) + r\ \frac{dE(r)}{dr} \tag{1.224}$$

$$\begin{aligned} R_{rr} &= \frac{1}{2E(r)}\ \frac{d^2E(r)}{dr^2} + \frac{1}{rE(r)}\ \frac{dE(r)}{dr} \\ &= \frac{1}{2rE(r)}\ \frac{d\ R_{\theta\theta}(r)}{dr} \end{aligned} \tag{1.225}$$

Thus, if $R_{\theta\theta} = 0$ then R_{rr} is automatically zero. Hence, from $R_{\theta\theta} = 0$ we get

$$\frac{d}{dr}(rE(r)) = 1 \tag{1.226}$$

so

$$rE(r) = r + \text{constant} = r + C \tag{1.227}$$

From (1.223) we then know the functions $E(r)$ and $F(r)$ in terms of the constant C, which can be fixed by means of the results $g_{00} \to -1 + 2GM/r$ obtained in the Newton limit, see eqs. (1.36) and (1.37).

Thus, from (1.191) we get $C = -2MG$, and hence

$$\begin{aligned} E(r) &= 1 - \frac{2MG}{r} \\ F(r) &= \frac{1}{1 - \frac{2MG}{r}} \end{aligned} \tag{1.228}$$

and the metric (1.191) becomes

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2MG}{r}} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1.229}$$

This is the Schwarzschild solution. The radial variable r has the property that the solution is asymptotically flat. Furthermore, the solution is singular at $r = 0$, as one would expect because of the presence of the mass. This singularity can be shown to be physical since e.g. the scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is singular only at $r = 0$. For an elaboration of this feature, see sections 1.25 and 1.26.

Remarks on the history of the Schwarzschild solution:

Karl Schwarzschild was a well-known astronomer, who had wide interests in all branches of physics and mathematics. When the war started in August 1914 he entered military service, where he calculated missile trajectories in France and later in Russia. Here he worked out his solution (explained above) of the newly proposed Einstein field equation. He send his paper to Einstein, who replied: "I had not expected that one could formulate the exact solution of the problem in such a simple way". Schwarzschild contracted an illness while in Russia. After having returned to Berlin in March 1916, he died two months later at the age of 42 years.

1.21 Birkhoff's theorem

We saw from $R_{tr} = 0$ that the function F is time-independent, and all time-derivatives disappeared. In principle the function E could depend on time,

$$E(r,t) = f(t)\left(1 - \frac{2MG}{r}\right) \tag{1.230}$$

but $f(t)$ can be removed by a redefinition of time, $t \to t'$ where

$$t' = \int^t \sqrt{f(t)}\, dt \tag{1.231}$$

Thus the Schwarzschild metric (1.229) always results, and we have Birkhoff's theorem saying that a spherically symmetric gravitational field in empty space must be static with a metric given by the Schwarzschild solution.

This theorem is similar to Newton's result that the gravitational field outside a spherically symmetric mass distribution behaves af if the mass was concentrated in the centre. It is a priori far from obvious that the similar result applies in general relativity, where the body needs not be static. A non-static body can in general emit gravitational waves. However, Birkhoff's theorem shows that no gravitational radiation can escape from a spherical body into empty spaces.

Birkhoff's theorem can also be applied if we have a spherical cavity in a spherical mass distribution. In this case there is no mass at the center, and hence $GM = 0$, and the metric is just flat Minkowski. Thus, if e.g. the universe was a spherical mass distribution, the metric would be flat, and gravitation would have no effect in a spherical cavity.

1.22 The general relativistic Kepler problem

We shall now discuss some implications of the Schwarzschild solution (1.229) for the solar system. In (1.229) it appears that there is a singularity for a radius $r = 2MG$, which is called the Schwarzschild radius. We shall discuss this phenomena later. Here we only remark that for the sun $2M_\odot G = 2.95$ km, which is deep in the solar interior. Hence this apparent singularity is not relevant for the solar system, since the solution (1.229) is only valid outside the massive bodies (for the earth the Schwarzschild radius is ≈ 1 cm).

In eqs. (1.205) , (1.208), (1.211), and (1.214) we have written down the equations of motion. Using that E and F can be taken to be time-independent, and using the result (1.228), we get

$$\frac{d}{d\tau}(r^2\dot{\theta}) = r^2 \sin\theta \ \cos\theta \dot{\varphi}^2 \tag{1.232}$$

$$\frac{d}{d\tau}(r^2 \sin^2\theta \dot{\varphi}) = 0 \tag{1.233}$$

$$\frac{d}{d\tau}\left[\left(1 - \frac{2MG}{r}\right)\dot{t}\right] = 0 \tag{1.234}$$

where the dot denotes differentiation with respect to τ. We also have an equation for $\ddot{r}$, which we replace by the line element (1.229),

$$1 = \left(1 - \frac{2MG}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2MG}{r}} - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) \tag{1.235}$$

Here we assumed that the particle is not a photon, since for a photon $d\tau = 0$. We shall return to the photon case later.

We thus have four equations for the four functions $t(\tau), r(\tau), \theta(\tau), \varphi(\tau)$, so we can study planetary motion by solving these equations. A great simplification occurs because this motion lies in a plane: It is easy to see that if we fix θ at some initial value of τ, then θ will be fixed to the same value for all other τ's. Taking e.g.

$$\theta = \frac{\pi}{2} \tag{1.236}$$

for some $\tau = \tau_0$, and $\dot{\theta} = 0$ for $\tau = \tau_0$, then eq. (1.232) implies that this remains true for all τ. Eq. (1.233) can then be integrated

$$r^2\dot{\varphi} = H \tag{1.237}$$

where H is a constant of integration. Also, eq. (1.234) gives

$$\left(1 - \frac{2MG}{r}\right)\dot{t} = L \tag{1.238}$$

where L is another constant. Substituting these results into (1.235) we get

$$1 = \frac{L^2}{1 - \frac{2MG}{r}} - \frac{\dot{r}^2}{1 - \frac{2MG}{r}} - \frac{H^2}{r^2} \tag{1.239}$$

In applications to orbital motion we are interested in the shape of the orbit, and we would therefore like to know r as a function of φ, instead of τ. We have

$$r' \equiv \frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} \tag{1.240}$$

From (1.237) we thus have

$$\dot{r} = \dot{\varphi} r' = \frac{H}{r^2}\, r' \tag{1.241}$$

Inserting this in (1.239) we get

$$1 - \frac{2MG}{r} = L^2 - \frac{H^2}{r^4}\, r'^2 - \frac{H^2}{r^2}\left(1 - \frac{2MG}{r}\right) \tag{1.242}$$

which is a differential equation for $r = r(\varphi)$.

In the classical Kepler problem it is convenient to introduce the new variable

$$u = \frac{1}{r} \tag{1.243}$$

It turns out that this substitution is also convenient in general relativity. From (1.243) we have $r' = -u'/u^2$, and eq. (1.242) gives

$$1 - 2MGu = L^2 - H^2 u'^2 - H^2 u^2 (1 - 2MGu) \tag{1.244}$$

which can be solved for u',

$$u'^2 = \frac{L^2 - 1}{H^2} + \frac{2MG}{H^2}\, u - u^2 + 2MGu^3 \tag{1.245}$$

This gives the exact result

$$\varphi = \varphi_0 + \int_{u_0}^{u} \frac{du}{\sqrt{\frac{L^2-1}{H^2} + \frac{2MG}{H^2} u - u^2 + 2MGu^3}} \tag{1.246}$$

The integral is of the elliptic type, and (1.246) gives $\varphi = \varphi(u)$, so the inverse function[8] provides us with $u = u(\varphi) = 1/r(\varphi)$. However, in practice this elliptic integral is not very useful. Although it can easily be evaluated numerically using e.g. Mathematica, the

[8] One can express u or r as functions of φ by inverting the integral (1.246) by means of Weierstrass' p-function $\wp$, $\frac{MG}{2r} = \frac{1}{12} + \wp(\varphi + \text{const.})$, where the Weirstrass invariants are given by $g_2 = \frac{1}{12} - \frac{M^2G^2}{H^2}$, $g_3 = \frac{1}{216} + \frac{M^2G^2}{6H^2} - \frac{M^2G^2L^2}{4H^2}$. For more discussion of this solution, see E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, Cambridge University Press (1937), pages 390-393.

main work is to express the constants L, H in terms of quantities that are measured by the astronomers.

To orient ourselves in this mess let us notice that the term $2MGu^3$ under the square root is expected to be very small relative to e.g. the u^2 term, since $2MGu \approx \frac{1}{3}10^{-7}$ is the Schwarzschild radius divided by the (average) planetary radius. As a matter of fact, if we did not have this term, the result (1.246) would be the Newtonian result, except that the constant H in (1.237) would be related to the usual time (not the proper time). This suggests that we expand the integral (approximately) around the Newtonian solution.

To do this systematically we notice that in the motion around the sun the orbit reaches a maximum (minimum) distance, called the *aphelia* (perihelia), where
$r = r_+(r_-)$ with $dr/d\varphi = 0$, i.e. for $u = u_\pm$ we also have $du/d\varphi = 0$. Eq. (1.244) then gives

$$u_\pm^2 - \frac{L^2}{H^2(1-2MGu_\pm)} = -\frac{1}{H^2} \tag{1.247}$$

which determines the integration constants L and H in terms of $u_\pm$, which are known from astronomical observations. This is a third order equation. To simplify the analysis we expand to second order

$$u_\pm^2 - \frac{L^2}{H^2}(1 + 2MGu_\pm + 4M^2G^2u_\pm^2) + \frac{1}{H^2} \approx 0. \tag{1.248}$$

Subtracting these two equations for u_+ and u_- we get

$$0 \approx (u_+ - u_-)\left[u_+ + u_- - 2MG\,\frac{L^2}{H^2}\,\left(1 + 2MG(u_+ + u_-)\right)\right]$$

or

$$2MG\frac{L^2}{H^2} \approx \frac{u_+ + u_-}{1 + 2MG(u_+ + u_-)} \tag{1.249}$$

The result (1.246) can be written

$$\varphi(r) = \varphi(r_-) + \int_{1/r_-}^{1/r} \frac{du}{\sqrt{1-2MGu}}\left[\frac{L^2}{H^2(1-2MGu)} - u^2 - \frac{1}{H^2}\right]^{-\frac{1}{2}} \tag{1.250}$$

where φ is measured from the perihelia. To proceed we expand

$$\frac{1}{1-2MGu} \approx 1 + 2MGu + 4M^2G^2u^2 + \cdots, \tag{1.251}$$

so

$$\frac{L^2}{H^2(1-2MGu)} - u^2 - \frac{1}{H^2} \approx \frac{L^2-1}{H^2} + \frac{2MGL^2}{H^2}u + \left(\frac{4M^2G^2L^2}{H^2} - 1\right)u^2 \tag{1.252}$$
$$= C(u_- - u)(u - u_+)$$

The last form follows from the fact that the left hand side is approximately a second order polynomial in u, which vanishes for $u = u_\pm$ according to eq. (1.248). Consider the term linear in u in (1.253) with coefficient $C(u_+ + u_-)$. It should equal $2MG\ L^2/H^2$ according to the left hand side of eq. (1.253). Hence

$$C(u_+ + u_-) = 2MG\frac{L^2}{H^2} = \frac{u_+ + u_-}{1 + 2MG(u_+ + u_-)}$$

where eq. (1.249) has been used. Thus

$$C \approx 1 - 2MG(u_+ + u_-). \tag{1.253}$$

Inserting this in (1.250) we get

$$\varphi(r) - \varphi(r_-) \approx \left(1 + MG\left(\frac{1}{r_+} + \frac{1}{r_-}\right)\right) \int_{\frac{1}{r_-}}^{\frac{1}{r}} \frac{(1 + MGu)\ du}{\sqrt{(u_- - u)(u - u_+)}} \tag{1.254}$$

where the constant factor outside the integral comes from expanding $1/\sqrt{C}$. The integral can be performed trivially by making the substitution

$$u = \frac{1}{r} = \frac{1}{2}\ (u_+ + u_-) + \frac{1}{2}\ (u_+ - u_-) \sin\psi \tag{1.255}$$

and we get

$$\varphi(r) - \varphi(r_-) = \left(1 + \frac{3}{2}\ MG\left(\frac{1}{r_+} + \frac{1}{r_-}\right)\right)(\psi + \frac{\pi}{2}) - \frac{1}{2}\ MG\left(\frac{1}{r_+} - \frac{1}{r_-}\right)\cos\psi \tag{1.256}$$

Now, when r goes from r_- to r_+ it follows from (1.255) that ψ goes from $\psi = -\pi/2$ to $\psi = \pi/2$. The total charge in φ per revolution is thus $2|\varphi(r_+) - \varphi(r_-)|$. For an ellipse this equals 2π, so the perihelia precesses an amount $\Delta\varphi$, where

$$\Delta\varphi = 2|\varphi(r_+) - \varphi(r_-)| - 2\pi \tag{1.257}$$

Thus,

$$\Delta\varphi = 3\pi\ MG\left(\frac{1}{r_+} + \frac{1}{r_-}\right) \text{radians/revolution.} \tag{1.258}$$

For Mercury this gives $\Delta\varphi = 0.1038''$ per revolution. In each century Mercury makes 415 revolutions, and observations go back to 1765. From general relativity we thus have

$$\Delta\varphi = 43.03\text{" per century} \tag{1.259}$$

whereas observational data give

$$(\Delta\varphi)_{\text{obs}} = 43.11 \pm 0.45\text{"per century} \tag{1.260}$$

This is the most important experimental verification of general relativity, since it is sensitive to the second order expansion of g_{00}. It should be pointed out that due to perturbations from other planets and due to the rotation of the earth, Newtonian gravity also gives a precession of Mercury,

$$(\Delta\varphi)_{\text{Newton}} = 5557.62 \pm 0.20\text{"} \tag{1.261}$$

whereas the actual observation gives

$$(\Delta\varphi)_{obs}^{Tot} = 5600.73 \pm 0.41\text{"} \tag{1.262}$$

so the value (1.260) is obtained as the difference $(\Delta\varphi)_{obs} = (\Delta\varphi)_{obs}^{TOT} - (\Delta\varphi)_{Newton}$. One might feel unhappy about the subtraction of two large numbers, because small systematic errors may influence (1.260) quite considerably. Some discussion of this point has been made in the literature. However, nobody has found any convincing evidence against the result (1.260).

1.23 Deflection of light by a massive body

So far we have considered the gravitational effects for a massive body moving around the sun. Now we wish to find these effects for light, which is characterized by $d\tau = 0$ (i.e., velocity of light is always one in the freely falling elevators). Therefore we cannot divide by $d\tau$, as we did in the previous section.

Let us consider the free motion along a curve described by $x^\mu = x^\mu(p)$, where p is some parameter. Proceeding as in section 1.3 the equivalence principle leads to

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\lambda\rho}\,\frac{dx^\lambda}{dp}\,\frac{dx^\rho}{dp} = 0, \tag{1.263}$$

since light moves in a straight line in the freely falling elevators. Changing to another parameter, $p \to q$, with $p = p(q)$ and $q = q(p)$, we obtain

$$\frac{d^2q}{dp^2}\,\frac{dx^\mu}{dq} + \left[\frac{d^2x^\mu}{dq^2} + \Gamma^\mu_{\lambda\rho}\,\frac{dx^\lambda}{dq}\,\frac{dx^\rho}{dq}\right]\left(\frac{dq}{dp}\right)^2 = 0 \tag{1.264}$$

Thus we see that we get again the geodesic equation of motion if and only if $d^2q/dp^2 = 0$, i.e. if q is a linear function of p. It, therefore, follows that the form of the geodesic equation remains only if we make linear transformations of the parameter, so we can take $d\tau^2 = \text{constant} \times dp^2$, and use p as another parameter. For a photon one then has $d\tau/dp = 0$, i.e. the constant of proportionality vanishes. The equations of motion in the last section should therefore be replaced by similar equations with τ replaced by p.

In most equations this change only leads to trivial alterations. However, eq. (1.235) now becomes

$$0 = \left(1 - \frac{2MG}{r}\right)\left(\frac{dt}{dp}\right)^2 - \frac{\left(\frac{dr}{dp}\right)^2}{1 - \frac{2MG}{r}} - r^2\left[\left(\frac{d\theta}{dp}\right)^2 + \sin^2\,\theta\left(\frac{d\varphi}{dp}\right)^2\right] \tag{1.265}$$

since $d\tau = 0$. Thus, introducing again integration constants as in eqs. (1.237) and (1.238) (with ($\dot\varphi \to d\varphi/dp$ and $\dot t \to dt/dp$, respectively), we get

$$0 = L^2 - \frac{H^2}{r^4}\,r'^2 - \frac{H^2}{r^2}\left(1 - \frac{2MG}{r}\right) \tag{1.266}$$

Proceeding as in sect. 22 we get the solution

$$\varphi(r) = \varphi(r_1) + \int_{1/r_1}^{1/r} \frac{du}{\sqrt{\frac{L^2}{H^2} - u^2(1 - 2MGu)}} \tag{1.267}$$

which is again a elliptic integral which should be expanded.

Let us consider a light ray passing the sun. The radial variable r originates at the sun and measures the distance to the light ray. At the point of closest approach $r = r_0$ one has $dr/d\varphi = 0$ ($r(\varphi)$ has its minimum), so from (1.266) we get

$$L^2 = \frac{H^2}{r_0^2}\left(1 - \frac{2MG}{r_0}\right) \tag{1.268}$$

Thus

$$\begin{aligned}\varphi(r) &= \varphi(\infty) + \int_0^{1/r} \frac{du}{\sqrt{u_0^2(1-2MGu_o) - u^2(1-2MGu)}} \\ &\approx \varphi(\infty) + \int_0^{1/r} \frac{du}{\sqrt{u_0^2 - u^2}} \left(1 + MG\,\frac{u_0^3 - u^3}{u_0^2 - u^2}\right)\end{aligned} \tag{1.269}$$

Using

$$\frac{u_0^3 - u^3}{u_0^2 - u^2} = u + \frac{u_0^2}{u_0 + u}$$

we get

$$\begin{aligned}\varphi(r) &\approx \varphi(\infty) + \int_0^{1/r} \frac{du}{\sqrt{u_0^2 - u^2}} \left(1 + MGu + \frac{MGu_0}{1 + \frac{u}{u_0}} + \cdots\right) \\ &= \varphi(\infty) + \sin^{-1}\left(\frac{u}{u_0}\right) + MGu_0\left(2 - \sqrt{1 - \frac{u^2}{u_0^2}} - \sqrt{\frac{1 - u/u_0}{1 + u/u_0}}\right) + \cdots\end{aligned} \tag{1.270}$$

Now $r = 1/u$ goes from ∞ to r_0 and then goes from r_o to ∞. The total change in φ is $2|\varphi(r_0) - \varphi(\infty)|$, because of the symmetry of the problem: Half of the bending takes place before the passage of $r = r_0$, and the other half after the passage of this point. If there was no deflection this would be π, so the deflection is $\Delta\varphi = 2|\varphi(r_0) - \varphi(\infty)| - \pi$. Hence, from (1.270) we get

$$\Delta\varphi = \frac{4MG}{r_0} \tag{1.271}$$

For the sun we get, with $r_0 =$ the solar radius $(= 6.95 \times 10^5$ km)

$$\Delta\varphi = 1.75'' \tag{1.272}$$

A recent observational value is $1.70 \pm 0.10''$ in good agreement with (1.272). The quantity $\Delta\varphi$ was first observed in 1919, and the excellent agreement with (1.271) brought Einstein's general relativity into the newspapers. However, $\Delta\varphi$ is in general very difficult to measure with high precision. The precession of perihelia in sect. 22 is far more accurate, both from an experimental and a theoretical point of view.

Instead of using the sun as the lense one can obtain much more spectacular effects by use of galactic lenses. We shall not discuss the details, but the principle is quite simple. Suppose we consider a source which is positioned in such a way that it lies on an straight line passing through the center of a galaxy to an observer. Just like in the case of the sun, light passing to the observer will be bend by the galaxy. Because of the very special geometry, there is cylindrical symmetry. Thus, if we take the source-galaxy-observer line to be the z-axis, the bending will result in an effect which is rotationally invariant in the $(x, y)-$plane. Therefore the observer will see a circle (where the maximal intensity is) around the galaxy. Such a circle is called an Einstein Ring, since it was first predicted by Einstein himself. If the geometry is not so symmetric, the ring degenerates to one or more arcs. These phenomena have been neatly observed.

1.24 Black holes

As already mentioned the Schwarzschild solution looks singular for $r = 2MG$. In our solar system this seems not to be relevant. However, it could be that somewhere in our universe there exists an object which is so small that $r = 2MG$ could be reached. The question is then what happens at the Schwarzschild radius?

We know from Birkhoff's theorem shown in sect. 21, that outside a (possibly time-dependent) spherically symmetric object Schwarzschild's solution is valid. Let us consider radial motion where r and t are functions of proper time τ. Since $\dot{\varphi} = 0$, we have from (1.237), (1.238) and (1.239) that $H = 0$, and

$$\begin{aligned} \dot{t} &= \frac{L}{1-\frac{2MG}{r}} \\ \dot{r}^2 &= L^2 - \left(1 - \frac{2MG}{r}\right) \end{aligned} \tag{1.273}$$

Now, let us consider a particle originating at large distance $r \to \infty$, where space is flat (Minkowski) and special relativity is valid. In the asymptotic system let the clock be chosen such that $dt/d\tau \to 1$ for $r \to \infty$. Then $L = 1$, and hence

$$\begin{aligned} \dot{t} &= \frac{1}{1-\frac{2MG}{r}} \\ \dot{r}^2 &= \frac{2MG}{r} \end{aligned} \tag{1.274}$$

Now we have two times: the proper time τ is the one measured by an observer falling freely in the gravitational field, and the time t is the time measured by an observer at rest at large distances, where space is flat. From the second equation (1.274) we get, since $\sqrt{r}\,\dot{r} = -\sqrt{2MG}$ (r decreases with τ),

$$\frac{2}{3\sqrt{2MG}}(r^{3/2} - r_0^{3/2}) = \tau_0 - \tau \tag{1.275}$$

with $r = r_0$ for $\tau = \tau_0$. The solution (1.275) is valid as long as we are outside the body, so if the radius of the body is less than $2MG$, it follows from (1.275) that the freely falling observer passes the Schwarzschild radius without observing any singularity. If the mass is just a point mass, this observer reaches $r = 0$ in a finite proper time

$$\tau_{r=0} = \tau_0 + \frac{2}{3\sqrt{2MG}}\, r_0^{3/2} \tag{1.276}$$

Incidentally it should be noticed that this conclusion could have been reached without any calculations, since the second equation (1.274) is the same as one would obtain in Newton gravity (why?), except that there proper time is replaced by Newton time. However, mathematics does not care about the time variable, it cares only about the differential equation, so from everyday intuition (=Newton gravity) we know that the point $r = 0$ would be reached in a finite time.

Let us now consider this motion from the point of view of the observer at rest at large distances from the massive object. From (1.274) we have

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = -\sqrt{\frac{2MG}{r}}\left(1 - \frac{2MG}{r}\right) \tag{1.277}$$

This differential equation can be solved exactly without any problem. However, it is more instructive to consider the two cases where r is much larger than the Schwarzschild radius or where $r \approx 2MG$. In the first case we can use $2MG/r \ll 1$, and hence (1.277) becomes approximately the same as the proper time result. Hence, at large distances there is approximately no difference between the proper time and the asymptotic time, which is of course quite reasonable. For $r \approx 2MG$ the situation is quite different. Here eq. (1.277) gives to a good approximation

$$\frac{dr}{dt} = -\frac{1}{r}\sqrt{\frac{2MG}{r}}\,(r-2MG) \approx \frac{-1}{2MG}(r-2MG) \tag{1.278}$$

with the solution

$$r \approx 2MG + \text{constant} \times e^{-\frac{t}{2MG}} \tag{1.279}$$

Thus we see that from the point of view of the asymptotic observer, the Schwarzschild radius is never reached (it takes an infinite time to get there).

This feature can be put into perspective if we think of the asymptotic observer as a human being on this earth, who observes a distant spherically symmetric body by sending a test body towards it and compare this with the situation as seen by an observer inside the test body. Seen from earth it takes an infinite time for the test body to get to the Schwarzschild radius, and during this time the entire evolution of the universe happens. However, the observer in the test body reaches the Schwarzschild radius in a finite time (on his watch), and then he proceeds. He thus sees what happens after $t = \infty$ (on our clock)!

A body which collapses to a sphere with radius equal to the Schwarzschild radius is called a **black hole**. The reason for this is that no particles (including photons) can be emitted from the surface. To give a full discussion of black holes, one needs also to solve the Einstein equations inside the mass distribution, which turns out to be a relatively complicated task, which we shall not attempt. However, the blackness is clear from (1.279), since if we reverse the time direction it follows that a particle emitted from the surface $r = 2MG$ would require an infinite time t to reach us. For light $d\tau = 0$, so radial motion of light satisfies

$$\left(1-\frac{2MG}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2MG}{r}} = 0$$

i.e.

$$\left|\frac{dr}{dt}\right| = 1 - \frac{2MG}{r} \qquad (r \geq 2MG) \tag{1.280}$$

so the velocity of light seen from the asymptotic system approaches zero as $r \to 2MG$ (!). Comparing with eq. (1.278) we see that if light should reach us from the Schwarzschild radius, an infinite time would be needed, so this can never happen. The body is black. Thus, if a body shrinks to its Schwarzschild radius, it disappears from our view. Since g_{00} is zero for $r = 2MG$, it follows from the discussion (1.38), (1.39),(1.40) that there is an infinite red shift. Therefore the shrinking body fades out of sight. The Schwarzschild radius is called the *horizon* of the black hole for obviuos reasons.

Do black holes exist? The stability of a star is determined by the balance between the pull of gravity and the pressure from radiation emitted by nuclear fusion. When the light nuclei have been used up, the fusion ceases, and the gravitational pull may win.

This requires masses larger than the mass of our sun, which will not end up as a black hole. In some cases it can happen that the radius asymptotically approaches $2MG$ (from above), and a black hole can be formed. There is now much observational evidence for the existence of black holes. For example, one can observe a star which moves around a point following Kepler's laws. However, in the point nothing is observed. This point is therefore expected to be a black hole. Also, X-ray radiation from falling charged particles has been observed. Quite recently (2001) it seems as if the existence of the horizon has been directly observed near the black hole candidate Cygnus XR-1. One tracks the ultraviolet emission from hot clumps of gas circling XR-1. In two cases the signature of the emission dims rapidly, before disappearing as it dips below the horizon. The light dims as it is stretched by gravity to ever-longer wavelengts, in accordance with what we saw above. We refer to the literature for a discussion of these interesting questions.

We end by some historical remarks: Black Holes were predicted from Newtonian gravity by Laplace (∇^2) in 1799, in a paper entitled "Proof of the theorem, that the attractive force of a heavenly body could be so large, that light could not flow out of it." The argument is quite simple. Let us consider light as consisting of particles with mass μ (ultimately you can take the limit $\mu \to 0$, since the result is independent of μ). In modern terminology we would call μ the photon mass. At the time when Laplace did his work, there were the competing theories of light, namely that it has a wave nature (Fresnel and others), or it has a particle nature (Newton). We know now that both pictures are right in a duality sense. Anyhow, consider a spherical body with mass M, and let us try to shoot a particle with mass μ away from this body. The condition for escape is that the kinetic energy exceeds the gravitational energy, so

$$\frac{1}{2}\mu v^2 > G\mu M/r. \tag{1.281}$$

Thus, after division by μ, we see that the critical radius is given by

$$r_{\rm crit} = 2GM/v^2. \tag{1.282}$$

Here Laplace mentions that v is the velocity on the surface of the body. In general it can depend on r, but if we assume that $v = c$, which we have taken to be one, the critical radius is just the Schwarzschild radius! Thus Laplace correctly predicted the existence of black holes, which is a most remarkable historical fact.

1.25 Kruskal coordinates

The Schwarzschild radius appears as a singularity in the metric. To investigate whether this is a real singularity, or whether it has been induced by our choice of coordinates, we should compute some invariant quantity. One obvious choice is given by $R_{\alpha\beta\gamma\delta}\ R^{\alpha\beta\gamma\delta}$ =invariant. The diligent reader can verify that for the Schwarzschild solution one gets $R_{\alpha\beta\gamma\delta}\ R^{\alpha\beta\gamma\delta} = 48M^2G^2/r^6$. Thus, this invariant is completely regular for $r = 2MG$, and is only singular for $r = 0$. This singularity is physically reasonable, since the point mass is situated at $r = 0$.

Motivated by the remarks above one may then ask if it is possible to find new coordinates without any singularity at $r = 2MG$, but singular for $r = 0$ (in this connection it should be remembered that in general relativity the coordinates are arbitrary). These new coordinates were found by Kruskal in 1960, and we shall motivate them in the following. Since the angular coordinates are not changed, we shall consider only the quantity

$$
\begin{aligned}
ds^2 &\equiv \left(1-\frac{2MG}{r}\right)\left(dt^2-\frac{dr^2}{\left(1-\frac{2MG}{r}\right)^2}\right) \\
&= \left(1-\frac{2MG}{r}\right)\left(dt-\frac{dr}{1-\frac{2MG}{r}}\right)\left(dt+\frac{dr}{1-\frac{2MG}{r}}\right).
\end{aligned} \tag{1.283}
$$

It is natural to try to "absorb" the singular factor associated with dr^2 by a new variable. To this end we notice that

$$
\frac{dr}{1-\frac{2MG}{r}} = d\left(r+2MG\ln\left(\frac{r}{2MG}-1\right)\right), \tag{1.284}
$$

where we have taken $r > 2MG$. We then have

$$
ds^2 = \left(1-\frac{2MG}{r}\right)\left(dt-dr-2MGd\ln\left(\frac{r}{2MG}-1\right)\right)\left(dt+dr+2MGd\ln\left(\frac{r}{2MG}-1\right)\right). \tag{1.285}
$$

This suggest that we introduce new variables $v_\pm$ by

$$
\begin{aligned}
\frac{t-r}{2MG} &= \ln\left(\frac{r}{2MG}-1\right)-v_-, \\
\frac{t+r}{2MG} &= -\ln\left(\frac{r}{2MG}-1\right)+v_+.
\end{aligned} \tag{1.286}
$$

From eq. (1.285) it follows that the trajectory of light moving radially in a Schwarzschild black hole is given by $v_+ = 0$ or $v_- = 0$. The metric in the new variables becomes

$$
ds^2 = -4M^2G^2\left(1-\frac{2MG}{r}\right)dv_-dv_+. \tag{1.287}
$$

This form of the metric still has a one-way membrane for $r = 2MG$ because of the bracket on the right hand side. To remove this factor by another change of coordinates we notice that from (1.286) it can be written in the two forms,

$$
\frac{r}{2MG}-1 = e^{v_-}\ e^{\frac{t-r}{2MG}} = e^{v_+}\ e^{-\frac{t+r}{2MG}}. \tag{1.288}
$$

Multiplying these two expressions together gives

$$
\left(\frac{r}{2MG}-1\right)^2 = e^{v_++v_-}\ e^{-\frac{r}{MG}}. \tag{1.289}
$$

Thus (1.287) becomes

$$
ds^2 = -\frac{8M^3G^3}{r}\ e^{-\frac{r}{2MG}}\ e^{\frac{1}{2}(v_++v_-)}\ dv_+dv_-. \tag{1.290}
$$

Thus the unwanted factor $1-2MG/r$ can be removed by the substitution

$$
\frac{1}{2}\ e^{\frac{1}{2}v_\pm}dv_\pm = ds_\pm, \tag{1.291}
$$

which amounts to

$$v_\pm = 2 \ln s_\pm \tag{1.292}$$

with (use (1.286))

$$s_+ s_- = \left(\frac{r}{2MG} - 1 \right) e^{r/2MG}, \quad \frac{s_+}{s_-} = e^{t/2MG}. \tag{1.293}$$

The metric then becomes

$$ds^2 = -32M^3G^3 \frac{1}{r} e^{-r/2MG} ds_+ ds_-, \tag{1.294}$$

which is singular only at $r = 0$, in agreement with the fact that the Riemann tensor is singular only in this point. In this metric r should be considered as a function of $s_\pm$. Usually this metric is written in terms of still other variables,

$$s_+ = t' + r', \quad s_- = r' - t'. \tag{1.295}$$

The metric then takes the final form

$$d\tau^2 = 32M^3G^3 \frac{1}{r} e^{-r/2MG} (dt'^2 - dr'^2) - r(r',t')^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{1.296}$$

where

$$\begin{aligned} t'^2 - r'^2 &= -\left(\frac{r}{2MG} - 1 \right) e^{r/2MG} \quad \text{(all } r\text{)}, \\ \frac{t'}{r'} &= \tanh \frac{t}{4MG} \quad (\text{for } r > 2MG), \quad \frac{t'}{r'} = \coth \frac{t}{4MG} \quad (\text{for } r < 2MG), \end{aligned} \tag{1.297}$$

as is seen from eq. (1.293). Here we have left it as an excersise for the reader to do the above calculations for the case $r < 2MG$. It turns out that the final result is the one exhibited in (1.297). Alternatively, we can continue the expression (1.296) analytically from $r > 2MG$ to $r < 2MG$.

In the Kruskal form of the metric (1.296) the variable r^2 in front of the angular terms is an implicit function of t', r', given by (1.297). The same is true for the factor in front of $(dt'^2 - dr'^2)$ in (1.296).

From (1.297) we see that the lines of constant r, t are formed by a set of hyperbolas intersected by the lines $t' = r' \tanh(t/4MG)$ in the t', r' plane. The singular point $r = 0$ is mapped to the hyperbola $t'^2 - r'^2 = 1$, and the horizon is mapped to the degenerate hyperbola $t'^2 - r'^2 = 0$ consisting of two straight lines. It should be noticed that (1.297) has two solutions for given t, r, corresponding to the two branches of the hyperbolas. These two solutions are bounded by the $r = 0$ hyperbola $t'^2 - r'^2 = 1$. The solutions lead to the same metric. One class of solutions can be identified with the standard asymptotically flat universe introduced originally in the discussion of the Schwarzschild solution. The second class of solutions correspond to the same metric, and is a "ghost" universe. These two universes are related through the Kruskal variables, and they are said to be "connected" by a Schwarzschild *"throat"*. This "connection" is not physical, since once a test body is inside the horizon, it will always hit the point mass.

Let us consider a light ray travelling radially in the r' coordinate. From (1.296) we see that it has the equation of motion $|dr'/dt'| = 1$, corresponding to a velocity of light equal one. If we study this in the old coordinates r, t we see from (1.297) that the light ray starts at some finite t with $r > 2MG$, then travels towards $r = 2MG$ corresponding to $t = \infty$, and crosses the line $t = \infty$ and goes into the interior of the horizon. In this continued motion r decreases, and most remarkably, t also decreases. This means that t is not a good time variable inside the horizon.

If the ray is emitted from the inside of the horizon, it will travel through increasing r's, but t is *decreasing*. When the ray crosses the horizon, $t = -\infty$. The motion, measured in the asymptotic time t, is thus time reversed.

1.26 Painlevé's version of the Schwarzschild metric

The Kruskal form of the Schwarzschild metric discussed in the previous section is somewhat complicated. Here we shall present a much simpler transformation of the Schwarzschild coordinates, which has no singularity for $r = 2MG$. It was originally found by Painlevé (1921), who used these coordinates to criticize Einstein gravity for allowing the singularities to come and go! As we know, this is a result of the coordinates beeing arbitrary in general relativity, so one ask for some scalar quantity $\phi(x)$ where a singularity cannot be transformed away due to $\phi'(x') = \phi(x)$ under $x \to x'$. Such a scalar was given in the beginning of the last section, $\phi = R_{\alpha\beta\gamma\delta}\ R^{\alpha\beta\gamma\delta} = 48M^2G^2/r^6$. This shows that there is no singularity in $r = 2MG$.

To go back to Painlevé, we start from

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2MG}{r}} - r^2 d\Omega^2, \tag{1.298}$$

where $d\Omega^2$ represents the angular part of the Schwartzschild metric, which is not changed in the following. A new time $t_\star$ is introduced by the equation

$$dt_\star = dt + \sqrt{\frac{2MG}{r}}\ \frac{dr}{1 - \frac{2MG}{r}}. \tag{1.299}$$

The variable r is unchanged. Integrating eq. (1.299) we obtain

$$t_\star = t + 2\sqrt{2MGr} + 2MG\ \ln \frac{\sqrt{r} - \sqrt{2MG}}{\sqrt{r} + \sqrt{2MG}}, \tag{1.300}$$

where $r > 2MG$. For $r < 2MG$ a similar (real) expression is valid.

Inserting eq. (1.299) in the metric (1.298) we get after some simple algebra

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt_\star^2 - 2\sqrt{\frac{2MG}{r}} dr dt_\star - dr^2 - r^2 d\Omega^2. \tag{1.301}$$

Thus we see that the dr^2-term in the new coordinates is not associated with a singularity, in accordance with the fact that $r = 2MG$ is not a singularity of the theory.

We still have $g_{00} \to 0$ for $r \to 2GM$, so at the horizon light is infinitely red-shifted. Thus, some charged matter falling towards the center and therefore emitting radiation, will be observed to produce ever increasing wave-lenghts, and will ultimately be unobservable.

From eq. (1.299), or

$$\dot{t}_\star = \dot{t} + \sqrt{\frac{2MG}{r}} \frac{\dot{r}}{1 - \frac{2MG}{r}}, \tag{1.302}$$

where a dot means derivative with respect to proper time τ, we obtain by use of eq. (1.274)

$$\dot{t}_\star = \frac{1}{1-\frac{2MG}{r}} - \frac{2MG}{r}\frac{1}{1-\frac{2MG}{r}} = 1. \tag{1.303}$$

This shows that for a freely falling test person the proper time observed by him (equivalence principle!) is equal to the Painlevé time $t_\star$ (plus a possible constant), $t_\star = \tau$+const. Measured on a "star -clock" ($t_\star$) the observer will reach the horizon in a finite time-however, if he communicates with us by means of radiation, this will get out of range for our detectors at large distance when $r \to 2MG$!!

1.27 Tidal forces and the Riemann tensor

So far we have discussed a point observer falling towards the black hole. However, if the observer has an extension, tidal forces will occur. In this section we shall study this effect in details. Let us consider the separation between two nearby geodesics, i.e. the trajectories of two freely falling observers,

$$x^\mu(\tau) \quad \text{and} \quad x^\mu(\tau) + \delta x^\mu(\tau). \tag{1.304}$$

These two coordinates can be thought of as the head and feets, respectively, of a freely falling observer (assuming that this observer is quite elastic, so that head and feets can move freely). Let us consider the covariant derivative of $\delta x^\mu(\tau)$, where this derivative was introduced in eq. (1.67), i.e.

$$\frac{D\delta x^\mu}{D\tau} = \frac{d\delta x^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}(x)\delta x^\beta \,\frac{dx^\alpha}{d\tau}. \tag{1.305}$$

This is a vector, since the coordinate difference δx^μ is a vector. We want to find the tidal "force", so inspired by Newtonean mechanics we take the second derivative of the coordinate difference, i.e.

$$\frac{D}{D\tau}\frac{D\delta x^\mu}{D\tau} = \frac{D}{D\tau}\left(\frac{d\delta x^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}(x)\delta x^\beta \,\frac{dx^\alpha}{d\tau}\right). \tag{1.306}$$

To simplify life we go to a freely falling elevator, where the Christoffel symbols (but *not* their derivatives) vanish in x^μ, so

$$\frac{D}{D\tau}\frac{D\delta x^\mu}{D\tau} = \frac{d}{d\tau}\left(\frac{d\delta x^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}(x)\delta x^\beta \,\frac{dx^\alpha}{d\tau}\right) + 0 = \frac{d^2\delta x^\mu}{d\tau^2} + \frac{\partial\Gamma^\mu_{\alpha\beta}}{\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau}\delta x^\beta. \tag{1.307}$$

The two neighbouring geodesics ("head" and "feets") satisfy

$$0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x(\tau))\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{d^2x^\mu}{d\tau^2} \tag{1.308}$$

and

$$\begin{aligned} 0 &= \frac{d^2(x^\mu+\delta x^\mu)}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x(\tau)+\delta x(\tau))\frac{d(x^\alpha+\delta x^\alpha)}{d\tau}\frac{d(x^\beta+\delta x^\beta)}{d\tau} \\ &= \frac{d^2x^\mu}{d\tau^2} + \frac{d^2\delta x^\mu}{d\tau^2} + \delta x^\nu\frac{\partial\Gamma^\mu_{\alpha\beta}(x(\tau))}{\partial x^\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + O((\delta x)^2), \end{aligned} \tag{1.309}$$

in the freely falling elevator in x^μ. Subtracting the two equations we get

$$\frac{d^2\delta x^\mu}{d\tau^2} = -\frac{\partial\Gamma^\mu_{\alpha\beta}(x(\tau))}{\partial x^\nu}\,\delta x^\nu(\tau)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{1.310}$$

Inserting this in eq. (1.307) we obtain

$$\frac{D}{D\tau}\frac{D\delta x^\mu}{D\tau} = \left(\frac{\partial\Gamma^\mu_{\alpha\beta}}{\partial x^\nu} - \frac{\partial\Gamma^\mu_{\alpha\nu}}{\partial x^\beta}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\beta = R^\mu{}_{\alpha\beta\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\beta. \tag{1.311}$$

Here we used the expression for the Riemann tensor in freely falling coordinates. However, eq. (1.311) is valid in any coordinate system, since the left hand side is a vector, and the product of the last three factors on the right is a tensor. Hence the first factor on the right must be the Riemann tensor in any frame of reference.

The result is thus that the tidal force on an extended observer, i.e. the second derivative of the difference between head and feets, is given by the Riemann tensor.

We can actually demonstrate that the acceleration (1.306) can be interpreted as the four-acceleration of a particle moving along the geodesic $x^\mu + \delta x^\mu$ as seen by an observer moving along the geodesic x^μ, so it is the acceleration of the feets seen from the head.

First we have to realize that a freely falling observer will claim that a vector field, V^μ say, which is defined along his geodesic (path of motion), is constant if it is carried into itself by parallel transport, i.e. if (remember that the $D/D\tau-$derivative is just the $d/d\tau-$derivative in the freely falling system located at x^μ)

$$\frac{DV^\mu}{D\tau} = 0. \tag{1.312}$$

Thus, in forming time derivatives she/he will compare the actual change δV^μ with the change for a parallel transported vector, and thus consider the derivative

$$\frac{\delta V^\mu - \delta V^\mu_{\text{parallel transport}}}{d\tau} = \frac{DV^\mu}{D\tau}. \tag{1.313}$$

Now, given the position vector δx^μ of the neighbouring particle (the distance to the feets seen from the head), he will define the velocity vector (the velocity with which the feets move away!)

$$\frac{D\delta x^\mu}{D\tau} \equiv V^\mu(\tau), \tag{1.314}$$

and by use of (1.313) the acceleration of feets seen from head is therefore precisely

$$a^\mu(\tau) = \frac{DV^\mu}{D\tau} = \frac{D}{D\tau}\frac{D\delta x^\mu}{D\tau}. \tag{1.315}$$

The above result (1.311) is completely general. In the next section we shall see what happens to the tidal force for a Schwarzschild black hole.

1.28 The Tidal force from the Schwarzschild solution

Let us consider the metric given by the Schwarzschild solution. An observer is in free fall towards the center following a geodesic $t = t(\tau)$ and $r = r(\tau)$ in the standard coordinates.

At time $\tau = \tau_0$ he/she passes the point $P_0 = (t_0, r_0)$. The coordinates in the *freely* falling system around P_0 are denoted (t', r'). The freely falling coordinates should be Lorentzian according to the principle of equivalence. Using that the metric transforms as a tensor, we have the conditions

$$-1 = g_{t't'}(P_0) = \left(\frac{\partial t}{\partial t'}(P_0)\right)^2 g_{tt}(P_0) + \left(\frac{\partial r}{\partial t'}(P_0)\right)^2 g_{rr}(P_0), \tag{1.316}$$

$$0 = g_{t'r'}(P_0) = \frac{\partial t}{\partial t'}(P_0)\ \frac{\partial t}{\partial r'}(P_0)\ g_{tt}(P_0) + \frac{\partial r}{\partial t'}(P_0)\ \frac{\partial r}{\partial r'}(P_0)\ g_{rr}(P_0), \tag{1.317}$$

and

$$+1 = g_{r'r'}(P_0) = \left(\frac{\partial t}{\partial r'}(P_0)\right)^2 g_{tt}(P_0) + \left(\frac{\partial r}{\partial r'}(P_0)\right)^2 g_{rr}(P_0). \tag{1.318}$$

The metric on the left hand side refers to the freely falling Lorentzian local system, and the metric $g_{\mu\nu}$ is the Schwarzschild metric. For later use, we shall not use the explicit form of this metric until the very end of this section. Thus, most of the following results are valid for any spherically symmetric metric.

To solve these equations we introduce the quantities

$$\Lambda_0{}^0 \equiv \frac{\partial t}{\partial t'}(P_0)\ \sqrt{-g_{tt}(P_0)},\quad \Lambda_1{}^0 \equiv \frac{\partial t}{\partial r'}(P_0)\ \sqrt{-g_{tt}(P_0)}, \tag{1.319}$$

and

$$\Lambda_0{}^1 \equiv \frac{\partial r}{\partial t'}(P_0)\ \sqrt{g_{rr}(P_0)},\quad \Lambda_1{}^1 \equiv \frac{\partial r}{\partial r'}(P_0)\ \sqrt{g_{rr}(P_0)}. \tag{1.320}$$

From eqs. (1.316)-(1.318) these quantities should satisfy

$$-1 = -\left(\Lambda_0{}^0\right)^2 + \left(\Lambda_0{}^1\right)^2,\ 0 = -\Lambda_0{}^0\Lambda_1{}^0 + \Lambda_0{}^1\Lambda_1{}^1,\ +1 = -\left(\Lambda_1{}^0\right)^2 + \left(\Lambda_1{}^1\right)^2. \tag{1.321}$$

These equations are precisely the defining equations of the Lorentz transformation (see almost any book on special relativity, or solve (1.321) directly), so we can solve by taking

$$\Lambda_0{}^0 = \gamma,\ \Lambda_0{}^1 = v\gamma,\ \Lambda_1{}^0 = v\gamma,\ \Lambda_1{}^1 = \gamma, \tag{1.322}$$

where

$$\gamma = 1/\sqrt{1 - v^2}. \tag{1.323}$$

Here v is a parameter restricted by $0 \le v \le 1$.

Next we can insert this solution for the Λ's in eqs. (1.319) and (1.320) to obtain

$$\frac{\partial t}{\partial t'}(P_0) = \frac{\gamma}{\sqrt{-g_{tt}(P_0)}},\quad \frac{\partial t}{\partial r'}(P_0) = \frac{v\gamma}{\sqrt{-g_{tt}(P_0)}}, \tag{1.324}$$

$$\frac{\partial r}{\partial t'}(P_0) = \frac{v\gamma}{\sqrt{g_{rr}(P_0)}},\quad \frac{\partial r}{\partial r'}(P_0) = \frac{\gamma}{\sqrt{g_{rr}(P_0)}}. \tag{1.325}$$

So far v is a free parameter, and we now ask for the physical meaning of it. The falling observer is at rest in his own freely falling system, so his actual trajectory satisfies

$$\frac{dt'}{d\tau} = 1 \quad \text{and} \quad \frac{dx'^\mu}{d\tau} = 0 \quad \text{for} \quad \mu = 1, 2, 3. \tag{1.326}$$

Transforming these conditions to the (r, t) coordinates we find

$$\frac{dt}{d\tau} = \frac{\partial t}{\partial t'}\frac{dt'}{d\tau} = \frac{\gamma}{\sqrt{-g_{tt}(P_0)}}, \tag{1.327}$$

and

$$\frac{dr}{d\tau} = \frac{\partial r}{\partial t'}\frac{dt'}{d\tau} = \frac{v\gamma}{\sqrt{g_{rr}(P_0)}}. \tag{1.328}$$

An observer *at rest* at the point P_0 (i.e. remaining at the value $r = r_0$ at all times) sees the freely falling observer move a distance (use the $dr^2/(1-2MG/r)-$term in the Schwarzschild metric!)

$$dl = \sqrt{g_{rr}(P_0)}\,\frac{dr}{d\tau}d\tau = v\gamma d\tau, \tag{1.329}$$

where we used eq.(1.328). For similar reasons this takes the time

$$dT = dt\sqrt{-g_{tt}(P_0)} = \gamma d\tau. \tag{1.330}$$

So in other words, this entirely fixed observer sees the nearby falling observer (his feets) pass him with a velocity $dl/dT = v\gamma/\gamma = v$. Thus, the parameter v can be interpreted as the velocity of the freely falling observer as seen by an observer who remains fixed at $r = r_0$. So v is the velocity of the feets as seen from the head.

We shall now use the general formula (1.311) to an observer falling radially in the spherically symmetric metric. In the observers freely falling coordinates the acceleration between head and feets is

$$a'^{\mu} = R'^{\mu}{}_{\nu\alpha\beta}\frac{dx'^{\nu}}{d\tau}\frac{dx'^{\beta}}{d\tau}\delta x'^{\alpha}. \tag{1.331}$$

This expression simplifies considerably, since obviously the observer is at rest in his freely falling system, so $dx'^{\nu}/d\tau = 0$ for $\nu = 1, 2, 3$. Also, $dx'^{\nu}/d\tau$ is a unit vector, and $g'_{\mu\nu} = \eta_{\mu\nu}$, so it follows that $dt'/d\tau = 1$. Therefore

$$a'^{\mu} = R'^{\mu}_{t\alpha t}\;\delta x'^{\alpha}. \tag{1.332}$$

Now, let us take the observer to be entirely in the radial direction, with the distance h between head and feet. We then have

$$a'^{\mu} = h\; R'^{\mu}{}_{trt}. \tag{1.333}$$

If we are only interested in the tidal acceleration between head and feet we have $\mu = r$, and consequently we only need to evaluate $R'^{r}{}_{trt}$.

To do this we first use that the primed system is flat (freely falling elevator) in P_0, so

$$R'^{r}{}_{trt}(P_0) = R'_{rtrt}(P_0) = \frac{\partial x^{\mu}}{\partial x'^{r}}\frac{\partial x^{\nu}}{\partial x'^{t}}\frac{\partial x^{\rho}}{\partial x'^{r}}\frac{\partial x^{\sigma}}{\partial x'^{t}}R_{\mu\nu\rho\sigma}(P_0). \tag{1.334}$$

Here we also used the transformation formula for the Riemann tensor. From eqs. (1.324) and (1.325) the transformation coefficients can be evaluated,

$$R'^{r}{}_{trt}(P_0) = \left(\frac{\partial t}{\partial r'}\frac{\partial r}{\partial t'} - \frac{\partial r}{\partial r'}\frac{\partial t}{\partial t'}\right)^2\; R_{trtr} = (\gamma^2 v^2 - \gamma^2)^2\; R_{trtr} = \; R_{trtr}. \tag{1.335}$$

Now

$$R_{rtrt}(P_0) = g_{rr}(P_0)\ R^r{}_{trt}(P_0) \tag{1.336}$$

by use of the (diagonal) Schwarzschild metric. The Riemann tensor can be evaluated by use of the list of non-vanishing Christoffel symbols in Section 1.18,

$$R^r{}_{trt} = -\frac{\partial \Gamma^r_{tt}}{\partial r} + \Gamma^\lambda_{tr}\Gamma^r_{t\lambda} - \Gamma^\lambda_{tt}\Gamma^r_{r\lambda} = -\frac{\partial \Gamma^r_{tt}}{\partial r} + \Gamma^r_{tt}\left(\Gamma^t_{tr} - \Gamma^r_{rr}\right). \tag{1.337}$$

Here we used that $\Gamma^r_{tr} = 0$, since $\partial F/\partial t = 0$. Therefore we find

$$R^r{}_{trt} = -\frac{\partial}{\partial r}\left(\frac{1}{2F}\frac{\partial E}{\partial r}\right) + \frac{1}{2F}\frac{\partial E}{\partial r}\left(\frac{1}{2E}\frac{\partial E}{\partial r} - \frac{1}{2F}\frac{\partial E}{\partial r}\right). \tag{1.338}$$

It should be noticed that this formula is valid in any spherically symmetric metric.

If we use $E = 1/F$, which is valid for the Schwarzschild metric, we obtain

$$R^r{}_{trt} = -\frac{1}{2}\ E\frac{\partial^2 E}{\partial r^2}. \tag{1.339}$$

Using this in (1.335) and (1.336) we get the result

$$R'^r{}_{trt} = -\frac{1}{2}\ \frac{\partial^2 E}{\partial r^2}. \tag{1.340}$$

Here we used that $g_{rr} = 1/E$. Also, from eq. (1.333) we get the final result

$$a'^r = -\frac{h}{2}\ \frac{\partial^2 E}{\partial r^2}. \tag{1.341}$$

Now we can specify to the Schwarzschild metric by taking $E = 1 - 2MG/r$, so eq. (1.341) gives

$$a'^r = \frac{2MGh}{r_0^3}. \tag{1.342}$$

Thus, in some arbitrary point r_0 this is the acceleration that an observer will see his/her feets move away (assuming that the feets fall first towards $r = 0$)! It is of some interest that this is also the result obtained in Newton gravity: The acceleration of the head is $-MG/r^2$, and the feets have acceleration $-MG/(r-h)^2$. Therefore the relative acceleration is $2MGh/r^3$, where we expanded in the (assumed) small quantity h/r.

The result (1.342) is in units where $c = 1$. In standard units we should rewrite (1.342) as

$$a'^r = \frac{2MGhc}{r_0^3} \tag{1.343}$$

It should be noticed that this acceleration does not behave in a special way at the Schwarzschild radius, and the only singularity occurs at $r_0 = 0$, as expected. Note that a'^r does not depend on the relative velocity v of head and feets.

If we take M to be the solar mass[9], then if r_0 is the Schwarzschild radius ≈ 2.95 km, and if $h = 2$ m, then

$$a' = \frac{h}{r^2_{\text{Schwarzschild}}}\ c^2 = \frac{2\ \text{m}}{(3\times 10^3\ \text{m})^2}\ (3\times 10^8\ \text{m/sec})^2 \approx 2\times 10^{10}\ \ \text{m/sec}^2. \tag{1.344}$$

[9]Actually our sun will not end up as a black hole. Instead the mass should be somewhat larger, perhaps a few solar masses, but this does not change the disasterous order of magnitude estimate essentially.

This is a rather disastous acceleration!

However, if we take one of the large black holes observed in the center of several galaxies (like the Milky Way) with masses of order 10^6 solar masses, the acceleration would only be

$$a' \approx h\ 10^{-2}\ \mathrm{m/sec}^2, \tag{1.345}$$

where h is measured in meters. Thus, large black holes are quite friendly!

1.29 The energy-momentum tensor for electromagnetism

In this section we shall construct the energy-momentum tensor for electromagnetism as a preliminary for finding the solution of the Einstein equations for a point mass with charge (the Reissner-Nordström solution). We start by considering special relativity, so the metric tensor is just $\eta_{\mu\nu}$. We assume that the reader is familiar (from the course on electromagnetism) with the energy density

$$T^{00} = \frac{1}{2}\ (\vec{E}^2 + \vec{B}^2) \tag{1.346}$$

and the Poynting vector

$$T^{0i} = (\vec{E} \times \vec{B})_i, \quad i = 1, 2, 3. \tag{1.347}$$

We have used that the Poynting vector is the momentum density, and hence is to be identified with T^{0i}.

We start by reminding the reader that the antisymmetric electromagnetic field tensor $F^{\mu\nu}$ can be written

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{1.348}$$

Let us notice that

$$F^{\alpha\beta}F_{\alpha\beta} = F^{0i}F_{0i} + F^{i0}F_{i0} + F^{ij}F_{ij} = 2(-\vec{E}^2 + \vec{B}^2). \tag{1.349}$$

Then we can write

$$T^{00} = \vec{E}^2 + \frac{1}{4}\ F^{\alpha\beta}F_{\alpha\beta}. \tag{1.350}$$

Now we can write $\vec{E}^2$ in a fancy way

$$\vec{E}^2 = \eta_{\alpha\beta}F^{0\alpha}F^{0\beta}, \tag{1.351}$$

and collecting results we thus have

$$T^{00} = \eta_{\alpha\beta}F^{0\alpha}F^{0\beta} - \frac{1}{4}\eta^{00}F^{\alpha\beta}F_{\alpha\beta}. \tag{1.352}$$

Next consider the Poynting vector. We have

$$T^{01} = E_2B_3 - E_3B_2 = F^{02}F^{12} + F^{03}F^{13} = F^{0\alpha}F_{1\beta}\eta_{\alpha\beta}. \tag{1.353}$$

This obviously generalizes to

$$T^{0i} = F^{0\alpha} F^{i\beta} \eta_{\alpha\beta}. \tag{1.354}$$

We can combine eqs. (1.352) and (1.354) to form the quantity

$$T^{\mu\nu} = F^{\mu\alpha} F^{\nu\beta} \eta_{\alpha\beta} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \tag{1.355}$$

which is a tensor in special relativity. In general relativity one therefore has from the principle of covariance

$$T^{\mu\nu} = F^{\mu\alpha} F^{\nu\beta} g_{\alpha\beta} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \tag{1.356}$$

We see that this energy-momentum tensor is symmetric, as it should be since it is the source on the right hand side of the Einstein equations.

In a domain where there is no current J^μ, this energy-momentum tensor is conserved in special relativity, and in general relativity it is covariantly conserved. If currents and charges are present, only the *total* energy-momentum tensor is conserved: The total $T^{\mu\nu}$ is given by the expression (1.355) above plus the energy-momentum tensor of the charge-current carrying particles. We leave it to the reader to work out the details (Problem 9), since we shal only need this tensor outside a point charge.

1.30 The Reissner-Nordström solution

In the case of the Schwarzschild solution we could take advantage of the fact that the energy-momentum tensor vanishes outside the spherically symmetric mass. This simplifies the mathematics considerably, since we could solve the empty space Einstein equations and take into account the mass by suitable boundary conditions at infinity. In the following we shall solve a somewhat more complicated problem, where we have a point mass with mass M and a charge q. Since the energy-momentum tensor of electromagnetism involves long-range forces, we cannot use the empty space Einstein equations any longer. The relevant electromagnetic energy-momentum tensor is given by eq. (1.356), and can be written in the form

$$T^\mu{}_\nu = F^{\mu\alpha}\ F_{\nu\alpha} - \frac{1}{4} \delta^\mu_\nu\ F_{\alpha\beta} F^{\alpha\beta}, \tag{1.357}$$

As in the case of the Schwarzschild solution, the obvious spherical symmetry of the problem allows us to write the metric in the form

$$d\tau^2 = A(r,t) dt^2 - B(r,t) dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{1.358}$$

Since we have a static point charge, there is no magnetic field, so $F^{ij} = 0,\ i,j = 1,2,3$. The electric field must be radial, because of the spherical symmetry, and hence the only non vanishing contravariant field is $F^{0r} = E_r \equiv E$. Similarly, the only covariant field is $F_{0r} = g_{0\alpha} g_{r\beta} F^{\alpha\beta} = g_{00} g_{rr} F^{0r} = -A(r,t) B(r,t) E$. From (1.357) we then easily obtain the energy-momentum tensor

$$\begin{aligned} T^\mu{}_\nu &= 0 \text{ for } \mu \neq \nu, \\ T^0{}_0 &= T^r{}_r = -\frac{1}{2}\ A(r,t) B(r,t)\ E(r,t)^2, \\ T^\theta{}_\theta &= T^\phi{}_\phi = \frac{1}{2}\ A(r,t) B(r,t)\ E(r,t)^2. \end{aligned} \tag{1.359}$$

In particular, we notice that the trace of the energy-momentum tensor vanishes,

$$T^{\alpha}{}_{\alpha} = 0. \tag{1.360}$$

Away from the point charge it follows from (1.106) that

$$\frac{\partial}{\partial x^{\nu}} \left(\sqrt{g}\ F^{\mu\nu}\right) = 0. \tag{1.361}$$

Since $g = A(r,t)B(r,t)r^4 \sin^2\theta$ from (1.358), this leads to the two equations

$$\frac{\partial}{\partial x^0} \left(\sqrt{A(r,t)B(r,t)} F^{r0}\right) = 0, \tag{1.362}$$

and

$$\frac{\partial}{\partial r} \left(r^2\sqrt{A(r,t)B(r,t)} F^{0r}\right) = 0. \tag{1.363}$$

We take the electric field to be time-independent[10], i.e. $F^{r0} = F^{r0}(r)$, so from (1.362) we obtain

$$A(r,t)B(r,t) = \text{time independent} \equiv f(r)^2. \tag{1.364}$$

From (1.363) we then obtain

$$E_r = \frac{\text{const.}}{r^2\sqrt{A(r,t)B(r,t)}} = \frac{\text{const.}}{r^2 f(r)}. \tag{1.365}$$

From the results (1.216), transcribed to the metric (1.358) by the replacements $E \to A$ and $F \to B$, and from $T^t{}_r = 0$, or $T_{tr} = 0$, we have

$$\frac{-1}{rB(r,t)} \frac{\partial B(r,t)}{\partial t} = 0, \tag{1.366}$$

so $B(r,t)$ is time independent. From (1.364) it then follows that the function $A(r,t)$ is also time independent, i.e.

$$A(r,t) = A(r) \ \text{ and } \ B(r,t) = B(r). \tag{1.367}$$

To proceed, we consider the Einstein equations (remember that the trace of the energy-momentum tensor vanishes, see (1.360))

$$R^t{}_t = -8\pi G T^t{}_t \ \text{ and } \ R^r{}_r = -8\pi G T^r{}_r. \tag{1.368}$$

Using $T^t{}_t = T^r{}_r$ we therefore have $R^t{}_t = R^r{}_r$. From (1.216) this means

$$R^t{}_t - R^r{}_r = \frac{1}{rA(r)B(r)} \frac{\partial A(r)}{\partial r} + \frac{1}{rB(r)^2} \frac{\partial B(r)}{\partial r} = \frac{1}{rB(r)} \left(\frac{\partial \ln A(r)}{\partial r} + \frac{\partial \ln B(r)}{\partial r} \right) = 0. \tag{1.369}$$

[10] This assumption is not necessary. The reader can show that the electric field is time-independent by doing the following calculations with an a priori time dependent electric field, and show that the result is the same as we shall obtain. In analogy with Birkhoff's theorem one finds that it is only consistent with the Einstein equations to have $A(r,t) = A(r)f(t)$, but $f(t)$ can be absorbed by a redefinition of t. The electric field is then constant in time.

This differential equation obviously has the solution $\ln B(r) = -\ln A(r) +$ const. The constant can be determined from the boundary condition that space is flat at infinity, i.e. $A(r)B(r) \to 1$ for $r \to \infty$. Therefore we have

$$B(r) = 1/A(r) \tag{1.370}$$

as in the Schwarzschild case. The electric field in (1.365) therefore simplifies,

$$E_r = \frac{\text{const.}}{r^2} = \frac{q}{4\pi r^2}. \tag{1.371}$$

Here we identified the constant with the electric charge. This is seen to correspond to the standard definition of the charge by means of Gauss' law.

To find the so far unknown function $B(r)$ in the metric we need the Einstein equation $R^\theta{}_\theta = -8\pi G T^\theta{}_\theta = -4\pi G E^2$. In using this equation the actual long range character of the energy-momentum tensor will play a crucial role. We have

$$R^\theta{}_\theta = \frac{1}{r^2}\left(\frac{\partial r A(r)}{\partial r} - 1\right) = -G\frac{q^2}{4\pi r^4}, \tag{1.372}$$

where we used (1.359). Equation (1.372) leads to the simple differential equation

$$\frac{\partial r A(r)}{\partial r} = 1 - G\frac{q^2}{4\pi r^2}, \tag{1.373}$$

with the solution

$$A(r) = \frac{1}{B(r)} = 1 + \frac{C}{r} + G\frac{q^2}{4\pi r^2}. \tag{1.374}$$

Here C is an constant of integration, which we determine by requiring that we get the Schwarzschild solution for the case $q = 0$. Thus $C = -2MG$, and the metric (1.358) becomes

$$d\tau^2 = \left(1 - \frac{2MG}{r} + \frac{Gq^2}{4\pi r^2}\right)dt^2 - \frac{dr^2}{1 - \frac{2MG}{r} + \frac{Gq^2}{4\pi r^2}} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.375}$$

which is the Reissner-Nordström solution.

The metric (1.375) displays the important feature that in Einstein's theory of gravity an electric charge influences the gravitational behavior. A similar feature is not true in Newton's gravity, which is completely decoupled from electromagnetism. It should be observed that the gravitational effect of the charge decreases faster at large r than the purely gravitational effect. However, for small r the situation is just the opposite, namely that the electric contribution $\propto 1/r^2$ dominates over pure gravity $\propto 1/r$, with several interesting consequences, as we shall see in the following.

The solution (1.375) has the feature that if the charge is large enough, there is no horizon. In this case the singularity at $r = 0$ is called a *naked* singularity. This feature can be seen by observing that the quantity

$$A(r) = 1 - \frac{2MG}{r} + \frac{Gq^2}{4\pi r^2} \tag{1.376}$$

in the metric goes to infinity for $r \to 0$, and approaches 1 for $r \to \infty$. In between these values the function $A(r)$ has a minimum. Requiring that $A'(r) = 0$ we find the value of r corresponding to the minimum

$$r_{\min} = \frac{q^2}{4\pi M}. \tag{1.377}$$

In this point A has the value

$$A(r_{\min}) = 1 - \frac{4\pi G M^2}{q^2}. \tag{1.378}$$

The minimum is positive if

$$\frac{q^2}{4\pi M^2} > G. \tag{1.379}$$

Thus, if this condition is satisfied there is no horizon. For a proton one has

$$\frac{q^2}{GM^2} \approx 10^{36} \gg 4\pi, \tag{1.380}$$

so a proton does not have a horizon! However, this argument should not be taken too seriously, since a proton cannot be descibed by classical physics.

If the condition (1.379) is violated, i.e. if the charge is not so large, we have a horizon like in the Schwarzschild case. This distance where the horizon is passed is given by the equation $A(r) = 0$, which has the solutions

$$r_{\pm} = MG\left(1 \pm \sqrt{1 - \frac{q^2}{4\pi M^2 G}}\right). \tag{1.381}$$

A freely falling observer, seen from an observer at large distance, passes out of view at $r = r_+$, which is the horizon. This follows because the metric (1.375) for photons, $d\tau = 0$, in radial motion ($\theta =$const and $\phi =$const.) leads to the velocity of light

$$\frac{dr}{dt} = 1 - \frac{2MG}{r} + \frac{Gq^2}{4\pi r^2} \to 0 \ \text{ for } \ r \to r_+, \tag{1.382}$$

as seen by an observer far away from the charged mass. For a material particle ($d\tau \neq 0$) we get the equations of motion (letting the clocks run identically at infinite distance)

$$\dot{t} = 1/\left(1 - \frac{2MG}{r} + \frac{Gq^2}{4\pi r^2}\right) \tag{1.383}$$

and (directly from the metric (1.375))

$$\dot{r}^2 = \frac{2MG}{r} - \frac{Gq^2}{4\pi r^2} = \frac{r_+ + r_-}{r} - \frac{r_+ r_-}{r^2}. \tag{1.384}$$

The last equation tells us how a freely falling observer, using proper time τ, sees the radial fall. Like in the Schwarzschild case, nothing special happens in passing the horizon r_+. However, the rest of the fall is very different, because at the distance

$$r_c = \frac{q^2}{8\pi M} = \frac{r_+ r_-}{r_+ + r_-}. \tag{1.385}$$

the velocity $\dot{r}$ vanishes. This means that the electric contribution to the metric overwhelms the pure gravity contribution, and the combined effect appears repulsive. Thus, although gravity is attractive outside the horizon, we see that this does not hold inside. Therefore it is not always true that gravity is attractive.

This distance (1.385) is below r_+ and r_-, as is easily seen from the last form of eq. (1.385). This means that an observer can pass through the original asymptotically flat region $r > 2MG$ to the region $r_+ > r > r_-$ and further into the region $r_- > r \geq r_c$ where the motion comes to rest. The singularity at $r = 0$ is therefore never reached!

To study what happens near r_c, let us find the acceleration by differentiating (1.384) with respect to the proper time τ. On both sides we get a factor $\dot{r}$, and disregarding the point where r is exactly equal to r_c, we get by dividing by $\dot{r}$

$$\ddot{r} = -\frac{r_+ + r_-}{2}\left(\frac{1}{r^2} - 2\frac{r_c}{r^3}\right). \tag{1.386}$$

We see that at large distances the acceleration is negative, as expected in a free fall. Coming from large distances we see that $-\ddot{r}$ increases until $r = 3r_c$. Then $-\ddot{r}$ decreases and $\ddot{r} = 0$ for $r = 2r_c$. Then $\ddot{r}$ actually becomes positive, corresponding to acceleration in the direction away from the singularity at $r = 0$. When r approaches r_c, corresponding to $\dot{r} \to 0$, the acceleration has the positive value $\ddot{r} = r_+ r_-/(2r_c^3)$. The exciting possibilty therefore exists that the observer can re-emerge into a region with $r > r_+$. However, since this observer has passed out of view the first time she went through $r = r_+$, and this took an infinite coordinate time t, the region with $r > r_+$ into which she re-emerges is an asymptotically flat region which is *different* from the asymptotically flat region in which this observer started the fall. This observer has passed through a *wormhole* produced by the charge. Alternatively, two observers living in different asymptotically flat universes can fall freely and meet at r_c. They can compare their informations, but they cannot communicate back to their original universes, since they both went out of sight for observers situated in the original universes. Unfortunately there is apparently no way in which an observer can travel from the original asymptotically flat universe to another one, and then back to the original universe to inform about what was seen on the other side of the wormhole.

It should be mentioned that the Reissner-Nordström metric actually has an infinite number of asymptotically flat spaces with the same metric. This can be seen from the following gedankenexperiment: First let the freely falling observer fall from some distance outside the horizon. In a finite proper time she will reach $r = r_c$, and then move out into another asymptotically flat space. As mentioned before, this space must be different from the original one, since she passed out of view (after an infinite coordinate time t, like in the Schwarzschild case) as seen from the observers living in the first asymptotically flat space. In the new space her velocity decreases as she moves outward in r, and by means of a suitable rocket she can turn around and fall again. Repeating this procedure, in sufficient (finite) proper time, she will travel to an arbitrarily large number of different asymptotically flat universes. Thus there exists an infinite number of such universes.

For the Reissner-Nordström solution coordinates analogous to the Kruskal coordinates have been found. These are more complicated due to the existence of the two "horizons" $r_\pm$, and two sets of coordinates are actually needed.

We end the discussion of the charged metric by calculating the tidal force in the Reissner-Nordström solution. The first change we have to make is notational, replacing

E and F by A and B. Using that $A = 1/B$ (like in Schwarzschild) we can use equation (1.341) which now reads

$$a'^r = -\frac{h}{2}\frac{\partial^2 A}{\partial r^2} = \frac{2MGh}{r^3}\left(1 - \frac{3r_c}{r}\right). \tag{1.387}$$

Here we inserted the explicit form for $A(r)$ and introduced the critical distance given in eq. (1.385). We should note that this result has no analogy in Newton gravity, which is uninfluenced by the charge.

The result (1.387) differs from the corresponding Schwarzschild result (1.342) by not being positive definite. For $r = 3r_c$ the acceleration vanishes, and for r below $3r_c$ it becomes negative, so the feet will now accelerate towards the head. Presumably the observer will become somewhat seasick. The fall will continue from $3r_c$ to r_c, where the motion stops as mentioned before. In r_c the negative acceleration acieves the value $-4MGh/r_c^3$.

1.31 The spherically symmetric solution in 2+1 dimensions

Einstein's field equations look the same way in all dimensions. It is therefore of interest to show that the solutions can differ remarkably by going to other dimensions than 3+1. Consequently let us look for a solution analogous to the Schwarzschild solution in two space and one time dimension. It turns out that this result is quite interesting from the point of view of principles, and is also quite different from the Schwarzschild solution, so for this reason we shall give the derivation. We start from the metric

$$d\tau^2 = dt^2 - e^{\eta(r)}(dr^2 + r^2 d\theta^2) = dt^2 - e^{\eta(\sqrt{x^2+y^2})}(dx^2 + dy^2). \tag{1.388}$$

Here we used the notation $r^2 = x^2 + y^2$. For a static metric, the most general rotationally invariant 2-dimensional metric is $A(r)dr^2 + B(r)(dr^2 + r^2 d\theta^2)$, as can be seen as in section 1.16. Introducing the new variable $r' = \exp\left(\int \frac{dr}{r}\sqrt{\frac{A+B}{B}}\right)$, one obtains the spatial part of (1.388).

Using the Euler-Lagrange variational principle, we easily obtain

$$\frac{d}{d\tau}(2\dot{t}) = 0, \quad \frac{d}{d\tau}(2e^\eta \dot{x}) = (\dot{x}^2 + \dot{y}^2)\, e^\eta \frac{\partial \eta}{\partial x}, \quad \frac{d}{d\tau}(2e^\eta \dot{y}) = (\dot{x}^2 + \dot{y}^2)\, e^\eta \frac{\partial \eta}{\partial y}. \tag{1.389}$$

The last equation follows by symmetry from the second by observing that the metric (1.388) is symmetric in x and y. Hence all results derived from this metric must be symmetric under interchange of x and y. The Christoffel symbols can then be easily read off by performing the τ−differentiations. For example, the second equation leads to

$$\ddot{x} + \frac{1}{2}\frac{\partial \eta}{\partial x}\dot{x}^2 - \frac{1}{2}\frac{\partial \eta}{\partial x}\dot{y}^2 + \frac{\partial \eta}{\partial y}\dot{x}\dot{y} = 0. \tag{1.390}$$

In this way we obtain

$$\Gamma^\lambda_{tt} = 0, \quad \Gamma^t_{\mu\nu} = 0, \quad \Gamma^x_{xx} = \frac{1}{2}\frac{\partial \eta}{\partial x}, \quad \Gamma^x_{yy} = -\frac{1}{2}\frac{\partial \eta}{\partial x}, \quad \Gamma^x_{xy} = \frac{1}{2}\frac{\partial \eta}{\partial y}. \tag{1.391}$$

The remaining symbols follow by the symmetry between x and y,

$$\Gamma^y_{yy} = \frac{1}{2}\frac{\partial\eta}{\partial y}, \quad \Gamma^y_{xx} = -\frac{1}{2}\frac{\partial\eta}{\partial y}, \quad \Gamma^y_{xy} = \frac{1}{2}\frac{\partial\eta}{\partial x}. \tag{1.392}$$

Also, for the logarithm of the metric tensor we have

$$\log\ g = 2\eta(\sqrt{x^2+y^2}) = 2\eta(r). \tag{1.393}$$

We are now in the position to compute the Ricci tensor $R_{\mu\kappa}$, for which we have the general formula

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \log\ g}{\partial x^\mu \partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\kappa\sigma} - \frac{1}{2}\Gamma^\sigma_{\mu\kappa}\frac{\partial \log\ g}{\partial x^\sigma}. \tag{1.394}$$

We then obtain

$$R_{tt} = 0, \quad R_{ti} = 0, \quad i = x, y. \tag{1.395}$$

Also

$$\begin{aligned} R_{xx} &= \frac{\partial^2\eta}{\partial x^2} - \frac{\partial\Gamma^\lambda_{xx}}{\partial x^\lambda} + \Gamma^\sigma_{x\lambda}\Gamma^\lambda_{x\sigma} - \Gamma^\sigma_{xx}\frac{\partial\eta}{\partial x^\sigma} \\ &= \frac{\partial^2\eta}{\partial x^2} - \frac{\partial\Gamma^x_{xx}}{\partial x} - \frac{\partial\Gamma^y_{xx}}{\partial y} + \Gamma^\sigma_{xx}\Gamma^x_{x\sigma} + \Gamma^\sigma_{xy}\Gamma^y_{x\sigma} - \Gamma^x_{xx}\frac{\partial\eta}{\partial x} - \Gamma^y_{xx}\frac{\partial\eta}{\partial y} \\ &= \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\eta\left(\sqrt{x^2+y^2}\right) = R_{yy}, \end{aligned} \tag{1.396}$$

where the last equation follows by symmetry (i.e. without doing any calculations). After a few calculations we also find

$$R_{xy} = \frac{\partial^2\eta}{\partial x \partial y} - \frac{\partial\Gamma^x_{xy}}{\partial x} - \frac{\partial\Gamma^y_{xy}}{\partial y} + \Gamma^\lambda_{x\sigma}\Gamma^\sigma_{y\lambda} - \Gamma^x_{xy}\frac{\partial\eta}{\partial x} - \Gamma^y_{xy}\frac{\partial\eta}{\partial y} = 0. \tag{1.397}$$

From these equations it now follows that

$$R = -R_{tt} + g^{xx}R_{xx} + g^{yy}R_{yy} = e^{-\eta(\sqrt{x^2+y^2})}\ \nabla^2\eta(\sqrt{x^2+y^2}), \tag{1.398}$$

where the operator multiplying $\exp(-\eta)$ in the last equation is the (flat) Laplacian acting on an $r-$dependent function.

Let us consider a point mass M at rest in the point $r = 0$. The corresponding energy density is given by

$$\rho = M\delta^2(x)/\sqrt{g}. \tag{1.399}$$

Here $\delta^2(x) = \delta(x)\ \delta(y)$ is the two dimensional delta function [11]. Thus, (1.399) is valid only in the $x - y$-coordinates, where $g = e^{2\eta(r)}$. The reason for the identification in eq. (1.399) is that

$$\int d^2x\sqrt{g}\rho = M. \tag{1.400}$$

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G\ T_{\mu\nu}, \tag{1.401}$$

[11]The reason for the $1/\sqrt{g}$ in (1.399) is that the density ρ should be a scalar. The delta function satisfies $\int d^2xF(x)\delta^2(x) = F(0)$ for some arbitrary function $F(x)$. However, d^2x is not invariant, but $d^2x\sqrt{g}$ is. Therefore $\delta^2(x)/\sqrt{g}$ is invariant. A similar result is valid in higher dimensions.

with $T_{00} = \rho$ and all other components $T_{\mu\nu} = 0$, therefore gives by contraction,

$$R = -16\pi G\, T_{00}, \tag{1.402}$$

therefore reduces to (use that in the $x - y$-coordinates $g = e^{2\eta}$)

$$\nabla^2\, \eta(r) = -16\pi GM\, \delta^2(x). \tag{1.403}$$

It is easy to solve eq. (1.403) if we use that

$$\nabla^2\, \log r = 2\pi\, \delta^2(x), \tag{1.404}$$

which shows that $\log r$ is the Greens function for the two dimensional Laplace operator. Eq. (1.404) can be shown by noticing that for $r \neq 0$, $\log r$ trivially satisfies eq. (1.404). In the neighbourhood of $r = 0$ we have

$$\frac{1}{2\pi}\int d^2x\, \nabla^2 \log r = \frac{1}{2\pi}\oint d\mathbf{n}\cdot\nabla \log r = \frac{1}{2\pi}\oint d\mathbf{n}\frac{\mathbf{r}}{r^2} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1, \tag{1.405}$$

as claimed in eq. (1.404). The Einstein equation (1.403) now has the solution

$$\eta(r) = -8GM \log r. \tag{1.406}$$

The metric (1.388) thus becomes

$$d\tau^2 = dt^2 - r^{-8GM}(dr^2 + r^2 d\theta^2). \tag{1.407}$$

Here r occuring in the power is measured in some arbitrary units.

The surprising feature concerning the metric (1.407) is that it can be transformed to a simpler form, looking like *flat* space! We can introduce a new radial variable r' by

$$(dr')^2 = r^{-8GM} dr^2, \text{ or } r' = r^{1-4GM}/(1-4GM). \tag{1.408}$$

By this substitution we get

$$d\tau^2 = dt^2 - (dr')^2 - (1-4GM)^2 (r')^2 d\theta^2 = dt^2 - (dr')^2 - (r')^2 (d\theta')^2, \tag{1.409}$$

where we introduced the *rescaled* angle θ' by

$$\theta' = (1-4GM)\,\theta. \tag{1.410}$$

It is important to notice that although this metric looks flat, the angle θ' is restricted by

$$0 \leq \theta' \leq 2\pi(1-4GM), \tag{1.411}$$

since the original standard angle θ is, of course, restricted to be between 0 and 2π.

Due to the form (1.409) of the metric it is clear that in any point (t, r', θ'), there are no pointwise effects of gravity. Thus there are no local gravitational fields.

However, due to the fact that the variation of the angle θ' is restricted to be less than 2π, there are global gravitational effects. This is illustrated in figure 1.2 with $\epsilon = 1-4GM$.

The two points P and P' are in the same distance from the origin. In between these points, for $2\pi\epsilon \leq \theta' \leq 2\pi$, there is a *forbidden region*. A space with a forbidden region of

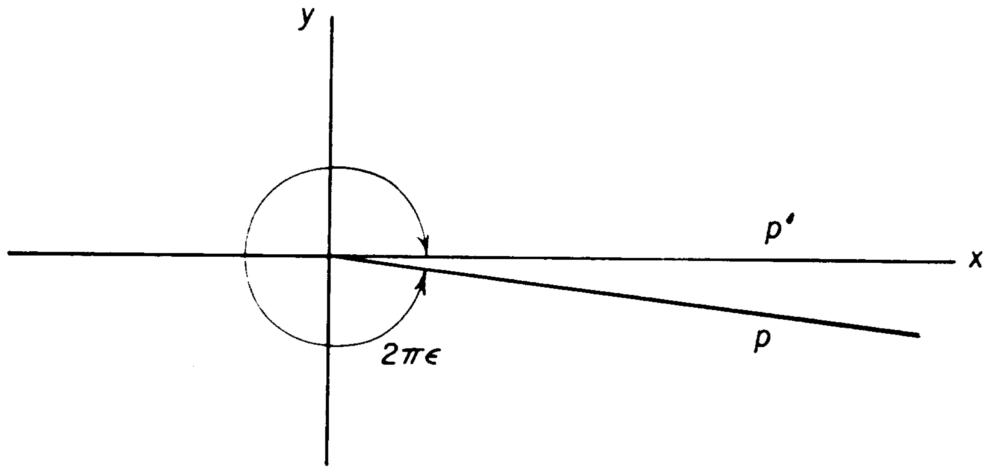

Figure 1.2:

this type is called *conical*. The two points are therefore the same point. So an observer "can be in P and P'" (in the following this point is called P/P'). The same applies to a source of light. In the conical coordinates, light follows a straight line. Thus, if light originates in P/P', it can move along two straight lines, which crosses one another in a single point. There then appears a double picture of the source in P/P'. In the r', θ' coordinates, a light ray coming from far away and going far away in the opposite direction corresponds to a change in polar angle $\Delta\theta' = \pi$, but the physical angle in the original standard coordinates is just

$$\Delta\theta = \pi/(1 - 4GM) \leq \pi. \tag{1.412}$$

We reemphasize that this solution is valid in two space and one time dimensions. Thus it is, of course, not directly possible to do an experiment testing the strange behaviour found above. However, in the next section we shall see that conical spaces can exist even in our three+one dimensional space.

1.32 Cosmic strings

Let us now consider 3+1 dimensions, where we take the metric to be

$$d\tau^2 = dt^2 - dz^2 - e^{\eta(r)}(dx^2 + dy^2), \text{ with } r^2 = x^2 + y^2. \tag{1.413}$$

The $x - y-$part of the metric is thus the same as the metric studied in the previous section. The metric tensor $g_{\mu\nu}$ having one or both indices equal to z is thus trivial. The calculation of the Ricci tensor $R_{\mu\nu}$ therefore proceeds like in the previous section. Using (1.395) and (1.413), we get for the tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\ R \tag{1.414}$$

that

$$G_{tt} = R_{tt} - \frac{1}{2}g_{tt}\ R = \frac{1}{2}R. \tag{1.415}$$

Also, it is easily shown from the results (1.396), (1.397) and (1.398) that

$$G_{xx} = 0,\ G_{yy} = 0,\ G_{zz} = -\frac{1}{2}R, \tag{1.416}$$

while all other components of $G_{\mu\nu}$ vanish. From (1.398) we see that

$$R = e^{-\eta(r)}\nabla^2\eta(r),\ \text{with}\ \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2. \tag{1.417}$$

The conclusion is thus that the tensor $G_{\mu\nu}$ vanishes, except for

$$G_{zz} = -G_{tt} = -\frac{1}{2}R. \tag{1.418}$$

The metric (1.413) is thus consistent with an energy-momentum tensor of the form

$$T_{tt} = -T_{zz},\quad T_{\mu\nu} = 0\quad \text{for all other } \mu \text{ and } \nu. \tag{1.419}$$

Furthermore, we take the non-vanishing components of $T_{\mu\nu}$ to be proportional to a two-dimensional delta function,

$$T_{tt} = -T_{zz} = \sigma\ \delta^2(x)\frac{1}{\sqrt{g}}. \tag{1.420}$$

The constant σ has dimension energy/length, and is called the *string tension.* The object described by the energy momentum tensor (1.419) and (1.420) is called a *cosmic string.* It is a line along the z-axis, with support only in the point $x = y = 0$. The pressure is negative, since T_{tt} by definition is positive. We shall later see that it is possible to construct such an object in certain field theories. In this case the cosmic string is a "defect", i. e. an object which contains false vacuum, giving rize to the negative pressure.

Proceeding as in the previous section, we can now solve the Einstein equations,

$$\eta(r) = -8G\sigma\ \log r. \tag{1.421}$$

The metric (1.413) can now be written in the two coordinate systems

$$d\tau^2 = dt^2 - dz^2 - r^{-8G\sigma}(dr^2 + r^2 d\theta^2) = dt^2 - dz^2 - (dr')^2 - (r')^2(d\theta')^2, \tag{1.422}$$

where r' and θ' are defined by

$$r' = r^{1-4G\sigma}/(1-4G\sigma),\ \theta' = (1-4G\sigma)\theta. \tag{1.423}$$

We thus see that again there appears a conical space, where the angle θ' has a range which is less than 2π.

The conical space leads to the occurence of double pictures of a given light source placed "behind" the cosmic string, seen from an observer in "front" of the string. This is shown in figures 1.3 and 1.4.

In fig. 1.3 we see the situation from the point of view of the (r,θ)-coordinates, so the light rays emitted from the source Q (e.g. a quasar) bend towards the observer O. In fig. 1.4 we see the situation from the point of view of the conical coordinates (r',θ'), where light moves in straight lines. From fig. 1.4 we get

$$\tan(\Delta\phi/2) = l\ \tan(\pi a)/(d+l),\ \text{where}\ a = 4G\sigma. \tag{1.424}$$

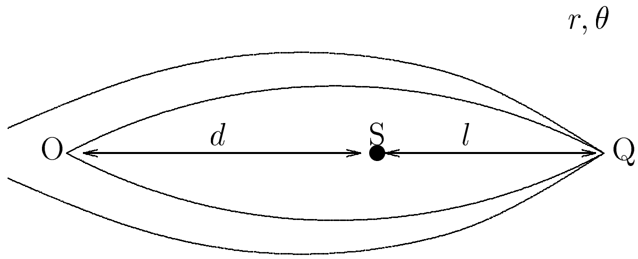

Figure 1.3:

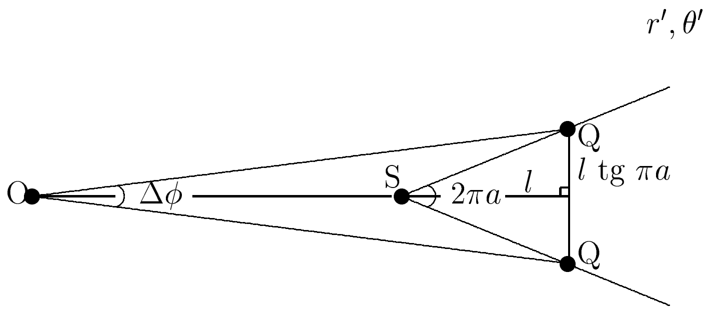

Figure 1.4:

Here $\Delta\phi$ is the angular separation between the two pictures, and d is the distance from the observer to the string, and l is the distance from the string to the source Q, which appears to be in two points in the conical coordinates, but these points are to be identified. In practice the quantity $4G\sigma$ is small relative to one, and hence we can expand eq. (1.424),

$$\Delta\phi \simeq 8\pi G\sigma l/(d+l). \qquad (1.425)$$

A typical cosmic string is generated in the "Grand Unification" (characterized by having the strengths of weak, electromagnetic, and strong interactions equal), which has an energy scale of order 10^{15} GeV, leading to $\sigma \sim 10^{30}$ GeV^2. The string tension then has the value

$$\sigma \approx 10^{22}\text{g/cm} \approx 10^{18}\text{tons/m} \ (\approx 10^{18}\text{elephants/m}) \qquad (1.426)$$

in ordinary units. From (1.425) this leads to an angular separation $\Delta\phi$ of order 2×10^{-5}, i.e. aproximately 4 arc seconds. This is clearly within the possibilty for observation. A cosmic string is, of course, an extended object, and hence should be observed by a concentration of many double pictures along the string, which itself cannot be observed directly. Also, the cosmic string does not have to be a straight line, but can have the form of a curve moving around in space-time. So far, there is no unambiguous observational evidence for cosmic strings, but there exist some observations of unusual concentration of double pictures along rather crumbled curves. These pictures are candidates for cosmic strings.

Chapter 2

Cosmology

2.1 The cosmological problem

Speculations about the nature of the Universe form an integral part of the history of mankind. Various mythologies and religions have several scenarios to offer. In modern physics the canonical scenario is based on astronomical observations described in terms of Einstein's general relativity. In physics it is always important to fix a relevant scale. If we wish to describe the solar system, there is no need to bother about the properties of the Universe at intergalactic scales. If, on the other hand, we wish to understand the large scale structure of the Universe, we do not need to be concerned with small details like our solar system. Then we can consider even the galaxies [1] to be "point particles". Now it turns out that galaxies tend to "cluster", so we have also clusters of galaxies [2], which on a very large scale may be considered to be particles. There are also clusters of clusters of galaxies [3] ("superclusters"), which when viewed at a large scale are also particles. This amount to "building up" the Universe from "particles" with diameter $10^8 - 10^9$ light years.

This situation is somewhat similar to considering a gas consisting of a large number of particles, where we do not keep track of the individual particles, but rather consider the large scale statistical properties. It therefore also follows that in the large scale view of the universe we cannot assign any "center". Every point is equivalent to any other point in the universe. This means a radical deviation from the medieval idea that the earth is the center of the universe. One can say that since it turned out that we do not live in the center of the universe, we then assume that nobody else has this honor.

The idea that all points in our (large scale) universe are equivalent is called the **cosmological principle**. It means that the universe is assumed to be homogeneous and isotropic (because for us, and hence for everybody else, the universe appears to be approximately isotropic) around any point.

One of the first proponents of this picture of the universe was Giordano Bruno (1548-1600), who went beyond the Copernican heliocentric theory, which still maintained a universe with a sphere of fixed stars. Instead, Bruno made the suggestion that the universe is infinite and the stars are solar systems like our own[4].

[1]With masses $\sim 10^{11} - 10^{12} M_\odot$.

[2]With masses $\sim 10^{12} - 10^{13} M_\odot$.

[3]With masses $\sim 10^{14} - 10^{15} M_\odot$.

[4]When Bruno refused to retract his views he was sentenced to death. He addressed his judges, saying: "Perhaps your fear in passing judgement on me is greater than mine in receiving it".

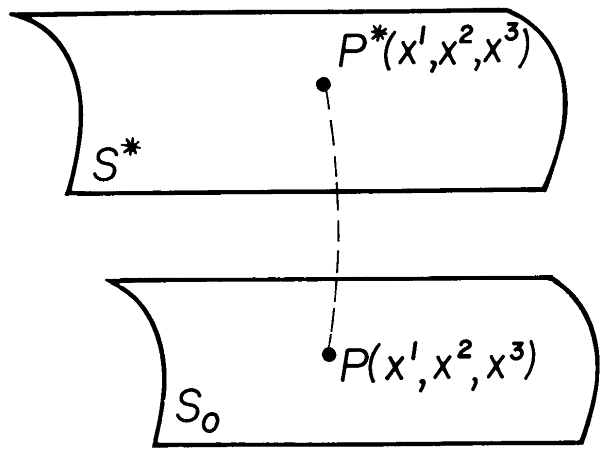

Figure 2.1: Gaussian coordinates for a co-moving system. S_0 and S^* are three dimensional spaces (hypersurfaces)

The extent to which the cosmological principle is right is not precisely known. In principle one could measure the energy-momentum tensor $T_{\mu\nu}(x)$ in all points of the universe and then solve the Einstein equations. However, astronomy does certainly not provide data enough for obtaining $T_{\mu\nu}(x)$. Therefore, the cosmological principle should be considered as a first aid for interpreting the observations, rather than as providing an extremely precise description of the universe.

2.2 The cosmological standard model

The cosmological "standard model" (not to be confused with the "standard model" in particle physics) is based on the cosmological principle, i.e. a homogeneous and isotropic universe, as well as the assumed existence of a "cosmic standard time". This time is intimately connected to the evolution of the universe. For example, the temperature is believed to decrease as the universe evolves, and the cosmic standard time could then be taken as a function of the temperature which increases monotonically as the temperature decreases. Instead of temperature one could use other scalars, the main point being that the cosmic time t increases, when the universe evolves. The existence of t is taken to mean that at a given fixed value of t, the matter of the universe is at rest in three dimensional space (just like in the formulation of the cosmological principle, this means in reality that on the **average** the matter of the universe is at rest in the three dimensional space). As time t increases, the matter of the universe "follows" this increasing t. This coordinate system is therefore called the **co-moving system**.

We can describe this co-moving system by means of so-called Gaussian coordinates, where we consider a geodesic along which time increases (see fig. 2.1).

The coordinates x^1, x^2, x^3 remain constant along any geodesic perpendicular to the initial surface S_o. Along such a geodesic we have $d\tau^2 = dt^2$, and hence $g_{00} = -1$. In order that the t-lines are orthogonal to the 3-dimensional hypersurface it is necessary that any four vector $(0, a, b, c)$ is orthogonal to the vector $(1, 0, 0, 0)$ which is tangent to the t-line, so in general we need $g_{01} = g_{02} = g_{03} = 0$ on the hypersurface. The line element thus has the form

$$\begin{aligned} d\tau^2 &= dt^2 - g_{ij}\, dx^i dx^j \qquad (i,j = 1,2,3) \\ x^i &\in S_0 \end{aligned} \tag{2.1}$$

Consider the equation for a geodesic,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\rho}\,\frac{dx^\lambda}{d\tau}\,\frac{dx^\rho}{d\tau} = 0 \tag{2.2}$$

When $d\tau^2 = dt^2$ and the coordinates x^1, x^2, x^3 are constants along the geodesic, we deduce that $\Gamma^i_{00} = 0$ for $i = 1,2,3$. This implies from (1.27) that

$$2\,\frac{\partial g_{0i}}{\partial t} = \frac{\partial g_{00}}{\partial x^i} \tag{2.3}$$

and since $g_{00} = -1$ on any geodesic, we obtain

$$\frac{\partial g_{0i}}{\partial t} = 0$$

along the geodesic t-lines. Thus, the elements g_{01}, g_{02}, g_{03} of the metric tensor, which are zero on the initial surface S_0, remain zero on all other surfaces. Thus the metric for a co-moving system has the form

$$d\tau^2 = dt^2 - g_{ij}(t, x^1, x^2, x^3) dx^i\, dx^j \tag{2.4}$$

over all space. Eq. (2.4) is thus a consequence of the assumed existence of a cosmic standard time.

The picture of the development of the universe presented above is thus that at a given cosmic time t the galaxies, clusters of galaxies etc. are (on the average) at rest. Thus, we can construct the three-dimensional space coordinates by putting numbers (x, y, z) on each galaxy. As time increases the coordinates remain the same but all distances can be scaled with a time-dependent scale factor.

Now let us consider an arbitrary but fixed point in space, around which we take spherical coordinates r, θ, φ. The requirement of isotropy in this point means that the line element has the form

$$d\tau^2 = dt^2 - f_1(t,r)dr^2 - f_2(t,r)r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{2.5}$$

since the only rotational invariant at this point are t, r, dr^2 , and $dr^2 + r^2\,(d\theta^2 + \sin^2\,\theta d\varphi^2)$. The functions f_1 and f_2 only depend on t and r, since a θ and/or φ-dependence of f_1 and f_2 would destroy rotational invariance. Since we are using co-moving coordinates, eq. (2.1) implies that there cannot be a $drdt$-term in eq. (2.5), even though it is rotational invariant. Thus eq. (2.5) is the most general form of the line element.

Now we must ensure that as time goes by, the only thing that happens is that distances are scaled. Thus, the galaxies are dragging along the coordinate mesh, so each galaxy keeps the same coordinates. The galaxies do move, but the distance between the galaxies move too, so the coordinates remain the same. This means that in (2.5) r, θ, and φ remain the same, and the only thing which can happen is that the overall scale changes. If, at a given but arbitrary time, we consider two points $r_1, \theta_1, \varphi_1,$ and r_2, θ_2, φ_2 with $d\theta =$

$d\varphi = 0$ at both points, then the ratio of the line elements $f_1(t, r_1)dr^2$ and $f_1(t, r_2)dr^2$ must remain the same at all times, since only the overall scale can change, i.e. $f_1(t, r_1)/f_1(t, r_2)$ is independent of time and equal some function $F(r_1, r_2)$, i.e.

$$f_1(t, r_1) = f_1(t, r_2)\ F(r_1, r_2)$$

Taking r_2 to be fixed, this implies

$$f_1(t, r) = f(t)\ L(r) \tag{2.6}$$

i.e. f_1 factorizes. The same can be shown for the function f_2 (take e.g. $dr = d\varphi = 0$ and $d\theta \neq 0$ at the two points). Thus we are led to the line element

$$d\tau^2 = dt^2 - f(t)\ L(r)\ dr^2 - g(t)\ H(r) r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{2.7}$$

This line element clearly has the property that the coordinates remain the same, and the only thing which happens is that lengths are scaled. Introducing $r' = \sqrt{H(r)}r$ we get the slightly simpler form for the line element

$$d\tau^2 = dt^2 - f(t)\ L(r)\ dr^2 - g(t)\ r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{2.8}$$

where we have dropped the prime on r.

Eq. (2.8) now represents a line element which is consistent with the standard cosmological model. It should be remembered, however, that in eq. (2.8) we selected an arbitrary but fixed point as the origin of the spherical coordinates. Later on we have to ensure that this choice of an arbitrary origin agrees with the assumption of a homogeneous space.

We can now compute the Christoffel symbols using the variational principle discussed in section 1.17,

$$\delta \int d\tau \left[\left(\frac{dt}{d\tau}\right)^2 - f(t)L(r)\left(\frac{dr}{d\tau}\right)^2 - g(t)r^2 \left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{d\tau}\right)^2 \right) \right] = 0 \tag{2.9}$$

After straightforward calculations we get:

$$\begin{aligned}
&\Gamma^t_{rr} = \tfrac{\dot{f}L}{2}, && \Gamma^t_{\theta\theta} = \tfrac{1}{2}\, r^2 \dot{g}, && \Gamma^t_{\varphi\varphi} = \tfrac{1}{2}\, r^2 \sin^2\theta \dot{g} \\
&\Gamma^r_{rr} = \tfrac{L'}{2L}, && \Gamma^r_{\theta\theta} = -\tfrac{rg}{fL}, && \Gamma^r_{\varphi\varphi} = -\tfrac{rg}{fL}\ \sin^2\theta \\
&\Gamma^r_{rt} = \Gamma^r_{tr} = \tfrac{\dot{f}}{2f} \\
&\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \tfrac{1}{r}, && \Gamma^\theta_{\theta t} = \Gamma^\theta_{t\theta} = \tfrac{\dot{g}}{2g}, \\
&\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \\
&\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \tfrac{1}{r}, && \Gamma^\varphi_{t\varphi} = \Gamma^\varphi_{\varphi t} = \tfrac{\dot{g}}{2g} \\
&\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta
\end{aligned} \tag{2.10}$$

The Ricci tensor can now be obtained from

$$R_{\mu\nu} = \frac{1}{2}\,\frac{\partial^2 \ln g}{\partial x^\mu\,\partial x^\nu} - \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\rho} + \Gamma^\alpha_{\rho\nu}\Gamma^\rho_{\mu\alpha} - \frac{1}{2}\,\Gamma^\rho_{\mu\nu}\,\frac{\partial \ln g}{\partial x^\rho} \tag{2.11}$$

For reasons which shall soon be clear, we start by computing R_{tr},

$$R_{tr} = \frac{\dot{g}}{rg} - \frac{\dot{f}}{rf} \tag{2.12}$$

where the *dot* indicates the time derivative. The Einstein equations read

$$R_{\mu\nu} = -8\pi\, G(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}T^\alpha{}_\alpha) + \Lambda\, g_{\mu\nu} \tag{2.13}$$

The energy-momentum tensor has the form of a perfect fluid (see sect. G14), since this means an isotropic and homogeneous universe if the pressure p and the density ρ are only functions of t, i.e.

$$T_{\mu\nu} = p(t)\; g_{\mu\nu} + (p(t) + \rho(t))U_\mu U_\nu \tag{2.14}$$

where U_μ is given in co-moving coordinates by

$$U^t = 1 \quad , \quad U^i = 0 \tag{2.15}$$

The last equation follows because matter is at rest. Thus R_{tr} must vanish, since T_{tr} and g_{tr} both vanish. Hence from (2.12)

$$\frac{\dot{g}}{g} = \frac{\dot{f}}{f} \tag{2.16}$$

so we get $g = f$ (up to a constant which can always be absorbed by trivial redefinitions). The line element is thus from (2.8)

$$d\tau^2 = dt^2 - f(t)\left[L(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{2.17}$$

Inserting $f = g$ the Christoffel symbols simplify somewhat (for example, $\Gamma^r_{\theta\theta}$ becomes time independent, etc.), and the Ricci tensor is more easily evaluated,

$$\begin{aligned}
R_{rr} &= -\frac{L'}{rL} - \frac{L\ddot{f}}{2} - \frac{L\dot{f}^2}{4f} \\
R_{\theta\theta} &= -1 + \frac{1}{L} - \frac{rL'}{2L^2} - \frac{r^2}{2}\,\ddot{f} - \frac{r^2}{4}\,\frac{\dot{f}^2}{f} \\
R_{\varphi\varphi} &= \sin^2\theta\; R_{\theta\theta} \\
R_{tt} &= \frac{3\ddot{f}}{2f} - \frac{3\dot{f}^2}{4f^2} \\
R_{\mu\nu} &= 0 \quad \text{for} \quad \mu \neq \nu
\end{aligned} \tag{2.18}$$

where *prime* denotes the derivative with respect to r. Since the metric is diagonal, we can easily construct the mixed Ricci tensor,

$$R^r{}_r = \frac{1}{fL}\, R_{rr} = -\frac{L'}{rfL^2} - \frac{\ddot{f}}{2f} - \frac{\dot{f}^2}{4f^2}$$

$$\begin{aligned}
R^\theta{}_\theta &= \frac{1}{r^2 f}\ R_{\theta\theta} = \frac{-1}{r^2 f} + \frac{1}{r^2 f L} - \frac{L'}{2 r f L^2} - \frac{1}{2}\frac{\ddot{f}}{f} - \frac{1}{4}\frac{\dot{f}^2}{f^2} \\
R^\varphi{}_\varphi &= \frac{1}{r^2 f \sin^2\theta}\ R_{\varphi\varphi} = R^\theta{}_\theta \\
R^t{}_t &= -\frac{3\ddot{f}}{2f} + \frac{3\dot{f}^2}{4f^2} \\
R^\mu{}_\nu &= 0 \quad \text{for} \quad \mu \neq \nu
\end{aligned} \tag{2.19}$$

The curvature scalar is then trivially obtained,

$$R = -\ \frac{3\ddot{f}}{f} - \frac{2}{r^2 f}\left(1 - \frac{1}{L}\right) - \frac{2L'}{r f L^2} \tag{2.20}$$

Now isotropy clearly requires that $R^r{}_r = R^\theta{}_\theta = R^\varphi{}_\varphi$, since if this is not the case there would be one (or more) preferred direction. This also follows from the energy-momentum tensor (2.14), of course. Consulting eqs. (2.19) we see that $R^r{}_r = R^\theta{}_\theta$ is a non-trivial requirement.

We also need to ensure that R^μ_ν is homogeneous in space, so it must not depend on any other variable but time. Again consulting (2.19) we see that in general this is not true, but has to be imposed by hand. This is due to the fact that originally we imposed isotropy relative to a definite point. We can repair this by requiring (see $R^r{}_r$)

$$\frac{L'}{r L^2} = 2k \tag{2.21}$$

where k is a constant, and (see $R^\theta{}_\theta$)

$$\frac{1}{r^2} - \frac{1}{r^2 L} + \frac{L'}{2 r L^2} = 2k \tag{2.22}$$

The constants on the right-hand side have to be the same because of $R^r{}_r = R^\theta{}_\theta$. Eq. (2.21) is trivially solved,

$$L = \frac{1}{c - k r^2} \tag{2.23}$$

where c is a constant. Inserting this in eq. (2.22) we get

$$\frac{1-c}{r^2} + 2k = 2k$$

i.e. $c = 1$. Thus we have found the explicit form of the spatial dependence of the line element, i.e.

$$d\tau^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\varphi^2)\right] \tag{2.24}$$

where we have replaced $f(t)$ by $a(t)^2$ for reasons which will be clear later.

From eq. (2.19) we see that the space part of the Ricci tensor,

$$\begin{aligned}
{}^3R_{ij} &= \frac{\partial\ {}^3\Gamma^k{}_{ki}}{\partial x^j} - \frac{\partial\ {}^3\Gamma^k{}_{ij}}{\partial x^k} + {}^3\Gamma^k{}_{li}\ {}^3\Gamma^l{}_{kj} - {}^3\Gamma^k{}_{ij}\,{}^3\Gamma^l{}_{kl} \\
{}^3R^i{}_j &= {}^3g^{ik}\ {}^3R_{kj}
\end{aligned} \tag{2.25}$$

is given by $-L'/ra^2L^2 = -2k/a^2$ for $R^r{}_r$. Here the 3 in front of R_{ij} and Γ^k_{lm} indicates that these quantities are computed in 3 dimensions (time fixed!). Thus the Ricci scalar in 3 dimensions is [5]

$$ {}^3R = -\frac{6k}{a(t)^2} \tag{2.26} $$

The Gaussian curvature K in 3 dimensions is given by ${}^3R = -6\ {}^3K$, so

$$ {}^3K = \frac{k}{a(t)^2} \tag{2.27} $$

This means that $a(t)$ is the radius of curvature. This suggests that the metric (2.24), which is called the Robertson-Walker metric, has a geometric interpretation which we shall discuss in the next section.

2.3 A geometric interpretation of the Robertson-Walker metric

The metric in co-moving coordinates has a trivial time part, whereas the space part is non-trivial. It is of interest that the three-dimensional part of the metric (2.24) has a simple geometric interpretation. Let us consider three-space as embedded in a fictitious four dimensional space, where the fourth coordinate is called z. The metric is assumed to be just Euclidean, so the element of length ds is given by Pythagoras' theorem,

$$ ds^2 = (d\vec{x})^2 + (dz)^2 \tag{2.28} $$

Let us consider a hypersphere,

$$ \vec{x}^2 + z^2 = R^2 \tag{2.29} $$

where R is the radius. Differentiating (2.29) we get

$$ dz^2 = 2z\ dz = -d(\vec{x}^2) \tag{2.30} $$

so we can eliminate dz^2 from (2.28),

$$ ds^2 = (d\vec{x})^2 + \frac{(d\vec{x}^2)^2}{4z^2} \tag{2.31} $$

In this relation z can be eliminated by use of (2.29)

$$ ds^2 = (d\vec{x})^2 + \frac{(d\vec{x}^2)^2}{4(R^2 - \vec{x}^2)} \tag{2.32} $$

Introducing polar coordinates we can write this as

$$ ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\varphi^2) + \frac{r^2 dr^2}{R^2 - r^2} $$

[5] Readers who are not familiar with differential geometry in three dimensions can skip these remarks, since in the next section we shall derive the result (2.27) from simple considerations.

i.e.

$$ds^2 = \frac{R^2\ dr^2}{R^2 - r^2} + r^2(d\theta^2 + \sin^2\theta\ d\varphi^2) \tag{2.33}$$

If we measure the distance r relative to the radius R, we obtain with $\rho = r/R$

$$ds^2 = R^2\left[\frac{d\rho^2}{1-\rho^2} + \rho^2(d\theta^2 + \sin^2\theta\ d\varphi^2)\right] \tag{2.34}$$

If, for a fixed time, we put $a(t) = R$, then we see that the metric (2.34) is precisely the space part of (2.24) with $k = +1$.

The curvature of the hypersphere is always $1/R^2$ (large radius means small curvature and vice versa, in agreement with intuition). Thus, at a fixed time this agrees with (2.27).

The interpretation of the Robertson-Walker metric given above has a particularly nice interpretation in terms of the framework of co-moving coordinates. One can picture co-moving coordinates the following way: Let us imagine that space is filled with a dense cloud ("ideal fluid") of **freely falling** particles. Each particle carries a small clock, and is given a fixed set of spatial coordinates, which are determined at a certain time, say $t = 0$. The space-time coordinate $t, \vec{x}$ of any event are defined by taking $\vec{x}$ to be the coordinate of the particle which is just passing by where the event occur, and t as the time read off from the particles clock. Alternatively one can say that a coordinate mesh is being dragged along by the cloud of particles, with time defined by clocks fixed on the mesh. The existence of a cosmic standard time (which is **assumed** in the standard model) means that we can choose $t = 0$ in such a way that all particles are at rest at that time. It then follows as in (2.1)–(2.4) that this remains true at all times.

In cosmology we can use the galaxies as the points dragging along the coordinate mesh. The co-moving trajectories are then just the paths of the galaxies. If we do that, the three-dimensional projection of the hypersphere (2.29) can then be regarded as a balloon, where at a given time the galaxies are marked by dots on the three-surface. As the balloon is inflated the dots move, but the distance between the dots will move too, so each dot keeps the same coordinates.

In this picture one can say that R is the radius of the universe, since from (2.29) it is clear that $\sqrt{\vec{x}^2}$ cannot exceed R. Thus, in eq. (2.34) ρ is always less than one.

If k in (2.24) is negative, we cannot use the hypersphere (2.29). Instead we can consider the surface

$$-\vec{x}^2 + z^2 = R^2 \quad , \quad ds^2 = (d\vec{x})^2 - dz^2 \tag{2.35}$$

which is called a pseudo-hypersphere. In this case space is obviously unlimited, and genuinely non-Euclidean (Gauss-Bólyai-Lobachevski geometry). Proceeding as before, one obtains the Robertson-Walker metric (2.24) with $k = -1$. In this case R does not have the interpretation as the radius of the universe, but it still sets the scale, and is therefore called the **cosmic scale factor**. The curvature is everywhere negative, and is given by $-1/R^2$.

The case $k = 0$ corresponds to a completely flat space with $ds^2 = d\vec{x}^2$, again with a time-dependent scale factor $a(t)$.

2.4 Hubble's law

We shall now investigate the frequency of light emitted by some distant object in a universe described by the Robertson-Walker metric (2.24). Since light has $d\tau = 0$ we get for radial motion from (2.24)

$$dt^2 = a(t)^2 \, \frac{dr^2}{1 - kr^2} \tag{2.36}$$

From this we get for a light wave leaving a galaxy at $r = r_1$ at time $t = t_1$, and arriving at our galaxy $r = 0$ at time $t = t_0$,

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} = \begin{cases} \sin^{-1} r_1 & (k = +1) \\ r_1 & (k = 0) \\ \sinh^{-1} r_1 & (k = -1) \end{cases} \tag{2.37}$$

In co-moving coordinates galaxies have constant coordinates (r, θ, φ), only the scale changes. Thus r_1 is time independent. We wish to relate (2.37) to the frequency of light, so let the periods be δt_0 and δt_1 as measured by us and on the emitting galaxy, respectively. Thus

$$\int_{t_1 + \delta t_1}^{t_0 + \delta t_0} \frac{dt}{a(t)} = \int_{t_1}^{t_0} \frac{dt}{a(t)} \tag{2.38}$$

so using that δt_0 is very small (10^{-14} sec for a typical light signal), we get

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)} \tag{2.39}$$

where we used the rather safe approximation $a(t + 10^{-14}\,\text{sec}) \approx a(t)$ (i.e., the universe evolves very little in 10^{-14} sec !). Eq. (2.39) then gives for the frequencies

$$\frac{\lambda_1}{\lambda_0} = \frac{\nu_0}{\nu_1} = \frac{\delta t_1}{\delta t_0} = \frac{a(t_1)}{a(t_0)} \tag{2.40}$$

This shows that the wave length shifts like the scale factor: The wave length is stretched or shortened at the same rate as the universe expands or contracts.

Conventionally this relations is expressed in terms of the red-shift parameter z,

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_1)} - 1 = \frac{a(t_0) - a(t_1)}{a(t_1)} \tag{2.41}$$

where λ is the wave length, and where we used $\lambda_0/\lambda_1 = \nu_1/\nu_o$.

From eq. (2.41) we see that since λ_o is the light observed by us after a long travel from the galaxy at $r = r_1$, then if $z > 0$, i.e. $\lambda_0 > \lambda_1$ (red shift) or if $z < 0$ (blue shift), it follows that $a(t_0) > a(t_1)$ (red shift) or $a(t_0) < a(t_1)$ (blue shift). Obviously $t_0 > t_1$ in both cases, since we observe the light later than it was emitted. Hence we have the result that **an expanding (contracting) universe is characterized by a red shift (blue shift) of the spectral lines**. For increasing $t_0 - t_1$ we expect that $a(t_0)$ will increase (decrease) more and more, and hence z will increase (decrease) more and more, so the larger (smaller) the value of z is, the farther is the object.

Suppose that the emitting galaxy is relatively close to us. We can then expand eq. (2.41) and obtain to lowest order

$$z \approx \frac{\dot{a}(t_0)}{a(t_0)}\,(t_0 - t_1) \tag{2.42}$$

where $t_0 - t_1$ is sufficiently small. Now, in the first approximation $t_0 - t_1$ is the distance measured by an astronomer (remember the velocity of light is one), as is clear intuitively as well as from eq. (2.37), which to lowest order implies that $(t_0 - t_1)/a(t_0) \approx r_1$, and for the metric (2.24) we know that $a(t_0)r_1$ is the physical distance for r_1 small (relative to the scale parameter). Thus (2.42) can be written

$$z \approx H_0 L \quad , \quad H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \tag{2.43}$$

where L is the distance measured by the astronomer. If L is not small, one needs to go back to the relation (2.41),

Let us notice that for r_1 small it follows from the Robertson-Walker line element that $\dot{a}(t_0)r_1$ is approximately the radial velocity v_r of the galaxy, and since $(t_0 - t_1)/a(t_0) \approx r$, eq. (2.43) can be written

$$z \approx v_r \approx H_0 L \tag{2.44}$$

This may be interpreted as a frequency shift due to the Doppler effect. Such an interpretation should, however, not be taken too seriously, since light is certainly influenced by the gravitational fields, as the derivation of eq. (2.43) shows.

The relation (2.43) is the famous **Hubble law** (1929). Observationally it is found that there is a red shift, which increases with the distance of the galaxy. The Hubble "constant" H_0 actually depends on time. Due to new observations it is becoming better known,

$$\frac{1}{H_0} \approx (1 - 1.5) \times 10^{10}\text{years.} \tag{2.45}$$

Hubble's discovery of "a roughly linear relation between velocities and distances" had a profound effect on cosmology, and in a sense it marks the beginning of modern cosmology.

A very short biography of Edwin Hubble 1889-1953:

Hubble[6] had an undergraduate degree in astronomy and mathematics. However, he then took a legal degree, and practized as an attorney in Kentucky, where he joined its bar in 1913. He served in the first world war, where he got the rank of major. He was bored with law, and went back to his studies in astronomy. In 1919 he began to work at Mt. Wilson Observatory in California, where he would work for the rest of his life. He was researching nebulae, fuzzy patches of light in the sky. In 1924, he announced the discovery of a Cepheid, or variable star, in the Andromeda Nebulae. Since the work of Henrietta Leavitt had made it possible to calculate the distance to Cepheids, he calculated that this Cepheid was much further away than anyone had thought and that therefore the nebulae was not a gaseous cloud inside our galaxy, like so many nebulae, but in fact, a galaxy of stars just like the Milky Way. Only much further away. Until now, people believed that the only thing existing ouside the Milky Way were the Magellanic Clouds. The

[6] Most of this material is taken from www.edwinhubble.com. Check also "hubble biography" in google, where much more material can be found.

universe was much bigger than had been previously presumed. By observing redshifts in the wavelengths of light emitted by the galaxies, he saw that galaxies were moving away from each other at a rate constant to the distance between them (Hubble's Law). The further away they were, the faster they receded. This led to the calculation of the point where the expansion began, and supported the big bang theory. When Einstein proposed his theory of gravity, he thought that the universe was static, and he introduced the cosmological constant to accomodate this. Following Hubble's discoveries, Einstein is quoted as having said that second guessing his original findings (i.e. the field equations without Λ, naturally leading to a non-static universe), was the biggest blunder of his life. He visited Hubble to thank him in 1931.

2.5 Higher order correction to Hubble's law

In recent years astronomical observations have become very precise, and the next order term in the Hubble law becomes important, as we shall discuss later. To obtain this expansion we should return to exact eq. (2.40). So let us start by the second order Taylor expansion

$$\frac{1}{a(t)} \approx \left[a(t_0) + \dot{a}(t_0)(t-t_0) + \frac{1}{2}\ddot{a}(t_0)(t-t_0)^2\right]^{-1} \approx \frac{1}{a(t_0)} - \frac{\dot{a}(t_0)}{a(t_0)^2}(t-t_0) + \frac{H_0^2}{2a(t_0)}(2+q_0)(t-t_0)^2, \tag{2.46}$$

where we expanded around t_0 and where

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)}, \quad q_0 = -\frac{\ddot{a}(t_0)a(t_0)}{\dot{a}(t_0)^2}. \tag{2.47}$$

The quantity q_0 is called the *deceleration parameter*. Going back to the exact equation (2.41) we therefore see that it can be expanded as

$$z = \frac{a(t_0)}{a(t_1)} - 1 \approx H_0 \Delta t + \frac{1}{2}H_0^2\ (2+q_0)(\Delta t)^2, \quad \Delta t = t_0 - t_1. \tag{2.48}$$

To lowest order Δt is the distance between the source and the observer. However, the universe is not static, so to be more accurate we should take into account the metric of the universe. Since light has $d\tau = 0$, we have for radial motion that the comoving distance is

$$\int_{t_1}^{t_0} \frac{dt}{a(t)}. \tag{2.49}$$

This is the constant distance which fits with the emission and observation times t_1 and t_0. However, the physical distance is the comoving distance multiplied by the scale factor, i.e.

$$L_0 = a(t_0)\int_{t_1}^{t_0} \frac{dt}{a(t)}. \tag{2.50}$$

By use of the expansion (2.46) we see that

$$L_0 \approx \Delta t + \frac{1}{2}H_0(\Delta t)^2 + O((\Delta t)^3). \tag{2.51}$$

The correction is consistent with intuition: If the universe expands, $H_0 > 0$, then the physical distance is also expanded relative to the naive distance Δt. Similarly if the universe contracts: The distance then becomes shorter than naively expected.

We can now re-express z given by (2.48) in terms of L_0 instead of Δt,

$$z \approx H_0 L_0 + \frac{1}{2} L_0^2 H_0^2 (1 + q_0). \tag{2.52}$$

Later we shall see that the second term in this expression has played an important role in modern cosmology.

2.6 Einstein's equations and the Robertson-Walker metric

So far the scale factor $a(t)$ has not been determined as a function of cosmic time. To do that, we shall return to the Einstein equations following from eqs. (2.13), (2.14), and (2.19). From (2.14), by use of the co-moving four velocity (2.15), we get

$$T^t{}_t - \frac{1}{2}\, T^\mu{}_\mu = -\frac{1}{2}\,(3p + \rho) \tag{2.53}$$

$$T^r{}_r - \frac{1}{2}\, T^\mu{}_\mu = -\frac{1}{2}\,(p - \rho) \tag{2.54}$$

with the $\theta\theta$ and $\varphi\varphi$ components given by the same expression as (2.54). Using the results (2.21) or (2.22) for the function $L(r)$ we then get $(f(t) = a(t)^2)$

$$\begin{aligned} R^t{}_t &= -3\frac{\ddot{a}}{a} \\ R^r{}_r &= R^\theta{}_\theta = R^\varphi{}_\varphi = -\frac{\ddot{a}}{a} - 2\,\frac{\dot{a}^2}{a^2} - \frac{2k}{a^2} \end{aligned} \tag{2.55}$$

so the Einstein equations read

$$3\ddot{a} = -4\pi G(\rho + 3p)a \tag{2.56}$$

$$a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G(\rho - p)a^2 \tag{2.57}$$

In these equations we did not include the cosmological constant. In modern cosmology this is unreasonable since Λ plays an important role, so we shall now show that there is a simple way to include the cosmological constant in the pressure and density. To see this, write the Einstein equation with the cosmological constant in the form,

$$R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu\, R = -8\pi G T^\mu{}_\nu + \Lambda\, \delta^\mu{}_\nu = -8\pi G \tilde{T}^\mu{}_\nu, \tag{2.58}$$

where the new energy-momentum tensor is given by

$$\tilde{T}^\mu{}_\nu = T^\mu{}_\nu - \frac{\Lambda}{8\pi G}\,\delta^\mu{}_\nu = \tilde{p}(t)\delta^\mu{}_\nu + (\tilde{p}(t) + \tilde{\rho}(t))\, U^\mu U_\nu, \tag{2.59}$$

with

$$\tilde{\rho}(t) = \rho(t) + \frac{\Lambda}{8\pi G}, \quad \tilde{p}(t) = p(t) - \frac{\Lambda}{8\pi G}. \tag{2.60}$$

Thus we see that *the cosmological constant can be included in the Einstein equations just by changing the meaning of the energy density and the pressure in accordance with eq.* (2.60). For $\Lambda > 0$ we see that the energy is always positive, whereas the pressure can be negative. In particular, if the cosmological constant dominates, the pressure becomes negative. The physical meaning of Λ is related to "vacuum", since Λ does not refer to any state of matter. Thus it is an inevitable consequence of having a vacuum energy-momentum tensor $T(\mathrm{vac})_{\mu\nu} = -\Lambda g_{\mu\nu}/8\pi G$ (note that this is the only tensor that can be formed in a vacuum, since by definition this state does not contain anything except virtuality) that the pressure is negative if the cosmological constant is positive, which is what is observed. Here we should remember that $g_{00} = -1$.

In the following we shall leave out the tilde in the density and pressure. The effect of the cosmological constant can then always be obtained in a simple way from eq. (2.60). Later we discuss the important effects of the cosmological constant, and we shall then return to the substitution (2.60).

In addition to the Einstein equation we also have to ensure that the energy-momentum tensor (2.14) is covariantly conserved. Using eq. (1.88) this means

$$T^{\mu\nu}{}_{;\nu} = \frac{\partial p}{\partial x^\nu}\, g^{\mu\nu} + \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^\nu}\left[\sqrt{g}(\rho+p)U^\mu U^\nu\right] + \Gamma^\mu{}_{\nu\lambda}(p+\rho)U^\nu U^\lambda = 0 \tag{2.61}$$

In co-moving coordinates we have from (2.2) that $\Gamma^\mu_{00} = 0$. Eq. (2.61) is then trivial for $\mu = r, \theta, \varphi$ (remember that due to the homogeneous space p and ρ can only depend on time, not on space), and we have

$$-\frac{dp}{dt} + \frac{1}{\sqrt{g}}\,\frac{d}{dt}[\sqrt{g}(p+\rho)] = 0 \tag{2.62}$$

The determinant for the Robertson-Walker metric is given by

$$g = a(t)^6\,\frac{r^4 \sin^2\theta}{1-kr^2} \tag{2.63}$$

In eq. (2.62) only the time dependence of g is relevant, and we get

$$a(t)^3\,\frac{dp(t)}{dt} = \frac{d}{dt}\left[a(t)^3\,(p(t)+\rho(t))\right] \tag{2.64}$$

Eq. (2.64) can also be written

$$\frac{d}{da}\,(\rho a^3) = -3pa^2 \tag{2.65}$$

If an equation of state $p = p(\rho)$ is given, eq. (2.65) can be solved for $\rho = \rho(a)$, and the evolution can then be computed from (2.56) and (2.57). For example, if the pressure is negligible, $p \ll \rho$, then

$$\rho \propto a(t)^{-3} \qquad (p \ll \rho) \tag{2.66}$$

If one has extreme relativistic particles (like photons) the universe is radiation dominated with $p = \frac{1}{3}\rho$, and

$$\rho \propto a(t)^{-4} \quad (p = \frac{1}{3}\rho) \tag{2.67}$$

We notice that the Einstein equations (2.56) and (2.50) can be simplified by eliminating $\ddot{a}$. Thus, inserting (2.56) in (2.57) we get

$$\dot{a}^2 + k = \frac{8\pi G}{3}\,\rho a^2 \tag{2.68}$$

which is called the Friedmann equation. [7]

The functional equations in the standard model are thus the equation of state $p = p(\rho)$, the energy conservation (2.65), and the Friedmann equation (2.68). The last two equations are easy to remember: the quantity a^3 is proportional to the "volume" of the universe (for $k = +1$), and $M = \rho a^3$ is proportional to the mass of the universe. Eq. (2.65) can then be written (multiply by da) $dM + pdV = 0$, which expresses energy balance during the evolution, since dM is the change in energy balanced by pdV, which is the work done against the pressure forces. The Friedmann equation (2.68) has an "almost Newtonian" interpretation: multiply by $\frac{1}{2}$,

$$\frac{1}{2}\,\dot{a}^2 - G\left(\frac{4\pi\rho a^3}{3}\right)\frac{1}{a} = -\frac{k}{2} \tag{2.69}$$

and the first term is the Newtonian kinetic energy of a test particle with unit mass and with coordinate proportional to the scale factor, the second term ($G\times$ mass of the universe $\times(1/a)$) is the Newtonian potential. The sum of the "kinetic energy" and the "potential energy" is then the constant total energy, which is negative (positive) for $k = +1 (k = -1)$ and vanishes for $k = 0$. These considerations should be taken purely as an aid for the memory.

2.7 The Big Bang

If the pressure p is positive, many features of the standard model can be understood without specification of the equation of state $p = p(\rho)$. This is in particular true in the early universe, where it will turn out that the cosmological constant is not of importance.

To proceed, we need to use the Friedmann equation (2.68), i.e.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} \tag{2.70}$$

and the energy conservation (2.65), i.e.

$$\frac{d}{da}(\rho a^3) = -3pa^2 \tag{2.71}$$

It is also convenient to use eq. (2.56),

$$3\ddot{a} = -4\pi G(\rho + 3p)a \tag{2.72}$$

which is a consequence of eqs. (2.70) and (2.71). We can now qualitatively derive the following results:

[7] It is easy to show that eqs. (2.56) and (2.57) are implied by eqs. (2.65) and (2.68). The reason we only have two independent equations is that $T^{\mu\nu}{}_{;\nu} = 0$ is already contained in $(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0$.

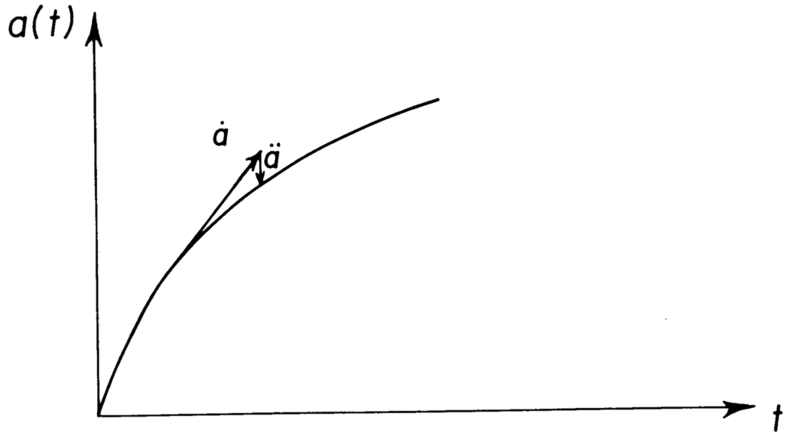

Figure 2.2: Plot of $a(t)$ with $\dot{a} > 0$ and $\ddot{a} < 0$

2.7.1 Existence of the big bang (the initial singularity)

If $p > 0$, or more generally, if $\rho + 3p$ is positive, it follows from (2.72) that $\ddot{a}/a < 0$, which means that gravity is *attractive*. Since we observe red shifts (and not blue shifts) it follows from the discussion in section 4 that at present the Hubble "constant" (2.43) is positive, so $\dot{a}/a > 0$. Thus, in a plot of $a(t)$ versus t the velocity $\dot{a}$ is positive, but the acceleration $\ddot{a}$ is negative, and we have a situation like the one shown in fig. 2.2. It is seen that sometime in the past, the function $a(t)$ has reached zero.

Conventionally we take this time to be $t = 0$. Clearly a vanishing scale factor means that the Robertson-Walker metric becomes singular, since the three-dimensional co-moving space shrinks to a single point as $t \to 0$. This singularity is the famous **big bang**. Such singularities are known to occur generically in most space-times under very general assumptions, [8] so it is not just a feature of the Robertson-Walker metric. The occurrence of the singularity means that the universe starts as an explosion, and $a(t)$ increases until it reaches its present value.

2.7.2 The age of the Universe according to Big Bang

The history of the present universe is thus that it starts with $a = 0$ and $a(t)$ increases, until it reaches its present value $a(t_o)$ where t_0 is the age of the universe.

As mentioned in the last subsection, $\ddot{a} < 0$ when $\rho + 3p > 0$. Thus we have that $\dot{a}$ is decreasing and therefore is smaller today than it was in the past,

$$\dot{a}(t_0) < \dot{a}(t) \quad \text{for} \quad 0 \leq t < t_0. \tag{2.73}$$

Using this inequality we obtain an upper limit for the age of the universe

$$t_0 = \int_0^{a(t_0)} \frac{da}{\dot{a}} < \int_0^{a(t_0)} \frac{da}{\dot{a}(t_0)} = \frac{a(t_0)}{\dot{a}(t_0)} = \frac{1}{H(t_0)} \tag{2.74}$$

[8] See e.g. S.W. Hawking and G.F.R. Ellis, The large-scale structure of space-time, Cambridge Univ. Press, 1973.

where $H(t_0)$ is the Hubble "constant" at present (see eq. (2.43)). From eq. (2.45) we see that the upper limit on the age of the universe is between 10 and 15 billion years,

$$t_0 \lesssim (10 - 15) \times 10^9 \text{ yr} \tag{2.75}$$

This limit is actually an important feature of the big bang model, since there are arguments based on geology, stellar evolution and the nuclear abundances, which put lower bounds on the age of the universe in a way which is independent of the big bang model. We shall return to this point later.

2.7.3 Discussion of the fate of the Universe

Having looked at the past, let us now turn to the future. Suppose $a(t)$ continues to grow and ultimately $a(t) \to \infty$. From eq. (2.71) we see that if $p \geq 0$, the quantity ρa^3 is constant or decreasing. Therefore, we must have $\rho a^2 \to 0$ for $a \to \infty$, and it follows that $\dot{a}^2 \to -k$ (at least as fast as $1/a$, since $\rho a^2 \to 0$ at least like $1/a$). Thus it is only possible for $a(t)$ to go to infinity if $k = -1$ or $k = 0$. For $k = -1$ we thus get for the "open universe"

$$a(t) \to t \quad \text{for} \quad t \to \infty \quad \text{and} \quad k = -1 \tag{2.76}$$

For $k = 0$ we have similarly for the "flat universe"

$$a(t) \to \infty \quad \text{slower than } t \quad \text{for} \quad t \to \infty \quad \text{and} \quad k = 0 \tag{2.77}$$

For $k = +1$ it is impossible for $a(t)$ to go to infinity, and from (2.70) we see that $\dot{a}(t)$ becomes zero when $a(t)$ has its maximum value a_{max}, where $\rho a_{\text{max}}^2 = 3/8\pi G$. Since $\ddot{a}$ is always negative, the scale parameter will then drop down again (with $\dot{a}$ negative and blue-shifted spectral lines) and reach $a(t) = 0$ in a finite time t. This is the "closed universe".

It therefore would be quite interesting to know the sign of k to see whether the universe expands forever, or whether we shall be heading for the disastrous future singularity. We shall discuss this in the next section.

2.8 Fitting parameters to observations

The general relativistic equations (2.70) and (2.71) are incomplete without the specification of an equation of state $p = p(\rho)$. In the standard model two such equations are used, namely for relativistic matter relevant for the early universe, and for non-relativistic matter relevant for the present universe.

Most of the matter (galaxies and clusters of galaxies) observed today appear to be non-relativistic with kinetic energy much smaller than the rest energy. In the kinetic theory of gases (here applied to galaxies) $p = \frac{1}{3}\rho\overline{v^2}$, where $\sqrt{\overline{v^2}}$ is the root-mean-square average velocity of the gas molecules (= the galaxies). Reintroducing the velocity of light, the pressure in the energy-momentum tensor enters like p/c^2, and hence $p/c^2 = \frac{1}{3}\rho\overline{v^2}/c^2$, and for galaxies observed now $\overline{v^2}/c^2 \ll 1$. Hence eq. (2.71) gives with $p = 0$

$$\rho = \frac{B}{a^3} \tag{2.78}$$

where B is a constant of integration. Eq. (2.78) means that the energy content of a co-moving volume a^3 remains constant when $a(t)$ changes.

In the early universe one usually assumes the existence of a hot gas of extreme relativistic particles with the kinetic energy much larger than the rest energy. For a relativistic ideal gas one has[9]

$$p = \frac{1}{3}\,\rho \tag{2.79}$$

and eq. (2.71) gives

$$\rho = \frac{D}{a^4} \tag{2.80}$$

where D is a constant of integration. The $1/a^4$ behavior is related to the fact that relativistically the energy in a co-moving volume "red shifts" like $1/a$, while the volume still scales as $1/a^3$.

Now at very early times matter was very hot, and as the universe expands and cools, the kinetic energy of the matter redshifts and ultimately it becomes non-relativistic.

We shall try to fit the various parameters to observations. In order to fit the crucial parameter k (open, flat, closed universe for $k = -1,\ k = 0,\ k = +1$, respectively) let us use eq. (2.70) in the form

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} \equiv \frac{8\pi G}{3}\,\rho_{\text{crit.}} \tag{2.81}$$

If $\rho = \rho_{crit}$, then the expansion is driven entirely by ρ, and $k = 0$. In eq. (2.81) H is the value of the Hubble "constant" at time t. From

$$H \approx \frac{h}{10^{10}\ \text{yr}} \quad , \quad 1 \geq h \geq 0.4 \tag{2.82}$$

today, we get

$$\rho_{\text{crit.}} \approx 2h^2 \times 10^{-29}\ g/cm^3 \tag{2.83}$$

Introducing $\Omega = \rho/\rho_{crit}$ eq. (2.81) becomes

$$\frac{k}{a^2} = \frac{8\pi G}{3}\,\rho_{\text{crit.}}(\Omega - 1) = H^2(\Omega - 1) \tag{2.84}$$

so $\Omega > 1, \Omega = 1, \Omega < 1$ correspond to $k = +1, k = 0, k = -1$, respectively.

Eq. 2.84 has a simple intuitive appeal: Suppose Ω is very large. Then we have a lot of energy, with a lot of gravitational attraction, which will counteract the expansion making a closed universe. Similarly, if Ω is very small, there is very little counteraction to expansion, and the universe is open.

Since ρ_{crit} is given in terms of the Hubble constant, and since Ω could in principle be determined by "measuring" the mass density in the universe, the Friedmann equation (2.84) allows a determination of the sign of k, and hence the fate of the universe.

If one considers luminous matter (stars) one finds

$$\Omega_{\text{lum.}} = \frac{\rho_{\text{lum.}}}{\rho_{\text{crit.}}} \simeq 0.04 \tag{2.85}$$

[9]For a relativistic particle one has energy $\approx$ |momentum|; the factor 1/3 arises because statistically particles can move in three directions.

so $\Omega \geq 0.04$. However, one can also estimate the masses of dynamical systems by asking for the gravitational forces necessary to explain the observed motions.

For a system with spherical symmetry one can use Kepler's third law $GM/r = v^2$, where M is the mass interior to the orbit of an object (a star) with velocity v and orbital radius r. By studying the orbital motion of stars in a galaxy at the radius where light has essentially disappeared (this radius is called the Holmberg radius), one can measure the mass of the luminous material and one gets (2.85). However, studies of orbits of stars in spiral galaxies (for elliptic galaxies similar results do not exist, perhaps because of the lack of "test bodies", i.e. stars outside luminous matter) beyond the Holmberg radius have revealed the interesting and stunning result that v does not go as $1/\sqrt{r}$, but instead stays approximately constant. This phenomenon is called "flat rotation curves", and it means that the mass M increases linearly with r, even though there is essentially no luminous matter to observe. This indicate that a large part of all matter is "dark". For galaxies one gets

$$\Omega_{\text{dyn.}} = \frac{\rho_{\text{dyn.}}}{\rho_{\text{crit.}}} \approx 0.25 \tag{2.86}$$

so $\Omega \gtrsim 0.3$. The discrepancy between (2.85) and (2.86) gives rise to the search (theoretically and experimentally) for dark matter. We shall, however, not enter this fascinating subject.

From geology, stellar evolution, and evolution of nuclear abundances one can make arguments which puts a lower limit of 10 billion years on the age of the universe. Now Ω is related to the age of the universe, as one can see qualitatively. If Ω is very large, we have a lot of gravitational interaction, slowing down the expansion, and thus the life time of the universe is small. If Ω is very small, the converse is true (if $\Omega = 0$ there is only expansion, no counteraction from gravity). This turns out to give the bound

$$\Omega \lesssim 3 \tag{2.87}$$

Thus, using this and (2.86) we have

$$0.3 \leq \Omega \leq 3 \tag{2.88}$$

Consequently we see that from these arguments we are not in a position to determine whether the universe is closed ($\Omega > 1$), flat ($\Omega = 1$), or open ($\Omega < 1$), and we cannot say whether the universe will eventually re-contract or expand forever. However, recently (1998-2001) new types of observations have been developed (supernovae and fluctuations in the microwave background) which give very strong indications that the universe is flat and expanding due to a cosmological constant. This will be briefly discussed in section 2.13.

From eq. (2.84) and the bounds (2.88) we get

$$\frac{8\pi G}{3} \cdot 2\ \rho_{\text{crit.}} \geq \frac{k}{a^2} \geq -\ \frac{8\pi G}{3}\ 0.9\ \rho_{\text{crit.}} \tag{2.89}$$

today. This limits the three-dimensional curvature k/a^2.

Finally we shall determine the parameter D in eq. (2.80). To get a bound on D, we need to consider the microwave background radiation.

2.9 The cosmic microwave radiation background

In the standard big bang model the universe starts out with a very small scale factor. Presumably the temperature is very high, and the universe is filled with highly relativistic particles. As the universe expanded, the temperature dropped, so matter and radiation cooled. At a temperature of around 4000^0K (the decoupling temperature), the free electrons joined the nuclei to bind into neutral atoms, and the thermal contact between matter and radiation was broken, since the absence of free charged particles means that the interactions of photons with other matter drop off steeply. The radiation existing at that time has since been much redshifted, but it should still fill space around us.

To describe the situation in very simple terms let us consider photons in thermal equilibrium. We found in eq. (2.40) that the frequency changes with the scale factor according to

$$\frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)} \tag{2.90}$$

Thus, if t_1 is the time of the decoupling, we can get the redshift today ($t = t_0$) from eq. (2.90). Also, because of eq. (2.90), the Boltzmann factor for a photon satisfies

$$e^{\frac{h\nu_1}{kT_1}} = e^{\frac{h\nu_0}{kT_0}} \tag{2.91}$$

where we have defined

$$T_0 \equiv T_1 \,\frac{a(t_1)}{a(t_0)} \tag{2.92}$$

If there is a thermal equilibrium, we can thus absorb the scale factor in the temperature. In particular we get at the present age of the universe the Planck black body radiation distribution of the photons,

$$\rho(\nu_0)d\nu_0 = \frac{8\pi h{\nu_0}^3 \; d\nu_0}{\exp\left(\frac{h\nu_0}{kT_0}\right) - 1} \tag{2.93}$$

Thus, the big bang predicts that there should be a thermal distribution of photons present today, left over from the time of the decoupling of photons from other matters. Presumably the most significant experimental result in cosmology since Hubble's discovery is that such a spectrum is indeed observed (Penzias and Wilson 1965).

In principle the temperature T_0 at present can be predicted from eq. (2.92), which was also done by a number of people (Gamow, Alpher, Herman, Dicke, Peebles, Roll and Wilkinson) before the discovery of the radiation. After the recombination the universe is matter dominated, and we have (2.78), i.e.

$$\rho = \frac{B}{a^3} \tag{2.94}$$

from which we get

$$\frac{a(t)}{a(t_0)} = \left(\frac{\rho(t_0)}{\rho(t)}\right)^{1/3} \tag{2.95}$$

Inserting the densities of baryons at t and at present ($\rho(t_0) \approx 10^{-6} g/cm^3$) the most careful prediction was $T_0 \approx 5^o$K. The experiments give a nice black body radiation spectrum

(2.93) with $T_0 = 2.75 \pm 0.05^o$K, in good agreement with the theoretical estimates. Recent satellite measurements give a very precise Planck spectrum.

We can now obtain a lower bound on the constant D in eq. (2.80) for ρ in the radiation dominated early universe, $\rho = D/a^4$. The presently observed microwave photons contribute a relativistic component ρ_{MW} to ρ, as they did in the relativistic era. Thus

$$\rho_{MW} = \frac{D_{MW}}{a^4} \quad \text{(all times)}$$

In the relativistic era the photons were not the only component to the energy density so

$$D > D_{MW} \tag{2.96}$$

Now using

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \tag{2.97}$$

we obtain by integrating (2.93) over the frequency

$$D > D_{MW} = \rho_{MW}\ a^4\Big|_{\text{today}} = (kT_0)^4\ \frac{8\pi^5 a^4}{15h^3}\Bigg|_{\text{today}} \tag{2.98}$$

From (2.84) and the bound (2.88) one gets a lower bound on $a(t)$, and inserting this we get for $\hbar = 1$

$$D \geq 6 \times 10^{114} \tag{2.99}$$

Thus the constant D in the radiation dominated density $\rho = D/a^4$ is huge. We shall return to this later.

2.10 The matter dominated era

We now consider the matter dominated era with negligible pressure, so $\rho = B/a^3$. Also, in this section for simplicity we ignore the cosmological constant, which can be taken into consideration at the cost of more complicated formulas. The Friedmann equation (2.70) becomes

$$\frac{\dot{a}^2}{a_0^2} = \frac{8\pi G}{3}\ \left(\rho_{\text{crit.}}\right)_0\ \Omega_0 \frac{a_0}{a} - \frac{k}{a_0^2} \tag{2.100}$$

where we divided by a_0^2 (a_0 is the scale parameter today) and used

$$\frac{\rho}{\rho_0} = \left(\frac{a_0}{a}\right)^3 \tag{2.101}$$

From eq. (2.84) we have

$$\frac{k}{a^2} = \frac{8\pi G}{3}\ \rho_{\text{crit}}(\Omega - 1) = H^2(\Omega - 1) \tag{2.102}$$

Using this today in eq. (2.100) we get

$$\frac{\dot{a}^2}{a_0^2} = H_0^2(1 - \Omega_0 + \Omega_0\ \frac{a_0}{a}) \tag{2.103}$$

This equation can easily be solved for time,

$$t = \frac{1}{H_0} \int_0^{a/a_0} \frac{dx}{\sqrt{1 - \Omega_0 + \frac{\Omega_0}{x}}} \tag{2.104}$$

In particular, the present age of the universe is

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{dx}{\sqrt{1 - \Omega_0 + \frac{\Omega_0}{x}}} \leq \frac{1}{H_0} \tag{2.105}$$

where the inequality (valid for $\Omega_0 > 0$) was already derived from the qualitative arguments in sect. 6. The integral in (2.104) or (2.105) can easily be performed. It is most convenient to distinguish the case $\Omega_0 > 1$, $\Omega_0 = 1$, and $\Omega_0 < 1$ (closed, flat, open universe, respectively). The intuitive appeal of eq. (2.105) should be emphasized. If Ω_0 is very large/small, gravity counteracts very much/little, and hence the life time is expected to be short/large. This is seen to be the case in the above integral.

2.10.1 The closed Universe $\Omega_0 > 1$

We define the "development angle" θ by

$$\sin^2 \frac{\theta}{2} = \frac{\Omega_0 - 1}{\Omega_0} \frac{a(t)}{a_0} \tag{2.106}$$

The integral (2.104) can then easily be performed and we get

$$H_0 t = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} (\theta - \sin\theta) \tag{2.107}$$

which together with eq. (2.106) shows that $a = a(t)$ is a cycloid. The scale parameter increases from zero at $\theta = 0,\ t = 0,$ and reaches a maximum at

$$\begin{aligned} \theta_{\max} &= \pi \\ t_{\max} &= \frac{\pi \Omega_0}{2H_0(\Omega_0 - 1)^{3/2}} \\ a_{\max} &= \frac{\Omega_0\, a_0}{\Omega_0 - 1} \end{aligned} \tag{2.108}$$

and then it returns to zero at $\theta = 2\pi,\ t = 2t_{\max}$. The present age of the universe is obtained by putting $a(t) = a_0$ in eq. (2.106),

$$t_0 = \frac{1}{H_0} f(\Omega_0) \tag{2.109}$$

$$f(\Omega_0) = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \left[\cos^{-1}\left(\frac{2}{\Omega_0} - 1\right) - \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1}\right] \tag{2.110}$$

for $\Omega_0 \geq 1$. The function $f(\Omega_0)$ is seen to be monotonically decreasing. For $\Omega_0 = 2$ one has $f(2) = \frac{\pi}{2} - 1 \simeq 0.57$, for $\Omega_0 \to \infty$ one has $f(\Omega_0) \to \pi/2\sqrt{\Omega_0}$. Thus, if Ω_0 is very large, the life time is very small, as mentioned before.

2.10.2 The flat Universe $\Omega_0 = 1$

Here eq. (2.104) is trivial and gives

$$\frac{a(t)}{a_0} = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{2.111}$$

and the age of the universe is

$$t_0 = \frac{2}{3}\, H_0^{-1} \approx \frac{2}{3h} \times 10^{10}\ \mathrm{yr} \tag{2.112}$$

with $0.4 \leq h \leq 1$. Since evidence from geology etc. shows that $t_0 > 10^{10}$ years, we see that if $\Omega_0 = 1$ then we cannot have $h = 1$, but h must be less than ≈ 0.67. The result (2.111) or (2.112) could also be derived from eqs. (2.106) and (2.107) by noticing that for $\Omega_0 \to 1$ it follows from eq. (2.106) that $\theta \to 0$. Expanding the trigonometric functions one then gets (2.111). The result (2.112) again has the form (2.109), so $f(1) = \frac{2}{3}$.

2.10.3 The open universe $\Omega_0 < 1$

Formally we can again use the substitution (2.106) and the result (2.107), except that the development angle is imaginary, $\theta = i\psi$. Hence we get

$$H_0 t = \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}(\sinh\psi - \psi) \tag{2.113}$$

with

$$\cosh\psi - 1 = 2\,\frac{1-\Omega_0}{\Omega_0}\,\frac{a(t)}{a_0} \tag{2.114}$$

The age of the universe is given by (2.109), but now

$$f(\Omega_0) = \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\ln\,\frac{2-\Omega_0+2\sqrt{1-\Omega_0}}{\Omega_0} \tag{2.115}$$

Again $f(\Omega_0)$ is monotonically decreasing with increasing Ω_0, and the maximum value is $f(0) = 1$.

2.10.4 Inclusion of the cosmological constant

We end this discussion by emphasizing that in view of newer observations the cosmological constant should be introduced in the formulas discussed above. Since this is straightforward we leave the details of the inclusion of Λ as an excersize for the reader. Generalizing the steps leading to eq. (2.105) one finds[10]

$$H_0 t_0 = \int_0^1 \frac{dx}{\sqrt{1-\Omega_T+\Omega_M/x+\Omega_\Lambda x^2}}, \tag{2.117}$$

[10] Here we have ignored radiation. If we denote the radiative Ω by Ω_R, this formula would read

$$H_0 t_0 = \int_0^1 \frac{dx}{\sqrt{1-\Omega_T+\Omega_R/x^2+\Omega_M/x+\Omega_\Lambda x^2}}, \tag{2.116}$$

where the radiative contribution is important only in the early universe.

where the Ω's refer to the present time t_0, and $\Omega_\Lambda = \Lambda/3H^2$. This integral is of an elliptic type and (2.117) can easily be integrated numerically. We give a few examples:

$$H_0 t_0 = 0.923 \;\; (\Omega_M = 0.35,\; \Omega_\Lambda = 0.65) \text{ and } H_0 t_0 = 1.014 \;\; (\Omega_M = 0.25,\; \Omega_\Lambda = 0.75). \tag{2.118}$$

In all these examples the total Ω_T is one. If this is the case, the integral in (2.117) can actually be performed in a simple way,

$$H_0 t_0 = \frac{2}{3}\int_0^1 \frac{dy}{\sqrt{\Omega_M + \Omega_\Lambda y^2}} = \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_M}} + \sqrt{1 + \frac{\Omega_\Lambda}{\Omega_M}}\right). \tag{2.119}$$

Here it should be remembered that $\Omega_M + \Omega_\Lambda = 1$.

2.10.5 Discussion of the life time of the Universe

It is clear from common sense that the universe must be older than the oldest stars. In principle, this can be checked by measuring the Hubble constant H or h very accurately. This is one of the main purposes of the Hubble space telescope. To illustrate the problem for the reader, let us mention a few numbers: The ages of the oldest globular clusters are $t_0 = 13.5 \pm 2.0$ Byr. This number depends on models as well as observations. However, recently (2001) it has been possible to find (so far only one) old star where U^{238} was found in the spectrum. From the life time of uranium one then finds $t_0 = 12.5 \pm 1.5$ Byr. Allowing for some additional time to create this star, the life time of the universe would then be something like 13.5 Byr.

As an example, suppose $t_0 \sim 15$ Byr and $\Omega_0 = 0.1$. Then $h = \frac{2}{3} \times 0.9 = 0.59$. This is below the reported values from the Hubble telescope ($h = 0.7 \pm 0.15$), which, however, has been much debated.

In general, it is easy to see that the small Ω (including dark matter) is not enough to provide sufficient life time from the cosmological model. To get out of this problem, the expansion should somehow be enhanced. This has led to the (re-)introduction of the cosmological constant, i.e. to add $-\Lambda g_{\mu\nu}$ on the left hand side of the Einstein equation. Later on, as we shall discuss at the end of this chapter, new observations on supernovae and fluctuations in the microwave background, have shown that there must be a cosmological constant (or something similar). Thus, the cosmological constant is not introduced as a convenient way to solve the life time problem, but is really needed to fit other observations.

We have seen in eq. (2.60) that it is easy to take into account the presence of a cosmological constant in the previously discussed cosmological Einstein equations. We just need to make the replacements

$$p \to \tilde{p} = p - \frac{\Lambda}{8\pi G}\,, \qquad \rho \to \tilde{\rho} = \rho + \frac{\Lambda}{8\pi G}\,. \tag{2.120}$$

To see that Λ is related to expansion, let us take the simple example of flat space $k = 0$, with no matter, $\rho = 0$ and $p = 0$. Thus, $\tilde{\rho} = \frac{\Lambda}{8\pi G}$. The Friedmann equation then gives

$$\dot{a}^2 = \frac{8\pi G}{3}\tilde{\rho} a^2 - k = \frac{\Lambda}{3} a^2\,, \tag{2.121}$$

with the solution $a(t) = e^{Ht}$, $H = \sqrt{\Lambda/3}$ for Λ positive. Thus, we see that if Λ is positive, it is possible to have expansion just from the cosmological constant.

In the older universe we see that the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{B}{a^3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{2.122}$$

implies that the different terms dominate at different times: In the beginning (after recombination) the matter term dominates, then the curvature (k/a^2) takes over as a increases, and ultimately the cosmological constant term dominates completely. For $\Lambda > 0$ and $k = -1$ the expansion continues forever. For $\Lambda < 0$ the expansion comes to a stop, since the left hand side of the Friedmann equation is always positive. In this case there is a maximum value of a obtained for $\dot{a} = 0$.

We have mentioned before that the cosmological constant does not play an important role in the early relativistic universe, where the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{D}{a^4} - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{2.123}$$

When a is very small, the first term dominates entirely, and the curvature ($-k/a^2$) and the cosmological terms are utterly unimportant. Thus our considerations on the big bang in sect. 2.6 are valid in spite of the fact that we did not explicitly take into account the cosmological constant.

If Λ is introduced, it is easy to solve any age-problem. An example: Let $t_0 = 13.5$ Byr and $h = 2/3$. With $\Omega_0 = 0.35$ and $\Omega_\Lambda = 0.65$ these numbers fit, as can be seen by use of (2.118). We end by remarking that the age problem has been there since Hubble's dicovery. Using his value for H, the life time of the universe was shorter than the life time of the planet earth! This led to a disbelief in the standard cosmological model. However, as mentioned before there are now observations, independent of the age problem, which requires a cosmological constant. Observations seem to converge to a consistent picture of the universe as beeing described by the standard model to a first approximation.

2.11 Causality structure of the big bang (The horizon problem)

The region of causal contact with a given event is determined by how far photons originating at that event can travel. We shall consider photons emitted at big bang and see how far they can propagate.

For photons $d\tau = 0$, so the Robertson-Walker metric gives $dt^2 = a(t)^2 dl^2$, where dl is the time-independent distance in co-moving coordinates. The physical causal distance is thus

$$d = a(t_1)l = a(t_1)\int_0^{t_1} \frac{dt}{a(t)} \tag{2.124}$$

where the time 0 is the big-bang time that the photon was emitted, whereas t_1 is the time it was received.

To obtain $a(t)$ let us consider the Friedmann equation (2.70),

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k \tag{2.125}$$

In the early universe $\rho = D/a^4$, and a is quite small. Consequently the term $\rho a^2 = D/a^2$ dominates over the curvature term, and we get quite easily by solving $\dot{a}^2 \simeq (8\pi G/3)D/a^2$ that

$$a(t) \approx \sqrt{2t\sqrt{\frac{8\pi}{3}\,GD}} \qquad (2.126)$$

so $a(t)$ is proportional to $\sqrt{t}$ in the early universe. From (2.124) we thus get for the physical distance,

$$d \approx 2t_1 \qquad (2.127)$$

Consequently we see that for t_1 small, i.e. in the early universe

$$\frac{d}{a(t_1)} \approx \sqrt{\frac{2t_1}{\sqrt{\frac{8\pi}{3}GD}}}. \ll 1 \qquad (2.128)$$

The conclusion from this is that **in the early universe the causal distance is much smaller than the scale factor**. This means that light emitted from a point cannot catch up with the expansion, which goes much faster than light[11]. This can be expressed by saying that close to the initial singularity the universe consists of many arbitrarily small causally separate regions. This is clearly very surprising, since the standard model assumes that the universe is, at any stage in its development, homogeneous. Although this is an assumption, and although one is free to make such an assumption, the situation does not appear very self-consistent. From a physical point of view one would like the homogeneous universe to be established by the contact between all points by physical light signals, so that physics in any point can establish itself to be equal to physics in any other point by means of causal contacts. This is not the case in the early universe, and this is clearly a difficulty with the standard model. This is the causality problem, also called the horizon problem. The latter concept refers to the fact that Einstein gravity produces a horizon beyond which one cannot observe.

In the matter dominated (later) universe the situation is different. For $k = 0$ we have from (2.125)

$$a = \left(\frac{3}{2}t\sqrt{\frac{8\pi}{3}\,GB}\right)^{2/3} \qquad (2.129)$$

and hence from

$$d = a(t_2)l = a(t_2)\int_{t_1}^{t_2}\frac{dt}{a(t)} \qquad (2.130)$$

we get

$$d = 3\left(t_2 - (t_1 t_2^2)^{1/3}\right) \approx 3t_2 \qquad (2.131)$$

for $t_2 \gg t_1$, so throughout the later evolution of the universe the largest regions of causal contact are of the order t, which is larger than $a(t)$.

[11] To give an analogy to this situation, suppose you want to bike 10 km to tell a friend how she should set her watch in order to synchronize your watches, assuming that biking is the fastest way of communication. Unfortunately, some evil person is changing the length scale much faster that you can bike, so you will never reach her to tell her the time. The standard model assumes that somehow she knows how you have set your watch! Admittedly, it could happen by accident that she selected the same setting of time as you did. However, in the proper analogy there are billions of such friends, so the probability of accidental coincidences would be essentially zero.

The microwave background discussed in section 8 is observed to be very isotropic. It turns out that the microwave photons originated in causally separate regions if their directions differ by more than a few degrees. Thus the data show that a truly striking degree of isotropy is present in the universe, and in the big bang standard model this could not have been produced by causal processes. In the big bang model this striking isotropy is not explained, but has to be put in "by hand" to fit the observations.

Finally, let us discuss another puzzle, "the flatness problem". Flat space, i.e. $\Omega = 1$, is an unstable equilibrium. If Ω is ever exactly equal to one, it remains equal to one. Now we know that today Ω is at least 0.1 - 0.3, which is not very far from 1. The Friedmann equation can be rewritten

$$k = \dot{a}^2(\Omega - 1). \tag{2.132}$$

Since k is a constant, and since $\dot{a}$ increases if we go backward in time, it follows that in early times Ω must have been extraordinarily closer to 1 than it is today. To give some examples: For Ω to be in the allowed range today, it must have been equal to 1 to an accuracy of 15 decimal places 1 sec after big bang (time of nucleosynthesis). At grand Unification Ω should be equal to 1 to an accuracy of 49 decimals. This fact is not explained by the Standard Model.

2.12 Inflation

We have seen that the standard model has many successful features, and gives a nice framework for discussing the rather few astronomical observations which are available. In the last section, however, we saw that certain features emerge from the standard model, which point to difficulties for the model if we want to understand it in more physical terms.

The standard model is therefore to be supplemented by a new scenario called "inflation" , where many of the physical difficulties have been removed. Inflation concerns itself with the early universe, where the difficulties arise . In the previous work we have assumed that the pressure is positive. Although this seems intuitively natural, it is not logically necessary. In quantum field theory one can have negative pressure.

We cannot give a full explanation of this, since this would require the introduction of fields. However, let us give an intuitive description of what happens. Suppose, at a very early time, the system is in a metastable state, called the "false vacuum": On short time scales the energy cannot be lowered, so "false" could be replaced by "temporary". Given enough time, the false vacuum decays to a stable "true" vacuum, which is the state with lowest possible energy. Also, let us assume that it takes a rather long time for the false vacuum to decay. This is, in short, the scenario in inflation.

Let us call ρ_f the energy density of the false vacuum (relative to the true vacuum). To find the pressure of the false vacuum, let us imagine that we enclose it in a cylindrical container with a movable piston. So inside the cylinder and piston, we have the energy density ρ_f. Outside we have true vacuum, with $\rho = p = 0$. Now move the piston outwards adiabatically, which means that there is no heat flow into or out of the cylinder. When the piston is pulled, the false vacuum keeps its energy density *everywhere*, also in the "new" volume. This is because this energy is (temporarily) the lowest possible one. So the change in energy dU is given by $dU = \rho_f dV$, where dV is the change in volume. However, in the adiabatic case we always have $dU + pdV = 0$. Hence,

$$p = -\rho_f < 0, \tag{2.133}$$

i.e. the pressure is negative.

You may wonder where the extra energy arising when we pull the piston comes from? It is, of course, supplied by the agent pulling the piston, who has to perform this work against negative pressure!

Another way of seeing the negative pressure related to vacuum is to ask for the energy-momentum tensor for a false or true vacuum. This $T^{\mu\nu}$ cannot depend on velocities etc., since the vacuum does not move or carry momentum. The only relativistically invariant tensor is therefore $T^{\mu\nu} = -\rho g^{\mu\nu}$, where ρ is the density of the true or false vacuum. The minus sign occurs because by definition the energy density is positive. This tensor therefore corresponds to the negative pressure $p = -\rho$.

The negative pressure has profound consequences. Consider eq. ((2.72))

$$\ddot{a} = -\frac{4\pi}{3} G(\rho + 3p)a. \tag{2.134}$$

Usually this means $\ddot{a} < 0$, i.e. gravity is *attractive*. Now, during the time the false vacuum dominates

$$\ddot{a} = +\frac{8\pi}{3} G\rho_f a \tag{2.135}$$

so gravity is now *repulsive*!! This means that during inflation, gravity does *not* counteract the expansion, on the contrary, because the solution of the attractive gravity equation is

$$a(t) = c_1 e^{\chi t} + c_2 e^{-\chi t}\ , \qquad \chi = \sqrt{\frac{8\pi}{3} G\rho_f} \tag{2.136}$$

where c_1, c_2 and χ are constants (remember that ρ_f is the constant energy density of the false vacuum). Let us estimate ρ_f. A simple model is to assume that it is the density relevant at the Grand Unification scale, with temperature 10^{15} GeV. In natural units ρ_f has dimension $(\text{mass})^4$, so

$$\rho_f \approx (10^{15}\text{GeV})^4 \tag{2.137}$$

This is enormous! To get this density in the sun, we would have to compress it to the size of a proton. We then get

$$\frac{1}{\chi} \approx 10^{-34}\text{sec} \tag{2.138}$$

and the Hubble constant (which is now a true constant) becomes

$$\frac{\dot{a}}{a} = \chi = 10^{34}\text{sec}^{-1}. \tag{2.139}$$

Let us consider the density during inflation,

$$\frac{d}{da}(\rho_f a^3) = -3pa^2 = 3\rho_f a^2 = \rho_f \frac{da^3}{da} \tag{2.140}$$

so $d\rho_f/da$ vanishes, and ρ_f is time independent as we knew beforehand. The Friedmann equation for $k = 0$ is then just

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3} G\rho_f} = H_{\text{inflation}}. \tag{2.141}$$

Let us now modify the standard big bang so that for $0 \leq t \leq t_1$ we have the usual radiation domination ($\rho = D_1/a^4$), for $t_1 < t \leq t_2$ we have negative pressure and the equation of state (2.133), for $t_2 < t \leq t_d$ we return to the relativistic equation of state, whereas for $t > t_d$ (t_d =time of decoupling of photons) we have the usual matter domination ($\rho = B/a^3$). Then the evolution looks like:

$0 < t < t_1$	radiation dominance	$a \sim \sqrt{t}$	$\rho = D_1/a^4$
$t_1 < t < t_2$	inflation	$a \sim e^{tH_{\text{infl.}}}$	$\rho = \text{const.}$
$t_2 < t < t_d$	radiation dominance	$a \sim \sqrt{t}$	$\rho = D_2/a^4$
$t_d < t$	matter dominance	$a \sim t^{2/3}$	$\rho = B/a^3$

The integration constant D_1 and D_2 are now determined by requiring consistency at the times t_1 and t_2, so

$$D_2 = D_1 \; e^{4H_{\text{inflation}}(t_2-t_1)} \tag{2.142}$$

The important feature is that even if D_1 is of order one, then D_2 can be very large. If $H_{infl.}(t_2-t_1) \geq 66$ then one can have $D_2 \geq 10^{114}$ with $D_1 = 0(1)$. Thus, the large value of D_2 can be obtained in a quite reasonable way from a decent value of D_1. Models, where the equation of state (2.133) is satisfied, are called "inflatory" models, and when $a(t)$ blows up exponentially the universe is said to be inflating. As the inflationary period proceeds all fields get red-shifted away exponentially fast. The inflationary period essentially wipes the slate clear of primordial fluctuations (from $0 < t < t_1$).

One may ask whether $H_{\text{infl}}(t_2 - t_1) > 66$ is reasonable. Since $\chi = H_{\text{infl}}$ we need $t_2 - t_1 \geq 10^{-32}$ sec, which is the right time for the inflationary period. In the phase transiton where false goes to true vacuum, the energy stored in the false vacuum is released in the form of new particles (in thermodynamics language the released energy is the "latent heat" of the phase transition). These new particles then come to thermal equilibrium at the temperature $T = 10^{14}$ GeV.

What happens to the causality problem? Start with the present universe, which is causally connected. Going backwards in time we encounter inflation, with its enormous contraction (we go backwards!), producing an incredibly small region out of our universe of size $2t$. In this small region photons can maintain causal contact, so there is no problem. To see this in more details, let us estimate the distance d travelled by light

$$d = a(t_2) \int_0^{t_2} \frac{dt'}{a(t')} = a(t_2) \left(\int_0^{t_1} \frac{dt'}{a(t')} + \int_{t_1}^{t_2} \frac{dt'}{a(t')} \right), \tag{2.143}$$

where the various times are defined above. In the inflationary time interval $t_1 < t < t_2$ we take

$$a(t) = a(t_1) \; e^{H(t-t_1)}. \tag{2.144}$$

In the interval from big bang to t_1 we have

$$a(t) = a(t_1)\sqrt{t/t_1}. \tag{2.145}$$

Hence

$$d \approx (2t_1 + 1/H) \; e^{H(t_2-t_1)}. \tag{2.146}$$

Therefore the causal distance is blown up by an exponential factor, quite compatible with the exponential blow up of the scale factor exhibited in (2.144), where $a(t_1)$ is the size of a *causal* domain just before inflation sets in, i.e.

$$a(t_1) = 2t_1. \tag{2.147}$$

It should be noticed that this scale factor is the size of a *physical* early universe and that it represents a tiny part of the causally non-connected early universe. Also, notice that eq. (2.147) would not work if there is no inflation, since the resulting older universe would be far too small to have anything to do with the universe we know.

After the end of inflation, the universe is radiation dominated for some time. At this stage the causal distance is again $2t$.

What about Ω? We still have Friedmann, $k = \dot{a}^2(\Omega - 1)$. During inflation, $\dot{a}$ increases extremely rapidly, so Ω is driven towards 1, so

$$\Omega_{\text{inflation}} = 1. \tag{2.148}$$

Finally, it should be said that inflation occurs in many versions, and some of them are somewhat different from the intuitive description given above.

2.13 Observational evidence for the cosmological constant

In 1998 evidence for a non-vanishing cosmological constant was found by observing $\approx$ 50 type Ia supernovae out to redshifts of order one[12]. The data was plotted in the magnitude-redshift Hubble diagram. The main point in using distant supernovae is that they are considered to be very good *standard candles*: they are assumed to be associated with the nuclear detonation of white dwarfs. It is completely outside the scope of these notes to describe the theory of white dwarfs[13]. However, because of the knowledge of the blow up mechanism, the intrinsic luminosity and the observed flux are known, and hence the (luminosity) distance can be computed from the ratio of these quantities. In particular, features of the distant sample of Ia's appear to be similar to a nearby sample. So far, there is no serious objection against Ia as a standard candle. Therefore, and because it has become possible to observe high-z objects, the old uncertainty concerning the determination of the distance in Hubble's relation seem overcome.

For nearby galaxies it is sufficient to use the simple Hubble "law", $z = v_r = HL$, where L is the distance and v_r is the radial velocity (see eq. (2.44)). In the standard model, distant galaxies should have larger velocities than predicted by the simple Hubble law, because gravitation slows the expansion. However, the supernovae observations showed precisely the opposite! The conclusion is therefore that the expansion speeds up.

This is in contrast to the standard model without a cosmological constant. However, introducing Λ we saw in connection with eq. (2.111) that the evolution of the velocity $\dot{a}$ changes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{2.149}$$

[12] A. Riess et al., Astrophys. J. 116 (1998) 1009-this paper is most easily found on the net, see astro-ph/9805201- and B. Schmidt et al., Astrophys. J. 507 (1998) 46-see astro-ph/9805200.

[13] See e.g. S. Weinberg, Gravitation and Cosmology, p. 317.

For a positive Λ, in a late universe the two first terms on the right hand side are subdominant, and the expansion is driven by the last term. Remembering that

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G} \quad \text{and} \quad H = \frac{\dot{a}}{a}, \tag{2.150}$$

we obtain the following form for the Friedmann equation,

$$\frac{k}{a^2} = H^2(\Omega_{\text{M}} + \Omega_\Lambda - 1). \tag{2.151}$$

Here $\Omega_{\text{M}} = \rho/\rho_{\text{crit}}$ is the matter density, and $\Omega_\Lambda = \Lambda/3H^2$ is the relative density associated with the cosmological constant, which can be interpreted as a vacuum energy, since it exists irrespective of matter.

By further expansion of $z = (a(t_0) - a(t_1))/a(t_1)$ in eq. (2.52) we obtained a generalization of the simple Hubble law,

$$z \approx H_0 L_0 + \frac{1}{2} H_0^2 L_0^2 (1 + q_0), \tag{2.152}$$

where q_0 is the deceleration parameter today, with

$$q = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{2.153}$$

The interesting thing is that q can be expressed in terms of the Ω's. In the later universe we can ignore the pressure and we have

$$\ddot{a} = -\frac{4\pi G}{3}\,\rho a + \frac{\Lambda}{3}\,a = -H^2 a q \tag{2.154}$$

from the definition (2.153) of q. If we use $H^2 = 8\pi G \rho_{\text{crit}}/3$ and $\Omega_{\text{M}} = \rho/\rho_{\text{crit}}$, eq. (2.154) becomes

$$q = \frac{1}{2}\,\Omega_{\text{M}} - \Omega_\Lambda = \frac{1}{2}\Omega_{\text{tot}} - \frac{3}{2}\Omega_\Lambda. \tag{2.155}$$

Here $\Omega_{\text{tot}} = \Omega_{\text{M}} + \Omega_\Lambda$ is the total energy (ignoring the small contribution from radiation). The name deceleration parameter refers to the expectation that $q > 0$, since $\ddot{a}$ was expected to be negative if there is no cosmological constant (if $\rho + 3p > 0$). However, if Ω_Λ is present, this need no longer be true, and we see that Λ tries to change the sign of q, so Λ produces positive acceleration.

The main point is that eqs. (2.152) and (2.155) supplies astronomers with a method of actually measuring Ω_Λ if Ω_{M} is known from other observations, provided the distance L_0 is known accurately enough. From the high-z supernovae observations mentioned above one finds

$$\Omega_\Lambda \approx 0.7. \tag{2.156}$$

This is consistent with a total energy

$$\Omega_{\text{tot}} \sim 1. \tag{2.157}$$

These numbers are of course expected to have some uncertainty. It is to be noticed that the last result is at least roughly consistent with inflation. For further discussion, we refer to the recent reviews to be found in astro-ph (http://xxx.lanl.gov/list/astro-ph/new).

More recently precise data have been obtained for the fluctuations in the microwave background. These lead again to $\Omega_{\text{tot}} \sim 1$, rougly consistent with inflation. These data are completely independent of the supernovae observations, and are thus not subject to any doubt concerning the mechanism behind supernovae production. Therefore the conclusion that the universe is at least approximately flat is indicated by independent set of observations[14]. It is of philosophical interest that even very precise future measurements will always have an uncertainty. Thus, if it turns out that $\Omega = 1.00000... \pm X$, then the existence of X, no matter how small, will always prevent one from making the statement that the universe is exactly flat. One can only say that with a high probability it is flat[15].

The conclusion is that the cosmological constant is necessary because of the recent observations. Cosmology seems to converge rapidly towards a very precise science where observations determine the parameters. With $\Omega_\Lambda \sim 2/3$ and $\Omega_{\text{M}} \sim 1/3$, the flat universe will go on expanding forever[16]. This gives rize to the problem that we live in a profitable time from the point of view of observations, because in the future astronomers will be able to observe fewer and fewer objects. Also, the cosmological constant gives rize to several other problems. For example, the matter density is of the same order as the density associated with Λ. However, these two densities have different origins (matter versus vacuum) and behave in a different way as functions of the scale parameter. Consequently it is not easy to find an explanation why Ω_{matter} is so close in value to Ω_Λ in our time.

There are several attempts to understand the cosmological constant without really having it (for example, by the use of suitable scalar theories), in such a way that there is temporarily something which looks like a cosmological constant. In this way perhaps the end is not an empty universe. On the other hand, the simplest possible explanation of the data is a constant Λ.

In quantum field theory a vacuum energy emerges naturally. This is precisely what is needed for understanding the cosmological constant. However, this has not led to a Λ of the right order of magnitude. This constitutes one of the most important problems in modern physics.

It is the effort of many human beings to ask for a "meaning" (of life, death, the universe,...). So in cosmology we have the opportunity to ask very precisely what is the meaning of the universe. The cosmological constant represents the vacuum, i.e. "nothing". Most of the energy ($\sim 70\%$) in the universe is thus empty space. So perhaps the answer to the question on the meaning of the universe is that there is no meaning!

In this connection it should be noted that the cosmological constant allowed by Einstein gravity is considered together with the assumptions of the standard model, namely a homogeneous and isotropic universe with an evolution characterized by the cosmological time. Although this may appear reasonable on an extremely large scale, these assumptions are definitely not valid on shorter scales where the universe is inhomogeneous. This may lead to a problem because of the non-linearity of the Einstein field equations which can make the averaging procedure much more subtle than one might naively think. Thus the equation (2.155) used to analyze the Supernovae data depends on the Ω's because of Einstein field equations. If we think of the various quantities in (2.155) as large scale

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[14] The data from the fluctuations in the microwave background actually have a statistical preference for a total Ω which is slightly larger than one, but Ω can be one within a standard deviation.

[15] This difficulty is due to the fact that $\Omega = 1$ is a borderline case. If, for example, it had turned out that $\Omega \approx 0.3$, no such problem of principles would occur.

[16] See, however, the previous footnote

averages, it is not obvious that there are no corrections coming from the non-linearities.

2.14 The end of cosmology?

Lawrence Krauss and Robert Scherrer (see arXiv:0704.0221, where further references can be found) have investigated some of the depressing consequences of the cosmological constant. As we have seen in eq. (2.149), ultimately there will be an expansion driven by the cosmological constant. In approximately 100 billion years almost all galaxies will be invisible, except the Local Group of six galaxies which are gravitationally bound and do not participate in the large scale expansion. Observers in this "island universe" will be fundamentaly incapable of determining the true nature of our universe, as Krauss and Scherrer say. Of course, there will be no such observers on earth, since its life time is only of the order 4.5 bilion years, but perhaps there are observers some other place in the Milky Way, perhaps having the same DNA as us.

These observers will have absolutely no way of knowing that the universe expands and will think that the universe is *static*. This was Einstein's starting point, corrected by astronomical observations. This was also the reason for his introduction of the cosmological constant!

In the island universe there is no way of recognizing that there was a Big Bang. Due to the expansion, the cosmic microwave background will disappear in the sense that the wavelenghts will be shifted and totally buried by radio noise in our galaxy. So perhaps these observers will spend their time producing a lot of papers on why there are six (as contrasted to any other number) galaxies in the whole universe.

Although the return of the static universe is not of immediate concen, philosophically the situation in the island universe is interesting, because the observers can have the right physics but are unable (no matter how good they are) to get the right cosmology because of lack of observational support. One may wonder if the same thing is happening today-are we also beeing deceived like the islanders (or like the man who came to Casablanca for the waters)?

2.15 An inhomogeneous universe without Λ?

Recently there has been much interest in the study of an inhomogeneous universe in order to see if it is possible to avoid the cosmological constant. Of course, from CMB we know that the early universe is to a high degree of accuracy homogeneous. Hopwever, at early times Λ plays no role.

For the universe at the present epoch one basic consideration is that even though the universe now appears to be homogeneous at very large scales, observationally there are *voids* of considerable size, surrounded by clusters of galaxies. It is estimated that 40-50 per cent of the present universe consists of voids with diameter of order $30/h$ Mpc. We refer the reader to a paper by David L. Wiltshire (Phys. rev. Letters **99**, 251101 (2007)), where further references can be found. The necessary formalism is somewhat complicated, but the physical idea is simple: It is necessary to average over the different structures that have different time rates, because the time depends on the gravitational field. Also, in the voids there is very little matter, so this part of the universe has a negative curvature metric ($k = -1$). Therefore volumes are larger, and hence densities are smaller than expected in the standard cosmological model. It then turn out that the deceleration parameter q is given by

$$q = -\frac{(1-f)(8f^3 + 39f^2 - 12f - 8)}{(4 + f + 4f^2)^2}, \tag{2.158}$$

where f is the void volume fraction. Here we see that in spite of having no cosmologaical constant, q changes sign. At early times, there are no voids, so $f = 0$, giving $q = 1/2$. But q changes sign for $f = 0.5867$, corresponding to acceleration (*not* due to Λ!). The apparent acceleration becomes maximal for $f = 0.7736$ with $q = -0.043$. At later times q approaches zero, i.e. the acceleration disappears and there will be no island universe!

Chapter 3

Problems

Problem 1: Consider cylindrical coordinates:

$$x = r\cos\phi \quad , \quad y = r\sin\phi \quad , \quad z = z$$

Find the metric expressed in terms of these coordinates. Find the Laplace operator in these coordinates.

Do the same for spherical coordinates:

$$x = r\cos\phi\sin\theta \quad , \quad y = r\sin\phi\sin\theta \quad , \quad z = r\cos\theta \quad .$$

Check the correctness of the resulting Laplace operator by comparing your result with some book (e.g. on quantum mechanics).

Problem 2: Let A_μ be a quantity consisting of four components, A_0, A_1, A_2, A_3. Assume that

$$A_\mu B^\mu = \text{invariant}$$

for an arbitrary vector B^μ. Show that then A_μ is a vector.

Generalize this result to arbitrary tensors.

The resulting theorem is called the "quotient–theorem".

Problem 3: Let $S_{\mu\nu} = S_{\nu\mu}$ be a symmetric tensor and $A_{\mu\nu}$ an antisymmetric tensor, $A_{\mu\nu} = -A_{\nu\mu}$. Show that $S_{\mu\nu}A^{\mu\nu}$ is identically zero.

Show that an arbitrary tensor with two indices can be written as the sum of a symmetric and an antisymmetric tensor.

Problem 4: Consider a two–dimensional surface of a sphere, defined by

$$x^2 + y^2 + z^2 = R^2 \quad .$$

What is the metric expressed in terms of x and y ?

Find the metric in spherical coordinates.

Problem 5: Assume that $V^\nu{}_{;\mu} = \partial V^\nu/\partial x^\mu + \Gamma^\nu_{\mu\sigma}V^\sigma$. Then show that $V_\rho = g_{\rho\nu}V^\nu$ satisfies eq. (1.61).

Problem 6: Use the energy-momentum tensor (1.157) in the differential equation (1.158) to derive the special relativistic fluid equation (1.159). Hint: Consider (1.158) for $\mu = 0$ and $\mu = 1, 2, 3$ and combine these results.

Problem 7: Consider the Einstein field equation with a positive cosmological constant. Assume spherical symmetry and $T_{\mu\nu} = 0$. Rewrite the field equation in the form $R_{\mu\nu} = ...$, where ... contains only Λ and $g_{\mu\nu}$. Hint: Compute the Ricci scalar in terms of Λ.

Because of spherical symmetry, the metric (1.191) can be used. The two functions $E(r, t)$ and $F(r, t)$ are then to be determined from the Einstein equation you just derived. Use the results in eq. (1.217) to determine the solution.

The result should be

$$E(r, t) = 1/F(r, t) = 1 - \frac{\Lambda}{3}\, r^2. \tag{3.1}$$

Discuss in details why the time dependence disappears. Hint: Look at R_{tr}. Please note that this metric does not become flat at $r \to \infty$. Where is it flat?

Consider a test particle which starts from $r = 0$ with some initial velocity and moves in a radial direction. Derive the relevant equations of motion by help of the Euler-Lagrange method. Consider r as a function of proper time τ and coordinate time t. Show that $r(\tau)$ ultimately becomes infinite for $\tau \to \infty$ whereas $r(t)$ approaches a finite value (which one?) for $t \to \infty$.

Problem 8: Consider the deflection of light around a black hole described by the Schwarzschild metric. Show that a photon (light particle) can move around in a circular orbit $r = 3GM$. Are there any other possible values of r?

Problem 9: *The de Sitter Universe:* In Problem 7 we found the solution of the Einstein equations with a cosmological constant Λ, assuming spherical symmetry. For an observer situated at $r = 0$ measuring the coordinate time t the universe looks rather empty: Show that light emitted from $r = 0$ will be infinitely red shifted at $r = \sqrt{\Lambda/3}$, and hence will practically speaking be unobservable after some finite time.

Show that the metric found in problem 7 can be expressed as

$$d\tau^2 = d\tilde{t}^2 - a(\tilde{t})^2(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2), \tag{3.2}$$

where $a(t)$ is called the scale factor. Hint: Use the transformations (t, r are the coordinates used in Problem 7)

$$t = \tilde{t} - \frac{1}{2\sqrt{\Lambda/3}}\, \ln\left(1 - \frac{\Lambda}{3}\, \tilde{r}^2 e^{2\sqrt{\Lambda/3}\,\tilde{t}}\right), \tag{3.3}$$

and

$$r = \tilde{r} e^{\sqrt{\Lambda/3}\,\tilde{t}}. \tag{3.4}$$

Find the scale factor. Discuss the resulting (de Sitter) universe.

Chapter 4

Some constants

General constants

Speed of light (This value is "exact" in the sense that the meter is defined - since 1983 - as the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 s).	c	$= 2.99\ 792\ 458\ \times 10^{10}$ cm s^{-1}
Planck's constant	h	$= 6.626\ 176(36) \times 10^{-27}$ erg s
	$\hbar$	$= h/2\pi = 1.054\ 5887(57) \times 10^{-27}$ erg s
		$= 6.582\ 173(17) \times 10^{-22}$ MeV s
Electron charge	e	$= 4.803242(14) \times 10^{-10}$ esu
		$= 1.6021892(46) \times 10^{-19}$ Coulomb
Conversion constant (useful in converting to "natural units" $\hbar = c = 1$)	$\hbar c$	$= 197.32858(51)$ MeV fm
Fine structure constant	α	$= e^2/\hbar c = 1/137.03604(11)$
	1 fm	$= 10^{-13}$ cm
	1 eV	$= 1.6021892 \times 10^{-12}$ erg
Boltzmann's constant	k	$= 1.381 \times 10^{-23}$JK^{-1}$= 8.617 \times 10^{-5}$eVK^{-1}

As of may 20. 2019 Planck's constant, the electron charge and Boltzmann's constant are defined as(see hyperlinks):

h = 6.626 070 15 *10^{-34} Js

e = 1.602 176 634 *10^{-19} C

k = 1.380 649 *10^{-23} JK^{-1}

Constants in General Relativity and Astronomy

Light year	1 l.y.	=	9.4605×10^{17} cm
Parsec	1 pc	=	$3.0856(1) \times 10^{18}$ cm
		=	3.2615 l.y.
Solar mass	$M_\odot$	=	$1.989(2) \times 10^{33}$ g
Solar radius	$R_\odot$	=	$6.9598(7) \times 10^{5}$ km
Earth mass	$M_\oplus$	=	$5.977(4) \times 10^{27}$ g
Earth equatorial radius	$R_\oplus$	=	$6.37817(4) \times 10^{3}$ km
Mean earth–sun distance			$1.495985(5) \times 10^{13}$ cm
Gravitational constant	G	=	$6.6732(31) \times 10^{-8}$ dyn cm^2 g^{-2}
	G/c^2	=	7.425×10^{-29} cm g^{-1}
Hubble constant	H_0	=	70 ± 7 km s^{-1} Mpc^{-1}
Hubble time	H_0^{-1}	=	$(4.4 \pm .4) \times 10^{17}$ s=14.0 ± 1.4 Gyr
Planck length		=	$\sqrt{G\hbar/c^3} = 1.616 \times 10^{-32}$cm
Planck mass		=	$\sqrt{\hbar c/G} = 2.177 \times 10^{-5}$g
Planck time		=	$\sqrt{G\hbar/c^5} = 5.391 \times 10^{-44}$s

Chapter 5

Some literature

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