

Notes on Conformal Field Theory

Lecture notes

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1 Motivation

Two dimensional conformal field theory (CFT) has been developed for about 10 years now, starting with the seminal paper by Belavin, Polyakov and Zamolodchikov, and has proven to be an extraordinarily rich field of mathematical physics. It is common to list three main areas of application.

1.1 String theory

In string theory, conformal field theory (CFT) plays a very important if not dominant role. Current formulations of string theory are not quite satisfactory, but they give an idea of the importance of CFT. When the string travels through space and time it sweeps out a world surface, a world sheet. One may imagine quantum string theory being formulated in terms of path integrating over histories of such world sheet embeddings. If we parametrize the world sheet by coordinates, (z_1, z_2) , we see that for any one embedding, the space time position, $X^\mu(z_1, z_2)$ of this point may be regarded as a two dimensional field, or, in d space-time dimensions, a d -component 2-dimensional field. Path integrating over this field should be more or less equivalent to summing over embeddings. Thus we are led to realize that a general string theory is formulated in terms of some 2-dimensional quantum field theory. Quantum states of the string (states with a certain number of vibrational phonons for example) are thought to provide models for elementary particles at the most fundamental level. This feeling is further supported by the fact that among these particles one always finds gauge bosons interacting with other string states as prescribed in gauge theories, much as it is observed in the Standard Model of elementary particle physics. Also one always finds gravitons interacting with other string states as prescribed by Einstein's general theory of relativity (in the "low energy limit", i.e. for energies much smaller than the Planck constant).

The precise set of gauge bosons and the infinite generalizations thereof carried by the other Planck mass string states, depends on the geometry of the space in which the string is taken to propagate, or equivalently, to the 2-dimensional quantum field theory describing the string coordinates as functions of world sheet coordinates. Thus one is led to realize the deep connection between these two things: (i) the geometry of the background embedding space, and (ii) the nature of the 2-dimensional quantum field theory. Complicated embedding spaces with curled up dimensions are known now to be capable of providing more or less realistic scenarios for a string theory describing Nature - although an enormous amount of work remains before such a statement can be made really believable. (As an aside: The connection between the geometry of the background and the CFT is not one-to-one; rather each CFT corresponds to two different so-called mirror manifolds; we shall not at all dwell on such subtleties).

In general relativity we are used to the situation where not all possible geometries are actually physically, or better classically, allowed. Only the ones satisfying Einstein's equations of motion are true classical solutions. In a string theory the analogous question provides the fundamental link to conformal field theory. Indeed it turns out, that demanding the background space to satisfy classical equations of motions, is equivalent to demanding the 2-dimensional quantum field theory to be a *conformal field theory*. What we do in string theory when we study conformal *quantum* field theory is thus equivalent to studying *classical solutions* to string theory, with quanta interacting according to tree

diagrams. Perturbative quantum corrections are dealt with by allowing the string world sheet to have a more and more complicated topology, the number of handles playing the role of the number of loops in ordinary field theory Feynman diagrams.

As we shall see and study in quite some detail, the conformal field theories are characterized by possessing an infinite dimensional symmetry algebra, the conformal or Virasoro algebra. A particular field theory has a Hilbert space providing a representation space of this algebra. The most important parameter characterizing the representation is the so-called *central charge* commonly denoted by the letter c . The numerical value of this number is fixed in string theory, but depends on the geometrical structure assumed for the world sheet. Thus for the simplest case of an ordinary differential geometry, the central charge has to be 26. A very primitive way of obtaining this number is to take the string to propagate in 26 flat space-time dimensions. This has led to the incorrect notion among non-experts, that string theory gives rise to such wildly unphysical dimensions. Today one knows of millions of other conformal field theories describing all sorts of complicated manifolds, some of which also possess 4 flat space-time dimensions. The point I want to make, however, is that the value of c plays a very important role in string theory. The value, 26, gets modified to different values if the world sheet geometry is more complicated than a simple Riemann geometry. Such generalized geometries currently attract much attention as promising candidates for better models and formulation.

In this course practically no more will be said about string theory, even though that provides my own personal motivation for being interested in CFT.

1.2 Two-dimensional critical systems.

The second traditional area of application of CFT is to 2-dimensional critical phenomena. The more recent applications to the Quantum Hall effect and other branches of condensed matter physics, I shall say nothing about. The relation to critical phenomena, however, will serve to illustrate most of the things we shall discuss. Part of the relation of course depends on the fact that euclidean quantum field theory and statistical mechanics employ identical mathematical formalisms, so that we shall often be sloppy about whether we use the relevant language for one or the other.

We shall illustrate CFTs on the critical behaviour of the two dimensional Ising model in particular. But one knows an infinity of “generalized Ising models” for which the different kinds of criticality have been beautifully classified by means of CFT.

Let us attempt an intuitive discussion of why a critical system – in particular in 2 dimensions – gives rise to conformal invariance.

Let us think in terms of an Ising like model formulated on a square lattice. As is well known, in the disordered phase, close to criticality one observes a characteristic correlation length, $\xi(T)$ depending on the temperature, T in such a way that when the temperature tends to a critical value, $T \rightarrow T_c$, the correlation length (measured in lattice units) diverges, $\xi(T) \rightarrow \infty$. The presence of a correlation *length* is equivalent to the presence of a *mass gap* in the theory, the smallest mass being the inverse of the correlation length. Such a theory of course is not scale invariant: there are features in the theory explicitly referring to something having a scale, either the correlation length or the mass.

Let us imagine a simulation of the Ising model on some huge computer monitor screen, with pixels white or black depending on the two possible values of the Ising spin. Close to criticality we would observe fluctuating patterns with “clumps” of the size of the

correlation length. *At* criticality, on the other hand there are fluctuations *at all scales*, no scale in particular being singled out, and indeed the fluctuating patterns look just the same whether we look at our (foot ball field size) television screen from a distance of many meters, or closer by: If not, there were a correlation length, a scale after all. Moving in on the television screen is a renormalization group transformation, and the statement that things look just the same, is the statement that we are at a renormalization group fixed point, i.e. beta functions are all zero, i.e. we have *scale invariance*.

This is the defining property of a conformal field theory. The fluctuating patterns look the same at all scales. Also it does not matter if we tilt our head either: we have rotational invariance (2-dimensional Lorentz invariance in the quantum field theory). This last property does not hold if we are so close we can distinguish the square lattice, but at distances much larger than that the lattice becomes irrelevant, exactly when the correlation length becomes infinite in lattice units.

The occurrence of scale invariance is unlike in a theory like QCD in which there is a non vanishing beta function and a “running coupling constant”, meaning that fluctuations interact with different effective strengths (the coupling constant) depending on the scale at which we observe them. In CFT’s all couplings are at fixed points, and we have this phenomena in the case of critical fluctuations in a system having a 2nd order phase transition.

Actually it should also be clear now, that we have a little more than that, or as it turns out, an infinity more than that. Indeed we have *local* scale and rotational invariance. The group of such transformations is exactly the conformal group: if we take a small (a local) object and subject it to a dilatation (shrink it or expand it) and a rotation, we have effectively mapped it so that it preserves its shape: it has been subject to a *conformal transformation*.

It is easy to see that the group of conformal transformations in two dimensions is just the group of analytic reparametrizations (or anti analytic reparametrizations). Strictly speaking, since any non-trivial (i.e. non constant) analytic function becomes singular at one or more points, these reparametrizations only have a local meaning. We shall often forget about this subtlety when we talk about them, but we do have to come back to this for example when we derive the fundamental conformal Ward identity.

Consider an analytic function, $z \mapsto f(z)$. It is easy to verify that locally this gives conformal transformations. Indeed, consider a point, z_0 together with two small increments dz_1, dz_2 . The three points, $z_0, z_0 + dz_1, z_0 + dz_2$ are mapped to the three new points, $f(z_0), f(z_0 + dz_1) = f(z_0) + f'(z_0)dz_1, f(z_0 + dz_2) = f(z_0) + f'(z_0)dz_2$. This shows that the triangle formed by the three points is obtained by dilating the original triangle by the real number $|f'(z_0)|$ and rotating it by the argument of $f'(z_0)$. A similar argument of course holds for analytic functions of the complex conjugate, \bar{z} , of z . We shall come back to the algebra of conformal transformations presently.

There can be no doubt that one of the reasons conformal field theories may be treated in such detail and with such exactitude is, that they have inherited many of the beautiful properties of the theory of analytic functions.

1.3 Applications to mathematics in general and group theory in particular.

No doubt this will be one of the lasting contributions of conformal field theory. Let us hope, it will not turn out to be the most important one.

At any rate, conformal field theory is closely related to the study of certain infinite dimensional Lie algebras, in particular the Virasoro algebra, which is the only one with which we shall be concerned. As such, the subject has given a boost to the mathematical understanding of infinite dimensional Lie algebras. In addition all sorts of beautiful connections have turned up, relations to Riemann surface theory, knot theory, integrable hierarchies, etc. etc.

At this point let us briefly say a few first features concerning the conformal algebra.

“Any standard field theory” which we might want to study, would often be required to be Poincaré invariant. In two dimensions this means invariance under (2-d) space-time translations, and invariance under (2-d) Lorentz transformations. Here we shall always think in terms of a rotation to euclidean (imaginary) time. In that case, the Lorentz transformations are simply the $U(1)$ transformations rotating the point $z = z_1 + iz_2$ by a real angle:

$$z \mapsto e^{i\phi} z$$

($\phi \in \mathbb{R}$). The set of space time translations is just the set

$$z \mapsto z + a$$

$a \in \mathbb{C}$.

Now that we are interested in a *conformal* field theory, we are furthermore interested in theories invariant under real dilations (scale transformations)

$$z \mapsto \lambda z$$

$\lambda \in \mathbb{R}$. We see that together with the requirement of Lorentz invariance these two may be expressed compactly as

$$z \mapsto bz$$

b complex.

The two kinds of transformations $z \mapsto z + a$ and $z \mapsto bz$ are the only conformal ones globally defined on the complex plane. We shall be interested (first) in theories defined on the *compactified* plane or *Riemann sphere*. One may think of it as being obtained by a stereographic mapping of the plane onto the sphere. On the Riemann sphere “the point at infinity” is just the “South Pole” and plays no particularly singular role. On the Riemann sphere, therefore, there is one more kind of globally defined transformation: The inversion

$$z \mapsto 1/z$$

If we combine all of those globally defined transformations we easily see that they may all be described by the transformations

$$z \mapsto \frac{az + b}{cz + d} \tag{1}$$

for a, b, c, d complex parameters. Of course this fraction is invariant under simultaneous multiplication by the numerator and denominator by a common constant. Thus we may impose a constraint such as

$$ad - bc = 1 \tag{2}$$

In other words, the transformations are characterized by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

with determinant 1. Furthermore, combining two such mappings each being described by a 2×2 matrix, we obtain a new one, described exactly by the 2×2 matrix which is the matrix product of the two first ones. (Show that!) Also it has determinant 1 again. We see therefore, that the set of globally defined conformal transformations on the sphere is the group, $SL(2, \mathbb{C})$, namely the group of 2×2 complex matrices which have determinant one: The Special (determinant = 1) Linear (matrices) group in 2 dimensions over Complex numbers. (Strictly speaking, the relevant group is $SL(2, \mathbb{C})/\mathbb{Z}_2$, since matrices differing by a factor -1 will correspond to the same transformation. For the most part one is sloppy about this point.)

The reader is invited to work out how the coordinate grid is changed under such a transformation and make several pretty drawings. Small squares in the grid are mapped into small squares again (a conformal transformation), but having different sizes and orientations.

Exercise

The group $SL(2, C)$ will play a special role for us. It is clearly a Lie group with 3 complex parameters and therefore 3 generators, which we shall soon learn to denote:

$$L_{-1}, L_0, L_1$$

They will form the 3-dimensional *algebra*, $sl(2, C)$, a sub algebra of the Virasoro algebra spanned by $\{L_n | n \in \mathbb{Z}\}$.

Let us get a first idea about this infinite dimensional algebra.

Consider representing “the group of conformal reparametrizations” on classical scalar functions (scalar fields), as follows:

Let $\phi(z)$ be a scalar field, and let $f : z \mapsto f(z)$ be a conformal (analytic) transformation. Also, let f^{-1} be the inverse transformation (locally defined): $f^{-1}(f(z)) = z = f(f^{-1}(z))$. Define the action on the field, ϕ , by

$$f : \phi \mapsto \phi^f$$

by

$$\phi^f(z) \equiv \phi(f^{-1}(z)) \tag{4}$$

The group property works like this:

$$\phi \xrightarrow{f} \phi^f \xrightarrow{g} (\phi^f)^g$$

Here

$$(\phi^f)^g(z) = \phi^f(g^{-1}(z)) = \phi(f^{-1}(g^{-1}(z))) = \phi((g \circ f)^{-1}(z))$$

Thus

$$(\phi^f)^g = \phi^{g \circ f} \tag{5}$$

To find the algebra, consider the infinitesimal transformation

$$z \mapsto f(z) = z + \epsilon(z)$$

with $f^{-1}(z) = z - \epsilon(z)$. Then

$$\phi^f(z) = \phi(z - \epsilon(z)) = \phi(z) - \epsilon(z)\partial\phi(z) = (I - \epsilon(z)\partial)\phi(z) \tag{6}$$

showing that the generator corresponding to the group element, f is

$$-\epsilon(z)\partial$$

(∂ denotes differentiation after the complex variable, z , not z_1 or z_2 , nor \bar{z}). Clearly the set of generators form a vector space, the conformal algebra: if $\epsilon_1(z)\partial$ and $\epsilon_2(z)\partial$ are generators, so is the sum. It is often convenient to choose a particular basis for the set of generators, Thus, if we consider functions, f , holomorphic in the neighbourhood of the unit circle, we may expand in a Laurent series

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1} \tag{7}$$

(the numbering being a matter of convention, but see below). In other words, we expand on the set of basic mappings, $z \mapsto z^{-n-1}$. Then our generator of the conformal transformations may be written as

$$-\epsilon(z)\partial = \sum_{n \in \mathbb{Z}} \epsilon_n \ell_n \tag{8}$$

where

$$\ell_n \equiv -z^{n+1}\partial \tag{9}$$

One easily verifies, that acting on scalar fields, these generators, satisfy the commutation relations

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n} \tag{10}$$

This algebra is often referred to as the *Loop algebra of reparametrizations of the circle*. Indeed, letting $z = e^{i\theta}$ on the unit circle, we see that the Laurent expansion is just the Fourier expansion there, and we are considering an element of the group of mappings of the unit circle into the real numbers (for ϵ_n real), or better, the real translation group.

The interesting point is that the Virasoro algebra, as we shall see, is of the following very similar form, referred to as the *central extension of the Loop algebra*:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \tag{11}$$

in which the parameter, c appears as the “central extension” (with a normalization chosen on the basis of convention).

One knows by now an infinity of *extensions* (not the above *central* extension, but non-trivial extensions involving an infinity of new generators) of the Virasoro algebra, containing the latter as a sub algebra. In many cases they have been analyzed and proven similarly (remarkably) tractable from a mathematical point of view. They play a dominant role in the development of string theory. We shall not however, be concerned with them in this course.

1.4 But also...

The above are the standard reasons usually put forward for being interested in CFTs.

Here I want to add a couple of additional points of a more general nature.

The traditional approach to quantum field theory takes the following form: One begins with a formulation of some *classical* field theory, and then one proceeds to *quantizing* it, for example by a “canonical quantization” or by a “path integral quantization”. These techniques are beautiful in themselves, but in fact are foreign to the concept of a quantum theory. All they ensure, really, is a procedure to obtain *a quantum theory* some way or other, and the *quantization procedure* implies, that in some appropriate classical limit, the quantized theory reduces to the classical theory from which we started.

There are strong conceptual arguments against considering such a procedure a fundamental one. For example one of the most beautiful and striking results in quantum theory is the “particle-wave duality”, saying among other things that a quantum field theory has (at least) two radically different “classical limits”: One in which the theory reduces to a classical field theory, and one in which it reduces to a classical particle theory. The quantization procedure breaks this symmetry, by focusing arbitrarily on one of those from the outset. One may of course focus on the other – like Feynman did when he proposed his rules for QED – but that does not really restore the symmetry *ab initio*. One is left with the impression that the quantum theory should be perceived in its own right without reference to any classical theory at all from which it was quantized. Quite on the contrary, it is the classical theory which should be considered as nothing but a certain limit of quantum theory.

In addition, as is well known, the quantum field theory obtained after “quantization”, usually has to have its very formulation reconsidered in terms of the renormalization programme, defining it in a still more subtle way in terms of a singular limiting procedure.

Here CFT’s come in as extremely instructive examples of quantum field theories often built from the beginning as *the final complete theory* without reference to any classical theory, nor to any quantization procedure, nor to any renormalization programme. The theories are simply there as full quantum theories. In addition, the CFT’s usually may be solved exactly (!) in several ways.

This being said, it is also often of crucial *practical* interest, nevertheless to think about them in more traditional ways as well, but they do in fact provide striking examples of what might perhaps be termed “a more fundamental paradigm”.

2 Fundamental Concepts in Conformal Field Theory

2.1 Conformal tensors and primary fields

In any quantum field theory there is usually an infinity of quantum field operators to worry about. Thus, if $\phi(z_1, z_2)$ is a scalar field, we also have to consider all possible derivatives of that field, $\partial_1^{n_1} \partial_2^{n_2} \phi(z)$. In a CFT, these operators are neatly grouped into infinite families forming representations of the conformal algebra, called highest weight representations. The highest weight member is called a *primary field*, the others *descendant fields*. In very many cases of interest, it may then happen that even though there is an infinity of field operators, there is only a finite number of such conformal families, each being characterized by its primary member. This is a tremendous simplification since as we shall see, knowledge of correlators among primary fields will suffice to calculate *any* correlator in the theory relatively easily in terms of the former.

Let us motivate the properties taken for the primary fields by first considering *conformal tensors*.

First consider general tensors of the kind considered in general relativity. Let

$$T = T_{i_1 i_2 \dots i_n}(z_1, z_2) dz^{i_1} dz^{i_2} \dots dz^{i_n}$$

be a rank- n tensor. Let

$$(z_1, z_2) \rightarrow (z'_1, z'_2)$$

be a general coordinate transformation, and let

$$T' = T'_{j_1 j_2 \dots j_n}(z'_1, z'_2) d(z')^{j_1} d(z')^{j_2} \dots d(z')^{j_n}$$

denote the tensor in the new coordinate system. Then

$$T = T'$$

from which follows

$$T_{i_1 i_2 \dots i_n}(z_1, z_2) = T'_{j_1 j_2 \dots j_n}(z'_1, z'_2) \frac{\partial z'_{j_1}}{\partial z_{i_1}} \dots \frac{\partial z'_{j_n}}{\partial z_{i_n}} \quad (12)$$

In CFT, it is convenient to introduce

$$\begin{aligned} z &\equiv z_1 + iz_2 \\ \bar{z} &\equiv z_1 - iz_2 \end{aligned} \quad (13)$$

denoted as holomorphic and anti holomorphic variables, or as left movers and right movers, or as light cone coordinates, depending on the context.

Under a *conformal transformation* we have

$$\begin{aligned} z' &= z'(z) \\ \bar{z}' &= \bar{z}'(\bar{z}) \end{aligned} \quad (14)$$

so that z' is an analytic (conformal) function of z , independent of \bar{z} (this is equivalent to the function being an analytic function of z), and similar for \bar{z}' . In this basis, it is customary to denote indices corresponding to these variables as z, \bar{z} also. Thus, a vector

field, $V(z_1, z_2)$ will have components, V_1, V_2 in the original basis, and components, $V_z, V_{\bar{z}}$ in the new basis.

Using the standard transformation rules, eq.(12), we get in a new coordinate system

$$\begin{aligned} V_z(z, \bar{z}) &= V'_z(z', \bar{z}') \frac{\partial z'}{\partial z} + V'_{\bar{z}}(z', \bar{z}') \frac{\partial \bar{z}'}{\partial z} \\ &= V'_z(z', \bar{z}') \frac{dz'}{dz} \end{aligned} \quad (15)$$

Similarly

$$V_{\bar{z}} = V'_{\bar{z}} \frac{d\bar{z}'}{d\bar{z}} \quad (16)$$

This feature is generic: the *sum* over indices j_m in eq.(12) reduces to a *single term*, all others being zero because the corresponding partial derivatives are zero for analytic functions. We see the crucial new feature, that the components individually, have nice transformation properties, not just the whole set of components.

Thus, if for a rank n tensor we consider the component with h z -like indices and \bar{h} \bar{z} -like indices, then it transforms as follows,

$$T_{z\dots z\bar{z}\dots\bar{z}} = T'_{z\dots z\bar{z}\dots\bar{z}} \left(\frac{dz'}{dz} \right)^h \left(\frac{d\bar{z}'}{d\bar{z}} \right)^{\bar{h}} \quad (17)$$

This component we shall call a *primary field of dimension (h, \bar{h})* . We shall often denote it as $\phi^{(h, \bar{h})}(z, \bar{z})$, and we may summarize, by stating that for such a primary field

$$\phi^{(h, \bar{h})}(z, \bar{z}) dz^h d\bar{z}^{\bar{h}}$$

is invariant under conformal transformations.

The above discussion applies to tensors for which the numbers, (h, \bar{h}) will always be non-negative integers. However, we shall generalize the concept of a primary field to include also the cases where the dimensions are any numbers at all. We shall mostly think in terms of real numbers, and we shall find that for unitary quantum field theories the dimensions will have to be non-negative. Also for so-called *rational* CFT's, the ones with only a finite number of primary fields, the dimensions will turn out to be positive rational numbers. Thus, in the case of the critical Ising model, we shall find the numbers, $0, \frac{1}{2}, \frac{1}{16}$. A particularly relevant and well known example is provided by a 2-dimensional spinor, for which $h = \bar{h} = \frac{1}{2}$, corresponding to the fact that a spinor field is only well defined on the double cover: it changes sign under a rotation by 2π . But in CFT's it makes good sense to consider arbitrary values of the dimensions.

Since we shall concentrate on the *conformal algebra* much of the time, we shall be interested in *infinitesimal* transformations. First consider a scalar field, i.e. one of dimension $(0, 0)$. It is unchanged under conformal transformations, but the coordinates corresponding to the event at which it is evaluated (the site in statistical mechanics) is changed. Let

$$\begin{aligned} z &= z' + \epsilon(z) \\ \bar{z} &= \bar{z}' + \bar{\epsilon}(\bar{z}) \end{aligned} \quad (18)$$

denote the infinitesimal transformation. Then

$$\phi(z, \bar{z}) = \phi'(z - \epsilon(z), \bar{z} - \bar{\epsilon}(\bar{z})) = \phi'(z, \bar{z}) - \epsilon(z)\partial\phi(z, \bar{z}) - \bar{\epsilon}(\bar{z})\bar{\partial}\phi(z, \bar{z}) \quad (19)$$

We summarize this in the following infinitesimal form for the transformation

$$\phi'(z, \bar{z}) - \phi(z, \bar{z}) \equiv \delta_\epsilon \phi(z, \bar{z}) = \epsilon(z)\partial\phi(z, \bar{z}) + \bar{\epsilon}(\bar{z})\bar{\partial}\phi(z, \bar{z}) \quad (20)$$

Notice that in this formulation the arguments of the fields used to define the variation, refer to the same values of the coordinates for both fields, not in fact to the same event (or site in stat. mech.). For a primary field of dimension h we have to work out also

$$\left(\frac{dz}{dz'}\right)^h = 1 + h\epsilon'(z)$$

($\epsilon'(z) \equiv \partial\epsilon(z)$) Thus we get for a primary field of dimension $(h, 0)$

$$\delta_\epsilon \phi(z) = \epsilon(z)\partial\phi(z) + h\partial\epsilon(z)\phi(z) \quad (21)$$

This is the most convenient form to remember for the conformal transformation of a primary field. If $\bar{h} \neq 0$ there are additional terms involving \bar{z} completely analogous to the above. Also we have written down the rule as if ϕ did not depend on \bar{z} at all. If this is the case, the field (which is not really a proper field, but just a very useful building block) is called a *chiral vertex operator*. Very often we shall be sloppy in our notation and only distinguish between these various things when it is essential for the context.

2.2 Examples. Free fields

Let us go over the examples of free scalar fields and free fermionic fields. These will provide useful and surprisingly interesting examples, that may in fact be taken to far greater generality than we shall have time to consider here.

A free scalar theory is given by the action

$$\begin{aligned} S[\phi] &= \frac{\sigma}{4} \int d^2z \partial_i \phi \partial_i \phi \\ &= \sigma \int d^2z \partial \phi \bar{\partial} \phi \end{aligned} \quad (22)$$

where we have switched to holomorphic and anti holomorphic coordinates in the last line, and where we have inserted a constant, σ , later to be disposed of by choice of normalization for the field. Our notation is that

$$\begin{aligned} \partial &= \frac{1}{2}(\partial_1 - i\partial_2) \\ \bar{\partial} &= \frac{1}{2}(\partial_1 + i\partial_2) \\ d^2z &= dz_1 \wedge dz_2 = \frac{i}{2} dz \wedge d\bar{z} \end{aligned} \quad (23)$$

The path integral corresponding to some external current, J , takes the form

$$\begin{aligned}
Z[J] &= \int \mathcal{D}\phi \exp\{-\sigma \int d^2z \partial\phi \bar{\partial}\phi + \int d^2z \phi \cdot J\} \\
&= \int \mathcal{D}\phi \exp\{\sigma \int d^2z \{[\phi + \frac{1}{2\sigma} J \partial^{-2}] \partial \bar{\partial} [\phi + \frac{1}{2\sigma} \partial^{-2} J]\} \\
&\quad \times \exp\{-\frac{1}{4\sigma} J \cdot \partial^{-2} \cdot J\}
\end{aligned} \tag{24}$$

Here we used functional notation ($J \cdot \partial^{-2} \cdot J = \int d^2x d^2y J(x) \partial^{-2}(x, y) J(y)$) and completed the square as usual in Gaussian integrals. We also introduced the propagator, ∂^{-2} , which is easily found to be related as usual to the 2-point function. For an appropriate choice of the normalization of the field, i.e. for an appropriate choice of the constant, σ , we obtain

$$\begin{aligned}
\langle \phi(z) \phi(w) \rangle &= \frac{\delta}{\delta J(z)} \frac{\delta}{\delta J(w)} Z[J] \\
&= -\partial^{-2}(z, w) = -\log|z - w|^2 = -\log(z - w) - \log(\bar{z} - \bar{w})
\end{aligned} \tag{25}$$

Indeed we may verify that acting with $\partial \bar{\partial}$ on the logarithm produces a delta function,

$$\bar{\partial}_{\bar{z}} \partial_z (\log(z - w) + \log(\bar{z} - \bar{w})) = \bar{\partial}_{\bar{z}} \frac{1}{z - w} \tag{26}$$

(This calculation is quite formal and has to be taken with care. For a safer treatment see the exercise below.) We want to show that this is a delta function situated at $z = w$. Of course it is identically zero whenever $z \neq w$. To find out whether there is a delta function distribution, we make use of Stoke's theorem

$$\frac{1}{\pi} \int_D d^2z \bar{\partial} f(z, \bar{z}) = \oint_{\partial D} \frac{dz}{2\pi i} f(z, \bar{z}) \tag{27}$$

(Exercise: verify that!) and Cauchy's theorem

$$\oint_{\partial D} \frac{dz}{2\pi i} f(z) = \text{sum of residues of } f \text{ in domain } D \tag{28}$$

where D is some (compact) domain in the complex plane and ∂D is the boundary, suitably oriented.

So we take a domain, D containing the point, w and integrate over the above

$$\frac{1}{\pi} \int_D d^2z \bar{\partial}_{\bar{z}} \frac{1}{z - w} = \oint_w \frac{dz}{2\pi i} \frac{1}{z - w} = 1 \tag{29}$$

proving the assertion.

Exercise: Make the above more precise by introducing a cut-off, for instance by taking the propagator to be

$$\log(|z - w|^2 + \epsilon^2)$$

and consider the limit $\epsilon \rightarrow 0$.

The result of this calculation will be of considerable interest to us in the following, namely that the two-point functions of ϕ and $\partial\phi$ may be obtained as

$$\begin{aligned}
\langle \phi(z) \phi(w) \rangle &= -\log(z - w) - \log(\bar{z} - \bar{w}) \\
\langle \partial\phi(z) \partial\phi(w) \rangle &= -\frac{1}{(z - w)^2}
\end{aligned} \tag{30}$$

We make two remarks:

(i) The above form of the 2-point function is an example of the phenomenon of *Operator Product Expansions* (OPE's) being singular. Indeed we may conclude that inside all correlators, the corresponding *operators* behave as

$$\partial\phi(z)\partial\phi(w) \sim -\frac{1}{(z-w)^2} \quad (31)$$

in the limit $z \rightarrow w$. For *classical* fields, such behaviour is impossible to understand: The product of the fields would behave as the product of functions depending of z and w respectively, and a singular behaviour in $(z-w)$ could never occur. In the *quantum* theory on the other hand this may easily happen as a result of violent fluctuations in the coincidence limit. In general we will now expect a quantum (or statistical mechanics) correlator like

$$\langle\phi(z)\phi_1(z_1)\dots\phi_n(z_n)\rangle$$

for different fields (not just the scalar field we have considered), for fixed values of z_1, \dots, z_n to be a smooth function of z as long as $z \neq z_i$, however with singular behaviours quite possibly developing at the points z_i . Indeed we shall be much concerned with these singularities.

(ii) The second remark, to be made more clear later, is that the above 2-point functions are special examples of a general result for primary fields: If $\phi_h(z)$ is a primary field with dimension, h , (and we only concentrate on the z -dependence), then the two-point function will be

$$\langle\phi_h(z)\phi_h(w)\rangle \propto \frac{1}{(z-w)^{2h}} \quad (32)$$

If we include the antiholomorphic piece and consider a 2-point function of a field having both dimensions equal: $h = \bar{h}$, then we similarly get

$$\langle\phi_{h,h}(z, \bar{z})\phi_{h,h}(w, \bar{w})\rangle \propto \frac{1}{(z-w)^{2h}} \frac{1}{(\bar{z}-\bar{w})^{2h}} = \frac{1}{|z-w|^{4h}} \quad (33)$$

Indeed, from the scalar field ϕ we may construct the vector, or dimension $(1,0)$ field $\partial\phi(z)$. We also see that for the “scalar” field, ϕ itself with dimension 0, the rule works if we then interpret the singularity $(z-w)^0$ as a logarithm.

Next we want similarly to consider the *free fermionic theory*. In 2 euclidean dimensions, we may choose the two Dirac gamma matrices simply to be the Pauli matrices, σ_1, σ_2 since these satisfy the relevant Clifford relation

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

Then the Dirac operator, $\not{\partial}$ becomes

$$\left(\begin{array}{cc} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{array} \right) = 2 \left(\begin{array}{cc} 0 & \partial \\ \bar{\partial} & 0 \end{array} \right)$$

Thus we may take the free fermi action in terms of chirality fields, $\psi, \bar{\psi}$ (obtained by the projection $\frac{1}{2}(1 \pm \sigma_3)$, notice $i\sigma_3 = \sigma_1\sigma_2$ plays the role of γ_5) as

$$S = \sigma_F \int d^2x (\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) \quad (34)$$

again with a constant we wish to dispose of by suitably normalizing. The 2-point functions again are related to the propagator, $\bar{\partial}^{-1}$ for ψ and ∂^{-1} for $\bar{\psi}$. The calculation above of ∂^{-2} showed that in fact we may take (up to normalization, thus defining that)

$$\begin{aligned}\langle\psi(z)\psi(w)\rangle &= \frac{1}{z-w} \\ \langle\bar{\psi}(\bar{z})\bar{\psi}(\bar{w})\rangle &= \frac{1}{\bar{z}-\bar{w}}\end{aligned}\tag{35}$$

This provides yet another example of an operator product expansion, an OPE again.

2.3 The energy momentum tensor

The most important object in CFT is the energy momentum tensor which will turn out to provide all the generators of conformal transformations. In fact, we shall be able to write for a primary field of dimension, h ,

$$\delta_\epsilon\phi(z) = [L_\epsilon, \phi(z)] = \epsilon(z)\phi'(z) + h\epsilon'(z)\phi(z)\tag{36}$$

where the generator operator, L_ϵ , is given as

$$L_\epsilon \equiv \oint_o \frac{dz}{2\pi i} \epsilon(z)L(z)\tag{37}$$

with $L(z) = T_{zz}(z)$ the relevant component of the energy momentum tensor.

For a general classical field theory defined on a manifold with a metric tensor, $g_{ij}(z, \bar{z})$, the energy momentum tensor may be conveniently defined as

$$T_{ij}(z, \bar{z}) \equiv \frac{\delta S}{\delta g_{ij}(z, \bar{z})}\tag{38}$$

For a generally covariant theory this defines a tensor object which is symmetric and which satisfies a covariant conservation equation, well known from general relativity:

$$D^i T_{ij}(z, \bar{z}) \equiv 0\tag{39}$$

In actual fact we shall really need a slight but important generalization of this object. In the cases mostly of interest, it will turn out, namely that the CFT energy momentum tensor, is not “quite a tensor”. It will not even be a primary conformal field. The transformation property will contain a term spoiling the tensor property. This term is the central charge term, given by the central charge, c , that we have mentioned.

Roughly speaking what we shall do is the following: First we notice that if $\delta g_{ij}(z, \bar{z})$ happens to be “a pure gauge”, i.e. be a change in the metric obtained just by a coordinate transformation, then the action in a generally covariant theory will be completely unchanged, so that the functional derivative after this change of g_{ij} is identically zero.

As we have emphasized, however, an infinitesimal conformal transformation in general is something that does not make sense globally: only for the $SL(2)$ subgroup are the transformations globally defined. For all others, singularities will be present somewhere, and instead we shall be interested in infinitesimal transformations of the following kind: $z \rightarrow z + \epsilon(z)$, where $\epsilon(z)$ is holomorphic inside some domain D bounded by ∂D and

identically zero outside. This transformation is conformal inside D and outside D , but exactly at the boundary something strange and discontinuous happens.

We might perhaps *define* a classical conformal field theory to be one for which the corresponding change in the action, $\delta_\epsilon S$ is given in terms of a boundary integral only, but has zero contribution from the (2-dimensional) integration which would usually be present for an arbitrary theory under such a transformation.

Consider the example of the free scalar field, and use the normalization (to be verified) so that we consider the variation with $\delta_\epsilon \phi = \epsilon \partial \phi$

$$\begin{aligned}
\delta_\epsilon S &= \delta_\epsilon \left[\frac{1}{2\pi} \int d^2 z \partial \phi \bar{\partial} \phi \right] \\
&= \frac{1}{2\pi} \int d^2 z [\partial(\epsilon \phi') \bar{\partial} \phi + \partial \phi \bar{\partial}(\epsilon \phi')] \\
&= \frac{1}{\pi} \int d^2 z \partial \phi \bar{\partial}(\epsilon \phi') \\
&= \frac{1}{\pi} \int d^2 z [\bar{\partial} \epsilon \phi' \phi' + \frac{1}{2} \epsilon \bar{\partial}(\phi' \phi')] \\
&= \frac{1}{2\pi} \int d^2 z \bar{\partial} \epsilon(\phi' \phi')
\end{aligned} \tag{40}$$

Here we used partial integrations without boundary terms, justified by ϵ vanishing identically outside the domain, D . We see that the integrand obtained vanishes everywhere ϵ is holomorphic, just as it should for a conformal field theory: We have verified that the free scalar theory is conformally invariant. The only contribution to the integral comes from an arbitrarily narrow strip, $S[\partial D]$, surrounding the boundary, since $\bar{\partial} \epsilon$ vanishes identically away from the boundary. Thus we may write the above as

$$\delta_\epsilon S = \frac{1}{2\pi} \int_{S[\partial D]} dz \bar{\partial}(\epsilon(z)(\phi' \phi')) \tag{41}$$

Here we added a term $\epsilon(z) \bar{\partial}(\phi' \phi')$ to the integrand, but since the area of the strip is infinitesimal and this added term is bounded, that is allowed. But now we may use Stoke's theorem. The boundary of the strip has two components. Along the outer component the orientation is that of ∂D but ϵ vanishes identically. Along the inner component the orientation is opposite to that of ∂D and we get

$$\delta_\epsilon S = \oint_{\partial D} \frac{dz}{2\pi i} \epsilon(z) T_\phi(z) \tag{42}$$

with

$$T_\phi(z) \equiv -\frac{1}{2} \partial \phi(z) \partial \phi(z) \tag{43}$$

We shall refer to $T(z)$ as the holomorphic part of the energy momentum tensor.

In a coordinate system where the metric in holomorphic, antiholomorphic variables is flat and therefore looks like

$$\begin{aligned}
g_{zz} &= g_{\bar{z}\bar{z}} = 0 \\
g_{z\bar{z}} &= g_{\bar{z}z} = \frac{1}{2}
\end{aligned} \tag{44}$$

(so that $ds^2 = |dz|^2 = dzd\bar{z}$), one finds for a symmetric energy momentum tensor (with indices 0, 1 here rather than 1, 2 as before)

$$\begin{aligned} T_{zz} &= \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \equiv T \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \equiv \bar{T} \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) \end{aligned} \tag{45}$$

As explained in Ambjørn's lectures, the response of the action to a pure scale transformation is the *trace* of the energy momentum tensor. Thus in a conformal field theory in which scale invariance is a principal concern, we see that (for a flat metric), the *trace of the energy momentum tensor vanishes*.

An entirely similar treatment of the free fermionic theory yields the energy momentum tensor for that:

$$\begin{aligned} \delta_\epsilon S_\psi &= \oint_{\partial D} \frac{dz}{2\pi i} \epsilon(z) T_\psi(z) \\ T_\psi(z) &= -\frac{1}{2} \psi(z) \partial \psi(z) \end{aligned} \tag{46}$$

This kind of relation we shall take as the defining property of a classical conformal field theory: Under a discontinuous conformal transformation vanishing outside a domain, D , the action changes by a contour integral of $\epsilon(z)$ times what is then defined to be the energy momentum tensor of the theory.

Using that the trace of T vanishes, the classical conservation for T may be obtained as

$$\begin{aligned} \bar{\partial} T(z, \bar{z}) &= 0 \\ \partial \bar{T}(z, \bar{z}) &= 0 \end{aligned} \tag{47}$$

As we shall see these relations have precise quantum generalizations in terms of the *conformal Ward identity*.

2.4 Field theory on the cylinder and on the plane. The conformal Ward identity.

As is well known, the relation between a path integral and the operator theory is, that correlators defined by the path integral become Greens functions, namely vacuum expectation values of *time ordered* products of the corresponding quantum operators. We therefore have to say a few words about the choice of time ordering.

Consider the 2-dimensional quantum field theory living on a cylinder, extending infinitely to either side. We may choose coordinates (σ, τ) in an obvious way so that $\sigma \in [0, 2\pi[$ parametrizes a point on the circle generating the cylinder, and $\tau \in \mathbb{R}$ parametrizes the “time like” position along the cylinder. We may think of this picture as a representation of the euclidean world sheet of a closed string propagating. And we may take “time” to mean this coordinate τ .

It is convenient to go between this cylinder representation and a conformal transformation to the plane as follows:

Define the complex variable, $\zeta = \tau + i\sigma$, and consider the conformal transformation to the complex plane given by

$$z = e^\zeta \tag{48}$$

Thus the “zero-time-surface”, $\tau = 0$ is mapped to the unit circle, and any other fixed time, “space-like” surface is mapped to concentric circles. At $\tau \rightarrow -\infty$ they shrink to the point, $z = 0$, the infinite past, and at $\tau \rightarrow +\infty$ they blow up to infinity, or the south pole on the Riemann sphere.

On a global conformal transformation, i.e. a projective or $SL(2, \mathbb{C})$ transformation, the points $z = 0$ and $z = \infty$ will be mapped to two other points, and the original circles will be mapped to other circles (yes, circles) surrounding these points and zooming in on them as $\tau \rightarrow \pm\infty$. The original straight line $\sigma = \text{constant}$ lines (fixed points in “space”), will be mapped into circular curves (yes circular) connecting the two images of $z = 0, z = \infty$.

I will suffice thus to concentrate on the “simple radial time ordering”, other correlators being obtained relatively easily by projective transformations.

Thus our correlators defined by path integrals will correspond to expectation values of the corresponding quantum field *operators* in “radial time ordering” in the complex plane.

Now we want to derive the crucial *conformal Ward identity*. Thus consider a generic conformal field theory which we may think of as defined in terms of some path integral,

$$\int \mathcal{D}\phi e^{-S[\phi]} \tag{49}$$

ϕ may denote a collection of fields. Eventually it will not matter whether we even know of any path integral representation of the theory, only the structure is useful for the derivation. Similarly we consider some local field operators, which in the path integral may be considered functions of the basic field,

$$\Phi_n(z, \bar{z}) = \Phi_n(\phi(z, \bar{z}))$$

We want to consider a typical Greens function for these

$$\langle \Phi_1(z_1) \dots \Phi_n(z_n) \rangle$$

(neglecting to write always the \bar{z}_i variables). We denote this correlator simply as $\langle F[\phi] \rangle$, implying an even more general form. When written like this it is understood that ϕ and the Φ 's are quantum *operators*, but we use the same letters for the c-number (or Grassmann valued) fields inside the path integrals. Thus

$$\langle F[\phi] \rangle = \int \mathcal{D}\phi e^{-S[\phi]} F[\phi] \tag{50}$$

Since the fields, ϕ are integrated over, we may replace the integration variable by any other name, like ϕ' and we may choose

$$\phi'(z) = \phi(z) + \delta_\epsilon \phi \tag{51}$$

corresponding to a conformal transformation defined by some infinitesimal function, $\epsilon(z)$ holomorphic inside some domain, D and zero outside. (Sometimes ϕ' means the derivative

of ϕ , but here it is just a new slightly different field). Since nothing at all changes under this, we get identically the above expression again, and by subtracting the two we get zero. Let us analyze this zero from the point of view of the right hand side

$$\begin{aligned}
0 &= \delta_\epsilon \int \mathcal{D}\phi e^{-S[\phi]} F[\phi] \\
&= \int \delta_\epsilon[\mathcal{D}\phi] e^{-S[\phi]} F[\phi] - \int \mathcal{D}\phi e^{-S[\phi]} \delta_\epsilon S F[\phi] + \int \mathcal{D}\phi e^{-S[\phi]} \delta_\epsilon F[\phi] \\
&= 0 - \oint_{\partial D} \frac{dz}{2\pi i} \epsilon(z) \langle T(z) F[\phi] \rangle + \langle \delta_\epsilon F[\phi] \rangle
\end{aligned} \tag{52}$$

Here we used, or rather introduced another defining property of CFT's namely that the *measure* $\mathcal{D}\phi$ is invariant under conformal transformations. If that was not the case we would talk about an anomaly being present and spoiling the conformal invariance of the classical theory. Second we used the previously introduced defining property

$$\delta_\epsilon S = \oint_{\partial D} \frac{dz}{2\pi i} \epsilon(z) T(z)$$

Finally we have derived **the conformal Ward identity** in the form

$$\langle \delta_\epsilon F[\phi] \rangle = \oint_{\partial D} \frac{dz}{2\pi i} \epsilon(z) \langle T(z) F[\phi] \rangle \tag{53}$$

Let us use this identity to derive a number of crucial and fundamental properties about a CFT. We imagine that $F[\phi]$ is of the form mentioned above:

$$F[\phi] = \Phi_1(z_1) \dots \Phi_n(z_n)$$

(i) Then on the right hand side we have the correlator

$$f(z) \equiv \langle T(z) F[\phi] \rangle$$

which is a function of the variable, z , for fixed values of z_i . We want first to show that this function is an analytic function of z , in general with singularities at the points, z_i , but nowhere else. To this end we simply first pick a domain, D , so that all the points, z_i lie *outside* D , so that $\epsilon(z_i) = 0$ for all $i = 1, \dots, n$. Then of course the left hand side,

$$\langle \delta_\epsilon F[\phi] \rangle$$

vanishes. In other words, for the function, $f(z)$ we deduce that

$$\oint_{\partial D} \frac{dz}{2\pi i} \epsilon(z) f(z) \equiv 0$$

for any contour, ∂D not containing any of the points, z_i and any function, $\epsilon(z)$ holomorphic inside D . This is exactly what is needed to conclude by Cauchy's theorem, that the function, $f(z)$ is holomorphic inside D . This proves the assertion.

(ii) An equivalent way of saying the same is to say that, $\bar{\partial} f(z) \equiv 0$, or, whenever the operator

$$\bar{\partial} T(z)$$

appears inside a correlator, that correlator vanishes, provided no field operators are present at the point, z . Thus we may write loosely

$$\bar{\partial}T(z) = 0$$

the meaning of which has to be made clear just as in the sentence above. Notice that this is exactly like the classical result, that the energy momentum tensor is conserved.

(iii) Indeed the above discussion implies that correlators may be extended into functions of *two independent sets* of variables, the z_i 's and the \bar{z}_i 's the latter of which may be taken to be fixed, for variable z_i 's. In the end the physical correlators have to do with the points where \bar{z}_i is the complex conjugate of z_i , but at intermediate steps in the analysis it is very convenient to abandon this constraint. An analytic function satisfying the conformal Ward identity is termed a *conformal block*. It is only a function of the z_i 's. Similarly one may build anti conformal blocks depending on the \bar{z}_i 's. Finally one may attempt to construct the physical correlator by combining the two sets of blocks. In so doing, one must make sure that the result satisfies the *monodromy requirement* that whenever we take one of the positions of a field, say the point z_i and move it in a path around some of the other points to finally return to where we started, then the correlator must be unchanged: Monodromy invariant. This property in general is *not* satisfied by the conformal blocks. It may happen for example that, regarded as a function of z_i , the conformal block has a square root branch point or some other branch point at the point z_j . Then there is a *non-trivial monodromy* and it becomes a non-trivial matter to figure out how this may be repaired by gluing blocks and anti blocks together. This problem has been much studied, but will not be elaborated here. In fact we shall often be concerned with the individual blocks, and often without explicitly making that clear.

(iv) The conformal Ward identity shows that the energy momentum tensor is the generator of conformal transformations. In fact, consider the case where the domain, D , only contains one of the points, say z_i . Further, for simplicity, consider the case where $\Phi_i(z_i)$ is a primary field. Then the change under the conformal transformation simply consists in the result of the conformal change in $\Phi_i(z_i)$ and for a primary field we know that this is just

$$\delta_\epsilon \Phi(z_i) = \epsilon(z_i) \partial \Phi_i(z_i) + h_i \partial \epsilon(z_i) \Phi(z_i)$$

where h_i is the conformal dimension of $\Phi_i(z)$. The conformal Ward identity shows that the same change must result from inserting the energy momentum tensor, $T(z)$ into the correlator and integrating the result around the point, z_i , namely the boundary, ∂D only encloses that point by assumption. This must hold in any correlator whatsoever, hence it makes sense to express the result in terms of the quantum operators themselves as

$$\oint_{z_i} \frac{dz}{2\pi i} \epsilon(z) T(z) \Phi_i(z_i) = \epsilon(z_i) \partial \Phi_i(z_i) + h_i \partial \epsilon(z_i) \Phi_i(z_i) \quad (54)$$

This rule enables us to obtain the first crucial general *operator product expansion* (OPE). Namely, we may conclude, that the *singularities* above must be of the form

$$T(z) \Phi_i(z_i) = \frac{\partial \Phi_i(z_i)}{z - z_i} + \frac{h_i}{(z - z_i)^2} \partial \Phi_i(z_i) + \text{non-singular terms in } (z - z_i) \quad (55)$$

This is almost immediately clear by Cauchy's theorem and derivatives thereof:

$$\oint_{z_i} \frac{dz}{2\pi i} \frac{f(z)}{(z - z_i)^p} = \frac{1}{(p - 1)!} f^{(p-1)}(z_i) \quad (56)$$

(v) The integral of the OPE, eq.(54) shows that we may identify the operator

$$L_\epsilon \equiv \oint_0 \frac{dz}{2\pi i} \epsilon(z) L(z) \quad (57)$$

as the *quantum generator* of the conformal transformation generated by the conformal reparametrization

$$z \rightarrow z + \epsilon(z).$$

Here we may conveniently think about the integration contour as a circle running around the origin, $z = 0$. When inserted in a correlator, the precise position of the contour is unimportant as long as we do not cross singularities: points where operators sit. This we shall have to come back to. Now we have written $L(z)$ rather than $T(z)$. Both names will be used without too much explanation for the energy momentum tensor. L is often preferred when we talk about the Virasoro generator, but that is just the energy momentum tensor as we shall soon see. In order to relate L_ϵ to the contour integral eq.(54) we first emphasize that whenever we write down a product of operators in an OPE, the meaning is slightly deceptive. We are lifting a general rule for correlators to an identity for operators. But then it is crucial to remember that inside the correlators everything is (radially) time ordered. Thus, the *order* of the operators in an OPE is not always what we write down. It is understood that if necessary, a radial time ordering is implied, so that the operator with argument having the largest absolute value must always be considered to stand to the left.

We now want to argue that the contour integral

$$\oint_{z_i} \frac{dz}{2\pi i} \epsilon(z) L(z) \Phi_i(z_i) = \epsilon(z_i) \partial \Phi_i(z_i) + h_i \partial \epsilon(z_i) \Phi_i(z_i) = \delta_\epsilon \Phi_i(z_i) \quad (58)$$

may be written

$$\delta_\epsilon \Phi_i(z_i) = [L_\epsilon, \Phi_i(z_i)] = L_\epsilon \Phi_i(z_i) - \Phi_i(z_i) L_\epsilon \quad (59)$$

which indeed would prove that L_ϵ is the conformal generator. The statement follows rather simply upon remembering the meaning of the notation. In the last difference we mean the usual Hilbert space product of operators, such as we should in order to talk about generators and commutators. So in the first term

$$L_\epsilon \Phi_i(z_i)$$

it must be understood that the contour defining L_ϵ in eq.(57) is taken along a circle running just *outside* the point $z = z_i$. Similarly in the last term

$$\Phi_i(z_i) L_\epsilon$$

it is understood that the contour is taken just *inside* that same point. Otherwise there would be a conflict between the Hilbert space way and the OPE way of writing the product of operators. But then the *difference* between the two (defining the commutator) is equivalent to taking the contour integral in a small loop around the point, z_i , itself, just as in eq.(57). Hence the statement that the energy momentum tensor is the generator of conformal transformations.

(vi) For any primary field, $\phi_h(z)$ of conformal dimension, h , there is an infinity of *descendant* fields, defined as all linear combinations of the class of fields that may be obtained by multiple application of some $L_{\epsilon_1}, L_{\epsilon_2}, \dots$:

$$[\dots[L_{\epsilon_2}, [L_{\epsilon_1}, \phi_h(z)]]\dots]$$

If we know a correlator among primary fields, it is rather easy from the OPE's to calculate any correlator among certain descendant fields. The result, as we shall see later, is obtained by applying a certain explicit differential operator to the correlator of the primaries.

We have considered the OPE between the energy momentum tensor and a primary field. How about the OPE between the energy momentum tensor and itself? It is rather clear by considering dimensions that the dimension of the the energy momentum tensor must be 2. This is also expected from the fact that it originated as a rank-2 tensor. If it was a primary conformal field (which it is not!), we would have the equivalent rules

$$\begin{aligned} T(z)T(w) &= \frac{2}{(z-w)^2}T(w) + \frac{\partial T(w)}{z-w} + n.s.t. \\ \delta_\epsilon T(w) &= [T_\epsilon, T(w)] = \epsilon(w)\partial T(w) + 2\partial\epsilon(w)T(w) \end{aligned} \quad (60)$$

where we shall often use the n.s.t to denote the possible appearance of non-singular terms, the form of which depend on the theory.

As it happens, $T(z)$ is not a primary, in fact from the above it is clear that $T(z)$ must be considered a descendant of the unit operator. Correspondingly we cannot be sure that it transforms as just stated. Indeed the correct form in a CFT is as follows:

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{\partial T(w)}{z-w} + n.s.t \\ \delta_\epsilon T(w) &= [T_\epsilon, T(w)] = \epsilon(w)\partial T(w) + 2\partial\epsilon(w)T(w) + \frac{c}{12}\epsilon'''(w) \end{aligned} \quad (61)$$

Here, again we have introduced the central charge, c . We shall soon see that the free field theories introduced above, precisely give rise to these OPE's and that the free scalar theory has $c = 1$ whereas the free fermi theory has $c = \frac{1}{2}$. The above form for the introduction of "the central extension" is the only one (up to trivial reformulations) which is mathematically consistent, as discussed in the various exercises. A given CFT is characterized by several things, but the most important of those is the value of the central charge.

Exercise:

1. Show that the two rules, eqs.(61), are equivalent.
2. Show that the commutator of two conformal transformations is given by

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\partial\epsilon_1\epsilon_2 - \epsilon_1\partial\epsilon_2}$$

both when the variations are applied to primary fields and when they are applied to an energy momentum tensor for some arbitrary value of the central charge, c .

2.5 The Virasoro Algebra

The energy momentum tensor is the generator of conformal transformations. Hence we expect it will satisfy the conformal algebra in some way or other. Indeed it does. It is called the Virasoro algebra. It may be formulated in several equivalent ways some of which are convenient for some purposes, others for other purposes. In fact, eqs.(61) are exactly two such formulations. However, we now show how to obtain from those some more conventional formulations.

For an ordinary Lie algebra, we are used to expressing it in terms of commutation relations between elements forming a basis for the vector space constituting the algebra. These are typically of the form

$$[T^a, T^b] = i f^{ab}_c T^c \quad (62)$$

where $a, b, c = 1, \dots, d$, the dimension of the algebra. The structure constants f^{ab}_c depend on the algebra *and on the choice of basis*. So there is a question of the choice of basis.

Let us first present a treatment pretty much free of basis choice. Indeed, for a given infinitesimal conformal transformation $z \rightarrow z + \epsilon(z)$ we may consider that as defining an element of the conformal algebra, and the corresponding element of the Virasoro algebra is L_ϵ . Thus we begin with

$$[L_{\epsilon_1}, L(z)] = \delta_{\epsilon_1} L(z) = \epsilon_1(z) \partial L(z) + 2\partial \epsilon_1(z) L(z) + \frac{c}{12} \epsilon_1'''(z) \quad (63)$$

Multiply this equation by $\epsilon_2(z)$ and do a contour integration around the origin to obtain

$$\begin{aligned} [L_{\epsilon_1}, L_{\epsilon_2}] &= \oint_0 \frac{dz}{2\pi i} \{ \epsilon_2(z) \epsilon_1(z) \partial L(z) + 2\epsilon_2(z) \partial \epsilon_1(z) L(z) + \frac{c}{12} \epsilon_2(z) \epsilon_1'''(z) \} \\ &= \oint_0 \frac{dz}{2\pi i} \{ L(z) (-\partial \epsilon_2(z) \epsilon_1(z) + \epsilon_2(z) \partial \epsilon_1(z)) + \frac{c}{12} \epsilon_2(z) \epsilon_1'''(z) \} \\ &= L_{\partial \epsilon_1 \epsilon_2 - \epsilon_1 \partial \epsilon_2} + \frac{c}{12} \oint_0 \frac{dz}{2\pi i} \epsilon_2(z) \epsilon_1'''(z) \end{aligned} \quad (64)$$

Notice that this calculation is very closely related to some of the exercises. This form may be considered a rather basis independent version of the Virasoro algebra.

Now let us obtain from that the standard form. It comes by choosing the particular basis corresponding to a complete set of infinitesimal conformal transformations being provided by the set $\{\epsilon_n | \epsilon_n(z) = z^{n+1}\}$. In fact any $\epsilon(z)$ holomorphic in the neighbourhood of the unit circle may be expanded in a Laurent series in terms of those. Also we may conveniently expand the energy momentum tensor itself in this way:

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad (65)$$

so that

$$L_{\epsilon_n} = \oint_0 \frac{dz}{2\pi i} z^{n+1} \sum_{m \in \mathbb{Z}} L_m z^{-m-2} = L_n \quad (66)$$

by Cauchy's theorem. Notice that $L_{k\epsilon} = kL_\epsilon$.

Now use the form of the Virasoro algebra above with $\epsilon_1(z) = z^{m+1}, \epsilon_2(z) = z^{n+1}$ so that

$$\begin{aligned}\partial\epsilon_1(z)\epsilon_2(z) - \epsilon_1(z)\partial\epsilon_2(z) &= (m-n)z^{m+n+1} = (m-n)\epsilon_{m+n}(z) \\ \oint_0 \frac{dz}{2\pi i} \epsilon_2(z)\epsilon_1'''(z) &= (m+1)m(m-1) \oint_0 \frac{dz}{2\pi i} z^{m+n-1} \\ &= m(m^2-1)\delta_{m+n,0}\end{aligned}\tag{67}$$

This finally gives the conventional form of the **Virasoro Algebra**:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\tag{68}$$

This is an infinitely dimensional Lie algebra with generators $\{L_n, n \in \mathbb{Z}, c\}$ with the central element, c commuting with all others and thus playing the role of a c -number. The structure constants in this basis are readily read off.

The Cartan subalgebra consisting of all those elements which may be simultaneously diagonalized (the maximally commuting subalgebra) consists of the two generators

$$L_0, c$$

They form a commuting subalgebra. Discarding the rather trivial c the only other finite dimensional subalgebra, is the $SL(2, \mathbb{C})$ we have met before generated by L_{-1}, L_0, L_1 . The reader should verify that they indeed constitute a subalgebra not involving the central charge, c .

2.6 Examples and suggested exercises for sects 1 and 2

2.6.1 Conformal transformations in d dimensions

We consider the space \mathbb{R}^d with flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ of signature (p, q) :

$$\eta_{\mu\nu} = \text{diag}(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q)\tag{69}$$

Conformal transformations are defined as those that leave the metric invariant up to a scale change:

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Omega(x)g_{\mu\nu}(x)\tag{70}$$

whereas in general we have

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)\tag{71}$$

(1A) Show that if the infinitesimal transformation $x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$ is conformal the scale change is

$$\Omega(x) = 1 - \frac{2}{d}\partial \cdot \epsilon\tag{72}$$

Hint, derive and make use of

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) \eta_{\mu\nu} \quad (73)$$

(1B) Show also that

$$(\eta_{\mu\nu} \square + (d-2) \partial_\mu \partial_\nu) \partial \cdot \epsilon = 0 \quad (74)$$

where the d'Alambertian is $\square = \partial^\mu \partial_\mu$. This identity shows that in two dimensions the situation is radically different from situations in $d > 2$ dimensions.

(1C) Show that in $d = 2$ dimensional Euclidean space the Cauchy-Riemann equations are fulfilled.

The finite conformal transformation group is known to be formed by the Poincare group, dilatations and special conformal transformations obtained by an inversion, a translation and a second inversion. The Poincare group is the semidirect product of translations and Lorentz transformations of flat space (parametrized by Λ_ν^μ), so we have

$$x^\mu \rightarrow x^\mu + a^\mu \quad (75)$$

$$x^\mu \rightarrow \Lambda_\nu^\mu x^\nu \quad (76)$$

$$x^\mu \rightarrow \lambda x^\mu \quad (77)$$

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2} \quad (78)$$

The inversion is

$$x^\mu \rightarrow \frac{x^\mu}{x^2} \quad (79)$$

(1D) Verify the form of the special conformal transformation and in all four cases find the corresponding infinitesimal transformations.

(1E) Show that the scale changes are

$$\Omega(x) = 1 \quad (80)$$

$$\Omega(x) = 1 \quad (81)$$

$$\Omega(x) = \lambda^{-2} \quad (82)$$

$$\Omega(x) = (1 + 2b \cdot x + b^2 x^2)^2 \quad (83)$$

Hint, derive and (in the case of the special conformal transformations) make use of

$$\Omega_{f \circ g}(x) = \Omega_f(g(x)) \Omega_g(x) \quad (84)$$

which shows that the composition of the two conformal transformations f and g is itself a conformal transformation. Find the corresponding infinitesimal expressions and compare these to (72).

2.6.2 The 2-point function in d dimensions

In this exercise we want to determine the 2-point functions up to normalization on the basis of the results found in exercise 1.

(2A) Use the transformation (71) to express the Jacobian of a conformal transformation as

$$\left| \frac{\partial x'}{\partial x} \right| = \Omega^{-d/2} \quad (85)$$

According to the results in (1E) this means that for dilatations and special conformal transformations the Jacobian is given respectively by

$$\left| \frac{\partial x'}{\partial x} \right| = \lambda^d \quad (86)$$

and

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 + 2b \cdot x + b^2 x^2)^d} \quad (87)$$

In a conformal field theory the (quasi-primary) fields transform under conformal transformations according to

$$\phi_j(x) \rightarrow \left| \frac{\partial x'}{\partial x} \right|^{\Delta_j/d} \phi_j(x') \quad (88)$$

where Δ_j is called the dimension of ϕ_j . The theory is covariant under conformal transformations in the sense that the correlation functions satisfy

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \prod_{i=1}^n \left| \frac{\partial x'}{\partial x} \right|_{x=x_i}^{\Delta_i/d} \langle \phi_1(x'_1) \cdots \phi_n(x'_n) \rangle \quad (89)$$

As we now shall see the covariance property under the conformal group imposes severe restrictions on 2-point functions, and also on the N-point functions.

(2B) Verify that translational, rotational and dilatation invariance demand the following functional dependence

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{C_{12}}{x_{12}^{\Delta_1 + \Delta_2}} \quad (90)$$

where C_{12} is a normalization constant and $x_{12} = |x_1 - x_2|$.

(2C) Use the special conformal transformation for x_{12} (express $(x'_{12})^2$ in terms of x_j) to justify

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \begin{cases} \frac{C_{12}}{x_{12}^{2\Delta}} & \Delta_1 = \Delta_2 = \Delta \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \quad (91)$$

2.6.3 The central extension

In this exercise we consider the Virasoro algebra with central charge c from an algebraic point of view as the only possible central extension of the Loop algebra respecting the Jacobi identities. The Virasoro algebra is

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \quad (92)$$

The Loop algebra found in subsection 1.3

$$[L_m, L_n] = (m - n)L_{m+n} \quad (93)$$

A central extension of this is defined to be an algebra

$$[L_m, L_n] = (m - n)L_{m+n} + c_{m,n} \quad (94)$$

$$[L_l, c_{m,n}] = 0 \quad (95)$$

(3A) Verify the identities

$$(m - n)c_{m+n,l} + (n - l)c_{n+l,m} + (l - m)c_{l+m,n} = 0 \quad (96)$$

$$c_{m,n} = -c_{n,m} \quad (97)$$

(3B) Use the transformations

$$L_n \rightarrow \tilde{L}_n = L_n + \frac{1}{n}c_{n,0} \quad n \neq 0 \quad (98)$$

$$L_0 \rightarrow \tilde{L}_0 = L_0 + \frac{1}{2}c_{1,-1} \quad (99)$$

whereby $\tilde{c}_{m,n}$ is defined as

$$[\tilde{L}_m, \tilde{L}_n] = (m - n)\tilde{L}_{m+n} + \tilde{c}_{m,n} \quad (100)$$

to find the relations

$$\tilde{c}_{m,n} = c_{m,n} - (m - n)\frac{1}{m+n}c_{m+n,0} \quad (101)$$

and

$$\tilde{c}_{m,-m} = c_{m,-m} - \frac{1}{2} \cdot 2mc_{1,-1} \quad (102)$$

Now we drop the tilde and note that

$$c_{n,0} = c_{0,n} = c_{1,-1} = 0 \quad (103)$$

(3C) Put $l = 0$ in (96) and deduce that $c_{m,n}$ must be proportional to $\delta_{m+n,0}$. Then put $l = -m - 1$ and $n = 1$ and obtain

$$(m - 1)c_{-m-1,m+1} + (m + 2)c_{m,-m} + (-2m - 1)c_{1,-1} = 0 \quad (104)$$

Show that this recursion relation reduces to the following valid for $m \geq 2$ (note that $c_{1,-1} = c_{0,0} = 0$)

$$c_{m+1,-m-1} = \frac{(m + 2)(m + 1)m}{6}c_{2,-2} \quad (105)$$

Finally use a suitable normalization of c to obtain

$$c_{m,n} = \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \quad (106)$$

We have by means of a few algebraic manipulations demonstrated that the Virasoro algebra (92) is the unique central extended Loop algebra as defined in (94-95). The only free parameter is the central charge c .

3 Basic properties of scaling operators

Consider the conformal transformation of a primary field of (holomorphic) conformal dimension, h :

$$[L_\epsilon, \phi_h(z)] = \epsilon(z)\partial\phi_h(z) + h\partial\epsilon(z)\phi_h(z) \quad (107)$$

and let us work this out for the standard basis, corresponding to the Laurent expansion,

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad (108)$$

for which

$$L_n = \oint_0 \frac{dz}{2\pi i} z^{n+1} L(z) \quad (109)$$

corresponds to $\epsilon(z) = z^{n+1}$. This gives

$$[L_n, \phi_h(z)] = z^{n+1}\partial\phi_h(z) + (n+1)z^n h\phi_h(z) \quad (110)$$

Of particular interest is the case where the point about which we do the Laurent expansion is the point where the field operator is taken, in this case $z = 0$. In that case we get

$$\begin{aligned} [L_{-1}, \phi_h(0)] &= \partial\phi_h(0) \\ [L_0, \phi_h(0)] &= h\phi_h(0) \\ [L_n, \phi_h(0)] &= 0 \text{ for } n > 0 \end{aligned} \quad (111)$$

The first of those say that L_{-1} acts like the derivative, the second that the conformal dimension is the eigenvalue of L_0 and the last constitute an important characterization of primary fields.

It will be very useful for us to think also in terms of the *states* of the theory in addition to thinking in terms of the fields. First we introduce the *vacuum*, $|0\rangle$. It should be rotational dilatational and translational invariant, and by now we know this means that

$$\begin{aligned} L_0|0\rangle &= 0 = \bar{L}_0|0\rangle \\ L_{-1}|0\rangle &= 0 = \bar{L}_{-1}|0\rangle \end{aligned} \quad (112)$$

Using the vacuum, we may form what is known as *highest weight states*. Concentrating on the holomorphic part, they are defined in terms of the primary fields as

$$|h\rangle \equiv \lim_{z \rightarrow 0} \phi_h(z)|0\rangle \quad (113)$$

Actually we want also to make sure that this works for the energy momentum tensor itself, in other words, that the limit exists for

$$T(z)|0\rangle = \sum_n z^{-n-2} L_n|0\rangle \quad (114)$$

For $n \leq -2$ this vanishes for $z \rightarrow 0$, but for $n \geq -1$ we get a singularity unless

$$L_n|0\rangle = 0 \text{ for } n \geq -1 \quad (115)$$

which we therefore insist must hold for a “good vacuum” in the theory.

This immediately show that we have

$$\begin{aligned} L_0|h\rangle &= h|h\rangle \\ L_n|h\rangle &= 0 \text{ for } n > 0 \end{aligned} \tag{116}$$

These are the equations characterizing a highest weight state of conformal dimension, h . Clearly there are similar equations for the anti holomorphic sector. For the very particular (but most important) primary field, which is nothing but the unit operator, we simply have equations for the vacuum itself. Thus the vacuum is a highest weight state of dimension zero, but it is also annihilated by L_{-1} due to translational invariance. The fact that the vacuum is annihilated, in particular by the 3 $sl(2, \mathbb{C})$ generators, L_{-1}, L_0, L_1 , gives rise to the term: The $SL(2, \mathbb{C})$ invariant vacuum.

The term *highest weight* comes from ordinary Lie algebra theory. Thus consider an $SU(2)$ multiplet $|j, m\rangle$ with spin, j being a positive integer or half integer, and with eigenvalue, m for J_3 with $m = -j, -j + 1, \dots, +j$. We may map between states of different values of m by means of the well known raising and lowering operators:

$$\begin{aligned} J_+|j, m\rangle &\propto |j, m + 1\rangle \\ J_-|j, m\rangle &\propto |j, m - 1\rangle \end{aligned} \tag{117}$$

The “magnetic quantum number” is a special example of what is called *the weight* in group theory, and there is a *highest weight*, $m = j$ for which

$$J_+|j, j\rangle = 0$$

In the case of the Virasoro algebra, J_3 is analogous to L_0 , and we have an infinity of raising operators, L_n with $n > 0$. Actually by a twist of convention, the “raising operators” *lower* the eigenvalue of L_0 as is easily seen by the algebra:

$$L_0(L_n|h\rangle) = L_n L_0|h\rangle + [L_0, L_n]|h\rangle = hL_n|h\rangle + (0 - n)L_n|h\rangle = (h - n)L_n|h\rangle \tag{118}$$

In the cylinder coordinates previously introduced, in which ‘time’ is related to the radial distance from the origin in the complex plane, the ket-states above may be considered “incoming states”, since $z = 0$ correspond to the infinite past. Similarly, we may conveniently construct outgoing states corresponding to the infinite future. Thinking of $z \rightarrow \infty$ in terms of $w \equiv 1/z \rightarrow 0$ with $\frac{dw}{dz} = -\frac{1}{z^2} = -w^2$, it becomes natural to define

$$\langle h| = \lim_{z \rightarrow \infty} \langle 0|\phi_h(z)z^{2h} \tag{119}$$

In the case of the energy momentum tensor itself, this rule leads to the following hermiticity properties

$$L_n^\dagger = L_{-n}$$

In particular we get for the vacuum-bra and -ket:

$$\langle 0|L_n = 0, \quad n \leq +1, \quad L_n|0\rangle = 0, \quad n \geq -1$$

3.1 The two point function

We want to argue that in a CFT, the form of the two point function is given uniquely up to normalization as

$$\langle \phi_{h_1, \bar{h}_1}(z_1, \bar{z}_1) \phi_{h_2, \bar{h}_2}(z_2, \bar{z}_2) \rangle = \frac{c_{12}}{(z_1 - z_2)^{h_1+h_2}} \frac{\bar{c}_{12}}{(\bar{z}_1 - \bar{z}_2)^{\bar{h}_1+\bar{h}_2}} \quad (120)$$

where $c_{12} \equiv 0$ unless $h_1 = h_2$ ($\bar{c}_{12} \equiv 0$ unless $\bar{h}_1 = \bar{h}_2$). Let us concentrate on the holomorphic part. From the transformation property we get

$$\langle \phi_{h_1}(z_1) \phi_{h_2}(z_2) \rangle = (f'(z_1))^{h_1} (f'(z_2))^{h_2} \langle \phi_{h_1}(f(z_1)) \phi_{h_2}(f(z_2)) \rangle \quad (121)$$

This equation may be derived by inserting conformal group operators around each field. One needs to be able to say that such a conformal group operator leaves the vacuum invariant. That, as we have seen, requires that the group operator in question is of the form

$$\exp\{a_{-1}L_{-1} + a_0L_0 + a_1L_1\}$$

in other words, is an $SL(2, \mathbb{C})$ transformation for which

$$f(z) = \frac{az + b}{cz + d}$$

These, and only these generators will annihilate both the bra- and the ket-vacuum.

Let us take $f(z) = \lambda z$ and $z_2 = 0$ to obtain

$$\langle \phi_{h_1}(z_1) \phi_{h_2}(0) \rangle = \lambda^{h_1+h_2} \langle \phi_{h_1}(\lambda z_1) \phi_{h_2}(0) \rangle \quad (122)$$

In particular, take $z_1 = 1$, and get

$$\langle \phi_{h_1}(\lambda) \phi_{h_2}(0) \rangle = \frac{1}{\lambda^{h_1+h_2}} \langle \phi_{h_1}(1) \phi_{h_2}(0) \rangle \quad (123)$$

This proves the first part of the assertion, since by translational invariance the two-point function can clearly only depend on the difference of the arguments. To see that the two-point function vanishes unless $h_1 = h_2$, we use the above to conclude that

$$\begin{aligned} \frac{c_{12}}{(z_1 - z_2)^{h_1+h_2}} &= \langle \phi_{h_1}(z_1) \phi_{h_2}(z_2) \rangle \\ &= c_{12} \frac{(f'(z_1))^{h_1} (f'(z_2))^{h_2}}{(f(z_1) - f(z_2))^{h_1+h_2}} \end{aligned} \quad (124)$$

It is not difficult to verify that this is a valid identity for projective mappings iff $h_1 = h_2$. Perhaps a simpler argument consists in noticing that by a projective transformation we may map any two points to ∞ and 0. If we do that, the two point function becomes proportional to

$$\langle h_1 | h_2 \rangle$$

Inserting $L_0 = L_0^\dagger$ and letting it act to the left and the right we get

$$h_1 \langle h_1 | h_2 \rangle = \langle h_1 | L_0 | h_2 \rangle = h_2 \langle h_1 | h_2 \rangle$$

The form of the two point function shows that for positive values of conformal dimension, the two-point function *decreases* with increasing distance between the scaling operators. This is perfectly sensible. On the other hand, negative conformal dimensions would lead to *growing* two-point functions as the distance between operators get larger. This is non meaningful. Thus we shall assume that for realistic critical systems, only *positive conformal dimensions are allowed*. This also justifies considering primary fields and highest weight representations of the Virasoro algebra. In such representations, namely, we get zero if we apply a generator L_n with $n > 0$ to the state. We may apply L_n 's with $n < 0$, but this only leads to even larger values of the eigenvalue of h as we have seen.

Exercise

Show from the above that the vacuum expectation value of the energy momentum tensor vanishes. Then find the two-point function

$$\langle 0|T(z)T(w)|0\rangle = \frac{c/2}{(z-w)^4}$$

in two ways: (i) by the OPE's, and (ii) by the Virasoro algebra.

4 The central charge

4.1 Doing free field theory

We saw in eq.(43) and in eq.(46) that the energy momentum tensors for a free scalar field and a free fermionic field respectively are

$$\begin{aligned} T_\phi(z) &= -\frac{1}{2}\partial\phi(z)\partial\phi(z) \\ T_F(z) &= -\frac{1}{2}\psi(z)\partial\psi(z) \end{aligned} \tag{125}$$

In a quantum field theory these expressions will have to be modified, indeed be given a precise meaning, since as we have emphasized, it does not make sense to consider the product of two operators at the same point, when there are singular OPE's. In our case we have for the quantum fields, as we have seen

$$\begin{aligned} \partial\phi(z)\partial\phi(w) &= -\frac{1}{(z-w)^2} + \text{n.s.t.} \\ \psi(z)\psi(w) &= \frac{1}{z-w} + \text{n.s.t.} \end{aligned} \tag{126}$$

One therefore introduces the concept of *normal ordering* of operators. It can be done in a variety of ways, all of which are equivalent for free fields, but may differ in more complicated situations. The idea is the same, however: We wish to subtract away the singular part of the OPE, and concentrate on the finite piece left over:

$$\begin{aligned} T_\phi(w) &= -\frac{1}{2} : \partial\phi(w)\partial\phi(w) : \equiv -\frac{1}{2} \lim_{z \rightarrow w} \left(\partial\phi(z)\partial\phi(w) + \frac{1}{(z-w)^2} \right) \\ T_F(w) &= -\frac{1}{2} : \psi(w)\partial\psi(w) : \equiv -\frac{1}{2} \lim_{z \rightarrow w} \left(\psi(z)\partial\psi(w) - \frac{1}{(z-w)^2} \right) \end{aligned} \tag{127}$$

Exercise:

Show that as a consequence of the Heisenberg equations of motions, the free operator fields may be written as a sum of terms depending on either z alone or \bar{z} alone. Introduce the following mode expansions for the holomorphic pieces

$$\begin{aligned}\partial\phi(z) &= i \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \\ \psi(z) &= \sum_{r \in \mathbb{Z} - \frac{1}{2}} \psi_r z^{-r - \frac{1}{2}}\end{aligned}\tag{128}$$

Show that the OPE's among the fields are equivalent to the following commutation relations:

$$\begin{aligned}[a_n, a_m] &= n\delta_{n+m,0} \\ \{\psi_r, \psi_s\} &= \delta_{r+s,0}\end{aligned}\tag{129}$$

Finally, show that the above normal ordering is equivalent to the rule of putting annihilation operators to the right, treating operators inside normal ordering signs as classical variables, (i.e. as Grassmann variables for the fermions). Here, *creation operators* are those with negative moding and *annihilation operators* those with positive moding.

Exercise:

Show that the above normal ordering prescription in the case of free scalars or free fermions is equivalent to the following rule

$$: A(w)B(w) : \equiv \oint_w \frac{dz}{2\pi i} \frac{A(z)B(w)}{z-w}\tag{130}$$

Whether we define the normal ordering one way or another, we may evaluate the OPE between normal ordered expressions by means of **Wick's Theorem**. Consider the OPE between normal ordered expressions of the form

$$A(z) \equiv: A_1(z)A_2(z)\dots :$$

and

$$B(w) \equiv: B_1(w)B_2(w)\dots :$$

The product of these two may be classified according to the number of *contractions*. The situation is simplest for free fields, which are the ones we consider here. A contraction between $A_i(z)$ and $B_j(w)$ is simply the vacuum expectation value of the OPE of the two, or the two-point function

$$\langle A_i(z)B_j(w) \rangle$$

The product of $A(z)$ and $B(w)$ may now be written as a sum of terms involving, 0, 1, 2, ... contractions. The term with zero contractions is the fully normal ordered term:

$$: A_1(z)A_2(z)\dots B_1(w)B_2(w)\dots :$$

It is part of the contribution usually written in OPE's as *n.s.t.*, the non-singular term.

Next follows the single contractions. These are obtained from the zero contraction term by removing in all possible ways, one $A_i(z)$ factor and one $B_j(w)$ factor, and replacing them with the contraction factor. In so doing, one must remember for fermi fields to count how many signs it takes to move the A factor up to the B factor.

The multiple contractions are similarly the ones obtained by collecting multiple $\langle A_i B_j \rangle$ pairs in all possible ways, working out always the signs if fermions are present.

As the name (Wick's Theorem) suggests, the proof is just like the first step in the proof of Feynman rules. It is worth while emphasizing though, that the calculation is completely exact. There is no perturbation theory involved here. In the present case of *free theories* this may seem like an empty statement, however, this way of calculating OPE's in a CFT holds beyond free field theory to some extent, and also even in the case of the free field theory there are in fact non-trivial aspects, to which we shall come back.

The best thing is to illustrate all this on our two examples.

4.1.1 Free scalar field

Let us check that $\partial\phi(z)$ is a primary conformal field of dimension, 1. Thus consider the OPE with the energy momentum tensor, and compare with eq.(55):

$$\begin{aligned}
T_\phi(z)\partial\phi(w) &= -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \\
&= -\frac{1}{2} (2\partial\phi(z)\langle\partial\phi(z)\partial\phi(w)\rangle) + \text{n.s.t.} \\
&= +\frac{1}{(z-w)^2}\partial\phi(z) + \text{n.s.t.} \\
&= \frac{1}{(z-w)^2}(\partial\phi(w) + (z-w)\partial[\partial\phi(w)] + \dots) + \text{n.s.t.} \\
&= \frac{1}{(z-w)^2}\partial\phi(w) + \frac{1}{z-w}\partial[\partial\phi(w)] + \text{n.s.t.} \tag{131}
\end{aligned}$$

The last result is what we wanted: The OPE shows that there are only two singular terms (like there should be for a primary field), a single pole the coefficient of which is the derivative of the primary field (times 1), and a double pole the coefficient of which is the primary field itself times the conformal dimension of it, in our case here: 1. Notice the importance of making sure that the fields on the right hand side are always taken at the point, w , the point of the primary field of the left hand side. This gave rise to certain Taylor expansions, of which we only needed a finite number of terms, since we are only interested in the *singularities* of the OPE here. Very often one meets with the practical problems that free scalar fields may be normalized differently from what we present here. Correspondingly the coefficient in front of the energy momentum tensor will then be changed. The best way of checking whether one has a consistent set of normalizations, is to do the above calculation and making sure that the coefficient of the single pole term is correct. Then and only then can one use the coefficient of the double pole term to read off the conformal dimension.

4.1.2 Free fermi field

Let us similarly check that the free fermi field is a primary field of conformal dimension, $\frac{1}{2}$:

$$\begin{aligned}
T_F(z)\psi(w) &= -\frac{1}{2} : \psi(z)\partial\psi(z) : \psi(w) \\
&= -\frac{1}{2}(\psi(z)\partial_z\langle\psi(z)\psi(w)\rangle - \langle\psi(z)\psi(w)\rangle\partial_z\psi(z)) + \text{n.s.t.} \\
&= +\frac{1}{2}\frac{1}{(z-w)^2}\psi(z) + \frac{1}{2}\frac{1}{z-w}\partial\psi(z) + \text{n.s.t.} \\
&= \frac{1}{2}\frac{1}{(z-w)^2}(\psi(w) + (z-w)\partial\psi(w) + \dots) + \frac{1}{2}\frac{1}{z-w}(\partial\psi(w) + \dots) + \text{n.s.t.} \\
&= \frac{\frac{1}{2}}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial\psi(w) + \text{n.s.t.} \tag{132}
\end{aligned}$$

Again this is the anticipated result, with a pole and a double pole both having the coefficients required for a primary of dimension $\frac{1}{2}$, and with no higher singularities.

Exercise:

Work out the singularities in the OPE between the relevant energy momentum tensors and $\partial^2\phi(w)$ and $\partial\psi(w)$ and show that these are not primary fields. What goes wrong in the OPE's? Show that the poles and the double poles are just as expected. Why is the dimensions "expected" to be 2 and 3/2 respectively?

4.1.3 The central charge in free field theory

We now come to the somewhat more complicated calculation of working out the OPE's of the energy momentum tensors with themselves. We shall see that T is *not* a primary field, that it *is* however, a field of dimension, 2, and we shall find the correct values of the central charge, c , for the two kinds of theories. We compare the result of course with eq.(61).

First the scalar field:

$$T_\phi(z)T_\phi(w) = \frac{1}{4} : \partial\phi(z)\partial\phi(z) : : \partial\phi(w)\partial\phi(w) : \tag{133}$$

It is best to consider the single and the double contractions in turn. Clearly this time there *are* some double contractions to worry about, and clearly there are no *more* than singles and doubles to worry about.

The single contractions give the result (we stop writing explicitly +n.s.t. all the time):

$$\begin{aligned}
&\frac{1}{4} \cdot 4\langle\partial\phi(z)\partial\phi(w)\rangle : \partial\phi(z)\partial\phi(w) : \\
&= -\frac{1}{(z-w)^2}(: \partial\phi(w)\partial\phi(w) : + (z-w) : \partial^2\phi(w)\partial\phi(w) : + \dots) \\
&= -\frac{1}{(z-w)^2} : \partial\phi(w)\partial\phi(w) : - \frac{1}{z-w}\frac{1}{2}\partial_w : \partial\phi(w)\partial\phi(w) : \\
&= \frac{2}{(z-w)^2}T_\phi(w) + \frac{1}{z-w}\partial T_\phi(w) \tag{134}
\end{aligned}$$

This result is exactly the result expected for a *primary field* of conformal dimension, 2. However, we are not through yet. We still have the double contractions to do. That one will “spoil” the primary nature of the energy momentum tensor and instead introduce a central charge. Quite generally the single contractions produce classical results, multiple contractions produce quantum corrections.

The double contractions give rise to the terms

$$\frac{1}{4} \cdot 2(\langle \partial\phi(z)\partial\phi(w) \rangle)^2 = \frac{\frac{1}{2}}{(z-w)^4} \quad (135)$$

This is exactly the term,

$$\frac{c/2}{(z-w)^4}$$

in the fundamental OPE, eq.(61), and it shows that the free scalar theory has a central charge of

$$c_\phi = 1$$

Finally we do the same calculation for the free fermi theory:

$$T_F(z)T_F(w) = \frac{1}{4} : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \quad (136)$$

First the single contractions:

$$\begin{aligned} & \frac{1}{4}(-\langle \psi(z)\psi(w) \rangle : \partial\psi(z)\partial\psi(w) : + \partial_w \langle \psi(z)\psi(w) \rangle : \partial\psi(z)\psi(w) : \\ & + \partial_z \langle \psi(z)\psi(w) \rangle : \psi(z)\partial\psi(w) : - \partial_z \partial_w \langle \psi(z)\psi(w) \rangle : \psi(z)\psi(w) :) \\ & = \frac{1}{4} \left\{ -\frac{1}{z-w} (: \partial\psi(w)\partial\psi(w) : + \dots) \right. \\ & + \frac{1}{(z-w)^2} (: \partial\psi(w)\psi(w) : + (z-w) : \partial^2\psi(w)\psi(w) : + \dots) \\ & - \frac{1}{(z-w)^2} (: \psi(w)\partial\psi(w) : + (z-w) : \partial\psi(w)\partial\psi(w) : + \dots) \\ & + \left. \frac{2}{(z-w)^3} (: \psi(w)\psi(w) : + (z-w) : \partial\psi(w)\psi(w) : + \frac{1}{2}(z-w)^2 : \partial^2\psi(w)\psi(w) : + \dots) \right\} \\ & = \frac{1}{4} \left\{ 0 + \frac{1}{(z-w)^2} : \partial\psi(w)\psi(w) : + \frac{1}{z-w} : \partial^2\psi(w)\psi(w) : \right. \\ & - \frac{1}{(z-w)^2} : \psi(w)\partial\psi(w) : + 0 + 0 + \frac{2}{(z-w)^2} : \partial\psi(w)\psi(w) : + \frac{1}{z-w} : \partial^2\psi(w)\psi(w) : \left. \right\} \\ & = -\frac{1}{(z-w)^2} : \psi(w)\partial\psi(w) : - \frac{1}{2} \frac{1}{z-w} : \psi(w)\partial^2\psi(w) : \\ & = \frac{2}{(z-w)^2} T_F(w) + \frac{1}{z-w} \partial T_F(w) \quad (137) \end{aligned}$$

(notice that $:\psi(w)\psi(w): \equiv 0$ etc. since operators inside normal ordering signs commute and anticommute like classical fields, so the square of any fermi field is zero.) Again we see the single contractions producing the expected result apart from the central charge

term. That one we get again from the double contractions:

$$\begin{aligned}
& \frac{1}{4} \{ -\langle \psi(z)\psi(w) \rangle \partial_z \partial_w \langle \psi(z)\psi(w) \rangle + \partial_w \langle \psi(z)\psi(w) \rangle \partial_z \langle \psi(z)\psi(w) \rangle \} \\
&= \frac{1}{4} \left\{ +\frac{2}{(z-w)^4} - \frac{1}{(z-w)^4} \right\} \\
&= \frac{\frac{1}{4}}{(z-w)^4}
\end{aligned} \tag{138}$$

Showing that the central charge of the free fermion theory is

$$c_F = \frac{1}{2}$$

It is rather clear that if we consider a theory built by tensoring several commuting free scalar fields and several (anti) commuting free fermi fields, then the total central charge will be the sum of the individual pieces. Therefore we may think of the central charge as somehow measuring the “effective number of degrees of freedom” in the theory, counting 1 for a free scalar boson and $\frac{1}{2}$ for a free fermion. We shall see later that this intuition may be taken further, and is thus rather useful.

4.2 The Schwartzian derivative

We have seen that under an infinitesimal conformal transformation the energy momentum tensor transforms as

$$\delta_\epsilon T(z) = \epsilon(z) \partial T(z) + 2\partial\epsilon(z)T(z) + \frac{c}{12}\epsilon'''(z) \tag{139}$$

where the last term in fact shows that we are not dealing with a primary field: the energy momentum tensor is not an honest tensor. If it had been a tensor, then under the finite conformal transformation

$$z \mapsto f(z)$$

it would transform as

$$T(z) \rightarrow (f'(z))^2 T(f(z))$$

We need to understand how to modify this rule in the presence of the extra central charge term. The answer turns out to be given by the *Schwartzian derivative*:

$$\{f, z\} = \frac{f'''(z)f'(z) - \frac{3}{2}f''(z)^2}{f'(z)^2} \tag{140}$$

and the correct finite transformation is

$$T(z) \rightarrow (f'(z))^2 T(f(z)) + \frac{c}{12} \{f, z\} \tag{141}$$

We shall present two arguments for this results: A straight forward mathematical one, and a perhaps instructive one based on an explicit consideration for the free scalar theory.

First consider the mathematical aspect.

From the infinitesimal transformation

$$\delta_\epsilon T = \epsilon T' + 2\epsilon' T + \frac{c}{12}\epsilon''' \tag{142}$$

corresponding to $f(z) = z + \epsilon(z)$ we have that

$$\{z + \epsilon, z\} = \epsilon''' + o(\epsilon^2) \quad (143)$$

while dimensional analysis tells us that $\{f, z\}$ is of dimension two. Two successive transformations $z \rightarrow u(z) \rightarrow w(u(z))$ result in

$$\begin{aligned} T(z) &= \left(\frac{du}{dz}\right)^2 T(u) + \frac{c}{12}\{u, z\} \\ &= \left(\frac{du}{dz}\right)^2 \left(\left(\frac{dw}{du}\right)^2 T(w) + \frac{c}{12}\{w, u\} \right) + \frac{c}{12}\{u, z\} \\ &= \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w, z\} \end{aligned} \quad (144)$$

so we must have

$$\{w, z\} = \left(\frac{du}{dz}\right)^2 \{w, u\} + \{u, z\} \quad (145)$$

This is the crucial result. The Schwartzian derivative is the solution to this equation consistent with the infinitesimal result. This is not difficult to verify. Actually, however, it is possible to argue rather directly for the form. From what we know, the anomalous term must vanish identically for the projective (or $SL(2, C)$) transformations. In particular, invariance under scaling transformations puts $\{\lambda u, u\} = 0$ from which we get

$$\{\lambda u, z\} = \{u, z\} \quad (146)$$

This means that $\{f, z\}$ must be invariant under $f \rightarrow \lambda f$. Combining the above arguments we deduce that the natural ansatz is

$$\{f, z\} = \frac{f'''(z)}{f'(z)} + a \underbrace{\left(\frac{f''(z)}{f'(z)}\right)^2}_{=0} + g(z) + \dots \quad (147)$$

We cannot allow any "free" z -dependence $g(z)$, because it would violate the infinitesimal version. Likewise, there is no "pure" $f(z)$ -dependence. The coefficient, 1, in front of the first term is compensated for by the conventional $\frac{c}{12}$ in (141). Furthermore, in the general ansatz, we cannot allow higher derivatives of f than one in the denominator because such terms would lead to divergencies in the infinitesimal version. It is impossible to do proper expansions if they are included, take e.g.

$$\left(\frac{f^{(n+1)}(z)}{f^{(n)}(z)}\right)^2 \quad n \geq 2 \quad (148)$$

Check that consistency with (145) demands $a = -\frac{3}{2}$.

This concludes our argumentation for the form of the anomalous term being the Schwartzian derivative.

Next consider the calculation based on our free scalar field theory example. In fact, naively it might be surprising that there should be any anomalous term, since for the scalar theory, we have

$$T_\phi(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) :$$

and the fields, $\partial\phi(z)$ are in fact primary and transform just into $f'(z)\partial\phi(f(z))$, so why does the product not? The answer lies in the regularization inherent in the introduction of the normal ordering signs. Let us introduce the normal ordering via a point splitting and do the transformation more carefully:

$$\begin{aligned}
T(z) &= -\frac{1}{2} \lim_{\delta \rightarrow 0} \left(\partial\phi(z + \frac{1}{2}\delta) \partial\phi(z - \frac{1}{2}\delta) + \frac{1}{\delta^2} \right) \\
&\rightarrow -\frac{1}{2} \lim_{\delta \rightarrow 0} \left(f'(z + \frac{1}{2}\delta) f'(z - \frac{1}{2}\delta) \partial\phi(f(z + \frac{1}{2}\delta)) \partial\phi(f(z - \frac{1}{2}\delta)) + \frac{1}{\delta^2} \right) \\
&= -\frac{1}{2} \lim_{\delta \rightarrow 0} \left(f'(z + \frac{1}{2}\delta) f'(z - \frac{1}{2}\delta) \partial\phi(f(z + \frac{1}{2}\delta)) \partial\phi(f(z - \frac{1}{2}\delta)) \right. \\
&\quad + \frac{f'(z + \frac{1}{2}\delta) f'(z - \frac{1}{2}\delta)}{(f(z + \frac{1}{2}\delta) - f(z - \frac{1}{2}\delta))^2} \\
&\quad - \frac{f'(z + \frac{1}{2}\delta) f'(z - \frac{1}{2}\delta)}{(f(z + \frac{1}{2}\delta) - f(z - \frac{1}{2}\delta))^2} \\
&\quad \left. + \frac{1}{\delta^2} \right) \\
&= (f'(z))^2 T(f(z)) - \frac{1}{2} \lim_{\delta \rightarrow 0} \left(-\frac{f'(z + \frac{1}{2}\delta) f'(z - \frac{1}{2}\delta)}{(f(z + \frac{1}{2}\delta) - f(z - \frac{1}{2}\delta))^2} + \frac{1}{\delta^2} \right) \tag{149}
\end{aligned}$$

where we used the appropriate point splitting for defining $T(f(z))$. So, the anomalous term evaluates to

$$\begin{aligned}
&\frac{1}{2} \frac{(f'(z) + \frac{1}{2}\delta f''(z) + \frac{1}{8}\delta^2 f'''(z))(f'(z) - \frac{1}{2}\delta f''(z) + \frac{1}{8}\delta^2 f'''(z))}{(\delta f'(z) + \frac{1}{24}\delta^3 f'''(z))^2} - \frac{1}{2\delta^2} \\
&= \frac{1}{2} \frac{f'^2 - \frac{1}{4}\delta^2 f''^2 + \frac{1}{4}\delta^2 f' f'''}{\delta^2 f'^2} \left(1 - \frac{1}{12}\delta^2 \frac{f'''}{f'}\right) - \frac{1}{2\delta^2} \\
&= \frac{1}{12} \frac{f' f''' - \frac{3}{2} f''^2}{f'^2} \tag{150}
\end{aligned}$$

which is the result we wanted to prove.

4.3 The central charge as the ground state energy

Let us consider the theory defined on the cylinder previously introduced, but let us keep an arbitrary circumference of ℓ for that, i.e. we use coordinates

$$\zeta = \tau + i\sigma = \frac{\ell}{2\pi} \log z \tag{151}$$

with $\sigma \in [0, \ell[$ and fields appropriately periodical to become well defined on the cylinder. Let us denote by $T_c(\zeta)$ and $T_p(z)$ the energy momentum tensors on the cylinder and on the plane. Using the finite transformation law including the schwartzian derivative, which we have just obtained, we get for

$$z = f(\zeta) = \exp\left\{\frac{2\pi}{\ell}\zeta\right\}$$

$$\begin{aligned}
\{f, \zeta\} &= \frac{\left(\frac{2\pi}{\ell}\right)^4}{\left(\frac{2\pi}{\ell}\right)^2} \left(1 - \frac{3}{2}\right) \\
&= -\frac{1}{2} \left(\frac{2\pi}{\ell}\right)^2 \\
\frac{df}{d\zeta} &= \frac{2\pi}{\ell} z \\
T_c(\zeta) &= \left(\frac{2\pi}{\ell}\right)^2 \left(z^2 T_p(z) - \frac{c}{24}\right)
\end{aligned} \tag{152}$$

The hamiltonian on the cylinder is the space integral of the time-time component of the energy momentum tensor

$$H_c = \frac{1}{2\pi} \int_0^\ell d\sigma (T_c(\sigma, 0) + \bar{T}_c(\sigma, 0)) \tag{153}$$

Then we get

$$\begin{aligned}
\frac{1}{2i\pi} \int d\zeta T_c(\zeta) &= \left(\frac{2\pi}{\ell}\right)^2 \oint \frac{dz}{2\pi i} \frac{\ell}{2\pi} \frac{1}{z} \left[z^2 T_p(z) - \frac{c}{24}\right] \\
&= \frac{2\pi}{\ell} L_0 - \frac{\pi c}{12\ell}
\end{aligned} \tag{154}$$

with L_0 being the Virasoro generator in the plane. Thus

$$H_c = \frac{2\pi}{\ell} (L_0 + \bar{L}_0) - \frac{\pi c}{6\ell} \tag{155}$$

We see that the energy on the cylinder get eigenvalues of the form

$$E = E_0 + \frac{2\pi}{\ell} \Delta \tag{156}$$

where

$$\Delta \equiv h + \bar{h}$$

is the scaling dimension of the primary state in question. We have argued that this has to be positive in a sensible theory. Further

$$E_0 \equiv -\frac{\pi c}{6\ell}$$

showing that indeed the central charge is related to the minimal energy in the system.

This result is closely related to the so called Casimir effect: that the infinite sum over harmonic oscillator zero modes, contributing to the vacuum energy, gets modified when the system is confined to a compact region, in particular in 1 dimension. This in a way is our problem here. The “space” of our problem is the interval $[0, \ell]$ with periodic boundary conditions. As is well known the free scalar theory gives rise to a hamiltonian, which is simply a sum of harmonic oscillators:

$$H_c = \sum_n \omega_n (a_n^\dagger a_n + \frac{1}{2})$$

where the a_n 's are standard harmonic oscillator annihilation operators, and where the energies are

$$\omega_n = \frac{2\pi}{\ell}n$$

We have omitted the contribution from the anti holomorphic parts, but it is completely analogous. Also, the normalization of the annihilation operators here is the one standard for the harmonic oscillators, and different from what we did in eq.(128), but the result of course is the same. We see that the zero point energy becomes the formal expression (counting also the anti holomorphic contribution)

$$E_0 = \frac{2\pi}{\ell} \sum_n n \quad (157)$$

One may think of defining this divergent sum in terms of a regularization. It is popular to use the zeta function regularization giving

$$\begin{aligned} \zeta(s) &= \sum_n \frac{1}{n^s} \\ \zeta(-1) &= -\frac{1}{12} \end{aligned} \quad (158)$$

which just gives the result we want in our case. However, it is worth while emphasizing that when we carry out a renormalization of a quantum field theory, there are always two steps to consider: (i) we have to do a regularization, and this involves a considerable arbitrariness. Then (ii) we have to control the arbitrariness by means of *renormalization conditions*, specifying for example that certain standard measurements must result in certain prescribed numerical results.

In the case at hand, the (tacit) renormalization condition is that the field theory we obtain should be a *conformal field theory*. This is what has allowed us to circumvent in a powerful way, talking about defining divergent sums.

4.4 The central charge as a conformal anomaly

We have emphasized that a defining property of conformal field theory is the vanishing of the trace of the energy momentum tensor: This is the response to scale transformations. In the conformal holomorphic/antiholomorphic coordinates we use, this means

$$\langle T_{z\bar{z}} \rangle = 0$$

However, this tracelessness is broken by an anomaly when we consider the theory defined on a curved background. Indeed, when curvature is present, described by

$$R(z, \bar{z}) = \frac{-2}{R_1 R_2}$$

with R_1 and R_2 being the principal curvatures of the 2-dimensional surface at the point we consider, one finds instead of a vanishing expectation value for the trace of the energy momentum tensor, the formula

$$\langle T_{z\bar{z}} \rangle = -\frac{c}{48\pi} R \quad (159)$$

This provides yet another physical significance for the central charge.

It is rather clear from simple dimensional considerations, that the trace anomaly will have to be proportional to R , and in an otherwise scale invariant theory this is the only thing it could depend on. Also, it was shown in Jan Ambjørns's lectures that for a free scalar field theory, the trace anomaly is the above result with the special value $c = 1$ in that case. Here we complete the statement by showing in a simple way that for a conformal field theory the trace anomaly must also be proportional to the central charge. In fact it is possible to perform the argument below carefully and completely derive eq.(159), but for simplicity we stop at the point where we obtain proportionality to c , and then refer to Ambjørn's lecture for the prefactor.

In fact, we shall be concerned with the variation of $\langle T_{\mu\nu} \rangle$, under a genuine geometrical shift, $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, not just the gauge variations we have mostly considered up until now. So we imagine a shift in which the reparametrization invariant curvature undergoes a genuine change. Under such a change, the classical action changes by

$$\delta S = -\frac{1}{4\pi} \int T_{\mu\nu} \delta g^{\mu\nu} d^2 z \quad (160)$$

which gives us first

$$\frac{\delta}{\delta g^{\mu\nu}(z)} \int \mathcal{D}\phi e^{-S} = +\frac{1}{4\pi} \langle T_{\mu\nu}(z) \rangle = \frac{1}{4\pi} \int \mathcal{D}\phi e^{-S} T_{\mu\nu}(z) \quad (161)$$

Actually this equation is just the statement that the partition function of our theory is in fact *invariant* under the metric deformation: The path integral in the old geometry with the old action is equal to the path integral in the new geometry with the new action:

$$e^{-S_{\text{new}}} = e^{-S_{\text{old}}} \left(1 + \frac{1}{4\pi} \int d^2 z T_{\mu\nu}(z) \delta g^{\mu\nu}(z) \right) \quad (162)$$

Similarly, the expectation value of the energy momentum tensor is invariant, but this invariance is again the result of two compensating changes: the trace of the energy momentum tensor changes, and the theory changes. Thus a second variation may be written as

$$\begin{aligned} 0 &= \delta \langle T_{\mu\nu}(z) \rangle \\ &= \langle \delta T_{\mu\nu}(z) \rangle + \frac{1}{4\pi} \int \mathcal{D}\phi e^{-S} T_{\mu\nu}(z) \int d^2 w T_{\lambda\sigma}(w) \delta g^{\lambda\sigma}(w) \end{aligned} \quad (163)$$

or

$$\langle T_{\mu\nu}(z) \rangle = -\frac{1}{4\pi} \int d^2 w \langle T_{\mu\nu}(z) T_{\lambda\sigma}(w) \rangle \delta g^{\lambda\sigma}(w) \quad (164)$$

We see that we are going to get the two point function for the energy momentum tensor involved. Let us change to the standard holomorphic/antiholomorphic basis. Then there is only a nonvanishing two point function between two $T = T_{zz}$'s or two $\bar{T} = T_{\bar{z}\bar{z}}$'s,

$$\langle T(z) T(w) \rangle = \frac{c/2}{(z-w)^4},$$

and we get

$$\langle \delta T(z) \rangle = -\frac{c}{8\pi} \int d^2 w \frac{\delta g^{zz}(w)}{(z-w)^4} \quad (165)$$

This expression is divergent and has to be regularized. That may be done in many ways and we adopt the following procedure with a small regulator, a

$$\langle \delta T(z) \rangle = -\frac{c}{8\pi} \int d^2w \delta g^{zz}(w) \frac{(\bar{z} - \bar{w})^4}{((z-w)(\bar{z}-\bar{w}) + a^2)^4}$$

This introduces a dependence on \bar{z} and a small calculation gives the result

$$\bar{\partial} \langle \delta T(z) \rangle = -\frac{c}{2\pi} \int d^2w \delta g^{zz}(w) \frac{a^2(\bar{z} - \bar{w})^3}{((z-w)(\bar{z}-\bar{w}) + a^2)^5}$$

We see that the factor multiplying δg^{zz} is a distribution vanishing everywhere except near $w = z$ as it should be since we took the derivative of $T(z)$ with respect to \bar{z} which vanishes outside singularities. Without loss of generality we may put $z = 0$ and evaluate the integral in polar coordinates to obtain

$$\frac{c}{4\pi} \int_0^\infty dr^2 \int_0^{2\pi} d\theta \delta g^{zz}(w) \frac{a^2 r^3 e^{-3i\theta}}{(r^2 + a^2)^5}$$

In this integral the angular integration will produce zero for a constant δg^{zz} . In fact to obtain a non vanishing contribution we must expand δg^{zz} to third order in w (as opposed to \bar{w}), giving

$$\begin{aligned} & \frac{a^2 c 2\pi}{3! 4\pi} \int_0^\infty dr^2 \partial^3 \delta g^{zz}(0) \frac{r^6}{(r^2 + a^2)^5} \\ &= \frac{a^2 c}{12} \partial^3 \delta g^{zz}(0) \int_0^\infty dx \frac{x^3}{(x + a^2)^5} \\ &= \frac{c}{48} \partial^3 \delta g^{zz} \end{aligned} \tag{166}$$

If we imagine we start from a flat situation and consider a small curvature, we may remove the δ 's. Also we may use the conservation of energy and momentum in the form

$$\bar{\partial} T_{zz} + \partial T_{z\bar{z}} = 0$$

to obtain

$$\langle T_{z\bar{z}} \rangle = -\frac{c}{48} \partial^2 g^{zz} \tag{167}$$

This in fact is just the result we wanted to obtain: The left hand side is the trace of the energy momentum tensor and the right hand side is proportional to c . That proportionality came about as a result of the universal form of the two point function for the energy momentum tensor. We do not yet quite have the wanted factor

$$-\frac{R}{48\pi}$$

multiplying c , but the reader may imagine how a more careful treatment of the geometrical aspects of the calculation will produce that. At this point we merely appeal to the calculation by Ambjørn, that for the free scalar theory with $c = 1$, that factor was obtained.

5 On the representations of the Virasoro algebra

Imagine some CFT. The corresponding Hilbert space will constitute a representation space for the Virasoro algebra. This statement is an obvious consequence of the fact that the energy momentum tensor is the generator of the Virasoro algebra, and it is allowed to act freely on the Hilbert space of the theory. Actually, the Hilbert space is in fact a representation space for the two commuting Virasoro algebras belonging to the independent holomorphic and antiholomorphic degrees of freedom. Even though this latter point is quite important in many ways, we shall mostly more or less ignore it here.

The fact that we may view the Hilbert space as a representation space implies that we can learn very much about CFT's by studying the representation theory of the Virasoro algebra. The rather thorough understanding of the representation theory of the Virasoro algebra, is partly the origin of the fact that in many cases CFT's may be solved exactly.

We shall only be interested in particular kinds of representations, those termed highest weight representations, as previously explained. We have already derived the defining property for those: There is a highest weight state (hws), $|h\rangle$ satisfying

$$\begin{aligned} L_0|h\rangle &= h|h\rangle \\ L_n|h\rangle &= 0, \quad \text{for } n > 0 \end{aligned} \tag{168}$$

Highest weight states are in 1 – 1 correspondence with primary fields: the former are obtained by acting with a primary field on the $SL(2)$ invariant vacuum. The reason we only want to study highest weight representations, is that we want the eigenvalue of L_0 to be bounded from below, in fact to be *positive* so that correlators *fall off* with distance rather than increase with distance. Indeed as we have seen, L_0 , is closely related to the hamiltonian of the system with “radial time”. Actually, for realistic quantum field theories we even want the representations to be *unitary*, and one may argue that also for most applications to critical phenomena this is required. It should be emphasized though, that even though the theory finally has to be unitary, there are by now very many constructions of such theories involving non-unitary representations at certain intermediate steps. However, here we shall mostly restrict ourselves to *unitary highest weight representations*.

It has turned out, that there are some rather amazing results about such representations. Below we shall try to explain and illustrate those result, but we shall not have time to provide complete proofs of everything.

5.1 The Verma module

The Verma module defines the simplest kind of representation we may think about. It is generically irreducible, but in fact we shall be mostly interested in those cases where it is *not*. Only in those cases will the representations turn out to be unitary if $c < 1$, and those are the (very famous) representations we shall mostly describe.

If we act on the highest weight state with an L_n with n positive, we get zero, so we only need to act with L_n 's with $n < 0$. Thus we may build states like

$$L_{-1}|h\rangle, L_{-1}L_{-1}|h\rangle, L_{-2}|h\rangle, L_{-2}L_{-1}|h\rangle$$

etc., etc. Quite generally, we may construct a state of the form

$$L_{\{-n\}_k}|h\rangle \equiv L_{-n_1}L_{-n_2}\dots L_{-n_k}|h\rangle \tag{169}$$

and using the Virasoro algebra, we may always arrange to convert such a state into a linear combination of states of this form satisfying some lexicographic ordering, like

$$n_1 \geq n_2 \geq \dots \geq n_k > 0$$

Also, from the Virasoro algebra follows

$$[L_0, L_{-n}] = nL_{-n} \quad (170)$$

i.e., the operator L_{-n} increases the conformal dimension by the amount n . Clearly the space spanned by the states eq.(169) form a representation space for the Virasoro algebra. To prove this we have to check that acting with any generator, L_{-N} , on $L_{\{-n\}_k}|h\rangle$ we get a linear combination of terms of the same kind. If $N \geq n_1$ this is trivial. Otherwise, we use the algebra to move up the generator L_{-N} to where it belongs in the lexicographic ordering. Every time we use the algebra, we get new terms of similar kind, but eventually we are done. Similarly, for an L_N with $N > 0$, we freely introduce the commutator of that with the product of all the other generators, and use the algebra to get rid of it.

The Verma module so constructed is graded according to the eigenvalue of L_0 :

$$V = \bigoplus_N V_N \quad (171)$$

where V_N is spanned by states $L_{\{-n\}_k}|h\rangle$ for which

$$n_1 + n_2 + \dots + n_k = N$$

For a fixed N , there is only a finite number of independent such states. The number N is called *the level*. Thus for $N = 1$ we only have

$$L_{-1}|h\rangle \quad (172)$$

for $N = 2$

$$\begin{aligned} L_{-1}L_{-1}|h\rangle \\ L_{-2}|h\rangle \end{aligned} \quad (173)$$

and for $N = 3$

$$\begin{aligned} L_{-1}L_{-1}L_{-1}|h\rangle \\ L_{-2}L_{-1}|h\rangle \\ L_{-3}|h\rangle \end{aligned} \quad (174)$$

so that $\dim V_1 = 1$, $\dim V_2 = 2$, $\dim V_3 = 3$. In fact

$$\dim V_N = P(N) \quad (175)$$

where $P(N)$ is the number of partitions of the integer, N : the number of ways N may be written as a sum of (lexicographically ordered) positive integers.

In the subspace V_N , L_0 has eigenvalue $h + N$.

It should not be surprising that generically the Verma module is irreducible. Recall, that a representation space of a group or an algebra is called *reducible* if it is possible

to find a (non-trivial) sub space which is mapped entirely into itself under the action of the generators, so that this subspace constitutes a sub representation by itself. It is well known that for finite dimensional Lie algebras, one constructs irreducible representations exactly in analogy to the construction of the Verma module: One starts by dividing all generators into three categories: raising generators, lowering generators and generators in the Cartan sub algebra. For $su(2)$ these are the well known, J^+, J^-, J^3 . A highest weight state is one which is annihilated by all the raising generators, analogous to our L_n with $n > 0$. The Cartan generator for us is simply L_0 (and c). For finite dimensional Lie algebras, one then acts on the highest weight state with lowering generators in all possible ways until one gets zero. This builds a finite dimensional, irreducible representation, quite analogous to the Verma module, except the Verma module is infinite dimensional.

The surprising thing is that in certain exceptional cases, of great practical importance, the Verma module turns out to be in fact reducible, in the most interesting cases even infinitely reducible. All the states of the form $L_{\{-n\}_k}|h\rangle$ are called *descendant states*, as opposed to the highest weight state, $|h\rangle$ itself. The condition for having a reducible representation is to be able to find at some level, N , in the Verma module a descendant state $|s\rangle$, which is itself also *highest weight*, in other words one for which

$$L_M|s\rangle = 0 \quad \text{for } M > 0 \quad (176)$$

in addition to

$$L_0|s\rangle = h + N$$

Any state satisfying eq.(176) is termed *singular*. Clearly such a singular state somewhere down in the Verma module will give rise to a new sub-Verma module, which is itself a sub representation space, so that if such singular vectors occur, the representation is reducible, or *degenerate* as it is often called.

We shall soon be much concerned with these singular states.

Let us first however, work out some very simple necessary conditions for representations to be *unitary*. Our unitarity is linked to the hermiticity condition,

$$L_n^\dagger \equiv L_{-n} \quad (177)$$

It is very easy to show that unitary representations can occur only for $h \geq 0$ and $c \geq 0$. In fact consider the state $L_{-N}|h\rangle$ for N some positive integer. Unitarity requires the norm of this state to be non-negative:

$$\langle h|L_N L_{-N}|h\rangle \geq 0$$

but we may use the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{n+m} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}$$

together with the highest weight condition, $L_N|h\rangle = 0$ to calculate the above as (taking $\langle k|h\rangle = 1$)

$$\langle h|L_N L_{-N}|h\rangle = \langle h|[L_N, L_{-N}]|h\rangle = \langle h|[2NL_0 + \frac{c}{12}N(N^2 - 1)]|h\rangle = 2Nh + \frac{c}{12}N(N^2 - 1) \geq 0 \quad (178)$$

Taking $N = 1$ we see that $h \geq 0$ and taking $N \rightarrow \infty$ we obtain $c \geq 0$.

5.2 The minimal unitary series

It is known that unitary representations exist for any value of $h \geq 0$ provided $c > 1$, and that is also not so hard to prove. However for $0 \leq c < 1$ an extremely interesting and most amazing situation occurs which we now describe. Later we come back to several sample calculations illustrating the situation (but not providing complete proofs).

In general the Verma module is irreducible, but for particular values of the conformal dimension, degeneracies occur. There is a famous formula, the Kac formula for these conformal dimensions, which reads:

$$h_{p,q}(c) = \frac{1}{48} \{ 12(p-q)^2 + (1-c)(p^2 + q^2 - 2) \pm (p^2 - q^2) \sqrt{(25-c)(1-c)} \} \quad (179)$$

Here p and q are any integers, and for any choice, the above describes a curve in the (c, h) plane as shown in fig.1. For conformal dimensions corresponding to these curves, degeneracies occur: The Verma module is reducible.

Further it is possible to show that for nearly *all* points in the region $h \geq 0$, $0 < c < 1$, the representation of the Virasoro algebra is *non-unitary*: there exist states in the representation space having negative norms. However, an important discovery made by Friedan, Qiu and Shenker (the existence part of which was later proved by Goddard, Kent and Olive) is that there exist magical isolated points for which the Verma module is highly (infinitely) reducible with many zero-norm states, but for which equivalence classes may be obtained to define *unitary representations*. These situations occur when *an infinity* of curves of the type in eq.(179) cross each other at a certain point. The following transformation allows one rather easily to realize that this can happen.

Define the new variable, m , rather than c by the transformation

$$c = 1 - \frac{6}{m(m+1)} \quad (180)$$

Inserting this parametrization of c in eq.(179) one obtains an equivalent, useful formula for the $h_{p,q}$'s:

$$h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad (181)$$

It is now rather obvious that whenever m is an integer (and it is enough to consider $m = 2, 3, 4, \dots$), an infinity of curves, will intersect at one point, namely all solutions, (p, q) , to the equation

$$(m+1)p - mq = \text{constant}$$

will produce identical values for $h_{p,q}(m)$.

The famous unitary series obtains when m is an integer and p and q are restricted as

$$p = 1, 2, \dots, m-1; \quad q = 1, 2, \dots, m$$

(When (p, q) are not restricted, the values of $h_{p,q}$ repeat periodically and symmetrically as functions of (p, q)). The case $m = 2$ is rather trivial. It gives $c = 0$ and $h_{1,1} = h_{1,2} = 0$.

The case $m = 3$ is already very interesting and we shall be much concerned with it. It turns out to describe the critical Ising model. From the formulas above we find $c = \frac{1}{2}$ together with the following *Kac-table* (we take p to label the columns and q to label the rows from below) of conformal dimensions:

1/2	0
1/16	1/16
0	1/2

Similarly $m = 4$ gives rise to $c = 7/10$ and the Kac-table:

3/2	7/16	0
3/5	3/80	1/10
1/10	3/80	3/5
0	7/16	3/2

The reader may easily verify the general presence of some obvious symmetries. This case, $m = 4$, describes the critical behaviour of the tricritical Ising model. Indeed, for all the higher values of integer m there are increasing tables. The conformal dimensions in the tables correspond to the presence of scaling operators of the corresponding dimensions, in some critical 2-dimensional system. And these critical 2-dimensional systems in turn are classified this way. In the figure, several curves are indicated and the points in the above tables marked out.

In the next subsection we shall try to provide some feeling for how such results can arise.

5.3 Singular states

First it is useful to remark that a singular state, or *null state* as it is often called, is orthogonal to any state in the representation (the highest weight module) which means (among other things) that the norm of the null state is zero. The proof is easy and left as an exercise.

To make the existence of null states more explicit, we now proceed to actually construct examples of such singular states.

On level 1, the only possible singular state is

$$L_{-1}|h, c\rangle \tag{182}$$

To check when it can be singular, we first let L_1 operate on it

$$L_1 L_{-1}|h, c\rangle = [L_1, L_{-1}]|h, c\rangle = 2h|h, c\rangle \tag{183}$$

This must vanish for a singular state. Hence we must have $h = 0$. But then by the algebra, it is trivial that L_N with $N \geq 2$ will also vanish on the state, so that in fact if the conformal weight of the hws is zero, $L_{-1}|h, c\rangle$ is a singular state. This corresponds to the Kac-curve, $h_{1,1}$.

On level 2 a possible singular state must be of the form

$$L_{-2}|h, c\rangle + aL_{-1}^2|h, c\rangle \tag{184}$$

Exercise:

Show by operating on it with L_1 and L_2 that the condition that these both give zero is

$$c = \frac{2h(5-8h)}{2h+1} \quad (185)$$

$$a = \frac{-3}{2(2h+1)} \quad (186)$$

Whenever c and h are related as indicated we have the singular state

$$\left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right) |h, c\rangle \quad (187)$$

Actually we should really check if *any* L_N , $N > 0$ vanishes on the singular state, but if L_1, L_2 vanish, then by the Virasoro algebra the rest do as well.

It is easy to solve for h in terms of c in eqs.(186), and one gets

$$h = \frac{1}{16} \{ (5-c) \pm \sqrt{(25-c)(1-c)} \}$$

This is exactly the formula for $h_{1,2}$ and $h_{2,1}$ above.

Of particular interest to us will be the fact that at the first non-trivial unitary value for c , namely $c = \frac{1}{2}$, corresponding to the Ising model, we find the possible h -values from the above as

$$h = 1/16, 1/2$$

in addition to the value $h = 0$ always present due to the presence of the unit operator (and its descendant, the energy momentum tensor) in any theory. These are exactly the values obtained in the $m = 3, c = \frac{1}{2}$ table above.

Quite generally, when $h = h_{p,q}(c)$, then there is a singular vector in the Verma module at level $N = pq$. For high values of the level, these can be quite laborious to find in the way indicated. But the above should give the reader an idea.

The existence of singular vectors is closely related to degeneracies in the *Kac determinant*. This determinant is the product of subdeterminants at individual levels. At each level we take the determinant of the bilinear form of the basis vectors.

Thus at level 1, we simply find

$$\langle h, c | L_1 L_{-1} | h, c \rangle = 2 \langle h, c | L_0 | h, c \rangle = 2h \quad (188)$$

We see that the Kac determinant at level 1 vanishes when $h = h_{1,1} = 0$.

Similarly at level 2, we obtain

$$\begin{aligned} \det M_2 &= \det \begin{pmatrix} \langle h | L_2 L_{-2} | h \rangle & \langle h | L_1 L_1 L_{-2} | h \rangle \\ \langle h | L_2 L_{-1} L_{-1} | h \rangle & \langle h | L_1 L_1 L_{-1} L_{-1} | h \rangle \end{pmatrix} \\ &= \det \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix} \\ &= 2(15h^3 - 10h^2 + 2h^2c + hc) \\ &= 32[h - h_{1,1}(c)][h - h_{1,2}(c)][h - h_{2,1}(c)] \end{aligned} \quad (189)$$

This expression shows that the shaded region in fig. 1 is excluded from unitarity, since the bilinear form of the Hilbert space metric has a negative determinant there. The reader may imagine how a systematic application of the Kac determinant may allow one to exclude the entire region, $h > 0, 1 < c < 1$ except for the discrete points given.

Figure 1: The portion, $0 \leq c < 1$ of the (c, h) plane with a few of the curves, $h = h_{p,q}(c)$ indicated, and some of the unitarity points given for $c = 0, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}$ corresponding to $m = 2, 3, 4, 5$.

5.4 Singular fields in correlators

In this subsection we want to obtain two closely related results:

(i) Whenever we have found a correlator (quantum theory Green function, or stat. mech. correlator) for any set of *primary* fields, it is trivial by means of the Conformal Ward Identity to find any correlator of *descendants* of the corresponding primaries. This fact reduces enormously the problem of obtaining the complete solution of a CFT: obtaining (in principle at least) all of its correlators.

(ii) Any correlator involving a primary field containing somewhere down in its Verma module a *singular* field, satisfies a partial differential equation. In certain simple cases, such as in the case of the Ising model, these differential equations will suffice to obtain exact expressions for certain correlators, and to obtain crucial information about the theory.

First we use the conformal Ward identity to derive

$$\langle T(z)\phi_1(w_1)\cdots\phi_M(w_M)\rangle = \sum_{j=1}^M \left(\frac{h_j}{(z-w_j)^2} + \frac{1}{z-w_j}\partial_j \right) \langle \phi_1(w_1)\cdots\phi_M(w_M)\rangle \quad (190)$$

Indeed, for any $\epsilon(z)$ in the conformal Ward identity, we may take a contour containing just the point, w_j , and get

$$\begin{aligned} & \oint_{w_j} \frac{dz}{2\pi i} \epsilon(z) \langle T(z)\phi_1(w_1)\cdots\phi_j(w_j)\cdots\phi_M(w_M)\rangle \\ &= \langle \phi_1(w_1)\cdots\delta_\epsilon\phi_j(w_j)\cdots\phi_M(w_M)\rangle \\ &= \oint_{w_j} \frac{dz}{2\pi i} \epsilon(z) \left(\frac{h_j}{(z-w_j)^2} + \frac{1}{z-w_j}\partial_j \right) \langle \phi_1(w_1)\cdots\phi_M(w_M)\rangle \end{aligned} \quad (191)$$

By summing similar contributions with contours surrounding the other points as well, equivalent to a single contour encircling all points, w_j , we obtain the result.

Let us now introduce a notation for the operator product expansion (OPE) between T and a primary field ϕ of dimension h

$$T(z)\phi(w) = \sum_{n \geq 0} (z-w)^{n-2} \phi^{(-n)}(w) \quad (192)$$

From the singular part we recognize

$$\phi^{(0)} = h\phi(w) \quad \phi^{(-1)}(w) = \phi'(w) \quad (193)$$

The fields $\phi^{(-n)}$ for $n > 0$ are the descendants of ϕ (or just secondary fields).

Clearly,

$$\phi^{(-n)}(w) = \hat{L}_{-n}(w)\phi(w) \quad (194)$$

where

$$\hat{L}_{-n}(w) \equiv \oint_w \frac{dz}{2\pi i} \frac{T(z)}{(z-w)^{n-1}} \quad (195)$$

corresponding to a Laurent expansion of $T(z)$ around the point, w , rather than the point, $w = 0$ previously considered. This is generalized as

$$\phi^{(-n_1, \dots, -n_M)}(w) = \hat{L}_{-n_1}(w) \cdots \hat{L}_{-n_M}(w)\phi(w) \quad (196)$$

where

$$\hat{L}_{-n_j}(w) = \oint_{C_j} \frac{dz_j}{2\pi i} \frac{T(z_j)}{(z_j - w)^{n_j-1}} \quad (197)$$

The contour C_j surrounds w as well as the integration variables z_i for $j < i \leq M$. Deforming the contour around the point, w_1 , so that it surrounds instead all the other points (with opposite orientation), — and such a deformation is legal since the correlator involved is nonsingular at infinity —, we may now work out that

$$\begin{aligned} & \langle \phi_1^{(-n)}(w_1) \phi_2(w_2) \cdots \phi_M(w_M) \rangle \\ = & - \sum_{j=2}^M \oint_{w_j} \frac{dz}{2\pi i} \frac{1}{(z - w_1)^{n-1}} \langle T(z) \phi_1(w_1) \cdots \phi_M(w_M) \rangle \\ = & (-1)^n \sum_{j=2}^M \left(\frac{h_j(n-1)}{(w_1 - w_j)^n} + \frac{1}{(w_1 - w_j)^{n-1}} \partial_j \right) \langle \phi_1(w_1) \cdots \phi_M(w_M) \rangle \end{aligned} \quad (198)$$

This calculation immediately generalizes to the case $\phi^{-n_1, -n_2, \dots}$ and further to the case, where there are several such descendant fields present. Thus we have completed the first part of our task: We have proven that correlators of descendant fields are obtained simply by applying certain explicitly known differential operators to the correlator of primary fields.

We now turn to the second part of our task: To demonstrate that correlators involving a primary field containing a singular field somewhere in its Verma module, satisfy a differential equation, which may be simply written down from the equation satisfied by the singular field.

We remember the correspondence

$$|h, c\rangle = \lim_{z \rightarrow 0} \phi_h(z) |0, c\rangle \quad (199)$$

between a hws and a primary field. For simplicity, let us concentrate on a particular example, and consider the singular state considered above in eq.(187). This is the case

$$h = h_{1,2} \text{ or } h_{2,1} : \frac{1}{16} \{ (5 - c) \pm \sqrt{(25 - c)(1 - c)} \}$$

Since our Hilbert space will be taken to contain only the *irreducible* part of the Verma module built on $|h, c\rangle$, it means that the singular vectors together with their entire Verma modules are removed from the physical Hilbert space. Technically this is achieved by taking the irreducible representation to consist of equivalence classes of states in the original Verma module: Two states are in the same class if they differ by a state in the Verma module of the singular state. For correlators this means, that correlators containing a singular field (or a descendant thereof), are put identically equal to zero in the theory, and this is consistent.

For the particular situation under consideration with $h = h_{1,2}$ or $h = h_{2,1}$, we then have

$$0 = \langle 0 | \phi_1(w_1) \cdots \phi_M(w_M) \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right) \phi_h(0) | 0 \rangle \quad (200)$$

and because the correlator is invariant under translations we have (renaming w_i for $w_i + z$)

$$0 = \langle \phi_1(w_1) \cdots \phi_M(w_M) \left(\hat{L}_{-2}(z) - \frac{3}{2(2h+1)} \hat{L}_{-1}^2(z) \right) \phi(z) \rangle$$

$$= \left[\sum_{j=1}^M \left(\frac{h_j}{(z-w_j)^2} + \frac{1}{z-w_j} \partial_{w_j} \right) - \frac{3}{2(2h+1)} \partial_z^2 \right] \langle \phi_1(w_1) \cdots \phi_M(w_M) \phi(z) \rangle \quad (201)$$

Here we used the above results to convert \hat{L}_{-2} into a differential operator. For \hat{L}_{-1} , however, we refrained from doing that, but instead simply used that this operator is equivalent to a differentiation.

If in particular *all* the primary fields are the same field, namely, $\phi_{1,2}$, then we may use a more convenient and general notation to write

$$\left[\sum_{j \neq i}^N \left(\frac{h_j}{(z_i - z_j)^2} + \frac{1}{z_i - z_j} \partial_j \right) - \frac{3}{2(2h+1)} \partial_i^2 \right] \langle \phi(z_1) \cdots \phi(z_N) \rangle = 0 \quad (202)$$

The reader may easily imagine how singular fields of different forms are made use of for deriving differential equations. In the next section we shall use the above differential equation as an example to obtain an exact result concerning one of the 4-point functions for the critical Ising model.

6 The critical 2-dimensional Ising model

6.1 Critical exponents of the Ising model

As is well known from previous parts of this course, the 2-dimensional Ising model is defined as a stat. mech. system living on a square lattice and with “spin” degrees of freedom, $\sigma_i = \pm 1$ at lattice site, i . The energy is located along a link and is minimal when the spins at the ends have the same value. This allows defining the energy density, ϵ_n , associated with the links emanating from the site, n .

The Ising model is an example of a system developing a 2nd order phase transition at some critical temperature. There is a disordered, high temperature phase with $\langle \sigma \rangle = 0$, and a low temperature, ordered phase with $\langle \sigma \rangle \neq 0$ (locally or with suitable boundary conditions), so that $\langle \sigma \rangle$ plays the role of an order parameter, the magnetization. In the high temperature phase, two point functions fall off exponentially with some characteristic correlation length, ξ . The presence of the length tells us that the theory is not conformal, and the inverse of the correlation length acts as a mass in the associated, massive euclidean quantum theory.

At the critical point, however, the correlation length diverges, the theory contains no scale and it becomes conformal, massless. This is the situation we want to understand further. At this point, correlation functions no longer fall off exponentially. Instead two point functions fall off only as a power of the distance. In general this power is parametrized for the magnetization in d dimensions as

$$\langle \sigma_n \sigma_0 \rangle \sim \frac{1}{|n|^{d-2+\eta}} \quad (203)$$

For the 2-dimensional Ising model, $d = 2$, and standard, famous treatments of that model yield the result

$$\eta = \frac{1}{4} \quad (204)$$

It is similarly possible to give meaning to the concept of a two point function for the energy density. We consider how the spin-frustrations near site no. n are correlated with the same spin-frustrations near site 0 (for reference):

$$\langle \epsilon_n \epsilon_0 \rangle \sim \langle \sigma_n \sigma_{n+1} \sigma_0 \sigma_1 \rangle \sim \frac{1}{|n|^{2(d-1/\nu)}} \quad (205)$$

defining in a conventional way the critical exponent, ν . For the $d = 2$ Ising model, the above mentioned famous calculations give the answer

$$\nu = 1 \quad (206)$$

At the critical point it becomes possible to define a continuum theory, and we know it will be conformal. Associated with the spin-density and the energy density, we shall have continuum fields, $\sigma(z, \bar{z})$ and $\epsilon(z, \bar{z})$ in addition to the unit operator, which we always have. The presence of these two fields already make us think of the $c = \frac{1}{2}$ theory in which we have seen there are exactly two primary fields, $\phi_{1,2}$ and $\phi_{1,3} = \phi_{2,1}$ with conformal dimensions $h_{1,2} = 1/16$ and $h_{1,3} = h_{2,1} = 1/2$. In addition to those, again of course there

is the unit operator. The correspondence is greatly strengthened by the numerical values for the critical exponents, which imply that the two point functions must behave like

$$\begin{aligned}\langle\sigma(z_1)\sigma(z_2)\rangle &= \frac{1}{|z_1 - z_2|^{1/4}} \\ \langle\epsilon(z_1)\epsilon(z_2)\rangle &= \frac{1}{|z_1 - z_2|^2}\end{aligned}\tag{207}$$

This is exactly the behaviour for primary fields of conformal dimensions $(1/16, 1/16)$ and $(1/2, 1/2)$, remembering the general rule:

$$\langle\phi_{h,\bar{h}}(z_1)\phi_{h,\bar{h}}(z_2)\rangle = \frac{1}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}}$$

for a conformal field of dimension (h, \bar{h}) .

In general, the identification between the critical behaviour of a particular stat. mech. theory and a corresponding conformal field theory, rests on the accumulation of a body of circumstantial evidences. One knows, that such a relation must exist. Thus, calculation of several critical exponents usually suffice to make the identification highly convincing. If in addition one can calculate the central charge from the statistical mechanics point of view, and if one knows from the general theory of CFT's that there are no other CFT's at that value of c , then the identification is unique. For the critical Ising model it is possible to calculate also c and find the value $\frac{1}{2}$, but we shall not do that here. Instead, we shall show in the next subsections how to obtain correlators in the corresponding CFT, and we shall study some of the things that may be learned from them.

6.2 The 4-point function of the spin field, $\sigma = \phi_{1,2}$

In this subsection we want to calculate the 4-point function

$$G(z_1, z_2, z_3, z_4) = \langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle$$

explicitly and exactly using the machinery we have accumulated. We put $c = \frac{1}{2}$ and have $h_\sigma = 1/16$. Thus the Green function satisfies the partial differential equation, eq.(202)

$$\left[\frac{4}{3}\partial_j^2 - \sum_{j \neq i}^4 \left(\frac{1/16}{(z_j - z_i)^2} + \frac{1}{z_j - z_i}\partial_i\right)\right] G(z_1, z_2, z_3, z_4) = 0\tag{208}$$

Our first task is to use projective invariance to convert this *partial* differential equation to an *ordinary* differential equation.

For a general projective transformation, $z \mapsto f(z)$, with

$$\begin{aligned}f(z) &= \frac{az + b}{cz + d} \\ ad - bc &= 1\end{aligned}\tag{209}$$

one easily proves the following useful formulas:

$$\begin{aligned}f'(z) &= \frac{1}{(cz + d)^2} \\ f(z_1) - f(z_2) &= \sqrt{f'(z_1)f'(z_2)}(z_1 - z_2)\end{aligned}\tag{210}$$

We have already seen that the form of the two point function is completely fixed by projective invariance and is non-vanishing only if the two fields are the same (at least have the same conformal dimension).

For the 3-point function, the form is similarly fixed. This is a consequence of the fact that a general projective transformation contains 3 complex parameters, which may be used to map the three points to given “standard” places. Thus the general value is given in terms of this particular value. In fact

$$\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3) \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_2+h_3-h_1}} \quad (211)$$

where

$$z_{ij} \equiv z_i - z_j$$

With the formulas above it is quite easy to verify that this indeed transforms as it should under a projective transformation. The constant, C_{123} is the only non-trivial thing about the 3-point function.

When we go on to the 4-point function, it is similarly clear, that since we may only use projective invariance to remove the dependence somehow on 3 points, the dependence on one point remains. This is the reason our partial differential equation is converted to an ordinary one. It is convenient to introduce the projectively invariant so-called double ratio

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad 1 - x = \frac{z_{14}z_{23}}{z_{13}z_{24}} \quad (212)$$

Using the formulas above, one easily verifies that this is an invariant under projective transformations. Furthermore, using the above formulas it is also very easy to verify that the following form

$$\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)\phi_{h_4}(z_4) \rangle = \prod_{i<j} z_{ij}^{-h_i-h_j+\sum_{\ell} h_{\ell}/3} F_0(x) \quad (213)$$

exactly transforms as required under projective transformations. (As often, we have left out all considerations of the antiholomorphic variables for the moment).

First let us write

$$\begin{aligned} G(z_1, z_2, z_3, z_4) &= (z_{13}z_{24})^{-1/8} Y(x) \\ Y(x) &= [x(1-x)]^{-1/24} F_0(x) \end{aligned} \quad (214)$$

Trivial differentiations allow us to work out $([\partial_x, f(x)] = f'(x))$

$$\begin{aligned} [\partial_1, (z_{13}z_{24})^{-1/8}] &= (z_{13}z_{24})^{-1/8} \left(-\frac{1}{8z_{13}} \right) \\ [\partial_2, (z_{13}z_{24})^{-1/8}] &= (z_{13}z_{24})^{-1/8} \left(-\frac{1}{8z_{24}} \right) \\ [\partial_3, (z_{13}z_{24})^{-1/8}] &= (z_{13}z_{24})^{-1/8} \left(+\frac{1}{8z_{13}} \right) \\ [\partial_4, (z_{13}z_{24})^{-1/8}] &= (z_{13}z_{24})^{-1/8} \left(+\frac{1}{8z_{24}} \right) \end{aligned} \quad (215)$$

Then the above partial differential equation for G :

$$\left[\frac{4}{3} \partial_3^2 - \frac{1}{16} \left(\frac{1}{z_{13}^2} + \frac{1}{z_{23}^2} + \frac{1}{z_{34}^2} \right) + \frac{1}{z_{13}} \partial_1 + \frac{1}{z_{23}} \partial_2 - \frac{1}{z_{34}} \partial_4 \right] G = 0 \quad (216)$$

gets converted into a partial differential equation for Y :

$$\left\{ \frac{4}{3} \partial_3^2 + \frac{1}{z_{13}} \partial_1 + \frac{1}{z_{23}} \partial_2 + \frac{1}{3z_{13}} \partial_3 - \frac{1}{z_{34}} \partial_4 - \frac{z_{24}^2}{16z_{23}^2 z_{34}^2} \right\} Y = 0 \quad (217)$$

Since Y only depends on x , however, we may use the chain rule

$$\begin{aligned} \partial_1 &= \frac{\partial x}{\partial z_1} \partial_x = + \frac{z_{34} z_{23}}{z_{13}^2 z_{24}} \partial_x \\ \partial_2 &= \frac{\partial x}{\partial z_2} \partial_x = - \frac{z_{34} z_{14}}{z_{13} z_{24}^2} \partial_x \\ \partial_3 &= \frac{\partial x}{\partial z_3} \partial_x = + \frac{z_{12} z_{14}}{z_{13}^2 z_{24}} \partial_x \\ \partial_4 &= \frac{\partial x}{\partial z_4} \partial_x = - \frac{z_{12} z_{23}}{z_{13} z_{24}^2} \partial_x \end{aligned} \quad (218)$$

to convert all differentiations into differentiations wrt x . Finally, we may use projective invariance to fix 3 of the 4 points. A standard, convenient choice is

$$\begin{aligned} z_1 &\rightarrow \infty \\ z_2 &\rightarrow 1 \\ z_4 &\rightarrow 0 \\ z_3 &\rightarrow x \end{aligned} \quad (219)$$

with

$$\begin{aligned} z_{12} &\sim z_{13} \sim z_{14} \sim \infty \\ z_{23} &\rightarrow 1 - x \\ z_{24} &\rightarrow 1 \\ z_{34} &\rightarrow x \end{aligned} \quad (220)$$

In that configuration several simplifications take place, and one finds the following ordinary differential equation for Y :

$$\left(\frac{4}{3} \frac{d^2}{dx^2} - \frac{1}{16} \frac{1}{x^2(1-x)^2} + \frac{1-2x}{x(1-x)} \frac{d}{dx} \right) Y = 0 \quad (221)$$

This equation is further simplified by the substitution

$$Y(x) = [x(1-x)]^{-1/8} F(x) \quad (222)$$

Using

$$[\partial_x, (x(1-x))^{-1/8}] = - \frac{1-2x}{8x(1-x)}$$

we finally find the differential equation for the function, $F(x)$,

$$\left[x(1-x)\frac{d^2}{dx^2} + \left(\frac{1}{2} - x\right)\frac{d}{dx} + \frac{1}{16}\right]F(x) = 0 \quad (223)$$

In general this kind of treatment for a 4-point function leads to hypergeometrical differential equations. However, the case under consideration is so simple that the solution may be expressed by elementary functions. One very simply verifies that in fact the two independent solutions are:

$$f_i(x) = \sqrt{1 \pm \sqrt{1-x}} \quad (224)$$

for $i = \pm 1$. Any solution of the linear second order differential equation must be a linear combination of these two.

Now the question arises: What linear combination of these two independent solutions is the one relevant for our physical Green function, G ? The answer is related to our neglect so far, of the antiholomorphic dependence. Indeed, what we have shown is that *for fixed values of \bar{z}_i* , the Green function is a solution of the above differential equations. But the *physical* Green function does *not* have z_i 's and \bar{z}_i 's independent of each other. We have used the conformal Ward identity to generalize the framework and consider the so called *conformal blocks* depending on one of the (sets of) variables z_i or \bar{z}_i alone. Clearly what we can say then is the following: (i) In addition to the conformal blocks we have obtained, there are similar anticonformal blocks depending on the complex conjugate variables. (ii) The physical Green function must then be of the form

$$G(z_1, z_2, z_3, z_4) = \left| \frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}} \right|^{1/4} \sum_{i,j=1}^2 a_{ij} f_i(x) f_j(\bar{x}) \quad (225)$$

So we are left with the problem of determining the coefficients, a_{ij} . These coefficients are determined by considerations different from the ones we have used so far.

First, it is a completely general requirement for a euclidean Green function in a quantum field theory (and similarly for a stat. mech. correlator), that they must be single valued in their variables. We shall not go further into this property which follows from very general requirements. This requirement is sometimes termed *monodromy invariance* (gr. *dromos*, way and *monos*, single): the value of the Green function, considered as a function of z_1 , say for fixed value of z_2, z_3, z_4 , should not depend on the history of z_1 . Thus if we start z_1 at one place, and let it wind its way around the other points somehow in the complex plane, to finally return to where it started, we must obtain the same value for the Green function. This requirement is evidently *not* satisfied by the conformal blocks, such as $f_i(x)$. These instead, are analytic functions with branch points. In our case, there are branch points at $x = 0, 1, \infty$, and the analytic function is *not* single valued in the complex plane. In fact each square root is single valued in a certain *cut* plane, or better on a two sheeted Riemann surface. In fact the two functions are nothing else than the continuation of each other to a second sheet. Thus, if we take the complex variable, x , on a small journey around the point $x = 1$ and let it return after one winding, then the roles of $f_{+1}(x)$ and $f_{-1}(x)$ are exactly interchanged.

It will thus be a requirement on the coefficients a_{ij} that they should somehow glue together the blocks and the anti blocks so that the non-trivial monodromy for each of them cancel out.

A second requirement is very simple in our case in which all four fields are identical. We simply must require that the Green function is completely symmetric in the 4 variables z_1, z_2, z_3, z_4 .

In the most general case, the complete analysis of such requirements can be rather complicated. In our simple case, however, one may rather easily verify that in fact they are satisfied by taking

$$\begin{aligned} a_{11} &= a_{22} \equiv a \\ a_{12} &= a_{21} = 0 \end{aligned} \quad (226)$$

giving

$$G(z_1, z_2, z_3, z_4) = a \left| \frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}} \right|^{1/4} \left(\left| \sqrt{1 + \sqrt{1-x}} \right|^2 + \left| \sqrt{1 - \sqrt{1-x}} \right|^2 \right) \quad (227)$$

Indeed, consideration of single valuedness around $x = 0$, implies $a_{12} = a_{21} = 0$, so that we obtain the numerical values indicated. Further, single valuedness under the above journey around $x = 1$ under which the roles of f_{+1} and f_{-1} are interchanged, requires $a_{11} = a_{22}$. We are left with the single constant, a , which we shall determine in the next subsection from normalization considerations.

Even so, it is not yet clear that our result eq.(227) is satisfactory. We still have to check that it leads to a Green function completely symmetric in the arguments. Since the prefactor multiplying F_0 is obviously symmetric, we have to understand whether F_0 itself is symmetric. From what we have obtained we get

$$F_0(x) = a|x(1-x)|^{-1/6} \left(\left| 1 + \sqrt{1-x} \right| + \left| 1 - \sqrt{1-x} \right| \right) \quad (228)$$

Let us consider just one example of a permutation: $z_2 \leftrightarrow z_3$, under which

$$x \rightarrow \frac{1}{x}$$

Then

$$\begin{aligned} F_0\left(\frac{1}{x}\right) &= a \left| \frac{1}{x} \left(1 - \frac{1}{x}\right) \right|^{-1/6} \left(\left| 1 + \sqrt{1 - \frac{1}{x}} \right| + \left| 1 - \sqrt{1 - \frac{1}{x}} \right| \right) \\ &= a|x|^{-1/6}|1-x|^{1/6} \left(\left| \sqrt{x} + \sqrt{x-1} \right| + \left| \sqrt{x} - \sqrt{x-1} \right| \right) \end{aligned} \quad (229)$$

It is not totally obvious that this is the same as $F_0(x)$. To verify that it is indeed, consider (by way of example again) the case where x has an infinitesimal positive imaginary part, and $x > 1$. Then we may work out

$$\begin{aligned} \sqrt{1-x} &= -i\sqrt{x-1} \\ F_0(x) &= a[x(x-1)]^{-1/6} \left(\left| 1 - i\sqrt{x-1} \right| + \left| 1 + i\sqrt{x-1} \right| \right) \\ &= 2a[x(x-1)]^{-1/6} \sqrt{1+(x-1)} \\ &= 2a[x(x-1)]^{-1/6} \sqrt{x} \end{aligned} \quad (230)$$

From the above calculation we find for x in the same interval:

$$\begin{aligned}
F_0\left(\frac{1}{x}\right) &= a[x(x-1)]^{-1/6} \left(\left| \sqrt{x} + \sqrt{x-1} \right| + \left| \sqrt{x} - \sqrt{x-1} \right| \right) \\
&= a[x(x-1)]^{-1/6} \left(\sqrt{x} + \sqrt{x-1} + \sqrt{x} - \sqrt{x-1} \right) \\
&= 2a[x(x-1)]^{-1/6} \sqrt{x}
\end{aligned} \tag{231}$$

The reader may now go over the remaining parts of the argument in similar ways.

6.3 The fusion algebra and its partial verification for the Ising model

We have argued somewhat loosely, that the 2-dimensional critical Ising model corresponds to a Hilbert space carrying the three irreducible representations of the Virasoro algebra corresponding to the 3 primary fields

$$I, \epsilon(z, \bar{z}), \sigma(z, \bar{z})$$

The statement that at $c = \frac{1}{2}$ these are the only unitary representations makes it clear that we can allow no more. (Of course we have not proven the unitarity theorem). Strictly speaking the Hilbert space carries representations of two copies of the Virasoro algebra, one corresponding to the holomorphic variables, and one corresponding to the antiholomorphic ones, but the situation for the latter is identical to the former. However, it is in fact highly non-trivial that any physical theory at all can exist. In order for this to be the case we must have *the fusion algebra* satisfied.

Let us explain what that means. Consider the OPE of two primary fields, and since we shall be particularly concerned with two σ 's, consider

$$\sigma(z_2, \bar{z}_2)\sigma(z_3, \bar{z}_3) = z_{23}^{-2/16} \bar{z}_{23}^{-2/16} + \text{n.s.t} \tag{232}$$

The singular piece is the one we have looked at before in connection with the two point function, and we have normalized the fields so that the coefficient is, 1. Now we shall be interested also in the next to leading piece: n.s.t. This term involves a certain operator, and the non-trivial requirement is that this operator is either the unit operator, I , (or one of its descendants), or the energy density, ϵ , (or one of its descendants)¹. *If* the new operator is something else, then we learn that multiplying two operators of the theory, namely two σ 's, we produce something *not lying in the theory*, in other words, there would be no theory with the Hilbert space just built out of the three unitary representations of the Virasoro algebra. Looked at from this point of view, it is really rather a miracle that we actually can have a consistent theory just with these three representations. In the present case we may express the result which does turn out to hold, as we shall soon verify, as the so called fusion relation

$$[\sigma] \times [\sigma] \sim [I] + [\epsilon] \tag{233}$$

¹A term involving σ itself would be ok, but for the Ising model, such a term cannot occur due to the rather obvious invariance of the theory under the transformation, $\sigma \rightarrow -\sigma$, remembering the physical significance of σ as being related to an average magnetization.

The content of this relation is as follows: If we consider the OPE of any two operators both belonging to the conformal families of the primary field, σ , then the operators appearing in this OPE will all belong either to the conformal family of the unit operator, or to the conformal family of the energy density operator.

The fusion algebra may be shown to be completed in the case of the Ising model into

$$\begin{aligned} [\sigma] \times [\epsilon] &\sim [\sigma] \\ [\epsilon] \times [\epsilon] &\sim [I] \end{aligned} \tag{234}$$

(leaving out the trivial ones like $[I] \times [\sigma] = [\sigma]$).

A CFT for which the Hilbert space consists of a *finite* number of irreducible representations of the Virasoro algebra and for which the corresponding fusion algebra closes, is called a *minimal model*. It is possible to show, that minimal models exist only for $c < 1$. Unitary, minimal CFT's then only exist for the discrete values of c we described above. For c larger than one, there are some very useful generalizations of these ideas. One knows of (infinitely) many CFT's having Hilbert spaces with only a finite number of representations of certain *extended chiral algebras* containing the Virasoro algebra as a sub algebra. These are then termed *rational CFT's*. But we shall not deal with them here.

Instead we shall be content with verifying as an example, the fusion rule eq.(233), using the 4-point function eq.(227).

First we consider the most general form for a fusion rule in terms of the OPE:

$$\phi_{h_1}(z)\phi_{h_2}(w) = (z-w)^{-h_1-h_2+h_3}\phi_{h_3}(w) + \dots \tag{235}$$

the strength of the singularity is as indicated, and this follows from simple considerations of the behaviour under scaling (dimensional analysis), (and in particular does not depend on the operators being primaries.) Thus we may learn about the conformal dimension of an operator occurring in a fusion rule by studying the nature of the singularities in the corresponding OPE.

Thus, remembering the conformal dimension of $(\frac{1}{2}, \frac{1}{2})$ for ϵ , we may rewrite the fusion rule we want to establish in a more precise way as

$$\sigma(z_2, \bar{z}_2)\sigma(z_3, \bar{z}_3) = \frac{1}{|z_{23}|^{1/4}} + C_{\sigma\sigma\epsilon}|z_{23}|^{3/4}\epsilon(z_3, \bar{z}_3) \tag{236}$$

We want to use our result for the 4-point function to verify that indeed the singularity is as indicated ², proving that the operator occurring must have conformal dimensions $(\frac{1}{2}, \frac{1}{2})$ consistent with ϵ . Therefore, let us insert the OPE (considering the limit $z_3 \rightarrow z_2$) in the 4-point function to see what we are looking for:

$$\begin{aligned} G \quad z_2 \xrightarrow{\sim} z_3 \quad &\langle \sigma(z_1) \left(\frac{1}{|z_{23}|^{1/4}} + C_{\sigma\sigma\epsilon}|z_{23}|^{3/4}\epsilon(z_3) \right) \sigma(z_4) \rangle \\ &= \frac{1}{|z_{23}|^{1/4}} \langle \sigma(z_1)\sigma(z_4) \rangle + |z_{23}|^{3/4} \langle \sigma(z_1)\epsilon(z_3)\sigma(z_4) \rangle C_{\sigma\sigma\epsilon} \\ &= \frac{1}{|z_{23}z_{14}|^{1/4}} \left(1 + C_{\sigma\sigma\epsilon}^2 \left| \frac{z_{23}z_{14}}{z_{13}z_{34}} \right| \right) \end{aligned} \tag{237}$$

² $z_{23}^{3/8}$ is a branch point singularity in the sense of analytic function theory, even though no infinity develops at $z_{23} = 0$

where we used the general result for a three point function eq.(211) ($h_1 = 1/16, h_3 = 1/2, h_4 = 1/16$, and with a similar contribution from the antiholomorphic part)

$$\langle \sigma(z_1)\epsilon(z_3)\sigma(z_4) \rangle = \frac{C_{\sigma\sigma\epsilon}}{|z_{13}||z_{14}|^{-3/4}|z_{34}|}$$

In the limit $z_3 \rightarrow z_2, x \rightarrow 1$, so

$$\left| \frac{z_{23}z_{14}}{z_{13}z_{34}} \right| \rightarrow |1-x|$$

and

$$G \sim \frac{1}{|z_{23}z_{14}|^{1/4}} \left(1 + C_{\sigma\sigma\epsilon}^2 |1-x| \right) \quad (238)$$

Now let us compare this with the result of our calculation above, eq.(227). In the same limit we obtain

$$\begin{aligned} G(z_1, z_2, z_3, z_4) &\sim a \frac{1}{|z_{23}z_{14}|^{1/4}} \frac{1}{|x|^{1/4}} \left(\left(1 + \frac{1}{2}\sqrt{1-x} \right) \left(1 + \frac{1}{2}\sqrt{1-\bar{x}} \right) \right. \\ &\quad \left. + \left(1 - \frac{1}{2}\sqrt{1-x} \right) \left(1 - \frac{1}{2}\sqrt{1-\bar{x}} \right) \right) \\ &\simeq 2a \frac{1}{|z_{23}z_{14}|^{1/4}} \left(1 + \frac{1}{4}|1-x| \right) \end{aligned} \quad (239)$$

This form is exactly of the expected structure, eq.(238). Thus we have (i) verified the fusion rule eq.(233), (ii) fixed the remaining normalization constant as

$$a = \frac{1}{2}$$

and finally (iii) obtained the fusion constant (up to a sign ambiguity)

$$|C_{\sigma\sigma\epsilon}| = \frac{1}{2} \quad (240)$$

7 Bibliography

The material in these notes is mostly based on the following references:

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A useful anthology is

"Conformal Invariance and Applications to Statistical Mechanics," Claude Itzykson, Hubert Saleur and Jean-Bernard Zuber, World Scientific (1988)

which contains about 50 important reprinted articles.