Quantum sensing under the effect of colored noise

Precision measurements in Ramsey Spectroscopy

Bachelor Thesis

Written by *Kasper H. Nielsen*

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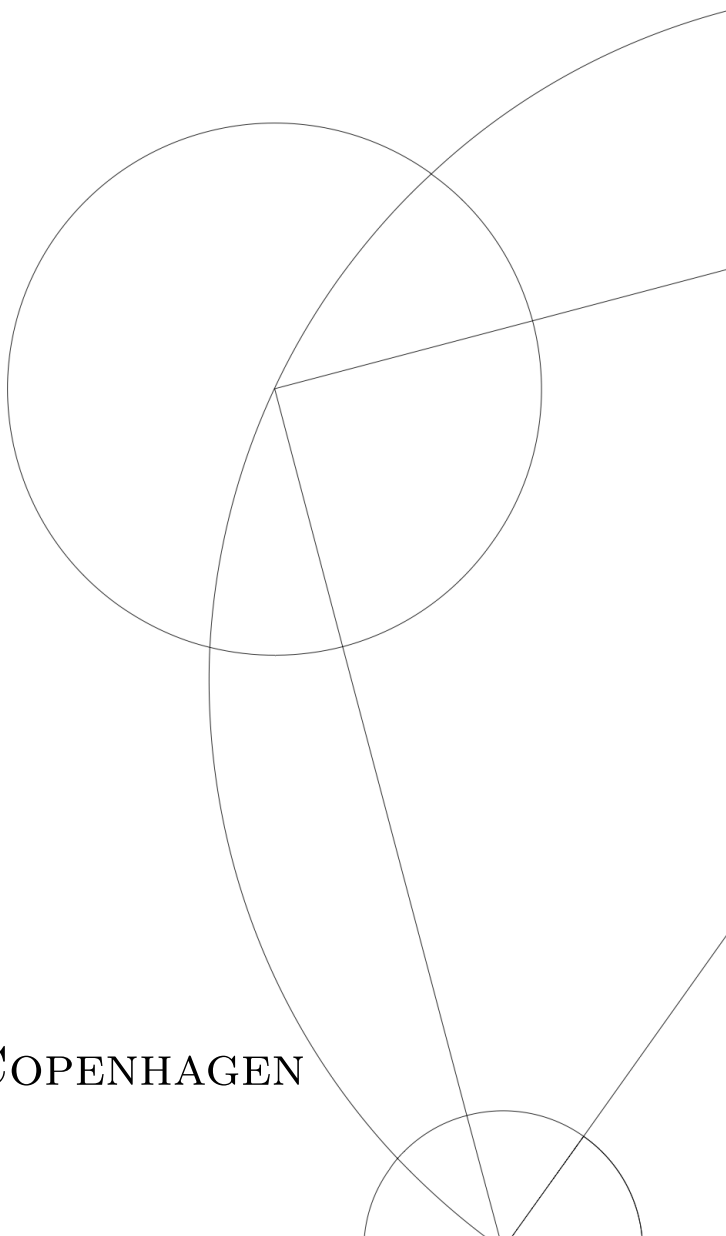
Supervised by
Prof. Anders Søndberg Sørensen
Ph.d. fellow Eva Maria Gonzalez Ruiz



University of Copenhagen

UNIVERSITY OF COPENHAGEN

Name of Institute:	Niels Bohr Institutet
Name of Department:	Theoretical Quantum Optics
Author(s):	Kasper H. Nielsen
Email:	kaw.nielsen@gmail.com

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Name _____

Signature _____

Date _____

Abstract

In this thesis we investigate the impact of noise on the experimental setup of Ramsey Spectroscopy.

In the first part of this thesis, we derive the standard quantum limit in a Markovian environment, where the presence of white noise dictates the precision of measurements. Our result matches previous findings, but is derived from a more general setting.

We move on to a non-Markovian environment. Here it has previously been shown that one can obtain a better precision than the standard quantum limit in terms of parameters T and L - being the total time of the experiment and number of atoms respectively.

In our model we choose a spectral density function consisting of white noise with colored noise added on top - where the colored noise is represented by a Gaussian.

We find that in our model it is not possible to surpass the fundamental scaling of the white noise limit. We also find that the precision of a measurement is limited by the peak value of the spectral density function, in the case where we assume pure colored noise.

Contents

Introduction

The precision of a measurement is always limited by the uncertainty of our measurement apparatus. Whether it be the measurement of length with a ruler or caliper, or it is the measurement of time with a stopwatch or an atomic clock, there is always a fundamental uncertainty bound to each single measurement.

The poor mans approach to improving his measurement, without investing in new and expensive equipment, would be to perform several independent measurements. Now with L measurements, he would be able to reduce the uncertainty on the average value by a factor of $\frac{1}{\sqrt{L}}$, because the poor man has read about The Central Limit Theorem in R. Barlow's priceless book on statistics[2]. We can now measure anything with in principle infinite precision by performing infinitely many measurements.

The also not so rich physicist however, is wondering if it would be possible to get an even better scaling in uncertainty than with repeated measurements. Performing an infinite number of measurements would take a long time, there must be a better way to get better precision. Would it be possible to get a scaling with more measurements of $\frac{1}{\sqrt{L^{3/2}}}$ or maybe even $\frac{1}{L}$?

The physicist knows of a world. The Quantum World. Where things do not behave the way we are used to, and phenomena that sounds fantastic are actually real. Would it be possible to exploit quantum effects to achieve better scaling?

Well the answer is; yes, no and maybe - it all depends on which environment you are in. In the uncorrelated (quantum) world the best achievable scaling is what is know as The Standard Quantum Limit or The Shot Noise Limit which is the familiar $\frac{1}{\sqrt{L}}$[3]. However if we instead use special entangled states, meaning we are now in a correlated world, it is shown that one can achieve a scaling of $\frac{1}{L}$, known as the Heisenberg limit.[3]

But all of this is in a world without noise, which is a good world, but not very realistic. Things change in the presence of noise, and depends on what type of noise is present. In the white noise case, it has been shown [7] that both using correlated and uncorrelated particles, the best possible scaling is $\frac{1}{\sqrt{L}}$ - where special entangled states yielding an improvement by a constant factor of $\sqrt{e}$ - to a fundamental limit $\sqrt{\frac{2\gamma}{LT}}$. This result seem to be more or less generally accepted.

In the presence of colored noise, the results are not as clear, and many different models and approaches are made in order to optimize precision. Some assume an ohmic spectral density function which has little to no noise at a given frequency and get a better scaling in number if atoms [4]. Others suggest quantum teleportation in order to avoid correlation with the environment and get better scaling in total time of the experiment [9].

Motivation

The intention of this thesis is to find the limit of uncertainty in a specific colored noise scenario and compare this to other results ([4],[9]) which claim to surpass the fundamental limit of white noise. This should not be possible in general, given the fundamental limit of white noise, but might be possible in some cases. The colored noise chosen in our model consist of white noise term, with a colored noise contribution added on top. However, the first step is to derive the fundamental limit of white noise in different and more general way than what is done in the article [7]. This also allows for a calibration of the model used in this theses, so results are comparable.

The setup is Ramsey Spectroscopy, which is a highly useful experiment. Ramsey spectroscopy is metrologically equivalent to Mach-Zehnder interferometry [3], and a central part of how an atomic clocks work. Mach-Zehnder interferometers are widely used, and gained attention recently as large interferometers are used as detectors of gravitational waves.

Until recently the SI-unit of a second was defined by: the ground-state hyperfine transition frequency of the cesium-133 atom using Ramsey Spectroscopy [1]. Atomic clocks are however still very applicable: for instance in GPS-satellites where corrections to the clocks in the satellites are constantly made due to time dilation effects of both special and general relativity.

1 Theory

1.1 Spectroscopy

This section is based on the theory part of Wineland et al. [12], but with additions based mostly on the course Quantum Mechanics 1 [6].

One purpose of spectroscopy is to measure the transition frequency ω_0 of a two-level system e.g. an atom with some energy levels or a spin-particle in a magnetic field. The difference in energy between two levels is related to the transition frequency by:

$$\omega_0 = \frac{E_e - E_g}{\hbar} = \frac{\Delta E}{\hbar}, \tag{1.1}$$

where E_e and E_g denote the energy of the excited- and ground state respectively, and $\hbar$ is the reduced Planck constant.

1.1.1 Population spectroscopy

By exposing L identical two level systems (the population) to an external field for a time T, and afterwards observe how the population is distributed in the two energy levels (e.g. how many is in the excited and the ground state), it is possible to determine the transition frequency. In this derivation we assume a spin-$\frac{1}{2}$ model, where particles are exposed to an external uniform magnetic field. Note that measuring either population will give the same answer because of conservation of particles $L = L_+ + L_-$.[1] Based on that measurement one can determine, with a certain precision (later denoted σ_Δ), the transition frequency ω_0.

In the spin-$\frac{1}{2}$ model, the magnetic field is applied in the z-direction $\boldsymbol{B} = B_0\hat{z}$, and each electron (spin-$\frac{1}{2}$ two-level system), has a magnetic dipole moment $\boldsymbol{\mu} = \gamma \boldsymbol{S}$. This means the Hamiltonian for each particle is:

$$H_0 = -\boldsymbol{\mu} \cdot \boldsymbol{B} = -\gamma B_0 S_z, \tag{1.2}$$

where no interaction between the atoms ir assumed. In the case of electrons $\gamma = \mu_B g_J/\hbar$, where μ_B is the Bohr Magneton and g_J is the Landé g-factor. In all cases $S_z = \hbar/2\sigma_z$ is the spin operator where σ_z is the z-Pauli matrix. The eigenstates of this Hamiltonian are $|m\rangle = \left|\pm\frac{1}{2}\right\rangle$, with the eigenvalue $S_z|m\rangle = \hbar m|m\rangle$. The eigenvalues for the full Hamiltonian are:

$$E_+ = \left\langle +\frac{1}{2}\middle| H_0 \middle| +\frac{1}{2}\right\rangle = -\gamma B_0 \hbar/2, \qquad E_- = \left\langle -\frac{1}{2}\middle| H_0 \middle| -\frac{1}{2}\right\rangle = +\gamma B_0 \hbar/2. \tag{1.3}$$

The transition frequency defined in equation (1.1) is related to the eigenenergies by:

$$\omega_0 = \frac{\gamma B_0 \hbar}{\hbar} = \gamma B_0, \tag{1.4}$$

which is known as the Larmor frequency. The spin precesses (Larmor Precession) around the B-field at this frequency, which will be shown in the Heisenberg Picture.

The Heisenberg Equation for the spin operator $\boldsymbol{S} = (S_x, S_y, S_z) = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$ can now be derived, starting form the Heisenberg Equation of motion:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \frac{1}{i\hbar}[A(t), H] + \frac{\partial A(t)}{\partial t}, \tag{1.5}$$

where the spin operator is independent of time, and hence the last term is zero. We are left with calculating the elements of the commutator. For spin operators the commutator relation is:

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k, \tag{1.6}$$

[1] Note that L, L_+, L_- denotes; the total number of particles, the number of particles in the excited state, and the number of particles in the ground state respectively. This has nothing to do with orbital angular momentum operators.

using the compact form with the Levi-Civita symbol ε_{ijk}. Note that H_0 is a constant times the S_z operator with the commutator:

$$\begin{aligned} \omega_0[\boldsymbol{S}, S_z] = \omega_0([S_x, S_z]\hat{x} + [S_y, S_z]\hat{y} + [S_z, S_z]\hat{z}) = \omega_0 i\hbar(-S_y\hat{x} + S_x\hat{y} + 0\hat{z}), \\ \frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \omega_0(-S_y\hat{x} + S_x\hat{y}). \end{aligned} \tag{1.7}$$

This expression looks very much like the the cross product between two vectors. One general 3-component vector, and one which is oriented in the z-direction. If we define a new vector $\boldsymbol{\omega}_0 = \omega_0\hat{z}$, and calculate the cross product:

$$\boldsymbol{\omega}_0 \times \boldsymbol{S} = \omega_0(-S_y\hat{x} + S_x\hat{y}) = \frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t}, \tag{1.8}$$

which is the Heisenberg equation of motion. It is clear the the spin $\boldsymbol{S}$ rotates in the xy-plane.

To treat a system of L particles the collective angular momentum operator is defined by the sum of the angular momentum of the individual L particles:

$$\boldsymbol{J} = \sum_{l=1}^{L} \boldsymbol{S}_l, \tag{1.9}$$

where $\boldsymbol{S}_l$ is the spin of the lth particle, and the angular momentum part ($J = L + S$)[2] of $\boldsymbol{J}$ is assumed to be zero. This is valid since we assume stationary spin-$\frac{1}{2}$ particles with no orbital angular momentum. It will be convenient to use the following basis: $|\boldsymbol{J}, \boldsymbol{M}\rangle$, to describe the entire system. Following equations 1.8 and 1.9, the Heisenberg equation for $\boldsymbol{J}$ is:

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{\omega}_0 \times \boldsymbol{J}. \tag{1.10}$$

In order to determine ω_0, a clock radiation field must be applied. This magnetic field rotates around the z-axis, following the precession of the spins. The applied field has the form:

$$\boldsymbol{B}_1 = B_1\left[\cos(\omega t + \theta)\hat{x} + \sin(\omega t + \theta)\hat{y}\right], \tag{1.11}$$

where $\omega \approx \omega_0$. The setup can now be described in a frame which rotates at frequency ω. Here the net B-field is given by the two contributions from the two fields:

$$\begin{aligned} \boldsymbol{B} &= B_r\hat{z} + B_1[\cos(\theta)\hat{x} + \sin(\theta)\hat{y}], \\ B_r &= B_0 + \omega/\gamma = B_0(\omega_0 - \omega)/\omega_0. \end{aligned} \tag{1.12}$$

The parameter θ can be chosen freely. If $\theta = \pi/2$ is assumed, the net field is $\boldsymbol{B} = B_r\hat{z} + B_1\hat{y}$. In the rotating frame, the Hamiltonian is given by:

$$\begin{aligned} H_r &= -\gamma\boldsymbol{J}\cdot\boldsymbol{B} = \boldsymbol{\omega'}\cdot\boldsymbol{J}, \qquad & \boldsymbol{\omega'} &\equiv \omega_r\hat{z} + \omega_1\hat{y}, \\ \omega_r &\equiv -\gamma B_r = \omega_0 - \omega, \qquad & \omega_1 &\equiv -\gamma B_1. \end{aligned} \tag{1.13}$$

Note that $\boldsymbol{J}$ is now in the rotating frame, but only it is only J_z which is of interest. Since the rotation is around the z-axis the value of J_z is the same in both the stationary and rotation frame. In the rotating frame, the Heisenberg equation for the angular momentum is:

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{\omega'} \times \boldsymbol{J}, \tag{1.14}$$

which means that $\boldsymbol{J}$ rotates around $\boldsymbol{\omega'}$, which is parallel to $\boldsymbol{B}$.

After a time t_f where the clock radiation has been applied, a detector measures the number of particles in the $\left|+\frac{1}{2}\right\rangle$ state. This is given by:

$$L_+(t_f) = J_z(t_f) + JI, \tag{1.15}$$

[2]Here L denotes orbital angular momentum, but from now on L only denotes number the of particles.

where I is the identity operator. To verify this expression, note that there are a total of L particles and L_+ particles are in the $\left|+\frac{1}{2}\right\rangle$ state. That means $L - L_+$ particles are in the $\left|-\frac{1}{2}\right\rangle$ state, which are the two states of $J_z(t_f)$. The identity just returns L times the absolute value of the spin. Let us write it generally where J_z can take values $\pm s$, and not just $\pm\frac{1}{2}$

$$L_+ = L_+ s + (L - L_+)(-s) + Ls = 2L_+ s = L_+, \qquad \text{for } s = \frac{1}{2}. \tag{1.16}$$

1.1.2 Ramsey method

The idea of Ramsey spectroscopy is to divide the complex precession around $\boldsymbol{B} = B_r\hat{z} + B_1\hat{y}$ into three stages:

1. A rotation around the y-axis of $\pi/2 = \Sigma_R t_{\pi/2}$,
2. A rotation (clock rotation) around the z-axis of $(\omega_0 - \omega)T$,
3. A rotation around the y-axis of $\pi/2\Sigma_R t_{\pi/2}$.

The two rotations around the y-axis are know as the Ramsey-pulses with frequency Ω_R known as the Rabi frequency, for a time which makes the angle of rotation exactly π/w. The Rabi frequency is high compared to $\omega_0 - \omega$, which means one can assume pure rotation around the y-axis in these periods. During the clock rotation the B-field which creates the Ramsey pulses is set to zero.

In the case where the system is prepared in the state: $\boldsymbol{J}(0) = -J_z(0)\hat{z}$, the first Ramsey-pulse rotates $\boldsymbol{J}$ to coincide with the x-axis. The clock radiation rotates the vector around the z-axis, starting from the positive x-direction (assuming $\Omega_R > 0$) by an angle of $\omega_r T$. The second Ramsey-pulse aligns $\boldsymbol{J}$ with the yz-plane. With this initial preparation, one finds that:

$$J_z(t_f) = -J_z(0)\cos(\omega_r T). \tag{1.17}$$

This means that the number of particles in the state $\left|+\frac{1}{2}\right\rangle$ - following equation (1.15):

$$\langle L_+(t_f)\rangle = J - \langle J_z(0)\rangle\cos(\omega_r T). \tag{1.18}$$

To describe a general vector $\boldsymbol{J}$ in this setup, the 3 rotations can be describe by 3-dimensional rotation matrices. The rotation describe in equation (1.14), can be split into the 3 individual rotations. In general $\boldsymbol{\omega}' \times \boldsymbol{J} = \boldsymbol{\Omega}$, where Ω is the frequency and holds information on the axis of rotation. For a vector $\vec{v}$ which undergoes rotation, the new position $\vec{v'}$ after the rotation is given by $\vec{v'} = R\vec{v}$. So the final state is the initial state times the change of that state - in this case rotations. Rotations around the y and z-axis can be describe by the 3d rotation matrices.

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.19}$$

The Ramsey-pulses are always around the y-axis at an angle $\theta = \pi/2$. The clock radiation is always around the z-axis at an time dependent angle $\theta = \omega_r T$. The matrix which describes the entire rotation is the product the rotations - note the matrices of rotation are applied from the right in the order they occur, although the first and last rotation are equivalent and therefore it does not make a difference here.

$$R_R = R_y(\pi/2)R_z(\omega_r T)R_y(\pi/2) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\omega_r T) & \sin(\omega_r T) \\ 0 & -\sin(\omega_r T) & \cos(\omega_r T) \end{bmatrix}. \tag{1.20}$$

In the case where we start in a general state $\boldsymbol{J}(0) = (J_x(0), J_y(0), J_z(0))$, the final state is

$$\begin{aligned} J_x(t_f) &= -J_x(0), \\ J_y(t_f) &= J_y(0)\cos(\omega_r T) + J_z(0)\sin(\omega_r T), \\ J_z(t_f) &= J_y(0)\sin(\omega_r T) - J_z(0)\cos(\omega_r T), \end{aligned} \tag{1.21}$$

where we recover the result from equation (1.17) in the case where the state is initially prepared in the $\boldsymbol{J}(0) = -J_z(0)\hat{z}$ state (also known as $\left|-\frac{1}{2}\right\rangle$).

2 White noise

A system subjected to white noise corresponds to the system being in a Markovian environment. This environment is characterized by a short memory time, actually infinitely short, meaning the noise is uncorrelated in time. The noise is only determined by the reservoirs present state and not the past state - which one would expect in a general non-Markovian reservoir. The correlation function of this environment is described by Dirac-Delta function, and therefore the spectral density function is a constant. That means the same noise for all frequencies. The correlation function and spectral density function are related by the Fourier transform.

2.1 Previous results

This section concludes the primary results from the article *Improvement of Frequency Standards with Quantum Entanglement* by S. F. Huelga, C. Macchiavello, T. Pellizzari, and A. K. Ekert [7].

Their model of dephasing, a Lindblading Master Equation:

$$\dot{\rho}(t) = i\Delta(\rho\,|1\rangle\langle 1| - |1\rangle\langle 1|\,\rho) + \gamma/2(\sigma_z\rho\sigma_z - \rho). \tag{2.1}$$

The expression of uncertainty on a measurement of Δ both for uncorrelated particles:

$$\sigma_\Delta = \sqrt{\frac{1-\cos^2(\Delta t)e^{-2\gamma t}}{nTte^{-2\gamma t}\sin^2(\Delta t)}}, \tag{2.2}$$

where Δ is the value we want to measure, the decay rate $\tau_{dec} = 1/\gamma$ - τ_{dec} is the decoherence time, n is the number of atoms (constant), T is the total time of the experiment (constant), t the time of a single run of the experiment (performed T/t times for all n atoms).

The minimal uncertainty for both uncorrelated and maximally entangled particles:

$$\sigma_{\Delta,opt} = \sqrt{\frac{2\gamma e}{LT}}. \tag{2.3}$$

They then show that it is possible to overcome the limit with certain partially entangled states, to achieve a new minimal value, which they say is not possible to overcome.

$$\sigma_{\Delta,opt} \geq \sqrt{\frac{2\gamma}{LT}}. \tag{2.4}$$

2.2 Hamiltonian

We start of the calculation the same way as any textbook question in quantum mechanics - with the Hamiltonian:

$$\hat{H} = (\Delta + F(t))\hat{S}_z, \tag{2.5}$$

where $\Delta = \omega_r = \omega_0 - \omega$ and $F(t)$ is the noise which is characterized by the correlation function:

$$\begin{aligned} \big\langle F(t)F(t')\big\rangle &= \alpha\gamma\delta(t-t'), \\ \langle F(t)\rangle &= 0, \end{aligned} \tag{2.6}$$

where the brackets denote the expectation value. Note that we also require the noise to be symmetric. $\gamma = 1/\tau_{dec}$ is the decay rate equal to the inverse of the decoherence time, and α is a constant which will be determined by calibration to the article [7], so results are comparable. This Hamiltonian is for 1 particle, later we will expand to several particles.

Our task is to measure Δ with the best precision possible, or rather to determine with what precision one can measure Δ. Actually we are interested in measuring ω_0, but we assume that ω - the frequency of both the clock radiation - is known with infinite precision. Thus a measurement of $\Delta = \omega_0 - \omega$ and ω_0 will have the same fundamental uncertainty.

In the full Hamiltonian for L particles, we assume the individual particles to uncorrelated. Also we can rewrite the spin operator $\hat{S}_z = \frac{1}{2}\sigma_z = \frac{1}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|)$ where $\hbar = 1$. The full Hamiltonain is thus:

$$\hat{H} = \frac{1}{2}\sum_{l=1}^{L}[(\Delta_l + F_l(t))\sigma_z(l)]. \tag{2.7}$$

2.3 Density matrix description

The density matrix is an operator which contains information both about the state and the Hamiltonian. It is defined as: *The definitions below are from the lecture notes to Quantum Optics 2 [11], but equivalent definitions can be found in [5].*

$$\hat{\rho} = \sum_{i,j} \rho_{ij} \left|i\right\rangle\!\left\langle j\right|, \qquad \rho_{ij} = \left\langle i\right| \hat{\rho} \left|j\right\rangle. \tag{2.8}$$

We will also need the master equation which is defined

$$\frac{\mathrm{d}\hat{\rho}_S}{\mathrm{d}t} = \frac{1}{i\hbar}\Big[\hat{H}_S, \hat{\rho}_S\Big] + \mathcal{L}(\hat{\rho}_S), \tag{2.9}$$

where $\mathcal{L}(\hat{\rho}_S)$ is the Liouvillian operator:

$$\mathcal{L}(\hat{\rho}_s) = -\frac{\gamma}{2}(\hat{c}^\dagger\hat{c}\hat{\rho}_s + \hat{\rho}_s\hat{c}^\dagger\hat{c} - 2\hat{c}^\dagger\hat{\rho}_s\hat{c}), \tag{2.10}$$

and $\hat{c}, \dagger\hat{c}$ is the annihilation and creation operator respectively.

We start out in the $L = 1$. In this model the Liouvillan is set to zero. The master equation in the 1 particle case is:

$$\begin{aligned}\Big[\hat{H}_S, \hat{\rho}_S\Big] &= \frac{1}{2}(\Delta + F(t))[\rho_{10}\left|1\right\rangle\!\left\langle 0\right| - \rho_{01}\left|0\right\rangle\!\left\langle 1\right| - (-\rho_{10}\left|1\right\rangle\!\left\langle 0\right| + \rho_{01}\left|0\right\rangle\!\left\langle 1\right|)], \\ \frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} &= -i(\Delta + F(t))(\rho_{10}\left|1\right\rangle\!\left\langle 0\right| - \rho_{01}\left|0\right\rangle\!\left\langle 1\right|),\end{aligned} \tag{2.11}$$

where the diagonal terms cancel out. This is the differential equation which describe the dynamics of the system which is solved by separation of variables. Let us resctrict our focus to just one of the off-diagonal terms.

$$\begin{aligned}\frac{\mathrm{d}\rho_{10(t)}}{\mathrm{d}t} &= -i(\Delta + F(t))\rho_{10}(t) \\ \int_{t_0}^{t_0+T} \frac{1}{\rho_{10}(t)}\mathrm{d}\rho_{10}(t) &= -i\int_{t_0}^{t_0+T} \Delta + F(t)\mathrm{d}t, \\ \rho_{10}(t_0+T) &= \rho_{10}(t_0)\exp\left(-i\int_{t_0}^{t_0+T} F(t)\mathrm{d}t - i\Delta T\right).\end{aligned} \tag{2.12}$$

It is clear that ρ_{10} of a present (or future) time is directly proportional to itself in the past, and the exponential part looks like a phase - partly due to signal (Δ) and partly due to the noise. Let us change notation to $\phi(t) = \int_{t_0}^{t_0+T} F(t)\mathrm{d}t$, and look at the expectation value of this matrix element.

$$\langle\rho_{10}(t_0+T)\rangle = \langle\rho_{10}(t_0)\exp(-i\phi(t))\exp(-i\Delta T)\rangle. \tag{2.13}$$

Because the environment is Markovian (uncorrelated in time) we can divide the expectation value into a product of 3 expectation values.

$$\langle\rho_{10}(t_0+T)\rangle = \langle\rho_{10}(t_0)\rangle\langle\exp(-i\phi(t))\rangle\langle\exp(-i\Delta T)\rangle. \tag{2.14}$$

Here $\rho_{10}(t)$ and $\exp(-i\Delta T)$ are just numbers, and therefore the expectation value is equal to that number.

2.3.1 Gaussian Probability Density Function

We will now look into the expectation value of the noise. We assume the probability density function for the noise to be a Gaussian distribution. This is a good assumption since the field is random stochastic field, and by the central limit theorem, one would expect a Gaussian distribution.

$$P(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big). \tag{2.15}$$

Here we assume the noise to be centered around 0 in ϕ-space, hence $\mu = 0$. Furthermore from statistics [2], we know that the expectation value of a function is the integral over all of space, of that function times the probability distribution $\langle f \rangle = \int_{-\infty}^{\infty} f(x)P(x)\mathrm{d}x$.

$$\langle \exp(-i\phi(t)) \rangle = \int_{-\infty}^{\infty} \exp(-i\phi) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{\phi^2}{2\sigma^2}\Big)\mathrm{d}\phi = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\Big(-\frac{\phi^2}{2\sigma^2} - i\phi\Big)\mathrm{d}\phi. \tag{2.16}$$

To solve this integral we use formula 18.75 in Schaum's: [8]

$$\int_{-\infty}^{\infty} \exp\big(-(ax^2+bx+c)\big) = \sqrt{\frac{\pi}{a}} \exp\Big(\frac{b^2-4ac}{4a}\Big). \tag{2.17}$$

In our case $a = \frac{1}{2\sigma^2},\ b = i,\ c = 0$. This gives the following result.

$$\langle \exp(-i\phi(t)) \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{\frac{\pi}{(2\sigma^2)^{-1}}} \exp\Big(\frac{i^2}{4(2\sigma^2)^{-1}}\Big) = \exp\Big(-\frac{\sigma^2}{2}\Big), \tag{2.18}$$

but the variance of an operator A is defined $\sigma^2 = \langle A^2 \rangle - \langle A \rangle^2$ [2], which is:

$$\sigma^2 = \langle \phi(t)^2 \rangle - \langle \phi(t) \rangle^2 = \Big\langle \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} F(t)F(t')\mathrm{d}t\mathrm{d}t' \Big\rangle = \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} \big\langle F(t)F(t') \big\rangle \,\mathrm{d}t\mathrm{d}t', \tag{2.19}$$

where $\langle \phi(t) \rangle = 0$ because of the symmetric condition on the noise. Note that this section is general for any symmetric stochastic field, and will be used during colored noise as well.

2.3.2 Back to density matrix

In the case of the correlation function of white noise defined in equation (2.6) used in equation (2.19) yields:

$$\sigma^2 = \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} \alpha\gamma\delta(t-t')\mathrm{d}t\mathrm{d}t' = \alpha\gamma T, \tag{2.20}$$

which inserted back in the density matrix element gives:

$$\rho_{10}(t_0+T) = \rho_{10}(t_0) \exp\Big(-\frac{\alpha}{2}\gamma T - i\Delta T\Big). \tag{2.21}$$

2.4 Calibration of model

In order to move forwards and get comparable results, we need to calibrate our model to the article[7]. Their model is given in (2.1), which we reduce step by step.

$$\begin{aligned} \rho\,|1\rangle\langle 1| - |1\rangle\langle 1|\,\rho = [\rho, |1\rangle\langle 1|] = \rho_{01}\,|0\rangle\langle 1| - \rho_{10}\,|1\rangle\langle 0|\,, \\ \sigma_z\rho\sigma_z - \rho = -2\rho_{10}\,|1\rangle\langle 0| - 2\rho_{01}\,|0\rangle\langle 1|\,. \end{aligned} \tag{2.22}$$

The matrix element ρ_{10} which we looked at in our model gives in their model:

$$\dot{\rho}_{10} = -(i\Delta + \gamma)\rho_{10}, \tag{2.23}$$

which is a very similar differential equation to equation (2.12). Solving it gives:

$$\rho_{10}(t_0+T) = \rho_{10}(t_0) \exp(-\gamma T - i\Delta T), \tag{2.24}$$

which is easily comparable to equation (2.21) and we conclude $\alpha = 2$ for the two models to agree. Which means that the correlation function for the noise model now reads $\langle F(t)F(t') \rangle = 2\gamma\delta(t-t')$.

2.5 The limit of uncertainty

With the model calibrated we want to express the noise as a Fourier Series. This is because we do not know anything about $F(t)$. We know what $\langle F(t)F(t')\rangle$ is, but we want to just look at the zeroth order component of the series. The definition of the Fourer series written in the exponential way is: [10]

$$s(x) = \sum_{n=-\infty}^{\infty} c_n \exp(i\omega_n x), \qquad c_n \ = \frac{1}{P}\int_P s(x)\exp(-i\omega_n x)\mathrm{d}x. \tag{2.25}$$

For a real valued function $c_n = c_{-n}$, and thus for $F(t)$ - which is real valued - the Fourier series is:

$$F(t) = c_0 + 2\sum_{n=1}^{\infty} c_n \exp(i\omega_n t). \tag{2.26}$$

Inserting this in the Hamiltonian gives:

$$\hat{H} = \frac{1}{2}[\Delta + c_0 + 2\sum_{n=1}^{\infty} c_n \exp(i\omega_n t)]\hat{\sigma_z}. \tag{2.27}$$

Now it is clear that the last part contains both dependence on t and c_n which one can tweak and change, but we can never get rid of the zeroth order component. We can therefore write a new Hamiltonian with a minimum of noise:

$$\hat{H}_{min} = \frac{1}{2}(\Delta + c_0)\hat{\sigma_z}. \tag{2.28}$$

Where if we want to measure Δ, we would use an operator $\hat{\Delta} = 2\hat{H}$, where we expect and expectation value of Δ, which it is $\left\langle\hat{\Delta}\right\rangle = 2\left\langle\hat{H}\right\rangle = \frac{1}{2}(\langle\Delta\rangle + \langle c_0\rangle) = \Delta$ - where we used the fact that the noise is symmetric. Also note that Δ and c_0 are both multiplied on the same $\hat{\sigma_z}$, thus for any state $|\psi\rangle$ the Pauli matrix will return a common constant to both Δ and c_0. The variance on this operator is then:

$$\sigma^2_{\hat{\Delta}} = \left\langle\hat{\Delta}^2\right\rangle - \left\langle\hat{\Delta}\right\rangle^2 = \left\langle(\Delta + c_0)^2\right\rangle - \langle\Delta\rangle^2 - \langle c_0\rangle^2 = \left\langle\Delta^2\right\rangle + \left\langle c_0^2\right\rangle + 2\left\langle\Delta c_0\right\rangle - \langle\Delta\rangle^2 = \left\langle c_0^2\right\rangle, \tag{2.29}$$

where we again used the fact that the noise is symmetric, and that the expectation value and square for a scalar A commutes: $\langle A\rangle^2 = \left\langle A^2\right\rangle$.

Now we just need to calculate the expectation value of the square of the noise.

$$\left\langle c_0^2\right\rangle = \left\langle \frac{1}{T^2}\int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} F(t)F(t')\mathrm{d}t\mathrm{d}t' \right\rangle = \frac{2\gamma}{T}. \tag{2.30}$$

From this we can conclude that the minimal uncertainty for the measurement of Δ a single particle is

$$\sigma_\Delta \geq \sqrt{\frac{2\gamma}{T}}. \tag{2.31}$$

Now we want to improve this by performing experiments with L particles instead of one.

2.5.1 L particle case

In the case of L particles, the Hamilton for minimal uncertainty reads:

$$\hat{H}_{min} = \frac{1}{2}\sum_{l=1}^{L}(\Delta_l + c_{0,l})\hat{\sigma_z}, \tag{2.32}$$

where $\sum_{l=1}^{L}\Delta_l = L\Delta$. Since all particles are exposed to the same field, a total of "L times the amount of signal" is present, but that is not the case for the noise which is assumed spatially uncorrelated

meaning $\langle F_l(t)F_{l'}(t')\rangle = 2\gamma\delta_{l,l'}\delta(t-t')$. Thus we can conclude that $\hat{\Delta} = \frac{2\hat{H}}{L}$. This results in an uncertainty of:

$$\begin{aligned} \sigma_{\hat{\Delta}} &= \frac{\left\langle (\sum_{l=1}^{L} c_0)^2 \right\rangle}{L^2} \\ \left\langle (\sum_{l=1}^{l} c_0)^2 \right\rangle &= \left\langle \sum_{l=1}^{L}\sum_{l'=1}^{L} \int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} F_l(t)F_{l'}(t')\mathrm{d}t\mathrm{d}t' \right\rangle = \frac{2\gamma L}{T}. \end{aligned} \tag{2.33}$$

Which means that the minimal uncertainty for L particles is given by:

$$\sigma_\Delta \geq \sqrt{\frac{2\gamma}{TL}}. \tag{2.34}$$

The same result as the article [7] and in agreement with the standard quantum limit. This calculation is however more general, since no assumption on entanglement vs uncorrelated is made. That is also expected since entangled and uncorrelated measurements are metrologically equivalent [4].

3 Colored noise

Colored noise is characterized by in principle any arbitrary correlation function. These environments have nonzero (not infinitely short) correlation times, but for times longer than the correlation time one would expect a white noise-like behavior. In order to incorporate both white noise and colored noise into our model, we choose our spectral density function wisely. Both to get a model which somewhat resembles what one might expect from a real world experiment, but also one where we have a chance of actually calculating something - read: not too complicated.

3.1 Previous results

Results in the case of non-Markovian dephasing (colored noise) goes far and wide, but many articles pursuit the same - beating the Standard Quantum Limit. Here we present two articles which claim to have done so. The presented scaling's are in terms of variance.

One article [4], by the same authors as the article on white noise, obtain a better scaling than SQL by a factor of $\frac{1}{\sqrt{L}}$. They using maximally entangled states in the case of an ohmic spectral density function, in comparison to a product state (uncorrelated).

Another article [9], achieve a better scaling by a factor of $1/T$ compared to SQL. They use quantum teleportation of the atoms around to different sites and thereby avoiding correlation with the environment.

The model of our choice is different. We stay far away from quantum teleportation, and chose another spetral density function than an ohmic. An ohmic function is a viable choice, but it tends to zero for low values, which we would like to avoid in our model.

3.2 Model

The noise model (spectral density function) we choose is a Gaussian centered around zero, but with the 2γ from the white noise result added on top. A sketch is shown in figure 1.

$$G(\omega) = \frac{A}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\omega^2}{2\sigma^2}\right) + 2\gamma, \tag{3.1}$$

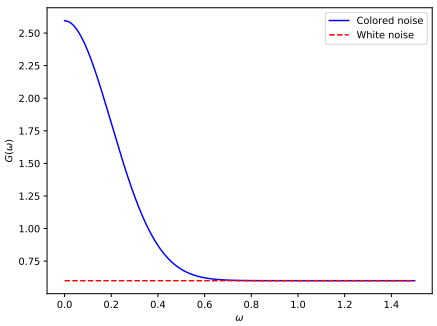

Figure 1: A sketch of our spectral density function. Here $A = 1,\ \gamma = 0.3,\ \sigma = 0.2$.

where the factor of A adds an extra degree of freedom to the otherwise regular normalized Gaussian. We also got the parameter σ which allows us to determine the width of the curve, note that it is NOT the uncertainty on

Δ. It is just a parameter in our function. Also the function is written in terms of frequency instead of time, but as discussed in the white noise section, it is just a matter of the Fourier transform. The full correlation function for this model is:

$$\left\langle F(\omega)F(\omega')\right\rangle = G(w)\delta(\omega+\omega'). \tag{3.2}$$

In this model we can not just integrate from start to end. Because the noise now depend on the past, we have to divide the integration into sections and add them up in the end. We make the following change:

$$\int_{t_0}^{t_0+T} \mathrm{d}t \to \sum_{n=1}^{N} \int_{t_n-1}^{t_n} \mathrm{d}t, \tag{3.3}$$

where $t_n - t_{n-1} = \delta t$ and $N\delta t = T$. So now it is δt which is the time of one integration period. Note that T is assumed constant, and not dependent of δt.

3.3 Ramsey spectroscopy

The idea is now to use the results of Ramsey Spectroscopy- where the noise is only present during the clock radiation. To good approximation the two Ramsey-pulses are so fast, that the noise does not really affect the rotation. Starting from equation (1.18) and using the results of $\boldsymbol{J}(t_f)$ from equation (1.21), and the expected number of particles in the $\left|+\frac{1}{2}\right\rangle$-state at t_f is:

$$\langle L_+(t_f)\rangle = J + \sum_{n=1}^{N}\left[\langle J_y(0)\rangle \sin\left(\Delta\delta t + \int_{t_{n-1}}^{t_n} F(t)\right) - \langle J_z(0)\rangle \cos\left(\Delta\delta t + \int_{t_{n-1}}^{t_n} F(t)\right)\right]. \tag{3.4}$$

If the particles are prepared in an initial state

$$\psi(0) = |J = L/2, M = -L/2\rangle\,, \tag{3.5}$$

where all particles are in the $\left|-\frac{1}{2}\right\rangle$ state - it has the following characteristics $\langle J_z(0)\rangle = -J$, $\langle J_x(0)\rangle = \langle J_y(0)\rangle = 0$, $\sigma_{J_z(0)} = 0$, $\sigma_{J_x}(0) = \sigma_{J_y}(0) = \sqrt{J/2}$. Then the expected number of particles in the up state is given by, where we again substitute the integral of the noise with $\phi_n(t)$:

$$\langle L_+(t_f)\rangle = \frac{L}{2}(1 + \sum_{n=1}^{N}\cos(\Delta\delta t + \phi_n(t))). \tag{3.6}$$

3.4 Method

The way we want to determine Δ as the slope of a straight line for $\Delta \to 0$. Cosine is not very good for this as the slope of the tangent to a cosine varies for small arguments. If we subtract a factor of $\pi/2$ we can transform the cosine to a sine. Then for small arguments we can approximate $\sin(\Delta\delta t) \approx \Delta\delta t$. Because of the noise, we assume the slope of the straight line to be off by a factor of β. This method is sketched in figure 2. The correction β must be the same at all times, since a straight line has the same slope between any two points. This yields:

$$L_+(t_f) = \frac{L}{2}(1 + \frac{\Delta T}{\beta}), \tag{3.7}$$

where $\sum_{n=1}^{N}\delta t = \delta t N = T$. If now we want to determine Δ based on the number of particles in the $\left|+\frac{1}{2}\right\rangle$ state, we can rewrite the expression.

$$\Delta = \frac{L_+(t_f) - L/2}{L} 2\beta. \tag{3.8}$$

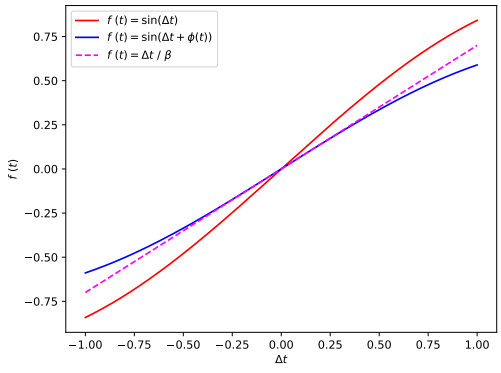

Figure 2: The shape of a sine with noise (blue), will vary slightly from a sine without noise (blue). The value of Δ can be found by a straight line fit for small values of ΔT.

If we are to express this in terms of an observable we can measure, we know from equation (1.15) that $\sum_{l=1}^{L} J_{z,l}(t_f) = L_+ - L/2$ for this exact state (3.5). Thus we can change we can rewrite the expression for $\hat{\Delta}$, which is an operator, just like in the white noise case:

$$\hat{\Delta} = \frac{\sum_{l=1}^{L} \hat{J_{z,l}}(t_f)}{LT} 2\beta. \tag{3.9}$$

Finding the expectation value of this operator, means finding the expectation value of $\hat{J_{z,l}}$ since the rest is just numbers, with trivial expectation values. But from equation (1.21) and the characteristics for our wavefunction in the initial state described by equation (3.5) we know what $J_{z,l}(t_f)$ gives. Note that we subtracted a factor of $\pi/2$ which must also be included in all expressions for the clock rotation. Thus:

$$\langle J_{z,l}(t_f) \rangle = 1/2 \sum_{n=1}^{N} \langle \sin(\Delta\delta t + \phi_n(t)) \rangle \,. \tag{3.10}$$

We want to expand the sine to exponential form taylor expand around $\Delta = 0$

$$\begin{aligned} \langle \sin(\Delta\delta t + \phi_n(t)) \rangle &= \left\langle \frac{\exp(i\Delta\delta t)\exp(i\phi_n(t)) - \exp(-i\Delta\delta t)\exp(-i\phi_n(t))}{2i} \right\rangle, \\ &= \frac{((1 + i\Delta\delta t)\langle \exp(+i\phi_n(t)) \rangle - (1 - i\Delta\delta t)\langle \exp(-i\phi_n(t)) \rangle}{2i}. \end{aligned} \tag{3.11}$$

Just like for white noise, the probability density function is assumed to be Gaussian. This means we repeat section 2.3.1, and therefore just write down the result. Note however that $\langle \exp(i\phi_n(t)) \rangle = \langle \exp(-i\phi_n(t)) \rangle$ because of the Gaussian integral (equation (2.17)). We thus have:

$$\langle \sin(\Delta\delta t + \phi_n(t)) \rangle \approx \Delta\delta t \exp\left(-\frac{\langle \phi_n(t)^2 \rangle}{2} \right). \tag{3.12}$$

This result is combined with the equation for the Δ operator, where nothing anymore depends on l and therefore the sum over l just gives a factor of L:

$$\left\langle \hat{\Delta} \right\rangle = \frac{L\beta}{LT} \sum_{n=1}^{N} \Delta\delta t \exp\left(-\frac{\langle \phi_n(t)^2 \rangle}{2} \right), \tag{3.13}$$

where it is clear that for $\beta = \exp\left(+ \langle \phi_n(t)^2 \rangle / 2 \right)$ we get the expected expectation value of Δ:

$$\left\langle \hat{\Delta} \right\rangle = \frac{\Delta LT}{LT} \exp\left(+\frac{\langle \phi_n(t)^2 \rangle}{2} \right) \exp\left(-\frac{\langle \phi_n(t)^2 \rangle}{2} \right) = \Delta. \tag{3.14}$$

Which is exactly the expectation value we want.

So far we have not used the correlation function. It will however be relevant now, when we need to look at the expectation value of the square of the operator - which we need in order to determine the uncertainty. We start this calculation from equation (3.9), or rather a squared slight modified version.

$$\hat{\Delta^2} = \frac{(2\beta)^2}{(TL)^2} \sum_{l=1}^{L} \sum_{l'=1}^{L} \sum_{n=1}^{N} \sum_{n'=1}^{N} \left\langle \hat{J_{z,l,n}}(t_f) \hat{J_{z,l',n'}}(t_f) \right\rangle. \tag{3.15}$$

Now we assume the individual particle spins to be uncorrelated, this means getting rid of one of the sums over l. Note that the remaining sum over l just gives a factor of L. Thus when we now look at the product of $J_z(t_f)$'s, we restrict ourselves to the case where $l = l'$ and just remove the subscript of l completely.

$$\begin{aligned} &\left\langle \hat{J_{z,n}}(t_f) \hat{J_{z,n'}}(t_f) \right\rangle, \\ &= \left\langle \hat{J_{z,n}}(0) \hat{J_{z,n'}}(0) \sin(\Delta\delta t + \phi_n) \sin(\Delta\delta t + \phi_{n'}) \right\rangle + \left\langle \hat{J_{y,n}}(0) \hat{J_{y,n'}}(0) \cos(\Delta\delta t + \phi_n) \cos(\Delta\delta t + \phi_{n'}) \right\rangle. \\ &+ \left\langle \hat{J_{z,n}}(0) \hat{J_{y,n'}}(0) \sin(\Delta\delta t + \phi_n) \cos(\Delta\delta t + \phi_{n'}) \right\rangle + \left\langle \hat{J_{y,n}}(0) \hat{J_{z,n'}}(0) \cos(\Delta\delta t + \phi_n) \sin(\Delta\delta t + \phi_{n'}) \right\rangle. \end{aligned} \tag{3.16}$$

First of all the two last terms will always be zero since $\langle J_y(0)\rangle = 0$. Let us divide the equation in to different cases, and start with the case where $n = n'$ and $n \neq n'$ afterwards. We remove the subscript of n since they are all the same.

$$\left\langle \hat{J}_z(t_f)^2 \right\rangle = \left\langle \hat{J}_z(0)^2 \sin^2(\Delta\delta t + \phi(t)) \right\rangle + \left\langle \hat{J}_y(0)^2 \cos^2(\Delta\delta t + \phi(t)) \right\rangle . \tag{3.17}$$

We now know that $\left\langle J_z(0)^2 \right\rangle = \left\langle J_y(0)^2 \right\rangle = \frac{1}{4}$, and $1 = \cos^2(x) + \sin^2(x)$,

$$\left\langle \hat{J}_z(t_f)^2 \right\rangle = \frac{1}{4}\left[\left\langle \sin^2(\Delta\delta t + \phi(t) \right\rangle + 1 - \left\langle \sin^2(\Delta\delta t + \phi(t) \right\rangle \right] = \frac{1}{4}. \tag{3.18}$$

Now for the case where $n \neq n'$. The second term gives zero for the same reason as the two last terms are zero. So here we just get $\frac{1}{4}\sin(\Delta\delta t + \phi_n)\sin(\Delta\delta t + \phi_{n'})$.The final result is then,

$$\left\langle \hat{\Delta}^2 \right\rangle = \frac{(\beta)^2}{T^2 L}\left[N + \sum_{n=1}^{N}\sum_{n'\neq n}^{N} \langle \sin(\Delta\delta t + \phi_n(t))\sin(\Delta\delta t + \phi_{n'}(t)) \rangle \right], \tag{3.19}$$

where the case of $n = n'$ gives a factor of N, since that happens N times when both variables are summed over from 1 to N. The two sums still over n, n' is only for $n \neq n'$.

Next up is the expectation value of the product of the sines.

$$\begin{aligned} &\langle \sin(\Delta\delta t + \phi_n)\sin(\Delta\delta t + \phi_{n'}) \rangle , \\ &= \frac{1}{2}\left[((\Delta\delta t)^2 - 1)\exp\left\langle -(\phi_n + \phi_{n'})^2/2 \right\rangle + ((\Delta\delta t)^2 + 1)\exp\left\langle -(\phi_n - \phi_{n'})^2/2 \right\rangle \right]. \end{aligned} \tag{3.20}$$

where the expansion is the same as in equation (3.11) and (3.12), with the Gaussian probability distribution. This expression can be simplified even further by the expansion of the correlation functions:

$$\left\langle (\phi_n \pm \phi_{n'})^2/2 \right\rangle = \frac{1}{2}(\left\langle \phi_n^2 \right\rangle + \left\langle \phi_{n'}^2 \right\rangle \pm 2\left\langle \phi_n\phi_{n'} \right\rangle) = \left\langle \phi_n^2 \right\rangle \pm \left\langle \phi_n\phi_{n'} \right\rangle , \tag{3.21}$$

because $\left\langle \phi_n^2 \right\rangle = \left\langle \phi_{n'}^2 \right\rangle$ as they are both summed from 1 to N. This allows us to rewrite

$$\langle \sin(\Delta\delta t + \phi_n)\sin(\Delta\delta t + \phi_{n'}) \rangle = \exp\left(-\left\langle \phi_n^2 \right\rangle\right)\left[(\Delta\delta t)^2 \cosh(\langle \phi_n\phi_{n'} \rangle) + \sinh(\langle \phi_n\phi_{n'} \rangle) \right]. \tag{3.22}$$

3.5 Correlation functions

It is clear that two different correlation functions are present: $\left\langle \phi_n^2 \right\rangle$ and $\langle \phi_n\phi_m \rangle$. Remember the model defined in section 3.2. The spectral density function is defined in frequency space, but the noise enters the equation in time-space (integrated in time in equation (3.4)). The noise is therefore Fourier Transformed as the first step following the definition: [10]

$$F(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{F}(\omega)\exp(i\omega t)\mathrm{d}\omega. \tag{3.23}$$

This means ϕ_n can be rewritten

$$\phi_n = \int_{t_{n-1}}^{t_n} F(t)\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{t_{n-1}}^{t_n}\int_{-\infty}^{\infty} \tilde{F}(\omega)\exp(i\omega t)\mathrm{d}\omega\mathrm{d}t. \tag{3.24}$$

Now the correlation functions can be calculated separately. Note that the only difference between them is the time integrals. In $\left\langle \phi_n^2 \right\rangle$ the two integrals in time are over the same interval, and in $\langle \phi_n\phi_{n'} \rangle$ they are not.

3.5.1 $\langle \phi_n^2 \rangle$

$$\langle \phi_n^2 \rangle = \frac{1}{(2\pi)} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\omega)\delta(\omega+\omega') \exp(i\omega t) \exp\big(i\omega' t'\big) \mathrm{d}\omega \mathrm{d}\omega' \mathrm{d}t \mathrm{d}t'. \tag{3.25}$$

The integral of the delta function in frequency space gives a factor of 1 when $\omega' = -\omega$.

$$\begin{aligned} \langle \phi_n^2 \rangle &= \frac{1}{(2\pi)} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{-\infty}^{\infty} G(\omega) \exp(i\omega t) \exp\big(-i\omega t'\big) \mathrm{d}\omega \mathrm{d}t \mathrm{d}t', \\ &= \frac{1}{(2\pi)} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{-\infty}^{\infty} G(\omega) \exp\big(i\omega(t-t')\big) \mathrm{d}\omega \mathrm{d}t \mathrm{d}t'. \end{aligned} \tag{3.26}$$

Now the integral in frequency space can be solved. Note that the result above is general for this model - independent on the spectral density function $G(\omega)$.

$$\int_{-\infty}^{\infty} G(\omega) \exp\big(i\omega(t-t')\big) \mathrm{d}\omega = \int_{-\infty}^{\infty} \Big(\frac{A}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{\omega^2}{2\sigma^2}\Big) + 2\gamma\Big) \exp\big(i\omega(t-t')\big) \mathrm{d}\omega \tag{3.27}$$

This integral of a sum can be split into two integrals. The result of the latter of the integrals is a delta function in time multiplied with the same constant [10]. (Delta functions can also be represented $\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(it(x-x'))\mathrm{d}t$). The other integral is again formula 18.75 in Schaum's [8]:

$$\langle \phi_n^2 \rangle = \frac{1}{(2\pi)} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} A \exp\Big(-(t-t')^2 \frac{\sigma^2}{2}\Big) + 2\pi 2\gamma\delta(t-t') \mathrm{d}t \mathrm{d}t' \tag{3.28}$$

This is again the (double) integral of a sum, which we can split. From the latter integral we just recover the white noise result $2\gamma\delta t$, where the factor of 2π is canceled by the factor of $1/2\pi$ in front of the integrals. The former is the double integral of a Gaussian where the first integration gives an error function, and the integral of an error function gives: $\int \mathrm{erf}(z)\mathrm{d}z = z\,\mathrm{erf}(z) + \exp\big(-z^2\big)/\sqrt{\pi}$[3],

$$\begin{aligned} &\int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} A \exp\Big(-(t-t')^2 \frac{\sigma^2}{2}\Big) \mathrm{d}t \mathrm{d}t', \\ &= A\left[\frac{-2}{\sigma^2} + \frac{\sqrt{2\pi}}{\sigma}(t_n - t_{n-1})\,\mathrm{erf}\Big(\frac{\sigma}{\sqrt{2}}(t_n - t_{n-1})\Big) + \frac{2}{\sigma^2}\exp\Big(-(t_n-t_{n-1})^2\frac{\sigma^2}{2}\Big)\right]. \end{aligned} \tag{3.29}$$

If we again use $t_n - t_{n-1} = \delta t$, and combine the results of all the integrals, we obtain the final expression:

$$\langle \phi_n^2 \rangle = \frac{A}{2\pi}\left[\frac{-2}{\sigma^2} + \frac{\sqrt{2\pi}}{\sigma}\delta t\,\mathrm{erf}\Big(\frac{\sigma}{\sqrt{2}}\delta t\Big) + \frac{2}{\sigma^2}\exp\Big(-\delta t^2\frac{\sigma^2}{2}\Big)\right] + 2\gamma\delta t. \tag{3.30}$$

3.5.2 $\langle \phi_n \phi_{n'} \rangle$

The only difference in this section is the integrals in time which are no longer over the same intervals. Thus we can reuse the result up to the integrals in time, which is equation (3.28).

Again the integral can be split in two, but this time the delta function gives 0, since the time intervals do not overlap. The boundaries of the integrals coincide a finite number of times, but it is only at a single point in time, and therefore give no net contribution will arise from it. The integral of the Gaussian part is:

$$\begin{aligned} &\int_{t_{n-1}}^{t_n} \int_{t_{n'-1}}^{t'_n} A \exp\Big(-(t-t')^2\tfrac{\sigma^2}{2}\Big) \mathrm{d}t' \mathrm{d}t = \\ &\tfrac{\sqrt{\pi}A}{\sqrt{2}\sigma}[-(t_{n'-1}-t_{n-1})\,\mathrm{erf}(\tfrac{\sigma}{\sqrt{2}}(t_{n'-1}-t_{n-1})) + (t_{n'}-t_{n-1})\,\mathrm{erf}(\tfrac{\sigma}{\sqrt{2}}(t_{n'}-t_{n-1})) + (t_n - t_{n'-1})\,\mathrm{erf}(\tfrac{\sigma}{\sqrt{2}}(t_n - t_{n'-1})) - (t_{n'}-t_n)\,\mathrm{erf}(\tfrac{\sigma}{\sqrt{2}}(t_{n'}-t_n))] \\ &+ \tfrac{A}{\sigma^2}[-\exp\Big(-(t_{n'-1}-t_{n-1})^2\tfrac{\sigma^2}{2}\Big) + \exp\Big(-(t_{n'}-t_{n-1})^2\tfrac{\sigma^2}{2}\Big) + \exp\Big(-(t_n-t_{n'-1})^2\tfrac{\sigma^2}{2}\Big) - \exp\Big(-(t_{n'}-t_n)^2\tfrac{\sigma^2}{2}\Big)]. \end{aligned} \tag{3.31}$$

[3] Found with the sympy integral package in Python

Which we can be rewritten by the definition of δt - which is the same between any two neighboring n, n''s. We can also define a new variable $n' - n = \delta n$. Combining it all gives the final expression:

$$\begin{aligned}\langle \phi_n \phi_{n'} \rangle = \\ \frac{A}{\sqrt{\pi}\sigma 2^{3/2}} \left[-2\delta t \delta n \operatorname{erf}(\frac{\sigma}{\sqrt{2}}\delta t \delta n) + \delta t(\delta n + 1)\operatorname{erf}(\frac{\sigma}{\sqrt{2}}\delta t(\delta n + 1)) + \delta t(\delta n - 1)\operatorname{erf}(\frac{\sigma}{\sqrt{2}}\delta t(\delta n - 1)) \right] \\ + \frac{A}{\sigma^2 2\pi} \left[-2\exp\left(-(\delta t \delta n)^2 \frac{\sigma^2}{2} \right) + \exp\left(-(\delta t(\delta n - 1))^2 \frac{\sigma^2}{2} \right) + \exp\left(-(\delta t(\delta n + 1))^2 \frac{\sigma^2}{2} \right) \right].\end{aligned} \tag{3.32}$$

with the new variable δn, the sums over n and n' are not very useful. They can be transformed to a single sum over the new single variable.

3.5.3 Rewriting the sums

To make our lives easier we changed from variables n and n' to $\delta n = n' - n$, which is a viable choice, but the we must also change the two sums over n and n'. One way to tackle this problem, is by imagining the two sums as a square matrix, where the rows correspond to n and columns to n'. We then sum over all the matrix elements. $\delta n = 0$ corresponds to the diagonal elements (of which there are N), but these should not be included in the sum. $\delta n = \pm 1$ corresponds to the diagonal "rows" just above and below the "center diagonal row", with $N - 1$ elements in each of these. Following the pattern yields a total number of elements $N - |\delta n|$ in the δn'th diagonal row.

Now the sums can be changed to δn, where we must include both negative and positive values of δn and exclude 0. But we can do even better by realizing that the $\langle \phi_n \phi_{n'} \rangle$ is symmetric around $\delta n = 0$ - which one would also expect since it should not make a difference whether we define $\delta n = n' - n$ or $\delta n = n - n'$. So we can change to only summing over positive values of δn and multiply by a factor of 2. Thus the entire sum can be changed to:

$$\sum_{n=1}^{N} \sum_{n' \neq n}^{N} \rightarrow 2 \sum_{\delta n=1}^{N-1} (N - \delta n). \tag{3.33}$$

3.6 Variance

We are now able to calculate the variance $\sigma_\Delta^2 = \left\langle \hat{\Delta}^2 \right\rangle - \left\langle \hat{\Delta} \right\rangle^2$. In general this can only be done numerically, since variance contain a sum of hyperbolic trigonometric functions, which arguments contain Gaussian functions and error functions - an impossible task analytically. The full expression is given by:

$$\left\langle \hat{\Delta}^2 \right\rangle = \frac{1}{T^2 L} \left[\frac{T}{\delta t} \exp\left(\left\langle \phi_n^2 \right\rangle \right) + (\Delta \delta t)^2 2 \sum_{\delta n=1}^{N-1} (N - \delta n) \cosh \langle \phi_n \phi_{n'} \rangle + 2 \sum_{\delta n=1}^{N-1} (N - \delta n) \sinh \langle \phi_n \phi_{n'} \rangle \right], \tag{3.34}$$

However, by investigating different limits the correlation functions can be simplified. Either by a Taylor expansion or if terms converge to some constant value. The two limits of interest are: the short time limit $\delta t << 1/\sigma$ and long time limit $\delta t >> 1/\sigma$. Besides these limits, a numeric calculation is done of the full expression. Both to get results which are valid in between the other models, but also to check for errors in the analytical approximations.

3.6.1 Short time limit

For $\left\langle \phi_n^2 \right\rangle$ this allows for a Taylor expansion of both the error function and the Gaussian around 0.

$$\exp\left(-\delta t^2 \frac{\sigma^2}{2} \right) \rightarrow 1 - \delta t^2 \frac{\sigma^2}{2}, \qquad \operatorname{erf}(\frac{\sigma}{\sqrt{2}}\delta t) \rightarrow \frac{\sqrt{2}\sigma}{\sqrt{\pi}}\delta t, \qquad \left\langle \phi_n^2 \right\rangle = \frac{A}{2\pi}\delta t^2 + 2\gamma \delta t. \tag{3.35}$$

In the case of $\langle \phi_n \phi_{n'} \rangle$ there is also the variable δn, which we must consider. For the first part with the error functions, the three terms will to good approximation cancel out for all δn. This is confirmed

by visual inspection. For the Gaussians we can do the Taylor expansion which gives $\frac{A}{2\pi}\delta t^2$ and is only valid for small δn. However by visual inspection, the behavior in δn for larger values, which the Taylor expansion does not describe, is just the same Gaussian once again. Hence the final approximation is.

$$\langle \phi_n \phi_{n'} \rangle = \frac{A}{2\pi}\delta t^2 \exp\left(-(\delta t \delta n)^2 \frac{\sigma^2}{2}\right). \tag{3.36}$$

In the short time limit, it is also required that $A/\sigma^2 > 1$. This translates to $1/\sigma^2 > 1/A$, means we have a lower bound on $1/\sigma$. But in the short time limit there is already a lower bound on $1/\sigma$, which is good since we then have no upper bound on $1/\sigma$. This means that there will always exist a σ (actually an infinite number of σ's), for which these assumptions are true and where the approximation is valid.

In relation to $\langle \phi_n \phi_{n'} \rangle$ this requirement is useful since allows $\langle \phi_n \phi_{n'} \rangle$ to be small number. With these requirements A can be arbitrarily large, but δt can also be arbitrarily small. If A is very large, then the optimal δt - the δt which minimizes the variance - is also very small. For the δt's of our interest (depends on A, σ, γ) $\langle \phi_n \phi_{n'} \rangle$ will be small in which case we can Taylor expand; $\sinh(x) = x$, $\cosh(x) = 1$.

This allows us to calculate the sums analytically. The sum independent of δn gives a factor of $N(N-1)/2$ which for N large is equal to $N^2/2$. In the other sum, δn is the argument of a Gaussian. We can approximate the sum with an integral since the steps in the sum is of unit length. Under the assumption that N is large - by large we mean large enough that the Gaussian has effectively died out at $\delta n = N$, we can change to an integral of a Gaussian to infinity. An integral of a Gaussian from 1 to ∞ cannot be evaluated exactly, but an integral from 0 to ∞ can. This introduces a small error, since we now find the entire area under the Gaussian, whereas the very center part for $|\delta n|$ should have been excluded. This error will however introduce more noise, and in that sense we are not "cheating" by removing noise. To recap:

$$\sum_{\delta n=1}^{N-1} [(N-\delta n)\exp\left(-(\delta n \delta t)^2 \frac{\sigma^2}{2}\right)] \to \int_0^\infty (N-x)\exp\left(-(x\delta t)^2 \frac{\sigma^2}{2}\right) \mathrm{d}x = \frac{N}{\delta t \sigma}\sqrt{\frac{\pi}{2}} - \frac{1}{\delta t^2 \sigma^2}, \tag{3.37}$$

where the integral can be solved using formula 18.72 and 18.77 in Schaum's [8].

We can now write the final expression for the variance in the short time limit. Remember the definition of the total time $T = N\delta t$ and in the limit of N large $(N-1) = N$

$$\sigma^2_{\hat{\Delta}} = \frac{1}{T^2 L}\left[\frac{T}{\delta t}\exp\left(\frac{A}{2\pi}\delta t^2 + 2\gamma \delta t\right) + T^2\Delta^2 + \frac{AT}{\sqrt{2\pi}\sigma} - \frac{A}{\pi\sigma^2}\right] - \Delta^2 \tag{3.38}$$

To find the minimum we can take the derivative with respect to δt and set equal to zero. Only two terms are dependent on δt, so in the derivative, only these two will contribute. This gives an expression with the general form $\frac{a}{x}\exp\big(bx^2 + cx\big)$, which has a minimum at $2abx^2 + acx - a = 0$, just a polynomial of second order - we can even divide a out of the equation, which makes sense since $a = 1/TL$, and we do not expect the optimal time to depend on number of atoms or the total time. Solving the polynomial and plugging in values yields the minimum:

$$\delta t_{opt} = \left(-\gamma + \sqrt{\gamma^2 + \frac{A}{\pi}}\right)\frac{\pi}{A} \tag{3.39}$$

At first it seems untrue, since one would expect the minimum to depend on σ. However the last two terms (the only ones with σ) are independent of δt, and therefore the derivative is independent of σ. By visual inspection of the numeric calculation, it's minimum was found to be invariant of σ in the short time limit, however for $A/\sigma^2 >> 1$ the minimum started to diverge.

To get an algebraic result for the minimum, we assume $\Delta = 0$ and $\gamma = 0$ and also that T is large. Then the variance for the optimal integration time gives:

$$\sigma^2_{\hat{\Delta},opt} = \frac{1}{TL}\left(\sqrt{\frac{A}{\pi}}\exp\left(\frac{1}{2}\right) + \frac{A}{\sqrt{2\pi}\sigma}\right) \tag{3.40}$$

In the limit $A/\sigma^2 >> 1$ means the latter of the terms is the dominant one, hence the same scaling as the standard quantum limit.

$$\sigma_{\hat{\Delta},opt} \propto \sqrt{\frac{1}{TL}\frac{A}{\sigma}}, \qquad \sigma_{\hat{\Delta},opt} \geq \sqrt{\frac{1}{TL}\frac{A}{\sqrt{2\pi}\sigma}} \tag{3.41}$$

3.6.2 Long time limit

We can rewrite the long time limit as $\delta t\sigma >> 1$. This allows for investigation of the Gaussian and error function

$$\exp\left(-\delta t^2\frac{\sigma^2}{2}\right) \to 0, \qquad \text{erf}(\frac{\sigma}{\sqrt{2}}\delta t) \to 1, \qquad \langle\phi_n^2\rangle = -\frac{A}{\pi\sigma^2} + \frac{A}{\sqrt{2\pi}\sigma}\delta t + 2\gamma\delta t \tag{3.42}$$

In the case of $\langle\phi_n\phi_{n'}\rangle$ we easily see that the error functions will again cancel out, and the Gaussians will be zero. However two of the Gaussians contain a $\delta n \pm 1$ so whenever $\delta n = \pm 1$ the entire argument will be zero and hence the function will return a nonzero result. So for $\delta n = \pm 1$ the function will return the value $\frac{A}{2\pi\sigma^2}$. Given that we restricted δn to only be positive integers in the sum, we can write the function as:

$$\langle\phi_n\phi_{n'}\rangle = \frac{A}{2\pi\sigma^2}\delta(\delta n - 1) \tag{3.43}$$

where we refer to the Dirac delta function of the argument $(\delta n - 1)$.

In the long time limit we again require that $A/\sigma^2 < 1$. Which now translates to no lower bound on $1/\sigma$ - following the same arguments as the short time limit. However $1/\sigma > 0$, but we can in principle get infinitely close to zero. This means that $\langle\phi_n\phi_{n'}\rangle < 1$ and we can Taylor expand; $\cosh(x) = 1$. Here both sums are independent of δn, however the last sum only gives a contribution for $\delta n = 1$. Thus the final result of variance is:

$$\sigma_{\hat{\Delta}}^2 = \frac{1}{T^2L}\left[\frac{T}{\delta t}\exp\left(-\frac{A}{\pi\sigma^2} + \frac{A}{\sqrt{2\pi}\sigma}\delta t + 2\gamma\delta t\right) + T^2\Delta^2 + 2\frac{T}{\delta t}\sinh\left(\frac{A}{2\pi\sigma^2}\right)\right] - \Delta^2 \tag{3.44}$$

Again the minimum can be found by taking the derivative. The dominant terms is the first, so we set $\sinh\big(A/(2\pi\sigma^2)\big) = 0$. This yields an expression of the general form $\frac{a}{x}\exp(bx + c)$, which has a minimum at $abx - a = 0$, with the almost trivial solution $x_0 = \frac{1}{b}$. Inserting values gives the minimum value:

$$\delta t_{opt} = \frac{1}{\frac{A}{\sqrt{2\pi}\sigma} + 2\gamma} \tag{3.45}$$

Again, an algebraic expression can be obtained with $\Delta = 0$ and $\gamma = 0$.

$$\sigma_{\hat{\Delta},opt}^2 = \frac{1}{TL}\left(\frac{A}{\sqrt{2\pi}\sigma}\exp\left(-\frac{A}{\pi\sigma^2} + 1\right) + 2\frac{A}{\sqrt{2\pi}\sigma}\sinh\left(\frac{A}{2\pi\sigma^2}\right)\right) \tag{3.46}$$

In the limit $A/\sigma^2 \to 0$ we can rewrite $\exp(-x + 1) = e$ and $\sinh(x) = x$, yielding the standard quantum limit as scaling.

$$\sigma_{\hat{\Delta},opt} \propto \sqrt{\frac{1}{TL}\frac{A}{\sigma}}, \qquad \sigma_{\hat{\Delta},opt} \geq \sqrt{\frac{e}{TL}\frac{A}{\sqrt{2\pi}\sigma}} \tag{3.47}$$

3.6.3 Modified long time limit

Actually, we found by visual inspection that the approximation above for $\langle\phi_n\phi_{n'}\rangle$ works well in the very long time limit, but when $A/\sigma^2 < 1$ the long time approximation does diverge a bit from the numeric graph around the minimum of the variance curve - which is the point we are mostly interested in. We therefore found that by adopting the $\langle\phi_n\phi_{n'}\rangle$ of the short time approximation, we can create a curve which follows the numeric in the limit $A/\sigma^2 < 1$ around the δt's that minimize variance. The value of δt that minimizes this new curve (under the requirement A/σ^2) is the same point, since the first term in the expression of the variance is still the most dominant. Hence the final result is:

$$\sigma_{\hat{\Delta}}^2 = \frac{1}{T^2L}\left[\frac{T}{\delta t}\exp\left(-\frac{A}{\pi\sigma^2} + \frac{A}{\sqrt{2\pi}\sigma}\delta t + 2\gamma\delta t\right) + T^2\Delta^2 + \frac{AT}{\sqrt{2\pi}\sigma} - \frac{A}{\pi\sigma^2}\right] - \Delta^2 \tag{3.48}$$

4 Discussion

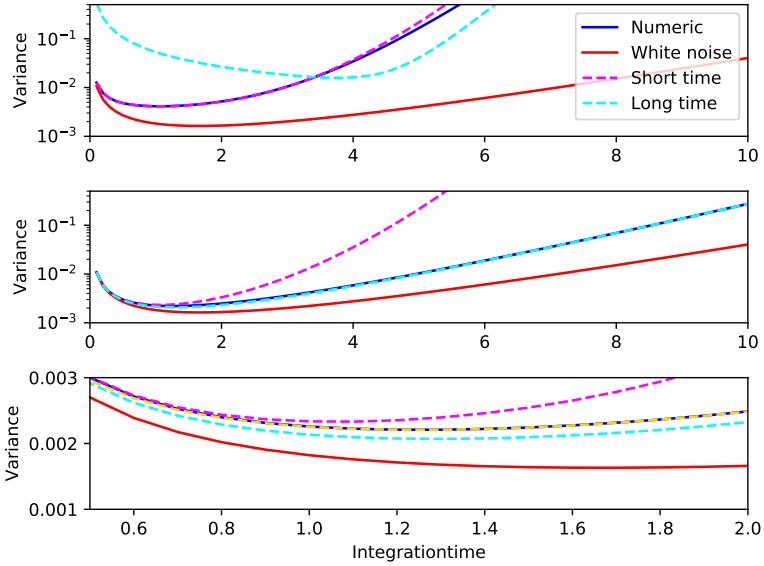

Figure 3: In the first plot $A/\sigma^2 = 25$, the second $A/\sigma^2 = 0.25$, and the third is a zoom of the second. On the third plot, a golden dashed line is added which is the *modified long time*. On the y-axis is variance, and on the x-axis is integration time δt.

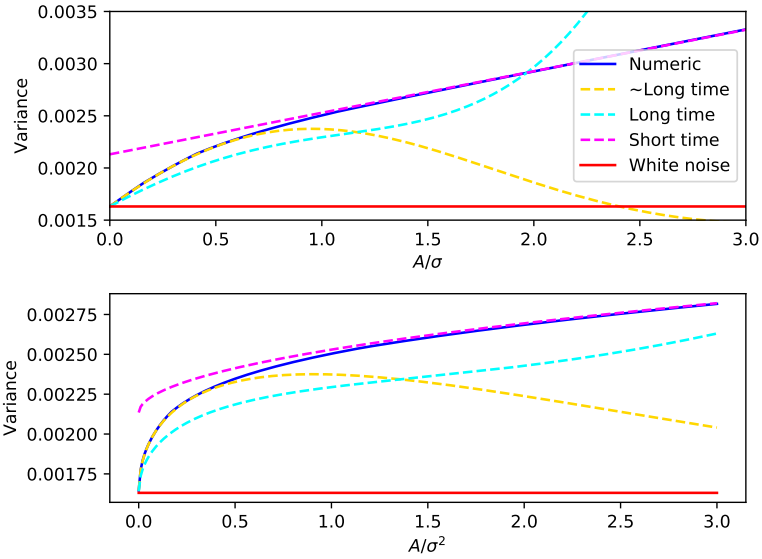

Figure 4: In both plots $\gamma = 0.3$. The graphs shows the minimal value of variance for the different limits. On the first plot the x-axis is A/σ and the second A/σ^2. The golden line is the *modified long time limit*. On the y-axis is variance.

In the case of white noise we find the exact same result as [7]. In our case we show that no matter what, the zeroth order term of the Fourier series will always be present. This therefore stands as a more general result since no assumptions are made on which states this result is valid for. It stand as a fundamental minimum which nothing, nor quantum entanglement, can surpass. An interesting question is now finding states which can achieve that scaling. The white noise articles proposes specific entangled states which in theory should be able to, but it is unclear if they actually managed to reach it. They definitely find improvements over the first limit $\sqrt{\frac{2\gamma e}{TL}}$, but never seem to reach the actual fundamental limit of $\sqrt{\frac{2\gamma}{TL}}$.

Given the fundamental limit of white noise, if more noise - white or colored - is added on top, logic would say that we can never surpass the fundamental limit. This is however not what [4] did. They assume an ohmic model which tends to zero for small and large frequencies. Given that our model is white noise with colored noise added on top, we obviously would not get directly comparable results. Our model in the case of only colored noise also tend to zero for large frequencies, but that behavior seems to not be relevant, since the peak value of our spectral density function, is what sets the limit of precision scaling.

Given that the models are not directly comparable, we could still investigate a few important features of our model. First of all that this model would not actually beat the SQL, and second to see how far from the pure white noise result we actually are.

To answer the first question, no we are never below fundamental limit. In the case where A/σ^2 is very small, we can get arbitrarily close to the limit $\sqrt{\frac{2\gamma e}{TL}}$, but never below it. This behavior is demonstrated in figure 4, where it seen that both the numeric and the two long time limits tend to the white noise limit when $A/\sigma^2 \to 0$ or $A/\sigma \to 0$. The answer to the second question is, that we are actually quite close. In both regimes of time the variance scales linearly as a function of $\frac{A}{\sigma}$, with two different slopes. This behavior is also shown on the first plot of figure 4, where the numeric curve (blue) changes from one linear regime to another, with what appears to be $\sqrt{A/\sigma}$ scaling in between.

From both plots in figure 4, it is clear that the short time limit works when $A/\sigma^{>}1$ and the long time limit works for $A/\sigma^2 < 1$, which is the assumption we made earlier. We can also see that modified

long time model, actually works better in terms of describing the minimum of the variance compared to the long time model. This is also clear on figure 3 where in the third plot, which is a zoom of the second, that the golden dashed line (modified long time model) fits the numeric model around the minimum.

Here the cyan dashed line (long time model) is actually lower than the numeric, which is also what we see on figure 4, where the long time model, is consequently lower than the numeric, at least in the regime of A/σ^2 where the model is valid. Note however on figure 3 on the second plot. For long times $\delta t > 1\sigma$ - which in this case is $\delta t > 4$ - the long time model fits the numeric model very well. In this regime of time, the modified long time model would not fit the numeric model as well.

On the first plot of figure 3 it is seen that the short time model fits the minimum of the numeric model good, whereas the long time model is far off. This is expected since that plot is for $A/\sigma^2 = 25$. However at $\delta t = 1/\sigma = 4$, the short time model starts to diverge, since we leave its regime in time. Had the plot gone wider we would see to long time model lining up with the numeric model for larger times, even though we are not in its regime of A/σ^2.

Another very interesting analytical result is in the long time limit for $\gamma = 0$ and $A/\sigma^2 \to 0$. The expression for variance is $\sigma^2_{\hat{\Delta}} \geq \frac{e}{TL}\frac{A}{\sqrt{2\pi}\sigma}$, which is exactly the same scaling as for white noise but with $2\gamma = \frac{A}{\sqrt{2\pi}\sigma}$, which is the maximal value of the spectra density function for $\omega = 0$. The scaling also has the factor of e as found by [7] in the case of white noise.

Similarly for the short time limit with $\gamma = 0$ and $A/\sigma^2 >> 1$. Then the analytic expression for variance is $\sigma^2_{\hat{\Delta}} \geq \frac{1}{TL}\frac{A}{\sqrt{2\pi}\sigma}$, but by an improvement of a factor of e. We are however still limited by the peak value of the spectral density function at $\omega = 0$. As seen by figure 4 the slope of the variance for A/σ^2 is lower in the region of $A/\sigma^2 > 1$ than in the region of $A/\sigma^2 < 1$. However if one were to describe the variance as a linear function of A/σ, it would have a relatively large offset in the limit of $A/\sigma^2 < 1$.

These two regimes; integration time and A/σ^2 are always relevant. Plotting the variance as a function of integration time (as in figure 3), the value of A/σ^2 determines which model fit at the minimum value of variance, but both models still fit in their respective time regimes related to the inverse of σ, no matter what the value of A/σ^2 is. Off cause the value of σ determines where the models fit. Changing to the minimal variance (as in figure 4), the integration time suddenly becomes irrelevant.

An interesting and very important question which we must address is how the colored noise model works in the case of pure white noise. In this case our model should be corrected. At the sum over n and n' - white noise only contributes when $n = n'$, since it is delta function correlated in time. This was also seen in the section of $\langle \phi_n \phi_{n'} \rangle$, where the white noise part gave no contribution, but it did in $\langle \phi_n^2 \rangle$. Thus the final expression of variance is:

$$\sigma^2_{\hat{\Delta}} = \frac{\exp(\langle 2\gamma \delta t \rangle)}{LT} \tag{4.1}$$

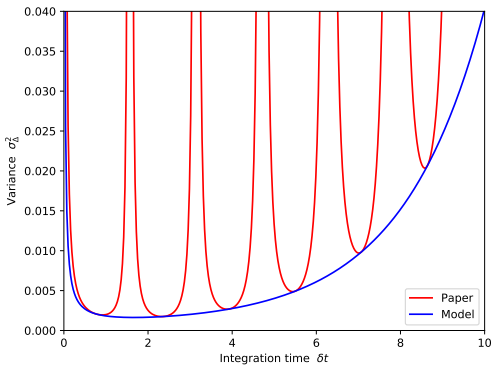

Figure 5: Variance of the colored noise model (blue) in the white noise case, compared to variance of the white noise article (red) for.

In figure 5 the variance of the white noise paper and the colored noise model are plotted. The uncertainty of the white noise model is given by equation (2.2) where we plot for $\Delta = 2$. Note how the local minima of the red curve follows the shape of the blue curve, but the red curve oscillates rapidly. This behavior is known as Ramsey-Fringes, and is due to the probability of measuring a particle in the excited he excited state is proportional to $\cos(\Delta T)$. This follows equation (1.18). In our model this behavior is killed by the approximation $\sin(\Delta\delta t + \phi_n) = \Delta\delta t/\beta$. Given that we are interested in the minimal value of variance, our model is still viable in spite of it not capturing the full behavior of Ramsey Spectroscopy.

5 Conclusion and outlook

These sections recap the results of this thesis, and proposes open questions which would be interesting to answer in the future.

5.1 Conclusion

In this thesis we have shown that the optimal precision, by it is possible to measure Δ, in the Hamiltonian $\hat{H} = 1/2(\Delta + F(t))\sigma_z$, where $F(t)$ is a symmetric field with the correlation function $\langle F(t)F(t')\rangle = 2\gamma\delta(t-t')$ is:

$$\sigma_{\Delta,opt} \geq \sqrt{\frac{2\gamma}{TL}} \tag{5.1}$$

where the delta function correlation in time, is what characterizes white noise.γ is the inverse of the decoherence time, T is the total time of the experiment and L is the number of atoms. This result stands as a fundamental limit which no measurement - uncorrelated or entangled - can surpass. The limit is in agreement with the result of others [7], however the result is derived from a more general point in this theses, with no assumptions made on the states used to perform the measurement.

We can also conclude that in a non-Markovian environment composed of white noise and colored noise, with the requirement on the spectral density function $G(\omega) \geq 2\gamma$ for all ω, one can not surpass the fundamental limit of white noise. Our spectral density function is $G(\omega) = \frac{A}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\omega^2}{2\sigma^2}\right) + 2\gamma$, and in the case of $\gamma = 0$ we found a scaling of the precision of:

$$\sigma_\Delta \propto \sqrt{\frac{1}{TL}\frac{A}{\sqrt{2\pi}\sigma}} \tag{5.2}$$

In the short time limit the scaling is a factor of e better than in the long time limit, but with an offset. Thus for pure colored noise $\gamma = 0$ in our model, it is still not possible to achieve a better scaling than the square root of the peak value of the spectral density function divided by total time and number of atoms. This means that precision in regards to total integration time T and total number of atoms L, does not surpass the standard quantum limit which scales as $1/\sqrt{LT}$. A scaling better than the standard quantum limit has been proposed by many: ([4],[9]), but in all cases their spectral density functions assume ohmic behavior.

We also derive two analytic approximations, which by visual inspection are found to describe the full numeric expression well in their respective limits of integration time δt and noise level A/σ^2 very well.

5.2 Outlook

It would be interesting to see if entanglement could improve the limits of uncertainty found by this model. In the paper on white noise [7] they find an improvement of a factor of $\sqrt{e}$ by using special entangled states. Our model - in the case of pure white noise - predicts $\sqrt{\frac{2\gamma e}{TL}}$, would it also gain the same improvement by using the exact same entangled states as proposed by the paper? And what uncertainty would these special entangled states yield on a measurement in the presence of general colored noise? Would other states do even better?

Another interesting numeric analysis would be to use an ohmic spectral density function like the on from [4] as $G(\omega)$ in our model and then compare results. One would expect very similar if not exactly the same result, but it is not given in advance.

Given our result in the case of $\gamma = 0$ it would also be interesting to take a Gaussian spectral density function again, but displacing it by ω_c is not centered around 0. This would mean that the peak value is no longer at $\omega = 0$ but $\omega = \omega_c$. Thus the full model would be:

$$G(\omega) = \frac{A}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\omega-\omega_c)^2}{2\sigma^2}\right) + 2\gamma \tag{5.3}$$

and see if the variance is determined by the peak value at $\omega = \omega_x$, or if it would be limited by that value at $\omega = 0$. Maybe some third option in between.

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