



PROPAGATION OF A PHOTON IN AN OPTICAL WAVE- GUIDE WITH MULTIPLE CHIRALLY COUPLED EMITTERS

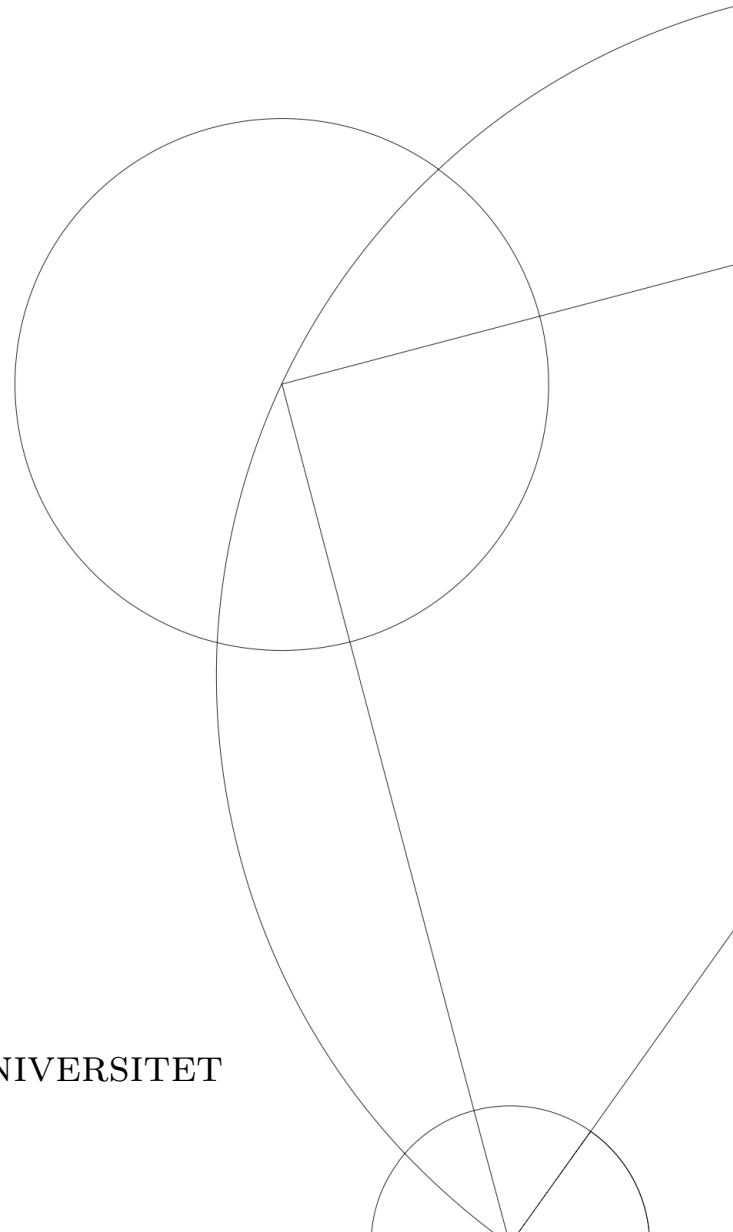
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Abstract

In this thesis a photon propagating in a 1D waveguide with multiple chirally coupled emitters is investigated. A general transfer matrix for multiple emitters is deduced. Based on this transfer matrix, numerical solutions for the reflectance and transmittance are plotted and compared with the nonchiral systems. It is found, that the chiral systems are much more prone to loss, which increases with the difference between the coupling to the fields propagating in the left and right directions. With the strong coupling the chiral systems has a smaller photonic band gap than the nonchiral system.

Contents

1	Introduction	2
2	Preliminaries	3
2.1	Quantization of electromagnetic field	3
2.2	Jaynes-Cummings Model	7
3	The Model	9
3.1	The System Hamiltonian	9
3.2	Time dependent Schrödinger Equation	11
3.3	Time independent Schrödinger Equation	14
4	Scattering of the field	16
4.1	Transfer Matrix	16
4.2	Scattering of ensemble of two-level atoms	18
5	Conclusion and outlook	20
	Appendix A Decay of single emitter initially in excited state	22
	Appendix B Transforming the Hamiltonian to real space	24

1 Introduction

The introduction of quantum mechanics has led to a whole new understanding of the world. The last few decades the applications of quantum mechanics in a more practical manner have flourished and the theory has become the foundation in the development of new technologies, such as quantum computation and quantum information. Light-matter interactions are essential for building such systems, where quantum optical systems are used to carry information[1].

Light-matter interactions in cavities are of great interest and has been investigated in many aspects in order to build quantum optical devices, which enable us to store optical information for a tunable amount of time[2]. In order to build such devices, it is essential that we understand the very basic properties of light-matter interactions.

One important aspect of light-matter interactions is the scattering of the photons[3]. The scattering can be changed by using arrays of atoms. Working with such a problem, one are dealing with a many-body problem. The usual approach to the solution is to solve the system for one atom, then try to generalize the system of one atom to a whole array of atoms. One method that has proven very effective is describing the system with a transfer matrix.[4]. This allows one to investigate important features of the interaction between photons and atoms. In [5] this method is used in order to look at the scattering properties of the system.

Another aspect is that due to the high velocities of the photons, the coupling between the light matter is often very weak. To exploit the full potential of light-matter interactions we need systems with strong coupling, which is slowly becoming a reality. Experimental achievements[6] has produced near unity coupling efficiency for a system with a quantum dot coupled to a photonic-crystal waveguide.

Most of the research on systems, with photons interacting with multiple atoms, is based on the assumption that the systems are nonchiral, i.e. that the interaction between the fields propagating in the left and right directions are the same. However, recent studies has led to the creation of chiral systems, where the interaction with the left and right going fields are not the same[7]. The effect of such propagation-direction-dependent emitters has not been investigated much and is the foundation of this thesis.

In this thesis, I shall be investigating the behaviour of a photon, travelling within a 1D waveguide with multiple chiral emitters and compare it with the nonchiral system. The main goal is to derive a general transfer matrix, in order to describe the scattering properties of the system.

2 Preliminaries

Throughout this thesis, I will assume the reader knows the basic notations and mathematics used for dealing with quantum mechanics. I will start with a description of the quantization of the electromagnetic field and continue to the description of the Jaynes-Cummings Model, which describes the interaction between an atom and a quantized field. This will be the basis for the derivation of the Hamiltonian operator for the system.

2.1 Quantization of electromagnetic field

I will be looking at a multimode field moving inside a waveguide with multiple emitters. I start by looking at a classical field, moving in free space without any sources of radiation. Using Maxwell's equations, the electric and magnetic field can be described in terms of a vector potential $\mathbf{A}(\mathbf{r},t)$ [8], where

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \quad (2.1a)$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t) \quad (2.1b)$$

The vector potential satisfies the differential equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \quad (2.2)$$

and the Coulomb gauge condition.

$$\nabla \cdot \mathbf{A}(\mathbf{r},t) = 0 \quad (2.3)$$

This choice of gauge requires the polarization to be orthogonal to the direction of propagation, which will simplify the later calculations. Assuming the field is moving within a cubic cavity with side length L , where $L \gg \frac{1}{k}$, allows me to impose a periodic boundary conditions. Requiring

$$e^{ik_x x} = e^{ik_x(x+L)}, \quad e^{ik_y y} = e^{ik_y(y+L)}, \quad e^{ik_z z} = e^{ik_z(z+L)} \quad (2.4)$$

allows only certain modes. Thus the wave vector can be described as

$$\mathbf{k} = (k_x, k_y, k_z) = \frac{2\pi}{L}(m_x, m_y, m_z) \quad (2.5)$$

where m_x , m_y and m_z are integers which can be both positive and negative. Distinct sets of these integers specify distinct normal modes of the field and $k = |\mathbf{k}| = \omega_k/c$. In the interval Δm_x , Δm_y and Δm_z , the total number of modes is given by

$$\Delta m = \Delta m_x \Delta m_y \Delta m_z = 2 \left(\frac{L}{2\pi} \right)^3 \Delta k_x \Delta k_y \Delta k_z \quad (2.6)$$

where the factor of two has been added in order to take the two independent polarizations into account. The vector potential will be defined as

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}, s} \mathbf{e}_{\mathbf{k}s} (A_{\mathbf{k}s}(t) e^{i\mathbf{k}\cdot\mathbf{r}} + A_{\mathbf{k}s}^*(t) e^{-i\mathbf{k}\cdot\mathbf{r}}) \quad (2.7)$$

where I sum over all the modes and the two independent polarizations. $A_{\mathbf{k},s}^{(*)}(t)$ are complex time dependent amplitudes and $\mathbf{e}_{\mathbf{k}s}$ is the polarization vector. From the gauge condition (2.3), the polarizations must be orthogonal to each other and to the direction of propagation. Thus

$$\mathbf{e}_{\mathbf{k}s} \cdot \mathbf{e}_{\mathbf{k}s'} = \delta_{ss'} \quad (2.8)$$

$$\mathbf{k} \cdot \mathbf{e}_{\mathbf{k}s} = 0 \quad (2.9)$$

Plugging (2.7) into (2.2) gives the wave equation for the complex amplitude

$$\frac{\partial^2 A_{\mathbf{k}s}}{\partial t^2} + \omega_k^2 A_{\mathbf{k}s} = 0 \quad (2.10)$$

which has the solution

$$A_{\mathbf{k}s}(t) = A_{\mathbf{k}s} e^{\pm i\omega_k t} \quad (2.11)$$

where $A_k = A_k(t=0)$ and $\omega_k = ck$. I only want solutions traveling along \mathbf{k} , which correspond only to the negative solutions. Using (2.7) in (2.1) gives me the electric and magnetic field

$$\mathbf{E}(\mathbf{r}, t) = i \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} (A_{\mathbf{k}s}(t) e^{i\mathbf{k}\cdot\mathbf{r}} - A_{\mathbf{k}s}^*(t) e^{-i\mathbf{k}\cdot\mathbf{r}}) \quad (2.12a)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{i}{c} \sum_{\mathbf{k}, s} \omega_k \left(\frac{\mathbf{k}}{|\mathbf{k}|} \times \mathbf{e}_{\mathbf{k}s} \right) (A_{\mathbf{k}s}(t) e^{i\mathbf{k}\cdot\mathbf{r}} - A_{\mathbf{k}s}^*(t) e^{-i\mathbf{k}\cdot\mathbf{r}}) \quad (2.12b)$$

where the term $\mathbf{k}/|\mathbf{k}|$ comes from the fact, that the two polarizations form a righthanded system, such that

$$\mathbf{e}_{\mathbf{k}1} \times \mathbf{e}_{\mathbf{k}2} = \frac{\mathbf{k}}{|\mathbf{k}|} \quad (2.13)$$

The field energy is given by

$$H = \frac{1}{2} \int_V dV \left[\varepsilon_0 |\mathbf{E}(\mathbf{r}, t)|^2 + \frac{1}{\mu_0} |\mathbf{B}(\mathbf{r}, t)|^2 \right] \quad (2.14)$$

Looking at the boundary conditions in (2.4) gives me

$$\int_V dV e^{\pm i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = \delta_{\mathbf{k}\mathbf{k}'} V \quad (2.15)$$

which shows, that only modes with the same integers or exactly opposites contribute to the energy. Thus the contribution from the electric field is

$$\begin{aligned} \frac{1}{2} \int_V dV \varepsilon_0 |\mathbf{E}(\mathbf{r}, t)|^2 &= \frac{\varepsilon_0 V}{2} \sum_{\mathbf{k}, s} \omega_k^2 \{ 2A_{\mathbf{k}s}(t) A_{\mathbf{k}s}^*(t) \\ &\quad - \mathbf{e}_{\mathbf{k}s} \cdot \mathbf{e}_{-\mathbf{k}s'} [A_{\mathbf{k}s}(t) A_{-\mathbf{k}s'}^*(t) + A_{\mathbf{k}s}^*(t) A_{-\mathbf{k}s'}(t)] \} \end{aligned} \quad (2.16)$$

I now look at the contribution from the magnetic field. The cross product can be separated into two parts

$$(\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}) \cdot (\mathbf{k} \times \mathbf{e}_{\mathbf{k}s'}) = \delta_{ss'} \quad (2.17)$$

$$(\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}) \cdot (-\mathbf{k} \times \mathbf{e}_{-\mathbf{k}s'}) = -\mathbf{e}_{\mathbf{k}s} \cdot \mathbf{e}_{\mathbf{k}s'} \quad (2.18)$$

Thus I get a change in sign when calculating the contribution from the magnetic field. I get

$$\begin{aligned} \frac{1}{2} \int_V dV \frac{1}{\mu_0} |\mathbf{B}(\mathbf{r}, t)|^2 &= \frac{\varepsilon_0 V}{2} \sum_{\mathbf{k}, s} \omega_k^2 \{ 2A_{\mathbf{k}s}(t) A_{\mathbf{k}s}^*(t) \\ &\quad + \mathbf{e}_{\mathbf{k}s} \cdot \mathbf{e}_{-\mathbf{k}s'} [A_{\mathbf{k}s}(t) A_{-\mathbf{k}s'}^*(t) + A_{\mathbf{k}s}^*(t) A_{-\mathbf{k}s'}(t)] \} \end{aligned} \quad (2.19)$$

Putting (2.16) and (2.19) into (2.14), the two last terms cancel out and I end up with

$$H = 2\varepsilon_0 V \sum_{\mathbf{k}, s} \omega_k^2 A_{\mathbf{k}s}(t) A_{\mathbf{k}s}^*(t) \quad (2.20)$$

From (2.20) it is apparent, that the energy only depends on each individual modes, since all the terms including crossterms cancel out. Thus, all modes are decoupled and do not interfere with each other. This feature gives rise to a canonical quantization of the field. Thus I wish to introduce the canonical momentum $p_{\mathbf{k}s}$ the canonical position $q_{\mathbf{k}s}$, that are both dependent on time. I define

$$A_{\mathbf{k}s}(t) = \frac{1}{2\omega_k \sqrt{\varepsilon_0 V}} (\omega_k q_{\mathbf{k}s} + ip_{\mathbf{k}s}) \quad (2.21a)$$

$$A_{\mathbf{k}s}^*(t) = \frac{1}{2\omega_k \sqrt{\varepsilon_0 V}} (\omega_k q_{\mathbf{k}s} - ip_{\mathbf{k}s}) \quad (2.21b)$$

Inserting this in the Hamiltonian in (2.20) and it becomes

$$H = \frac{1}{2} \sum_{\mathbf{k}, s} (p_{\mathbf{k}s}^2 + \omega_k^2 q_{\mathbf{k}s}^2) \quad (2.22)$$

which is just a sum of multiple independent harmonic oscillators. Looking at Hamiltons equations of motion[9] I get

$$\dot{q}_{\mathbf{k}s} = \frac{\partial H}{\partial p_{\mathbf{k}s}} = p_{\mathbf{k}s} \quad (2.23a)$$

$$\dot{p}_{\mathbf{k}s} = -\frac{\partial H}{\partial q_{\mathbf{k}s}} = -\omega_k^2 q_{\mathbf{k}s} \quad (2.23b)$$

from which it is apparent, that the dynamics of the new variables are the same as what I got from Maxwell's equation, since

$$\ddot{q}_{\mathbf{k}s} + \omega_k^2 q_{\mathbf{k}s} = 0 \quad \text{and} \quad \ddot{p}_{\mathbf{k}s} + \omega_k^2 p_{\mathbf{k}s} = 0 \quad (2.24)$$

and are thus canonical conjugate variables. This means that I can proceed the quantization by changing the canonical variables to the canonical operators $\hat{q}_{\mathbf{k}s}$ and $\hat{p}_{\mathbf{k}s}$, which satisfy the canonical commutation relations[10]

$$[\hat{q}_{\mathbf{k}s}, \hat{q}_{\mathbf{k}'s}] = [\hat{p}_{\mathbf{k}s}, \hat{p}_{\mathbf{k}'s}] = 0 \quad (2.25a)$$

$$[\hat{q}_{\mathbf{k}s}, \hat{p}_{\mathbf{k}'s}] = i\hbar\delta_{\mathbf{k}\mathbf{k}'} \quad (2.25b)$$

This allows me to describe (2.20) as a Hamiltonian operator

$$\hat{H} = \frac{1}{2} \sum_{\mathbf{k},s} (\hat{p}_{\mathbf{k}s}^2 + \omega_k^2 \hat{q}_{\mathbf{k}s}^2) \quad (2.26)$$

which is just the quantum mechanical description of a harmonic oscillator. Working with the harmonic oscillator it is convenient to introduce the annihilation(creation) operators, $\hat{a}_{\mathbf{k}}^{(\dagger)}$. These operators will be defined as

$$\hat{a}_{\mathbf{k}s} = \frac{1}{\sqrt{2\hbar\omega_k}} (\omega_k \hat{q}_{\mathbf{k}s} + i\hat{p}_{\mathbf{k}s}) \quad (2.27a)$$

$$\hat{a}_{\mathbf{k}s}^\dagger = \frac{1}{\sqrt{2\hbar\omega_k}} (\omega_k \hat{q}_{\mathbf{k}s} - i\hat{p}_{\mathbf{k}s}) \quad (2.27b)$$

which fulfils the commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}] = 0 \quad (2.28a)$$

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \quad (2.28b)$$

This allows me to rewrite (2.26) as

$$\hat{H} = \sum_{\mathbf{k},s} \hbar\omega_k \left(\hat{a}_{\mathbf{k}s}^\dagger \hat{a}_{\mathbf{k}s} + \frac{1}{2} \right) \quad (2.29)$$

The amplitudes in (2.21) can now be described as operators in terms of the annihilation and creation operators defined in (2.27)

$$\hat{A}_{\mathbf{k}s}^{(*)} = \left(\frac{\hbar}{2\omega_k \varepsilon_0 V} \right)^{1/2} a_{\mathbf{k}s}^{(\dagger)} \quad (2.30)$$

Thus the electric field can be described as an operator

$$\hat{\mathbf{E}}(\mathbf{r}) = i \sum_{\mathbf{k},s} \left(\frac{\hbar\omega_k}{2\varepsilon_0 V} \right)^{1/2} \mathbf{e}_{\mathbf{k}s} \left[\hat{a}_{\mathbf{k}s} e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}_{\mathbf{k}s}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \quad (2.31)$$

where $\mathcal{E}_0 = i \left(\frac{\hbar\omega_k}{2\varepsilon_0 V} \right)^{1/2}$, where V is the volume of quantization.

Since the strength of the magnetic field is in order $1/c$ compared to the electric field strength, the effects of the magnetic field are negligible compared to the ones from the electric field and will not be investigated further.

In this thesis the electromagnetic field is interacting with multiple emitters within a 1D waveguide, which will lead to guided and unguided modes. Thus the electromagnetic field can be described as a sum over all guided modes plus a sum over all unguided modes of the waveguide.

$$\mathbf{E} = \sum_k (\text{guided}) + \sum_k (\text{unguided}) \quad (2.32)$$

The only modes that will interact with the atoms are the ones supported by the waveguide. I will assume the electric field is focused around the atomic frequency ω_0 with a flat profile with cross section A and extend to some length $L \gg 1/k$. The effective volume of quantization will thus be described as $V = AL$. To simplify the calculations, I will be assuming the field will be polarized in the x-direction and propagating in the z-direction. Thus, the guided modes can be described by the quantized field

$$\hat{E}_x(z) = \sum_k \mathcal{E}_0 \frac{1}{\sqrt{L}} \left[\hat{a}_k e^{ikz} - \hat{a}_k^\dagger e^{-ikz} \right] \quad (2.33)$$

where $\mathcal{E}_0 = i \left(\frac{\hbar\omega_k}{2\varepsilon_0 A} \right)^{1/2}$, which is the definition I will be using in the later calculations. Later I will take the unguided modes into account, by adding a term to my Hamiltonian, which describes the decay to unguided modes.

2.2 Jaynes-Cummings Model

The interaction between an atom and a quantized electric field is described by the Jaynes-Cummings model[8].

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{int} \quad (2.34)$$

where \hat{H}_a is the free atomic Hamiltonian, \hat{H}_f is the free field Hamiltonian and \hat{H}_{int} is the interaction Hamiltonian, that describes the interaction between the field and the atom.

I am considering a single-mode field in a cavity and a two level atom. The ground state shall be denoted as $|g\rangle$ and the excited state as $|e\rangle$. Assuming the dipole moment and the polarization of the electric field is in the same direction, the interaction Hamiltonian is given by

$$\hat{H}_{int} = -\hat{d} \cdot \hat{E}_x(z_i) \quad (2.35)$$

where \hat{d} is the dipole moment operator and \hat{E} is the one found in (2.33) evaluated at the position of the atom z_i . I use an index, as I shall later be looking at an array of atoms.

Looking at parity considerations, one realizes that the dipole moment is odd under parity transformation, so only off-diagonal terms are non-zero. Hence the dipole moment operator can be written as, in the basis of the atom states

$$\hat{d} = |e\rangle \langle e| \hat{d} |g\rangle \langle g| + |g\rangle \langle g| \hat{d} |e\rangle \langle e| \quad (2.36)$$

Defining $d = \langle e| \hat{d} |g\rangle$, $d^* = \langle g| \hat{d} |e\rangle$ and $\hat{\sigma}_{ij} = |i\rangle \langle j|$, where i and j can be either e or g , (2.36) can be written as

$$\hat{d} = (d\hat{\sigma}_{eg} + d^*\hat{\sigma}_{ge}) \quad (2.37)$$

Inserting (2.37) and (2.33) in (2.35)

$$\begin{aligned} \hat{H}_{int} &= -\sum_k \mathcal{E}_0 \frac{1}{\sqrt{L}} (d\hat{\sigma}_{eg} + d^*\hat{\sigma}_{ge}) \left(\hat{a}_k e^{ikz_i} + \hat{a}_k^\dagger e^{-ikz_i} \right) \\ &= -\hbar \sum_k \frac{1}{\sqrt{L}} \left(g\hat{\sigma}_{eg}\hat{a}_k e^{ikz_i} + g\hat{\sigma}_{eg}\hat{a}_k^\dagger e^{-ikz_i} + g^*\hat{\sigma}_{ge}\hat{a}_k e^{ikz_i} + g^*\hat{\sigma}_{ge}\hat{a}_k^\dagger e^{-ikz_i} \right) \end{aligned} \quad (2.38)$$

where $\hat{\sigma}_{eg}$ is the operator that takes the atom from the ground to the excited state and $\hat{\sigma}_{ge}$ from the excited to the ground state and $g^{(*)} = d^{(*)}\mathcal{E}_0/\hbar$ is the coupling between the atom and the field. \mathcal{E}_0 depends on the cross section of the effective quantization volume, which makes sense physically. If A is numerically large, the chance that the field will interact with the atom is much smaller, since the field has more "free" space to propagate in.

Looking at (2.38), it is apparent that $\hat{\sigma}_{eg}\hat{a}^\dagger$ corresponds to the atom going from the ground to the excited state by emission of a photon and $\hat{\sigma}_{ge}\hat{a}$ corresponds to the atom going from the excited to the ground state by absorption of a photon. Hence they do not conserve energy and will be left out. This can also be shown using the rotating wave approximation, where it can be shown that since I am looking at near resonance frequencies, these two terms will vary in time with a much higher frequency, and can thus be neglected. I end up with the following interaction Hamiltonian

$$\hat{H}_{int} = -\hbar \sum_k \frac{1}{\sqrt{L}} \left(g\hat{a}_k\hat{\sigma}_{eg} e^{ikz_i} + g^*\hat{a}_k^\dagger\hat{\sigma}_{ge} e^{-ikz_i} \right) \quad (2.39)$$

The free atom Hamiltonian describes the energy between the ground and the excited state and is given by

$$\hat{H}_a = \hbar\omega_0\hat{\sigma}_{ee} \quad (2.40)$$

where $\hbar\omega_0$ is the energy between the ground and excited state. Inserting (2.39), (2.40) and (2.29), without the zero-point energy term, in (2.34) I arrive at

$$\hat{H} = \hbar\omega_0\hat{\sigma}_{ee} + \hbar \sum_k \frac{1}{\sqrt{L}} \omega_k \hat{a}_k^\dagger \hat{a}_k - \hbar \sum_k \left(g\hat{a}_k\hat{\sigma}_{eg} e^{ikz_i} + g^*\hat{a}_k^\dagger\hat{\sigma}_{ge} e^{-ikz_i} \right) \quad (2.41)$$

3 The Model

In this section I will start deducing my Hamiltonian describing the system I am looking at. After that I will solve the time dependent Schrödinger equation in order to look at the behavior of an atom interacting with an electromagnetic field. Lastly I will solve the time independent Schrödinger equation, in order to see how the electromagnetic field scatters when interacting with an atom.

3.1 The System Hamiltonian

I shall be investigating a photon incident from the right into a waveguide with an emitter. In section 2.1 I made a detailed description of the quantized field inside the waveguide. In order to take the decay to modes outside of the waveguide, I will be adding a complex term, \hat{H}_S , given by

$$\hat{H}_S = -\frac{i\hbar\Gamma_S}{2}\hat{\sigma}_{ee} \quad (3.1)$$

which describes the decay to unguided modes. This added term will be justified later in section 3.2.

I am investigating near-resonant frequencies and can thus treat the photons propagating in the left and right direction as separate fields, defining

$$\hat{a}_k e^{ikz} = \begin{cases} \hat{a}_{L,k} e^{-ikz} & \text{for } k < 0 \\ \hat{a}_{R,k} e^{ikz} & \text{for } k > 0 \end{cases} \quad (3.2)$$

which allows me to rewrite (2.41) as

$$\begin{aligned} \hat{H} = \hbar\omega_0\hat{\sigma}_{ee} + \hbar\sum_k\omega_k\hat{a}_k^\dagger\hat{a}_k - \hbar\left[\sum_{k<0}\frac{1}{\sqrt{L}}(g_{L,k}\hat{a}_{L,k}\hat{\sigma}_{eg}e^{-ikz_i} + g_{L,k}^*\hat{a}_{L,k}^\dagger\hat{\sigma}_{ge}e^{ikz_i}) \right. \\ \left. + \sum_{k>0}\frac{1}{\sqrt{L}}(g_{R,k}\hat{a}_{R,k}\hat{\sigma}_{eg}e^{ikz_i} + g_{R,k}^*\hat{a}_{R,k}^\dagger\hat{\sigma}_{ge}e^{-ikz_i}) \right] \end{aligned} \quad (3.3)$$

where $g_L^{(*)}$ is the coupling between the atom and the field propagating in the left direction and $g_R^{(*)}$ is the coupling to the field propagating in the right direction. The fact that I use direction-dependent couplings, is what makes the system chiral. In a nonchiral system, one would assume that $g_R = g_L$. In section 3.2 this will lead to two distinct decay rates to modes propagating in the left and right directions. I wish to go from the Schrödinger picture to the interaction picture. I define the following Hamiltonian and state vector

$$\hat{H}_0 = \hat{H}_a + \hbar\omega_0\sum_k\hat{a}_k^\dagger\hat{a}_k \quad (3.4)$$

$$|\Psi'\rangle = e^{-\frac{i\hat{H}_0 t}{\hbar}}|\Psi\rangle \quad (3.5)$$

This allows me to rewrite the Hamiltonian in the Interaction picture

$$\begin{aligned}\hat{H}' &= e^{\frac{i\hat{H}_0 t}{\hbar}} (\hat{H} - \hat{H}_0) e^{-\frac{i\hat{H}_0 t}{\hbar}} \\ &= \hbar \sum_k \Delta_k \hat{a}_k^\dagger \hat{a}_k - \hbar \left[\sum_{k < 0} \frac{1}{\sqrt{L}} (g_L \hat{a}_{L,k} \hat{\sigma}_{eg} e^{-ikz_i} + g_L^* \hat{a}_{L,k}^\dagger \hat{\sigma}_{ge} e^{ikz_i}) \right. \\ &\quad \left. + \sum_{k > 0} \frac{1}{\sqrt{L}} (g_R \hat{a}_{R,k} \hat{\sigma}_{eg} e^{ikz_i} + g_R^* \hat{a}_{R,k}^\dagger \hat{\sigma}_{ge} e^{-ikz_i}) \right]\end{aligned}\quad (3.6)$$

where $\Delta_k = \omega_k - \omega_0$, is the detuning. From now on I will be working in the Interaction picture and will write \hat{H}' as \hat{H} .

I wish to change the sum to an integral. From (2.5) I get the separation between points in 1D k-space, which is

$$\Delta k = \frac{2\pi}{L} \quad (3.7)$$

I define the continuous creation and annihilation operators as

$$\hat{a}_k^{(\dagger)} = \frac{1}{L} \int_{-L/2}^{L/2} dz \hat{E}^{+(-)}(z) e^{\pm ikz} \quad (3.8a)$$

$$\hat{a}^{(\dagger)}(k) = \frac{1}{\sqrt{2\pi L}} \int_{-\infty}^{\infty} dz \hat{E}^{+(-)}(z) e^{\mp ikz} = \sqrt{\frac{L}{2\pi}} \hat{a}_k^{(\dagger)} = \frac{1}{\sqrt{\Delta k}} \hat{a}_k^{(\dagger)} \quad (3.8b)$$

where $\hat{E}^+(z) = \hat{a}_k e^{ikz}$ and $\hat{E}^-(z) = [\hat{E}^+(z)]^\dagger$. Looking at the commutator for $\hat{a}^{(\dagger)}(k)$ I get

$$[a(k), a^\dagger(k')] \Delta k = [a_k, a_k^\dagger] = \delta_{kk'} \quad (3.9)$$

If I sum (3.9) over all values of k it will equal one. Assuming the separation of modes gets negligibly small, compared to the full length, the sum goes to an integral and Δk may be approximated by the differential dk . Thus I get

$$\sum_k [a(k), a^\dagger(k')] \Delta k = 1 \rightarrow \int dk [a(k), a^\dagger(k')] = 1 \quad (3.10)$$

The right hand side can only be true if the commutator of the continuous operators equals the dirac delta function.

$$[\hat{a}(k), \hat{a}^\dagger(k')] = \delta(k - k') \quad (3.11)$$

I multiply the right hand side of (3.6) with $\frac{\Delta k}{\Delta k}$

$$\begin{aligned}\hat{H} &= \hbar \frac{1}{\Delta k} \sum_k \Delta_k \hat{a}_k^\dagger \hat{a}_k \Delta k - \hbar \left[\frac{1}{\sqrt{2\pi} \sqrt{\Delta k}} \sum_{k < 0} (g_L \hat{a}_{L,k} \hat{\sigma}_{eg} e^{-ikz_i} + g_L^* \hat{a}_{L,k}^\dagger \hat{\sigma}_{ge} e^{ikz_i}) \Delta k \right. \\ &\quad \left. + \frac{1}{\sqrt{2\pi} \sqrt{\Delta k}} \sum_{k > 0} (g_R \hat{a}_{R,k} \hat{\sigma}_{eg} e^{ikz_i} + g_R^* \hat{a}_{R,k}^\dagger \hat{\sigma}_{ge} e^{-ikz_i}) \Delta k \right]\end{aligned}\quad (3.12)$$

Using (3.8) and assuming the separation gets negligibly small compared to the full length, such that the sum goes to an integral, I can rewrite (3.12) to

$$\begin{aligned} \hat{H} = \hbar \int dk \Delta_k \hat{a}^\dagger(k) \hat{a}(k) - \frac{\hbar}{\sqrt{2\pi}} \left\{ \int dk (g_L \hat{a}_L(k) \hat{\sigma}_{eg} e^{-ikz_i} + g_L^* \hat{a}_L^\dagger(k) \hat{\sigma}_{ge} e^{ikz_i}) \right. \\ \left. + \int dk (g_R \hat{a}_R(k) \hat{\sigma}_{eg} e^{ikz_i} + g_R^* \hat{a}_R^\dagger(k) \hat{\sigma}_{ge} e^{-ikz_i}) \right\} \end{aligned} \quad (3.13)$$

I now wish to describe it in real space. I define the creation and annihilation operator in real space, as the fourier transform of the ones in k -space.

$$\hat{c}_R(z) = \frac{1}{\sqrt{2\pi}} \int dk \hat{a}_R(k) e^{i(k-k_0)z} \quad \text{and} \quad \hat{a}_R(k) = \frac{1}{\sqrt{2\pi}} \int dz \hat{c}_R(z) e^{-i(k-k_0)z} \quad (3.14)$$

$$\hat{c}_L(z) = \frac{1}{\sqrt{2\pi}} \int dk \hat{a}_L(k) e^{-i(k-k_0)z} \quad \text{and} \quad \hat{a}_L(k) = \frac{1}{\sqrt{2\pi}} \int dz \hat{c}_L(z) e^{i(k-k_0)z} \quad (3.15)$$

which have the following commutation relations

$$[\hat{c}_L(z), \hat{c}_L^\dagger(z')] = [\hat{c}_R(z), \hat{c}_R^\dagger(z')] = \delta(z - z') \quad (3.16a)$$

$$[\hat{c}_L(z), \hat{c}_R^\dagger(z')] = [\hat{c}_R(z), \hat{c}_L^\dagger(z')] = 0 \quad (3.16b)$$

I am assuming the photons are near resonant with the frequency of the atom, thus I can approximate Δ_k as

$$\Delta_k \approx v_g(k - k_0)$$

This allows me to transform (3.13) into real space and my finale Hamiltonian is thus given by

$$\begin{aligned} \hat{H} = -\frac{i\hbar\Gamma_S}{2} \hat{\sigma}_{ee} + \hbar \left(v_g \int dz \left(i \hat{c}_L^\dagger \frac{\partial}{\partial z} \hat{c}_L(z) - i \hat{c}_R^\dagger(z) \frac{\partial}{\partial z} \hat{c}_R(z) \right) \right. \\ \left. - \int dz \delta(z - z_i) \left\{ g_L \hat{c}_L(z) \hat{\sigma}_{eg} e^{-ik_0z} + g_L^* \hat{c}_L^\dagger(z) \hat{\sigma}_{ge} e^{ik_0z} \right\} \right. \\ \left. - \int dz \delta(z - z_i) \left\{ g_R \hat{c}_R(z) \hat{\sigma}_{eg} e^{ik_0z} + g_R^* \hat{c}_R^\dagger(z) \hat{\sigma}_{ge} e^{-ik_0z} \right\} \right) \end{aligned} \quad (3.17)$$

where I have added the side term from (3.1), which describes the decay to unguided modes. The second term in the first line describes the propagation of modes in the left and right going directions. The two last lines describe the coupling between the left- and right-going quantized field and the transition $|g\rangle \leftrightarrow |e\rangle$. For a more detailed derivation see Appendix B.

3.2 Time dependent Schrödinger Equation

I wish to solve the time dependent Schrödinger Equation, in order to look at the dynamics of the atom.

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle \quad (3.18)$$

I use the Hamiltonian found in section 2.2, equation (3.17), described in the interaction picture and where the rotating wave approximation has been used. I define the state vector

$$|\Psi\rangle = f_e(t) |e\rangle |\emptyset\rangle + \int dz \left(f_R(z,t) \hat{c}_R^\dagger(z) + f_L(z,t) \hat{c}_L^\dagger(z) \right) |g\rangle |\emptyset\rangle \quad (3.19)$$

where $|\emptyset\rangle$ is the vacuum state. I wish to find solutions that fulfil

$$f_R(z < 0, t) = 0 \quad \text{or} \quad f_L(z > 0, t) = 0 \quad (3.20)$$

The left hand side of (3.18) is

$$i\hbar \frac{\partial |\Psi\rangle}{\partial t} = i\hbar \left(\dot{f}_e(t) |e\rangle |\emptyset\rangle + \int dz \left(\dot{f}_R(z,t) \hat{c}_R^\dagger + \dot{f}_L(z,t) \hat{c}_L^\dagger(z) \right) |g\rangle |\emptyset\rangle \right) \quad (3.21)$$

Using the commutation relations in (B.4a), the right hand side becomes

$$\begin{aligned} \hat{H} |\Psi\rangle = \hbar \int dz \left\{ \left(-iv_g \frac{\partial}{\partial z} f_R(z,t) - \delta(z - z_i) f_e(t) g_R^* e^{-ik_0 z} \right) \hat{c}_R^\dagger(z) \right. \\ \left. + \left(iv_g \frac{\partial}{\partial z} f_L(z,t) - \delta(z - z_i) f_e(t) g_L^* e^{ik_0 z} \right) \hat{c}_L^\dagger(z) \right\} |g\rangle |\emptyset\rangle \\ - \hbar \left\{ \frac{i\hbar \Gamma_S}{2} f_e(t) + \int dz \left(g_L f_L(z,t) e^{-ik_0 z} \delta(z - z_i) \right. \right. \\ \left. \left. + g_R f_R(z,t) e^{ik_0 z} \delta(z - z_i) \right) \right\} |e\rangle |\emptyset\rangle \end{aligned} \quad (3.22)$$

I multiply both sides of (3.18) with $\langle \emptyset | \langle e |$ and $\langle \emptyset | \langle g |$ and arrive at

$$\begin{aligned} \dot{f}_e(t) &= -\frac{\Gamma_S}{2} f_e(t) + i \int dz \delta(z - z_i) \left(g_L f_L(z,t) e^{-ik_0 z} + g_R f_R(z,t) e^{ik_0 z} \right) \\ &= -\frac{\Gamma_S}{2} f_e(t) + i \left(g_L f_L(z_i, t) e^{-ik_0 z_i} + g_R f_R(z_i, t) e^{ik_0 z_i} \right) \end{aligned} \quad (3.23a)$$

$$\begin{aligned} \int dz \left(\dot{f}_R(z,t) + \dot{f}_L(z,t) \right) &= \int dz \left\{ \left(-v_g \frac{\partial}{\partial z} f_R(z,t) + i\delta(z - z_i) f_e(t) g_R^* e^{-ik_0 z} \right) \right. \\ &\quad \left. + \left(v_g \frac{\partial}{\partial z} f_L(z,t) + i\delta(z - z_i) f_e(t) g_L^* e^{ik_0 z} \right) \right\} \end{aligned} \quad (3.23b)$$

From (3.23b) I get two differential equations

$$\dot{f}_R(z,t) + v_g \frac{\partial}{\partial z} f_R(z,t) = i\delta(z - z_i) f_e(t) g_R^* e^{ik_0 z} \quad (3.24a)$$

$$\dot{f}_L(z,t) - v_g \frac{\partial}{\partial z} f_L(z,t) = i\delta(z - z_i) f_e(t) g_L^* e^{-ik_0 z} \quad (3.24b)$$

I start looking at (3.24a). I assume the solution takes the form $f_R(z,t) = G_R(z,\tau_R)$, where $\tau_R = t - \frac{z}{v_g}$. The left hand side of (3.24a) becomes

$$\begin{aligned} \dot{f}_R(z,t) + v_g \frac{\partial}{\partial z} f_R(z,t) &= \frac{\partial}{\partial \tau_R} G_R(z,\tau_R) + v_g \left(\frac{\partial}{\partial z} G_R(z,\tau_R) - \frac{1}{v_g} \frac{\partial}{\partial \tau_R} G_R(z,\tau_R) \right) \\ &= v_g \frac{\partial}{\partial z} G_R(z,\tau_R) \end{aligned} \quad (3.25)$$

The same method can be used for $f_L(z,t)$, assuming the solution takes the form $f_L(z,t) = F_L(z,\tau_L)$ where $\tau_L = t + \frac{z}{v_g}$. Using this and the conditions in (3.20), I get the solutions

$$G_R(z,\tau_R) = \frac{i\theta(z-z_i)g_R^*}{v_g} f_e(t) e^{-ik_0 z_i} \quad (3.26a)$$

$$G_L(z,\tau) = \frac{i\theta(-z+z_i)g_L^*}{v_g} f_e(t) e^{ik_0 z_i} \quad (3.26b)$$

Plugging (3.26a) and (3.26b) into (3.23a) and assuming the atom is placed at the position $z_i = 0$, I have to evaluate $\theta(0)$, which is undefined since the Heaviside function is only defined for values less or greater than zero. To get a result, I will have to make a choice, thus I will assume $\theta(0) = \frac{1}{2}$. At the end one will see that this choice gives the same result as I arrive at, if I only look at an atom, with no photons present, see Appendix A, where this approximation has not been used. I get the following differential equation

$$\dot{f}_e(t) = -\frac{\Gamma_L + \Gamma_R + \Gamma_S}{2} f_e(t) \quad (3.27)$$

where $\Gamma_{R(L)} = |g_{R(L)}|^2/v_g$. (3.27) has the solution.

$$f_e(t) = e^{-\frac{\Gamma}{2}t} \quad (3.28)$$

where $\Gamma = \Gamma_R + \Gamma_L + \Gamma_S$, is the decay rate of the atom. Γ_R and Γ_L are the decay of modes propagating in either the left or right direction inside the waveguide and Γ_S is the decay rate to unguided modes, as shown in Figure 1. The fact that I used direction-dependent couplings in section 3.1, is what gives rise to chiral decay rates.

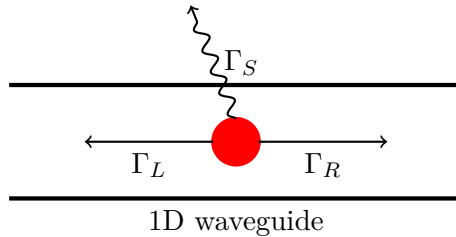


Figure 1: An atom (red dot) in a 1D waveguide, which may decay in left (Γ_L) or right (Γ_R) going guided modes or unguided modes (Γ_S).

The results from (3.26a) and (3.26b) justify the added side term to my Hamiltonian. I could have solved the time dependent Schrödinger Equation using the sum over all modes in my Hamiltonian instead, leading to the results I just found, a decay into left and right going guided modes plus a term of the decay into unguided modes.

3.3 Time independent Schrödinger Equation

I now wish to solve the time independent Schrödinger Equation in order to look at the interaction between the quantized field and the atom.

$$\hat{H}|\Psi\rangle = E|\Psi\rangle \quad (3.29)$$

As before, I will use the Hamiltonian found in section 2.2, equation (3.17) and the state vector (3.19). The Hamiltonian is described in the interaction picture and the rotating wave approximation has been used. The energy is $E = E_{\text{photon}} - E_{\text{atom}}$. Plugging into (3.29)

$$\begin{aligned} E \left[f_e |e\rangle |\emptyset\rangle + \int dz \left(f_R(z) c_R^\dagger c_R^\dagger(z) + f_L(z) c_L^\dagger(z) \right) |g\rangle |\emptyset\rangle \right] \\ = \hbar \int dz \left\{ \left(-iv_g \frac{\partial}{\partial z} f_R(z,t) - \delta(z - z_i) f_e(t) g_R^* e^{-ik_0 z} \right) \hat{c}_R^\dagger(z) \right. \\ \left. + \left(iv_g \frac{\partial}{\partial z} f_L(z,t) - \delta(z - z_i) f_e(t) g_L^* e^{ik_0 z} \right) \hat{c}_L^\dagger(z) \right\} |g\rangle |\emptyset\rangle \\ - \hbar \left\{ \frac{i\hbar\Gamma_S}{2} f_e(t) + \int dz \delta(z - z_i) \left(g_L f_L(z,t) e^{-ik_0 z} + g_R f_R(z,t) e^{ik_0 z} \right) \right\} |e\rangle |\emptyset\rangle \end{aligned} \quad (3.30)$$

Multiplying with $\langle\emptyset| \langle g|$ and $\langle\emptyset| \langle e|$ on both sides and I get

$$E f_e = -\hbar \left(g_L f_L(z_i) e^{-ik_0 z} + g_R f_R(z_i) e^{ik_0 z} + \frac{i\Gamma_S}{2} f_e \right) \quad (3.31a)$$

$$\begin{aligned} E \int dz \left(f_R(z) + f_L(z) \right) = \hbar \int dz \left\{ \left(-iv_g \frac{\partial}{\partial z} f_R(z) - \delta(z - z_i) f_e g_R^* e^{-ik_0 z} \right) \right. \\ \left. + \left(iv_g \frac{\partial}{\partial z} f_L(z) - \delta(z - z_i) f_e g_L^* e^{ik_0 z} \right) \right\} \end{aligned} \quad (3.31b)$$

From (3.31b) I get differential equations

$$\frac{\partial}{\partial z} f_R(z) = \frac{iE}{\hbar v_g} f_R(z) + \frac{ig_R^*}{v_g} f_e e^{-ik_0 z} \delta(z - z_i) \quad (3.32a)$$

$$\frac{\partial}{\partial z} f_L(z) = -\frac{iE}{\hbar v_g} f_L(z) - \frac{ig_L^*}{v_g} f_e e^{ik_0 z} \delta(z - z_i) \quad (3.32b)$$

When looking at the scattering, I am only interested in looking at what happens around the atom at $z = z_i$, and not how the field evolves between the atom. Thus I integrate from $z_i - \epsilon$ to $z_i + \epsilon$,

where z_i is the position of the atom and ϵ is a small distance which I will let go to zero at the end.

$$\int_{z_i-\epsilon}^{z_i+\epsilon} dz \frac{\partial}{\partial z} f_R(z) = \frac{iE}{\hbar v_g} \int_{z_i-\epsilon}^{z_i+\epsilon} dz f_R(z) + \frac{i g_R^*}{v_g} f_e \int_{z_i-\epsilon}^{z_i+\epsilon} dz e^{ik_0 z} \delta(z - z_i) \quad (3.33)$$

Doing so, the middle term will go to zero in the limit $\epsilon \rightarrow 0$. The last integral on the RHS will just be $e^{ik_0 z_i}$, since z_i is included in the area of integration. The left hand side will just be $f_R(z)$ evaluated at two end points. Defining $z_i^+ = z_i + \epsilon$ and $z_i^- = z_i - \epsilon$ and rearranging I get

$$f_R(z_i^+) = f_R(z_i^-) + \frac{i g_R^*}{v_g} f_e e^{-ik_0 z_i} \quad (3.34a)$$

$$f_L(z_i^-) = f_L(z_i^+) + \frac{i g_L^*}{v_g} f_e e^{ik_0 z_i} \quad (3.34b)$$

where $f_R(z_i^-)$ and $f_L(z_i^-)$ are to be considered as the left and right going fields a distance $-\epsilon$ from the atom and $f_R(z_i^+)$ and $f_L(z_i^+)$ are to be considered as the left and right going fields a distance $+\epsilon$ from the atom.

Since both $f_R(z)$ and $f_L(z)$ has to be continuous at z_i I can write

$$f_{R(L)}(z_i) = \frac{1}{2} \left(f_{R(L)}(z_i^+) + f_{R(L)}(z_i^-) \right) \quad (3.35)$$

Using the result from (3.34a) and (3.35) I can rewrite (3.31a) as

$$E f_e = -\hbar \left[g_L f_L(z_i^+) e^{-ik_0 z_i} + \frac{i \Gamma_L}{2} f_e + g_R f_R(z_i^-) e^{ik_0 z_i} + \frac{i \Gamma_R}{2} f_e + \frac{i \Gamma_S}{2} f_e \right] \quad (3.36)$$

where $\Gamma_{R(L)} = \frac{|g_{R(L)}|^2}{v_g}$ are the decay rates of the left and right going fields. If I rearrange I get an expression for f_e

$$f_e = \frac{-\hbar \left(g_L f_L(z_i^+) e^{-ik_0 z_i} + g_R f_R(z_i^-) e^{ik_0 z_i} \right)}{E + \frac{i \hbar \Gamma}{2}} \quad (3.37)$$

where I have defined the total decay $\Gamma = \Gamma_R + \Gamma_L + \Gamma_S$. Plugging this into (3.34a) I get my two functions, describing the scattering of the fields

$$f_R(z_i^+) = t_R(E) f_R(z_i^-) + r(E) e^{-i2k_0 z_i} f_L(z_i^+) \quad (3.38a)$$

$$f_L(z_i^-) = t_L(E) f_L(z_i^+) + r(E) e^{i2k_0 z_i} f_R(z_i^-) \quad (3.38b)$$

where I have defined

$$t_R(E) = \left(1 - \frac{i \hbar \Gamma_R}{E + \frac{i \hbar \Gamma}{2}} \right), \quad t_L(E) = \left(1 - \frac{i \hbar \Gamma_L}{E + \frac{i \hbar \Gamma}{2}} \right), \quad r(E) = -\frac{i \hbar \sqrt{\Gamma_R \Gamma_L}}{E + \frac{i \hbar \Gamma}{2}} \quad (3.39)$$

For the nonchiral system one would get $t_R = t_L$, thus the probability for decays to modes propagating in the left and right directions would be the same. It is important to notice, that the reflection coefficient is not chiral, and thus the probability for the field to get reflected is the same for both directions.

4 Scattering of the field

In this section I will be investigating the scattering of the field. Using the transfer matrix method[4] I will deduce a general transfer matrix, describing the scattering of the field from a whole ensemble of atoms. I then plot numerical solutions based on the transfer matrix, where I compare the chiral system with a nonchiral system.

4.1 Transfer Matrix

In section 3.3 I found the equations (3.38) describing the scattering of the field. I now wish to describe it in matrix form such that

$$\begin{pmatrix} f_R(z_i^+) \\ f_L(z_i^+) \end{pmatrix} = \mathbf{T} \begin{pmatrix} f_R(z_i^-) \\ f_L(z_i^-) \end{pmatrix} \quad (4.1)$$

where I on the LHS have the fields just right of an atom and on the RHS I have the field just left to the atom multiplied by some transfer matrix \mathbf{T} . Rearranging (3.38b) and inserting in (3.38a) gives me the matrix

$$\mathbf{T} = \mathbf{D}^{-1}\mathbf{M}\mathbf{D} \quad (4.2)$$

where

$$\mathbf{M} = \begin{pmatrix} \left(t_R - \frac{r^2}{t_L}\right) & \frac{r}{t_L} \\ -\frac{r}{t_L} & \frac{1}{t_L} \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} e^{ik_0z_i} & 0 \\ 0 & e^{-ik_0z_i} \end{pmatrix} \quad (4.3)$$

It is important to notice, that the matrix \mathbf{D} depends on the position of the atom, z_i , hence the transfer matrix for each atom will be different. If I want to look at an ensemble of N two-level atoms, I first assume that the atoms are arranged periodically with some lattice constant d and that the first atom is located at $z_1 = 0$. The n 'th atom will thus be located at $z_n = (n-1)d$. This allows me to write the transfer matrix for the n 'th atom as

$$\mathbf{T}_n = \mathbf{D}_0^{-(n-1)}\mathbf{M}\mathbf{D}_0^{(n-1)} \quad (4.4)$$

where I have defined

$$\mathbf{D}_0 = \begin{pmatrix} e^{ik_0d} & 0 \\ 0 & e^{-ik_0d} \end{pmatrix} \quad (4.5)$$

Thus, the transfer matrix for the first atom is just $\mathbf{T}_1 = \mathbf{M}$, since $\mathbf{D}_0^0 = \mathbf{I}$, where \mathbf{I} is the identity matrix. Now if I am looking at an ensemble of N atoms, the full length of the system is $L = (N-1)d$, and the complete transfer matrix, \mathbf{T}_E is

$$\mathbf{T}_E = \mathbf{T}_N\mathbf{T}_{N-1}\cdots\mathbf{T}_1 = \mathbf{D}_0^{-(N-1)}\mathbf{M}\mathbf{D}_0^{N-1}\mathbf{D}_0^{-(N-2)}\mathbf{M}\mathbf{D}_0^{(N-2)}\cdots\mathbf{D}_0^{-1}\mathbf{M}\mathbf{D}_0\mathbf{M} \quad (4.6)$$

which can be simplified to

$$\mathbf{T}_E = \mathbf{D}_0^{-N} (\mathbf{D}_0 \mathbf{M})^N \quad (4.7)$$

Thus the output can be written in matrix form in the following way

$$\begin{pmatrix} f_R(L^+) \\ f_L(L^+) \end{pmatrix} = \mathbf{T}_E \begin{pmatrix} f_R(z_1^-) \\ f_L(z_1^-) \end{pmatrix} \quad (4.8)$$

where L^+ denotes the position right after the last atom in the ensemble as illustrated on Figure 2.

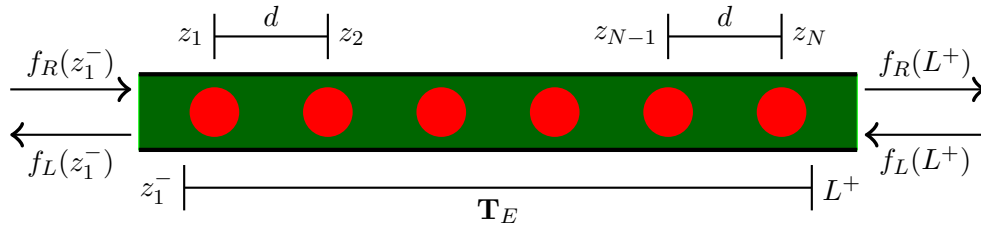


Figure 2: A 1D waveguide (green area) with N two-level atoms (red dots) separated with the lattice constant d . The first atom is located at $z_1 = 0$ and the n 'th atom is located at $z_n = (n-1)d$. The full length of the system is $L = (N-1)d$. The field on the RHS is described by the field on the LHS multiplied with the Transfer matrix \mathbf{T}_E .

To find the transmittance and reflectance for the left going field $f_L(z)$ interacting with N atoms, I will assume to have light incident from the left and (4.8) reads

$$\begin{pmatrix} f_R(L^+) = a \\ f_L(L^+) = b \end{pmatrix} = \mathbf{T}_E \begin{pmatrix} f_R(z_1^-) = 0 \\ f_L(z_1^-) = c \end{pmatrix} \quad (4.9)$$

where b is the incident field, a is the reflected field and c is the transmitted field. \mathbf{T}_E will have some matrix elements

$$\mathbf{T}_E = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad (4.10)$$

One can now easily find the transmittance and reflectance in terms of matrix elements, by looking at the ratio between the incident, reflected and transmitted fields. Thus I get

$$T_L(E) = \frac{|c|^2}{|b|^2} = \frac{1}{|T_{22}|^2} \quad \text{and} \quad R_L(E) = \frac{|a|^2}{|b|^2} = \frac{|T_{12}|^2}{|T_{22}|^2} \quad (4.11)$$

To find the reflectance and transmittance for the right going field, one just has to switch around Γ_R and Γ_L .

It can be confirmed, that (4.11) gives the same result for one atom, as one would get calculating the transmittance and reflectance from (3.38).

4.2 Scattering of ensemble of two-level atoms

In section 4.1 I have just derived equations which allow me to calculate the transmittance and reflectance from the Transfer matrix of the ensemble, equation (4.11). It can certainly be done analytically, but for high values of N the results gets very ugly and does not give much information. Instead I have made numerical solutions.

The chiral system is highly dependent on Γ_S and will in general have a higher loss than the nonchiral system, see Figure 3. The loss increases with the difference between coupling to the left and right going fields. The transmittance goes to zero when the number of atoms are increased. This is due to bragg scattering, reflected fields with a phase $2kd = n\pi$ will have constructive interference, creating a photonic band gap(PBG), where certain modes cannot pass through[11]. In the nonchiral system, this will lead to complete reflection at resonance and no loss, where the chiral system will have no transmittance, but always some loss.

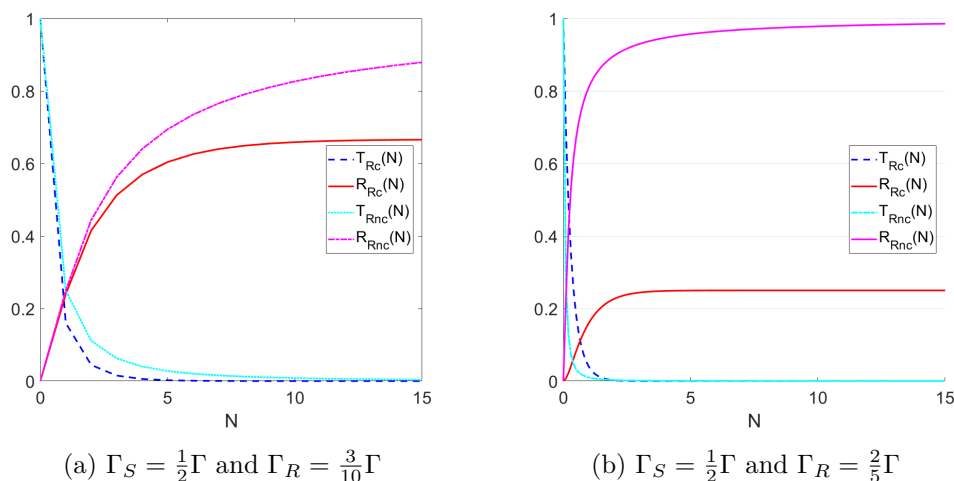


Figure 3: The transmittance and reflectance plotted as functions of the number of atoms in the ensemble, with $k_0d = \pi$ and $E = 0$. Both chiral($\Gamma_R \neq \Gamma_L$) and the nonchiral($\Gamma_R = \Gamma_L$) systems, denoted with c (chiral) and nc (nonchiral), have been plotted. The relation between Γ_S , Γ_R and Γ is noted under the plots. Both plots are between strong and weak coupling. In all cases T_R goes to zero and R_R converges to one for the nonchiral system and to some number, depending on the difference between Γ_R and Γ_L . The nonchiral system has less loss than the chiral system. It is the same picture one sees for the strong and weak coupling.

From Figure 3 it is apparent that the chiral system will always be connected to some loss, whereas the nonchiral system will end up as a Bragg Mirror for high values of N for certain frequencies. When the system is in the weak coupling regime, the PBG is wider in the chiral system, and

the reflected field is almost nonexistent, thus greatly restricting the allowed frequencies for the system, see Figure 4. In this case the nonchiral system seems to be more efficient and has more properties. Looking at the strong coupling regime, the picture changes. Suddenly the PBG is much wider for the nonchiral system and the loss of the chiral system has decreased dramatically. The chiral system even allows transmittance of frequencies, where the reflectance of the nonchiral system reaches unity. Though this is an extreme case with $\Gamma_S = 0.01\Gamma$, it is not an unrealistic picture, since systems with very strong coupling are being developed[6]. I have looked at the system where the atoms were separated by a lattice constant such that $2k_0d = \pi/4$. If the atoms instead were separated by a lattice constant such that $2k_0d = n\pi$ and the system had the strong coupling, the transmittance of the nonchiral system would almost go to zero with just two atoms, whereas transmittance for the chiral system would be almost unity and then slowly decrease as N would increase.

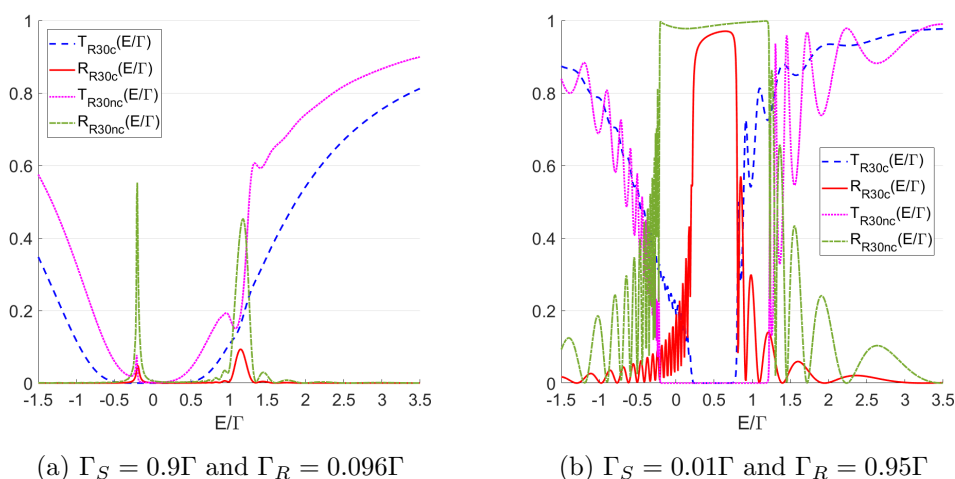


Figure 4: The transmittance and reflectance plotted as a function of E/Γ , with $k_0d = \pi/4$ and $N = 30$. Both chiral($\Gamma_R \neq \Gamma_L$) and the nonchiral($\Gamma_R = \Gamma_L$) systems, denoted with c (chiral) and nc (nonchiral), have been plotted. The relation between Γ_S , Γ_R and Γ is noted under the plots. (a) is the weak coupling regime and (b) is the strong coupling regime. For the weak coupling regime, the PBG is bigger for the chiral system, whereas for the strong coupling regime it is opposite.

5 Conclusion and outlook

In this thesis I have investigated how a photon interacts with a chiral ensemble of atoms within a 1D waveguide.

I have solved the time dependent Schrödinger equation, in order to look at the dynamics of the atom. It was shown that the atom will decay into a left or right going guided modes or into unguided modes and thus leaving the system.

Next the interaction between the field and the atom was investigated. By solving the time dependent Schrödinger equation, I found that the incoming fields would result in a superposition of a reflected and transmitted field.

Based on the result from the time independent Schrödinger equation, the Transfer matrix method was used to develop a transfer matrix, describing the interaction for one atom. This was used to develop a general transfer matrix describing the interaction of one photon with a whole ensemble of N emitters, based on the assumption that the atoms were arranged periodically with some lattice constant d .

The general transfer matrix for the ensemble of atoms was then used to produce numerical solutions for the system. The chiral system was compared with the nonchiral systems. It was shown, that the nonchiral system would have almost no loss as the number of atoms would increase, whereas the chiral system would have a huge loss. The loss of the chiral system would increase with the difference between the coupling between the atom and the left and right going fields.

Differences between the strong and weak coupling regimes were investigated, where it was shown, that the photonic band gap, relating to the Bragg scattering of the system, would decrease for the chiral system, as the coupling would increase.

The chiral systems shows a different behaviour than the nonchiral systems and it will be interesting to see if such systems can be used to construct arrays of atoms with the desired properties to build effective quantum systems. On a theoretical manner, this would be easy to investigate further, now that a general transfer matrix for a chiral system has been developed. One could imagine looking at an ensemble of atoms with different chirality. Furthermore, I have assumed that the atoms are placed periodically with a periodic lattice constant. In practice, this is very hard to accomplish, hence it would be more useful to analyse the system, where the atoms are randomly placed in the waveguide.

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Appendix A Decay of single emitter initially in excited state

I wish to describe the decay of a single emitter, which initially starts in the excited state, with no photons present. I define the state vector

$$|\psi(t=0)\rangle = |e\rangle |\emptyset\rangle, \quad |\psi(t)\rangle = f_e(t) |e\rangle |0\rangle + \sum_k f_k(t) \hat{a}_k^\dagger |g\rangle |\emptyset\rangle \quad (\text{A.1})$$

I am only interested in solving the system in order to see how the atom will decay. I could solve it for all modes and polarizations, but for simplicity, I shall only be looking at guided modes for $k > 0$ polarized in the x -direction. The Hamiltonian describing the system is given by

$$\hat{H} = \hbar \sum_{k>0} \left\{ v_g(k - k_0) \hat{a}_k^\dagger \hat{a}_k - g_R \hat{a}_k \hat{\sigma}_{eg} - g_R^* \hat{a}_k^\dagger \hat{\sigma}_{ge} \right\} \quad (\text{A.2})$$

where $\hat{a}_k^{(\dagger)}$ is the creation(annihilation) operator, $\hat{\sigma}_{eg}$ is the operator taking the atom from the excited to the ground state, $\hat{\sigma}_{ge}$ is the operator taking the atom from the ground to the excited state and $g_R^{(*)} = id^{(*)}/\hbar \left(\frac{\hbar\omega_k}{2\epsilon_0 V} \right)^{1/2}$ is the coupling between the field and the atom. The Hamiltonian is described in the Interaction picture and the rotating wave approximation has been made. I have furthermore assumed, that the system will have frequencies, ω_k close to the atomic frequency, ω_0 and thus written $(\omega_k - \omega_0) = v_g(k - k_0)$. I wish to solve the time dependent Schrödinger equation

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H} |\psi(t)\rangle \quad (\text{A.3})$$

Plugging in (A.1) and (A.2)

$$\begin{aligned} \dot{f}_e |e\rangle |0\rangle + \sum_k \dot{f}_k \hat{a}_k^\dagger |g\rangle |\emptyset\rangle &= i \sum_{k>0} \left\{ g_R f_k(t) |e\rangle |\emptyset\rangle \right. \\ &\quad \left. + (g_R^* f_e(t) - f_k(t) v_g(k - k_0)) \hat{a}_k^\dagger |g\rangle |\emptyset\rangle \right\} \end{aligned} \quad (\text{A.4})$$

giving me two coupled differential equations

$$\dot{f}_e = ig_R \sum_{k>0} f_k(t) \quad (\text{A.5a})$$

$$\dot{f}_k = ig_R^* f_e(t) - if_k(t) v_g(k - k_0) \quad (\text{A.5b})$$

(A.5b) is a linear first order differential equations, which has the solution

$$f_k(t) = ig_R^* \int_0^t e^{-iv_g(k-k_0)(t-t')} f_e(t') dt' \quad (\text{A.6})$$

Inserting this in (A.5a) gives me

$$\dot{f}_e = -|g_R|^2 \sum_{k>0} \int_0^t e^{-iv_g(k-k_0)(t-t')} f_e(t') dt' \quad (\text{A.7})$$

Assuming the separation between modes in k -space is negligibly small compared to the full length, Δk can be approximated with the differential dk and the sum goes to an integral

$$\frac{1}{\Delta k} \sum_{k>0} \Delta k \rightarrow \frac{1}{\Delta k} \int_0^\infty dk$$

With $\Delta k = 2\pi/L$ (A.9) can be written as

$$\dot{f}_e = -|\tilde{g}_R|^2 \frac{1}{2\pi} \int_0^\infty \int_0^t e^{-iv_g(k-k_0)(t-t')} f_e(t') dt' dk \quad (\text{A.8})$$

where $\tilde{g}_R^{(*)} = id^{(*)}/\hbar \left(\frac{\hbar\omega_k}{2\varepsilon_0 A} \right)^{1/2}$. In order to solve the above equation, I will be using Weisskopf-Wigner approximation[12]. Since the field will be focused around the atomic transition frequency, ω_0 , I will assume, that the excited state amplitude $f_e(t')$ varies on a rate $\Gamma \ll \omega_0$, and thus approximate it as $f_e(t') \approx f_e(t)$ and take it out of the integral. Now, the time of interest will be $t \gg 1/\omega_0$ and the upper limit of integration can be taken to infinity. I get

$$\dot{f}_e \approx -|\tilde{g}_R|^2 \frac{1}{2\pi} f_e(t) \int_0^\infty \int_0^\infty e^{-iv_g(k-k_0)\tau} d\tau dk \quad (\text{A.9})$$

where $\tau = (t - t')$. The integral over $d\tau$ can now be rewritten as

$$\int_0^\infty e^{-iv_g(k-k_0)\tau} d\tau = \int_{-\infty}^\infty H(\tau) e^{-iv_g(k-k_0)\tau} d\tau \quad (\text{A.10})$$

which is just the Fourier transform of the Heaviside function, which is

$$\int_{-\infty}^\infty H(\tau) e^{-iv_g(k-k_0)\tau} d\tau = \frac{\pi}{|v_g|} \delta(k - k_0) - i\mathcal{P}(k) \quad (\text{A.11})$$

The last term $\mathcal{P}(k)$ is the Cauchy Principal part. Because it is complex, it will lead to a change in frequency, the so called "Lamb-shift", which will diverge and I will thus assume that $-i\mathcal{P}(k) = 0$. Inserting back in (A.9) and I get

$$\dot{f}_e = -\frac{|\tilde{g}_R|^2}{2|v_g|} f_e(t) \int_0^\infty \delta(k - k_0) dk = -\frac{\Gamma_R}{2} f_e(t) \quad (\text{A.12})$$

where I have defined $\Gamma_R = \frac{|\tilde{g}_R|^2}{v_g}$. The solution is, assuming $f_e(t = 0) = 1$

$$f_e(t) = e^{-\frac{\Gamma_R}{2}t} \quad (\text{A.13})$$

Appendix B Transforming the Hamiltonian to real space

In section 3.1 I arrive at the following Hamiltonian operator for the system described in k -space.

$$\begin{aligned} \hat{H} = \hbar \int dk \Delta_k \hat{a}^\dagger(k) \hat{a}(k) - \frac{\hbar}{\sqrt{2\pi}} \left\{ \int dk (g_L \hat{a}_L(k) \hat{\sigma}_{eg} e^{-ikz_i} + g_L^* \hat{a}_L^\dagger(k) \hat{\sigma}_{ge} e^{ikz_i}) \right. \\ \left. + \int dk (g_R \hat{a}_R(k) \hat{\sigma}_{eg} e^{ikz_i} + g_R^* \hat{a}_R^\dagger(k) \hat{\sigma}_{ge} e^{-ikz_i}) \right\} \end{aligned} \quad (\text{B.1})$$

In order to transform the Hamiltonian operator in (B.1) I define the creation and annihilation operators in real space, as the fourier transform of the ones in k -space.

$$\hat{c}_R(z) = \frac{1}{\sqrt{2\pi}} \int dk \hat{a}_R(k) e^{i(k-k_0)z} \quad \text{and} \quad \hat{a}_R(k) = \frac{1}{\sqrt{2\pi}} \int dz \hat{c}_R(z) e^{-i(k-k_0)z} \quad (\text{B.2})$$

$$\hat{c}_L(z) = \frac{1}{\sqrt{2\pi}} \int dk \hat{a}_L(k) e^{-i(k-k_0)z} \quad \text{and} \quad \hat{a}_L(k) = \frac{1}{\sqrt{2\pi}} \int dz \hat{c}_L(z) e^{i(k-k_0)z} \quad (\text{B.3})$$

which has the following commutation relations

$$[\hat{c}_L(z), \hat{c}_L^\dagger(z')] = [\hat{c}_R(z), \hat{c}_R^\dagger(z')] = \delta(z - z') \quad (\text{B.4a})$$

$$[\hat{c}_L(z), \hat{c}_R^\dagger(z')] = [\hat{c}_R(z), \hat{c}_L^\dagger(z')] = 0 \quad (\text{B.4b})$$

I am assuming the photons are near resonant with the frequency of the atom, thus I can approximate Δ_k as

$$\Delta_k \approx v_g(k - k_0)$$

Looking at the first term in (B.1) I get

$$\begin{aligned} \int dk \Delta_k \hat{a}^\dagger(k) \hat{a}(k) &= v_g \int dk (k - k_0) (\hat{a}_R^\dagger(k) \hat{a}_R(k) + \hat{a}_L^\dagger(k) \hat{a}_L(k)) \\ &= \frac{v_g}{\sqrt{2\pi}} \int (k - k_0) \left(\int dz \hat{c}_R^\dagger(z) e^{i(k-k_0)z} \hat{a}_R(k) + \int dz \hat{c}_L^\dagger(z) e^{-i(k-k_0)z} \hat{a}_L(k) \right) \\ &= v_g \int dz \left(\hat{c}_R^\dagger(z) \frac{1}{\sqrt{2\pi}} \int dk (k - k_0) \hat{a}_R(k) e^{i(k-k_0)z} \right. \\ &\quad \left. + \hat{c}_L^\dagger(z) \frac{1}{\sqrt{2\pi}} \int dk (k - k_0) \hat{a}_L(k) e^{-i(k-k_0)z} \right) \\ &= v_g \int dz \left(i \hat{c}_L^\dagger \frac{\partial}{\partial z} \hat{c}_L(z) - i \hat{c}_R^\dagger(z) \frac{\partial}{\partial z} \hat{c}_R(z) \right) \end{aligned} \quad (\text{B.5})$$

where I between the last two lines have used the fact that

$$-i \frac{\partial}{\partial z} \hat{c}_R(z) = \frac{1}{\sqrt{2\pi}} \int dk (k - k_0) \hat{a}_R(k) e^{i(k-k_0)z} \quad (\text{B.6})$$

Looking at the second term in (B.1) I get

$$\begin{aligned}
 & \frac{1}{\sqrt{2\pi}} \int dk (g_L \hat{a}_L(k) \hat{\sigma}_{eg} e^{-ikz_i} + g_L^* \hat{a}_L^\dagger(k) \hat{\sigma}_{ge} e^{ikz_i}) \\
 &= \frac{1}{\sqrt{2\pi}} \int dk \left\{ g_L \hat{\sigma}_{eg} e^{-ikz_i} \left(\frac{1}{\sqrt{2\pi}} \int dz \hat{c}_L(z) e^{i(k-k_0)z} \right) \right. \\
 & \quad \left. + g_L^* \hat{\sigma}_{ge} e^{ikz_i} \left(\frac{1}{\sqrt{2\pi}} \int dz \hat{c}_L^\dagger(z) e^{-i(k-k_0)z} \right) \right\} \\
 &= \int dz \left\{ g_L \hat{\sigma}_{eg} e^{ik_0z} \frac{1}{2\pi} \int dk e^{-ik(z-z_i)} + g_L^* \hat{c}_L^\dagger(z) \hat{\sigma}_{ge} e^{-ik_0z} \frac{1}{2\pi} \int dk e^{ik(z-z_i)} \right\} \\
 &= \int dz \delta(z - z_i) \left\{ g_L \hat{c}_L(z) \hat{\sigma}_{eg} e^{-ik_0z} + g_L^* \hat{c}_L^\dagger(z) \hat{\sigma}_{ge} e^{ik_0z} \right\} \tag{B.7}
 \end{aligned}$$

and in the same way it can be shown that

$$\begin{aligned}
 & \frac{1}{\sqrt{2\pi}} \int dk (g_R \hat{a}_R(k) \hat{\sigma}_{eg} e^{ikz_i} + g_R^* \hat{a}_R^\dagger(k) \hat{\sigma}_{ge} e^{-ikz_i}) \\
 &= \int dz \delta(z - z_i) \left\{ g_R \hat{c}_R(z) \hat{\sigma}_{eg} e^{ik_0z} + g_R^* \hat{c}_R^\dagger(z) \hat{\sigma}_{ge} e^{-ik_0z} \right\} \tag{B.8}
 \end{aligned}$$

Putting (B.5), (B.7) and (B.8) together and I arrive at my final Hamiltonian in real space

$$\begin{aligned}
 \hat{H} = & -\frac{i\hbar\Gamma_S}{2} \hat{\sigma}_{ee} + \hbar \left(v_g \int dz \left(i\hat{c}_L^\dagger \frac{\partial}{\partial z} \hat{c}_L(z) - i\hat{c}_R^\dagger(z) \frac{\partial}{\partial z} \hat{c}_R(z) \right) \right. \\
 & - \int dz \delta(z - z_i) \left\{ g_L \hat{c}_L(z) \hat{\sigma}_{eg} e^{-ik_0z} + g_L^* \hat{c}_L^\dagger(z) \hat{\sigma}_{ge} e^{ik_0z} \right\} \\
 & \left. - \int dz \delta(z - z_i) \left\{ g_R \hat{c}_R(z) \hat{\sigma}_{eg} e^{ik_0z} + g_R^* \hat{c}_R^\dagger(z) \hat{\sigma}_{ge} e^{-ik_0z} \right\} \right) \tag{B.9}
 \end{aligned}$$