



PHOTON-PHOTON INTERACTIONS IN WAVEGUIDES

MASTER'S THESIS

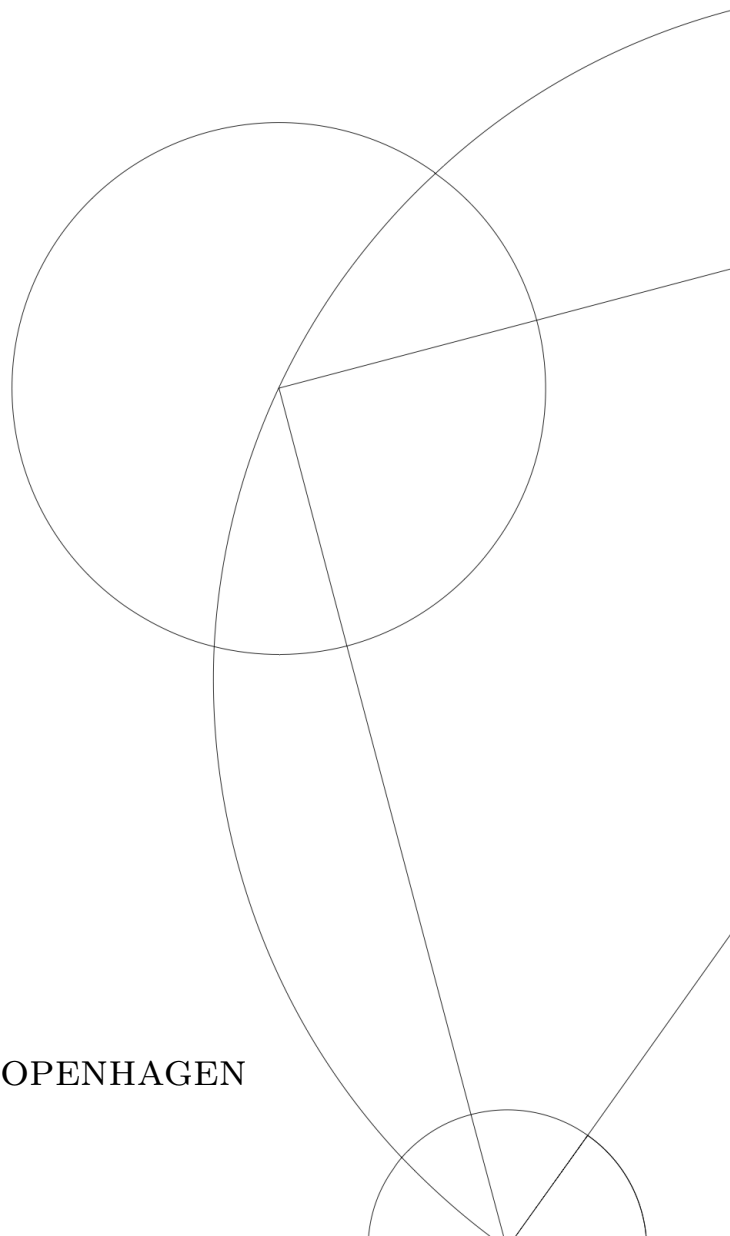
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Abstract

We investigate the optical properties of a system consisting of a waveguide coupled to a partially chiral infinite array of equidistant two-level emitters. In the first step, we employ a transfer matrix formalism, and are able to solve single photon transport for any combination of emitter position and chirality. We build upon this with the use of an effective Hamiltonian of the system to study the behavior of two photon bound states. We find that these come in two varieties, depending on the two photon momentum. One of these states is long-lived, whereas one decays in time via coupling to plane wave photons. We present a systematic way of deriving such states and a number of their properties, such as their dispersion, are studied. We also discuss further utilization of this systematic approach to bound states, and suggest developments to the presented theory.

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Chapter 1

Introduction

The last two centuries saw an unprecedented explosion in human understanding of both the inner workings of nature, as well as technological leaps that made reality out of previously far fetched science fiction ideas. The interplay between new understanding, leading to new technologies and these technologies, leading to new ways of probing nature has resulted in the world we inhabit today.

The computer, as one example of this development, was first made using big and unreliable vacuum tubes, which made a small scale commercial computers unthinkable (interestingly, this unreliability lead to classical error-correcting code, mirroring the modern study of quantum error-correction [1]). Only with the advent of the transistor did the computer, and most modern electric devices, as we now know it begin to be a reality. The transistor itself was a result of human endeavors to understand the fundamental physics of the nanoscopic, quantum mechanics, a field that had been highly esoteric only half a century earlier.

Today one particularly interesting challenge is posed by the emergence of the quantum computer. From a theoretical standpoint, it is known that it implies many new possibilities, both in efficiently simulating quantum systems, which may yield exciting results not only in basic research, but also in the “tabletop” fields, such as chemistry and medicine due to better understanding of the formation and interaction of molecules. Famously it allows for a fast way of factorizing large numbers, as was shown by Peter Shor in his seminal paper [2]. This capability may well render classical cryptography obsolete within the next century, as these types of are build on the current difficulties of large number factorization. On the other end of the spectrum, quantum technologies also have the promise of allowing completely safe encryption as was pointed out by Bennett and Brassard by use of the BB84 protocol [3].

The practical difficulties with making a quantum computer suggest that we may yet be dealing with the quantum mechanical vacuum tube, and new technologies will be needed. One component of this large machinery could be found by utilizing the recent discovery of the chiral light-matter interactions with waveguides [4]. Such systems may enable us to have useful, controlled interactions between photons. That photons, which have only seriously been considered to exist since the miracle year 1905 [5], are now controllable down to the single photon, is itself quite an achievement. This thesis investigates some of the properties of such a system.

That chiral light-matter interactions can even exist with high efficiency was only experimentally realized within the last decade [6]. However, there is always a slight coupling to both the right and left propagating electric field in a waveguide. As such, it is import for further understanding of the field to also gain insight into the dynamics of the partially chiral coupling, which will serve as the framework of this thesis.

In particular the thesis carries out research on the properties of single and two-photon properties of an infinite waveguide coupled to the partially chiral two-level emitters. We study the two photon bound states. The bound state is a state that has the property that photons tend to stick together, their separation roughly following an exponential decay [8] [7].

This is only a small part of a much larger and new field of research committed to studying properties of so called photonic molecules. While this is a new field, a number of interesting results have already been produced in the area of research. One is that there has already been suggestions that crystallization of strongly interacting photons [9].

We further study resonances of scattering light that arise as a signature of the two photon bound states in the waveguide. These results are interesting when considering recent results in the field [8].

While a two-level system may seem to “simple” to the outsider, but it has many applications in the real world for implementing quantum gates, such as the CNOT gate [6]. Further, such systems are currently being constructed in a number of ways, for example in Rydberg atoms [10] [11].

Most of the results of the thesis will be based on an approach with its origin in an effective Hamiltonian for the system we study. Such an approach to research is at the moment actively used in the field of theoretical quantum optics [12]. We use the so called electric field elimination [13] to focus only on the atomic excitations, making the study of the properties of our system considerably easier to study than it would be if we studied both electric field and atomic excitations at once.

The thesis will be divided into six chapters. The overview is as follows:

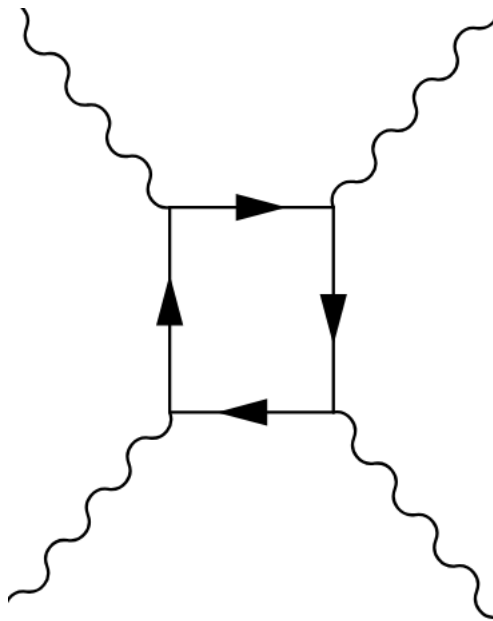


Figure 1.1: Feynman diagram showing the scattering of two photons with each other. [14] We shall not analyze the problem from a field theoretical point, but instead study interactions mediated by a coupling to auxiliary atoms.

Chapter 1: Introduction.

Chapter 2: Review of Basic Concepts. The chapter establishes some of the most relevant physical concepts that will be used in the rest of the thesis.

Chapter 3: Model. Here we will have the first discussion of the model we use to describe the system that is the basis of this thesis. Some preliminary results are derived.

Chapter 4: Bound States and the Effective Hamiltonian. Here we investigate the properties of the system using an effective Hamiltonian. Particular focus is placed on the two photon bound states that arise in the system.

Chapter 5: Conclusion. The main results of the thesis are discussed and put into the context of current research.

Chapter 6: Outlook. A discussion of relevant further paths of research is given.

Chapter 2

Review of Basic Concepts

2.1 Electrodynamics

2.1.1 The Maxwell Equations and Auxiliary Fields

In the beginning, there was James Clerk Maxwell. He made a coherent formulation of the equations that are still used - with an important notational update due to Oliver Heaviside - to this day in classical optics and electrodynamics. [16]

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho / \epsilon_0 \\ \nabla \times \mathbf{E} &= - \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}\end{aligned}\tag{2.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, ρ the charge density, $\mathbf{J}$ the current density, ϵ_0 the electric permittivity and μ_0 the magnetic permeability. Further, it is possible to define a so called vector field, $\mathbf{A}$, with the property that

$$\nabla \times \mathbf{A} = \mathbf{B}.\tag{2.2}$$

Note that according to the above, $\mathbf{A}$ is not unique, as the addition of any field $\mathbf{a}$, with the property that $\nabla \times \mathbf{a} = 0$ yields the same physics, due to the fact that the curl operator is linear. This is what is known as gauge invariance. We can then ask what the divergence of such a field is? This again does not have one unique answer, but it does have a few canonical ones, the usefulness of each depending on the problem at hand.

The two most common ones are perhaps the Coulomb and Lorentz gauge. The Coulomb gauge is defined by

$$\nabla \cdot \mathbf{A} = 0 \quad (2.3)$$

Furthermore, there is the potential field, ϕ . In the case of electrostatics, the electric field can be derived from the potential by $-\nabla\phi = \mathbf{E}$. However, the curl of a gradient is always 0, but Maxwell's Equations tell us that for a time varying field, this cannot be sufficient, as the curl of $\mathbf{E}$ is not 0. A possible remedy to this problem is subtracting the time derivative of $\mathbf{A}$, which indeed turns out to be right,

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}. \quad (2.4)$$

This is useful when studying the behavior of electromagnetic fields in a vacuum, leading to what was historically the great triumph of Maxwell's theory, the fact that light is electromagnetic waves.

2.1.2 The Vacuum and the Electromagnetic wave

In vacuum everything is simpler. Stones follow parabolas, planets follow ellipses and electromagnetic fields follow each other.

Consider Maxwell's Equations in the vacuum. Here ρ and $\mathbf{J}$ are both zero so the equations take on the form

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \quad (2.5)$$

Assuming $\phi = 0$, it is possible to frame the problem in terms of the vector potential instead. Noting that $\nabla \times \mathbf{A} = \mathbf{B}$ and $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ and using the Coulomb gauge

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A} = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \quad (2.6)$$

this is a wave equation, which has the known solution of the form

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \quad (2.7)$$

where $\mathbf{A}_0$ is the amplitude, $\mathbf{k}$ is a wave vector and ω is the angular frequency. The speed of the wave is $\omega/k = 1/\sqrt{\epsilon_0\mu_0} = c$, that is, the speed of light in a vacuum. This result was historically one of the strongest arguments for light being an electromagnetic wave.

From the vector potential it is then possible to derive the electric and magnetic fields,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = i\omega \mathbf{A}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad (2.8)$$

$$\mathbf{B} = \nabla \times \mathbf{A} = i\mathbf{k} \times \mathbf{A}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad (2.9)$$

It is now possible to derive a number of standard results from optics. First, from $\nabla \cdot \mathbf{E} = 0$ it can be derived that the electric field is perpendicular to the direction of propagation,

$$\nabla \cdot \mathbf{E} = -\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = \omega \mathbf{k} \cdot \mathbf{A} = 0 \quad (2.10)$$

this is only true if $\mathbf{k}$ and $\mathbf{A}_0$ are perpendicular, and $\mathbf{E}$ is parallel with $\mathbf{A}_0$. Using (2.8) to express $\mathbf{B}$ in terms of $\mathbf{E}$

$$\mathbf{B} = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E} \quad (2.11)$$

which shows that the magnetic field is perpendicular not only to the direction of propagation but also the electric field.

2.1.3 Electromagnetic Fields in free Space

One might imagine the universe as an infinite series of cubic boxes with side length l , each one an exact copy of all the others. Less colorful language would lead one to simply imply that the universe has periodic boundary conditions. If the universe is indeed in this form, l would have to be pretty big. Also, any object, field or event at a point, (x, y, z) , in space-time would have to be the same at $(x + n_x l, y + n_y l, z + n_z l)$, where n_x, n_y, n_z are any set of three integers. Any plane wave solution must then have the constraint on the wave vector that

$$\mathbf{k} = (2\pi n_x/l, 2\pi n_y/l, 2\pi n_z/l) \quad (2.12)$$

We denote the volume of the box $V = l^3$, and note that there is an orthogonality relation as follows,

$$\int_V dV e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = V \delta_{\mathbf{k},\mathbf{k}'} \quad (2.13)$$

It is possible to write any vector potential as a linear combination of these orthogonal functions,

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}, p} \hat{e}_p (A_{\mathbf{k}, p} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + A_{\mathbf{k}, p}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}) \quad (2.14)$$

where p is the polarization index, and $\hat{e}_p$ is a unit vector in the direction of p and $A_{\mathbf{k}, p}$ are coefficients of the field. The addition of $A_{\mathbf{k}, p}^*$ ensures a vector potential that is real. As $\nabla \cdot \mathbf{A} = 0$, $\mathbf{k} \cdot \hat{e}_p = 0$ and so p is a sum over two indices, with $\hat{e}_1 \cdot \hat{e}_2 = 0$. The electromagnetic fields then take the form

$$\mathbf{E}(\mathbf{r}, t) = i \sum_{\mathbf{k}, p} \omega_k \hat{e}_p (A_{\mathbf{k}, p} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} - A_{\mathbf{k}, p}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}) \quad (2.15)$$

$$\mathbf{B}(\mathbf{r}, t) = i \sum_{\mathbf{k}, p} (\mathbf{k} \times \hat{e}_p) (A_{\mathbf{k}, p} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} - A_{\mathbf{k}, p}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}) \quad (2.16)$$

The energy of the electromagnetic field within the cube can be expressed as

$$E = \frac{\epsilon_0}{2} \int dV (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \quad (2.17)$$

where the integration is over the volume of one cube. To express the energy in terms of the vector field, first note

$$\begin{aligned} \mathbf{E} \cdot \mathbf{E} = & - \sum_{\mathbf{k}, \mathbf{k}', p, p'} \omega_k \omega_{k'} (\hat{e}_{\mathbf{k}, p} \cdot \hat{e}_{\mathbf{k}', p'}) (A_{\mathbf{k}, p} A_{\mathbf{k}', p'} e^{i((\mathbf{k} + \mathbf{k}') \cdot \mathbf{r} - (\omega_k + \omega_{k'}) t)} \\ & + A_{\mathbf{k}, p}^* A_{\mathbf{k}', p'}^* e^{-i((\mathbf{k} + \mathbf{k}') \cdot \mathbf{r} - (\omega_k + \omega_{k'}) t)} \\ & - A_{\mathbf{k}, p}^* A_{\mathbf{k}', p'} e^{i((-\mathbf{k} + \mathbf{k}') \cdot \mathbf{r} - (-\omega_k + \omega_{k'}) t)} \\ & - A_{\mathbf{k}, p} A_{\mathbf{k}', p'}^* e^{i((\mathbf{k} - \mathbf{k}') \cdot \mathbf{r} - (\omega_k - \omega_{k'}) t)}) \end{aligned} \quad (2.18)$$

When carrying out the integration, we may use the orthogonality of (2.13), to get

$$\begin{aligned} \int dV \mathbf{E} \cdot \mathbf{E} = & 2V \sum_{\mathbf{k}, p} \omega_k^2 (A_{\mathbf{k}, p}^* A_{\mathbf{k}, p}) \\ & - V \sum_{\mathbf{k}, p, p'} \omega_k^2 ((\hat{e}_{\mathbf{k}, p} \cdot \hat{e}_{-\mathbf{k}, p'}) (A_{\mathbf{k}, p} A_{-\mathbf{k}, p'} e^{-2i\omega_k t} + A_{\mathbf{k}, p}^* A_{-\mathbf{k}, p'}^* e^{2i\omega_k t})) \end{aligned} \quad (2.19)$$

For the $\mathbf{B} \cdot \mathbf{B}$ term, the following identity is useful

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= \varepsilon_{ijk} \varepsilon_{mnk} A_i B_j C_m D_n \\
&= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) A_i B_j C_m D_n \\
&= A_i B_j C_i D_j - A_i B_j C_j D_i \\
&= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})
\end{aligned} \tag{2.20}$$

where Einstein summation is implied. Then

$$(\mathbf{k} \times \hat{\mathbf{e}}_{\mathbf{k},p}) \cdot (\mathbf{k}' \times \hat{\mathbf{e}}_{\mathbf{k}',p'}) = (\mathbf{k} \cdot \mathbf{k}')(\hat{\mathbf{e}}_{\mathbf{k},p} \cdot \hat{\mathbf{e}}_{\mathbf{k}',p'}) - (\mathbf{k} \cdot \hat{\mathbf{e}}_{\mathbf{k}',p'})(\mathbf{k}' \cdot \hat{\mathbf{e}}_{\mathbf{k},p}) \tag{2.21}$$

When integrating over the volume, the second term in the above equation is equal to 0, as $\mathbf{k} \cdot \hat{\mathbf{e}}_{\pm \mathbf{k},p} = 0$, so

$$\begin{aligned}
\int dV \mathbf{B} \cdot \mathbf{B} &= 2V \sum_{\mathbf{k},p} k^2 (A_{\mathbf{k},p}^* A_{\mathbf{k},p}) \\
&\quad + V \sum_{\mathbf{k},p,p'} k^2 ((\hat{\mathbf{e}}_{\mathbf{k},p} \cdot \hat{\mathbf{e}}_{-\mathbf{k},p'}) (A_{\mathbf{k},p} A_{-\mathbf{k},p'} e^{-2i\omega_k t}) + A_{\mathbf{k},p}^* A_{-\mathbf{k},p'}^* e^{2i\omega_k t})
\end{aligned} \tag{2.22}$$

Using the fact that $ck = \omega_k$, the time-dependent terms in both the equations cancel each other and the energy can be written

$$E = 2\varepsilon_0 V \sum_{\mathbf{k},p} \omega_k^2 A_{\mathbf{k},p} A_{\mathbf{k},p}^* = \frac{1}{2} \sum_{\mathbf{k},p} (\omega_k^2 q_{\mathbf{k},p}^2 + p_{\mathbf{k},p}^2) \tag{2.23}$$

where $A_{\mathbf{k},p} = \frac{1}{2\omega_k \sqrt{\varepsilon_0 V}} (\omega_k q_{\mathbf{k},p} + ip_{\mathbf{k},p})$. This energy has the same structure as that of a harmonic oscillator, a fact that will be of importance when we shortly will consider electromagnetic fields in the context of quantum mechanics.

2.2 Quantum Mechanics

2.2.1 The Schrödinger Picture

At the core of one approach to non-relativistic quantum mechanics is the Schrödinger equation, [18]

$$i\hbar \frac{\partial}{\partial t} |\Psi(\mathbf{r}, t)\rangle = H(\mathbf{r}, t) |\Psi(\mathbf{r}, t)\rangle = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, t) \right) |\Psi(\mathbf{r}, t)\rangle \tag{2.24}$$

where $\hbar$ is Dirac's constant, $|\Psi(\mathbf{r}, t)\rangle$ is the state of the system and H is the system Hamiltonian, an operator describing the energy of the system.

The most general case can in principle of the Schrödinger equation can be solved using

$$|\Psi(\mathbf{r}, t)\rangle = e^{-i\frac{H(t-t_0)}{\hbar}}|\Psi(\mathbf{r}, t_0)\rangle \quad (2.25)$$

which is the solution that one would naively expect if H was simply a scalar. It is useful from a theoretical standpoint, but often not directly possible to evaluate.

In the case of a time-independent Hamiltonian, the problem is reduced to solving the time-independent Schrödinger equation,

$$E|\psi(\mathbf{r})\rangle = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right)|\psi(\mathbf{r})\rangle \quad (2.26)$$

where E is the energy of the state. For these states, the time evolving state is

$$|\Psi(\mathbf{r}, t)\rangle = e^{-iEt/\hbar}|\psi(\mathbf{r})\rangle \quad (2.27)$$

For any Hamiltonian, H , there will in general be an infinite number of linearly independent solutions, $|\psi_n\rangle$ to the Schrödinger equation, each with an associated energy E_n , which may or may not be degenerate. There are, however, many cases where the Hamiltonian only has a finite number of states, such as the case of spin, and in any case it is often an excellent approximation to make a truncation of the Hilbert Space to a finite number of relevant states. This will be done later, when considering numerical solutions to a Hamiltonian.

For an arbitrary state, $|\phi\rangle$, it is possible to construct the state as a linear combination of the eigenstates of the Hamiltonian,

$$|\phi\rangle = \sum c_n |\psi_n\rangle \quad (2.28)$$

where the coefficients c_n can be found by taking the inner product of the state with an eigenstate, $c_n = \langle\psi_n|\phi\rangle$. According the so called Born interpretation, Originally proposed by Max Born, the probability of measuring a state is equal to the absolute square of the coefficient, $|c_n|^2$. The probability of measuring *something* must be unity, putting the following constraint on the coefficients,

$$\sum |c_n|^2 = 1. \quad (2.29)$$

This leads to an important fact about quantum mechanical states. They must be normalized

$$\langle\phi|\phi\rangle = 1 \quad (2.30)$$

This is a requirement if a given state is to be measurable. However, a given eigenstate of a Hamiltonian can in general be non-normalizable, but this makes the eigenstate an impossible actual state. This is surprisingly normal, being the case for free particles in infinite space [17].

2.2.2 Heisenberg Picture

An alternative formulation of quantum mechanics takes the operator, not the state, to be that which evolves over time. One can see how this works by taking the expectation value of some operator A in the Schrödinger picture

$$\langle A \rangle(t) = \langle \Psi | e^{iHt/\hbar} A e^{-iHt/\hbar} | \Psi \rangle \quad (2.31)$$

we are then at liberty to simply understand the state as constant in time while the time dependent operator becomes

$$A(t) = e^{iHt/\hbar} A(0) e^{-iHt/\hbar} \quad (2.32)$$

We can take the time derivative of the above and get

$$\dot{A}(t) = \frac{i}{\hbar} (HA(t) - A(t)H) = \frac{i}{\hbar} [H, A](t) \quad (2.33)$$

which is the Heisenberg equation [18]. The Schrödinger and Heisenberg picture do not contain any different physics, they simply represent different formalisms, the utility of each depending on the problem at hand.

2.2.3 Harmonic Oscillator

An example of particular importance is the harmonic oscillator, with the Hamiltonian

$$H = \frac{1}{2m} (p^2 + (m\omega x)^2) \quad (2.34)$$

One particularly elegant way of realizing the solution of the harmonic oscillator is by reframing it in terms of the annihilation operator,

$$a = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega x + ip) \quad (2.35)$$

and the creation operator, $a^\dagger$, which is the hermitian conjugate of the annihilation operator. These operators obey the commutation relation

$$[a, a^\dagger] = 1 \quad (2.36)$$

which follows directly from the canonical commutation relation $[x, p] = i\hbar$. The Hamiltonian is then

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \quad (2.37)$$

Assuming an eigenstate $|n\rangle$, with eigenenergy E_n , it is easy to show that $a^N|n\rangle$ and $(a^\dagger)^N|n\rangle$ are also eigenstates with eigen energy $E_n \mp N\hbar\omega$. To decide the ground state, one can impose the condition that $a|0\rangle = 0$, where $|0\rangle$ is the ground state. Using the representation of p in the x -basis, $p = -i\hbar\frac{d}{dx}$, the x -basis representation of the ground state is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \quad (2.38)$$

Using the commutation relations above, it is straightforward to show by induction that

$$[a^\dagger a, (a^\dagger)^n] = n(a^\dagger)^n \quad (2.39)$$

Effectively rendering us able to determine the normalization constant of an eigenstate, with

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \quad (2.40)$$

so we see that a state can be created by successively applying the creation operator. This hints at the so called second quantization, where it is not energy, but the particles themselves that are created and annihilated by operators.

2.2.4 Second Quantization and the Electric Field

So far we have only dealt with the classical electric field. We would now like to make a build a quantum mechanical theory of these fields. For this the work on the harmonic oscillator. Using the results in section 2.1.3 and 2.2.3 one can impose the canonical quantization relation [1]

$$[q_{\mathbf{k},p}, p_{\mathbf{k}',p'}] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}\delta_{p,p'}. \quad (2.41)$$

This is very reminiscent of the harmonic oscillator we briefly studied above. Using our intuition as a guide leads to a Hamiltonian of the form

$$H = \sum_{\mathbf{k},p} \hbar\omega_k (a_{\mathbf{k},p}^\dagger a_{\mathbf{k},p} + \frac{1}{2}) \quad (2.42)$$

where $a_{\mathbf{k},p} = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k q_{\mathbf{k},p} + ip_{\mathbf{k},p})$ is the annihilation operator and $a_{\mathbf{k},p}^\dagger$ is the creation operator. These derive their names from the fact that they create (annihilate) a photon in a given k state. If one applies a specific creation operator to a state n times, n photons will be occupying that state. Now it is a simple matter to make different states of particular excitation number in the electromagnetic field, one simply has to “create” the photons by applying the creation operator to the vacuum,

$$|\Psi\rangle = \prod_{i=0}^{\infty} \frac{1}{\sqrt{N_i!}} (a_i^\dagger)^{N_i} |0, 0, 0, \dots\rangle = |N_0, N_1, N_2, \dots\rangle \quad (2.43)$$

where the constants are inserted for the sake of normalization. In fact, the above is somewhat limited, as it is a well defined number state, where actual field can be a superposition of various number states.

2.2.5 Light-Matter Interaction

Let us consider a system consisting of a cavity with an atom and a field. We consider the atom to be a two level system, or two level emitter (TLE). The Hamiltonian associated with an atom shall be denoted by $H_A = \hbar\omega_0\sigma_z$. Further, the Hamiltonian associated with the field is $H_F = \hbar\omega a^\dagger a$. The atom and field are coupled, according to the Jaynes-Cummings model[1], by the interaction Hamiltonian $H_I = g(\sigma_+ + \sigma_-)(a^\dagger + a)$, with g acting as coupling constant. Making the rotating wave approximation (RWA), we can drop the non-energy conserving terms, $\sigma_+ a^\dagger$ and $\sigma_- a$, yielding a total Hamiltonian of the form

$$H = \hbar\omega_0\sigma_z + \hbar\omega a^\dagger a + g(\sigma_- a^\dagger + \sigma_+ a). \quad (2.44)$$

As we shall see in section 2.3, this model will have a slight modification to allow for partially chiral interaction between atom and field.

2.2.6 Approximating the Field Hamiltonian

In general the field Hamiltonian in a one dimensional wave guide can be described as

$$\sum_k \hbar\omega_k a_k^\dagger a_k \quad (2.45)$$

where ω_k is the dispersion relation of the light within a given medium. This thesis is concerned with light that is close to the transition frequency, ω_0 of some TLE. This

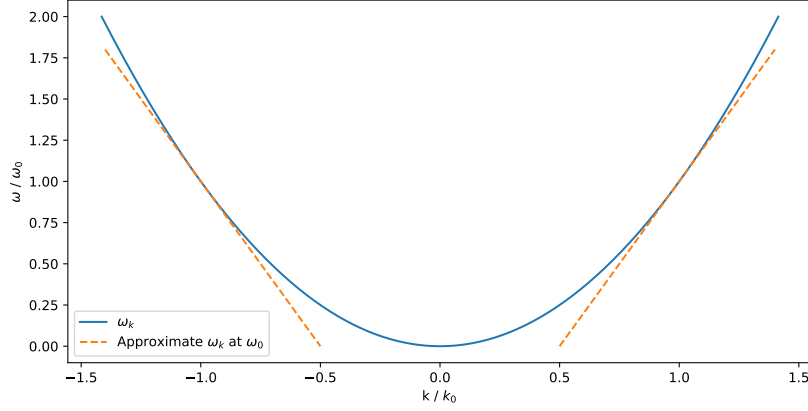


Figure 2.1: A symmetric dispersion can be approximated as linear in the close vicinity of some frequency, ω_0 .

transition frequency is associated with a wave number k_0 . We therefore make an approximation of ω_k by expanding it around the transition frequency as shown in figure 2.1,

$$\omega_k \approx \omega_0 + (k - k_0) \left. \frac{\partial \omega_k}{\partial k} \right|_{k=k_0} \quad (2.46)$$

The quantity $\frac{\partial \omega_k}{\partial k}$ is the group velocity, which shall be denoted v_g . It defines the speed at which a wavepacket, situated around k_0 in momentum space, will propagate. We can redefine our zero point energy to cancel the constant term that arisen due to ω_0 .

The Hamilton is represented in momentum space, but can be rewritten in spacial coordinates as

$$-i\hbar v_g \int dx [c_R^\dagger \frac{\partial}{\partial x} c_R - c_L^\dagger \frac{\partial}{\partial x} c_L] \quad (2.47)$$

where $c(x)$ is the real space annihilation operator of the electric field. Note that we are now writing explicitly whether we are dealing with right and left going modes to deal with the absolute sign that is introduced due to the linear approximation.

2.3 The Chiral Interaction, a Heuristic Approach

In this thesis, the partially chiral interaction is of particular interest. At first glance, the chiral coupling does not seem to make sense. It asks for a Hamiltonian of the form

$$H = V_R\sigma_{+c_R} + V_L\sigma_{+c_L} + H.c. \quad (2.48)$$

that is, the coupling between light and atom should be different depending on whether one sends in light in a right or left mode. This seems to violate time reversal symmetry; if one was to turn back time, surely there would be no difference in the interaction.

Considering the electric field right outside a waveguide, evanescent behavior leads us to describe the field in the following way,

$$\mathbf{E} = \mathbf{E}_0 e^{-\kappa z + i k x - i \omega t} \quad (2.49)$$

that is, an electric field propagating along the x -axis and decaying exponentially along the z -axis. As there is no charge in the system, the divergence of the field must be zero, yielding

$$\nabla \cdot \mathbf{E} = i k E_z - \kappa E_x = 0 \quad (2.50)$$

from which one can gather that $\mathbf{E}_0$ must take the form

$$\mathbf{E}_0 = E_0 e^{i \omega t} \begin{pmatrix} 1 \\ 0 \\ -\frac{i \kappa}{k} \end{pmatrix} \quad (2.51)$$

where E_0 is a measure of the strength of the electric field. The y component of the field is simply zero.

$\mathbf{E}_0$ is complex so to get the actual electric field, we take the real part,

$$\text{Re}(\mathbf{E}_0) = E_0 \begin{pmatrix} \cos(\omega t) \\ 0 \\ \frac{\kappa}{k} \sin(\omega t) \end{pmatrix} \quad (2.52)$$

so we find that the electric field rotates in a plane that is parallel with and intersecting the wave guide. An electric wave propagating in the opposite direction will then rotate in the opposite direction.

Using the wave equation we get the following relation between k and κ

$$k^2 - \kappa^2 = \frac{\omega^2}{c^2} \quad (2.53)$$

as ω is not zero, the magnitude of k and κ can not quite be unity, $\kappa/k \neq 1$, so the light will not be perfectly circular outside of the waveguide. However, in the limit $ck \gg \omega$, it is a good approximation. This further motivates the study of partially chiral systems.

If one places an atom in the vicinity of the wave guide, the light will interact with two different states of the atom. Further, one can produce a magnetic field to induce a splitting in the energies of the two states. This allows for a strong coupling to one of the states, while neglecting the influence of the other state. Such interactions can be tuned, and thus arises the Hamiltonian of the form stated above. Neglecting this magnetic field in the Hamiltonian makes the system appear to contradict time reversal symmetry, but as a matter of fact time reversal would reverse the direction of the magnetic field, thus making the photon switch the mode with which it interacts more strongly, which is exactly what one would expect.

Chapter 3

Model

3.1 Single Excitation, Transmission and Reflection

We are now ready to study the features of an infinite waveguide coupled to partially chiral TLEs. The model is an extension of [15] by allowing for different coupling strengths between left and right going modes of the electric field and the atom.

The atom is an effective two-level system, with a ground state $|g\rangle$ and excited state $|e\rangle$, with the states being orthogonal. The outer product states that are formed by these states will be denoted by $\sigma_{ij} = |i\rangle\langle j|$. These have the commutation relations

$$[\sigma_{ij}, \sigma_{mn}] = \delta_{jm}\sigma_{in} - \delta_{in}\sigma_{mj} \quad (3.1)$$

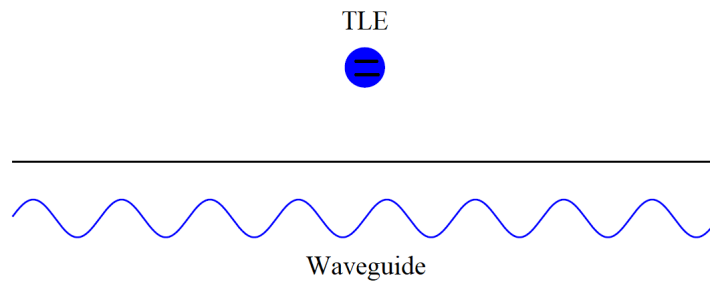


Figure 3.1: System considered. A single two level emitter coupled to a waveguide.

and the electric field operators $c(z)$ have commutator relation

$$[c_i(z), c_j^\dagger(z')] = \delta_{ij}\delta(z - z') \quad (3.2)$$

The Hamiltonian of the atom coupled to a wave guide is described in the form

$$\begin{aligned} H = & \hbar\omega\sigma_{ee} + \int dz((c_R(z)V_R + c_L(z)V_L)\sigma_{eg} + (c_R^\dagger(z)V_R + c_L^\dagger(z)V_L)\sigma_{ge})\delta(z - z_0) \\ & - i\hbar v_g \int dz(c_R^\dagger(z)\frac{\partial}{\partial z}c_R(z) - c_L^\dagger(z)\frac{\partial}{\partial z}c_L(z)). \end{aligned} \quad (3.3)$$

where $\omega = (E_e - E_g)/\hbar$ is the transition frequency of the two-level emitter, V_R (V_L) is the coupling of the right (left) going mode, and v_g is the group velocity of the modes.

The Hamiltonian has translational symmetry, with the exception of a discontinuity at z_0 . We know that an eigenstate of an operator that commutes with the Hamiltonian must also be an eigenstate of the Hamiltonian, so a trial eigenstate could take the form

$$|k, R\rangle = \int dz[e^{ikz}(\theta(z_0 - z) + t_k^R\theta(z - z_0))c_R^\dagger + e^{-ikz}r_k^R\theta(z_0 - z)c_L^\dagger + e_k^R\delta(z - z_0)\sigma_{eg}]|0, g\rangle \quad (3.4)$$

where $|0, g\rangle = a_g^\dagger|\emptyset\rangle$. The physical interpretation of (3.4) is that some incoming wave from the right, which scatters off the atom. The part of the wave that is transferred through the atom has amplitude t_k^R and the reflected part has amplitude r_k^R . A similar left-going incoming wave state, $|k, L\rangle$ could also be used. It is then straightforward to prove that this is an eigenstate by applying the Hamiltonian

$$H|k, R\rangle = E|k, R\rangle \quad (3.5)$$

leading to the equations of motion

$$Ee_k^R = \hbar\omega e_k^R + \frac{1}{2}V_L r_k^R e^{-ikz_0} + \frac{1}{2}V_R(1 + t_k^R)e^{ikz_0} \quad (3.6)$$

$$E = \hbar v_g k \quad (3.7)$$

$$e_k^R V_L = i\hbar v_g e^{-ikz_0} \quad (3.8)$$

$$e_k^R V_R = i\hbar v_g (t_k^R - 1)e^{ikz_0} \quad (3.9)$$

Solving these equation yields

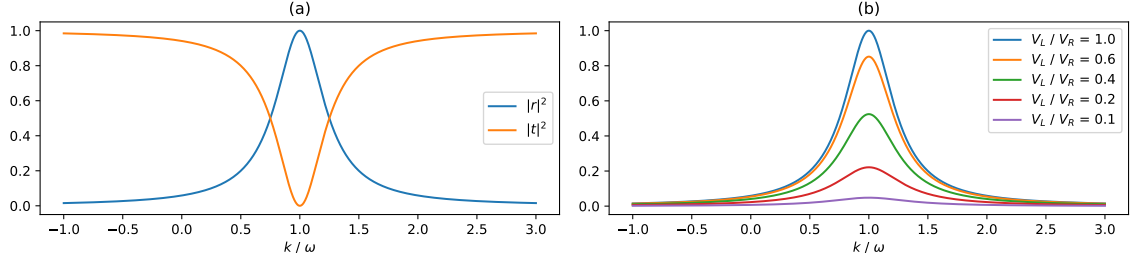


Figure 3.2: (a) shows the transmittance and reflectance for a completely non-chiral wave guide ($V_R = V_L$). At resonance the atom reflects any incoming wave completely. (b) shows the reflectance for systems of increasing chirality.

$$e_k^R = \frac{V_R e^{ikz_0}}{\hbar(v_g k - \omega) + \frac{i}{2\hbar v_g}(V_R^2 + V_L^2)} \quad (3.10)$$

$$r_k^R = -i \frac{e_k^R V_L e^{ikz_0}}{\hbar v_g} = -i \frac{V_R V_L e^{2ikz_0}}{\hbar^2 v_g (v_g k - \omega) + \frac{i}{2}(V_R^2 + V_L^2)} \quad (3.11)$$

$$t_k^R = 1 - i \frac{e_k^R V_R e^{-ikz_0}}{\hbar v_g} = \frac{\hbar^2 v_g (v_g k - \omega) + \frac{i}{2}(V_L^2 - V_R^2)}{\hbar^2 v_g (v_g k - \omega) + \frac{i}{2}(V_R^2 + V_L^2)} \quad (3.12)$$

where we note that $|r_k^R|^2 + |t_k^R|^2 = 1$, as is required. For an incoming left going wave, simply change the R to L and change the sign of the k in the exponent.

It is worth while to consider what exactly the above tells us about the physics of the system. For the completely non-chiral case (that is, $V_R = V_L$), the incoming wave will be reflected if it is on resonance with the atom. In the completely chiral case, the atom only couples to either the left or right going mode. As such it can also only emit in the same direction as it absorbed. This means that in this regime, all waves will be completely transferred, no matter the frequency and direction. Figure 3.3 shows these facts. Furthermore, for a completely chiral coupling, an incoming wave will get a π -shift in its phase. This can be used to implement a CNOT gate [6].

3.1.1 Transfer Matrix

It is possible to define a transfer matrix for the above problem [17], in principle for an arbitrary chain of chiral atoms. The matrix that tells one how a state propagates from

a position x to x' , must have the property

$$M(x', x) \begin{pmatrix} c_R(x) \\ c_L(x) \end{pmatrix} = \begin{pmatrix} c_R(x') \\ c_L(x') \end{pmatrix} \quad (3.13)$$

For the free propagation, without atoms, the transfer matrix simply takes the form

$$M_{free}(x', x) = \begin{pmatrix} e^{ik(x-x')} & 0 \\ 0 & e^{-ik(x-x')} \end{pmatrix} \quad (3.14)$$

For the propagation through an atom, the properties of reflection and transmission derived in section 3.1 will be needed. What is propagating to the right immediately to the right of the atom, c_R^+ , must be a combination of transmitted part of the right moving field immediately to the left of the atom, $t_R c_R^-$, and the reflected part of the left propagating field immediately to the right of the atom, $r_L c_L^+$. Similar arguments for the right propagating field immediately to the left of the atom yields the following equations,

$$\begin{aligned} c_R^+ &= t_R c_R^- + r_L c_L^+ \\ c_R^- &= t_L c_L^+ + r_R c_R^- \end{aligned} \quad (3.15)$$

where k has been omitted from the transmission and reflection coefficients. These equations can be rewritten in matrix form, allowing us to determine the transfer matrix for the atom,

$$M_{atom} \begin{pmatrix} c_R^- \\ c_L^- \end{pmatrix} = \begin{pmatrix} c_R^+ \\ c_L^+ \end{pmatrix} \quad (3.16)$$

where

$$M_{atom} = \frac{1}{t_L} \begin{pmatrix} t_R t_L - r_L r_R & r_L \\ -r_R & 1 \end{pmatrix} \quad (3.17)$$

In principle, the two kinds archetypal matrices above allow us to make any kind of single photon transfer with TLEs at arbitrary positions and transition frequencies. In the following a particular system consisting of an infinite array of identical TLEs with the distance d to each other. The “unit” transfer matrix of this system is $A = M_{free}(d, 0)M_{atom}$. It can be rewritten in diagonal form $A = SDS^{-1}$, where

$$D = \begin{pmatrix} e^{iK_0 d} & 0 \\ 0 & e^{iK_1 d} \end{pmatrix} \quad (3.18)$$

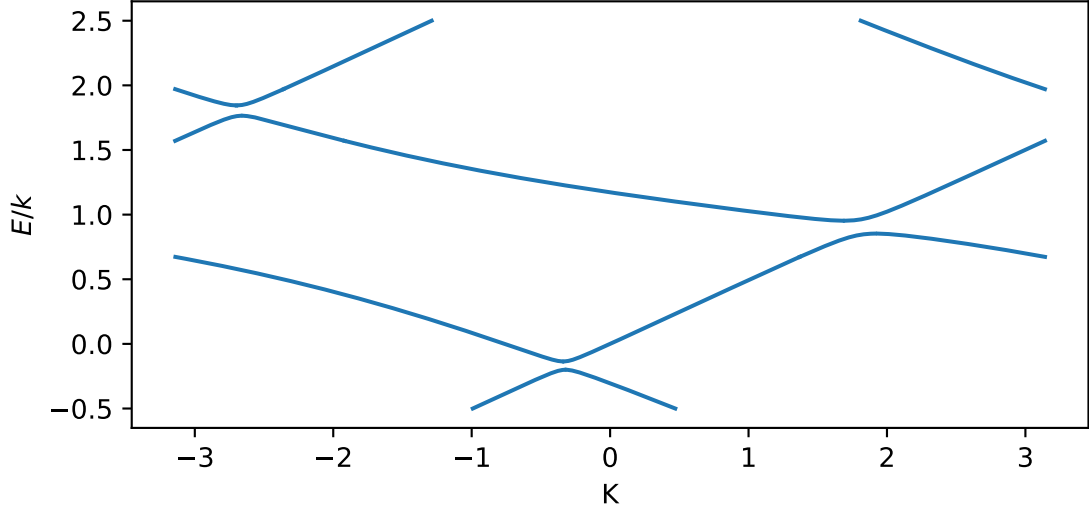


Figure 3.3: Photonic band gap arising in an infinite system of TLEs coupled to a waveguide. Here $\hbar = v_g = \omega = 1$, $d = 2$, $V_R = 0.1$ and $V_L = 0.9$.

but the eigen values of D are the same as those of A , so the allowed wave numbers can be determined simply by determining the eigen values of A and taking the imaginary part of the logarithm.

A region of interest is that of high v_g . An example of this regime is shown in figure 3.4.

3.2 Two Photon transport

We would now like to be able to deal with the case of two photon scattering in a partially chiral system. A systematic, successful process for dealing with this problem was unfortunately not achieved. However, some of the work done in this field is described in section 6.1.

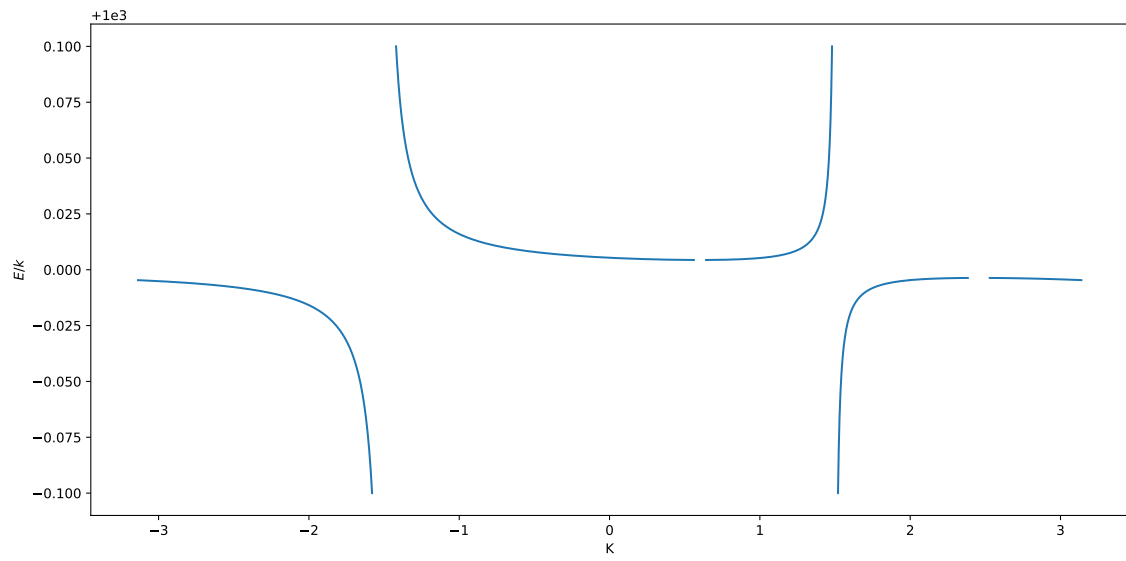


Figure 3.4: Photonic band gap in the case of high v_g . In this regime the wavevector is practically constant. Here $\hbar = 1$, $d = 6$, $v_g = 100$, $\omega = 10$, $V_R = 0.2$ and $V_L = 0.8$.

Chapter 4

Bound States and the Effective Hamiltonian

4.1 Derivation of Effective Hamiltonian

We would now like to study the photon transport from a different point of view. We start with the Hamiltonian that has been studied in the last chapter.

$$H = \sum_{j=-\infty}^{\infty} \{ \hbar\omega\sigma_{ee}^j + \int dz((c_R(z)V_R + c_L(z)V_L)\sigma_{eg}^j + (c_R^\dagger(z)V_R + c_L^\dagger(z)V_L)\sigma_{ge}^j)\delta(z - z_j) \} \\ - i\hbar v_g \int dz(c_R^\dagger(z)\frac{\partial}{\partial z}c_R(z) - c_L^\dagger(z)\frac{\partial}{\partial z}c_L(z)). \quad (4.1)$$

where j is an index over the array of TLEs. An effective Hamiltonian, where we effectively eliminate the field, only focusing on the atomic excitation, can be derived [13]. This is a particularly nice way of phrasing the problem at hand for a certain number of problems.

To determine the equations of motion of the system, we make use of the Heisenberg equation

$$\dot{A} = \frac{i}{\hbar}[H, A] \quad (4.2)$$

where A is an operator and H is the Hamiltonian of the system. 3.1 and 3.2 still hold, but with the addition that

$$[\sigma_{mn}^i, \sigma_{rs}^j] = 0 \text{ for } i \neq j \quad (4.3)$$

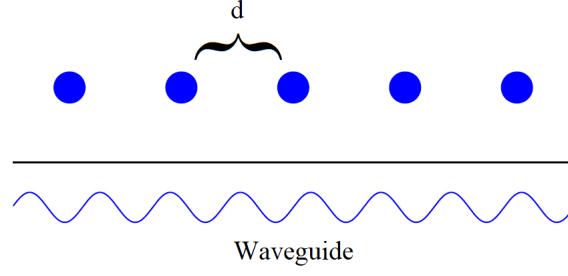


Figure 4.1: System considered. An infinite series of two level emitters with uniform spacing d coupled to a waveguide.

which leads to the equations of motion

$$\begin{aligned}
 \dot{\sigma}_{ge}^j(t) &= -i\omega\sigma_{ge}^j + \frac{i}{\hbar}(\sigma_{ee}^j - \sigma_{gg}^j)[V_R c_R(z_j) + V_L c_L(z_j)] \\
 \dot{c}_R(z, t) &= -v_g \frac{\partial}{\partial z} c_R(z) - \frac{iV_R}{\hbar} \sum_k \sigma_{ge}^k \delta(z - z_k) \\
 \dot{c}_L(z, t) &= +v_g \frac{\partial}{\partial z} c_L(z) - \frac{iV_L}{\hbar} \sum_k \sigma_{ge}^k \delta(z - z_k)
 \end{aligned} \tag{4.4}$$

the latter two equations can be rewritten as

$$\begin{aligned}
 \left(\frac{1}{v_g} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) c_R(z, t) &= -\frac{iV_R}{\hbar v_g} \sum_k \sigma_{ge}^k \delta(z - z_k) \\
 \left(\frac{1}{v_g} \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) c_L(z, t) &= -\frac{iV_L}{\hbar v_g} \sum_k \sigma_{ge}^k \delta(z - z_k)
 \end{aligned} \tag{4.5}$$

these equations can be solved exactly, with solutions

$$\begin{aligned}
 c_R(z, t) &= -\frac{iV_R}{\hbar v_g} \sum_k \theta(z - z_k) \sigma_{ge}^k \left(t - \frac{z - z_k}{v_g} \right) \\
 c_L(z, t) &= -\frac{iV_L}{\hbar v_g} \sum_k \theta(z_k - z) \sigma_{ge}^k \left(t - \frac{z_k - z}{v_g} \right)
 \end{aligned} \tag{4.6}$$

Inserting this into the equation of motion of σ yields a Hamiltonian of the form

$$H = -i \sum_i \left(\frac{\Gamma_R + \Gamma_L}{2} \sigma_i^\dagger \sigma_i + \Gamma_R \sum_{j < i} e^{ik_R|z_i - z_j|} \sigma_i^\dagger \sigma_j + \Gamma_L \sum_{i < j} e^{ik_L|z_i - z_j|} \sigma_i^\dagger \sigma_j \right) \tag{4.7}$$

Where i runs over all atoms. Here Γ_R and Γ_L , collectively $\Gamma_{tot} = \Gamma_R + \Gamma_L$, are the decay rates into right and left propagating modes respectively. We have set $\hbar = 1$ from here on.

4.2 Single Excitation

We now wish to study the eigenvalues of the Hamiltonian in the single excitation sector. To this end, we make the Ansatz that the eigenstate of this Hamiltonian in the single excitation subspace takes the form

$$|\psi\rangle = \sum_a e^{ikda} \sigma_a^\dagger |0\rangle \quad (4.8)$$

Clearly, this is not a normalized state, but it has translational invariance under a translation of d up to a phase, which is expected to be necessary for an eigenstate, as the Hamiltonian has this invariance. Letting the Hamiltonian act on the state yields

$$\begin{aligned} H|\psi\rangle = -i \sum_a \bigg(& \Gamma_R \sum_{j<i} e^{ik_R|z_i-z_j|+ikda} \sigma_i^\dagger \sigma_j \sigma_a^\dagger |0\rangle \\ & + \Gamma_L \sum_{i<j} e^{ik_L|z_i-z_j|+ikda} \sigma_i^\dagger \sigma_j \sigma_a^\dagger |0\rangle \\ & + \frac{\Gamma_R + \Gamma_L}{2} \sum_i e^{ikda} \sigma_i^\dagger \sigma_i \sigma_a^\dagger |0\rangle \bigg) \end{aligned} \quad (4.9)$$

Since we are working with two-level systems, σ is a fermionic operator, with the anti commutator relation $\{\sigma_i^\dagger, \sigma_j\} = \delta_{ij}$. Then we arrive at

$$\begin{aligned} H|\psi\rangle = -i \sum_i \bigg(& \Gamma_R \sum_{j<i} e^{ik_R d(i-j)+ikdj} \sigma_i^\dagger |0\rangle \\ & + \Gamma_L \sum_{i<j} e^{ik_L d(j-i)+ikdj} \sigma_i^\dagger |0\rangle \\ & + \frac{\Gamma_R + \Gamma_L}{2} e^{ikdi} \sigma_i^\dagger |0\rangle \bigg) \end{aligned} \quad (4.10)$$

Making the substitution $\tilde{j}_R = i - 1 - j$ for the first sum and $\tilde{j}_L = j - i - 1$ for the second yields

$$\begin{aligned}
H|\psi\rangle = -i \sum_i \left(\Gamma_R \sum_{\tilde{j}_R=0}^{\infty} e^{ikdi-i(k+k_R)d(1+\tilde{j}_R)} \sigma_i^\dagger |0\rangle \right. \\
+ \Gamma_L \sum_{\tilde{j}_L=0}^{\infty} e^{ikdi+i(k+k_L)d(1+\tilde{j}_L)} \sigma_i^\dagger |0\rangle \\
\left. + \frac{\Gamma_R + \Gamma_L}{2} e^{ikdi} \sigma_i^\dagger |0\rangle \right)
\end{aligned} \tag{4.11}$$

This then leads to

$$H|\psi\rangle = \omega|\psi\rangle = -i \left(\frac{\Gamma_R e^{-i(k-k_R)d}}{1 - e^{-i(k-k_R)d}} + \frac{\Gamma_L e^{i(k+k_L)d}}{1 - e^{i(k+k_L)d}} + \frac{\Gamma_R + \Gamma_L}{2} \right) |\psi\rangle. \tag{4.12}$$

Noting that

$$\text{Re} \left(\frac{e^{i\theta}}{1 - e^{i\theta}} \right) = -\frac{1}{2}, \tag{4.13}$$

we find that the imaginary part of ω is 0 for any choice of parameters. In total, we get

$$\omega = \frac{\Gamma_R \sin(k_0 - k)}{2(1 - \cos(k_0 - k))} + \frac{\Gamma_L \sin(k_0 + k)}{2(1 - \cos(k_0 + k))}. \tag{4.14}$$

This equation tells us that there is a divergence for $k = \pm k_0$. This divergence can be understood classically as the Bragg condition, where the light interferes destructively with itself. A concrete example of single excitation dispersion relation can be seen in figure 4.2.

From here on, we shall define $d = 1$ for brevity and notational ease.

4.2.1 Band Gap

As can be seen in figure 4.2, a band gap appears around $\omega = 0$. This band gap closes as the system becomes progressively more chiral, as can be seen in figure 4.4. To determine the size of the band gap, we first study the properties of the derivative of the energy,

$$\frac{\partial \omega}{\partial k} = \frac{\Gamma_R}{2(1 - \cos(k_0 - k))} - \frac{\Gamma_L}{2(1 - \cos(k_0 + k))} \tag{4.15}$$

so for $\frac{\partial \omega}{\partial k} = 0$, k must obey

$$\frac{\Gamma_R}{\Gamma_L} = \frac{1 - \cos(k_0 - k)}{1 - \cos(k_0 + k)}. \tag{4.16}$$

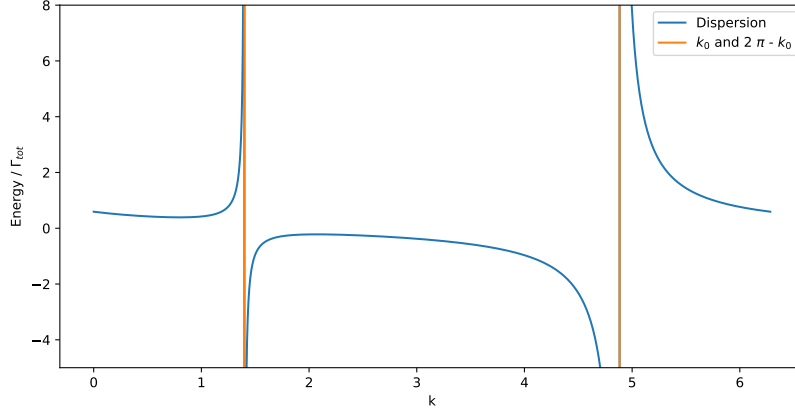


Figure 4.2: Dispersion in the single excitation sector for $k_0 = 1.4$ and $\Gamma_L/\Gamma_R = 9$.

In the limit where the waveguide couples weakly to the right-going modes, the above expression can be simplified by realizing that k is close to k_0 , we expand the cosine of the difference to second order and get

$$k = k_0 \pm \sqrt{\frac{2\Gamma_R}{\Gamma_L}(1 - \cos(2k_0))} = k_0 \pm \epsilon \quad (4.17)$$

Inserting this in equation 4.14 we get

$$\omega = \frac{\Gamma_R \sin(\mp\epsilon)}{2(1 - \cos(\mp\epsilon))} + \frac{\Gamma_L \sin(2k_0 \pm \epsilon)}{2(1 - \cos(2k_0 \pm \epsilon))} \quad (4.18)$$

To first order in ϵ this is

$$\omega_{\mp} = \frac{\Gamma_L \sin(2k_0)}{2(1 - \cos(2k_0))} \mp \sqrt{\frac{2\Gamma_R\Gamma_L}{1 - \cos(2k_0)}} \quad (4.19)$$

While it should apparently only hold true in the low Γ_R limit, it seems perfectly reliable even for $\Gamma_R/\Gamma_L = 1$, when compared to numerical values of the band gap, as can be seen in figure 4.3, a fact that asks for a better understanding of what exactly goes on. The size of the band gap is

$$\omega_{BG} = 2\sqrt{\frac{2\Gamma_R\Gamma_L}{1 - \cos(2k_0)}} \quad (4.20)$$

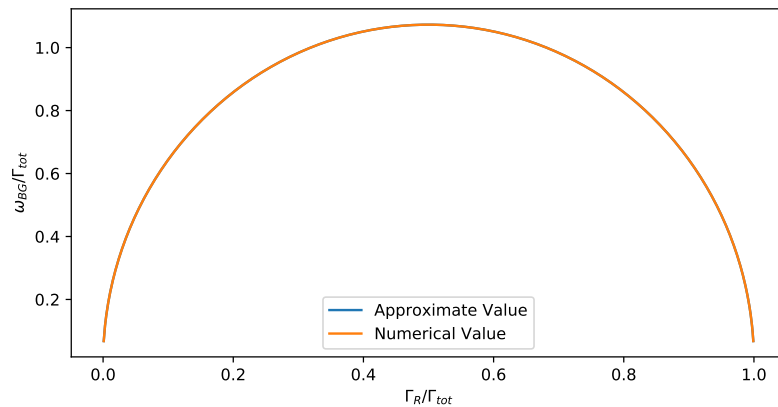


Figure 4.3: Comparison between numerical and approximate evaluation of band gap for single excitation (equation 4.20). As is apparent from the fact that only a single line is visible, the agreement is excellent.

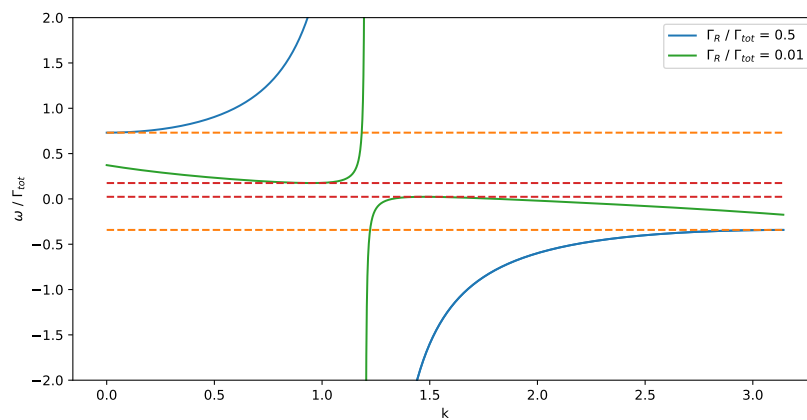


Figure 4.4: As the system becomes progressively more chiral, the band gap closes.

4.3 Double Excitation

Now that we understand the single excitation sector to a great degree, we are ready to tackle the problem of two excitations. Here we will finally be able to see interactions between the excitations, one of the goals of this thesis. In general, we have $[\sigma_i^\dagger, \sigma_j] = \delta_{ij}\sigma_j^z$, so it follows that

$$\sigma_i^\dagger \sigma_j \sigma_a^\dagger \sigma_b^\dagger = -\delta_{aj} \sigma_i^\dagger \sigma_a^z \sigma_b^\dagger - \delta_{bj} \sigma_i^\dagger \sigma_a^\dagger \sigma_b^z \quad (4.21)$$

at first sight, this looks strange, but note that the σ^z operator yields a minus sign for nonexcited emitters, so effectively, the above becomes

$$\sigma_i^\dagger \sigma_j \sigma_a^\dagger \sigma_b^\dagger = \delta_{aj} \sigma_i^\dagger \sigma_b^\dagger + \delta_{bj} \sigma_i^\dagger \sigma_a^\dagger \quad (4.22)$$

if a and b are different, and 0 if they are the same. Now, consider the double excitation state $\sigma_a^\dagger \sigma_b^\dagger |0\rangle$ where $a \neq b$,

$$\begin{aligned} H \sigma_a^\dagger \sigma_b^\dagger |0\rangle &= -i((\Gamma_R + \Gamma_L) \sigma_a^\dagger \sigma_b^\dagger \\ &+ \sum_i (\Gamma_R (e^{ik_0(i-a)} \sigma_i^\dagger \sigma_b^\dagger \theta(i-a) + e^{ik_0(i-b)} \sigma_i^\dagger \sigma_a^\dagger \theta(i-b)) \\ &+ \Gamma_L (e^{ik_0(a-i)} \sigma_i^\dagger \sigma_b^\dagger \theta(a-i) + e^{ik_0(b-i)} \sigma_i^\dagger \sigma_a^\dagger \theta(b-i))) |0\rangle \end{aligned} \quad (4.23)$$

Where $\theta(x)$ is the Heaviside function, where we define $\theta(0) = 0$.

In an effort to simplify the problem, we shall work with states that are well defined in their momentum per photon, K , and with a well defined spacing between the excitations, Δ . These states take the following form,

$$|K, \Delta\rangle = \sum_x e^{2iKx} \sigma_{x-\Delta/2}^\dagger \sigma_{x+\Delta/2}^\dagger |0\rangle \quad (4.24)$$

Note that x can be any positive integer divided by 2, to allow for excitations that have an odd number separation. We are also interested in using these states as we study two photon bound states, which are states where the excitations are localized near each other.

These states are orthonormal for fixed K ,

$$\langle K, \Delta' | K, \Delta \rangle = \delta_{\Delta, \Delta'} \quad (4.25)$$

Applying the Hamiltonian to $|K, \Delta\rangle$ yields

$$\begin{aligned}
H|K, \Delta\rangle = & -i(\Gamma_R + \Gamma_L)|K, \Delta\rangle \\
& -i \sum_x e^{2iKx} \sum_i (\Gamma_R(e^{ik_0(i-(x-\Delta/2))})\sigma_i^\dagger\sigma_{x+\Delta/2}^\dagger\theta(i-(x-\Delta/2)) \\
& + e^{ik_0(i-(x+\Delta/2))}\sigma_i^\dagger\sigma_{x-\Delta/2}^\dagger\theta(i-(x+\Delta/2))) \\
& + \Gamma_L(e^{ik_0(x-\Delta/2-i)}\sigma_i^\dagger\sigma_{x+\Delta/2}^\dagger\theta(x-\Delta/2-i) + e^{ik_0(x+\Delta/2-i)}\sigma_i^\dagger\sigma_{x-\Delta/2}^\dagger\theta(x+\Delta/2-i)))|0\rangle
\end{aligned} \tag{4.26}$$

The Heaviside functions arise when compared eliminating the sums over j . We make a change of variables to $x' = (a+b)/2$ and $\Delta' = |a-b|$. We then get

$$\begin{aligned}
H|K, \Delta\rangle = & -i(\Gamma_R + \Gamma_L)|K, \Delta\rangle \\
& -i \sum_{\Delta'} (\Gamma_R(e^{i(k_0-K)(\Delta+\Delta')}|K, \Delta'\rangle + e^{i(k_0-K)(\Delta'-\Delta)}|K, \Delta'\rangle\theta(\Delta'-\Delta)) \\
& + e^{i(k_0-K)(\Delta-\Delta')}|K, \Delta'\rangle\theta(\Delta-\Delta')) + \Gamma_L(e^{i(k_0+K)(\Delta'-\Delta)}|K, \Delta'\rangle\theta(\Delta'-\Delta) \\
& + e^{i(k_0+K)(\Delta'+\Delta)}|K, \Delta'\rangle + e^{i(k_0+K)(\Delta-\Delta')}|K, \Delta'\rangle\theta(\Delta-\Delta'))
\end{aligned} \tag{4.27}$$

where Δ' runs over one to infinity. We therefore find that the effective Hamiltonian can be written in the following form, $H_{\text{eff}}^K = \sum_{\Delta, \Delta'} \mathcal{H}_{\Delta\Delta'}^K |\Delta\rangle\langle\Delta'|$ with

$$\mathcal{H}_{\Delta, \Delta'}^K = -i\Gamma_R \sum_{\varepsilon=\pm 1} e^{i(k_0-K)|\Delta+\varepsilon\Delta'|} - i\Gamma_L \sum_{\varepsilon=\pm 1} e^{i(k_0+K)|\Delta+\varepsilon\Delta'|} \tag{4.28}$$

4.3.1 Solving for Bound State as an Eigenvalue Problem

Now that we have the Hamiltonian, we are able to make a truncation of the Hamiltonian to only a relatively small number of $\Delta \sim 100$, in order to derive the photon bound state of the system. This is in order to make the problem solvable numerically. The reason this works is that the bound state must almost exclusively consists of the first few $|K, \Delta\rangle$ states, as it would otherwise be neither bound nor normalizable. Therefore the high Δ states will effectively be thrown away, as they are zero anyway.

The resulting matrix is diagonalized. For a matrix of size $N \times N$ this yields N eigenvectors, or eigenstates, each with an associated eigenvalue, or eigenenergy. The bound states are determined from the N solutions by taking the state that has the lowest expectation value of excitation separation, $\langle\Delta\rangle$. The resulting dispersion relation can be seen in Figures 4.6 and 4.7.

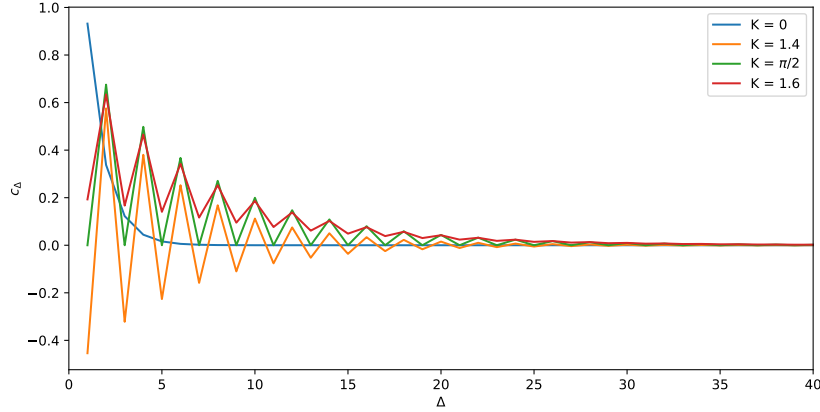


Figure 4.5: Eigenstates for a number of different values of K . Here the eigenstates are for a non-chiral system with $k_0 = 1.2$. Note that the values are real, which we would not necessarily expect.

Doing this reveals that there only exists bound states with imaginary value 0 for $K = 0$ and $k_0 < K < \pi - k_0$ as can be seen in figures 4.6 and 4.7. An imaginary value different from 0 indicates that the state decays, which we for the moment are not interested in. In principal the problem could be that a too small matrix has been chosen, but an increase in the size of our matrix does not seem to make any change to this stable region. That this is not simply a flaw in the code, but a result that can be motivated on physical grounds, will be discussed in the following section.

The eigenstates take on quite different forms. The bound state at $K = 0$ takes on a very nice shape as it is exponentially decreasing in Δ , whereas the states in stable region take on a more aggressive shape, while also decreasing in a somewhat exponential way. This is illustrated in figure 4.5.

4.3.2 Why the limited Region?

Why should it only be in the region $\pi - k_0 > K > k_0$ be the only one where there are true bound states? Part of the explanation may be found by looking at what may be termed the free states. The free state is comprised of two of the single excitation eigenstates, with wave number k_1 and k_2 , where $k_1 + k_2 = 2K$. We shall assume that $k_1 \geq K$ and $k_2 \leq K$ without loss of generality. What is special about the stable regions? The analytical dispersion relation of the single excitation sector has been found in the

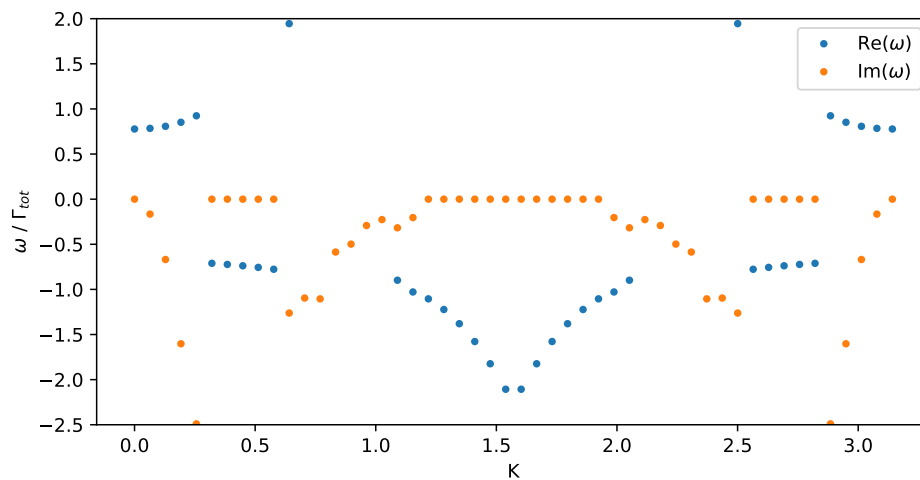
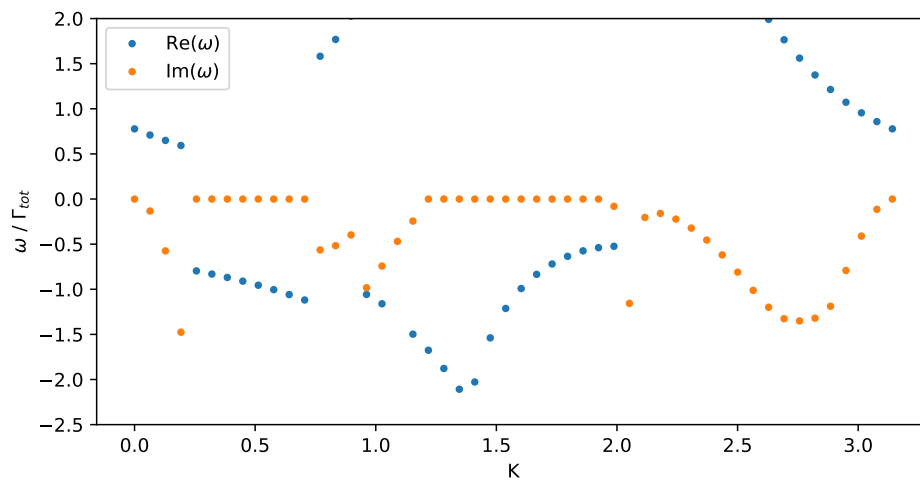


Figure 4.6: Eigenvalues for the bound states in the non-chiral regime.

Figure 4.7: Eigenvalues for the bound states in the partially chiral regime, $\Gamma_R/\Gamma_{tot} = 0.25$.

previous section, and it was found that there is an upper and lower excitation band, which is important, as the free states will come in three types: one with both excitations from the lower band, one with both excitations in the upper band, and one with an excitation from each.

Two lower band Excitations

As the lower band has a maximum value, but no lowest possible value, this type of excitation forms a band that has an uppermost value and goes to minus infinity.

Two upper band Excitations

As the upper band has a minimum value, but no greatest possible value, this type of excitation forms a band that has a lowermost value and goes to infinity.

One from each

The most interesting case is when the two excitations come from different bands. A very special case is that of $K = 0$. Here $k_1 = -k_2$ so there are in fact no excitations that come from different bands. As such the sum of the two excitations approaches infinity as $k_1 \rightarrow k_0$, then they make a discontinuous flip to minus infinity as k_1 just passes k_0 . This then leads to a band gap in the two free excitations spectrum. This is in agreement with the fact that there exists a stable bound state at exactly $K = 0$ and in fact its energy is within this band gap.

As soon as K changes even ever so slightly this symmetry disappears. Let us assume that $K > 0$ and small compared to k_0 . As $k_1 \rightarrow k_0$ the energy diverges towards infinity. As k_1 passed k_0 the energy makes a discontinuous jump to minus infinity. However, k_2 is now approaching $-k_0$, so in a very short range, the energy goes all the way from minus infinity to infinity. This tells us that there is no band gap and therefore we do not expect there to be any stable bound states. However, for very small values of K , the range of k_1 and k_2 that cover all energies is small, leading to a small density of state. We therefore expect the bound states to be long lived which translates into a small imaginary value of the eigenstate. Indeed, this is what we find, as the imaginary value behaves quadratic in K for small K .

The other interesting case is that of $\pi - k_0 > K > k_0$. Here there is a flip from minus infinity to infinity, but the next flip is from infinity to minus infinity, which means that we do not continuously scan over all possible energy values. There is, in other words, a

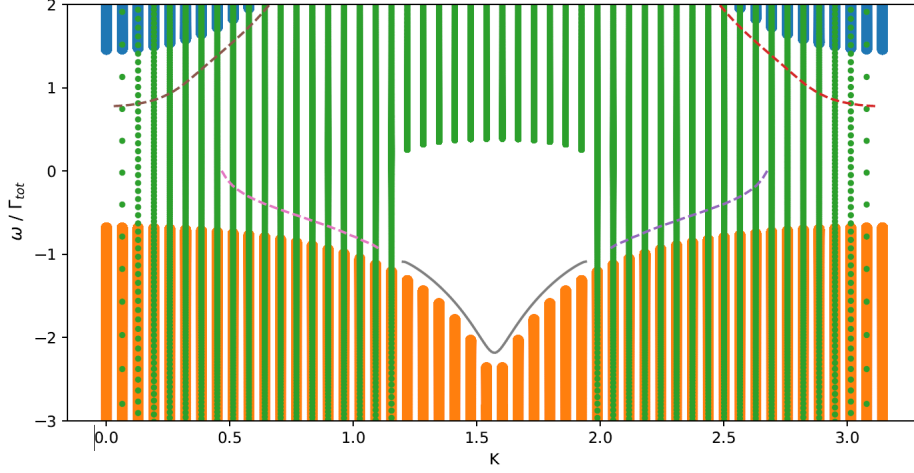


Figure 4.8: Figure showing the band gap arising in the two single excitation sector. Blue indicates two excitations from upper branch, orange two from lower and green one from each. The line in the middle shows the analytical solutions of energy in the bound region as described in section 4.4. Indeed, this is in the gap. The dotted lines in the unstable regions are also described in this section.

band gap in the possible energies for these values of K , which suggests that it is one can find stable bound states with these energies. This can be seen in figure 4.8.

4.4 Analytical Solution

Guided by the results of the numerical solution, it seems plausible that the eigenstates of the Hamiltonian are exponentially decaying in their Δ -occupation. We therefore apply the Ansatz that

$$|z\rangle = \sum_{\Delta=1}^{\infty} z^{\Delta} |\Delta\rangle \quad (4.29)$$

where we leave the K implied. In order for the state to be normalizable, we must have $|z| \leq 1$. We shall begin by considering what happens when the Hamiltonian in 4.28 is multiplied with this Ansatz function. The result is

$$H_{\text{eff}}|z\rangle = \sum_{\Delta\Delta'\Delta''} \mathcal{H}_{\Delta',\Delta''} z^{\Delta} |\Delta'\rangle \langle\Delta''|\Delta\rangle = \sum_{\Delta\Delta'} \mathcal{H}_{\Delta',\Delta} z^{\Delta} |\Delta'\rangle \quad (4.30)$$

where we drop the superscript K , leaving it implied in the rest of the section. Considering first the effect of the first term in 4.28 we have

$$-i\Gamma_R \sum_{\Delta\Delta'} \sum_{\varepsilon=\pm 1} z_-^{|\Delta+\varepsilon\Delta'|} z^\Delta |\Delta'\rangle = -i\Gamma_R \sum_{\Delta'} \sum_{\Delta} (z_-^{|\Delta+\Delta'|} + z_-^{|\Delta-\Delta'|}) z^\Delta |\Delta'\rangle \quad (4.31)$$

where $z_\pm = e^{i(k_0 \pm K)}$. We then focus on the sum over Δ . In order to take the absolute value in the exponent into account, we break up into a sum from 1 to Δ' and one from $\Delta' + 1$ to infinity

$$\begin{aligned} & \sum_{\Delta=1}^{\infty} (zz_-)^\Delta z_-^{\Delta'} + \sum_{\Delta=1}^{\Delta'} \left(\frac{z}{z_-}\right)^\Delta z_-^{\Delta'} + \sum_{\Delta=\Delta'+1}^{\infty} (zz_-)^\Delta z_-^{-\Delta'} \\ &= \frac{z_-^{\Delta'}}{(1/zz_-) - 1} + \frac{z_-^{\Delta'}}{1 - (z/z_-)} + \frac{z^{\Delta'}}{1 - (z_-/z)} + \frac{z^{\Delta'}}{(1/zz_-) - 1} \end{aligned} \quad (4.32)$$

so in total 4.31 reads

$$-i\Gamma_R \left(\frac{1}{1 - (z_-/z)} + \frac{1}{(1/zz_-) - 1} \right) |z\rangle + g(z) |z_-\rangle. \quad (4.33)$$

Including the second term in 4.28 then yields

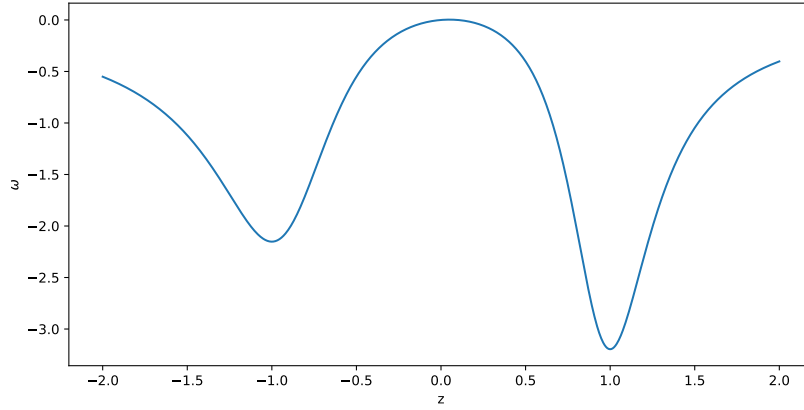
$$H_{\text{eff}}|z\rangle = \omega(z)|z\rangle + g_+(z)|z_+\rangle + g_-(z)|z_-\rangle \quad (4.34)$$

here

$$\begin{aligned} \omega(z) &= -i \left(\frac{\Gamma_L}{\frac{1}{zz_+} - 1} + \frac{\Gamma_L}{1 - \frac{z_+}{z}} + \frac{\Gamma_R}{\frac{1}{zz_-} - 1} + \frac{\Gamma_R}{1 - \frac{z_-}{z}} \right) \\ g_+(z) &= -i\Gamma_L \left(\frac{1}{\frac{1}{zz_+} - 1} + \frac{1}{\frac{z_+}{z} - 1} \right) \\ g_-(z) &= -i\Gamma_R \left(\frac{1}{\frac{1}{zz_-} - 1} + \frac{1}{\frac{z_-}{z} - 1} \right) \end{aligned} \quad (4.35)$$

or, written in a sometimes more transparent way,

$$\begin{aligned} \omega(z) &= \frac{2z\Gamma_L \sin(k_0 + K)}{1 + z^2 - 2z \cos(k_0 + K)} + \frac{2z\Gamma_R \sin(k_0 - K)}{1 + z^2 - 2z \cos(k_0 - K)} \\ g_+(z) &= 2iz\Gamma_L \frac{z - \cos(k_0 + K)}{1 + z^2 - 2z \cos(k_0 + K)} \\ g_-(z) &= 2iz\Gamma_R \frac{z - \cos(k_0 - K)}{1 + z^2 - 2z \cos(k_0 - K)} \end{aligned} \quad (4.36)$$

Figure 4.9: ω with up to four possible solutions

These states have the property that they are related to eigenstates, but on their own they are not. However, $\omega(z)$ is in general degenerate for a given value as seen in figure 4.9. In particular, the states that we would want are states with $z < 1$, as these are truly bound states. The objective is then to take a number of different states $|z\rangle$, and piece these together in such a way that the coefficients of $|z_{\pm}\rangle$ cancel, and what we are left with is a true eigenstate of the Hamiltonian. However, we shall first discuss some special cases, where it turns out that it is particularly easy to construct the analytical solution.

4.4.1 The Special Case $K = 0$

We have already found that $K = 0$ is special in that it is a point at which there is always a bound state. We note that for $K = 0$ it is the case that $z_+ = z_- = e^{ik_0}$. Then $g_+ \propto g_-$, so finding a value of z that makes $g_{\pm} = 0$ is unusually easy. To be concrete, we choose $g_+ = 0$, yielding the equation

$$\frac{1}{\frac{1}{ze^{ik_0}} - 1} = \frac{-1}{\frac{e^{ik_0}}{z} - 1} \quad (4.37)$$

This equation technically has two solutions. One is $z = 0$, which we can discard because it does not allow for a physically realizable state. The other, which is important, is

$$z = \cos(k_0). \quad (4.38)$$

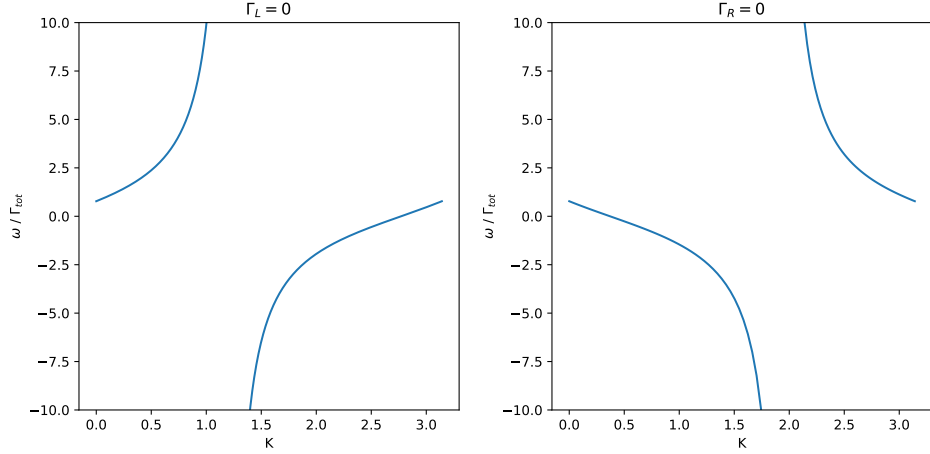


Figure 4.10: Comparison between the two extreme cases of chirality. Not surprisingly, these are mirror images of one another.

So we are able to express both energy and state in a closed form for any choice of parameters, except $k_0 = 0, \pi/2, \pi$. The eigenstate is therefore always exponentially decreasing in Δ , as would have to be the case for a bound state.

4.4.2 The Special Case of Chiral Waveguide

For the Chiral system it is the case that either g_+ or g_- is 0, so here it is also quite simple to satisfy the bi-condition. The procedure is then very similar to what was done for $K = 0$ and the result is

$$z = \cos(k_0 \pm K) \quad (4.39)$$

where it is a plus in the case of $\Gamma_R = 0$ and a minus in the case of Γ_L . The resulting dispersion relation for each of the two cases can be seen in fig. 4.10

This result seems to be upsetting. After, we have just spend the last pages convincing ourselves that stable bound states are only found within a special region of K , but the above result seems to suggest that in the chiral case, any K will do. The solution to this riddle is found in the decay rate of the unstable bound states.

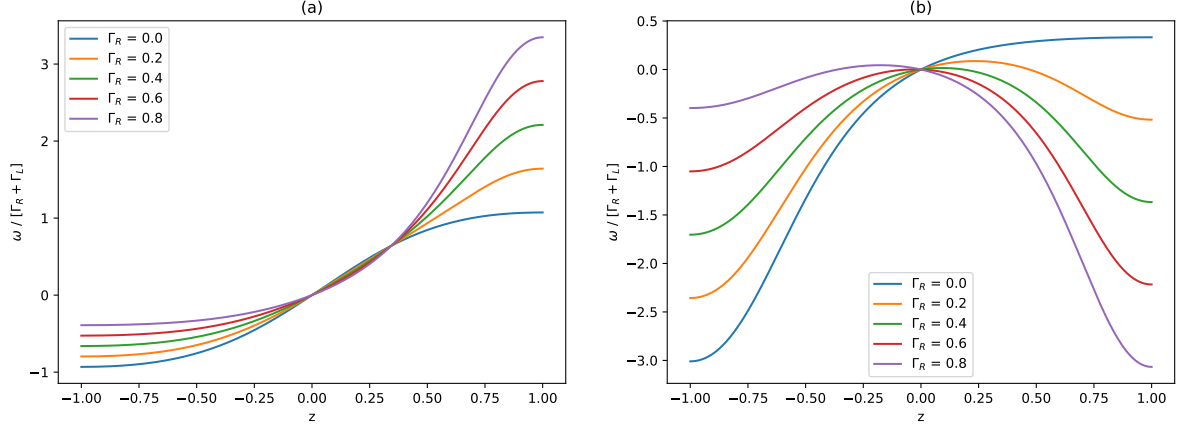


Figure 4.11: Energy spectra $\omega(z)$ for changing level of chirality. (a) shows the case in the unstable region with $k_0 = 1, K = 0.5$. As can be seen, there are no degenerate energies in the unstable region. (b) shows the case in the stable region with $k_0 = 1, K = 1.5$. In this case, all levels of chirality give rise to degenerate energies, except the completely chiral.

4.4.3 Gluing together an Eigenstate - the Stable Case

In the stable region, between k_0 and $\pi - k_0$ we have found that there exists a true bound state for any K . Within the region, it is also the case that a range of values of $\omega(z)$ are degenerate in z , with both values having the property that $|z| \leq 1$ (see figure 4.11). The value of z for a given ω can be found as a solution to a quartic equation, as explained in B. Denoting these values of z as z_1 and z_2 , an approach to finding the Eigenstate is to pick the state

$$|\psi\rangle = A|z_1\rangle + B|z_2\rangle \quad (4.40)$$

by then applying the Hamiltonian we readily get

$$H_{\text{eff}}|\psi\rangle = \omega|\psi\rangle + (Ag_+(z_1) + Bg_+(z_2))|z_+\rangle + (Ag_-(z_1) + Bg_-(z_2))|z_-\rangle \quad (4.41)$$

So the problem at hand is picking the right energy and the right ratio between A and B to satisfy the equations

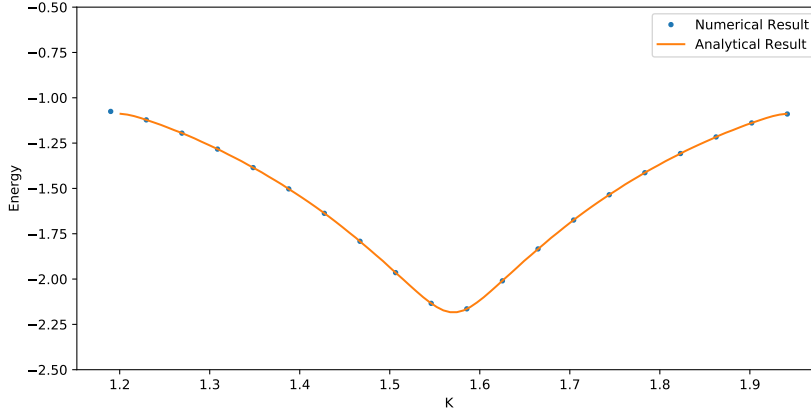


Figure 4.12: Comparison of numerical and analytical results for the case of a nonchiral system

$$\begin{aligned}
 \omega(z_1) - \omega(z_2) &= 0 \\
 Ag_+(z_1) + Bg_+(z_2) &= 0 \\
 Ag_-(z_1) + Bg_-(z_2) &= 0
 \end{aligned} \tag{4.42}$$

We Assume that A is different from zero and are therefore able to divide the last two equations by A and define a new quantity $R = B/A$. Isolating it in the first of the two equations, we get $R = -\frac{g_+(z_1)}{g_+(z_2)}$. It is then possible to reduce the problem to a single equations by substituting this R into the last equation, which yields

$$g_-(z_1) - \frac{g_+(z_1)g_-(z_2)}{g_+(z_2)} = 0 \tag{4.43}$$

so the problem has been reduced to finding the solution of a single equation in a single parameter, ω . It is then a simple matter to scan over the possible values of ω and find the value of that satisfies the above relation. We find that the results are then in very good agreement with the numerical results derived earlier as can be seen in Figures 4.12 and 4.13.

4.4.4 Gluing together an Eigenstate - the Unstable Case

In the unstable region there are no degenerate solutions to $\omega(z)$ for $|z| < 1$. We shall denote the single solution b . However, solutions still exist and can be combined with two

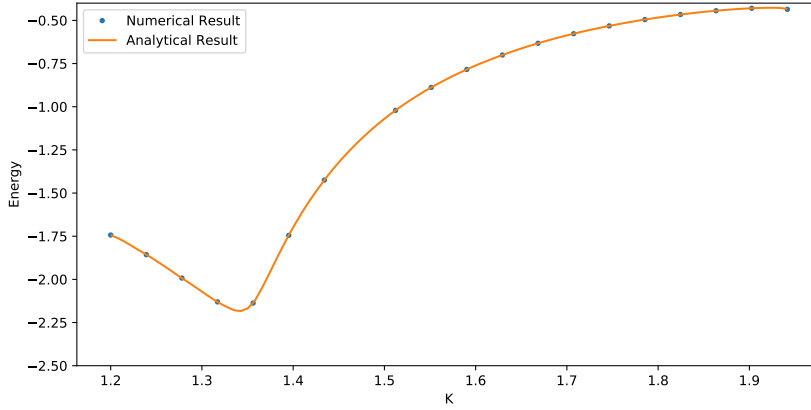


Figure 4.13: Comparison of numerical and analytical results for the case of a system with $\Gamma_L/\Gamma_R = 4$.

other solutions of the form q and q^* (that is, the two solution are complex conjugates of one another and with norm 1). This then leads us to suggest an eigenstate in the form of

$$|\psi\rangle = |b\rangle_N + A|q\rangle + B|q^*\rangle \quad (4.44)$$

where $|b\rangle_N = \sqrt{(1-b^2)/b^2}|b\rangle$ is the normalized $|b\rangle$. These states are different from the stable states in two important ways. First, they are not normalizable and therefore they are not physically realizable states. Second, they have solutions for an entire continuum of values of ω .

A similar procedure as that of the stable case leads to the two equations

$$\begin{aligned} \sqrt{\frac{1-b^2}{b^2}}g_+(b) + Ag_+(q) + Bg_+(q^*) &= 0 \\ \sqrt{\frac{1-b^2}{b^2}}g_-(b) + Ag_-(q) + Bg_-(q^*) &= 0 \end{aligned} \quad (4.45)$$

where the coefficients A and B can readily be found. These coefficients then give us a measure of the "boundedness" of a bound state with a given energy. We then study the behavior of $C = 1/(|A|^2 + |B|^2)$. In general we find that this quantity diverges at the minimal and maximal energy for the possible bound states, where $z = \pm 1$. As this would imply a bound state of infinite range, we discard the possibility of a localized

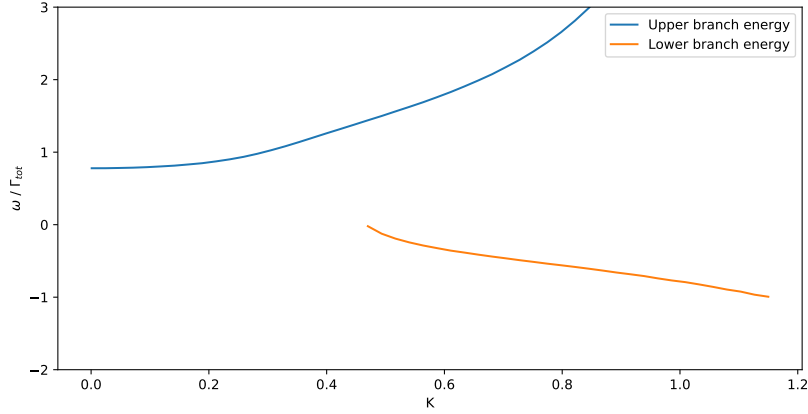


Figure 4.14: Energy of two branches derived by finding the peak of resonances. As shown, the lower branch appears “out of the blue”. Here $k_0 = 1.2$ and for nonchiral system.

bound state at these values. More interesting is the fact that in between these values we find either one or two resonance peaks. This fact is illustrated in 4.14.

The lower branch only emerges for certain values of K . As can be seen, defining a decay rate as the width of the resonance curve does not make much sense as one can hardly define a width, whether it be FWHM or a Gaussian fit. We therefore have to accept a level of ambiguity in this case.

4.4.5 Phase Shift of Resonances

In general, the scattering of a bound state is associated with a phase change. We therefore study the behavior of the phase of the coefficients A and B . Indeed, we find an associated phase shift, as illustrated in figure 4.17.

4.4.6 Convergence of Resonances

We find that in the vicinity of the analytically solved values of K (that is, for $K = 0$ and $k_0 < K < \pi - k_0$) there is a resonance with a peak converging these analytical values. This behavior is demonstrated in figure 4.18

We suspect the peak to converge with that of the nonchiral case, when we choose a sufficiently small Γ_R or Γ_L . We first consider the case of small Γ_R in the region $0 < K < k_0$ as the results from the completely nonchiral system exhibits no divergent

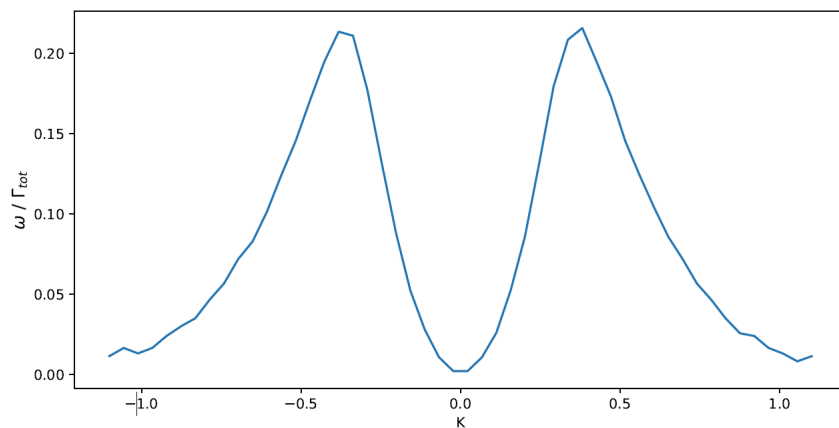


Figure 4.15: The width of the upper branch, showing quadratic behavior at $K = 0$. $\Gamma_R = \Gamma_L$ and $k_0 = 1.2$

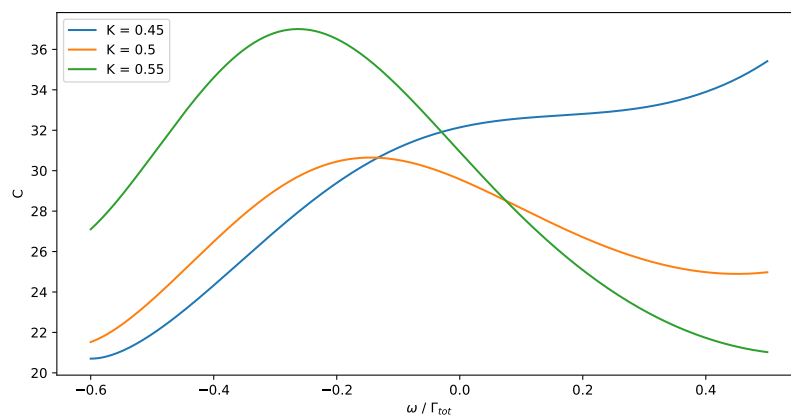


Figure 4.16: The lower branch peak appearing as K is tuned up. Notice that the upper branch for $K = 0.45$ has a much greater value of C at the right end of the figure, than the other two curves. This is the tail of the upper branch energy.

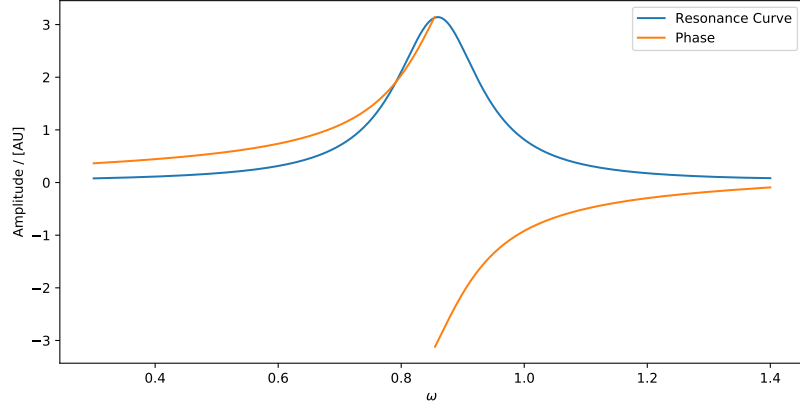


Figure 4.17: Resonances and phase plotted. As can be seen, the phase switches over at the peak of the resonance, suggesting a scattering process.

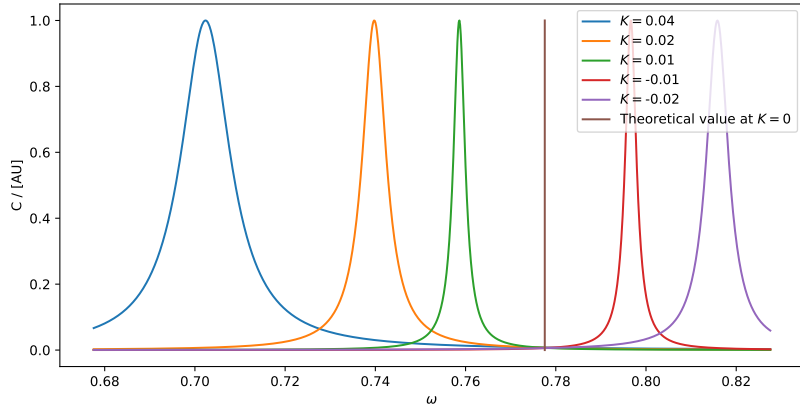


Figure 4.18: Resonances in the vicinity of $K = 0$. $\Gamma_L/\Gamma_R = 4$, $k_0 = 1.2$. As K approaches 0 the peak of the resonance converges towards the theoretical value of $K = 0$. The width of the resonance decreases linearly with K , reflecting a longer lifetime. All curves have been normalized to peak value 1 for convenience.

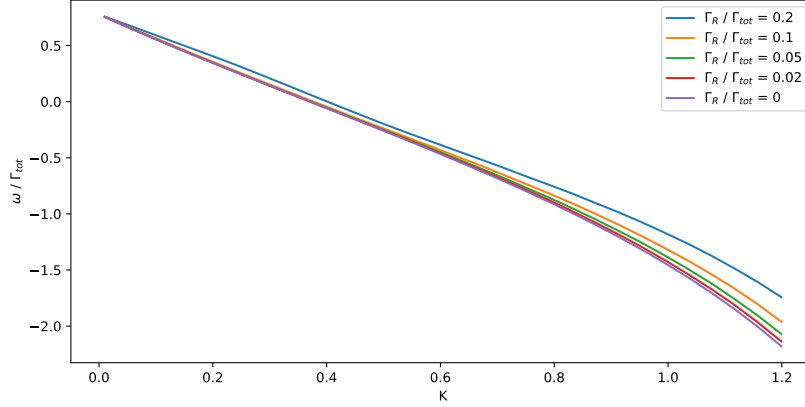


Figure 4.19: Dispersion relation in the unstable region. Dispersion derived by choosing peak value of resonance curve. The behavior of the dispersion curve converges that of the completely chiral. The bottom curve is simply a part of the curve show in 4.10. Here $k_0 = 1.2$

behavior in this region, as can be seen in figure 4.10. Figure 4.19 shows that there is indeed a convergent behavior in this region. This is completely analogue to the case of small Γ_L in the region $\pi - k_0 < K < \pi$.

4.4.7 Behavior of ω in Border Region

If we study the behavior of

$$\omega(z) = \frac{2z\Gamma_L \sin(k_0 + K)}{1 + z^2 - 2z \cos(k_0 + K)} + \frac{2z\Gamma_R \sin(k_0 - K)}{1 + z^2 - 2z \cos(k_0 - K)} \quad (4.46)$$

in the crossover region between stable and unstable bound states. First we define $k_0 - K = \delta K$ and study the behavior of ω for δK small but positive. Here we find that for the first term on the RHS of 4.46 tends towards a well defined continuous analytical function

$$\frac{2z\Gamma_L \sin(2k_0)}{1 + z^2 - 2z \cos(2k_0)} \quad (4.47)$$

However, expanding the sine and cosine of the second term to second order in δK yields

$$\frac{2z\Gamma_R \delta K}{1 + z^2 - 2z + \delta K^2} = \frac{2z\Gamma_R \delta K}{(z - 1)^2 + \delta K^2} \quad (4.48)$$

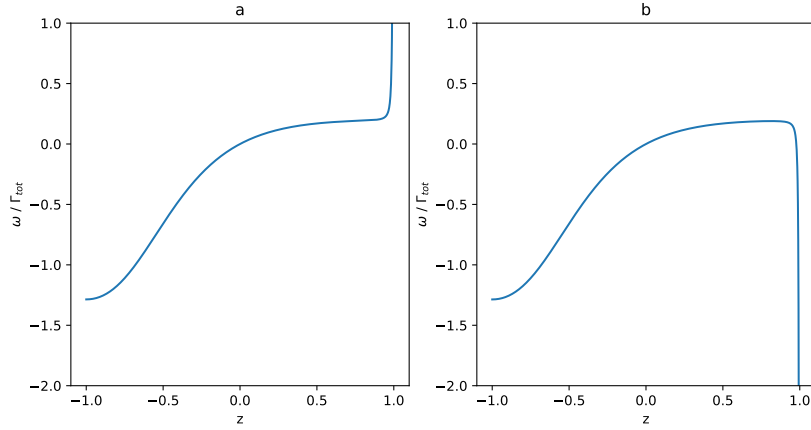


Figure 4.20: (a): Behavior of ω for K slightly smaller than k_0 . (b): Behavior of ω for K slightly bigger than k_0 . Notice that they are identical except for the wildly different behavior near $z = 1$.

which is heavily suppressed in comparison the the first term, except at $z = 1$, where it diverges. While this divergence is canceled when one expands to second order in δK , provided it is not exactly zero, we still see a very violent peak produced by this term right around $z = 1$, which then allows for practically any upper value of ω in the unstable region, provided that δK is small enough. When δK is negative, we find ourselves in the stable region. Here the peak also exists, but it is negative. A crude of ω right at k_0 in the stable region is then the curve described by 4.47 with a straight line cutting down at $z = 1$. This explains why there are always two solutions inside the stable region where $|z| < 1$, but only one outside. This is illustrated in figure 4.20

Motivated by wanting the g_+ and g_- terms to cancel, we attempt to make $g_+ = 0$ at $K = k_0$. This can be done by choosing $z = \cos(2k_0)$. Surprisingly, this turns out to give the right answer energy when compared with the results, though the reason is not entirely clear. The energy at $K = k_0$ is

$$\omega = 2\Gamma_L \cot(2k_0) \quad (4.49)$$

An example of this is shown in figure 4.21.

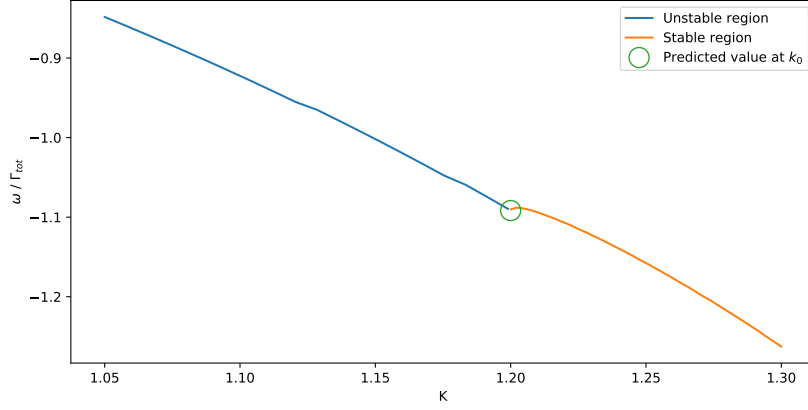


Figure 4.21: Dispersion around $K = k_0$. Here $k_0 = 1.2$ and the system is nonchiral. Note that while the line is indeed continuous, the derivative is not. This is not entirely surprising, as some phase change is taking place in this region.

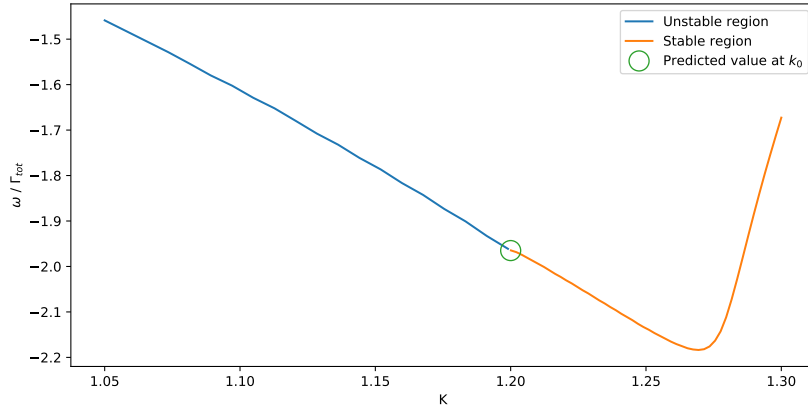


Figure 4.22: Dispersion around $K = k_0$. Here $k_0 = 1.2$ and $\Gamma_R/\Gamma_{\text{tot}} = 0.1$. Note that in this case, the curve at this points is more smooth, compared with the nonchiral case.

Chapter 5

Conclusion

5.1 Chapter 3

We have establish that it is possible to construct scattering matrices for one photon scattering processes. While we in this thesis have mainly used this to study two “extreme” cases, that of the single TLE and that of an infinite array of identical TLEs, it could in principle be applied to an arbitrary array of arbitrary TLEs.

5.2 Chapter 4

We have made a comprehensive study of the behavior of bound states in partially chiral system. We have found that there are stable states with well defined energies as well as unstable states that show up as resonances in the energy spectrum. These resonances continuously join the energy of the stable states as we vary K . We further interpreted the width of these resonance curves to be the decay rate of the states, with the width of the resonance curve tending to zero as K tends towards a stable region.

Chapter 6

Outlook

We have found some very interesting properties of the two photon bound state. However, corners will always be cut and results left hanging. A short discussion of future vistas is in order.

The section will be dedicated to these vistas.

6.1 Two Photon transport

The following is some preliminary work that was done on the two photon scattering matrix problem discussed briefly in chapter 3. It is possible to make a unitary transformation of the operators, which in effect decouples one of the modes from the atom, which is

$$c_e^\dagger(z) = \frac{V_R c_R^\dagger(z) + V_L c_L^\dagger(-z)}{\sqrt{V_R^2 + V_L^2}} \quad (6.1)$$

$$c_o^\dagger(z) = \frac{V_L c_R^\dagger(z) - V_R c_L^\dagger(-z)}{\sqrt{V_R^2 + V_L^2}} \quad (6.2)$$

the Hamiltonian is then identical to the one in [15], with the exception of $V = \sqrt{V_R^2 + V_L^2}$, and all the rest of the machinery moves right over.

$$c_R^\dagger(z) = \frac{V_R c_e^\dagger(z) + V_L c_o^\dagger(z)}{\sqrt{V_R^2 + V_L^2}} \quad (6.3)$$

$$c_L^\dagger(z) = \frac{V_L c_e^\dagger(-z) - V_R c_o^\dagger(-z)}{\sqrt{V_R^2 + V_L^2}} \quad (6.4)$$

Take the case of an incoming two-photon state

$$|in\rangle = |k, p\rangle_{RR} = \int \int dz dz' (e^{ikz+ipz'} + e^{ikz'+ipz}) c_R^\dagger(z) c_R^\dagger(z') |0, -\rangle \quad (6.5)$$

then the double operator can be rewritten in terms of the decoupled operators

$$c_R^\dagger(z) c_R^\dagger(z') = \frac{V_R^2 c_e^\dagger(z) c_e^\dagger(z') + V_L^2 c_o^\dagger(z) c_o^\dagger(z') + V_R V_L (c_e^\dagger(z) c_o^\dagger(z') + c_o^\dagger(z) c_e^\dagger(z'))}{V_R^2 + V_L^2} \quad (6.6)$$

The double c_e term is then, after scattering

$$t_k t_p |k, p\rangle_{ee} + \int \int dk' dp' \delta(E_1 - E_2) B |k', p'\rangle_{ee} \quad (6.7)$$

with B the same as in [15]. The double c_o term is left unchanged, and the $c_e c_o$ terms are then

$$\int \int dz dz' (t_k e^{ikz+ipz'} + t_p e^{ikz'+ipz}) c_e^\dagger(z) c_o^\dagger(z') |0, -\rangle \quad (6.8)$$

It will be interesting to see if this approach yield results in the future.

We would like this in particular, as it seems to be a much more direct way to approach the problem for making prediction in the laboratory, which, in some way, is what physics is all about.

6.2 One, Two, Many... Dimensions and Photons

This thesis has had the 1D waveguide as its focus. There should, however, not be any problems with expanding the work to more dimensions. For practical considerations, three dimensions suffices. Now K would have to be a vector adding to the complexity of the problem. However, a numerical approach should be well within the grasp of our current method, with a scaling in time that, while great now that one would have to solve a $N^2 \times N^2$ matrix in two dimensions, is not insurmountable with the current programming approach.

A different interesting feature of our system to expand is the increase of photons, or excitations, in our system. Here the main obstacle seems to find a proper way of reducing the problem to a single body system. In the thesis the Δ states had this property.

6.3 Completing the (Complex) Circle for Resonances

In the main thesis, we only considered resonances for values of z on the real axis from -1 to 1 . There is, however, nothing in principle wrong with having a complex z , so long as $|z| < 1$. Exploring this could very well yield different resonances than the ones that are discussed in this thesis.

6.4 Decay out of Waveguide

The thesis has considered a very idealized system, where any photon in our waveguide stays in the waveguide. Expanding the model by allowing for photons to decay out of the waveguide would of course be a logical step, allowing for a more general physical setup.

6.5 Other Kinds of Bound States?

While we have argued that there are indeed photon bound states allowed in our system, our approach hinges on the use of the $|K, \Delta\rangle$ states. We have, however, not shown that these are the only kinds of bound states that can occur in such a system.

6.6 Nailing down the Decay Rate

While we have a clear expectation that the width of the resonance curve is proportional to the decay rate of the state. The behavior is very similar to that of a localized state coupled to a continuum [?]

$$H = \epsilon_{loc} c_{loc}^\dagger c_{loc} + \sum_k \epsilon_k c_k^\dagger c_k + \sum_k g_k c_k^\dagger c_{loc} + H.c \quad (6.9)$$

Somehow finding a way of rewriting our Hamiltonian in this form would therefore be a great step toward putting our model on more sturdy theoretical ground.

Appendix A

The real ω and imaginary $g_{\pm}$

It is postulated that

$$\omega(z) = -i \left(\frac{\Gamma_L}{\frac{1}{zz_+} - 1} + \frac{\Gamma_L}{1 - \frac{z_+}{z}} + \frac{\Gamma_R}{\frac{1}{zz_-} - 1} + \frac{\Gamma_R}{1 - \frac{z_-}{z}} \right) \quad (\text{A.1})$$

is real. Assuming that z is real it is a simple matter to show that $\omega(z)$ must in fact be real. Taking the complex conjugate amounts to changing the overall sign due to the i in front, and letting $z_{\pm} \rightarrow 1/z_{\pm}$. This last change makes the first and second term on the right hand side as well as the third and fourth term change into each with a switch in sign that cancels the overall sign from the i . So $\omega(z) = \omega(z)^*$ and must therefore be real.

Similarly for $g_{\pm}(z)$

$$\begin{aligned} g_+(z) &= -i\Gamma_L \left(\frac{1}{\frac{1}{zz_+} - 1} + \frac{1}{\frac{z_+}{z} - 1} \right) \\ g_-(z) &= -i\Gamma_R \left(\frac{1}{\frac{1}{zz_-} - 1} + \frac{1}{\frac{z_-}{z} - 1} \right) \end{aligned} \quad (\text{A.2})$$

These equations are left unchanged by $z_{\pm} \rightarrow 1/z_{\pm}$, so the change of sign due to i is not canceled. This leads to $g_{\pm}(z) = -g_{\pm}(z)^*$ and must therefore be imaginary.

Appendix B

Solving ω as a Quartic Equation

The standard quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \quad (\text{B.1})$$

is known to have four solutions in x for any choice of coefficients. We are able to recast A.1 as a quartic equation by multiplying both sides by $(1 - zz_+)(1 - zz_-)(1 - z/z_+)(1 - z/z_-)$. The resulting coefficients have the values

$$\begin{aligned} a &= \omega \\ b &= -2(\omega(c_+ + c_-) + \Gamma_L s_+ + \Gamma_R s_-) \\ c &= 2(\omega(1 + 2c_+c_-) + 2(\Gamma_L s_+c_- + \Gamma_R s_-c_+)) \\ d &= b \\ e &= a \end{aligned} \quad (\text{B.2})$$

where $s_{\pm} = \sin(k_0 \pm K)$ and $c_{\pm} = \cos(k_0 \pm K)$. The solutions, while not very pretty, are well known and have been for almost four centuries.

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