# Chiral photon interaction mediated by a superconducting circuit QED 

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Master's Thesis
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## Abstract

Chiral quantum optics and superconducting circuits have lately risen the attention within the quantum information research field. Indeed, providing a system with a directional dependent response, together with the high precision and reliability from superconducting qubits, potential quantum information achievements can be harnessed. In this work we suggest a theoretical proposal which predicts high fidelity photon chiral interactions and decays brought by a time reversal symmetry breaking induced by a coupled microwave circulator to a waveguide. Based on the theoretical predictions, we further design a superconducting circuit QED architecture and present the results for two different approaches regarding the circulator design: a three Cooper Pair Box symmetric circulator and a Cooper Pair reservoir coupled to two superconducting islands. In the first case highly chiral decays are observed within the sweet spot, whereas in the second case chirality is completely lost unless particle-antiparticle symmetry is broken. Gate voltages preserving charge stability other than the sweet spot have been obtained, in which chiral decays and scattering amplitudes survive. In all cases it is observed that highly chiral scattering amplitudes and decays can be turned on and off accordingly tuning an external magnetic field through the circulator.

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## Chapter 1

## Introduction

The presence of a directional dependency of transmission modes in a waveguide has risen a wide variety of interests in the Quantum Optics and Quantum Information fields during the past two decades. Indeed chirality is an important concept, which describes the asymmetric property of a system emerging from a mirror symmetry breaking. Many works within the Quantum Optics field are based under the assumption of chiral waveguides which present a high dependency on the direction photons take. For instance, many applications and research fields can harness a chiral behavior among which we can find photon bound state realizations [1], long distance quantum communications [2], entanglement protocols [3], non-reciprocal reflection and transmission coefficients [4] and many other practical implementations in chiral quantum optics [5]. Many experimental researches achieved physical realizations through chiral materials $[6,7,8,9]$, and more recent works which incorporate superconducting circuits [10].

During the last decade, superconducting qubits gained leadership among other existent qubit realizations. Thanks to its engineered artificial nature they are relatively easier to design and control than other natural qubits $[11,12]$, being able to provide a flexible inbuilt parameter control in a very wide range. That makes them promising for scalable quantum systems such as quantum memories [13, 14] or quantum processors. Superconducting systems provide also promising experimental noise-less results thanks to their low-temperature state disabling thermal fluctuations. Since the first experimental realization of a charge qubit in 1999 [15], artificial qubits have taken the lead and even to the first evidence of quantum supremacy using a programmable superconducting processor [16]. Its simple architecture makes the theoretical functionality easy to understand and exploit to create further advanced models and quantum gate designs. Superconducting qubits are simply based on the anharmonicity presented by replacing an inductance for a non-linear Josephson Junction in an artificial Quantum Harmonic Oscillator. High quality circuit QED systems are routinely manufactured using Josephson Junctions, creating superconducting circuits of resonators and qubits [17, 18]. Radiating microwave photons, a two level system with free-charge fluctuations can be accessed and in that matter one can create a qubit state. They are also easy to control and manipulate, being able to create single qubit quantum
gates, and even capacitively couple two qubits to realize multiple qubit quantum gates. Their control is performed through superconducting microwave resonators which enable their optical manipulation trough waveguides.

In this work we propose another possible design to create a tunable chiral waveguide through the coupling of a superconducting circuit QED architecture. The latest, being conformed by a superconducting ring geometry, provides a time-reversal symmetry breaking by a magnetic flux going through which can be tuned. The superconducting architecture would be brought by an on-chip microwave circulator scheme [19, 20, 21, 22, 23]. Its superconducting-ring three-coupled qubit structure provides the novel low-temperature superconducting benefits and its widely tunable artificial nature.

Unless it is stated, through all this work natural units have been assumed, mainly affecting $\hbar$ used to have the numeric value 1 .

The structure of this thesis is divided in 6 main blocks:

- Chapter 1: Introduces the the current situation within the superconducting quantum computation and states the utility of building chiral waveguides (current chapter).
- Chapter 2: Presents the reader with useful tools and explanations regarding the quantum mechanical description of superconducting qubits, ending up introducing the time-reversal symmetry breaking of the microwave circulator.
- Chapter 3: Describes the starting point of this thesis with a theoretical model, presenting the directional decay probabilities and a fidelity we will exploit to measure the chiral behavior we are interested in.
- Chapter 4: Illustrates the design of the superconducting circuit in which the model we built is based on, further quantizing and solving one analytic approach of the problem.
- Chapter 5: Solves an alternative approach of the problem introduced in the previous chapter, obtaining a more deep understanding of the results. Finally compares both approaches.
- Chapter 6: Concludes the work, giving a global overview of the problem and providing possible future work related proposals.


## Chapter 2

## Preliminaries

Since we will cover many superconducting qubit fields throughout this work, in this first introductory chapter we will provide a brief outline to the quantum mechanical description of superconducting qubits, and introduce basic tools we will later need to use. We start showing the Quantum LC Oscillator and explaining how its harmonic energy spectrum does not provide a good quantum system for a qubit. Later on, we argument that introducing an anharmonic potential by replacing the Inductance by a Josephson Junction in the circuit indeed introduces an unequal energy spacing suitable to isolate a two level quantum system. We also outline two different regimes a qubit can operate in: the Transmon regime and the Cooper Pair Box. Finally, we introduce the idea of the quantum circulator by coupling three Cooper Pair Box qubits in a superconducting ring, which we observe that displays a time-reversal symmetry violation which will be harnessed to design a chiral photon interacting quantum device.

In this chapter we will not use Natural Units (explicitly writing $\hbar \neq 1$ ) to provide a global overview of the preliminary concepts.

### 2.1 Quantum LC oscillator

A classical realization of the Harmonic Oscillator can be achieved within an electronic circuit by connecting an inductance $L$ in parallel with a capacitance $C$ sketched in Figure 2.1 (a). The classical Hamiltonian description will be given by the energy term brought by the inductance and an energy term coming from the capacitance.

$$
\begin{equation*}
\mathcal{H}=\frac{Q^{2}}{2 C}+\frac{\Phi^{2}}{2 L} \tag{2.1}
\end{equation*}
$$

This artificial oscillator will be provided by an oscillating frequency $\omega=1 / \sqrt{L C}$. The LC circuit though, can be built into the quantum word, no longer obeying the classical mechanics given by (2.1). The quantization of the LC Hamiltonian can be straight provided by the quantization of both charge and flux operators, fulfilling the following canonical commutation relations.

$$
\begin{equation*}
[\hat{\Phi}, \hat{Q}]=i \hbar \quad \rightarrow \quad \Phi=\hat{\Phi} \quad, \quad Q=\hat{Q}=-i \hbar \frac{\partial}{\partial \Phi} \tag{2.2}
\end{equation*}
$$

The quantized Hamiltonian then will be exactly the same as the classical Hamiltonian in (2.1) but writing the hats on top of the charge and flux variables to indicate they are now operators. The commutation relations in (2.2) remind us of the canonical commutation relations of the well-known quantized position $\hat{x}$ and momentum $\hat{p}$ variables. Thus, we can re-write the charge and flux operators in terms of creation and annihilation operators as,

$$
\begin{equation*}
\hat{a}=\sqrt{\frac{m \omega}{2 \hbar}}\left(\hat{x}+i \frac{\hat{p}}{m \omega}\right) \quad \rightarrow \quad \hat{a}=\sqrt{\frac{C \omega}{2 \hbar}}\left(\hat{\Phi}+i \frac{\hat{Q}}{C \omega}\right) \tag{2.3}
\end{equation*}
$$

Where the capacitance $C$ is straight mapped from the mass of the harmonic oscillator, $m$. The creation operator $\hat{a}^{\dagger}$ can be straight obtained by complex conjugating $\hat{a}$. Therefore, we can re-write the quantum LC circuit Hamiltonian as follows.

$$
\begin{equation*}
\mathcal{H}=\hbar \omega\left(\hat{a}^{\dagger} \hat{a}+\frac{1}{2}\right) \tag{2.4}
\end{equation*}
$$

This is the well-known Quantum Harmonic Oscillator Hamiltonian, whose eigenstates will be determined by number states $|n\rangle$ with equal energy level spacing $\Delta E=\hbar \omega$. However, this energy level distribution will not be favorable for creating qubit states. To do that, we would be interested to access a two level system, but in this case, due to the equal level spacing, it is not possible. Apart from that, it can be demonstrated that at a certain temperature, the density of states of the Harmonic Oscillator can be described by a real-valued diagonal matrix, not becoming a pure state at finite temperature and therefore losing all possible related quantum effects.

### 2.2 Superconducting qubits

A solution to the previous situation to be able to create a qubit state would be to break the harmonicity of the energy levels of the LC circuit. This breaking can be achieved by replacing the inductance by a Josephson Junction, playing the role of a non-linear inductance (see Figure 2.1 (b)). The latest is just constructed by a superconductor-isolator-superconductor junction, which has the special feature to


Figure 2.1: Schematic representation of the circuit architecture of the LC circuit (a) and a superconducting qubit (b). See the only difference is that in the second, we replaced the inductance by a non-linear inductance brought by a Josephson Junction.
display anharmonic energy levels. We can then quickly re-build the superconducting qubit Hamiltonian from the quantum LC circuit and replace the inductance energy term by the Josephson Junction energy giving:

$$
\begin{equation*}
\mathcal{H}=\frac{Q^{2}}{2 C}+E_{J}(1-\cos (\phi)) \quad \text { where }: \quad E_{J}=\frac{I_{0} \Phi_{0}}{2 \pi} \tag{2.5}
\end{equation*}
$$

Where we introduced the Josephson Junction energy term which includes the superconducting states phase difference between both superconductors in the junction $\phi=\delta_{2}-\delta_{1}$ and the Josephson Energy $E_{J}$. This Hamiltonian defines the Superconducting Qubit [24] quantum mechanical dynamics. Observe now that we can replace the flux variable in the LC Hamiltonian by $\Phi \rightarrow \phi \Phi_{0} / 2 \pi$ which has also magnetic flux dimensions, and using the second Josephson law $V=\dot{\phi} \Phi_{0} / 2 \pi$ we can identify the following potential and kinetic energy terms.

$$
\begin{equation*}
U(\phi)=E_{J}(1-\cos (\phi)) \quad K(\dot{\phi})=\frac{C V}{2}=\left(\frac{\Phi_{0}}{2 \pi}\right)^{2} \frac{C \dot{\phi}^{2}}{2} \tag{2.6}
\end{equation*}
$$

See that the potential energy term basically depends on the phase difference variable, which can be related (as we have already mentioned) to the magnetic flux, and the kinetic term depends uniquely on the phase difference time derivative, related also with the flux momentum, canonically speaking. It can be observed how the latest term can be mapped to the capacitance energy term in $\mathcal{H}$ by switching $V=C Q$, recovering the same Hamiltonian in (2.5) by just adding the kinetic and potential energy terms. Finally, note that the kinetic term is defined by the electric potential. Although this crossed notation, the kinetic and potential energy terms definitions are based on the Josephson phase dependency, $\phi$, and the canonical variables $\hat{x}$ and $\hat{p}$ mapping we previously mentioned.

From here, the quantization of (2.5) is straight forward derived by converting the Josephson phase into an operator, $\hat{\phi}$ just as we did with the quantized magnetic flux. However, as in this case we are working with a superconducting circuit, the charges will appear grouped in Cooper Pairs with charge $2 e$. The charge variable will be then directly quantized in the number basis as the number of Cooper


Figure 2.2: Energy representation of the three discussed artificial setups. Starting from the left, the Harmonic Oscillator (H.O.) energy spectrum is displayed. See te energy separation between levels is always $\hbar \omega$. In the middle we illustrate the energy representation of the Transmon Qubit. The anharmonic potential given by the cosine function provokes unequal energy splittings between levels. The last figure on the right side shows the Cooper Pair Box (CPB) Qubit, where the potential can be approximated by consecutive parabolas. The sweet spot is defined by $N_{g}=1 / 2$, where the ground and first excited state are at their closest point, and the charge fluctuations are null.

Pairs given by $\hat{Q}=2 e \hat{n}$, where $\hat{n}=\hat{a}^{\dagger} \hat{a}$ is the number operator. All in all, the full quantized version of the superconducting qubit Hamiltonian will be given by:

$$
\begin{equation*}
\mathcal{H}=4 E_{c} \hat{n}^{2}-E_{J} \cos (\hat{\phi}) \tag{2.7}
\end{equation*}
$$

Where the Charging Energy $E_{c}=e^{2} / 2 C$ has been introduced. Taking a close look to the potential energy term, by expanding the cosine function for small $\hat{\phi}$ values up to $4^{\text {th }}$ order, we end up with a second order term $\hat{\phi}^{2} / 2$ minus a forth order term $\hat{\phi}^{4} / 24$. Since we can relate the Josephson phase with the inductance magnetic flux, we can directly map the kinetic term plus the lowest order term in the cosine expansion to re-create the Harmonic Oscillator Hamiltonian previously derived in (2.4). The second lower term in the cosine expansion is then added as a perturbation, ending up with the following Hamiltonian.

$$
\begin{equation*}
\mathcal{H}=\hbar \omega\left(\hat{n}+\frac{1}{2}\right)-2 E_{c}\left(\hat{a}+\hat{a}^{\dagger}\right)^{4} \quad \text { where }: \quad \hbar \omega=\sqrt{8 E_{J} E_{c}} \tag{2.8}
\end{equation*}
$$

The creation and annihilation operators where also mapped from (2.3) using the new oscillation frequency and expressing all results in terms of $E_{c}$ and $E_{J}$. Treating the last term as a perturbation to the Harmonic Oscillator behavior, going up to first order in perturbation theory we can treat it as $\langle n|\left(\hat{a}+\hat{a}^{\dagger}\right)|n\rangle=6\left(n^{2}+n+1 / 2\right)$. Now the equally separated energy spectrum of the Harmonic Oscillator is broken thanks to the anharmonic cosine term as we can see in Figure 2.2. Therefore, we are now able to access individual energy levels, meaning that we can create two level qubit states. The treatment of the $\hat{\phi}^{4}$ term in perturbation theory is only applicable when $E_{J} \gg E_{c}$, regime which we will shortly talk about.

The behavior of a superconducting qubit will differ significantly depending on the relation between
the charging energy $E_{c}$ and the Josephson energy $E_{J}$. By looking at (2.6) we realize that $E_{J}$ is only related to the potential energy contribution, while $E_{c}$ appears only in the kinetic energy term. Here we can anticipate at least two different regimes a qubit can operate. If $E_{J} / E_{c} \gg 1$ the dynamics will be mainly given by the potential energy term which dominates in the Hamiltonian and we say that the superconducting qubit operates as a Transmon qubit. Otherwise, if $E_{J} / E_{c} \ll 1$, the kinetic energy term will dominate in the Hamiltonian description and we say that the superconducting qubit operates under the Cooper Pair Box (CPB) regime. Following next we will expose both regimes and briefly explain the involved dynamics and main important results.

### 2.2.1 Transmon qubit

A superconducting qubit is said to operate in the Transmon regime when the Josephson energy term dominates over the charging energy. We have already seen, looking at the anharmonic character of the superconducting qubit Hamiltonian, that for $E_{c} \ll E_{J}$, a particle is trapped in an anharmonic potential. Effectively, the most dominant term is provided by the potential energy of $\mathcal{H}$. This makes possible the isolation of a two energy level system which we can operate with, defining a qubit. These qubit states should not be understood as individual extra Cooper Pairs occupying the superconducting islands since, as $E_{c}$ is considered sufficiently small, the island capacitance should be big enough thanks to its inverse proportional relation. A big superconducting island means the presence of multiple exceeding Cooper Pair charges. The energy states, as we have just seen, come directly from the Hamiltonian in (2.8), where charge and flux operators can be directly mapped to the H.O. creation and annihilation operators. We can already infer that the excited states of a Transmon qubit come defined by oscillating excitations in the living Cooper Pair charges in the islands. Although their low charge fluctuations and high coherence times, the Transmon qubit will not fit into the main purposes of this work. Hence, it will not be highly discussed and we will aim at a different regime (Cooper Pair Box).

### 2.2.2 Cooper Pair Box qubit

In the limit where $E_{c} \gg E_{J}$, the kinetic energy term in the Hamiltonian dominates in front of the anharmonic potential in which states were confined and the Cooper Pair Box (CPB) regime is defined (also known as Charge Qubit [25]). This regime requires a very small island capacitance, which is inversely proportional to the charging energy. The small island design may allow only a limited amount of extra Cooper Pairs to access the superconducting island. Typically, in this regime we only allow a maximum of 1 exceeding Cooper Pair per island, thereafter the CPB regime name comes from this feature. It is also very common to apply a gate voltage to control the qubit, which we will now explicitly take into account. Therefore, in the CPB regime, the Hamiltonian in (2.7) can be re-written in the number basis as follows.

$$
\begin{equation*}
\mathcal{H}=4 E_{c} \sum_{N}\left(N-N_{g}\right)^{2}|N\rangle\langle N|-\frac{E_{J}}{2} \sum_{N}(|N\rangle\langle N-1|+|N-1\rangle\langle N|) \tag{2.9}
\end{equation*}
$$

This re-structured form can be directly obtained knowing that the phase and number operators are related by $\langle\phi \mid n\rangle=e^{i n \phi} / \sqrt{2 \pi}$. Expanding this relation in the number of charge states one ends up seeing that $e^{i \phi}=|n+1\rangle\langle n|$. In (2.9) we just substituted the exponential functions from the cosine


Figure 2.3: Illustration of two CPB capactively coupled with a Josephson Junction as a mediator. The latest will allow Cooper Pair charges hoppings from one island to another.
function in (2.7) in the charge number basis. A part form that, we also introduced the applied gate voltage through $N_{g}=V_{g} / 2 e C_{g}$. As we previously said, we can truncate the number of Cooper Pair charges in the island to be $N=0,1$. This makes it possible for us to re-write the Hamiltonian in (2.9) in terms of the familiar Pauli matrices.

$$
\begin{equation*}
\mathcal{H}=4 E_{c}\left(1-2 N_{g}\right)^{2} \sigma_{z}-\frac{E_{J}}{2} \sigma_{x} \tag{2.10}
\end{equation*}
$$

Within the CPB regime, since the potential energy becomes less dominant, the system turns more susceptible to charge fluctuations. Experimentally, eliminating the charge noise is a must to be able to functionally operate. Setting $N_{g}=1 / 2$, namely called the sweet spot, will suppress the charge noise. See that in that case, the Hamiltonian in the CPB limit will be uniquely defined by the $\sigma_{x}$ matrix. The eigenstates in that case will be given by $|+\rangle$ and $|-\rangle$, with an energy separation of $E_{J}$. We can see in Figure 2.2 how this two level system is defined within the energy spectrum of the CPB qubit. Higher energy states can be accessed if we allow more than one Cooper Pair charge living in the superconducting island. Compared with the Transmon regime, the CPB regime seems more suitable for situations where charge states require to be fixed and individually localized in the islands.

### 2.3 Coupling two CPB qubits

In this section we will discuss and describe the capacitive coupling between superconducting qubits in the CPB limit with a mediating Josephson Junction, illustrated in Figure 2.3. The starting point will be present the Hamiltonian description of the system, i.e., state the potential and kinetic terms, as we have already seen in (2.6), which come defined by the Josephson Junction and capacitance energies respectively. The first one will be straight written in the cosine form as the Josephson Junction will not be affected by the coupling. However, since we are actually describing a capacitive coupling, there will be some accumulated charges $(q)$ in the circuit due the coupling, which should be optimized in the Hamiltonian to reach the more stable configuration. These charges will only appear in the capacitance
energy term, so we can start focusing on this. Following Figure 2.3, the capacitance energy term will come defined by:

$$
\begin{equation*}
\mathcal{H}=\frac{\left(Q_{1}-q\right)^{2}}{2\left(C_{c}+C_{g}\right)}+\frac{\left(Q_{2}+q\right)^{2}}{2\left(C_{c}+C_{g}\right)}+\frac{q^{2}}{2 C_{J}} . \tag{2.11}
\end{equation*}
$$

Where the $q$ charge indicates the induced charge accumulations due to the capacitive coupling. For simplicity, we ignored the gate charges introduced by the gate potential. These will merely add an offset value to the island charges $Q_{1} \rightarrow Q_{1}-Q_{1, g}$ and $Q_{2} \rightarrow Q_{2}-Q_{2, g}$, and will not play any crucial role for now. The final charge distribution will be given by the following $q$ value, which will minimize the total system energy.

$$
\begin{equation*}
\frac{\partial \mathcal{H}}{\partial q}=0 \quad \rightarrow \quad q=\frac{\tilde{C}}{C_{c}+C_{g}} Q_{1}-\frac{\tilde{C}}{C_{c}+C_{g}} Q_{2} \tag{2.12}
\end{equation*}
$$

Where $\tilde{C}^{-1}=\left(2 /\left(C_{c}+C_{g}\right)+1 / C_{J}\right)$. Substituting this back into the original capacitance Hamiltonian, one gets:

$$
\begin{equation*}
\mathcal{H}=\sum_{i=1,2} \frac{1}{2\left(C_{c}+C_{g}\right)}\left[2\left(\frac{\tilde{C}}{C_{c}+C_{g}}\right)^{2}+\left(1-\frac{\tilde{C}}{C_{c}+C_{g}}\right)^{2}\right] Q_{i}^{2}+\frac{1}{C_{J}}\left[2-3\left(\frac{\tilde{C}}{C_{c}+C_{g}}\right)^{2}\right] Q_{1} Q_{2} . \tag{2.13}
\end{equation*}
$$

Quantizing now the charge variables $Q_{1}$ and $Q_{2}$ into the charge number basis as we did in (2.9), we finally obtain:

$$
\begin{align*}
& \mathcal{H}=\sum_{i=1,2} 4 E_{c}\left(N_{i}-N_{g, i}\right)^{2}\left|N_{i}\right\rangle\left\langle N_{i}\right|+4 E_{c, 12}\left|N_{1} N_{2}\right\rangle\left\langle N_{1} N_{2}\right|- \\
& -\frac{E_{J}}{2} \sum_{i}\left(\left|N_{i}\right\rangle\left\langle N_{i}-1\right|+\left|N_{i}-1\right\rangle\left\langle N_{i}\right|\right) \tag{2.14}
\end{align*}
$$

Where we defined the charging energies:

$$
\begin{array}{ll}
E_{c}=\frac{4 e^{2}}{2\left(C_{c}+C_{g}\right)}\left[2\left(\frac{\tilde{C}}{C_{c}+C_{g}}\right)^{2}+\left(1-\frac{\tilde{C}}{C_{c}+C_{g}}\right)^{2}\right] & \rightarrow \frac{4 e^{2}}{2\left(C_{\Sigma}+C_{J}\right)} \frac{C_{\Sigma}^{2}+2 C_{J}^{2}}{\left(C_{\Sigma}^{2}-C_{J}^{2}\right)}  \tag{2.15}\\
E_{c, 12}=\frac{4 e^{2}}{C_{J}}\left[2-3 \frac{\tilde{C}}{C_{c}+C_{g}} \frac{\tilde{C}}{C_{c}+C_{g}}\right] & \rightarrow \frac{4 e^{2}}{C_{J}}\left[2-3 \frac{C_{J}^{2}}{\left(C_{\Sigma}+C_{J}\right)^{2}}\right]
\end{array}
$$

On the right side of (2.15) we introduced the total island capacitance defined as $C_{\Sigma} \equiv C_{c}+C_{g}+C_{J}$ in this case. Proceeding equally as we did studying the CPB qubit, we will re-write the Hamiltonian in (2.14) in terms of the Pauli matrices and also add the Josephson Junction term. In the sweet spot free of charge fluctuations, it can be re-written as follows.

$$
\begin{equation*}
\mathcal{H}=E_{c, 12} \sigma_{z}^{(1)} \sigma_{z}^{(2)}-\frac{E_{J}}{8}\left(\sigma_{+}^{(1)} \sigma_{-}^{(2)}+\sigma_{-}^{(1)} \sigma_{+}^{(2)}\right) \tag{2.16}
\end{equation*}
$$

Where $\sigma_{+}^{(j)}$ and $\sigma_{-}^{(j)}$ are the rising and lowering operators respectively, acting on the $j$ island. See how the Hamiltonian allows transitions of Cooper Pair charges from one island to another through the Josephson Junction. Later on in this work, we will use three coupled CPB qubits with three Josephson Junctions which will allow Cooper Pairs to travel from one island to another. As the circuit architecture will form a superconducting ring, a magnetic flux will go through, adding a phase difference to the exchanging term in the Hamiltonian. This phase will be controlled by the magnetic flux, and thanks to that, the circuit will have a tunable chiral behavior when coupled to a waveguide. This superconducting ring architecture is best known as Circulator, and one of the most interesting features is that it shows a time symmetry breaking in its Hamiltonian description, in which the chiral outputs are naturally based.

### 2.4 Circulators and time reversal symmetry

As exposed at the end of the previous section, a microwave circulator can be constructed with a superconducting ring circuit built form the coupling of three CPB qubits based on the scheme in Figure 2.3. As we have seen in (2.16), the Hamiltonian description of two coupled CPB will result in two terms: A charging energy term, and a Cooper Pair hoping term. From this idea, a theoretical model definition of a circulator can be built taking into account this two terms, and extending the idea to three coupled islands [19]. We will define $\hat{b}_{j}^{\dagger}, \hat{b}_{j}$ as the creation and annihilation operators of a Cooper Pair charge in the island $j$ respectively. We will also have to take into account the magnetic flux going through the superconducting ring, which will contribute in adding a phase to each hoping element. At the end of the day, the Hamiltonian definition of a Circulator can be effectively sketched as:

$$
\begin{equation*}
\mathcal{H}=\hbar \omega \sum_{j}^{3} \hat{b}_{j}^{\dagger} \hat{b}_{j}+J e^{i \varphi}\left(\hat{b}_{1}^{\dagger} \hat{b}_{2}+\hat{b}_{2}^{\dagger} \hat{b}_{3}+\hat{b}_{3}^{\dagger} \hat{b}_{1}\right)+\text { H.c. } \tag{2.17}
\end{equation*}
$$

To see how this Hamiltonian violates time reversal symmetry, let us first introduce the Timereversal operator. The Time-reversal operator $(\hat{T})$ is defined in such a way that, as its name indicates, reverses the direction of time. Assuming $\hat{U}^{\prime}(t)$ to be a unitary time translation operator, the time reversal operator will yield $\hat{T} \hat{U}^{\prime}(t) \hat{T}^{\dagger}=\hat{U}^{\prime}(-t)$. However, expanding the time translation operator infinitesimally, we end up having an anti-unitary behavior of $\hat{T}$, since $-\hat{T} i \mathcal{H}=i \mathcal{H} \hat{T}$. This problem can be solved by decomposing $\hat{T}=\hat{K} \hat{U}$ in a complex conjugation operator $\hat{K}$ and a unitary part given by $\hat{U}$ which reverts the direction of time. With that definition, a system is said to preserve the time reversal symmetry if $[\mathcal{H}, \hat{T}]=0$.

In (2.17), if no magnetic flux is applied through the circulator (meaning $\varphi=0$ ), one could always choose a gauge transformation of the $\hat{b}_{j}$ operators which makes the Hamiltonian real valued and therefore, time reversal symmetric. However, the presence of $\varphi$ adds up a condition, making the Hamiltonian only T-symmetric for certain values. See for example, choosing $\hat{b}_{j} \rightarrow \hat{b}_{j} e^{i \phi_{j}}$ as a gauge transformation, $\mathcal{H}$ will be time reversal symmetric only if:

$$
[\mathcal{H}, \hat{T}]=0 \quad \Rightarrow \quad\left\{\begin{array}{l}
\sin \left(\varphi+\phi_{2}-\phi_{1}\right)=0  \tag{2.18}\\
\sin \left(\varphi+\phi_{3}-\phi_{2}\right)=0 \\
\sin \left(\varphi+\phi_{1}-\phi_{3}\right)=0
\end{array} \quad \rightarrow \quad \varphi=\frac{\pi}{3} n \quad \text { for }: \quad n=0,1,2, . .\right.
$$

We see here that the time reversal symmetry is maintained as long as $\varphi=n \pi / 3$.. Indeed, the presence of a magnetic field induces the symmetry violation as one would have expected, since still in Maxwell Equations one can find that the time-reversal operator applied to the current yields $\hat{T} \mathbf{J} \hat{T}^{-1}=$ $-\mathbf{J}$, and a reversed current implies a reversed magnetic field.

The time-reversal symmetry violation presented in the circulator can be harnessed to induce waveguides with chiral properties. Indeed, the time reversal operator acting on the momentum operator will reverse its direction as $\hat{T} \mathbf{P} \hat{T}^{-1}=-\mathbf{P}$. This implies that a Hamiltonian which does not preserve the time reversal symmetry will actually treat differently modes depending on their direction. In this thesis we will see how coupling two different systems such as the waveguide and the circulator (the first being time reversal symmetric and the second not) will provide the resulting state with two different phases (each coming from the waveguide and circulator contributions) which we will be able to tune in order to adjust a chiral response from the system.

## Chapter 3

## Theoretical model

Our starting point will be based on a theory proposal in [19], where a superconducting qubit violating time reversal symmetry is suggested. The presented model in this chapter will be based on two resonating cavities, $A$ and $B$, coupled to a waveguide with a breaking of time symmetry induced by a magnetic field in a superconducting circuit. Each of the resonating cavities will be coupled to one port of a three-port microwave circulator through which a magnetic field is applied. The third port will be left coupled to the ground, which is mainly needed now basically to give an area where the magnetic flux can go through. The presence of the circulator will play an important role as a secondary actor behind the scenes, as we will not work directly with it. It will provide the necessary breaking of time reversal symmetry which will drive effective photon interactions in the waveguide to become chiral. The coupling through the circulator will be idealized as an effective superconducting coupling between resonators introducing a phase difference, $\varphi$, imitating the circulator Hamiltonian introduced in Section 2.4.

With this setup, the main idea will be based on the presence of two different paths a photon can take when entering our device. The simplest option would be to be absorbed by a resonator, and straight emitted back into the waveguide, traveling a distance equal to the separation between the resonators $A$ and $B, \Delta L$. Through this path, the photon will acquire a global phase $e^{i k_{o} \Delta L}$, directly depending on the distance it run. The other path a photon can take is, when absorbed by a resonator, for example the $A$, the photon can excite the superconducting circuit state, which will decay into the cavity $B$, able to straight decay back into the waveguide. Through this path, the output state will earn a global phase proportional to the phase difference acquired through the effective superconducting coupling, $e^{i \varphi}$. This phase will depend mainly of the flux going through the circulator. See that we generated two different paths from which a photon state can acquire two different phases. One now can play with the magnetic flux and the separation between both resonators to change the value of the phase of the output state. This feature can be used to create outputs with different decay rates depending mainly on the direction they take.

We will start building the Hamiltonian of a waveguide coupled to two resonators and obtain their equations of motion. Later on, the effects of the circulator will be systematically introduced. We will observe analytically the probabilities a photon has to decay whether to the right or to the left direction and which are the main parameters we can adjust to make this happen. Finally, a way to obtain a fidelity of our device will idealized, with which it will be easy to see the regimes where the chiral behavior to its maximum potential can be exploit.

### 3.1 Coupling two resonating cavities to a waveguide

### 3.1.1 Hamiltonian description

We will begin working on a simple model of having two resonating cavities coupled to a waveguide. The first step would be to build a familiar Hamiltonian description for the electric field traveling through the waveguide. This will be based on microwave narrow-band photons with a central frequency mode determined by $\omega_{o}$. Moreover, we can assume a linear dispersion of the electric field, ending up with the following Hamiltonian.

$$
\begin{equation*}
\mathcal{H}_{E}=\int d k \omega_{k} \hat{a}_{k}^{\dagger} \hat{a}_{k}=\int d k \omega_{o} \hat{a}_{k}^{\dagger} \hat{a}_{k}+\int d k v_{g}\left(k-k_{o}\right) \hat{a}_{k}^{\dagger} \hat{a}_{k}=\mathcal{H}_{\omega}+\mathcal{H}_{p r o p} \tag{3.1}
\end{equation*}
$$

Here we introduced the splitting of the description of the electric field between an oscillatory behavior given by $\mathcal{H}_{\omega}$, and the propagation of the field within $\mathcal{H}_{\text {prop }}$. In the latest we also introduced the group velocity, $v_{g}$, of the field. Staying with the last term of the Hamiltonian description, we will write the electric fields in terms of the right and left propagating modes. For that, first we need to define our quantized electric field in terms of the creation and annihilation operators and split it in the right and left parts.

$$
\begin{equation*}
E(z, t)=e^{i\left(k_{o} z-\omega_{o} t\right)} E_{R}(z, t)+e^{-i\left(k_{o} z+\omega_{o} t\right)} E_{L}(z, t) \tag{3.2}
\end{equation*}
$$

Where here we defined:

$$
\begin{align*}
& E_{R}(z, t)=\frac{1}{\sqrt{2 \pi}} \int d k \hat{a}_{k} e^{i\left(k-k_{o}\right) z} e^{i \omega_{o} t}  \tag{3.3}\\
& E_{L}(z, t)=\frac{1}{\sqrt{2 \pi}} \int d k \hat{a}_{-k} e^{-i\left(k-k_{o}\right) z} e^{i \omega_{o} t}
\end{align*}
$$

Note that the main difference between the right and left electric fields remains in the sign of $k-k_{o}$. Going further with the definitions we propose, one can re-write then the propagating part of the electric field Hamiltonian and obtain:

$$
\begin{equation*}
\mathcal{H}_{\text {prop }}=i v_{g} \int d z\left(E_{L}^{\dagger} \frac{\partial E_{L}}{\partial z}-E_{R}^{\dagger} \frac{\partial E_{R}}{\partial z}\right) \tag{3.4}
\end{equation*}
$$

Having a full Hamiltonian description for the electric field going through the waveguide, the next step will be to obtain now a similar Hamiltonian description for the resonators. The cavity operators will be defined in the number basis as $\hat{b}_{i}$, for $i=A, B$. Since we will mainly work near the resonance frequency $\omega_{o}$, it may seem reasonable to express their frequencies related to the energy difference respect to the resonance point, namely $\Delta_{j}$. Note that in that case, the detuning between both resonating cavities will be given by $\Delta=\left|\Delta_{A}-\Delta_{B}\right|$. We should then accommodate our calculations by switching to the interaction picture for the resonance frequency.

$$
\begin{equation*}
\mathcal{H}_{c a v}=\sum_{j=A, B}\left(\omega_{o}+\Delta_{j}\right) \tilde{b}_{j}^{\dagger} \tilde{b}_{j} \quad \text { where }: \quad \tilde{b}_{j}=\hat{b}_{j} e^{i \omega_{o} t} \tag{3.5}
\end{equation*}
$$

Both resonators and waveguide are now fully represented by their respective Hamiltonian descriptions. Finally, the interaction between the resonators and the waveguide will be modeled with the well known dipole interaction [26]. We will assume a coupling strength $g_{j}$ between the cavities $j=A, B$ and the transmission line.

$$
\begin{equation*}
\mathcal{H}_{i n t}=-\sqrt{2 \pi} \sum_{j} \int d z\left[e^{i k_{o} z} E_{R}(z, t)+e^{-i k_{o} z} E_{L}(z, t)\right] \tilde{b}_{j}^{\dagger} g_{j} \delta\left(z-z_{j}\right)+\text { H.c. } \tag{3.6}
\end{equation*}
$$

The full Hamiltonian of the whole system will be built from (3.4), (3.5) and (3.6). With these three terms we have a full description of the theoretical model we will work on.

### 3.1.2 Equations of motion

The equations of motion of the operators previously introduced will be presented in this section. We will start by looking at the electric field equations. From our electric field definitions in (3.3), we can derive the following main commutation relations.

$$
\begin{align*}
& {\left[E_{L}(z), E_{L}^{\dagger}\left(z^{\prime}\right)\right]=\left[E_{R}(z), E_{R}^{\dagger}\left(z^{\prime}\right)\right]=\delta\left(z-z^{\prime}\right)} \\
& {\left[E_{L}(z), E_{R}^{\dagger}\left(z^{\prime}\right)\right]=\left[E_{R}(z), E_{L}^{\dagger}\left(z^{\prime}\right)\right]=0} \tag{3.7}
\end{align*}
$$

Note that these relations suggest the defined electric fields to have dimensions of $[1 / \sqrt{L}]$, where by $L$ we mean distance. Observe that this dimension definitions forces the coupling strengths $g_{j}$ to have unit dimensions of $[E \sqrt{L}]$, for $E$ being energy, in order to preserve unit coherence in (3.6). Thanks to the presented the commutation relations, we end up with the following equations of motion.

$$
\begin{align*}
& \left(\frac{\partial}{\partial t}+v_{g} \frac{\partial}{\partial z}\right) E_{R}(z, t)=i \sqrt{2 \pi} e^{i k_{o} z} \sum_{j=A, B} g_{j} \tilde{b}_{j} \delta\left(z-z_{j}\right)  \tag{3.8}\\
& \left(\frac{\partial}{\partial t}-v_{g} \frac{\partial}{\partial z}\right) E_{L}(z, t)=i \sqrt{2 \pi} e^{-i k_{o} z} \sum_{j=A, B} g_{j} \tilde{b}_{j} \delta\left(z-z_{j}\right)
\end{align*}
$$

In order to solve these equations, one should define a re-scaled time coordinate different for the right and left cases: $\tau_{R}=t-z / v_{g}$ and $\tau_{L}=t+z / v_{g}$ respectively. One should also take into account here that the setup size is assumed to be much smaller than the group velocity. After substituting these new variables in (3.8), and integrating over the waveguide length $\Delta L$, the final equations of motion can be read as:

$$
\begin{align*}
& E_{R}(z, t)=\frac{i \sqrt{2 \pi}}{v_{g}} \sum_{j=A, B} e^{-i k_{o} z_{j}} g_{j} \tilde{b}_{j} \theta\left(z-z_{j}\right)+E_{R, I N}  \tag{3.9}\\
& E_{L}(z, t)=\frac{i \sqrt{2 \pi}}{v_{g}} \sum_{j=A, B} e^{i k_{o} z_{j}} g_{j} \tilde{b}_{j} \theta\left(z_{j}-z\right)+E_{L, I N}
\end{align*}
$$

Note that, assuming $v_{g} t \gg \Delta L$, after substituting in (3.8) both re-scaled time variables become the same as the original time variable. This means that the spatial fluctuations will be ignored, having as a result a stationary electrical field. We will store this result in a compact matrix form that we can exploit later, easing the way for our purposes. Hence, let us define:

$$
\begin{align*}
& \vec{E}_{\text {OUT }} \equiv\binom{E_{R}(z, t)}{E_{L}(z, t)} \quad \vec{E}_{I N} \equiv\binom{E_{R, I N}}{E_{L, I N}} \\
& \vec{b} \equiv\binom{\tilde{b}_{A}}{\tilde{b}_{B}} \quad M_{\theta}^{\dagger} \equiv \frac{1}{\sqrt{2}}\left(\begin{array}{cc}
\sqrt{\gamma_{A}} \theta\left(z-z_{A}\right) e^{-i k_{o} z_{A}} & \sqrt{\gamma_{B}} \theta\left(z-z_{B}\right) e^{-i k_{o} z_{B}} \\
\sqrt{\gamma_{A}} \theta\left(z_{A}-z\right) e^{i k_{o} z_{A}} & \sqrt{\gamma_{B}} \theta\left(z_{B}-z\right) e^{i k_{o} z_{B}}
\end{array}\right) . \tag{3.10}
\end{align*}
$$

This way, the final equations of motion of the electrical fields can me resumed in the following matrix expression.

$$
\begin{equation*}
\vec{E}_{O U T}=\vec{E}_{I N}+\frac{i}{\sqrt{v_{g}}} M_{\theta}^{\dagger} \vec{b} \tag{3.11}
\end{equation*}
$$

Here we must admit that we played with advantage since the decay rates of both cavities, $\gamma_{A}$ and $\gamma_{B}$ have been introduced, before actually seeing their derivation. We will be able to see shortly in the equations of motion of the resonators that the actual decay rates are defined to be $\gamma_{j}=4 \pi g_{j}^{2} / v_{g}$, for $j=A, B$. Note as well that one can get rid of the Heaviside step functions by locating the observing measure in the suitable position for each electric field component. In the right electric field case, the measure must be located on the right side of the waveguide, i.e. $z>z_{j}$, and on the left side for the left-going electric field component, i.e. $z<z_{j}$. This way, in a near future we will be able ignore the
effect of the step functions in (3.11) as we are only locally working with the $z$ that makes $\theta\left(z-z_{j}\right)=1$ for $E_{R}$ and $\theta\left(z_{j}-z\right)=1$ for $E_{L}$. However, one must keep track of them when introducing the electric field in (3.11) in the resonator equations of motion, so for now, we will keep the explicit notation in the matrix form.

Following up, we are going to obtain now the equations of motion for the resonator operators $\tilde{b}_{j}$. We consider the operators in the number basis to obey the canonical commutation relations $\left[\tilde{b}_{j}, \tilde{b}_{k}^{\dagger}\right]=\delta_{j k}$ and, from the full Hamiltonian description together with using the results in (3.11), the $\tilde{b}_{j}$ equations of motion will yield:

$$
\begin{align*}
\Rightarrow \dot{\tilde{b}}_{A} & =-i\left[\left(\omega_{o}+\Delta_{A}\right)-i \frac{\gamma_{A}}{2}\right] \tilde{b}_{A}-\frac{\sqrt{\gamma_{A} \gamma_{B}}}{2}\left[e^{-i k_{o} \Delta L} \theta\left(z_{A}-z_{B}\right)+e^{i k_{o} \Delta L} \theta\left(z_{B}-z_{A}\right)\right] \tilde{b}_{B}+ \\
& +i \sqrt{2 \pi}\left[e^{i k_{o} z_{A}} \tilde{E}_{R, I N}+e^{-i k_{o} z_{A}} \tilde{E}_{L, I N}\right] g_{A} \\
\Rightarrow \dot{\tilde{b}}_{B} & =-i\left[\left(\omega_{o}+\Delta_{B}\right)-i \frac{\gamma_{B}}{2}\right] \tilde{b}_{B}-\frac{\sqrt{\gamma_{A} \gamma_{B}}}{2}\left[e^{i k_{o} \Delta L} \theta\left(z_{B}-z_{A}\right)+e^{-i k_{o} \Delta L} \theta\left(z_{A}-z_{B}\right)\right] \tilde{b}_{A}+ \\
& +i \sqrt{2 \pi}\left[e^{i k_{o} z_{B}} \tilde{E}_{R, I N}+e^{-i k_{o} z_{B}} \tilde{E}_{L, I N}\right] g_{B} \tag{3.12}
\end{align*}
$$

The decay rates $\gamma_{A}$ and $\gamma_{B}$ have been defined. Now, since we kept the $\theta$ functions explicitly in the electric fields expressions, and assuming that the $A$ resonator is situated to the left side of $B$, i.e., $z_{B}>z_{A}$, only the $\theta\left(z_{B}-z_{A}\right)$ terms will survive. We also assumed $\theta(0)=1 / 2$. Also, the length between resonators has been defined as $\Delta L=z_{B}-z_{A}>0$. With that said, we can re-write both equations of motion in a compact matrix form.

$$
\begin{equation*}
\dot{\vec{b}}=M^{\prime} \vec{b}+i \sqrt{v_{g}} M \vec{E}_{I N} \tag{3.13}
\end{equation*}
$$

Where the $M^{\prime}$ matrix has been defined to be:

$$
M^{\prime} \equiv\left(\begin{array}{cc}
-i\left(\omega_{o}+\Delta_{A}\right)-\frac{\gamma_{A}}{2} & -\frac{\sqrt{\gamma_{A} \gamma_{B}}}{2} e^{i k_{o} \Delta L}  \tag{3.14}\\
-\frac{\sqrt{\gamma_{A} \gamma_{B}}}{2} e^{i k_{o} \Delta L} & -i\left(\omega_{o}+\Delta_{B}\right)-\frac{\gamma_{B}}{2}
\end{array}\right)
$$

Note in (3.13) we introduced $M$ which is equal to $M_{\theta}$ but evaluating all $\theta$ functions to unity. An important remark one needs to take into account here is that, actually, both matrices we defined in the equations of motion, $M^{\prime}$ and $M$, are related. This relation can be trivially observed when expressing the Hamiltonian in terms of matrices. By that we actually mean to use the vector definitions $\vec{b}$ and $\vec{E}_{\text {OUT }}$ and try to write the Hamiltonian in this compact notation. To see that, let us now consider only the terms concerning the resonating cavity operators, that is, only (3.5) and (3.6).

$$
\begin{align*}
& \mathcal{H}_{\text {int }}+\mathcal{H}_{\text {cav }}=-\sqrt{v_{g}} \int d z\left[\vec{E}_{O U T}^{\dagger} M_{\delta}^{\dagger} \vec{b}+\vec{b}^{\dagger} M_{\delta} \vec{E}_{\text {OUT }}\right]+\vec{b}^{\dagger} \mathcal{H}_{\Delta} \vec{b} \\
& M_{\delta}^{\dagger} \equiv \frac{1}{\sqrt{2}}\left(\begin{array}{cc}
\sqrt{\gamma_{A}} \delta\left(z-z_{A}\right) e^{-i k_{o} z} & \sqrt{\gamma_{B}} \delta\left(z-z_{B}\right) e^{-i k_{o} z} \\
\sqrt{\gamma_{A}} \delta\left(z_{A}-z\right) e^{i k_{o} z} & \sqrt{\gamma_{B}} \delta\left(z_{B}-z\right) e^{i k_{o} z}
\end{array}\right) \quad \int_{-\infty}^{\infty} d z M_{\delta}=M \tag{3.15}
\end{align*}
$$

Where $\mathcal{H}_{\Delta}$ has been introduced as a diagonal matrix with terms like $\omega_{o}+\Delta_{j}$, and the $M_{\delta}$ matrix which integrated over $z$ results in the $M$ matrix. We can now operate with the defined vector operators as normal operators which fulfill identical commutation relations. Note though that, in the $M_{\delta}$ matrix, the delta functions will evaluate the step functions in $\vec{E}_{O U T}$ when substituting, in such a way that the final result will be equivalent to:

$$
\begin{equation*}
\dot{\vec{b}}=\left(-i \mathcal{H}_{\Delta}-\int d z M_{\delta} M_{\theta}^{\dagger}\right) \vec{b}+i \sqrt{v_{g}} \int d z M_{\delta} \vec{E}_{I N} \tag{3.16}
\end{equation*}
$$

We must be careful though with this term and pay attention to the step functions. Before evaluating them, the matrix product yields diagonal terms like $\theta\left(z-z_{A}\right)+\theta\left(z_{A}-z\right)=1$ (same for $z_{B}$ ) and off-diagonal terms which involve step function terms like $\theta\left(z_{A}-z_{B}\right) e^{i k_{o} \Delta L}+\theta\left(z_{B}-z_{A}\right) e^{-i k_{o} \Delta L}$ for the equation of motion of $\tilde{b}_{A}$. Observe that in this case, we work locally on $z=z_{A}$, and the only surviving term in the off-diagonal mentioned term yields $e^{i k_{o} \Delta L}$, since we are assuming that $z_{B}>z_{A}$. The same happens in the $\tilde{b}_{B}$ equation of motion. Straight comparing (3.13) and (3.16) one easily sees that, at the end of the day, one gets $M^{\prime}=-i \mathcal{H}_{\Delta}-\int d z M_{\delta} M_{\theta}^{\dagger}$.

We have been defining all kind of matrices, rigorously keeping all terms as strict as possible. Henceforth, it will be safer to simplify slightly the expressions, and all the $M_{\theta}$ matrices coming from (3.11), which are not together with $M_{\delta}$ inside an integral, will be renamed to $M$, where the $\theta$ functions have been evaluated to unity. This is a valid simplification, since all step functions end up accompanied by right and left electric fields which, according to the direction they are being measured, they evaluate to 1 .

We now have a full quantum mechanical description of the cavities and the electric field. Following next, we will introduce in a systematic way a superconducting coupling term between both resonators. After that, we will see how the chiral decay rates and outputs begin to become an important matter.

### 3.1.3 Effective superconducting coupling

To be able to observe a chiral response we need to couple both resonating cavities to a three-port circulator, composed by a superconducting circuit. The coupling will be through both resonators, each to a different port, leaving a third port coupled to the ground. In these scheme, photons absorbed by the cavity resonators will be able to excite the superconducting circuit. The excitations will be idealized as combinations of excess Cooper Pair charges distributed within the circulator islands. Between each of the circulator islands, superconducting Josephson Junctions will mediate the charge hopping between islands, which will incorporate a phase difference $e^{i \varphi}$. Effectively, in our model we can idealize the immediate effects this setup will have in the Hamiltonian as an exchanging term between resonators with a certain phase difference, adding the same term we previously introduced in
(2.17) from Chapter 2. Therefore, the extra term that will join our previous Hamiltonian description will be the following.

$$
\begin{equation*}
\mathcal{H}_{S C}=J e^{-i \varphi} \tilde{b}_{A}^{\dagger} \tilde{b}_{B}+J e^{i \varphi} \tilde{b}_{B}^{\dagger} \tilde{b}_{A} \tag{3.17}
\end{equation*}
$$

Where $J$ is the superconducting coupling strength, which will be mainly defined by the Josephson Energy. We can easily now implement this term in the previous equations of motion, as we displayed them in a matrix notation easy to handle. The circulator is assumed to contain a finite number of exceeding Cooper Pairs which in principle should not change the results. In this model, our system is merely based on two resonating cavities governed by the operators $\hat{b}_{j}$. Therefore, it seems fair to think of the ground state being given by having both resonators empty, and excited states by having one photon in one of the resonators, which will be able to hop from one to another through the superconducting coupling introducing the phase $\varphi$. The circulator is basically a mediating system between both resonators which, containing a finite number of exceeding Cooper Pair charges, will be excited by the coupling with the resonators, and decay back into one of both resonators.

The implementation of this term will directly interfere on the off-diagonal elements in (3.16). More precisely, we can straight add them in the $\mathcal{H}_{\Delta}$ matrix. Note that the phases in (3.17) have the opposite sign, whereas the $e^{i k_{o} \Delta L}$ terms incorporated by the waveguide coupling are equal. This small difference in the phases will allow us to tune an interference through the magnetic flux $\Phi$ going through the superconducting ring (we assume for now that $\varphi \sim \Phi$ ) which will be the main responsible for the chiral outcomes. Therefore, the next step will be to directly obtain the spontaneous decay rates to the right and left directions of the resonators with the superconducting circuit already implemented. Then, we will obtain the probabilities to decay to both directions.

### 3.2 Decay probabilities and chirality

A good observable we can already reproduce with the equations of motion are the decay probabilities to the right and to the left direction. The phases incorporated by the waveguide coupling and the superconducting circuit between the resonators will allow us to create and manipulate a phase interference with the two possible paths a photon can take when decaying from a resonating cavity. We can briefly sketch them, assuming a photon exciting the $A$ resonator (located in the right side), it can chose between the following two options.

- It can directly decay into the waveguide, to the right or left directions. In the first case, the photon will output with no phase acquired, whereas in the second case, the photon will travel through the waveguide, acquiring a phase equal to $e^{i k_{o} \Delta L}$.
- It can create a virtual excitation in the superconducting circuit, traveling through it to the $B$ resonator and acquiring a phase $e^{i \varphi}$. Then, it will decay back into the waveguide to the right or left directions. In the first case, the output state will also gain a phase $e^{-i k_{o} \Delta L}$, and in the last case though, the output state will end up not obtaining any additional phase.

One sees here that there is an actual phase difference between the output to the right and to the left, being $e^{i\left(\varphi \pm k_{o} \Delta L\right)}$. Theoretically, a photon could be able to choose a wider selection of paths involving
loops. It can be absorbed and emitted continuously between both resonators and going backward and forward, or even keep exciting the superconducting system. All these options will contribute in having the same effect on the output states, and they are also implicitly included in our equations of motion.

To obtain the output probabilities, we must first solve the equations of motion of the resonator operators in (3.16) with the superconducting terms in (3.17) implemented in the off-diagonal terms in $\mathcal{H}_{\Delta}$. We will also assume that the input field is null, and consider a weak resonator-waveguide coupling. This will be directly manifested with low values on the decay rates to the waveguide, compared to the superconducting coupling strength. For simplicity, and without loss of generality, we will consider from now on equal conditions on both cavities, i.e. $\gamma_{A}=\gamma_{B} \equiv \gamma$. Hence, the regime we will work on will be conditioned by having $\gamma \ll J$. This also means that we will consider the energy splitting between cavities is null $\left(\Delta_{A}=\Delta_{B}=0\right)$. The off-diagonal terms in $M^{\prime}$ will be dominated by the superconducting coupling. We can sketch this within the following matrix separation.

$$
M^{\prime}=\left(\begin{array}{cc}
-i \omega_{o}-\frac{\gamma}{2} & -i J e^{i \varphi}  \tag{3.18}\\
-i J e^{-i \varphi} & -i \omega_{o}-\frac{\gamma}{2}
\end{array}\right)+\left(\begin{array}{cc}
0 & -\frac{\gamma}{2} e^{i k_{o} \Delta L} \\
-\frac{\gamma}{2} e^{i k_{o} \Delta L} & 0
\end{array}\right)
$$

Note that the first matrix contains in the diagonal terms the decay rates of the resonators and the off-diagonal terms the superconducting coupling between the islands. We will consider then the time evolution on the cavities to be mainly determined by the first matrix compared with the second, which in the $\gamma \ll J$ regime will not play an important role. We will then diagonalize the first matrix and, in the new eigenbasis we will transform the second matrix, which will be treated as a perturbation to the eigensystem. As we said, all the terms included in the first matrix are related to the circulator-resonator system. Thus, for future convenience, let us re-name the new eigenbasis to be the circulator eigenbasis. We will also define the operators in the new eigensystem to be $\tilde{c}_{j}$ and the matrix representation of the equations of motion of the new operators will be determined by:

$$
\begin{align*}
& \dot{\vec{c}}=D \vec{c} \quad \text { with }: \quad D=\left(\begin{array}{cc}
-\frac{\Gamma_{1}}{2}-i E_{1} & -i \frac{\gamma}{2} e^{i k_{o} \Delta L} \sin (\varphi) \\
i \frac{\gamma}{2} e^{i k_{o} \Delta L} \sin (\varphi) & -\frac{\Gamma_{2}}{2}-i E_{2}
\end{array}\right)  \tag{3.19}\\
& \tilde{c}_{1}^{\dagger}|00\rangle=-\frac{1}{\sqrt{2}}\left(e^{i \varphi}|10\rangle-|01\rangle\right) \quad \tilde{c}_{2}^{\dagger}|00\rangle=\frac{1}{\sqrt{2}}\left(e^{i \varphi}|10\rangle+|01\rangle\right)
\end{align*}
$$

The $D$ matrix is the $M^{\prime}$ matrix in the eigenbasis of the first matrix in (3.18), i.e., in the circulator eigenbasis. We will ignore the resulting off-diagonal terms in (3.19) as they can be ignored in the weak coupling regime. We also show the Schrodinger representation of each eigenstate in the resonator number basis. Each eigenstate represents two different entangled state combinations of a photon living in the resonator $A$ or $B$. Sending a photon trough the waveguide, depending on the photon frequency, this one will be able to excite the system to one of the excited states. It is interesting to remark here that, tuning accordingly the magnetic flux within $\varphi$, one can create all sort of entangled states with both resonating cavities. In principle, the number of exceeding Cooper Pairs in the superconducting circuit is conserved and does not perturb the global behavior as we presented before. The ground state of the system will be defined by $|00\rangle$, meaning that the resonators are empty, and that the circulator
is found in its ground state. The last statement is implicit in the notation. From this eigenstates, we can directly obtain their equations of motion.

$$
\begin{equation*}
\tilde{c}_{1}(t)=\tilde{c}_{1}(0) e^{-\frac{\Gamma_{1}}{2} t-i E_{1} t} \quad \tilde{c}_{2}(t)=\tilde{c}_{2}(0) e^{-\frac{\Gamma_{2}}{2} t-i E_{2} t} \tag{3.20}
\end{equation*}
$$

Each eigenstate has an oscillation component mainly governed by the Josephson coupling and another decaying component, both given by:

$$
\begin{align*}
\frac{\Gamma_{1}}{2} & =\frac{\gamma}{2}\left(1-\cos \left(k_{o} \Delta L\right) \cos (\varphi)\right) & \frac{\Gamma_{2}}{2} & =\frac{\gamma}{2}\left(1+\cos \left(k_{o} \Delta L\right) \cos (\varphi)\right)  \tag{3.21}\\
E_{1} & =\omega_{o}-J-\frac{\gamma}{2} \sin \left(k_{o} \Delta L\right) \cos (\varphi) & E_{2} & =\omega_{o}+J+\frac{\gamma}{2} \sin \left(k_{o} \Delta L\right) \cos (\varphi)
\end{align*}
$$

As we previously said, the obtained eigenstates in the Heisenberg picture show two possible excited states the circulator can be found in. One can see that the energy splitting between both excited states will be governed by the Josephson Energy, being $E_{2}-E_{1} \simeq 2 J$. We are working though in the $\gamma \ll J$ regime, stating that the energy separation must be big enough to avoid degenerate transitions to both possible excited states a photon can be absorbed to. That is mainly why we can ignore the decay rates when being compared with $J$.

The decay rates from each excited state, $\Gamma_{j}$, mainly depend on the different phases obtained through the paths a photon can take in our setup (through the superconducting circuit or the waveguide), apart from the decay rate $\gamma$ of each resonator to the waveguide. Since the diagonalization was given by a change of basis through unitary transformation, the results will be the same in both basis. Hence, we will work in the basis we feel more comfortable with, which in this case is the circulator eigenbasis. With the solutions on the cavity operators, we can now directly substitute (3.20) in (3.11) in the eigenbasis. The probability of outputting a photon from a decay transition to the right and left directions will be given by the following expressions.

$$
\begin{equation*}
P_{R}=v_{g} \int_{0}^{\infty} E_{R}(t) E_{R}^{\dagger}(t) d t \quad P_{L}=v_{g} \int_{0}^{\infty} E_{L}(t) E_{L}^{\dagger}(t) d t \tag{3.22}
\end{equation*}
$$

The group velocity was incorporated in the definitions by unit convention. The electric field commutators in (3.7) state that $E_{R}$ and $E_{L}$ have units of $1 / \sqrt{L}$. Integrating over time must be payed with a time unit on the denominator, and also a length unit to compensate the electric field dimensions. Therefore we end up with velocity dimensions which can be payed off with the characteristic velocity of our system, being the group velocity $v_{g}$.

To obtain the decay probabilities, let us first write the electric field operators in the eigenbasis $\tilde{c}$.

$$
\begin{align*}
\hat{E}_{R}(t) & =\frac{i}{2} \sqrt{\frac{\gamma}{v_{g}}} e^{-i k_{o} z_{B}}\left(1-e^{i k_{o} \Delta L} e^{i \varphi}\right) \tilde{c}_{1}(t)+\frac{i}{2} \sqrt{\frac{\gamma}{v_{g}}} e^{-i k_{o} z_{B}}\left(1+e^{i k_{o} \Delta L} e^{i \varphi}\right) \tilde{c}_{2}(t) \\
\hat{E}_{L}(t) & =\frac{i}{2} \sqrt{\frac{\gamma}{v_{g}}} e^{i k_{o} z_{B}}\left(1-e^{-i k_{o} \Delta L} e^{i \varphi}\right) \tilde{c}_{1}(t)+\frac{i}{2} \sqrt{\frac{\gamma}{v_{g}}} e^{i k_{o} z_{B}}\left(1+e^{-i k_{o} \Delta L} e^{i \varphi}\right) \tilde{c}_{2}(t) \tag{3.23}
\end{align*}
$$

Substituting now these expressions in (3.22) one ends up obtaining the following decay probabilities to the right and left directions.

$$
\begin{align*}
& P_{R}=\frac{\sin ^{2}\left(\frac{k_{o} \Delta L+\varphi}{2}\right)}{1-\cos \left(k_{o} \Delta L\right) \cos (\varphi)} \tilde{c}_{1}^{2}(0)+\frac{\cos ^{2}\left(\frac{k_{o} \Delta L+\varphi}{2}\right)}{1+\cos \left(k_{o} \Delta L\right) \cos (\varphi)} \tilde{c}_{2}^{2}(0)  \tag{3.24}\\
& P_{L}=\frac{\sin ^{2}\left(\frac{k_{o} \Delta L-\varphi}{2}\right)}{1-\cos \left(k_{o} \Delta L\right) \cos (\varphi)} \tilde{c}_{1}^{2}(0)+\frac{\cos ^{2}\left(\frac{k_{o} \Delta L-\varphi}{2}\right)}{1+\cos \left(k_{o} \Delta L\right) \cos (\varphi)} \tilde{c}_{2}^{2}(0)
\end{align*}
$$

We express them in terms of the initial eigenstate conditions, so they will mainly depend on the initial state of the system. They also depend on the phases $k_{o} \Delta L$ and $\varphi$ as we predicted. More interestingly, we can set these phases with a certain value which will completely null the output to one of the directions. Thanks to the probability normalization, the spots which suppress the decays to one direction will show a probability 1 of decaying to the other direction. Those spots will provide the outputs with the most chiral behavior, since the probability distance from decays to the right and to the left is at its maximum point, and the ones we are actually more interested in.

In Figure 3.1 we can see sketched the output probabilities for a given distance between resonators. The first case, displaying the situation where $k_{o} \Delta L=\pi / 2$, shows a more symmetrical behavior towards where we can find the maximum chiral behavior. The maximum chiral decays are observed in $\varphi=\pi / 2$ and $\varphi=3 \pi / 2$ for both possible excited states, with the probabilities inverted. This means that if the first excited state at the maximum chiral spot decays to the right, the second excited state will always decay to the left. This would suppose a problem to the chiral behavior if both excited states had a similar energy. However, we are assuming a sufficiently big energy splitting given by $2 J$, which ensures that the symmetry in the system is broken and chirality holds. See also that, in the maximum chiral spot, if the system decays to the right direction for example, the decay to the left is forbidden, as we previously assumed.

We also show the decay probabilities for $k_{o} \Delta L=\pi / 3$. The results are very similar than before, but in this case the maximum chiral behavior will depend on the initial state of the system. If initially the system is found in the eigenstate corresponding to $\tilde{c}_{1}$, the maximum chiral outputs will be observed in $\varphi=\pi / 3$ and $\varphi=2 \pi-\pi / 3$. On the other hand, if the system is initially in the eigenstate corresponding to $\tilde{c}_{2}$, the maximum chiral outputs will be seen if $\varphi=2 \pi / 3$ and $\varphi=2 \pi-2 \pi / 3$.

Apart from these two cases, it is important to mention that we can always find a spot with maximum chiral outputs, i.e., the system will always decay to a certain direction. The only situation where the outputs are totally symmetrical for any superconducting phase differences is when $k_{o} \Delta L=$ $n \cdot \pi$, for $n=0,1,2, \ldots$ See also that, for any distance between the resonators, if $\varphi=n \cdot \pi$ the outputs become symmetrical.


Figure 3.1: The decay probabilities are sketched in this figure, for a fixed distance between resonators. The top figures show the case $k_{o} \Delta L=\pi / 2$ and the figures below the case $k_{o} \Delta L=\pi / 3$. The right plots show the case where the system is initially in the $\tilde{c}_{1}$ eigenstate, and the left plots the case of having the system initially on the $\tilde{c}_{2}$ state. We can clearly see that the maximum chirality will be observed for values of superconducting phase difference $\varphi=\pi / 2$ and $\varphi=3 \pi / 2$ in the top plots. The bottom plots show a similar behavior, but the maximum chiral decays will be obtained for $\varphi=n \pi / 3$, being $n=0,1,2, \ldots$ depending on the initial state of the system.

For now we have managed to see that our system actually shows a potential chiral behavior when emitting photons from the resonators excited states. The direction probability of the decays can be modified and manipulated in a flexible way thanks to the quantum interference between the phases acquired through the different paths. We are still missing though a good measure to display that the device is actually working at the most optimal chiral behavior. Following next we will define a fidelity measure based on the scattering amplitudes of our system.

### 3.3 Scattering and Fidelity

The decay directional probabilities we have seen are a good signal that the theoretical model we are building has a chiral response towards decaying into photons. We need though now to see how the system reacts when there is an input field applied and if the scattering amplitudes will also show chiral outputs. As input field we understand single photons traveling through the waveguide. These can be described with $\vec{E}_{I N}$ present in both equations of motion (3.11) and (3.16). For now we have ignored them as we have only been concerned about the decay rates. However, it is now time to begin taking them into account to describe how the circulator reacts with incoming single photons.

In this section, we will first obtain the S-matrix out of the electric field and resonating cavities equations of motion. As we have mentioned, we will work with single photons as inputs. Finally we will design a good fidelity measure of our system, which we will be able to directly obtain with the scattering amplitudes. The fidelity will show in which regimes the scattering amplitudes become more chiral, and we will take a close analytic look to these working regimes.

### 3.3.1 S-Matrix

Previously we obtained the equations of motion of the resonator operators $\tilde{b}_{j}$ in (3.16) and the electric fields $E_{R}$ and $E_{L}$ in (3.11). We will now recover them, this time without ignoring the input field $\vec{E}_{I N}$, and also considering the superconducting coupling which will enter in the off-diagonal terms on $\mathcal{H}_{\Delta}$. This time we will not work on the eigenbasis, but instead, we will solve equation (3.16) by switching into Fourier space of the time variable. That will allow us to work with the incoming photon frequency instead, and introduce it as a variable.

We can directly switch the time variable to Fourier space by mapping the time derivative to $\partial_{t} \rightarrow-i \omega$, where $\omega$ is the photon frequency. In that case, the $\tilde{b}_{j}$ equations of motion will become:

$$
\vec{b}=\sqrt{v_{g}}\left(i \mathcal{H}_{\Delta}^{S C}-i \omega \mathbb{I}+\int d z M_{\delta} M_{\theta}^{\dagger}\right)^{-1} M \vec{E}_{I N} \quad \text { with : } \quad \mathcal{H}_{\Delta}^{S C}=\left(\begin{array}{cc}
\omega_{0} & J e^{i \varphi}  \tag{3.25}\\
J e^{-i \varphi} & \omega_{0}
\end{array}\right)
$$

Where $\mathcal{H}_{\Delta}^{S C}$ is the $\mathcal{H}_{\Delta}$ matrix with the superconducting coupling in the off-diagonal terms (and assuming no detuning between cavities in resonance $\Delta_{A}=\Delta_{B}=0$ ), and $\mathbb{I}$ is the $2 \times 2$ identity matrix.

Directly from this result, we can substitute equation (3.25) in the electric field equation of motion in (3.11) and finally obtain the S -matrix as the relation between the input and output electrical fields.

$$
\begin{equation*}
\vec{E}_{O U T}=S \vec{E}_{I N} \quad \text { with }: \quad S=1-M^{\dagger}\left(i \mathcal{H}_{\Delta}^{S C}-i \omega \mathbb{I}+\int d z M_{\delta} M_{\theta}^{\dagger}\right)^{-1} M \tag{3.26}
\end{equation*}
$$

The unitary shape of $S,[27,28]$, can be demonstrated by operating:

$$
\begin{align*}
& S S^{\dagger}=1+M^{\dagger}\left(\mathcal{H}_{\Delta}^{S C}-\omega \mathbb{I}-i \int d z M_{\delta} M_{\theta}^{\dagger}\right)^{-1} M M^{\dagger}\left[\left(\mathcal{H}_{\Delta}^{S C}-\omega \mathbb{I}-i \int d z M_{\delta} M_{\theta}^{\dagger}\right)^{-1}\right]^{\dagger} M- \\
& -i M^{\dagger}\left\{\left[\left(\mathcal{H}_{\Delta}^{S C}-\omega \mathbb{I}-i \int d z M_{\delta} M_{\theta}^{\dagger}\right)^{-1}\right]^{\dagger}-\left(\mathcal{H}_{\Delta}^{S C}-\omega \mathbb{I}-i \int d z M_{\delta} M_{\theta}^{\dagger}\right)^{-1}\right\} M=1 \tag{3.27}
\end{align*}
$$

Thanks to the Hermitic shape of $\mathcal{H}_{\Delta}^{S C}$, the last equality is fulfilled only if:

$$
\begin{equation*}
M M^{\dagger}=\int d z\left[M_{\delta} M_{\theta}^{\dagger}+M_{\theta} M_{\delta}^{\dagger}\right] \tag{3.28}
\end{equation*}
$$

Which is actually true, and the defined S-matrix is unitary. See that we needed to keep track of the step functions to ensure the unitaricity of the S-matrix. Indeed we need to always know to which direction is each term being measured to avoid overlaps, in which case the results would not give chiral results. Following next, we will finally obtain a fidelity measure to quantize the quality of the directional dependency the system presents.

### 3.3.2 Fidelity

As a final and conclusive result of the presented model, we will build a fidelity measure that indicates the regime in which the whole system scatters completely chiral results. In a real setup, single photons would be sent to the circulator and, depending on the inner configuration and the direction they come from, these would be scattered with different amplitudes. The most interesting spot is in which the difference between amplitudes related to the direction of the inputs is more significant. This spot will be given by the decay probabilities we have seen in Figure 3.1, more exactly when the decay probability to the right is at its highest point, 1, and the decay probability to the left is null (and the other way around). It will be then when the circulator behaves completely different depending on the direction we are looking at, i.e., at its most chiral regime. Knowing that, we will obtain the scattering amplitudes of a photon being outputted to the same direction it was absorbed. Since the scattered photon will mainly excite the eigensystem of the circulator to an excited energy level, we will be interested in working with the eigenbasis. Therefore, we will first re-write the S-matrix in (3.26), but now switching on the eigenbasis. Thanks to the weak coupling to the waveguide, $\gamma \ll J$, we
can ignore the terms containing decay rates compared to the energy splitting between excited states. Knowing also the matrix property $(A B C)^{-1}=C^{-1} B^{-1} A^{-1}$, we can write the scattering matrix in the following way.

$$
\begin{equation*}
S=1-\left(M^{\dagger} U\right)(-D-i \omega \mathbb{I})^{-1}\left(M^{\dagger} U\right)^{\dagger} \tag{3.29}
\end{equation*}
$$

Where $U$ is the unitary transformation responsible of changing to the eigenbasis. In the weak coupling regime, we can expand the $D$ matrix in Taylor Series assuming the off-diagonal terms are much smaller than the diagonal terms, ending up expressing the denominator in a Dyson Series form, from which we can consider only the first order terms to matter. In other words, we can ignore the off-diagonal terms in $D$. From the S-matrix we will only be interested in the $S_{11}$ and $S_{22}$ elements (diagonal terms). We will harness the definition of the output probabilities defined in (3.24) but differentiating between decays coming from both the different eigenstates as $P_{R}=P_{R}^{(1)} c_{1}^{2}(0)+P_{R}^{(2)} c_{2}^{2}(0)$ and $P_{L}=P_{L}^{(1)} c_{1}^{2}(0)+P_{L}^{(2)} c_{2}^{2}(0)$. Then, the scattering amplitudes we are seeking will become:

$$
\begin{align*}
& S_{11}=1-\frac{2 P_{R}^{(1)}}{1+i x_{1}}-\frac{2 P_{R}^{(2)}}{1+i x_{2}} \\
& S_{22}=1-\frac{2 P_{L}^{(1)}}{1+i x_{1}}-\frac{2 P_{L}^{(2)}}{1+i x_{2}} \tag{3.30}
\end{align*}
$$

Where $x_{1}=2\left(E_{1}-\omega\right) / \Gamma_{1}$ and $x_{2}=2\left(E_{2}-\omega\right) / \Gamma_{2}$. We now have a description of the scattering amplitudes, which also incorporate the information regarding the excited level a photon accesses when being absorbed. Thanks to the weak coupling assumption, the excited energies splitting will be mainly given by $E_{2}-E_{1}=2 J \gg \gamma$, and therefore we can assume that only one level is accessed without being perturbed by the second one. Assuming we tune our photon to access the first excited level, that is $\omega-\omega_{0} \simeq-J$, and tuning our circulator at its most chiral regime, for example in $k_{o} \Delta L=\varphi=\pi / 2$ with $P_{R}^{(1)}=1$ and $P_{L}^{(1)}=0$, see how the scattering amplitudes will evaluate to $S_{11}=-1$ and $S_{22}=1$. In that situation, we expect that the system does not interact with photons traveling to the left, which is exactly what $S_{22}$ is saying. However, at the most chiral spot, the right going photon will interact with the circulator, acquiring a phase $e^{i \pi}$. These outcomes are typical of gyrators [29], where the circulator acts as a $\pi$-gate only to photons traveling to a certain direction.

Knowing the behavior of the system at its highest chiral point, we can design now a fidelity measure able to indicate the cases when we reach this regime. We will use the idea of how a Mach-Zehnder interferometer (MZI) works to build a fidelity measure. A MZI consists of two $50 / 50$ beam splitters and two mirrors. A single photon will be sent to the first beam splitter, made of semi-transparent mirrors which will reflect or transmit a photon with equal probability. Both mirrors are placed in such a way that both paths created by the first beam splitter will re-unite together into the second beam splitter, which will again give the photon the option to be transmitted or reflected. The interesting physics behind the MZI is that the presence of both possible paths may create an state interference on the outputs, which changes when an object obstructs one arm. A normal object would block the way from both directions, but the chiral response from our model will bring a noticeable difference depending on the photon coming direction. Therefore, if we place the circulator in one arm of the MZI, the output state will be totally different whether if the photon is scattered by our system or not.


Figure 3.2: Illustration of a Mach-Zehnder Interferometer with a circulator blocking one of the arms. This implementation was introduced in [29] and also used in [22]. We show the two possibilities we take to obtain the fidelity measure of our system. The right figure (a) shows a photon entering from the port 0 and exiting from 3 , and the second (b) a photon going from 3 to 1 . Thanks to the presence of the circulator (square box with a circular arrow), at its highest chiral behavior, the probabilities $P_{0 \rightarrow 3}$ and $P_{3 \rightarrow 1}$ will be equal to 1 .

We will proceed in calculating the output states of the MZI and see clearly how it works with the circulator blocking one arm [30]. Looking at Figure 3.2, we have illustrated a MZI with a circulator placed in of the arms. In (a) we have two possible inputs ( 0 and 1 ) and two possible outputs ( 2 and $3)$. We will name $\hat{a}_{j}$ the photon states going trough the port $j=0,1,2,3$. Assuming that we send a photon into the MZI through 0, the output states will be:

$$
\begin{equation*}
\hat{a}_{2}=\frac{i}{2}\left(1+S_{11}\right) \hat{a}_{0} \quad \hat{a}_{3}=\frac{1}{2}\left(1-S_{11}\right) \hat{a}_{0} . \tag{3.31}
\end{equation*}
$$

Where we introduced $S_{11}$ as the scattering amplitude when a photon comes with the right direction and outputs from the circulator to the same direction, which is obtained form the S-matrix in equation (3.26). We see that, if the scattering amplitude $S_{11}=1$, meaning that the input field coming to the right is equal to the output field going to the same direction (photon does not see the circulator), then the output state will be certain, coming out from port 2. Also, if the photon is scattered from the circulator acquiring a $e^{i \pi}$ phase (meaning $S_{11}=-1$ ), the output will become certain but this time with probability 1 of coming out of port 3 . If we now propose the same experiment, but sending now a photon through port 3 , the output state will be given by:

$$
\begin{equation*}
\hat{a}_{0}=\frac{1}{2}\left(1-S_{22}\right) \hat{a}_{3} \quad \hat{a}_{1}=\frac{i}{2}\left(1+S_{22}\right) \hat{a}_{3} . \tag{3.32}
\end{equation*}
$$

Now here the $S_{22}$ scattering matrix element is the amplitude of having an output field going to the left with an input field coming from the same direction. We see very similar results as above, but now the scattering amplitude from the circulator is different. We can measure the different behavior of the circulator treating photons differently depending on their incoming direction with certainty, taking the cases where the outputs come out from a port with probability 1. The maximum chiral behavior will be given when $S_{11}=-1$ and $S_{22}=1$ (or $S_{22}=-1$ and $S_{11}=1$ ), as a photon traveling to the right direction will be scattered always with a phase $e^{i \pi}$ and a photon going to the left direction will
be ignored. Hence, we can build a fidelity measure taking into account the probability of a photon going from 0 to 3 , and from 3 to 1 , as it is sketched in Figure 3.2.

$$
\begin{equation*}
F=P_{0 \rightarrow 3} \times P_{3 \rightarrow 1}=\frac{1}{16}\left(1+S_{11}\right)\left(1+S_{11}^{*}\right)\left(1-S_{22}\right)\left(1-S_{22}^{*}\right) \tag{3.33}
\end{equation*}
$$

That way, the maximum value the fidelity will take (equal to 1 ) will happen when the circulator behaves completely different throughout photons coming from different directions. We may note here that this fidelity measure is relatively flexible. With the fidelity we have defined we are considering the maximum chirality happening when the circulator applies a phase $e^{i \pi}$ to photons traveling to the right. However, if the circulator applies the $e^{i \pi}$ phase only to left-going photons, the chiral behavior is equally interesting, but in equation (3.33) the fidelity yields 0 . To take that into account we could instead define the fidelity to be the probability of a photon to go from $0 \rightarrow 2$ and $2 \rightarrow 1$, so the maximum fidelity would appear when the phase $e^{i \varphi}$ is applied only to photons going to the left direction. We should take into account this relative definition when interpreting the fidelity results. Sticking with our decisions, in order to observe good fidelity values, we will look for maximum decay probabilities only to the right direction. Although, we must remember that the maximum chirality will be also applicable to decays to the left direction. The circulator will behave differently in both cases, but the nature of the chiral outputs will be the same.

Looking back at the scattering terms derived in (3.30), we see the decay probabilities introduced in the previous section will play an important role determining the fidelity of the system. We can therefore guide ourselves by looking again at the plots in Figure 3.1. The most chiral reactions occur for values of $k_{o} \Delta L=\pi / 2$ with $\varphi=\pi / 2$ and $\varphi=3 \pi / 2$. The only difference between the last two values is that in the first $(\varphi=\pi / 2)$, the decay from the circulator eigenstate $\tilde{c}_{1}$ will output a photon to the right direction while decaying from $\tilde{c}_{2}$ will output a photon to the left. However, when $\varphi=3 \pi / 2$, the left photon will be outputted from the decay of $\tilde{c}_{1}$ and the right photon from $\tilde{c}_{2}$. We expect then the fidelity to show a very similar behavior as well in these cases. It will not be the case for the other values we have seen in Figure 3.1. For $k_{o} \Delta L=\pi / 3$ the most chiral spots are observed to happen when $\varphi=\pi / 3$ (right decay from $\tilde{c}_{1}$ ), $\varphi=2 \pi / 3$ (left decay from $\tilde{c}_{2}$ ), $\varphi=4 \pi / 3$ (right decay from $\tilde{c}_{2}$ ) and $\varphi=5 \pi / 3$ (left decay from $\tilde{c}_{1}$ ). We will look closer at these regimes in the following section, and we will discuss the meaning of the results we observe.

### 3.4 Model results and conclusions

A full theoretical model has been built from the coupling of a waveguide to two resonating cavities, coupled at the same time through a superconducting circuit described by a microwave circulator. Our theory predicts a high chiral behavior decaying from the eigenstates both resonating cavities can be found in. We have also obtained a fidelity measure built form the scattering amplitudes of individual microwave photons come through the waveguide from the right and left directions. We present this fidelity to be a trustful clear measure to characterize the chiral behavior of the circulator. In this section we will see the results obtained straight from the fidelity written in (3.33) and we will discuss and explain why the circulator seems to express a chiral behavior in certain regimes with the tools we have already presented.

### 3.4.1 Fidelity results

Guiding ourselves with the results observed in Figure 3.1, we will illustrate the fidelity behavior of those cases in Figure 3.3. We will express it in terms of the incoming photon frequency. The peaks appear near the energy of the involved eigenstate driving the excitations and photon emissions. The observed peaks in the fidelity respond to photon emissions to the right direction, while for photons been emitted to the left, the fidelity yields null. This result may be confusing, as it seems like we are discriminating the chiral behavior for left-going photons and we only consider chirality for right-going photons. As we argued above, this issue comes from the relative definition of the fidelity through blocking one arm of a MZI. One may note that blocking the other MZI arm, the fidelity measure will output good values for photons being chirally scattered this time to the left direction. However, sticking with out starting fidelity definition, we clearly see in Figure 3.3 how the decays to the right direction show good fidelity values. For instance, the small peaks for $k_{o} \Delta L=\pi / 3$ with $\varphi=2 \pi / 3$ and $\varphi=5 \pi / 3$ would have reached 1 if we had blocked the other MZI arm with the circulator, as they describe transitions of photons decaying to the left direction (see Figure 3.1).

All fidelity peaks are found around the photon frequencies corresponding to both possible excited states energy transitions from the ground state. We show with dashed vertical color lines which eigenstate energy corresponds the main transitions providing with the chiral behavior. In blue continuous lines we show the fidelity obtained through the weak coupling assumption, assuming $10 \gamma=J$. With the same values, with blue dashed lines we show the fidelity but without the weak coupling approximations (i.e., without ignoring $\gamma$ compared to $J$ ). See that both continuous and dashed blue lines coincide in all plots, indicating that the taken decay rate values adjust properly to the weak coupling regime. However, in black continuous lines we show the fidelity through the weak coupling approximations and with dashed black lines the fidelity without taking these approximations, having $\gamma=J$. Since these values do not fulfill the weak coupling assumptions, the results obtained with the approximations deviate significantly form the non-approximated results, which at the same time, show worst fidelity results. Thus, assuming a weak coupling regime to the waveguide it is a good assumption to observe an optimal chiral behavior.

On the other hand, in Figure 3.3 we can also see how in the $k_{o} \Delta L=\pi / 2$ case, the peaks coincide perfectly with the dashed color lines, being the transition energies. For $k_{o} \Delta L=\pi / 3$ though, the peaks seem to be slightly deviated from the eigenenergies. This small deviation may come from the decay probabilities in Figure 3.1. Unlike the $k_{o} \Delta L=\pi / 2$ case, for $k_{o} \Delta L=\pi / 3$, when the decay from an eigenstate goes to the right with probability 1 , the decay from the other eigenstate to the same direction is not totally zero, giving a contribution then to the fidelity which may slightly swipe the peak of the maximum chiral performance.

The results show that the circulator behaves differently highly depending on the traveling direction the scattered photons. The phases $\varphi=\pi / 2$ and $k_{o} \Delta L=\pi / 2$ appear to be a good choice of chiral performance. Following up, we will take a closer look at the analytic behavior of the fidelity and how it approaches to its maximum value.


Figure 3.3: Fidelity results for the cases observed with the decay probabilities. We outline two different cases, having $J / \gamma=10$ (blue line), and $J / \gamma=1$ (black line) with $J=1 \mathrm{GHz}$. We can observe if these cases fit into the weak coupling regime, as in dashed lines we can see the non-approximated model predictions and in continuous lines within the weak coupling approximations. Observe how the blackcontinuous lines move far apart from the black-dashed lines as indeed show a case which a weak coupling is not fulfilled. See also how the $J / \gamma=10$ case shows good fidelity results fitting in with the weak coupling assumptions. We also display with vertical dashed lines the energy of the eigenstates from which the excitations and decays happen. The red dashed line ( $\omega-\omega_{o}=E_{2}$ ) corresponds to the energy of the $\tilde{c}_{2}$ state and the green dashed line $\left(\omega-\omega_{o}=E_{1}\right)$ corresponds to the energy of the $\tilde{c}_{1}$ state. If the fidelity peak falls on top of one of these lines, the photon scatterings excite the system state to the corresponding eigenstate, then that eigenstate is going to provide the main chiral decays. We see the maximum fidelity is achieved in all cases where the excited state decays to the right direction. The cases $k_{o} \Delta L=\pi / 3$ with $\varphi=2 \pi / 3$ and $k_{o} \Delta L=5 \pi / 3$ we know they produce maximum chiral decays to the left, as we can see in Figure 3.1. Our fidelity measure reflects a bad results (within the weak coupling regime) due to its relative definition, but we actually know that they are also good chiral choices.

### 3.4.2 Maximum fidelity behavior

As a final step in the discussion of the results of our theory, we will look at the fidelity analytic behavior for photon frequencies close to the transition energy. To do that, we will take $k_{o} \Delta L=\pi / 2$ and $\varphi=\pi / 2$ and $\varphi=3 \pi / 2$ to be an optimal regimes. Then, the fidelity can be read as:

$$
\begin{equation*}
\left.F\right|_{\varphi=\frac{\pi}{2}}=\left.\frac{x_{1}^{2}}{\left(1+x_{1}^{2}\right)\left(1+x_{2}^{2}\right)} \quad F\right|_{\varphi=\frac{3 \pi}{2}}=\frac{x_{2}^{2}}{\left(1+x_{1}^{2}\right)\left(1+x_{2}^{2}\right)} \tag{3.34}
\end{equation*}
$$

Note that for the $\varphi=\pi / 2$ case, tuning the photon frequency to be $\omega=E_{2}$, the system will be excited to $\tilde{c}_{2}$ and the decay will be always to the right. However, if we send photons with frequency $\omega=E_{1}$, the system will be instead excited to $\tilde{c}_{1}$ and the decays will be always go to the left. In this last case we see as $x_{1}=0$ that the fidelity will be zero, even though behaving totally chiral. The same is observed in the $\varphi=3 \pi / 2$ case, only considering good fidelity for decays to the right. This is not an error, but merely an issue related to the fidelity definition as we already discussed.

Knowing that our fidelity definition works for decays to the right, we will then consider $\omega=E_{1}$ for $\varphi=\pi / 2$ and $\omega=E_{2}$ for $\varphi=3 \pi / 2$. That way, both fidelity values become:

$$
\begin{equation*}
\left.F\right|_{\varphi=\frac{\pi}{2}}=\left.F\right|_{\varphi=\frac{3 \pi}{2}} \simeq 1-\frac{\gamma^{2}}{16 J^{2}} \tag{3.35}
\end{equation*}
$$

Where we assumed again $\gamma \ll J$. The scattering becomes chiral as long as the energy difference between excited states (proportional to $J$ ) is big enough so the photon does not have the chance to access the second excited state by accident. See in Figure 3.1 that the decays from the second excited states have the exact opposite decay direction probabilities than the firs excited state. This means that for a certain value of $\varphi$ and $k_{o} \Delta L$, if the first excited state decays to a certain direction, the second excited state will decay to the opposite direction. Hence, if a photon is able to access to both excited states, the decay will no longer be chiral. This argument directly forbids the degeneracy of eigenstates for a chiral response to exist.

With the same process, we will now look at the $k_{o} \Delta L=\pi / 3$ case. The fidelity will output a value close to unity if the decay goes always to the right so, looking at Figure 3.1, the $\tilde{c}_{1}$ eigenstate decays to the right when $\varphi=\pi / 3$ and $\tilde{c}_{2}$ decays to the right if $\varphi=4 \pi / 3$. We have the freedom to choose one configuration, and in this case we will choose the latest. See this case is not as simple as before, as now, the decays form $\tilde{c}_{2}$ will go to the right with probability one, but from $\tilde{c}_{1}$ we will have that $P_{R}^{(1)}=1 / 5$ and $P_{L}^{(1)}=4 / 5$ (values obtained by using the formula in (3.25)). The fact that $\tilde{c}_{1}$ does not decay at its maximum chiral point perturbs the decay of $\tilde{c}_{2}$ in such a way that the maximum fidelity it not longer in $\omega=E_{2}$. In a general way, one can see the maximum fidelity for:

$$
\begin{equation*}
F_{M A X}=\frac{x_{1}^{2}\left(P_{R}^{(1)}+1\right)^{2}+\left(x_{1}^{2}+\left(P_{R}^{(1)}\right)^{2}\right)^{2}}{\left(x_{1}^{2}+1\right)^{2}} \quad \text { for }: \quad x_{2}=\frac{P_{R}^{(1)}}{x_{1}} \tag{3.36}
\end{equation*}
$$

The presented $x_{1}-x_{2}$ relation implies that the maximum fidelity will be given by photons with frequency:

$$
\begin{equation*}
\omega=\frac{1}{2}\left( \pm \sqrt{\left(E_{1}-E_{2}\right)^{2}+4 \Gamma_{1} \Gamma_{2} P_{R}^{(1)}}+E_{1}+E_{2}\right) \tag{3.37}
\end{equation*}
$$

Which at the $\gamma \ll J$ limit yields $\omega \simeq E_{1}, E_{2}$, depending on which excited level decays to the right. Also, in that limit, the maximum frequency from (3.36) will take the approximate value of:

$$
\begin{equation*}
F \simeq 1-\frac{\gamma^{2}}{16 J^{2}}\left(1-2 P_{R}^{(1)}\right) \tag{3.38}
\end{equation*}
$$

Remember we are here tuning the decays from the second excited state, so we see that if the other possible excited state is also able to decay to the right, the fidelity will be modified by the presence of other states. However, if the tuned excited level decays to the right, the other eigenstate will always prefer to decay to the left as we have previously seen in Figure 3.1, so in this case $P_{R}^{(1)} \leq 0.5$. Therefore, it is expected to even improve the fidelity by having a secondary excited which does display symmetric decay rates. However, this variation will not be significant in the $\gamma \ll J$ region we operate.

The theoretical model we have built provided a set of two eigenstates of the system composed by two resonant cavities coupled through a superconducting circuit (circulator) and each coupled to a waveguide. Thanks to the presence of the circulator, and the ability of tuning the phase difference with a magnetic field, we have been able to see chiral decays from all the excited states. The next step would be to realize how would a real superconducting circuit QED imitate this behavior. In the following chapter we will idealize the setup of a microwave circulator coupled to a superconducting transmission line, quantize it and see if we are able to observe similar results.

## Chapter 4

## Quantum Circulator: system quantization and $3-\mathrm{CPB}$ approach

A classical circulator is a non-reciprocal electronic device used to drive electric pulses through a microelectronic circuit. Its most usual design incorporates three ports, 1,2 and 3 , which can emit and absorb pulses in a not-arbitrary order, being the emitting port highly dependent on which port absorbed the pulse. If an electric pulse inputs through the port 1 , the circulator will output a pulse from 2 ; if a pulse enters through port 2 , the circulator will output a pulse from 3 ; and if a pulse inputs in 3 , one can guess that the output will come from port 1. A circulatory behavior of pulses being driven inside the circulator can be deduced to achieve the given input-output relations. Thanks to its non-reciprocal nature, the fraction of energy flowing to one direction will not be the same as the fraction going to the opposite direction, creating an asymmetry in the system. We will harness the asymmetric nature of a circulator building a superconducting circuit QED architecture able to imitate the behavior of a classical circulator, which interacts with microwave photons instead than electric pulses. The circulator we will design will be based on the qubit coupling scheme outlined in Chapter 2.

In the theory we built in Chapter 3 we designed a model where we had two resonating cavities, both coupled to each other through a superconducting circuit and also to a waveguide. The superconducting coupling was straight mapped from the circulator model, introduced at the end of Chapter 2. The system decay rates and scattering amplitudes promised a highly chiral behavior for certain regimes, thanks to the presence of two different paths a photon can follow. Both paths apply a different phase to a quantum state which, systematically, at the end of the day, add up to create a quantum interference pattern in the output, different in right and left directions. To be able to re-create the same model we will design a quantum circulator based on a superconducting ring. This will be formed by three superconducting islands coupled through Josephson Junctions forming a ring (outlined in [20]). Two of the three islands will at the same time be capacitively coupled to a transmission line
with superconducting microwave resonators through which photons will be able to scatter and create excitations within the circulator [31]. The main difference from the theoretical model in the previous chapter here will be that we will no longer operate on the resonating cavities, but instead directly on the circulator islands. This fact will allow us to observe in a more direct way the time reversal symmetry breaking behavior, as well as clarify the possible circulator excited states. In our terminology, we may call quantum circulator and microwave circulator to the same device.

We will obtain a fully global quantized picture of the whole system obtaining a full Hamiltonian description, working in parallel with the model from the previous chapter. Staying in the Cooper Pair Box limit within all three circulator islands, we will analyze the energy configuration of the circulator and the chiral character of the decay amplitudes to the waveguide.

### 4.1 Isolated circulator: Superconducting ring

As we mentioned in the beginning, we will start by designing a quantum circulator resembling the functionality of a classical circulator. The main design will be based on three superconducting qubits capacitively coupled in a circular architecture with Josephson Junctions as mediators. The main circuit design is illustrated in Figure 4.1. Its shape may resemble a three junction Superconducting Quantum Interference Device (SQUID) [32, 33]. The superconducting islands, sketched as black dots, will be characterized by their charge $Q_{A}, Q_{B}$ and $Q_{C}$. Each of these charges incorporates an offset provided by the gate voltage applied to each island, $Q_{g}=V_{g} / C_{g}$. However, to ease the notation we will ignore them while developing further the system, but we will need to take them into account in a later point. To each island we connect a capacitance $C_{A}, C_{B}$ and $C_{C}$, named as charging capacitances which will directly interact as the output ports of the circulator. We also couple three gate capacitances $C_{A, g}$, $C_{B, g}$ and $C_{C, g}$ to each island to control the applied gate voltage $V_{g}$.

The Josephson Junction capacitances, or Josephson capacitances, will be $C_{A B}, C_{B C}$ and $C_{C A}$. An accumulated charge will stack in each of these capacitances which we denote with lower case $q$ charges. The main Hamiltonian description will be formed by two differentiated elements: The Josephson energy terms and the capacitance energies. Since we know we can implement the first as an anharmonic potential applied to the system, we will start by describing the latest, writing the energy elements that form the Hamiltonian description of the system. We will locate the charge variables in the corresponding islands they belong, and from there, we will write the energy terms brought by each capacitance in the circuit. Note we have a total of 9 capacitances giving 9 energy terms in the Hamiltonian. However, see that the gate capacitances are connected in parallel with the charging capacitances, meaning that we can write both together in the denominators, merging the 6 of them in 3 different elements. At the end of the day, one obtains the following capacitance Hamiltonian description of the circulator.

$$
\begin{align*}
& \mathcal{H}=\frac{\left(Q_{A}+q_{A B}-q_{C A}\right)^{2}}{2\left(C_{A}+C_{A, g}\right)}+\frac{\left(Q_{B}+q_{B C}-q_{A B}\right)^{2}}{2\left(C_{B}+C_{B, g}\right)}+\frac{\left(Q_{C}+q_{C A}-q_{B C}\right)^{2}}{2\left(C_{C}+C_{C, g}\right)}+  \tag{4.1}\\
& +\frac{q_{A B}^{2}}{2 C_{A B}}+\frac{q_{B C}^{2}}{2 C_{B C}}+\frac{q_{C A}^{2}}{2 C_{C A}}
\end{align*}
$$

We end up with 6 different energy terms. The first three (top row of equation (4.1)) correspond to the energy contribution from the gate and charging capacitances. The second row shows the


Figure 4.1: Sketched plan of the circulator architecture we will design and work with. The main three input-output ports are represented by the arms coupled to the superconducting islands (black dots) and each brings a capacitance which will mediate in the interaction strength with the exterior. The three Josephson Junctions are shown as crossed boxes drawn on top of each intermediate capacitance between the islands. All three Josephson energies are assumed to be equal throughout all this work. The superconducting phase difference given between the islands will be driven by the magnetic flux coming through the superconducting ring the circulator forms. This will determine mainly the direction and phase in which the circulator scatter photons, giving live to its promising chiral behavior.
energy contribution form the Josephson capacitance elements between the superconducting islands. See these terms mainly depend on the accumulated charges $q$. To provide an accurate description, we must find the Hamiltonian description providing the minimum energy configuration the accumulated charges can form. The optimization of (4.1) will give a system of three differential equations. We can anticipate here that in the next section we will be forced to re-define a new Hamiltonian when coupling the circulator to a waveguide, so the problem will scale from a $3 \times 3$ system of coupled differential equations to a $5 \times 5$. To avoid massive and complicated expressions, we will re-write (4.1) in a compact matrix form, ending up with:

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}^{T} C^{-1} \vec{Q}+\frac{1}{2} \vec{q}^{T} \Upsilon \vec{q}+\vec{Q}^{T}\left(C^{-1}-R^{-1}\right) \vec{q} \tag{4.2}
\end{equation*}
$$

Together with the charge vectors $\vec{Q}^{T} \equiv\left(Q_{A}, Q_{B}, Q_{C}\right)$ and $\vec{q}^{T} \equiv\left(q_{A}, q_{B}, q_{C}\right)$, we defined the $\Upsilon, C$ and $R$ capacitance matrices, written in detail in Appendix A. Note we strictly defined the transposed charge vectors as row vectors. In (4.2) we differentiate between three main terms. The first is mainly dependent on the island charge variables $\vec{Q}$ containing the capacitance terms in $C^{-1}$, and the other depending on the charge accumulations $\vec{q}$ with the capacitance terms in $\Upsilon$. The third term yields the main dependency between the charge accumulations and the island charges. It includes the diagonal matrix $C^{-1}$ and $R^{-1}$. As a mere curiosity, the last matrix corresponds to a rotation matrix which rotates a three dimensional vector an angle of $\pi / 3$ around the rotation axis situated in $1 / \sqrt{2}(1,1,1)$. We see here manifested the mathematical rotation nature inside of the circulator, which drives the accumulating charges around the three islands. This is brought by the circular architecture of the capacitive coupling, yet the Josephson Junctions still have not been introduced in the Hamiltonian description.

We can proceed now on optimizing the accumulations of charges in (4.2) and, thanks to its compact vector definition, we finally obtain:

$$
\begin{equation*}
\frac{\partial \mathcal{H}}{\partial \vec{q}}=0 \quad \rightarrow \quad \vec{q}=-\Upsilon^{-1}\left(C^{-1}-R^{-1}\right)^{T} \vec{Q} \tag{4.3}
\end{equation*}
$$

Substituting now the optimized accumulating charge vector back in (4.2), one finally ends up with the following Hamiltonian.

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}^{T}\left[C^{-1}-\left(C^{-1}-R^{-1}\right) \Upsilon^{-1}\left(C^{-1}-R^{-1}\right)^{T}\right] \vec{Q} \tag{4.4}
\end{equation*}
$$

The Hamiltonian mechanics provide a diagonal term given by $C^{-1}$ with a mixing term which brings the interaction between the island charges. See how the latest is built from the matrix $\Upsilon^{-1}$, which was highly attached to the time evolution of the accumulated charges, transformed through $C^{-1}-R^{-1}$ which is based on a rotation of $\pi / 3$. Basically, what we are reading here is that the time evolution of the island charges will be brought by the circular architecture of the circulator, giving only fluctuations around the superconducting ring. Shortly we will see how Josephson Energy term can be introduced as a potential manipulating the charge fluctuations. For now, let us simplify further our results by considering the same conditions on all the islands. That is, we will assume all charging capacitances to be equal, $C_{A}=C_{B}=C_{C} \equiv C_{q}$, and same for the gate capacitances $C_{A, g}=C_{B, g}=C_{C, g} \equiv C_{g}$.

Also, the Josephson capacitances between islands will be re-named to be $C_{A B}=C_{B C}=C_{C A} \equiv C_{J}$. This way, we realize that we can re-write the Hamiltonian in (4.4) and define:

$$
\begin{align*}
& \qquad C^{-1}-\left(C^{-1}-R^{-1}\right) \Upsilon^{-1}\left(C^{-1}-R^{-1}\right)^{T}=\frac{1}{\left(C_{q}+C_{g}\right) C_{J}} \Upsilon^{-1} \equiv \tilde{C}^{-1}  \tag{4.5}\\
& \text { with : } \tilde{C}^{-1}=\left(\begin{array}{lll}
\xi_{1} & \xi_{2} & \xi_{2} \\
\xi_{2} & \xi_{1} & \xi_{2} \\
\xi_{2} & \xi_{2} & \xi_{1}
\end{array}\right) \text { for : }\left\{\begin{array}{l}
\xi_{1}=\frac{\left(C_{\Sigma}-C_{J}\right)}{\left(C_{\Sigma}+C_{J}\right)\left(C_{\Sigma}-2 C_{J}\right)} \\
\xi_{2}=\frac{C_{J}}{\left(C_{\Sigma}+C_{J}\right)\left(C_{\Sigma}-2 C_{J}\right)}
\end{array}\right.
\end{align*}
$$

Where we incorporated the island capacitance $C_{\Sigma} \equiv 2 C_{J}+C_{q}+C_{g}$. We have now the capacitance energy term fully described. The full Hamiltonian description can be completed by straight adding the Josephson Energy term as follows.

$$
\begin{align*}
& \mathcal{H}_{\mathrm{circ}}=\frac{1}{2} \vec{Q}^{T} \tilde{C}^{-1} \vec{Q}+V_{J} \quad \text { with : } \\
& V_{J}=-E_{J} \cos \left[\frac{2 \pi}{\Phi_{o}}\left(\phi_{A}-\phi_{B}-\frac{\Phi}{3}\right)\right]-E_{J} \cos \left[\frac{2 \pi}{\Phi_{o}}\left(\phi_{B}-\phi_{C}-\frac{\Phi}{3}\right)\right]-  \tag{4.7}\\
& -E_{J} \cos \left[\frac{2 \pi}{\Phi_{o}}\left(\phi_{C}-\phi_{A}-\frac{\Phi}{3}\right)\right]
\end{align*}
$$

Where $2 \pi \phi_{j} / \Phi_{o}$, for $j=A, B, C$, are the superconducting phases in each island, and $\Phi$ is the magnetic flux going through the superconducting ring. We will not enter in much detail in the Josephson Energy term in equation (4.7) as it is beyond the scope of this work. The main interesting feature we were looking for is that the actual phase difference between islands becomes affected by $\Phi$. However, we must state that $\Phi$ will not be entirely the external magnetic flux going through the superconducting ring since the charge fluctuations and currents going around the circulator will also contribute, [34].

We have ended up with a Hamiltonian description for an isolated circulator, which we now need to couple to a waveguide following up with the first theoretical model we built. However, it will not be trivial to start from this point and couple to $\mathcal{H}_{\text {circ }}$ directly. We need to infer from the basics the coupling term that will play the interaction part in the full Hamiltonian description. The result in (4.7) will not remain useless though, as will serve as a guide to follow up in the next section.

### 4.2 Circulator coupled to a transmission line

As we just mentioned, coupling to a transmission line will not be a trivial step when obtaining the full Hamiltonian description of the whole system (circulator plus transmission line). Unlike the theory we built, for simplicity, we will get rid of the resonating cavities, operating directly on the the circulator islands, and coupling the islands to the transmission line [35]. Since we will add a transmission line


Figure 4.2: Full illustration of the whole system built from a circulator capacitively coupled to a microwave transmission line through two, $A$ and $B$, the the three ports. We leave the third port, $C$, coupled to the ground. The circulator schematic architecture is exactly the same as in Figure 4.1. We introduce a transmission line built from a set of consecutive inductances connected in series and coupling capacitors in each node to the ground.
to our circulator capacitively coupled through two ports, there will appear two additional fluctuating charges which will modify the energy optimization from before. The coupling strength need to be inferred from the Hamiltonian interaction term, which we can not straight include. Therefore, in this section we will re-obtain the circulator Hamiltonian again, but this time, we will couple a transmission line to both ports $A$ and $B$ in Figure 4.1. The transmission line will be modeled as a consecutive set of inductances, $L_{j}$, and capacitances, $C_{j}$, connected in series, as it is illustrated in Figure 4.2. However, we will work with the capacitance and inductance per length, $C_{j}=\delta z c_{j}$ and $L_{j}=\delta l c_{j}$, instead. To both endings of the transmission line we connect two capacitances $C_{L}$ and $C_{R}$. At some point we will assume the transmission line to be sufficiently large to consider infinitesimal distances between the elements, that is $\delta z$. For now we will stick with a finite total amount $N$ of transmission line elements.

The Hamiltonian description will be very similar as in the previous section, but now we will have to take into account the fluctuating charges provided by the transmission line coupling: $q_{l}$ from the coupling to $C_{B}$ and $q_{m}$ from the coupling to $C_{A}$. There will also be defined located charges in the transmission line nodes coupled to the circulator, $Q_{j}$. Apart from that, we can no longer merge the gate capacitances $C_{A, g}$ and $C_{B, g}$ terms with the charge capacitances $C_{A}$ and $C_{B}$ as now they are connected to the transmission line, and we will not be able to assume a parallel connection between capacitances. At the end of the day, we will end up having 14 energy terms contributing to the following Hamiltonian description:

$$
\begin{align*}
& \mathcal{H}=\frac{\left(Q_{A}+q_{A B}-q_{C A}-q_{m}\right)^{2}}{2 C_{A, g}}+\frac{\left(Q_{B}+q_{B C}-q_{A B}-q_{l}\right)^{2}}{2 C_{B, g}}+\frac{\left(Q_{C}+q_{C A}-q_{B C}\right)^{2}}{2\left(C_{C}+C_{C, g}\right)}+ \\
& +\frac{\left(Q_{m}+q_{m}\right)^{2}}{2 \delta z c_{m}}+\frac{\left(Q_{l}+q_{l}\right)^{2}}{2 \delta z c_{l}}+\frac{q_{A B}^{2}}{2 C_{A B}}+\frac{q_{B C}^{2}}{2 C_{B C}}+\frac{q_{C A}^{2}}{2 C_{C A}}+\frac{q_{m}^{2}}{2 C_{A}}+\frac{q_{l}^{2}}{2 C_{B}}+  \tag{4.8}\\
& +\frac{Q_{L}^{2}}{2 C_{L}}+\frac{Q_{R}^{2}}{2 C_{R}}+\sum_{i=1, i \neq m, l}^{N} \frac{Q_{i}^{2}}{2 \delta z c_{i}}+\sum_{i=1}^{N} \frac{\left(\phi_{i}-\phi_{i-1}\right)^{2}}{2 \delta z l_{i}}
\end{align*}
$$

The first three terms are the gate capacitance terms. The last of these together with the last two in the second row come from the charge capacitance terms (note how we can still merge $C_{C, g}$ and $C_{C}$ terms in one as we keep leaving the third circulator port coupled to the ground). The first two terms on the second row come from the capacitances $c_{m}$ and $c_{l}$ in the transmission line, coupled to the circulator and to the ground. The following three terms belong to the Josephson capacitances. The last row corresponds entirely to the transmission line. The first two terms come from the capacitances at each ending (right and left), the third term collects all capacitances coupled to the ground belonging in the transmission line, except the ones coupled to the nodes connected to the circulator (those are written in the two first terms on the second row); and the last term counts the contribution form the inductances connected in series. Remember we introduced $c$ and $l$ to be the transmission line capacitance and inductance per unit length, and $\delta z$ the separation between these elements.

The last row in (4.8) as we just said contains terms belonging uniquely to the transmission line. In fact, we can ignore the $C_{L}$ and $C_{R}$ since we can always set these to be big enough to neglect them. We identify the last two terms to be the Transmission-Line Hamiltonian, but with the $Q_{m}$ and $Q_{l}$ terms missing (as we took them out because each contains a fluctuating charge which makes them different than the rest). We will define then $\mathcal{H}_{t l}^{\prime}$ to be the Transmission Line Hamiltonian with the $m$ and $l$ elements missing.

We have obtained a Hamiltonian with 5 located charge variables $Q$ and 5 fluctuating charges $q$, similarly as in the previous section. In fact, we can re-write (4.8) in a compact matrix notation, just in the same way we defined the circulator Hamiltonian, ans obtain:

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}^{T} \hat{\mathbb{C}}^{-1} \vec{Q}+\frac{1}{2} \vec{q}^{T} \hat{\mathbb{P}} \vec{q}+\vec{Q}^{T}\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right) \vec{q}+\mathcal{H}_{t l}^{\prime} \tag{4.9}
\end{equation*}
$$

The overall structure is very similar than (4.2). We have introduced the vector definition of $\vec{Q}^{T} \equiv\left(Q_{A}, Q_{B}, Q_{C}, Q_{m}, Q_{l}\right)$ and $\vec{q}^{T} \equiv\left(q_{A B}, q_{B C}, q_{C A}, q_{m}, q_{l}\right)$. The matrices $\mathbb{P}, \mathbb{C}$ and $\mathbb{R}$ can be written as blocks of matrices containing $\Upsilon, C$ and $R$ (see Appendix B). The results are then expected to be very similar as in the previous section, but now the interaction terms are hidden in these new matrices. Just as we proceeded previously with the isolated circulator, we will optimize the fluctuating charge configuration in (4.9) using the handy compact vector notation we introduced.

$$
\begin{equation*}
\frac{\partial \mathcal{H}}{\partial \vec{q}}=0 \quad \rightarrow \quad \vec{q}=-\left(\hat{\mathbb{P}}^{-1}\right)^{T}\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right)^{T} \vec{Q} \tag{4.10}
\end{equation*}
$$

Thanks to the symmetric shape of $\mathbb{P}$, we can then introduce the optimized fluctuating charge back
in (4.9) and obtain the full Hamiltonian as follows:

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}^{T}\left[\hat{\mathbb{C}}^{-1}-\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right)\left(\hat{\mathbb{P}}^{-1}\right)^{T}\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right)^{T}\right] \vec{Q}+\mathcal{H}_{t l}^{\prime} \tag{4.11}
\end{equation*}
$$

We have ended up with a very similar Hamiltonian as in (4.4), plus the Transmission Line Hamiltonian. The first term though is hiding the circulator Hamiltonian description, expected to be quite similar than $\mathcal{H}_{\text {circ }}$ in (4.7) (without the $V_{J}$ term), together with the interaction with the transmission line. We will then now dissect this first term and separate the terms concerning the circulator, the transmission line and the interaction between both. To do that, we will divide now the vector charge variables in circulator charges, $\vec{Q}_{c i r c} \equiv\left(Q_{A}, Q_{B}, Q_{C}\right)$, and transmission line charges, $\vec{Q}_{t l} \equiv\left(Q_{m}, Q_{l}\right)$. Following up with the block matrix definition (Appendix B), we re-write (4.11) as:

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}_{c i r c}^{T} \tilde{C}_{c i r c}^{-1} \vec{Q}_{c i r c}+\frac{1}{2} \vec{Q}_{t l}^{T} \tilde{C}_{t l}^{-1} \vec{Q}_{t l}+\frac{1}{2} \vec{Q}_{c i r c}^{T} C_{i n t}^{-1} \vec{Q}_{t l}+\frac{1}{2} \vec{Q}_{t l}^{T}\left(C_{i n t}^{T}\right)^{-1} \vec{Q}_{c i r c}+\mathcal{H}_{t l}^{\prime} \tag{4.12}
\end{equation*}
$$

Where:

$$
\hat{C}_{\text {circ }}^{-1}=\left(\begin{array}{ccc}
\alpha & \chi & \zeta  \tag{4.13}\\
\chi & \alpha & \zeta \\
\zeta & \zeta & \sigma
\end{array}\right) \quad \hat{C}_{t l}^{-1}=\left(\begin{array}{cc}
\varrho & \eta \\
\eta & \varrho
\end{array}\right) \quad C_{i n t}^{-1}=\left(\begin{array}{cc}
\rho & \epsilon \\
\epsilon & \rho \\
\kappa & \kappa
\end{array}\right)
$$

For a full complete definition of the matrix terms, address to Appendix B. Observe the symmetry presented in the $A$ and $B$ ports displayed in $\hat{C}_{\text {circ }}^{-1}$, basically due the consideration of same conditions on both coupled ports to the transmission line. For further simplifications, we will keep assuming a weak coupling in this scenario, by now considering that the charging capacitances coupled to the transmission line $C_{q}$ are very small. However, we still need to consider that the distance between the transmission line capacitances and inductances, $\delta z$ is also low. In fact, both $C_{q}$ and $\delta z c$ magnitudes may have reasonably comparable sizes. Since we may need to take that into account, we will define $\tau$ as a dimensionless fractional relation between both.

$$
\begin{equation*}
\frac{\delta z c}{C_{q}} \equiv \tau \quad \text { with }: \quad \delta z c=\varepsilon \tau \quad C_{q}=\varepsilon \tag{4.14}
\end{equation*}
$$

Both quantities are small, so we introduced $\varepsilon$, with capacitance dimensions, which will be infinitesimally small compared to the rest of capacitances. With that in mind, we can approximate the matrix terms in (4.13) to the following.

$$
\left\{\begin{align*}
\alpha & =\left(1+\frac{C_{J}\left(C_{c}+C_{g}+3 C_{J}\right)}{\xi^{2}+\frac{\tau}{1+\tau} \varepsilon\left(C_{c}+C_{g}+2 C_{J}\right)}\right) \frac{1}{\left(C_{g}+3 C_{J}\right)+\frac{\tau}{1+\tau} \varepsilon}  \tag{4.15}\\
\zeta & =\frac{C_{J}}{\xi^{2}+\frac{\tau}{1+\tau} \varepsilon\left(C_{c}+C_{g}+2 C_{J}\right)} \\
\chi & =\frac{C_{J}\left(C_{c}+C_{g}+3 C_{J}\right)}{\xi^{2}+\frac{\tau}{1+\tau} \varepsilon\left(C_{c}+C_{g}+2 C_{J}\right)} \frac{1}{\left(C_{g}+3 C_{J}\right)+\frac{\tau}{1+\tau} \varepsilon} \\
\sigma & =\frac{C_{J}+C_{g}+\frac{\tau}{1+\tau} \varepsilon}{\xi^{2}+\frac{\tau}{1+\tau} \varepsilon\left(C_{c}+C_{g}+2 C_{J}\right)} \\
\eta & =\frac{1}{(1+\tau)^{2}} \chi \\
\varrho & =\left(1+\frac{1}{\varepsilon} \frac{(1+\tau)\left(C_{g}+3 C_{J}\right) \xi^{2}+\varepsilon C_{J}\left(C_{c}+C_{g}+3 C_{J}\right)}{(1+\tau) \xi^{2}+\tau \varepsilon\left(C_{c}+C_{g}+2 C_{J}\right)}\right) \frac{1}{(1+\tau)\left(C_{g}+3 C_{J}\right)+\tau \varepsilon}
\end{align*}\right.
$$

Which assuming a weak coupling, i.e., $\varepsilon \rightarrow 0$, end up being:

$$
\left.\begin{array}{ll}
\alpha=\left(1+\frac{C_{J}\left(C_{c}+C_{g}+3 C_{J}\right)}{\xi^{2}}\right) \frac{1}{\left(C_{g}+3 C_{J}\right)} & \zeta=\frac{C_{J}}{\xi^{2}}
\end{array} \quad \eta=\frac{1}{(1+\tau)^{2}} \chi\right] \text { ( } \begin{array}{ll}
\chi=\frac{C_{J}}{\xi^{2}} \frac{\left(C_{c}+C_{g}+3 C_{J}\right)}{\left(C_{g}+3 C_{J}\right)} & \sigma=\frac{C_{J}+C_{g}}{\xi^{2}}
\end{array} \varrho=\frac{1}{\varepsilon(1+\tau)}
$$

And:

$$
\begin{equation*}
\rho=\frac{1}{1+\tau} \alpha \quad \epsilon=\frac{1}{1+\tau} \chi \quad \kappa=\frac{1}{1+\tau} \zeta \tag{4.17}
\end{equation*}
$$

Where we defined the quantity $\xi^{2} \equiv C_{g}\left(C_{c}+C_{g}+3 C_{J}\right)+C_{c} C_{J}$ with square capacitance dimensions for a compact display. Note that all the terms have dimensions of the inverse of capacitance, which fits perfectly in our Hamiltonian. We also see in (4.17) how within the weak coupling regime, the interacting terms can be actually expressed in the purely circulator Hamiltonian terms, with a $1 /(1+\tau)$ factor in front. In fact, to avoid an incoherence in our theory and stick in this regime, we still need to assume that the interaction terms are much smaller than the terms in the circulator Hamiltonian, forcing us to work with values of $\tau \gg 1$, i.e., $C_{q} \ll \delta z c$. This is basically telling us that the capacitance coupling between the circulator and the transmission line is much weaker than the capacitance coupling inherent in the transmission line elements as a whole, which sounds reasonable. We may state that from now on we will stick only to linear terms in $1 / \tau$. See then how we can ignore the $\eta$ element in (4.15), as brings a second order $1 / \tau$ term.

A remarkable point we note is that, in the $C_{C} \rightarrow 0$ regime, we recover the exact same solution we obtained back in (4.5). One sees this if realizes that the island capacitances are defined by $C_{\Sigma, A}=$ $C_{\Sigma, B}=C_{q}+C_{g}+2 C_{J}$ and $C_{\Sigma, C}=C_{C}+C_{g}+2 C_{J}$. Since $C_{q} \rightarrow 0$, and right now we are also considering $C_{C} \rightarrow 0$, using this island capacitance definitions one ends up with the same capacitance terms obtained in the isolated circulator case. In fact, within this regime, we treat the transmission line as a perturbation to the circulator, leaving $\mathcal{H}_{\text {circ }}$ intact and just adding an interaction term. We will see in Section 4.4 how this regime provides an analytic solution of the circulator eigensystem, operating within the CPB regime.

We have reached a point in which we simplified as much as possible only using two basic assumptions: weak coupling between the transmission line and the circulator and equal capacitance conditions for both ports $A$ and $B$. Before writing the whole Hamiltonian, we realize that in the weak coupling the second term in (4.12) can be re-written as:

$$
\begin{equation*}
\frac{1}{2} \vec{Q}_{t l}^{T} \tilde{C}_{t l}^{-1} \vec{Q}_{t l} \simeq \frac{1}{2} \frac{Q_{m}^{2}}{\delta z c}+\frac{1}{2} \frac{Q_{l}^{2}}{\delta z c} \quad \rightarrow \quad \mathcal{H}_{t l}=\mathcal{H}_{t l}^{\prime}+\frac{1}{2} \frac{Q_{m}^{2}}{\delta z c}+\frac{1}{2} \frac{Q_{l}^{2}}{\delta z c} \tag{4.18}
\end{equation*}
$$

Therefore, based in our starting definition of the $\mathcal{H}_{t l}^{\prime}$ term, we can complete the transmission line Hamiltonian including the missing $m$ and $l$ terms and that now we dispose. Finally we will write the Hamiltonian separating the three main important terms: the circulator Hamiltonian $\mathcal{H}_{\text {circ }}$, the transmission-line Hamiltonian $\mathcal{H}_{t l}$ and the interaction term $\mathcal{H}_{\text {int }}$. In the fist mentioned term, we will also include the Josephson Energy terms directly just as we did previously in equation (4.7).

$$
\begin{align*}
& \mathcal{H}=\mathcal{H}_{\text {circ }}+\mathcal{H}_{t l}+\mathcal{H}_{\text {int }} \\
& \left\{\begin{array}{l}
\mathcal{H}_{\text {circ }}=\frac{1}{2} \vec{Q}_{\text {circ }}^{T} \tilde{C}_{\text {circ }}^{-1} \vec{Q}_{\text {circ }}+V_{J} \\
\mathcal{H}_{t l}=\sum_{i=1}^{N} \frac{Q_{i}^{2}}{2 \delta z c_{i}}+\sum_{i=1}^{N} \frac{\left(\phi_{i}-\phi_{i-1}\right)^{2}}{2 \delta z l_{i}} \\
\mathcal{H}_{\text {int }}=\frac{\alpha}{1+\tau}\left(Q_{A} Q_{m}+Q_{B} Q_{l}\right)+\frac{\chi}{1+\tau}\left(Q_{A} Q_{l}+Q_{B} Q_{m}\right)+\frac{\zeta}{1+\tau} Q_{C}\left(Q_{m}+Q_{l}\right)
\end{array}\right. \tag{4.19}
\end{align*}
$$

Observe how the interaction Hamiltonian takes into account the direct interaction between the $A-m$ and $B-l$ circulator transmission line coupling mediated by $\alpha /(1+\tau)$, which we will refer as direct coupling term. The cross-reference $A-l$ and $B-m$ interaction will be mediated by $\chi /(1+\tau)$, which we will refer as crossed coupling term. The third term, $\zeta /(1+\tau)$ contributes to the coupling strength between the $C$ island and the transmission line. Observe that this coupling goes over a Josephson capacitance and then the charge capacitance coupled to the transmission line. Interestingly enough, the crossed coupling brought by $\chi /(1+\tau)$ also goes over a Josephson capacitance and a charge capacitance in a same way as $\zeta /(1+\tau)$ does. Since their coupling is defined by the same capacitances one could imagine that they are actually equal. In fact, they are, but only when $C_{C} \rightarrow 0$. We see that $C_{C}$ plays an effective role defining the $\zeta$ coupling terms. All in all, we see that the $C$ island makes an appearance and, together with the crossed coupling brought by $\chi /(1+\tau)$, they start playing an important role in the front scene that we did not take into account in the theoretical model from Chapter 3. Since with the current regime we can't totally neglect them, we will keep it and see if it somehow perturbs the chiral results we are going after.

### 4.3 Full Hamiltonian quantization

In the previous section we obtained from scratch the full Hamiltonian description of the whole system composed by a quantum three-port circulator coupled to a microwave transmission line by two ports. The Hamiltonian in (4.19) is given in terms of the three located charges ( $Q_{A}, Q_{B}, Q_{C}$ ) in the superconducting islands forming the calculator and the two charges in the transmission line ( $Q_{m}, Q_{l}$ ). In this section we will proceed in quantize these charge variables in the charge number basis, in terms of creation and annihilation operators. The quantization of $\mathcal{H}$ will be given differently depending on the position of charge variables. As we already said, we will assume the superconducting island charges to be localized, and the charges in the transmission line to be un-localized. The aim of attacking the problem from this perspective yields on the nature of the transmission line and the superconducting islands themselves. The charges traveling trough the transmission line should be understood as electric field fluctuations, i.e., photons, whereas the charges in the superconducting islands are assumed be defined by Cooper Pairs, creating excited states of the circulator which will chiraly decay to its ground state. In this section we will proceed to quantize the full Hamiltonian, starting from the transmission line term.

### 4.3.1 Transmission line quantization

We will begin with the quantization of the transmission line Hamiltonian. For a full and complete understanding and basis formation, we will start presenting the Transmission Line Lagrangian density [36].

$$
\begin{equation*}
\mathcal{L}(z, t)=K-U=\frac{c}{2}\left(\frac{\partial \Phi(z, t)}{\partial t}\right)^{2}-\frac{1}{2 l}\left(\frac{\partial \Phi(z, t)}{\partial z}\right)^{2} \tag{4.20}
\end{equation*}
$$

Where we took $K$ as the kinetic energy and $U$ as the potential energy densities of the transmission line. See we are working with $c$ and $l$ to be the capacitance and inductance per unit length and $\Phi$ to be the magnetic flux. The equations of motion one can directly derive from (4.20) yield the following charge density.

$$
\begin{equation*}
q(z, t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}(z, t)}=c V \tag{4.21}
\end{equation*}
$$

Where $V=\dot{\Phi}$ is the electric potential. Using this now, we can convert the Lagrangian in (4.20) into the following Hamiltonian description.

$$
\begin{equation*}
\mathcal{H}_{t l}=\int_{\Delta L} d z\left[\frac{q(z)^{2}}{2 c}+\frac{1}{2 l}\left(\frac{\partial \Phi(z)}{\partial z}\right)^{2}\right] \tag{4.22}
\end{equation*}
$$

This is just the same Transmission Line Hamiltonian presented in (4.19) in the continuum limit, i.e., assuming $\delta z \rightarrow 0$. The integration goes over the total Transmission Line length $\Delta L$. As we
already discussed, we are not interested in define the transmission line charges to be localized but through propagating modes of the right and left going electric fields. With that in mind, we will write the following Fourier space charge and flux quantities.

$$
\begin{align*}
& q(z)=\int d k\left(q_{k} e^{i k z}+q_{k}^{*} e^{-i k z}\right) \\
& \Phi(z)=\int d k\left(\Phi_{k} e^{i k z}+\Phi_{k}^{*} e^{-i k z}\right) \tag{4.23}
\end{align*}
$$

For now the $q_{k}$ and $\Phi_{k}$ amplitudes will be considered arbitrary. The important remark here is that we can solve the equations of motion now provided by the Hamiltonian in (4.22) giving:

$$
\begin{equation*}
\ddot{\Phi}=\frac{k^{2}}{c l} \Phi \tag{4.24}
\end{equation*}
$$

The solutions yield the Harmonic Oscillator equations of motion of the flux, with a linear order in $k$ oscillation frequency $\omega_{k}=k / \sqrt{c l}$. Since in this study we will work with narrow-band microwave photons, we can assume that the total frequency is given by the most populated mode $\omega=k_{o} v_{g}$, where we introduced the group velocity $v_{g}=1 / \sqrt{c l}$. We will store this result to use it shortly. Let us now re-define the Fourier space definitions for $q$ and $\Phi$, defining the following amplitude.

$$
\begin{equation*}
A_{k}=\int[\alpha q(z)+i \beta k \Phi(z)] e^{-i k z} d z \tag{4.25}
\end{equation*}
$$

Where $\alpha$ and $\beta$ are here constants yet to be defined and adjusted. Together with its complex conjugate component, we can turn the relation (4.25) around and express the charge density and flux magnitudes as follows.

$$
\begin{align*}
& q(z)=\frac{1}{2 \alpha} \int\left(A_{k} e^{i k z}+A_{k}^{*} e^{-i k z}\right) d k  \tag{4.26}\\
& \Phi(z)=\frac{1}{2 i \beta k} \int\left(A_{k} e^{i k z}-A_{k}^{*} e^{-i k z}\right) d k
\end{align*}
$$

See that these definitions are very similar than (4.23), but now we introduced the quantities $\alpha$ and $\beta$ which will give us more flexibility when playing around with them. We also introduced an imaginary term together with the flux. Thanks to it, we can see how $A_{k}^{*}=A_{-k}$. Thanks to this relation we can write:

$$
\begin{align*}
& \int d z \frac{q(z)^{2}}{2 c}=\frac{1}{c(2 \alpha)^{2}} \int d k\left(A_{k} A_{k}^{*}+A_{k}^{*} A_{k}\right)  \tag{4.27}\\
& \int d z \frac{1}{2 l}\left(\frac{\partial \Phi(z)}{\partial z}\right)^{2}=\frac{1}{l(2 \beta)^{2}} \int d k\left(A_{k} A_{k}^{*}+A_{k}^{*} A_{k}\right)
\end{align*}
$$

And finally, evaluating $\alpha=\sqrt{1 / c}$ and $\beta=\sqrt{1 / l}$, we will substitute these terms back in the Hamiltonian in (4.22) giving the final and simple following form.

$$
\begin{equation*}
\mathcal{H}_{t l}=\int d k\left(A_{k} A_{k}^{*}+A_{k}^{*} A_{k}\right) \tag{4.28}
\end{equation*}
$$

We managed to re-write the transmission line Hamiltonian first given in (4.19) in a simple form in terms of the amplitudes introduced in (4.25). The quantized version of these amplitudes though, might respect the commutation relations given by the canonical variables in the Hamiltonian, which in this case correspond to the charge density and flux. Converting these canonical variables in operators, their commutation relations yield the following values.

$$
\begin{align*}
& {\left[\hat{\Phi}(z), \hat{q}\left(z^{\prime}\right)\right]=i \delta\left(z-z^{\prime}\right)} \\
& {\left[\hat{\Phi}(z), \hat{\Phi}\left(z^{\prime}\right)\right]=\left[\hat{q}(z), \hat{q}\left(z^{\prime}\right)\right]=0} \tag{4.29}
\end{align*}
$$

This commutation relations imply that the operator versions of the $A_{k}$ amplitudes fulfill:

$$
\begin{align*}
& {\left[\hat{A}_{k}, \hat{A}_{k^{\prime}}^{\dagger}\right]=\omega_{k} \delta\left(k-k^{\prime}\right)} \\
& {\left[\hat{A}_{k}, \hat{A}_{k^{\prime}}\right]=\left[\hat{A}_{k}^{\dagger}, \hat{A}_{k^{\prime}}^{\dagger}\right]=0} \tag{4.30}
\end{align*}
$$

Where we introduced the photon frequency obtained back when solving the equations of motion of the transmission line Hamiltonian in (4.24). Note that we still can define the $\hat{A}_{k}$ with relative freedom. Taking advantage of this flexibility, we realize that if we evaluate $\hat{A}_{k}=\sqrt{\omega_{k}} \hat{a}_{k}$ we can directly introduce the $\hat{a}_{k}$ bosonic operators in the number basis, which if substituted in (4.30) will yield the characteristic canonical commutation relations. Substituting now the new operator amplitudes back in the Transmission Line Hamiltonian in (4.28) we obtain the well-known quantum harmonic oscillator in the number basis in Fourier space.

$$
\begin{equation*}
\mathcal{H}_{t l}=\int d k \omega_{k}\left(\hat{a}_{k}^{\dagger} \hat{a}_{k}+\frac{1}{2} \delta\left(k-k^{\prime}\right)\right) \tag{4.31}
\end{equation*}
$$

The $\delta\left(k-k^{\prime}\right)$ has been introduced due to the commutation relations in (4.30). As we previously stated, we will mainly work with narrow band microwave photons, meaning that we can assume we
work only with the most populated mode $\omega$. Moreover, we will be interested in separate the charge modes in the transmission line in their right and left going mode components. For that reason, we will introduce the Right and Left going modes.

$$
\begin{equation*}
\hat{a}_{R} \equiv \int_{k>0} d k \hat{a}_{k} e^{i z\left(k-k_{o}\right)} \quad \hat{a}_{L} \equiv \int_{k<0} d k \hat{a}_{k} e^{i z\left(k-k_{o}\right)} \tag{4.32}
\end{equation*}
$$

A quick dimensional analysis suggests that the directional transmission line operators have dimensions of $[1 / \sqrt{L}]$, where $L$ represents a distance dimension, since $\hat{a}_{k} \sim[\sqrt{L}]$. See that the electric fields defined in (3.3) in the theoretical model from Chapter 3 have the same dimensions due to the commutation relations in (3.7). Reverting the relation expressing $\hat{a}_{k}$ in terms of these directional modes, and introducing it back into (4.26) inside the $\hat{A}_{k}$ amplitudes, one obtains the final quantized form for the $Q_{m}$ and $Q_{l}$ charges.

$$
\begin{align*}
& \frac{\hat{Q}_{m}}{\Delta L}=\int_{\Delta L} d z \frac{\sqrt{c \omega}}{2} {\left[\left(\hat{a}_{R} e^{i k_{o} z} \delta\left(z-z_{A}\right)+\hat{a}_{L} e^{-i k_{o} z} \delta\left(z_{A}-z\right)\right)+\right.} \\
&\left.+\left(\hat{a}_{R}^{\dagger} e^{-i k_{o} z} \delta\left(z-z_{A}\right)+\hat{a}_{L}^{\dagger} e^{i k_{o} z} \delta\left(z_{A}-z\right)\right)\right]  \tag{4.33}\\
& \frac{\hat{Q}_{l}}{\Delta L}=\int_{\Delta L} d z \frac{\sqrt{c \omega}}{2} {\left[\left(\hat{a}_{R} e^{i k_{o} z} \delta\left(z_{B}-z\right)+\hat{a}_{L} e^{-i k_{o} z} \delta\left(z-z_{B}\right)\right)+\right.} \\
&\left.+\left(\hat{a}_{R}^{\dagger} e^{-i k_{o} z} \delta\left(z-z_{B}\right)+\hat{a}_{L}^{\dagger} e^{i k_{o} z} \delta\left(z_{B}-z\right)\right)\right]
\end{align*}
$$

Note we integrated out the charge and flux densities over the transmission line length, and we located both charge variables to the position they are being measured, i.e., in the coupling positions they correspond to. Also, it is important to look out that the unit dimensions on the right side of the equal signs is $[Q / L]$, being $Q$ charge and $L$ distance, i.e., charge density. That is why the $\Delta L$ was introduced within the denominator on the left side of the equal sign. This $\Delta L$ will soon appear again when writing down the interaction Hamiltonian. The $Q_{m}$ charge is the charge measured in the coupling point to the $A$ port of the circulator, and $Q_{l}$ is the charge measured in the coupling point to the $B$ port. Finally, we will write the transmission line Hamiltonian in (4.31) in terms of the directional mode operators introduced in (4.32).

$$
\begin{equation*}
\mathcal{H}_{t l}=i v_{g} \int d z\left(\hat{a}_{L}^{\dagger} \frac{\partial \hat{a}_{L}}{\partial z}-\hat{a}_{R}^{\dagger} \frac{\partial \hat{a}_{R}}{\partial z}\right) \tag{4.34}
\end{equation*}
$$

Note we recovered the waveguide Hamiltonian we introduced in Chapter 3 to describe the electric field propagation. Instead of working with the quantized electric field here, we will operate with the directional quantized modes, which effectively should make no difference.

### 4.3.2 Circulator quantization: Transmon regime

We have now a quantized form of the transmission line expressed through the directional photon modes in (4.32). Following next, in this two following sections we will proceed to quantize the charge variables into charge operators in the circulator superconducting islands. We will analyze two different regimes a superconducting qubit can be treated. Depending on the charging-Josephson energy relations, we can locate out model in the Transmon regime ( $E_{J} / E_{c} \gg 1$ ) or in the Cooper Pair Box (CPB) regime $\left(E_{J} / E_{c} \ll 1\right)$, both outlined in the introductory chapter. We will start looking at the first option.

The Charge and Josephson Energies are defined to be the energy terms composing the superconducting circuit, which is this case we named it to be the circulator Hamiltonian $\mathcal{H}_{\text {circ }}$ given in (4.19). The charge energy $E_{c}$ corresponds at the energy term going together with the capacitance part of the Hamiltonian. This part describes the energy contribution of the system to localize the charges in the superconducting islands thanks to the capacitances. The Josephson Energy $E_{J}$ has already been introduced in $V_{J}$. It describes the energy contribution of the charges to travel and move through the Josephson Junctions to a superconducting island to another. If $E_{J}$ dominates in front of $E_{c}$, the charges will travel more freely than being fixed and localized in the islands, making the energy behavior resemble to an harmonic oscillator with an anharmonic potential. This regime is known as the Transmon Regime. Here we can quantize the charge and flux variables as we did with the transmission line charges, following a similar prescription as we indicated in (2.3) on the first introductory chapter. In fact, using the definitions in (4.26), introducing the $\hat{b}_{j}$ operator to be the superconducting island version of the $\hat{a}_{k}$ transmission line mode operators in the position space localized in $z_{j}$, at the end of the day we can derive the following relation.

$$
\begin{equation*}
e^{i \frac{2 \pi}{\Phi_{o}} \phi_{j}} \sim 1+i \frac{2 \pi}{\Phi_{o}} \phi_{j}=1+\frac{2 \pi}{\Phi_{o}} \frac{v_{g}}{2} \sqrt{\frac{l}{\omega}}\left(\hat{b}_{j}-\hat{b}_{j}^{\dagger}\right) \tag{4.35}
\end{equation*}
$$

This exponential expression will enter in the cosine functions in $V_{J}$. This way, we can express it with parabolas, resembling the harmonic oscillator as we previously mentioned, obtaining each cosine term to be:

$$
\begin{equation*}
\cos \left[\frac{2 \pi}{\Phi_{o}}\left(\phi_{j}-\phi_{k}-\frac{\Phi}{3}\right)\right] \sim\left(1+\left(\frac{2 \pi}{\Phi_{o}}\right)^{2} \phi_{j} \phi_{k}\right) \cos (\varphi)+\frac{2 \pi}{\Phi_{o}}\left(\phi_{j}-\phi_{k}\right) \sin (\varphi) . \tag{4.36}
\end{equation*}
$$

Where $\varphi=\frac{2 \pi}{\Phi_{o}} \frac{\Phi}{3}$. Writing this terms now with the $\hat{b}_{j}$ position operators, we obtain:

$$
\begin{equation*}
V_{J} \sim E_{J} \cos (\varphi) \sum_{j \neq k}\left[1-\left(\frac{2 \pi}{\Phi_{o}} \frac{v_{g}}{2}\right)^{2} \frac{l}{\omega}\left(\hat{b}_{j}^{\dagger} \hat{b}_{k}+\hat{b}_{j} \hat{b}_{k}^{\dagger}\right)\right] . \tag{4.37}
\end{equation*}
$$

The most important result we can observe is that the phase difference $\varphi$ is acquired not as a global phase in the quantum state, but inside a cosine factor which will be globally equal for a charge traveling to any direction. The charge transitions between pots inside the circulator will not involve any phase difference, making the phase interference with the transmission line path be null. The chiral
behavior is expected to be lost in the Transmon Regime. In fact, this result was sort of expected, as the charges are not traveling from an island to another, but following an oscillating behavior.

### 4.3.3 Circulator quantization: Cooper Pair Box regime

We have seen that assuming an oscillating charge regime in the circulator as the Transmon Regime vanishes the chiral behavior because no phase difference is outputted through the circulator. We imagine that the localization of charges is an important factor to take into account. In the present section we will quantize the circulator Hamiltonian in the Cooper Pair Box (CPB) regime. That is, assuming that $E_{c} \gg E_{J}$, and that the energy behavior is dominated by the capacitance term localizing the charges in the superconducting islands. We will assume then that we can re-write the charges and flux in the following number basis.

$$
\begin{equation*}
\hat{Q}_{j}=2 e \hat{N}_{j}=2 e \sum_{N_{j}}\left(N_{j}-N_{j, g}\right)\left|N_{j}\right\rangle\left\langle N_{j}\right| \quad e^{i \hat{\phi}_{j}}=\sum_{N_{j}}\left|N_{j}-1\right\rangle\left\langle N_{j}\right| \tag{4.38}
\end{equation*}
$$

Where $N_{j}=N_{A}, N_{B}, N_{C}$ are the number of exceeding Cooper Pair (CP) charges in the $A, B$ and $C$ islands respectively, and $N_{j, g}=N_{A, g}, N_{B, g}, N_{C, g}$ are the number of exceeding CP charges provided by the gate voltage. We brought back here the gate charges we ignored in our notation throughout all the previous operations. Since these are merely an offset applied to the actual charge variables, to which no hard operation like derivatives has been applied, seems fair to freely bring them out now in our notation. The circulator Hamiltonian with these definitions can be re-written as follows.

$$
\begin{align*}
& \mathcal{H}_{\text {circ }}=\sum_{j} \sum_{N_{j}} 4 E_{c_{j}}\left(N_{j}-N_{j, g}\right)^{2}\left|N_{j}\right\rangle\left\langle N_{j}\right|-\frac{E_{J}}{2} \sum_{j \neq k}\left[e^{-i \varphi}|0\rangle_{j}\left\langle\left. 1\right|_{j} \otimes \mid 1\right\rangle_{k}\left\langle\left. 0\right|_{k}+\text { H.c. }\right]+\right.  \tag{4.39}\\
& +\sum_{j \neq k} \sum_{N_{j}} \sum_{N_{k}} 4 E_{c_{j k}}\left(N_{j}-N_{j, g}\right)\left(N_{k}-N_{k, g}\right)\left|N_{j} N_{k}\right\rangle\left\langle N_{j} N_{k}\right|
\end{align*}
$$

Where we have again introduced the phase $\varphi=\frac{2 \pi}{\Phi_{o}} \frac{\Phi}{3}$ and also defined the following charging energies:

$$
\begin{array}{lll}
E_{c_{A}} & =\frac{e^{2}}{2} \alpha & E_{c_{B}}=\frac{e^{2}}{2} \alpha
\end{array} \quad E_{c_{C}}=\frac{e^{2}}{2} \sigma \quad \begin{array}{lll}
E_{c, \alpha} & =\frac{e^{2}}{2} \alpha & E_{c, \chi}=\frac{e^{2}}{2} \chi  \tag{4.40}\\
E_{c_{B C}}=\frac{e^{2}}{2} \zeta & E_{c_{C A}}=\frac{e^{2}}{2} \zeta
\end{array} .
$$

Since most of these charging energies have the same value, we define the right side charging energies which are the ones we will work with from now on. Throughout this work, we may also refer to $E_{c, \chi}$ and $E_{c, \zeta}$ as interacting charging energies (referred to the Coulomb interaction between islands). The superconducting islands are assumed to be designed such that, in this regime the number of exceeding

CP charges rarely exceeds $N=1$. In addition with the charging energy domination, stating that the charges will prefer to be located in the islands rather than traveling through the Junctions, it is safe to assume that $N_{j}=0,1$ and no more than one CP is allowed to occupy the same island. On the other had, interacting charging energy values comparable to $E_{J}$ will still preserve the CPB assumption within the islands [37]. That way we can finally write the circulator Hamiltonian in the CPB regime as:

$$
\begin{equation*}
\mathcal{H}_{c i r c}=\sum_{N_{A}=0}^{1} \sum_{N_{B}=0}^{1} \sum_{N_{C}=0}^{1}\left\langle N_{A} N_{B} N_{C}\right| E_{N_{A} N_{B} N_{C}}\left|N_{A} N_{B} N_{C}\right\rangle-\frac{E_{J}}{2} \sum_{j \neq k}\left[\frac{e^{-i \varphi}}{4} \sigma_{+}^{(j)} \sigma_{-}^{(k)}+\text { H.c. }\right] \tag{4.41}
\end{equation*}
$$

Where $\sigma_{+}$and $\sigma_{-}$are the rising and lowering operators in the number basis, and $E_{N_{A} N_{B} N_{C}}$ are the energies of having $N_{j}$ exceeding CP in the $j$ island. Now there is a phase $e^{ \pm i \varphi}$ which will apply a difference to a exceeding CP charge within the circulator, depending on the island-island hopping direction. This indicates that the chiral behavior may still exist within this regime. Thus, sticking to the CPB regime, we will work on the circulator Hamiltonian derived in (4.41). Also, note how the circulator charge variables $Q_{A}, Q_{B}$ and $Q_{C}$ were quantized in (4.38), contrasting with the transmission line charges quantization, which were introduced with traveling directional modes. We are still missing the interaction Hamiltonian, $\mathcal{H}_{\text {int }}$, quantization which should take into account both quantization schemes. Before that though, we will find the eigenstates of $\mathcal{H}_{\text {circ }}$, thus the possible excited CP state configurations. Then, within the circulator eigenbasis we will re-write $\mathcal{H}_{\text {int }}$ and proceed to its quantization following the processes we just carried out for both island and transmission line charges.

### 4.4 Circulator Eigenstates

Starting from the Hamiltonian description of the circulator in the CPB regime written in (4.41), in this section we will derive the possible eigenstates that can be occupied by an excitation. From $\mathcal{H}_{\text {circ }}$, we are presented with a Hamiltonian in the number basis, with $2^{3}=8$ basis states. First of all, let us sketch the basis order we will use to diagonalize the system.

$$
\begin{equation*}
\left|N_{A} N_{B} N_{C}\right\rangle \equiv\{\underbrace{|000\rangle}_{(0)}, \underbrace{|001\rangle,|010\rangle,|100\rangle}_{(1)}, \underbrace{|011\rangle,|101\rangle,|110\rangle}_{(2)}, \mid \underbrace{|111\rangle}_{(3)}\} \tag{4.42}
\end{equation*}
$$

The 8 basis states can be divided in 4 different groups, depending on the number of exceeding CP charges living within the circulator. If the circulator is found without any exceeding CP, and all the islands are empty, it will be found in the $|000\rangle$ state. If there is one exceeding CP , the circulator will be found in a certain combination of the $|001\rangle,|010\rangle$ and $|100\rangle$ states. If there are two extra CP charges occupying the circulator, its state will be given by a combination of the $|011\rangle,|101\rangle$ and $|110\rangle$ states. Finally, if all islands are occupied with one extra CP, there is only one possible combination given by the $|111\rangle$ state. In fact, we can represent (4.41) in a matrix shape within the basis given in (4.42) (see Appendix C). It is there where we realize that we can straight diagonalize the whole system by dividing it the presented 4 different sub-spaces, each being determined by the total amount of extra CP charges living in the circulator islands. To be able to provide easy understandable analytic
solutions, we will need to assume that $C_{C} \rightarrow 0$, meaning that the $C$ island will be weakly coupled to the ground. With this approach, all three islands are assumed to be equal within the CPB regime. Thus, we name it the $3-C P B$ approach. Looking at (4.16), we see how in this regime $E_{c, \alpha}=E_{c, \sigma}$ and $E_{c, \chi}=E_{c, \zeta}$ (meaning that the Coulomb interaction, as well as the charging energy, will be equal for all three islands). In fact, interestingly enough, we recover the isolated circulator Hamiltonian description we derived in (4.7). Indeed, assuming a weak capacitive coupling from the circulator islands to the exterior will merely mean considering the circulator to be isolated and any external interaction as a perturbation..

The obtained eigenstates and energies will strongly depend on the total magnetic flux going through the circulator, provided within the phase $\varphi$. It is commonly often to work with $N_{g}=1 / 2$ (sweet spot), where the charge fluctuations can be ignored, as well as providing the system with a particleantiparticle symmetry. Having that, the sub-systems (1) and (2) present the same Hamiltonian definition, whereas (0) and (3) present a degenerate excited state with an energy difference from the rest of sub-spaces proportional to the interacting charging energy $E_{c, \chi}$. We will be able to define the whole system by just focusing on the first two sub-systems, assuming that we have only one or two exceeding CP charges. Right below, we present the solution of the whole eigensystem in the 3-CPB approach.

$$
\begin{align*}
& E_{0}=E_{7}=3\left(E_{c, \alpha}+E_{c, \chi}\right)  \tag{4.43}\\
& \text { for : }\left\{\begin{array}{r}
\left|\Omega_{0}\right\rangle=|000\rangle \\
\left|\Omega_{7}\right\rangle=|111\rangle
\end{array}\right\} \\
& E_{1}=E_{4}=\frac{E_{J}}{2} \cos (\varphi)-\frac{E_{J}}{2} \sqrt{3} \sin (\varphi)+ \\
& +3 E_{c, \alpha}-E_{c, \chi} \\
& E_{2}=E_{5}=\frac{E_{J}}{2} \cos (\varphi)+\frac{E_{J}}{2} \sqrt{3} \sin (\varphi)+ \\
& +3 E_{c, \alpha}-E_{c, \chi} \\
& \text { for }:\left\{\begin{array}{l}
\left|\Omega_{1}\right\rangle=\frac{1}{\sqrt{3}}|001\rangle-\frac{e^{i \pi / 3}}{\sqrt{3}}|010\rangle-\frac{e^{-i \pi / 3}}{\sqrt{3}}|100\rangle \\
\left|\Omega_{4}\right\rangle=\frac{1}{\sqrt{3}}|011\rangle-\frac{e^{i \pi / 3}}{\sqrt{3}}|101\rangle-\frac{e^{-i \pi / 3}}{\sqrt{3}}|110\rangle
\end{array}\right\} \\
& \text { for }:\left\{\begin{aligned}
\left|\Omega_{2}\right\rangle & =\frac{1}{\sqrt{3}}|001\rangle-\frac{e^{-i \pi / 3}}{\sqrt{3}}|010\rangle-\frac{e^{i \pi / 3}}{\sqrt{3}}|100\rangle \\
\left|\Omega_{5}\right\rangle & =\frac{1}{\sqrt{3}}|011\rangle-\frac{e^{-i \pi / 3}}{\sqrt{3}}|101\rangle-\frac{e^{i \pi / 3}}{\sqrt{3}}|110\rangle
\end{aligned}\right\} \\
& \text { for : }\left\{\begin{aligned}
\left|\Omega_{3}\right\rangle & =\frac{1}{\sqrt{3}}|001\rangle+\frac{1}{\sqrt{3}}|010\rangle+\frac{1}{\sqrt{3}}|100\rangle \\
\left|\Omega_{6}\right\rangle & =\frac{1}{\sqrt{3}}|011\rangle+\frac{1}{\sqrt{3}}|101\rangle+\frac{1}{\sqrt{3}}|110\rangle
\end{aligned}\right\} \\
& E_{3}=E_{6}=-E_{J} \cos (\varphi)+3 E_{c, \alpha}-E_{c, \chi}
\end{align*}
$$



Figure 4.3: Graphical representation of the eigenenergies as functions of the superconducting phase $\varphi=\frac{2 \pi}{\Phi_{o}} \frac{\Phi}{3}$. We show three different cases in the (1) and (2) sub-spaces. (a) is showing the energy spectrum in the discussed regime, $C_{C} \rightarrow 0$ and $N_{g}=1 / 2$. The displayed energy functions are $E_{1}, E_{2}$ and $E_{3}$ which are shown analytically in (4.43). Observe how the ground state and the two different excited states change depending on the region $\varphi \in(n \pi / 3,(n+1) \pi / 3)$ we are looking at. Also, see that there is a degeneracy point every $n \pi / 3$ for $n=0,1,2, .$. in which two of three eigenenergies belonging to the same sub-space cross each other. In (b) we show a similar case but slightly apart from the sweet spot $\left(N_{g}=0.5-\varepsilon\right)$. This breaks CP-hole symmetry, hence splitting the energetic degeneracy between (1) and (2). The fact that now the energies split (due to considering $C$ island different from $A$ and $B$ ), makes the energy expressions in (4.43) hardly analytic. Instead, we show the real ground state ( $\mathbf{e}_{0}$ ) and the two excited states ( $\mathbf{e}_{1}$ and $\mathbf{e}_{2}$ ) in the (1) subspace (continuous lines) and in the (2) sub-space (dashed lines). Finally in (c) we illustrate a case further away from the sweet spot plus $C_{C} \rightarrow 0$. The energy splittings become more significant and the $\varphi$ dependency slightly vanishes.

In that case, we could say that there is no parallelism between both sub-spaces, but an antiparallelism! Observe that the (1) eigenstates describe the quantum mechanical state of one CP charge within the circulator, whereas the (2) eigenstates describe similar states, but for a CP hole instead. As we have argumented in Chapter 2, the time reversal symmetry operator ( $\hat{T}=\hat{U} \hat{K}$ ) is composed from a unitary transformation that switches the time direction around $(\hat{U})$ and an anti-unitary part formed by a complex conjugation operator $(\hat{K})$. Assuming a CP hole to be the antiparticle version of a CP charge (i.e., its charge conjugated version), applying the time reversal and charge conjugation $\hat{C}$ operator, defining $\hat{\Theta}=\hat{C} \hat{T}$, observe that, $\hat{\Theta}\left|\Omega_{1}\right\rangle=-e^{-i \pi / 3}\left|\Omega_{4}\right\rangle$, and $\hat{\Theta}\left|\Omega_{4}\right\rangle=-e^{i \pi / 3}\left|\Omega_{1}\right\rangle$. Hence, $\hat{\Theta}^{2}\left|\Omega_{1}\right\rangle=\left|\Omega_{1}\right\rangle$ (CT-symmetry). This means that $\left|\Omega_{1}\right\rangle$ is the charge-conjugated time-reversed version of $\left|\Omega_{4}\right\rangle$. Moreover, since within the sweet spot, both $\left|\Omega_{1}\right\rangle$ and $\left|\Omega_{4}\right\rangle$ are degenerated, $\left[\hat{\Theta}, \mathcal{H}_{\text {circ }}\right]=0$. Therefore, for $N_{g}=1 / 2$ we have an antiparticle-particle symmetry (as we previously commented).

The circulator energies are illustrated in Figure 4.3 where we show the energy spectrum in the present approach. We also sketch the energetic configuration if we slightly move aside from $N_{g}=1 / 2$ and $C_{C} \rightarrow 0$. The parameters we use to describe the latest case are $\varepsilon \equiv N_{g}-1 / 2$ and the fractional relations between $\sigma / \alpha$ and $\zeta / \chi$, which become the unity then $C_{C} \rightarrow 0$ as it was discussed before. The energy levels start flattening when moving away from the $C_{C} \rightarrow 0$ regime, and going further, the $\varphi$ dependency is lost. Also, the degeneracy from (1) and (2) sub-spaces is destroyed when $N_{g}$ is no longer in the sweet spot. The energies in the (a) sub-figure are straight taken form the analytic representation in (4.43). See that the ground state (and the excited states) changes strongly depending on the $\varphi$ region. For example, for $0<\varphi<\pi / 3$, the ground state of the circulator is given by $\left|\Omega_{3}\right\rangle$ and the first and second excited states by $\left|\Omega_{1}\right\rangle$ and $\left|\Omega_{2}\right\rangle$ respectively. Moving to $\pi / 3<\varphi<2 \pi / 3$ though, the ground and first excited states switch positions, so that $\left|\Omega_{1}\right\rangle$ becomes the ground state and $\left|\Omega_{3}\right\rangle$ the first excited state. This oscillating energy configuration goes on through all regions of $n \pi / 3<\varphi<(n+1) \pi / 3$. Therefore, the analytic energy excitations and decays will strongly depend on which region of $\varphi$ we are looking at.

As a final remark, note that we ended up working with the (1) and (2) sub-spaces, which describe the circulator with 1 and 2 extra CP charges respectively. The eigenstates describing the circulator empty (with no extra CP charges) and full (with 3 extra CP charges), in the 3-CPB approach, belong to a highly excited degenerated energy level, only if the Coulomb interaction between islands is large enough $\left(E_{c, \chi} \gg E_{J}\right)$. Our Hamiltonian does not provide any communication between sub-spaces, meaning that the number of exceeding CP charges may be controlled by an external supply (charge conservation). Indeed we see that one or two CP charges are required to provide decays and having a CP-hole symmetry, the physics provided by both cases are the same. The absorbed photons will excite the circulator state, but will not introduce any extra CP in the total number of charges. To do that, one would need to input the CP charges externally and regulate the total CP number. Assuming $E_{c, \chi} \gg E_{J}$, the lower energy states will correspond to both sub-spaces (1) and (2), therefore leaving the system within the (0) or (3) sub-paces will be unlikely for the system.

Following up, we will look at the interaction term in the Hamiltonian in (4.19) which now we should be able to directly quantize and express in the circulator eigenbasis. Then, we will obtain the decay rates analytically and directly see how chirality manifests.

### 4.5 Interaction term and decay rates

The derived eigensystem in the previous section and the given energy configuration suggests that the basic energy transitions can be summarized from 3 different states, the order of which will depend on
the $\varphi$ region we find ourselves. The Hamiltonian transmission line, $\mathcal{H}_{t l}$ and the circulator, $\mathcal{H}_{\text {circ }}$, terms have been quantized with two different methods. The interaction term, $\mathcal{H}_{\text {int }}$ collects both localized charges in the circulator and the directional transmission line modes. With the tools we developed during the past sections, we are now ready to quantize $\mathcal{H}_{\text {int }}$, which we will treat in the circulator eigenbasis to later express clearly the energy decays. In this section we will only look at the states from sub-space (1) with one extra CP in the circulator, but they are also applicable in sub-space (2).

First of all, in order to avoid confusions with the energy states ordering depending on $\varphi$, we will define $\mathbf{e}_{0}$ to represent the ground state energy, $\mathbf{e}_{1}$ the first excited state energy and $\mathbf{e}_{2}$ the second excited state energy. These energies will take the values of $E_{1}, E_{2}$ or $E_{3}$ depending on the $\varphi$ region. With the same idea, we will define $\left|\Psi_{0}\right\rangle$ to be the ground state with energy $\mathbf{e}_{0},\left|\Psi_{1}\right\rangle$ the first excited state with energy $\mathbf{e}_{1}$ and $\left|\Psi_{2}\right\rangle$ the second excited state with energy $\mathbf{e}_{2}$. The eigenstates will take the analytic form of $\left|\Omega_{1}\right\rangle,\left|\Omega_{2}\right\rangle$ and $\left|\Omega_{3}\right\rangle$, again depending on the region in $\varphi$. With these definitions, we can write the located charge variables in the circulator as follows.

$$
\begin{equation*}
\hat{Q}_{k}=2 e|1\rangle\left\langle\left. 1\right|_{k} \rightarrow \hat{Q}_{k}=2 e \sum_{i j} Q_{k, i j} \mid \Psi_{i}\right\rangle\left\langle\Psi_{j}\right| \quad \text { for }: \quad k=A, B, C \tag{4.44}
\end{equation*}
$$

We have switched from the charge number basis to the calculator eigenbasis, being $Q_{k, i j}$ unit-less matrix elements that define $\hat{Q}_{k}$ in the eigenbasis. By writing $|1\rangle_{k}$ we are taking the states that contain one extra cooper pair charge in the $k$ island. For example, $|1\rangle_{A}=|100\rangle+|110\rangle+|111\rangle$, but since we can divide the Hamiltonian in the three sub-spaces we already mentioned and here we will just be looking at (1) we can simply write $|1\rangle_{A}=|100\rangle$. Thanks to the $\left|\Psi_{j}\right\rangle$ eigenstate definitions, the states now will always be ordered, meaning that the ground state will always be $\left|\Psi_{0}\right\rangle$ and the two excited states $\left|\Psi_{1}\right\rangle$ and $\left|\Psi_{2}\right\rangle$. That way, and thanks to its Hermitic shape, we can straight perform the RWA (Rotating Wave Approximation) in $\mathcal{H}_{i n t}$, which will be basically translated to separating $\hat{Q}_{k}$ in upper and lower diagonal terms $\hat{Q}_{k}+\hat{Q}_{k}^{\dagger}$. The upper diagonal terms, given by $\hat{Q}_{k}^{\dagger}$, will contribute in exciting the circulator from the eigenstate $\left|\Psi_{i}\right\rangle$ to $\left|\Psi_{j}\right\rangle$, as long as $i<j$, and the lower diagonal terms, $\hat{Q}_{k}$, will contribute in the decay from an excited state to a lower energy state. The $\hat{Q}_{k}$ diagonal terms will vanish right after the RWA, and will not play any important role in the dynamics. The validity of the RWA applied in this case can be discussed in more detail, but it is not the aim of this work to dig deep further in this matter. We will just remark that we are applying the RWA right after turning $\mathcal{H}_{\text {int }}$ in the eigensystem of the circulator already diagonalized, which should strengthen the RWA application [38]. This way, we can re-write the interaction term in (4.19) as follows.

$$
\begin{aligned}
& \mathcal{H}_{i n t} \stackrel{R \underline{W} A}{=} \int d z \\
& \frac{\Delta L \sqrt{2 \alpha c}}{2 e} \frac{\sqrt{E_{c, \alpha} \omega}}{1+\tau}\left[\hat{Q}_{A}^{\dagger}\left(e^{i k_{o} z} \hat{a}_{R}+e^{-i k_{o} z} \hat{a}_{L}\right) \delta\left(z-z_{A}\right)+\hat{Q}_{B}^{\dagger}\left(e^{i k_{o} z} \hat{a}_{R}+e^{-i k_{o} z^{2}} \hat{a}_{L}\right) \delta\left(z-z_{B}\right)+\text { H.c. }\right]+ \\
& +\frac{\Delta L \sqrt{2 \chi c}}{2 e} \frac{\sqrt{E_{c, \chi} \omega}}{1+\tau}\left[\hat{Q}_{A}^{\dagger}\left(e^{i k_{o} z} \hat{a}_{R}+e^{-i k_{o} z} \hat{a}_{L}\right) \delta\left(z-z_{B}\right)+\hat{Q}_{B}^{\dagger}\left(e^{i k_{o} z} \hat{a}_{R}+e^{-i k_{o} z} \hat{a}_{L}\right) \delta\left(z-z_{A}\right)+\text { H.c. }\right]+ \\
& +\frac{\Delta L \sqrt{2 \zeta c}}{2 e} \frac{\sqrt{E_{c, \zeta} \omega}}{1+\tau}\left[\hat{Q}_{C}^{\dagger}\left(e^{i k_{o} z} \hat{a}_{R}+e^{-i k_{o} z} \hat{a}_{L}\right) \delta\left(z-z_{B}\right)+\hat{Q}_{C}^{\dagger}\left(e^{i k_{o} z} \hat{a}_{R}+e^{-i k_{o} z} \hat{a}_{L}\right) \delta\left(z-z_{A}\right)+\text { H.c. }\right]
\end{aligned}
$$

Observe how we quantized the localized charges in the circulator within the CPB regime and the transmission line charges as directional modes. Remember these incorporate the $\Delta L$ factor, as we saw in (4.33). After the quantization, we applied the RWA dropping the fast oscillating terms
and ending up with (4.45). The remaining terms represent the transitions involving a photon being absorbed and exciting the circulator, or the circulator decaying to a lower energy state emitting a photon to the transmission line. The RWA was also applied in the theoretical model from Chapter 3 , and (4.45) resembles the introduced dipole-interaction in (3.6). Note that here though, by looking at the eigenstate definitions in (4.43) one realizes that the sub-spaces with different number of CP charges are not connected though the $Q_{k, i j}$ elements in (4.44), not even in the interaction term. This means that charge will be conserved, and the excitations brought by the photons in the transmission line will not create more CP charges in the circulator, but excite the ground state configuration (as expected).

From (4.45), and taking into account (4.44), we can directly obtain the decay transitions from an eigenstate $\left|\Psi_{j}\right\rangle$ to $\left|\Psi_{i}\right\rangle$ as long as $i<j$, meaning that the energy from the first state is higher than the second, i.e., $\mathbf{e}_{i}<\mathbf{e}_{j}$. We will introduce a bra-ket notation for the right and left going modes: $|R\rangle,|L\rangle,|0\rangle$ indicating that there is a photon going to the right, a photon going to the left, and no photon fields (void) in the transmission line respectively. These states will interact with the creation and annihilation operators $\hat{a}_{R}$ and $\hat{a}_{L}$, and we will be able to write them combined with $\left|\Psi_{j}\right\rangle$ circulator eigenstates. For example, we will start looking at the decay transition from an excited eigenstate $\left|\Psi_{j}\right\rangle \otimes|0\rangle=\left|\Psi_{j}, 0\right\rangle$ to a lower energy eigenstate and a photon going to the right $\left|\Psi_{i}, R\right\rangle$ or to the left $\left|\Psi_{i}, L\right\rangle$, obtaining the following transition amplitudes.

$$
\begin{align*}
& \mathcal{H}_{i n t}\left|\Psi_{j}, 0\right\rangle=\frac{\sqrt{2 \Delta L c \omega}}{1+\tau}\left[\left(\sqrt{\alpha E_{c, \alpha}} Q_{A, i j}+\sqrt{\chi E_{c, \chi}} Q_{B, i j}+\sqrt{\zeta E_{c, \zeta}} Q_{C, i j}\right) e^{-i k_{o} z_{A}} \theta\left(z-z_{A}\right)+\right. \\
& \left.+\left(\sqrt{\alpha E_{c, \alpha}} Q_{B, i j}+\sqrt{\chi E_{c, \chi}} Q_{A, i j}+\sqrt{\zeta E_{c, \zeta}} Q_{C, i j}\right) e^{-i k_{o} z_{B}} \theta\left(z-z_{B}\right)\right]\left|\Psi_{i}, R\right\rangle+ \\
& \frac{\sqrt{2 \Delta L c \omega}}{1+\tau}\left[\left(\sqrt{\alpha E_{c, \alpha}} Q_{A, i j}+\sqrt{\chi E_{c, \chi}} Q_{B, i j}+\sqrt{\zeta E_{c, \zeta}} Q_{C, i j}\right) e^{i k_{o} z_{A}} \theta\left(z_{A}-z\right)+\right.  \tag{4.46}\\
& \left.+\left(\sqrt{\alpha E_{c, \alpha}} Q_{B, i j}+\sqrt{\chi E_{c, \chi}} Q_{A, i j}+\sqrt{\zeta E_{c, \zeta}} Q_{C, i j}\right) e^{i k_{o} z_{B}} \theta\left(z_{B}-z\right)\right]\left|\Psi_{i}, L\right\rangle \equiv \\
& \equiv R_{i j}^{\theta}\left|\Psi_{i}, R\right\rangle+L_{i j}^{\theta}\left|\Psi_{i}, L\right\rangle
\end{align*}
$$

We can straight evaluate the resulting step functions from integrating the delta functions in (4.45) at the decay direction, evaluating them to the unity (see Section 3.1.2). Also, we introduced $R_{i j}^{\theta}$ and $L_{i j}^{\theta}$ as the decay amplitudes from $\Psi_{j}$ to $\Psi_{i}$ states with the explicit $\theta$ function dependency to exploit it in the next chapter. We can then directly use Fermi's Golden Rule with (4.46) and obtain the right, left and total decay rates.

$$
\begin{equation*}
\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{(R)}=2 \pi R_{i j}^{*} R_{i j} \quad \Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{(L)}=2 \pi L_{i j}^{*} L_{i j} \quad \Gamma_{\Psi_{j} \rightarrow \Psi_{i}}=\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{(R)}+\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{(L)} \tag{4.47}
\end{equation*}
$$

Note we ignored the $\theta$ super-index since the definition of the decay rates imply a given direction, so at the end of the day, we evaluate all step functions to the unity. Simple and direct analytic results will no longer be available since our results are based on the RWA in (4.45) and the basis transformation in (4.44) strongly depends on the region where $\varphi$ is found.

Similar decay probabilities as the ones we observed in Chapter 3 can be already obtained. Here we will define the output probabilities as follows:


Figure 4.4: Directional output probabilities of photons decaying from the excited states $\left|\Psi_{1}\right\rangle$ and $\left|\Psi_{2}\right\rangle$, for $\varphi \in(0, \pi / 3)$. See that, as the latest can now decay to the first excited state, the total decay probabilities to the ground state become re-normalized being lower than the decays form the first excited state. The probabilities shape will not change even if we increase the crossed coupling term, since the third island coupling will be equal and compensate the crossed decays which would destroy the asymmetry.

$$
\begin{equation*}
P_{R}^{i j} \equiv \sum_{k, k<j} \frac{\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{(R)}}{\Gamma_{\Psi_{j} \rightarrow \Psi_{k}}} \quad P_{L}^{i j} \equiv \sum_{i, k<j} \frac{\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{(L)}}{\Gamma_{\Psi_{j} \rightarrow \Psi_{k}}} \tag{4.48}
\end{equation*}
$$

Observe the definition of the output probabilities now takes into account intermediate decays (deeper discussion to be found in the next chapter). The possible decays will come from the transitions $\left|\Psi_{1}\right\rangle \rightarrow\left|\Psi_{0}\right\rangle,\left|\Psi_{2}\right\rangle \rightarrow\left|\Psi_{0}\right\rangle$, and also from $\left|\Psi_{2}\right\rangle \rightarrow\left|\Psi_{1}\right\rangle$. The latest corresponds to a transition that was not considered in the first theoretical model we built, and we will have to take into account now. It will bring out a lower energetic photon than the input, result which we are not interested in. The decay rate probabilities are illustrated in Figure 4.4. See that now the output probabilities form the second excited state take into account the transitions to the first excited state, and that re-normalizes the output probabilities.

The most chiral spot in Figure 4.4 appears for $k_{o} \Delta L=\pi / 3$. As long as the Coulomb interaction between island is kept equal for all three (i.e., $E_{c, \chi}=E_{c, \zeta}$ ), the decaying chiral behavior remains constant. The crossed decays from the islands $A$ and $B$ to the transmission line are compensated by the decay rates from the $C$ island. In that case, the direct decays always dominate and chirality stabilizes. All this is thanks to the three equal island architecture presented by the circulator. Apart from that, the chirality is also constant as long as $\varphi \in(n \pi / 3,(n+1) \pi / 3)$, being $n=0,1,2, \ldots$. Translated to the applied magnetic flux, since $\varphi=2 \Phi \pi / 3 \Phi_{0}$, this means that for $\Phi / \Phi_{0} \in(n / 2,(n+1) / 2)$ the decay probabilities will remain with the same value. The decay probabilities illustrated in Figure 4.4 are taken for $n=0$, but one can see that the same result will be observed for $n=0,2,4, \ldots$, and the inverse probabilities will be observed for $n=1,3,5, \ldots$. We will take a close look to these probabilities in the next chapter.

The 3 - CPB regime $\left(C_{c} \rightarrow 0\right)$, which basically implies a circulator with equal superconducting islands capacitively coupled to a transmission line, shows an optimal and tunable chiral behavior towards decaying to the transmission line. The quantization of a real superconducting microwave circulator coupled through two ports, yields an interacting term which contains not only a direct coupling between the islands and the transmission line, but a crossed coupling (through the Josephson

Junction), and also a third coupling term which couples the third island to the transmission line (also through the Josephson Junctions). In this chapter, we decided to solve the whole system in the concrete regime assuming equal islands in the circulator (3-CPB). However, we can still imagine a second alternative approach that may be of interest. That is considering that the third circulator island is converted into a CP reservoir, which would effectively mean $C_{c} \rightarrow \infty$, but we will analyze this further in the following chapter.

## Chapter 5

## Quantum Circulator: CP-Reservoir approach

In Chapter 3 we have developed a theoretical model based on the coupling of two resonating cavities with an induced time reversal symmetry breaking through an additional effective coupling to a microwave circulator. The results promised a chiral behavior through scattering narrow band microwave photons. The operators we work with were defined on the resonating cavities, which were able to decay to the waveguide. The two resonators system could achieve two excited eigenstates to which a photon could be absorbed and emitted through the energy decays. The photons decay rates and scattering amplitudes yield a tunable chiral response through the magnetic flux and the distance between resonators.

With that idea in mind, in Chapter 4 we developed the superconducting circuit architecture of a microwave circulator coupled to a transmission line and obtained its Hamiltonian description. The strict assumptions we needed to follow were based on a weakly circulator-transmission line coupling and the superconducting islands belonging within the CPB regime, as the Transmon Regime showed a failure on the chiral results. From there, we decided to assume the third island capacitance coupling to the ground to be small enough, $C_{C} \rightarrow 0$, resulting in having all islands equal with the same conditions. The motivation to study that regime was merely based on the possibility of obtaining analytic expressions of the circulator eigenstates. The presented symmetry on the circulator showed that the decay probabilities to both directions remained constant within the magnetic flux window $\Phi / \Phi_{0} \in(n / 2,(n+1) / 2)$, for $n=0,1,2, \ldots$, and no crossed coupling dependence on the chiral behavior.

In the present chapter, we will look to a different regime, where we assume that the $C$ superconducting island is big enough to consider it a CP reservoir. In that case, $C_{C} \rightarrow \infty$ and the CPB regime will only be applicable to the $A$ and $B$ islands only. This regime would be the most similar to the theoretical model exposed in Chapter 3, where we only had two resonators coupled to the waveguide.


Figure 5.1: Schematic superconducting circuit architecture of the CP-Reservoir approach. Now the circulator disposes of a huge CP reservoir $\left(C_{C} \rightarrow \infty\right)$ instead of an island as the third port. We still dispose of a region where the magnetic flux can be set, which is the key of the chiral behavior.

We will systematically derive the Hamiltonian description which will be highly based on (4.19) and from there we will study which regimes are potentially favorable for the chiral behavior. We will derive the equations of motion following a parallelism with Chapter 3, and obtain the fidelity from the measure we initially designed. Finally, we will compare both circulator approaches and see also the fidelity values given by the $3-\mathrm{CPB}$ regime.

### 5.1 Hamiltonian description

The physical system we will describe is illustrated in Figure 5.3. In this case we just took the $C_{C}$ capacitance and we made it big enough to be considered a CP charge reservoir, i.e., $C_{C} \rightarrow \infty$. In this case, we are only allowing $A$ and $B$ to be within the CPB regime, letting a maximum of 1 extra CP charge to enter the superconducting islands. See now we still have a flux going through a closed superconducting circuit thanks to the Josephson coupling between the islands and the reservoir. We will have $N_{C}>1$ charges living in the reservoir, which will be able to access the islands.

We will start by writing down the Hamiltonian description of the new circulator architecture. It will be based on the matrix representation in Appendix C. Now though, we will allow $N_{C}$ to be higher than 1. Assuming that the total number of charges in the whole circulator is $N$, we can relate $N_{C}=N-N_{A}-N_{B}$, where $N_{A}$ and $N_{B}$ are the number of charges sitting in the superconducting islands $A$ and $B$ respectively. With that in mind, see that the basis in which we will operate in this case is the following.

$$
\begin{equation*}
\left|N_{A} N_{B}, N_{C}\right\rangle \equiv\{|00, N\rangle,|10, N-1\rangle,|01, N-1\rangle,|11, N-2\rangle\} \tag{5.1}
\end{equation*}
$$

Knowing now that $C_{C} \rightarrow \infty$, the capacitance coupling terms in (4.16) will yield $\zeta=\sigma=0$, and:

$$
\begin{equation*}
\alpha=\frac{C_{\Sigma}}{C_{\Sigma}^{2}+C_{J}^{2}} \quad \chi=\frac{C_{J}}{C_{\Sigma}^{2}+C_{J}^{2}} \quad C_{\Sigma}=C_{g}+2 C_{J} . \tag{5.2}
\end{equation*}
$$

We introduced $C_{\Sigma}$ as the total island capacitance. Within the new basis in (5.1) the circulator Hamiltonian will become:

$$
\mathcal{H}_{c i r c}=\left(\begin{array}{cccc}
E_{00} & -\frac{E_{J}}{2} e^{-i \varphi} & -\frac{E_{J}}{2} e^{i \varphi} & 0  \tag{5.3}\\
-\frac{E_{J}}{2} e^{i \varphi} & E_{10} & -\frac{E_{J}}{2} e^{-i \varphi} & -\frac{E_{J}}{2} e^{i \varphi} \\
-\frac{E_{J}}{2} e^{-i \varphi} & -\frac{E_{J}}{2} e^{i \varphi} & E_{01} & -\frac{E_{J}}{2} e^{-i \varphi} \\
0 & -\frac{E_{J}}{2} e^{-i \varphi} & -\frac{E_{J}}{2} e^{i \varphi} & E_{11}
\end{array}\right)
$$

Here we introduced the diagonal elements stating the corresponding energies, $E_{N_{A} N_{B}}$ of having $N_{A}$ and $N_{B}$ cooper pairs in the $A$ and $B$ islands. This Hamiltonian is very similar as in [39], but here acknowledging the exchange between $A$ and $B$ through a Josephson Junction. The energy values will be determined by the following expressions.

- $E_{00}$ is the energy of the system when the $A$ and $B$ islands are empty. This energy corresponds to:

$$
\begin{equation*}
E_{00}=4 N_{g}^{2}\left(2 E_{c, \alpha}+E_{c, \chi}\right) \tag{5.4}
\end{equation*}
$$

- $E_{10}$ is the energy of the system when the $A$ island is filled with one CP and $B$ is empty. The energy corresponds to:

$$
\begin{equation*}
E_{10}=4 E_{c, \alpha}\left[\left(1-N_{g}\right)^{2}+N_{g}^{2}\right]-4 E_{c, \chi}\left(1-N_{g}\right) N_{g} \tag{5.5}
\end{equation*}
$$

- $E_{01}$ is the energy of the system when the $A$ island is empty and $B$ is filled with one CP. The energy corresponds to:

$$
\begin{equation*}
E_{01}=4 E_{c, \alpha}\left[\left(1-N_{g}\right)^{2}+N_{g}^{2}\right]-4 E_{c, \chi}\left(1-N_{g}\right) N_{g} \tag{5.6}
\end{equation*}
$$

- $E_{11}$ is the energy of the system when both the $A$ and $B$ islands are filled with one CP. The energy corresponds to:

$$
\begin{equation*}
E_{11}=4\left(1-N_{g}\right)^{2}\left(2 E_{c, \alpha}+E_{c, \chi}\right) \tag{5.7}
\end{equation*}
$$

Observe that now, in contrast with the previous circulator Hamiltonian description when $C_{C} \rightarrow 0$, with the present basis we can not define different sub-spaces to straight diagonalize $\mathcal{H}_{\text {circ }}$. This happens due to the fact that now we are working with a fixed amount of CP charges $N$ and $C$ is no longer in the CPB regime, but instead is transformed in a reservoir. See how we are not able to straight
map the basis from (4.43) in (5.1) without having to introduce the $|11, N-2\rangle$ element which in the Hamiltonian appears to be connected to the other states. This makes the possible state combinations to form eigenstates highly connected, making the diagonalization of $\mathcal{H}_{\text {circ }}$ hardly analytic. We will then abandon the path of trying to find an analytic solution. However, we have to remember that we play with advantage since we already solved the $C_{C} \rightarrow 0$ regime, in which the present approach is based.

### 5.1.1 CT symmetry breaking and charge stability

As we saw in the Hamiltonian description in the 3-CPB approach, the sweet spot shows a charge-hole symmetry, which here is reflected in the energy expressions we show in (5.4) to (5.7). These have the exact same value when $N_{g}=1 / 2$. However, conserving charge conjugation symmetry in this situation may no be the most optimal solution. We will start arguing how the given Hamiltonian presents a CT-symmetry, and that in this case, this fact annihilates asymmetric decays. Due to the $4 \times 4$ symmetric architecture of the circulator Hamiltonian description in (5.3), we can try to find a gauge transformation within the $\left|N_{A} N_{B}, N_{C}\right\rangle$ states which preserves the time symmetry. However, this election comes conditioned by $\varphi$. For example, making $\left|N_{A} N_{B}, N_{C}\right\rangle \rightarrow e^{i \theta_{j}}\left|N_{A} N_{B}, N_{C}\right\rangle$, the circulator Hamiltonian will always commute with the time reversal operator $\hat{T}=\hat{U} \hat{K}$ if:

$$
[\mathcal{H}, \hat{T}]=0 \quad \Rightarrow \quad\left\{\begin{array}{l}
\sin \left(\varphi+\theta_{2}-\theta_{1}\right)=0  \tag{5.8}\\
\sin \left(\varphi+\theta_{3}-\theta_{2}\right)=0 \\
\sin \left(\varphi+\theta_{1}-\theta_{3}\right)=0 \\
\sin \left(\varphi+\theta_{4}-\theta_{3}\right)=0 \\
\sin \left(\varphi+\theta_{2}-\theta_{4}\right)=0
\end{array}\right.
$$

The first three equations already give a restriction to $\varphi$ to be proportional to $\pi / 3$ (as we have already previously seen in this thesis). Thus, since $\varphi=2 \pi \Phi / 3 \Phi_{0}$, the Hamiltonian in (5.3) breaks T-symmetry as long as the magnetic flux $\Phi \neq n \Phi_{0} / 2$. On the other hand, in the sweet spot $N_{g}=1 / 2$ we also have charge conjugation (particle-hole) symmetry. One can clearly see this fact as the energies $E_{00}=E_{11}$ and $E_{10}=E_{01}$ become equal. Guided from the results we previously obtained within the 3-CPB approach, having a CT-symmetric Hamiltonian description may seem an optimal condition. However, it is observed that chirality is not preserved anymore. Looking at the decay rate definitions in (4.46) and (4.47), see how applying the $\hat{\Theta}=\hat{C} \hat{T}$ operator (being $\hat{C}$ the charge conjugation operator) to the decay amplitude to the right $R_{i j}$ one obtains:

Energy fluctuations


| $E_{c, \chi} / E_{c, \alpha}$ | $N_{g}^{\text {stable } 1}$ |
| :---: | :--- |
| 0.0 | 0.507 |
| 0.1 | 0.506 |
| 0.2 | 0.5318 |
| 0.3 | 0.5522 |
| 0.5 | 0.5708 |
| 0.5 | 0.5882 |


| $E_{c, \chi} / E_{c, \alpha}$ | $N_{g}^{\text {stable 2 }}$ |
| :---: | :--- |
| 0.0 | 0.507 |
| 0.1 | 0.526 |
| 0.2 | 0.547 |
| 0.3 | 0.566 |
| 0.4 | 0.584 |
| 0.5 | 0.600 |

Figure 5.2: Transition energy fluctuations with respect to the reduced gate voltage $N_{g}$. The illustrated energy fluctuations are taken for $E_{J}=0.1 E_{c, \alpha}$ (being $E_{J} \sim 1 \mathrm{GHz}$ ). We only consider the energy transitions from the first excited level, as we have seen that higher energy state decays to intermediate levels may reduce chirality. Apart form $N_{g}=0.5$ sweet spot, other $N_{g}$ depending on $E_{c, \chi}$ can yield stable charge distributions. In vertical dotted lines we show the closer $N_{g}^{\text {stable } 1}>0.5$ values to the sweet spot where charge is still stable. Dashed vertical lines indicate the farthest $N_{g}^{\text {stable } 2}>0.5$ charge stable spots form the sweet spot. These values are listed in the table on the right side, with the corresponding $E_{c, \chi} / E_{c, \alpha}$ relation.

$$
\begin{align*}
& R_{i j}=\left\langle\Psi_{i}, R\right| \mathcal{H}_{i n t}\left|\Psi_{j}, 0\right\rangle=\left\langle\Psi_{i}, R\right| \hat{\Theta}^{\dagger} \hat{\Theta} \mathcal{H}_{i n t} \hat{\Theta}^{\dagger} \hat{\Theta}\left|\Psi_{j}, 0\right\rangle \sim \\
& \sim \\
& \quad\left\langle\Psi_{i}, R\right| \hat{\Theta}^{\dagger} \hat{\Theta} \sqrt{\alpha E_{c, \alpha}}\left(\hat{Q}_{A}+\hat{Q}_{B} e^{i k_{o} \Delta L}\right) \hat{a}_{R}^{\dagger} e^{i k_{o} z_{A}} \hat{\Theta}^{\dagger} \hat{\Theta}\left|\Psi_{j}, 0\right\rangle+ \\
& \quad+\left\langle\Psi_{i}, R\right| \hat{\Theta}^{\dagger} \hat{\Theta} \sqrt{\chi E_{c, \chi}}\left(\hat{Q}_{A} e^{i k_{o} \Delta L}+\hat{Q}_{B}\right) \hat{a}_{R}^{\dagger} e^{i k_{o} z_{A}} \hat{\Theta}^{\dagger} \hat{\Theta}\left|\Psi_{j}, 0\right\rangle=  \tag{5.9}\\
& = \\
& \quad \lambda_{i}^{*} \lambda_{j}\left\langle\Psi_{i}, R\right| \sqrt{\alpha E_{c, \alpha}}\left(-Q_{A, i j}-Q_{B, i j} e^{-i k_{o} \Delta L}\right) e^{-i k_{o} z_{A}}\left|\Psi_{i}, R\right\rangle+ \\
& \quad+\lambda_{i}^{*} \lambda_{j}\left\langle\Psi_{i}, R\right| \sqrt{\chi E_{c, \chi}}\left(-Q_{A, i j} e^{-i k_{o} \Delta L}-Q_{B, i j}\right) e^{-i k_{o} z_{A}}\left|\Psi_{i}, R\right\rangle \sim \\
& \sim-\lambda_{i}^{*} \lambda_{j} L_{i j}
\end{align*}
$$

Which is nothing else than the left decay amplitude. The $\lambda_{j}= \pm 1$ factors are the charge conjugation operators eigenvalues of $\left|\Psi_{j}\right\rangle$. Thus, we can not make $\Gamma^{(R)} \sim R_{i j}^{*} R_{i j}$ different from $\Gamma^{(L)} \sim L_{i j}^{*} L_{i j}$, and chirality is lost. Within the 3-CPB approach, in the sweet spot each state is degenerated, which means that not necessarily each state is an eigenstate of $\hat{\Theta}$ (as we have already seen in the previous chapter). Hence, (5.9) is not fulfilled and we were able to observe clear chiral decays.

The death of the chiral behavior in the sweet spot may indicate that the CP-Reservoir approach is not an optimal regime where to operate the circulator. However, $N_{g}=1 / 2$ may not be the only stable charge point. In Figure 5.2 we show how apart from the sweet spot, other $N_{g}$ values close to 0.5 preserve stable charge configurations, at the same time breaking CT-symmetry. The $N_{g}^{\text {stable }} \neq 0.5$


Figure 5.3: Energy levels for each stable charge spots pointed out in Figure 5.2. The Josephson Energy taken was $E_{J}=0.1 E_{c, \alpha}$ (being $E_{J} \sim 1 \mathrm{GHz}$ ). The top plots correspond to the gate voltage values closer to the sweet spot $\left(N_{g}^{\text {stable }}{ }^{1}\right)$ and the bottom plots to the farthest $\left(N_{g}^{\text {stable } 2}\right)$. Rising the interacting charging energy between islands increases the forth eigenstate energy, while leaving the rest with a relative energy difference $\sim E_{J}$.
values appear to be very similar, depending on the interacting charging energy relation with the island charging energy. Note that the curves in Figure 5.2 cross the origin from the y -axis more than once, even for $N_{g}>0.5$. We will test the chiral behavior towards decaying for two kind of the found $N_{g}^{\text {stable }}$ : those closer to the sweet spot ( $N_{g}^{\text {stable } 1}$ ) and the farthest ( $\left.N_{g}^{\text {stable } 2}\right)$. The first ones may describe stabler charge configurations since the fluctuations are flatter around the zero point. However, slightly higher values for the reduced gate voltage move further away from the CT-symmetric spot, hence improving the chiral behavior.

### 5.1.2 Energy spectrum

In Figure 4.3 we saw that, when $C_{C}$ began to grow, the energy levels split in three different unconnected levels. In that situation we disposed of two sub-spaces (1) and (2) with three eigenstates each providing a total of 6 energy levels which were not connected from one sub-space to another. In the sweet spot, both sub-spaces showed exactly the same energies, but slightly modifying the reduced gate voltage split such degenerated energies. In the present case we need to go away from the sweet spot to break CT-symmetry, since we do not dispose of unconnected degenerate levels anymore. Following up with the stable $N_{g}$ spots outlined above, we show the energy spectrum in each case, in Figure 5.3. Rising the interacting energy only affects the highest excited state, making it inaccessible for low energy transitions. The other three states stay together, split by an energy difference proportional to the Josephson Energy. In the farthest stable $N_{g}$ cases from the sweet spot, the oscillatory behavior from the three lower energies shows a very similar behavior than Figure 4.3, within the 3-CPB approach.

There all three islands were considered equal, allowing the entrance to only one CP charge. What we see here is a competition between the gate voltage and the interacting charging energy. Increasing the gate voltage will allow CP charges to stay together in the superconducting islands, but if we rise at the same time the interacting charging energy, CP-charges will repel each other from one island to another. Looking at the energy configurations, the farthest $N_{g}$ charge stable spots from $N_{g}=0.5$ yield more similar results than the 3 -CPB approach, which we already know shows chiral decays.

In the following sections we will derive the equations of motion for this approach. Our goal is to obtain a global fidelity measure for both approaches, starting on the CP-Reservoir, but later being able to map it to the 3-CPB approach. In the present case, since the sweet spot forbids chiral decays, we will consider the given charge stable spots, both the farthest and nearest to the sweet spot, for testing the fidelity of chirality.

### 5.2 Equations of motion

To begin finding on the equations of motion, we should define the operators we will work on. To create a certain parallelism with the theoretical model built in Chapter 3, the main two operators should be the directional mode operators in the transmission line brought by $\hat{a}_{R}$ and $\hat{a}_{L}$, and number operators counting the charges in the circulator islands. While the first two have already been mentioned and obtained through the quantization of the transmission line, the latest have not appeared in a familiar operator form yet. Since we are working in the CPB limit, we started defining the number states in each island and then switched to the circulator eigenbasis. In this section we will end up with an operational form of such states. First of all though, let us switch the notation of the interaction Hamiltonian in (4.45) to a handling matrix form. To do that now remember that, as $C_{C} \rightarrow \infty$, then $\zeta=0$ and the coupling between the reservoir and the transmission line is not acknowledged anymore. Also, knowing the $N_{C}=N-N_{A}-N_{B}$ relation, we can reduce the number of charge variables from 3 to only 2 being $Q_{A}$ and $Q_{B}$. Then, the interaction Hamiltonian can be re-written as follows.

$$
\begin{align*}
& \mathcal{H}_{\text {int }}=\sqrt{v_{g}}\left[\frac{\vec{Q}^{\dagger}}{2 e}\left(M_{\theta}^{\alpha}+M_{\theta}^{\chi}\right) \vec{a}+\vec{a}^{\dagger}\left(M_{\theta}^{\alpha}+M_{\theta}^{\chi}\right)^{\dagger} \frac{\vec{Q}}{2 e}\right] \quad \text { with }: \\
& M_{\theta}^{\alpha}=\frac{\Delta L \sqrt{2 c \omega}}{1+\tau} \sqrt{\frac{\alpha E_{c, \alpha}}{v_{g}}}\left(\begin{array}{cc}
e^{i k_{o} z_{A}} \theta\left(z-z_{A}\right) & e^{-i k_{o} z_{A}} \theta\left(z-z_{A}\right) \\
e^{i k_{o} z_{B}} \theta\left(z-z_{B}\right) & e^{-i k_{o} z_{B}} \theta\left(z-z_{B}\right)
\end{array}\right)  \tag{5.10}\\
& M_{\theta}^{\chi}=\frac{\Delta L \sqrt{2 c \omega}}{1+\tau} \sqrt{\frac{\chi E_{c, \chi}}{v_{g}}}\left(\begin{array}{cc}
e^{i k_{o} z_{B}} \theta\left(z-z_{B}\right) & e^{-i k_{o} z_{B}} \theta\left(z-z_{B}\right) \\
e^{i k_{o} z_{A}} \theta\left(z-z_{A}\right) & e^{-i k_{o} z_{A}} \theta\left(z-z_{A}\right)
\end{array}\right)
\end{align*}
$$

Where now $\vec{Q}=\left(Q_{A}, Q_{B}\right)$. Here we integrated over the delta functions in $\mathcal{H}_{\text {int }}$ and resulted in step functions. In our notation we denote the presence of step functions in a matrix by writing a sub-index $\theta$. With this notation, we can straight find the equations of motion of the $\vec{a}$ directional mode operators. Using the transmission line Hamiltonian derived in (4.34) and proceeding exactly as we did in the first chapter with the electric field equations of motion, we end up with:

$$
\begin{equation*}
\vec{a}=-\frac{i}{\sqrt{v_{g}}}\left(M_{\theta}^{\alpha}+M_{\theta}^{\chi}\right)^{\dagger} \frac{\vec{Q}}{2 e}+\vec{a}_{I N} \tag{5.11}
\end{equation*}
$$

Note we separated the matrix notation in terms related to the direct coupling ( $\alpha$ ) and terms coming from the crossed coupling $(\chi)$. In Chapter 3 we obtained the same equations of motion in (3.11) for the electric fields. The $2 e$ in the denominator comes from the charge operator definitions in the number basis in (4.44). If we now compare directly with the results in the first model, we can map from here the decay rates of the $A$ and $B$ islands. Observe that in the first theoretical model we only considered the direct coupling term, but now in the equations of motion a second coupling element appears due to the crossed couplings with the transmission line. Even though, assuming equal conditions in both islands, this last coupling term will introduce a directional difference in both decays in the number basis. Merging both $M^{\alpha}$ and $M^{\chi}$ and comparing with (3.10) we obtain:

$$
\begin{align*}
& \left\{\begin{array}{l}
\gamma_{A, R}=\gamma_{B, L}=\frac{4 \pi}{v_{g}} \frac{2 \Delta L}{(1+\tau)^{2}} \frac{\omega}{E_{c, t l}}\left|E_{c, \alpha}+E_{c, \chi} e^{i k_{o} \Delta L}\right|^{2} \\
\gamma_{A, L}=\gamma_{B, R}=\frac{4 \pi}{v_{g}} \frac{2 \Delta L}{(1+\tau)^{2}} \frac{\omega}{E_{c, t l}}\left|E_{c, \alpha}+E_{c, \chi} e^{-i k_{o} \Delta L}\right|^{2} \quad \text { having : }
\end{array}\right.  \tag{5.12}\\
& \mathcal{Q}_{\gamma}=\frac{\gamma}{\omega}=\frac{4 \pi}{v_{g}} \frac{2 \Delta L}{(1+\tau)^{2}} \frac{1}{E_{c, t l}}\left|E_{c, \alpha}+E_{c, \chi} e^{ \pm i k_{o} \Delta L}\right|^{2} \simeq\left(\frac{4 \pi}{1+\tau}\right)^{2} \frac{\Delta L}{\lambda} \frac{E_{c, \alpha}}{E_{c, t l}} \frac{E_{c, \alpha}}{\omega}
\end{align*}
$$

Where $\mathcal{Q}_{\gamma}$ is the quality factor of the superconducting islands towards decaying to the transmission line. This factor should be relevant compared to other island decaying factors through, for example, coupling to the environment, mainly because we need the chiral decays to dominate against other decay sources. Thanks to the linear dispersion relation in $\omega_{k}$, we can assume the group velocity $v_{g}$ to be equal to the photon phase velocity $v_{p}$, hence $v_{g}=v_{p}=\lambda \omega / 2 \pi$. We also defined the transmission line charging energy as $E_{c, t l}=e^{2} / 2 C_{t l}$, and the total transmission line capacitance given by $C_{t l}=c \Delta L$. See $\gamma_{A}$ and $\gamma_{B}$ are the decay rates from the individual superconducting islands to the transmission line. Recalling back the decay rates from the first theoretical model we built, we can straight map them with our results, and obtain the coupling strength with the transmission line as follows:

$$
\gamma_{A}=\gamma_{B}=\frac{4 \pi}{v_{g}} g^{2} \rightarrow\left\{\begin{array}{l}
g_{d}=\frac{1}{1+\tau} \sqrt{\frac{2 \Delta L \omega}{E_{c, t l}}} E_{c, \alpha}  \tag{5.13}\\
g_{c}=\frac{1}{1+\tau} \sqrt{\frac{2 \Delta L \omega}{E_{c, t l}}} E_{c, \chi}
\end{array}\right.
$$

Where we defined the direct coupling strength as $g_{d}$, and the crossed coupling strength as $g_{c}$. The total coupling strength will be given by $g=g_{d}+e^{ \pm i k_{o} \Delta L} g_{c}$. The $\pm$ sign in the crossed coupling strength will depend on the island which is decaying to the transmission line. Note as well that the coupling strengths depend explicitly on the photon frequency.

Next, we will find the equations of motion for operators similar than $\hat{b}_{j}$, which we used in Chapter
3. These are related to the $\vec{Q}$ operators as, since both are expressed in the charge number basis, $\hat{b}_{j}^{\dagger} \hat{b}_{j} \sim Q_{j}$. However, the implicit $\varphi$ dependence is hidden within the $Q_{k, i j}$ amplitudes in (4.44), and we need them to harness the chiral behavior. We will then work in the circulator eigenbasis $\left|\Psi_{j}\right\rangle$. The operators we will now define describe the eigenstate transitions in the circulator. Since we split $\vec{Q}$ within the RWA in creation and annihilation parts in (4.45), we will define the following circulator eigenoperators.

$$
\begin{equation*}
\hat{\Psi}_{i j} \equiv\left|\Psi_{i}\right\rangle\left\langle\Psi_{j}\right| \quad \hat{\Psi}_{i j}^{\dagger} \equiv\left|\Psi_{j}\right\rangle\left\langle\Psi_{i}\right| \quad \text { with : } \quad i<j \tag{5.14}
\end{equation*}
$$

The operator $\hat{\Psi}_{i j}$ takes the eigenstate $\left|\Psi_{j}\right\rangle$ and makes it decay to $\left|\Psi_{i}\right\rangle$, and the operator $\hat{\Psi}_{i j}^{\dagger}$ takes the eigenstate $\left|\Psi_{i}\right\rangle$ and excites it to $\left|\Psi_{j}\right\rangle$, for $i<j$. We have already seen these operators in the charge operator definitions in the eigenbasis, in (4.44). Before re-writing everything in terms of these eigenoperators, we need to take into account the following assumptions.

- We will only consider individual photon interactions with the circulator, i.e., only one photon at a time will excite the circulator.
- The circulator is always assumed to be found in the ground state, $\left|\Psi_{0}\right\rangle$, before any photon scatters.
- Once the circulator has been excited by a scattering photon, the decays we are interested in are the ones outputting a photon with the same frequency as the initial one.

With these considerations, we will group the eigenoperators in a vector to introduce it into the vector notation we are using.

$$
\begin{equation*}
\vec{\Psi} \equiv\left\{\hat{\Psi}_{01}, \hat{\Psi}_{02}, \quad \hat{\Psi}_{03}, \quad \hat{\Psi}_{12}, \quad \hat{\Psi}_{13}, \quad \hat{\Psi}_{23}\right\} \tag{5.15}
\end{equation*}
$$

Since the circulator will be always found in the ground state by an incoming photon and we are interested in the decays outputting same frequency photons, these will always end up back to the ground state. These decays are described by the first three operators in (5.15). With the vector notation now, knowing that these operators are highly related to the charge operators $\vec{Q}$, we will re-write the interaction Hamiltonian as follows.

$$
\mathcal{H}_{\text {int }}=\sqrt{v_{g}} \int d z\left[\vec{\Psi}^{\dagger} M_{\delta} \vec{a}+\vec{a}^{\dagger} M_{\delta}^{\dagger} \vec{\Psi}\right] \quad \text { with }: \quad M_{\delta}^{\dagger}=\frac{1}{\sqrt{2}}\left(\begin{array}{cccccc}
R_{01}^{\delta} & R_{02}^{\delta} & R_{03}^{\delta} & R_{12}^{\delta} & R_{13}^{\delta} & R_{23}^{\delta}  \tag{5.16}\\
L_{01}^{\delta} & L_{02}^{\delta} & L_{03}^{\delta} & L_{12}^{\delta} & L_{13}^{\delta} & L_{23}^{\delta}
\end{array}\right)
$$

The $M_{\delta}^{\dagger}$ matrix already appeared back in (3.16) in Chapter 3, but expressed on the circulator eigenbasis of the current approach. Using the notation we previously introduced in (4.47), we expressed the interaction Hamiltonian in terms of the decay amplitudes, being:

$$
\begin{align*}
& R_{i j}^{\delta}=\sqrt{\frac{2}{v_{g}}}\left[Q_{A, i j}\left(g_{d} e^{-i k_{o} z} \delta\left(z-z_{A}\right)+g_{c} e^{-i k_{o} z} \delta\left(z-z_{B}\right)\right)+\right. \\
& \left.+Q_{B, i j}\left(g_{d} e^{-i k_{o} z} \delta\left(z-z_{B}\right)+g_{c} e^{-i k_{o} z} \delta\left(z-z_{A}\right)\right)\right]  \tag{5.17}\\
& L_{i j}^{\delta}=\sqrt{\frac{2}{v_{g}}}\left[Q_{A, i j}\left(g_{d} e^{i k_{o} z} \delta\left(z_{A}-z\right)+g_{c} e^{i k_{o} z} \delta\left(z_{B}-z\right)\right)+\right. \\
& \left.+Q_{B, i j}\left(g_{d} e^{i k_{o} z} \delta\left(z_{B}-z\right)+g_{c} e^{i k_{o} z} \delta\left(z_{A}-z\right)\right)\right]
\end{align*}
$$

The $M_{\delta}$ contains the decay rate information from each eigenstate to the transmission line. Remark that we are stretching the use of $R_{i j}^{\delta}$ and $L_{i j}^{\delta}$ as good values to operate with. Indeed they have a powerful meaning. First of all, see that they contribute to the definition of the decay rates as we wrote in (4.47). Also, note they bring the $Q_{k, i j}$ amplitudes which where introduced as the $\hat{\Psi}_{i j}$ transition amplitudes in (4.44). Thanks to the Hermetic nature of the $\hat{Q}$ operators, these amplitudes fulfill $Q_{k, i j}=Q_{k, j i}^{*}$, the first being the probability amplitude a charge in $k$ has to go from $\left|\Psi_{j}\right\rangle$ to $\left|\Psi_{i}\right\rangle$, and the second the reversed transition. Grouping all together, we realize that assuming $j>i$, the $R_{i j}^{\delta}$ and $L_{i j}^{\delta}$ values actually describe the amplitudes of decaying from $\left|\Psi_{j}\right\rangle$ to $\left|\Psi_{i}\right\rangle$ emitting to the right or left respectively, and their complex conjugate describe the reversed transitions absorbing from the right or form the left. This is a powerful notation since we can now describe all kind of transitions and know their transition rates by just combining the two values in (5.17).

Before obtaining the equations of motion for the directional transmission line modes $\vec{a}$, we will take a close look at the defined eigenoperators $\hat{\Psi}_{i j}$. Since we will consider only one photon interactions with the circulator, we need to be careful with the commutation relations with the interaction Hamiltonian. For example, let us start looking at the following commutator.

$$
\begin{align*}
{\left[\hat{\Psi}_{01}, \mathcal{H}_{i n t}\right]=} & \int d z\left[R_{01}^{\delta} \hat{\Psi}_{00}-R_{01}^{\delta} \hat{\Psi}_{11}-R_{02}^{\delta} \hat{\Psi}_{21}-R_{03}^{\delta} \hat{\Psi}_{31}\right] \hat{a}_{R}+ \\
& +\left[L_{01}^{\delta} \hat{\Psi}_{00}-L_{01}^{\delta} \hat{\Psi}_{11}-L_{02}^{\delta} \hat{\Psi}_{21}-L_{03}^{\delta} \hat{\Psi}_{31}\right] \hat{a}_{L} \simeq  \tag{5.18}\\
& \simeq \int d z\left(R_{01}^{\delta} R_{01}^{\theta}+L_{01}^{\delta} L_{01}^{\theta}\right) \hat{\Psi}_{01}=\tilde{\Gamma}_{01} \hat{\Psi}_{01}
\end{align*}
$$

In the last step we inputted the result form (5.11) and expressed $\vec{Q}$ in terms of the eigenoperators in (5.15). Due to the assumptions we made, the only transition available left in the commutator will be the decay from $\left|\Psi_{1}\right\rangle$ to the ground state $\left|\Psi_{0}\right\rangle$. Obviously, if only one photon is absorbed and it excites the system to the first excite state, the only option this state has is to decay to the ground state with a decay rate $\tilde{\Gamma}_{01}$. However, we will shortly see that the latest is not precisely a decay rate. Since the system will be initialized always from the ground state, we are also interested in the following commutators.

$$
\begin{align*}
& {\left[\hat{\Psi}_{02}, \mathcal{H}_{i n t}\right]=\int d z\left[R_{02}^{\delta} \hat{\Psi}_{00}-R_{02}^{\delta} \hat{\Psi}_{22}-R_{01}^{\delta} \hat{\Psi}_{12}-R_{03}^{\delta} \hat{\Psi}_{32}+R_{12}^{\delta} \hat{\Psi}_{01}\right] \hat{a}_{R}+} \\
& +\left[L_{02}^{\delta} \hat{\Psi}_{00}-L_{02}^{\delta} \hat{\Psi}_{22}-L_{01}^{\delta} \hat{\Psi}_{12}-L_{03}^{\delta} \hat{\Psi}_{32}+L_{12}^{\delta} \hat{\Psi}_{01}\right] \hat{a}_{L} \simeq \\
& \simeq \int d z\left(R_{02}^{\delta} R_{02}^{\theta}+L_{02}^{\delta} L_{02}^{\theta}+R_{12}^{\delta} R_{12}^{\theta}+L_{12}^{\delta} L_{12}^{\theta}\right) \hat{\Psi}_{02}=\left(\tilde{\Gamma}_{02}+\tilde{\Gamma}_{12}\right) \hat{\Psi}_{02} \\
& {\left[\Psi_{03}, \mathcal{H}_{\text {int }}\right]=\int d z\left[R_{03}^{\delta} \hat{\Psi}_{00}-R_{03}^{\delta} \hat{\Psi}_{33}-R_{01}^{\delta} \hat{\Psi}_{13}-R_{02}^{\delta} \hat{\Psi}_{23}+R_{13}^{\delta} \hat{\Psi}_{01}+R_{23}^{\delta} \hat{\Psi}_{02}\right] \hat{a}_{R}+} \\
& +\left[L_{03}^{\delta} \hat{\Psi}_{00}-L_{03}^{\delta} \hat{\Psi}_{33}-L_{01}^{\delta} \hat{\Psi}_{13}-L_{02}^{\delta} \hat{\Psi}_{23}+L_{13}^{\delta} \hat{\Psi}_{01}+L_{23}^{\delta} \hat{\Psi}_{02}\right] \hat{a}_{L} \simeq \\
& \simeq \int d z\left(R_{03}^{\delta} R_{03}^{\theta}+L_{03}^{\delta} L_{03}^{\theta}+R_{13}^{\delta} R_{13}^{\theta}+L_{13}^{\delta} L_{13}^{\theta}+R_{23}^{\delta} R_{23}^{\theta}+L_{23}^{\delta} L_{23}^{\theta}\right) \hat{\Psi}_{03}=\left(\tilde{\Gamma}_{03}+\tilde{\Gamma}_{13}+\tilde{\Gamma}_{23}\right) \hat{\Psi}_{03} \tag{5.1}
\end{align*}
$$

Again, since we are only interested in decays emitting photons with the same frequency, the only terms left in these commutators are those describing transitions ending to the ground state. Observe that these involve decay rates to the intermediate levels, which where not included in the theoretical model from Chapter 3. All in all, we see that actually, the resulting operators from the interaction Hamiltonian can be resumed in the three eigenoperators $\hat{\Psi}_{01}, \hat{\Psi}_{02}, \hat{\Psi}_{03}$. These operators describe the transitions from the three excited states to the ground state. They have exactly the same function as the $\tilde{c}$ operators had in the theory we built in Chapter 3. For this reason, seems fair to re-name our eigenoperators to $\hat{c}_{1}, \hat{c}_{2}, \hat{c}_{3}$, which fulfill the following commutation relations.

$$
\begin{equation*}
\left[\hat{c}_{j}, \mathcal{H}_{\text {int }}\right]=\sum_{k}^{k<j} \Gamma_{k j} \hat{c}_{j} \quad\left[\hat{c}_{i}, \hat{c}_{j}\right]=0 \tag{5.20}
\end{equation*}
$$

Now we have all the tools to straight map the theoretical model we started building to the present approach, but having three possible excited states instead of two. Let us first re-write again the interacting plus circulator Hamiltonian terms as follows.

$$
\begin{equation*}
\mathcal{H}_{c i r c}+\mathcal{H}_{\text {int }}=\vec{c}^{\dagger} \mathcal{H}_{\Delta} \vec{c}+\sqrt{v_{g}} \int d z\left[\vec{c}^{\dagger} M_{\delta} \vec{a}+\vec{a}^{\dagger} M_{\delta}^{\dagger} \vec{c}\right] \tag{5.21}
\end{equation*}
$$

Where now:
$\mathcal{H}_{\Delta}=\left(\begin{array}{ccc}E_{1}-E_{0} & 0 & 0 \\ 0 & E_{2}-E_{0} & 0 \\ 0 & 0 & E_{3}-E_{0}\end{array}\right) \quad M_{\delta}^{\dagger}=\left(\begin{array}{ccc}R_{01}^{\delta} & R_{02}^{\delta} & R_{03}^{\delta} \\ L_{01}^{\delta} & L_{02}^{\delta} & L_{03}^{\delta}\end{array}\right) \quad \vec{c}=\left(\begin{array}{c}\hat{c}_{1} \\ \hat{c}_{2} \\ \hat{c}_{3}\end{array}\right) \quad \vec{a}=\binom{\hat{a}_{R}}{\hat{a}_{L}}$

With this Hamiltonian, and taking into account the commutation relations in (5.18) and (5.19),
we are now ready to find the equations of motion of the $\vec{c}$ and $\vec{a}$ operators. In the latest case, we will need also to use the transmission line Hamiltonian in (4.35).

$$
\begin{equation*}
\vec{a}=-\frac{i}{\sqrt{v_{g}}} M_{\theta}^{\dagger} \vec{c}+\vec{a}_{I N} \quad \dot{\vec{c}}=\left(-i \mathcal{H}_{\Delta}-\bar{\Gamma}\right) \vec{c}-i \sqrt{v_{g}} M_{\theta} \vec{a}_{I N} \tag{5.23}
\end{equation*}
$$

See that we are keeping the explicit $\theta$ function dependency in the matrix notation. We also introduced the decay rates of each eigenstate in the following compact matrix form.

$$
\bar{\Gamma}=\int d z M_{\delta} M_{\theta}^{\dagger} \simeq\left(\begin{array}{ccc}
\tilde{\Gamma}_{01} & 0 & 0  \tag{5.24}\\
0 & \tilde{\Gamma}_{02}+\tilde{\Gamma}_{12} & 0 \\
0 & 0 & \tilde{\Gamma}_{03}+\tilde{\Gamma}_{13}+\tilde{\Gamma}_{23}
\end{array}\right) \quad \text { with : } \quad \tilde{\Gamma}_{i j}=\int d z R_{i j}^{\delta *} R_{i j}^{\theta}+L_{i j}^{\delta *} L_{i j}^{\theta}
$$

We said previously that the $\tilde{\Gamma}$ are not exactly the decay rates. In fact, they are complex values whose imaginary part will contribute in slightly modifying the eigenenergies. However, since we are operating in the weak coupling limit and we assume that the decay rates are much smaller than the Josephson Energy, which governs the energy difference between levels, we can ignore the imaginary part on $\bar{\Gamma}$. On the other hand, one can find that the real part of these decay rates obey the following relation.

$$
\begin{equation*}
\tilde{\Gamma}_{i j}+\tilde{\Gamma}_{i j}^{*}=2 \operatorname{Re}\left[\tilde{\Gamma}_{i j}\right]=R_{i j}^{*} R_{i j}+L_{i j}^{*} L_{i j}=\Gamma_{i j} \tag{5.25}
\end{equation*}
$$

Therefore, as we can ignore the imaginary part of $\tilde{\Gamma}$, we can approximate it to be just $\tilde{\Gamma} \simeq \Gamma_{i j} / 2$, which is half of the decay rate. Note that this is exactly how we proceeded in Chapter 3, ignoring the imaginary parts on the decay rates which slightly swift the energy levels of the excited states.

We managed to completely map the first theoretical model proposal to the present approach. For now, the equations of motion are very similar, but now our interaction Hamiltonian allows transitions not only to the ground state, but to intermediate states, as we can see in the decay rates in (5.24). We already commented at the end of Chapter 4 that these intermediate level transitions will emit photons with undesired frequency, which we will consider as losses. This losses will highly affect the chiral decays from higher excited levels.

### 5.3 Decay probabilities

We will now proceed obtaining the decay probabilities from the possible excited levels of the circulator to the transmission line. We will start solving the equations of motion for the $\vec{c}$ operators, considering we have no input field.

$$
\begin{equation*}
\dot{\hat{c}}_{j}=\left[-i\left(E_{j}-E_{0}\right)-\frac{1}{2} \sum_{i<j} \Gamma_{i j}\right] \hat{c}_{j} \quad \rightarrow \quad \hat{c}_{j}(t)=\hat{c}_{j}(0) e^{-i\left(E_{j}-E_{0}\right) t-\frac{1}{2} \sum_{i<j} \Gamma_{i j} t} \tag{5.26}
\end{equation*}
$$

Note that the decay rates $\tilde{\Gamma}_{i j}$ have been substituted by $\Gamma_{i j} / 2$. We are only considering excitations starting from the ground state, but the decays can not only end up in the ground state, but on other intermediate excited states. The fact that the device can emit photons with undesired frequency forces us to take into account photon losses. These will be given by processes involving a photon exciting the system to a certain energy level, but decaying with more than one step to the ground state, giving a result multiple photons with inferior frequency as the initial one. All in all, the decay probabilities will be given by the following expressions.

$$
\begin{equation*}
v_{g} \int_{0}^{\infty} a_{R}^{\dagger} a_{R} d t=P_{R}+P_{R}^{\text {loss }} \quad v_{g} \int_{0}^{\infty} a_{L}^{\dagger} a_{L} d t=P_{L}+P_{L}^{\text {loss }} \quad \text { where : } \tag{5.27}
\end{equation*}
$$

- The probability of having a decay to the right will be:

$$
\begin{equation*}
P_{R}=\sum_{i}^{i<j} \sum_{j} \frac{\Gamma_{\Psi_{j} \rightarrow \Psi_{0}}^{R}}{\left(\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{R}+\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{L}\right)}\left|\hat{c}_{j}(0)\right|^{2} \tag{5.28}
\end{equation*}
$$

- The probability of having a decay to the left will be:

$$
\begin{equation*}
P_{L}=\sum_{i}^{i<j} \sum_{j} \frac{\Gamma_{\Psi_{j} \rightarrow \Psi_{0}}^{L}}{\left(\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{R}+\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{L}\right)}\left|\hat{c}_{j}(0)\right|^{2} \tag{5.29}
\end{equation*}
$$

- The probability of having a photon loss to the right will be:

$$
\begin{equation*}
P_{R}^{\text {loss }}=\sum_{i}^{i<j} \sum_{k \neq 0}^{k<j} \sum_{j} \frac{\Gamma_{\Psi_{j} \rightarrow \Psi_{k}}^{R}}{\left(\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{R}+\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{L}\right)}\left|\hat{c}_{j}(0)\right|^{2} \tag{5.30}
\end{equation*}
$$

- The probability of having photon loss to the left will be:

$$
\begin{equation*}
P_{L}^{\text {loss }}=\sum_{i}^{i<j} \sum_{k \neq 0}^{k<j} \sum_{j} \frac{\Gamma_{\Psi_{j} \rightarrow \Psi_{k}}^{L}}{\left(\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{R}+\Gamma_{\Psi_{j} \rightarrow \Psi_{i}}^{L}\right)}\left|\hat{c}_{j}(0)\right|^{2} \tag{5.31}
\end{equation*}
$$

Due to photon losses, high energetic excited states will loose their chiral behavior, as we have seen in Figure 4.4 from the $3-\mathrm{CPB}$ approach. At this point, we are only interested in decays from the first excited state to the ground state, totally avoiding decays to intermediate levels. On the other hand, the fidelity measure we defined in Chapter 3, more specifically in (3.33), showed a clear discrimination to left-going decays. Its maximum is only reachable through right-going decays, but as we initially argued, it is merely due to its definition, and we can also define it to show optimal values for left-going decays instead. Grouping all together, in order to observe good fidelity results, we will show now the decay probabilities to the right direction form the first excited state. All $N_{g}$ cases, with the corresponding $E_{c, \chi} / E_{c, \alpha}$ relation, will be illustrated.


Figure 5.4: Map representation of the decay probabilities to the right from the first excited level. We assumed $E_{c, \alpha} / E_{J}=10$. Yellow regions indicate that the decay probability to the right is one, and as the regions get dark, this probability reduces to zero (meaning decays to the left). Note every $\varphi=n \pi / 3$ the decays show a glide-reflectional symmetry, a straight reference of the T-symmetry breaking from the magnetic field. With red crosses we mark random $\varphi$ and $k_{o} \Delta L$ which yield probability one towards decaying to the right.

Starting with the $N_{g}$ charge stable values closer to the sweet spot, we show in Figure 5.4 the rightgoing decay probabilities to the transmission line. Since we are only looking at the first excited state decays, no photon losses will interfere. Therefore, as yellow regions indicate decays entirely to the right direction, the darkest zones indicate decays to the left. In case of zero interacting charging energy, the decays show a clear chiral behavior, which will be almost completely annihilated as soon as we turn on the interaction between islands. Remember that, as we increase the interacting charging energy, we also should increase $N_{g}$ to avoid charge fluctuations, according to Figure 5.2. As we keep increasing the interaction and the gate voltage, chiral regions appear to be more dominant. The competition between interaction and gate voltage appears again reflected in this case. The first contributes in gathering all charges together in the superconducting islands, whereas the second repels charges in different islands. Following up with this argument, low $N_{g}$ values are indeed required to fulfill the CPB regime in islands $A$ and $B$.

To provide a more wide view, let us look at the decay probabilities corresponding to farthest $N_{g}$ charge stable cases form the sweet spot (Figure 5.5). In this case we see in all situations a better chiral resolution from the decay probabilities. The gate voltage energy here dominates more in front of the interacting charging energy, compared with the Figure 5.4 cases. Indeed CP charges have now more freedom in accessing the islands. At some point, for high $E_{c, \chi}$ and $N_{g}$ values, the decay probabilities emphasize more the sudden jumps every $\varphi=n \pi / 3$. The energy configuration illustrated in Figure 5.3 for the farthest stable $N_{g}$ from the sweet spot show an equal oscillating shape than in Figure 4.3 from the 3 -CPB approach.

All in all, chirality remains optimal through alternative stable charge $N_{g}$ choices other than the sweet spot. The best chiral decay probabilities are observed for the farthest gate voltages from $N_{g}=0.5$. We want to stress here that the $N_{g}$ stable charge choices were mare for $N_{g}>0.5$. However,


Figure 5.5: Output probability map from the first excited state to the right direction. In this case we consider the $N_{g}$ stable charge choices farthest away from the sweet spot. With red crosses we mark random optimal chiral decay spots to later test the fidelity with.
as we have seen in Figure 5.2, below the sweet spot we can find symmetrically the same stable charge spots. We have seen that the best chirtality is observed for the farthest $N_{g}$ values from the sweet spot, but that also entitles the farthest $N_{g}$ values below the sweet spot as good chiral candidates. The only difference between $N_{g}>0.5$ or $N_{g}<0.5$ will be the direction of the dominant decay. What we are trying to remark is that, whenever choosing high $N_{g}$ values may destroy the CPB regime, one can look at the low $N_{g}<0.5$ values farthest form the sweet spot. Even so, in all cases we managed to find $\varphi$ and $k_{o} \Delta L$ which yield a total dominance to decay to the right direction. However, the decay probabilities may be deceptive measures, since we still need the total decay rate to be high enough to matter. In Figure 5.4 and Figure 5.5 we marked with red crosses random phase choices which make the device entirely decay to the right. Following up, we will re-obtain the S-matrix and fidelity we saw in Chapter 3, and test it with the marked choices.

### 5.4 Scattering and fidelity

With the observations we have already made we are ready to guess a good regime in order to obtain an optimal fidelity value. We will use the fidelity definition we introduced in Chapter 3, concretely in (3.34). Remember that the fidelity measure we defined was relative to which arm of the interferometer we are blocking. Out starting choice was a fidelity measure which counts the chiral outputs to the right as optimal values. That means that we will need to take the most chiral regions where the main decay is to the right direction. We will start by obtaining the scattering matrix of the system in the same way we proceeded in the second chapter. We will begin by solving the equations of motion for $\vec{c}$ in (5.23) in the Fourier space for the time variable, meaning $\partial_{t} \rightarrow-i \omega$.

$$
\begin{equation*}
\vec{c}=\sqrt{v_{g}}\left(\mathcal{H}_{\Delta}-\omega \mathbb{I}-i \bar{\Gamma}\right)^{-1} M \vec{a}_{I N} \tag{5.32}
\end{equation*}
$$

Where the $M$ matrix is the $M_{\theta}$ matrix but evaluating all $\theta$ functions to the unity. We will replace this solution in the equations of motion of $\vec{a}$ in (5.23) and finally obtain the S -matrix.

$$
\begin{equation*}
\vec{a}=S \vec{a}_{I N} \quad \text { with }: \quad S=1-M^{\dagger}\left(i \mathcal{H}_{\Delta}-i \omega \mathbb{I}+\bar{\Gamma}\right)^{-1} M \tag{5.33}
\end{equation*}
$$

The unitaricity of this matrix can be shown, thanks to our results in (3.28), by just demonstrating the following equality.

$$
\begin{equation*}
M M^{\dagger}=\int d z\left[M_{\delta} M_{\theta}^{\dagger}+M_{\theta} M_{\delta}^{\dagger}\right] \tag{5.34}
\end{equation*}
$$

The equality is fulfilled when assuming a weak coupling between the circulator and the transmission line, since we can ignore the off-diagonal terms on the left side (see (5.25)). The fidelity of the system will be given by the diagonal terms of the scattering matrix. We can represent them analytically, by still assuming the weak coupling regime, as follows.

$$
\begin{align*}
& S_{11}=1-i\left(\frac{\Gamma_{\Psi_{1} \rightarrow \Psi_{0}}^{R}}{\Delta E_{1}-\omega-i \Gamma_{01} / 2}+\frac{\Gamma_{\Psi_{2} \rightarrow \Psi_{0}}^{R}}{\Delta E_{2}-\omega-i\left(\Gamma_{02}+\Gamma_{12}\right) / 2}+\frac{\Gamma_{\Psi_{3} \rightarrow \Psi_{0}}^{R}}{\Delta E_{3}-\omega-i\left(\Gamma_{03}+\Gamma_{13}+\Gamma_{23}\right) / 2}\right) \\
& S_{22}=1-i\left(\frac{\Gamma_{\Psi_{1} \rightarrow \Psi_{0}}^{L}}{\Delta E_{1}-\omega-i \Gamma_{01} / 2}+\frac{\Gamma_{\Psi_{2} \rightarrow \Psi_{0}}^{L}}{\Delta E_{2}-\omega-i\left(\Gamma_{02}+\Gamma_{12}\right) / 2}+\frac{\Gamma_{\Psi_{3} \rightarrow \Psi_{0}}^{L}}{\Delta E_{3}-\omega-i\left(\Gamma_{03}+\Gamma_{13}+\Gamma_{23}\right) / 2}\right) \tag{5.35}
\end{align*}
$$

Where $\Delta E_{j}=E_{j}-E_{0}$ are the energy differences between excited states and the ground state. See we also substituted $\tilde{\Gamma}_{i j}=\Gamma_{i j} / 2$. We will re-write these terms by defining the following variables.

$$
\begin{equation*}
x_{1}=\frac{2\left(\omega-\Delta E_{1}\right)}{\Gamma_{01}} \quad x_{2}=\frac{2\left(\omega-\Delta E_{2}\right)}{\Gamma_{02}+\Gamma_{12}} \quad x_{3}=\frac{2\left(\omega-\Delta E_{3}\right)}{\Gamma_{03}+\Gamma_{13}+\Gamma_{23}} \tag{5.36}
\end{equation*}
$$

Therefore, the scattering amplitudes that will enter in the fidelity will be re-written as:

$$
\begin{align*}
& S_{11}=1-\frac{2 P_{R}^{(1)}}{1+i x_{1}}-\frac{2 P_{R}^{(2)}}{1+i x_{2}}-\frac{2 P_{R}^{(3)}}{1+i x_{3}}  \tag{5.37}\\
& S_{22}=1-\frac{2 P_{L}^{(1)}}{1+i x_{1}}-\frac{2 P_{L}^{(2)}}{1+i x_{2}}-\frac{2 P_{L}^{(3)}}{1+i x_{3}}
\end{align*}
$$

Which are exactly the same scattering amplitudes we wrote in (3.30) on Chapter 3. Before making any plots, we must be careful and be sure we fulfill the prerequisites to belong in the regime in which we based all our approximations. The most important restrictions we need to consider come form


Figure 5.6: Fidelity representations for each $N_{g}$ stable charge point close to the sweet spot case. The exposed examples are the $\varphi$ and $k_{o} \Delta L$ cases marked with red crosses in Figure 5.4. We show two $\tau$ cases in each situation: $\tau=200$ in continuous lines and $\tau=500$ in dashed lines. We also assumed $\lambda / \Delta L \sim 10$ and $E_{c, \alpha} / E_{c, t l} \sim 10^{3}$. Dashed vertical lines indicate the energy transitions form the first and second excited state to the ground state.
the CPB regime and the weak coupling regime. In the first case, we require that the charging energy fulfills $E_{c, \alpha} \gg E_{J}$, and the second case basically states that the decay rates are much smaller than the energy differences between eigenstates, meaning $\gamma_{k, R / L} \ll E_{J}$. However, our decay rate expressions, in (5.12), contain the charging energy in their definition, apart from the photon frequency, which will be comparable to the transition energy, i.e., the Josephson Energy ( $\omega \sim E_{J}$ ). It is here where the transmission line-charge coupling capacitances relation provided by $\tau$ will play an important role. Indeed, assuming $E_{c, \chi} \ll E_{c, \alpha}$ and using (5.12), the limit we are considering needs to fulfill:

$$
\begin{equation*}
\gamma \ll E_{J} \ll E_{c, \alpha} \quad \rightarrow \quad \frac{E_{c, t l}}{E_{c, \alpha}} \frac{\lambda}{\Delta L} \frac{E_{J}}{E_{c, \alpha}}\left(\frac{1+\tau}{4 \pi}\right)^{2} \gg 1 \gg \frac{E_{J}}{E_{c, \alpha}} \tag{5.38}
\end{equation*}
$$

Therefore, we need $\tau=\delta z c / C_{q}$ to be sufficiently big to allow a CPB be weakly coupled to the transmission line. In order to fulfill (5.38), let us assume $E_{J} \sim 1 \mathrm{GHz}$ and $E_{c, \alpha} \sim 10 \mathrm{GHz}$. By approximating $E_{c} \sim\left(e^{2} \varepsilon_{o} \pi\right) L^{-1}$ (where $L$ is the capacitor length and $\left(e^{2} / \varepsilon_{o} \pi\right) \sim \mathrm{GHz} \cdot \mu \mathrm{m}$ ), we can relate length dimensions with their corresponding charging energies. Assuming a typical $\lambda \sim$ 1 cm microwave length and an overlap region length $\Delta L \sim 100 \mu \mathrm{~m}$ between islands (also being the transmission line length between the coupling points), one sees that $E_{c, t l} / E_{c, \alpha} \sim 10^{-3}$ and $\lambda / \Delta L \sim$ $10^{2}$. Hence, having $\tau>100$ will be enough to fit in (5.38).

Knowing now the restrictions we need to follow to be able to enter the considered regimes, we will calculate the fidelity from the marked choices in Figure 5.4 and Figure 5.5 with red crosses. We will start by showing the $N_{g}$ stable charge cases closer to the sweet spot in Figure 5.6. We illustrate two different $\tau$ choices in each plot: $\tau=200$ with continuous lines and $\tau=500$ with dashed lines. Observe how decreasing the capacitive coupling between the circulator and the transmission line sharpens the


Figure 5.7: Fidelity plots for the farthest charge stable points. The exposed cases are taken from $\varphi$ and $k_{o} \Delta L$ values marked with red crosses in Figure 5.5. Compared with Figure 5.6, the peaks appear to be much wider (higher decay rates), still reaching $F=1$ by increasing $\tau$.
fidelity peaks improving its maximum closer to the unity. Since we are testing the fidelity from totally dominating decays to the right from the first excited state, the peaks concentrate around $\omega=E_{1}-E_{0}$ as expected. However, they do not always end up exactly on top of the transition energy. This is mainly because the nearest excited state is also able to decay to the same direction, slightly deviating the peaks position (see the derivation in Chapter 3, concretely in (3.36)). We see much sharper and thinner fidelity peaks in $E_{c, \chi} / E_{c, \alpha}=0.4$ case for example. The fidelity width is directly proportional to the absolute magnitude of the decay rate. Except in $E_{c, \chi} / E_{c, \alpha}=0.0$ and $E_{c, \chi} / E_{c, \alpha}=0.3$, all other random choices were picked from shrinked right decay regions which appear to be significantly more dominant than the left decays. However, the chosen spots show a very low value on the decay rates. Due to the scarce availability of high chiral zones, optimal $\varphi$ and $k_{o} \Delta L$ choices will have a high probability of ending up displaying low decay rates, hence thinner fidelity peaks. Having significantly low decay rates, already within the $\gamma \ll E_{J}$ regime, may decrease the quality factor $\mathcal{Q}_{\gamma}$ even ending up with other decaying sources dominating over the directional dependent decays. In that case, we would loose chirality.

On the other hand, in Figure 5.7 we illustrate the fidelity representations for the farthest $N_{g}$ charge stable values from the sweet spot. The exposed cases are straight taken from the random choices marked as red crosses in Figure 5.5. The main difference between having a big and small $N_{g}$ appears in the availability of chiral regions in the decay probability maps. Also, the decay rates are higher for the farthest $N_{g}$ choices from the sweet spot, providing wider fidelity peaks, which can turn thinner by increasing $\tau$, i.e., trough weaken the coupling.

The CP-Reservoir approach turned out to be more complicated than expected, since we needed to break CP-hole symmetry to observe chirality. Although having observed better fidelity results from the farthest $N_{g}$ charge stable choices from the sweet spot, one can always manage to find optimal chiral regions to operate in both cases (even if decay rates become significantly low). However, we have already mentioned that the exact same decay probabilities, but inverted, may be found looking at $N_{g}$ charge stable choices below the sweet spot. Those would solve the possibility that the CPB
regime is destroyed by choosing high values of $N_{g}$. To end this chapter, we will provide a comparison between both studied approaches, as well as providing a fidelity result for the 3-CPB approach using the same methodology.

### 5.5 3-CPB and CP-Reservoir: approach comparison

The time reversal symmetry breaking due the application of a magnetic flux that strongly interacts with our superconducting circuit has shown very similar results for both circulator architectures we analyzed. The main differences between both situations reside in the treatment of the third superconducting island, while both islands coupled to the transmission line remain intact within the CPB regime. The first case, where we assumed a third island small enough to be considered equal as the other two, we saw that the overall system could be resumed as a circulator perturbed by a transmission line. The weak coupling between the islands $A$ and $B$ to the transmission line, and the weak ground coupling with the $C$ island allowed us to merely treat all three islands equally since all are defined by the same island capacitance $C_{\Sigma}=C_{g}+2 C_{J}$. This description brought an eigensystem defined by the possibility of having up to 3 exceeding CP living inside the circulator, which where conserved by the Hamiltonian. Therefore, all excitations and decays would basically change the eigenstate configuration. The definitions of the eigenstates which describe the circulator inner state had an implicit dependency with the superconducting phase difference $\varphi$, since their analytic description changes every $\varphi=\pi / 3$. This dependency brings out the time reversal symmetry breaking that the circulator Hamiltonian displays. In fact, at the end of the previous chapter we have only seen the decay probabilities along the $k_{o} \Delta L$ axis for a given $\varphi$ value. If we plot a more general view of these decay probabilities, we will be able to see a clear jumps in probability every $\varphi=\pi / 3$ (see Figure 5.8). This abrupt changes occur because the ground and excited states suffer a sudden change. Due to the three-island symmetric architecture, the eigenstates do not show an explicit $\varphi$ dependence, but it is only present in their energies.

The second case we have analyzed shows a more similar architecture than the theoretical model we initially presented. In Chapter 3, remember we defined a system built of two resonant cavities coupled each to a waveguide, and also coupled each to an island of a circulator. The operators we used where defined in the resonant cavities, $\hat{b}$, and the waveguide $\hat{a}$. The circulator acted merely as a secondary character which mediates the excitations between resonators through a superconducting circuit brought by a Josephson Junction. The circulator brought a superconducting loop through which a tuned magnetic field was applied to introduce the phase difference $\varphi$ and break the time reversal symmetry to result in a chiral behavior towards the transmission line. In the CP-Reservoir approach though, we assumed a big ground coupling with the third island on the circulator, translated in $C_{C} \rightarrow \infty$. Unlike in Chapter 3, here we removed the resonating cavities and directly coupled the transmission line to the circulator islands through superconducting microwave resonators. Therefore, our state was described not on the cavities, but on each superconducting island forming the circulator, brought by the charge operators in the number basis $\hat{Q}$. Since we assumed a global fixed amount of CP charges $N$, and the $C$ island resulted in a reservoir, we were left only with two islands, $A$ and $B$, coupled to the transmission line.

In the beginning of this chapter though, we realized that the chiral decays within the CP-Reservoir architecture in the sweet spot will die. Hopefully, we managed to get around the problem by finding


Figure 5.8: Decay probability maps from the first theoretical model and the 3-CPB approach. Since the chirality is higher on the lower excited states and they are easily accessible, we just illustrate the decay probabilities from the first excited state to the right (top plots) and to the left (bottom plots). In (a) we show the first theoretical model decay probabilities, given analytically in (3.24). In (b) the decay probabilities from the three symmetric island circulator weakly coupled to the transmission line $\left(C_{C} \rightarrow 0\right)$.
other alternative $N_{g}$ values which preserve charge stability. Due to the multiple gate voltage choices, depending on the interacting charging energy, we chose 12 different spots with different interacting charging energies divided in two groups: 6 examples of $N_{g}$ values close to the sweet spot and 6 examples of $N_{g}$ at the farthest charge stable spot from the sweet spot. Although, all examples yield a chiral response towards decaying, and excellent fidelity values can be obtained, $N_{g}$ cases further away from $N_{g}=1 / 2$ exposed a wider chiral region in the decay probabilities with higher decay rates.

Looking at Figure 5.8 we can get a more general view of the decay probabilities belonging to the first theoretical model and the 3-CPB approach. Starting on the theoretical model formed by two resonators weakly coupled to a transmission line and to the circulator (Figure 5.8 (a)), we see that the probabilities show a glide-reflectional symmetry periodically every $\varphi=\pi$, and a total symmetry every $\varphi=2 \pi$. Since the circulator is not playing a direct role in the equations of motion in this case, but instead appears merely as an extra coupling effective term, the decay probabilities do not completely follow the time reversal symmetry conservation rule every $\varphi=\pi / 3$ we wrote on the introductory chapter. We are not directly operating on the circulator islands but on the resonating cavities, so the operators describing the main system dynamics are not the same as in the following approaches, and the phase $\varphi$ will be slightly different. Hence, it is normal that the asymmetric decaying behavior shown by the theoretical model is not strictly attached to the expected behavior form the circulator. On the other hand, quantizing a real superconducting circuit imitating the circulator, weakly coupled to a transmission line, and moving the operators forming our equations of motion to the superconducting islands, the time reversal symmetry conservation spots clearly appear in Figure 5.8 (b) (3-CPB approach). There we illustrate the decay probabilities already pre-viewed in Figure 4.4, but now from a more general perspective for all possible $\varphi$ values. See the decay probabilities abruptly change every $\varphi=\pi / 3$, since as we saw in Chapter 4 , the eigenstate ordering changes as well. Here is clearly manifested the time reversal symmetry conservation points predicted in the introductory chapter, appearing as glide-reflectional symmetric spots. This is basically because we are properly describing effectively an isolated circulator weakly coupled to (perturbed by) a transmission line. Also, the three equal islands symmetry provided a constant chirality level independent on the crossed coupling term, since the third island coupling cancels out the crossed decays which would destroy the asymmetric decays. Comparing with the decay probabilities from the CP-Reservoir approach (Figure 5.4 and Figure 5.5), all three cases have in common that for $k_{o} \Delta L=n \pi$ the decay probabilities are symmetric to both directions, and that the maximum chiral spots are found between $k_{o} \Delta L=n \pi$ and $k_{o} \Delta L=(n+1) \pi$, for $n=0,1,2, \ldots$.

With the first theoretical model and the CP-reservoir case we have obtained a fidelity measure, which appears to have a global definition only depending on the decay probabilities to both directions and the energy differences between the decaying excited state with the ground state.

$$
F=\frac{1}{16}\left(1-S_{11}\right)\left(1-S_{11}^{*}\right)\left(1+S_{22}\right)\left(1+S_{22}^{*}\right) \quad \text { where : }\left\{\begin{array}{l}
S_{11}=1-\sum_{j}^{n_{e}} \frac{2 P_{R}^{j}}{1+i x_{j}}  \tag{5.39}\\
S_{22}=1-\sum_{j}^{n_{e}} \frac{2 P_{L}^{j}}{1+i x_{j}}
\end{array}\right.
$$

Being $x_{j}=2\left(\omega-\Delta E_{j}\right) / \sum_{i=0}^{i<j} \Gamma_{i j}$ and $n_{e}$ the number of possible excited states. With this general expression we can directly now obtain the fidelity values for the 3 - CPB case. Thanks to the crossed coupling independence on the decay probabilities in that case, we can choose a couple of values for $k_{o} \Delta L$ and $\varphi$ which displays a maximum chiral behavior and see how the fidelity changes as we increase


Figure 5.9: Fidelity values for $\varphi=\pi / 4$ and $k_{o} \Delta L=\pi / 3$, within the 3-CPB approach ( $C_{C} \rightarrow 0$ ). As long as $\varphi \in(0, \pi / 3)$ and the island separation remains constant, the fidelity should not change its shape. See how the peak does not move from the energy difference form the first excited state to the ground state, as we increase the dominance of the crossed coupling. This term only sharpens the peak, making it narrower, since it enters the decay rates within $x_{1}$.
the crossed coupling strength. In Figure 5.9 we illustrate the fidelity for the circulator with equal islands coupled to a transmission line case. The peak fidelity does not move or gets lower than the unity as the crossed coupling strength increases. This fact is mainly because the decay probabilities remain unaffected under the effects of the crossed coupling $g_{c}$, being only the direct coupling $g_{d}$ the main directional decay responsible. However, $E_{c, \chi}$ still plays a relative role in the global decay, since as we increase the interacting charging energy between islands, the decay rates decrease, showing shrinked fidelity peaks. Also, within the 3-CPB approach, the fidelity peak will always appear exactly on the transition energy difference, as the second excited state in the most chiral spots shows a zero decay probability to the main decaying direction from the first excited state (see Chapter 3, concretely in (3.36)).

All in all, we observe how the main difference between the 3 -CPB and CP-reservoir cases is the symmetry provided by the circuit architecture which forbids chiral decays on the latest staying in the sweet spot. The 3 - CPB regime shows a more robust response in front of magnetic flux fluctuations (within $\varphi$ ) and, not only the directional decay probabilities hold under high crossed coupling values, but we also require a high Coulomb interaction between islands ( $E_{c, \chi} \gg E_{J}$ ) to ensure a good CP state preparation. This will also change the shape of the fidelity, making the photon frequency peaks thinner (hence decay rates lower) as we can see in Figure 5.9. A good CP preparation is required, since we are only interested in having $N=1$ or $N=2$ exceeding CP charges living inside the circulator islands, and having $N=0$ or $N=3$ will destroy the chirality (the circulator has only one eigenstate, not being able to be excited or show decays). On the other hand, the CP-reservoir regime may seem a more elegant and controllable setup, since the overall number of CP playing in the circulator does not strictly matter as they are automatically controlled by the reservoir. However, in this situation one needs to pay attention to the $N_{g}$ and $E_{c, \chi}$ choice to ensure chirality and stable charge configurations. Our study yields wider chiral regions for farthest $N_{g}$ values from the sweet spot, but fidelity holds excellent chiral results in all cases, accordingly picking the maximum chiral spots within the decay probability maps.

## Chapter 6

## Conclusions and Outlook

### 6.1 Conclusions

In this thesis we present a model which predicts highly tunable chiral device-photon interactions within a transmission line, by weakly coupling a microwave superconducting circulator, which introduces a tunable time reversal symmetry breaking through an applied magnetic field. The main focus on this work has been addressed to the circulator architecture and finding the regimes where the chiral effects are at their highest level. We started presenting a theoretical model based on the microwave circulator mechanics suggested in [19]. The model was based on the weak coupling of two resonating cavities to a waveguide. After finding the equations of motion of the quantum operators defined on the resonating cavities and on the waveguide, we introduced an additional effective coupling brought by connecting a three-port microwave circulator, each resonator connected to one port, leaving the third coupled to the ground. To present the most basic and simple results, we totally ignored the circulator superconducting mechanics, only taking into account the additional effective coupling between resonators, which introduced a tunable phase difference responsible of the time reversal symmetry breaking. With that theoretical set-up, we obtained the decay probabilities to both right and left directions on the waveguide which clearly manifested a chiral behavior which could be turned on and off depending on the magnetic flux through the circulator and the separation distance between resonators. Furthermore, we presented a fidelity measure based on a MZI architecture which could be obtained through the S-matrix. In its most chiral behavior, we have observed that the system acts a gyrator, applying a phase $e^{i \pi}$ to a photon going to a certain direction, and not interacting with a photon going to the opposite direction. Finally, we saw that the fidelity hold values highly close to the unity as long as the circulator is weakly coupled to the waveguide.

Once the theoretical model has been exposed, we started working on the design and setup of a superconducting circuit able to imitate the obtained results. In this case, and for simplicity, we
completely removed the resonating cavities form the setup, working instead directly with the superconducting islands forming the circulator. We start designing and presenting the Hamiltonian description of an isolated circulator with three coupled superconducting islands, obtaining the same results presented in [19]. Form there, we moved towards coupling capacitively two circulator islands to a transmission line and again obtain the overall Hamiltonian description from the whole system. Since we needed a weak coupling to obtain good chiral fidelity values, as the theory we built predicted, the capacitances responsible of the coupling needed to be small enough. However, the capacitance coupling the third island to the ground (namely $C_{C}$ ) remained untouched, which will gave us the freedom to explore two different approaches: the 3-CPB $\left(C_{C} \rightarrow 0\right)$ and the CP-Reservoir $\left(C_{C} \rightarrow \infty\right)$. The results show the appearance of three coupling terms between the circulator and the transmission line, giving the first two terms the names of direct coupling $\left(g_{d}\right)$ and crossed coupling $\left(g_{c}\right)$. Apart from that, in order to obtain similar results observed in the first theoretical model, the superconducting islands needed to fit within the CPB regime as a phase difference needs to be introduced form one island to another, and the Transmon limit showed an overall equal phase due to its oscillating behavior. From that point, we started looking at the regime where all three superconducting islands are found within the CPB limit (3-CPB approach), displaying analytic eigenstates and energies with a strong implicit magnetic flux dependency. Within this regime, the system is effectively understood as an isolated circulator perturbed by a transmission line. Therefore, the time reversal symmetry conservation spots coincide precisely with the predictions in [19]. Accordingly choosing a certain island separation ( $k_{o} \Delta L=\pi / 3$, for instance), the chiral behavior seems to be stable at its maximum point under magnetic field fluctuations as long as $\Phi / \Phi_{0} \in(n / 2,(n+1) / 2)$, for $n=0,1,2, \ldots$. Depending on the $n$ value, the chirality will be at its maximum spot applying a $\pi$ phase to photons traveling to the right or to the left. Also, thanks to the equal island symmetry, the directional dependency will not be destroyed by the crossed coupling term, as the third island coupling will null its attempts to correct the anti-symmetric decays. In fact, high $g_{c}$ values will involve high $E_{c, \chi}$ interacting charging energies (Coulomb interaction between islands), and having $E_{c, \chi} \gg E_{J}$ will ensure a CP state preparation in the circulator to fill only the sub-spaces belonging to having only 1 or 2 CP charges living in the circulator. Else, completely filling the circulator with 3 extra CP charges, or leaving it empty with 0 CP charges will not provide the desired chiral response. The fidelity peak holds stable under $g_{c} \sim g_{d}$, turning sharper as long as the crossed coupling approaches the direct coupling.

Finally, we analyze a different approach, no longer assuming a three-island symmetry, but contemplating how the system behaves when the third island (not coupled to the transmission line) is treated as a CP reservoir (CP-Reservoir approach). This situation shows a more similar behavior than the first theoretical model behavior since we only work with two independent regions where excitations can be stored (in this case two superconducting islands, while in the first theoretical model we were considering two resonating cavities). The coupling to a CP reservoir may ease the control over the total number of intervening CP charges. However, it is observed that in the sweet spot ( $N_{g}=1 / 2$ ), the chiral behavior is completely lost. The time reversal symmetry breaking provided by the circulator architecture, together with the particle-antiparticle symmetry in the sweet spot, destroys the asymmetric decays. The presence of the same symmetry within the 3-CPB approach did not destroy the chiral decays thanks to its three equal island architecture which provides unconnected (i.e., with different number of CP charges) degenerated states. Hence, in the CP-Reservoir approach we were forced into looking for alternative routes. Hopefully, we found other $N_{g} \neq 1 / 2$ values which still preserve charge stability, and provide chiral results. We obtained the decay probabilities and fidelity values for 12 different $N_{g}$ values, with different interacting charging energies $E_{c, \chi}$. The results yield a good chiral behavior in all cases, emphasizing the farthest $N_{g}$ charge stable values from the sweet spot, which provide a wider chiral region within the decay probabilities and highly similar results than the 3-CPB approach. We finally present a comparative explanation of both approaches, providing fidelity results also for the 3 - CPB approach, with the help of a global definition only dependent on the number of possible eigenstates the circulator can reach.

### 6.2 Outlook and proposals

Within this work we managed to present a robust setup with two possible interesting approaches providing a transmission line with a tunable chiral interactions with photon. We provide the reader with the needed theoretical tools to develop the experimental setups and a fidelity testing measure. Recently, superconducting qubits throughout the quantum information and quantum computation fields have gained popularity among experimental realization. Experimental realizations on microwave circulators have been successfully produced with optimal expected results. As we have proposed, a weak coupling to a superconducting transmission line introduces the chiral effects and results we presented which may provide useful solutions to a very wide list of quantum information challenges.

A further detailed study in depth on the CT-symmetry breaking nature in all proposed setups may provide a global understanding. Apart from that, an analysis beyond the RWA would provide a more complete model which may introduce other unexpected broken symmetries in the qubit potential or forbidden transitions we completely ignored [40, 41]. Finally, in our study we only allowed the system to interact with one photon at a time. However, providing the system with a decay rate sufficiently long and sending two photons instead of single ones may give rise to a photon-photon effective chiral interaction which could be highly harnessed within the quantum optics and quantum information fields.

## Appendix A

## Matrix notation: Isolated Circulator Hamiltonian

In this appendix we will derive and present the matrices shown in Section 4.1. The isolated circulator Hamiltonian description is given by the following expression (also written in (4.1)).

$$
\begin{align*}
& \mathcal{H}=\frac{\left(Q_{A}+q_{A B}-q_{C A}\right)^{2}}{2\left(C_{A}+C_{A, g}\right)}+\frac{\left(Q_{B}+q_{B C}-q_{A B}\right)^{2}}{2\left(C_{B}+C_{B, g}\right)}+\frac{\left(Q_{C}+q_{C A}-q_{B C}\right)^{2}}{2\left(C_{C}+C_{C, g}\right)}+  \tag{A.1}\\
& +\frac{q_{A B}^{2}}{2 C_{A B}}+\frac{q_{B C}^{2}}{2 C_{B C}}+\frac{q_{C A}^{2}}{2 C_{C A}}
\end{align*}
$$

To begin forming its matrix representation, we will define the following charge vectors.

$$
\begin{equation*}
\vec{Q} \equiv\left(Q_{A}, Q_{B}, Q_{C}\right) \quad \vec{q} \equiv\left(q_{A B}, q_{B C}, q_{C A}\right) \tag{A.2}
\end{equation*}
$$

Then we can group up all the terms in (A.1) within the following matrices.

$$
\Upsilon=\left(\begin{array}{ccc}
\frac{1}{C_{A}+C_{A, g}}+\frac{1}{C_{B}+C_{B, g}}+\frac{1}{C_{A B}} & -\frac{1}{C_{B}+C_{B, g}} & -\frac{1}{C_{A}+C_{A, g}} \\
-\frac{1}{C_{B}+C_{B, g}} & \frac{1}{C_{B}+C_{B, g}}+\frac{1}{C_{C}+C_{C, g}}+\frac{1}{C_{B C}} & -\frac{1}{C_{C}+C_{C, g}} \\
-\frac{1}{C_{A}+C_{A, g}} & -\frac{1}{C_{C}+C_{C, g}} & \frac{1}{C_{C}+C_{C, g}}+\frac{1}{C_{A}+C_{A, g}}+\frac{1}{C_{C A}}
\end{array}\right)
$$

$$
C=\left(\begin{array}{ccc}
C_{A}+C_{A, g} & 0 & 0  \tag{A.3}\\
0 & C_{B}+C_{B, g} & 0 \\
0 & 0 & C_{C}+C_{C, g}
\end{array}\right) \quad R=\left(\begin{array}{ccc}
0 & C_{B}+C_{B, g} & 0 \\
0 & 0 & C_{C}+C_{C, g} \\
C_{A}+C_{A, g} & 0 & 0
\end{array}\right)
$$

Writing the Hamiltonian in (A.1) with the introduced vectors and matrix notation, we obtain:

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}^{T} C^{-1} \vec{Q}+\frac{1}{2} \vec{q}^{T} \Upsilon \vec{q}+\vec{Q}^{T}\left(C^{-1}-R^{-1}\right) \vec{q} . \tag{A.4}
\end{equation*}
$$

Furthermore, we simplify the expressions by assuming equal conditions for the three superconducting islands and Josephson Junctions. Thus, we define the charging capacitances $C_{q}$, the gate capacitances $C_{g}$ and the Josephson capacitances $C_{J}$ to be:

$$
\left\{\begin{array}{l}
C_{q} \equiv C_{A}=C_{B}=C_{C}  \tag{A.5}\\
C_{g} \equiv C_{A, g}=C_{B, g}=C_{C, g} \\
C_{J} \equiv C_{A B}=C_{B C}=C_{C A}
\end{array}\right.
$$

And with these, we re-write the $\Upsilon, C$ and $R$ matrices written in (A.3) as follows.

$$
\begin{gather*}
\Upsilon=\frac{1}{\left(C_{c}+C_{g}\right) C_{J}}\left(\begin{array}{ccc}
2 C_{J}+C_{c}+C_{g} & -C_{J} & -C_{J} \\
-C_{J} & 2 C_{J}+C_{c}+C_{g} & -C_{J} \\
-C_{J} & -C_{J} & 2 C_{J}+C_{c}+C_{g}
\end{array}\right)  \tag{A.6}\\
C^{-1}=\frac{1}{C_{c}+C_{g}}\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \quad R^{-1}=\frac{1}{C_{c}+C_{g}}\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right)
\end{gather*}
$$

With these matrix definitions, one can simply re-write the isolated circulator Hamiltonian as in (4.5), defining as well the island capacitance $C_{\Sigma} \equiv 2 C_{J}+C_{q}+C_{g}$. It is later discovered in Section 4.2 that the same isolated Hamiltonian description is found within the $C_{C} \rightarrow 0$, basically because all islands are considered equally, with the assumptions we made here in (A.6).

## Appendix B

## Matrix notation: Circulator coupled to a transmission line Hamiltonian

In order to build the Hamiltonian description in (4.9) to describe the circulator coupled to a transmission line in a compact form, we defined the following vector charge variables.

$$
\begin{equation*}
\vec{Q} \equiv\left(Q_{A}, Q_{B}, Q_{C}, Q_{m}, Q_{l}\right) \quad \vec{q} \equiv\left(q_{A B}, q_{B C}, q_{C A}, q_{m}, q_{l}\right) \tag{B.1}
\end{equation*}
$$

These are exactly the same vector charges defined in equation (A.2), but adding two extra charges brought by the coupling with the transmission line. The Hamiltonian now will take the following compact from.

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}^{T} \hat{\mathbb{C}}^{-1} \vec{Q}+\frac{1}{2} \vec{q}^{T} \hat{\mathbb{P}} \vec{q}+\vec{Q}^{T}\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right) \vec{q}+\mathcal{H}_{t l}^{\prime} \tag{B.2}
\end{equation*}
$$

And the accumulated charge distribution optimized form:

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}^{T}\left[\hat{\mathbb{C}}^{-1}-\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right)\left(\hat{\mathbb{P}}^{-1}\right)^{T}\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right)^{T}\right] \vec{Q}+\mathcal{H}_{t l}^{\prime} \tag{B.3}
\end{equation*}
$$

In order to be able to express the Hamiltonian with the compact matrix notation, we needed to define the matrices in equation (B.4).

Note we divided the $5 \times 5$ matrices in blocks of matrices: a $3 \times 3$ matrix in top left (interacting with the circulator island charges $Q_{A}, Q_{B}, Q_{C}$ ), a $2 \times 2$ matrix on bottom right (interacting with the transmission line charges $Q_{m}, Q_{l}$ ) and $2 \times 3$ and $3 \times 2$ matrices in the off-diagonals (interacting with both circulator island and transmission line charges). Note that the diagonal blocks go together only with the circulator and transmission line charges separately, but the diagonal terms mix these, defining the interacting behavior. With this distribution we can now simplify significantly the expressions. We

will assume same conditions on charging, gate and Josephson capacitances just like in Appendix B, but we will leave the $C_{C}$ charging capacitance to be different, as it is coupled to the ground and not to the transmission line. We will also assume the latest to contain identical capacitance elements $c_{j}$.

$$
\left\{\begin{array}{l}
C_{q} \equiv C_{A}=C_{B}  \tag{B.5}\\
C_{g} \equiv C_{A, g}=C_{B, g}=C_{C, g} \\
C_{J} \equiv C_{A B}=C_{B C}=C_{C A} \\
\delta z c=\equiv c_{1}=c_{2}=\ldots=c_{N-1}=c_{N}
\end{array}\right.
$$

Therefore, the block matrices can be re-defined as follows:

$$
\begin{gather*}
\hat{\mathbb{P}}=\left(\begin{array}{c|c}
\hat{\Upsilon} & \left(\hat{S}^{-1}-\hat{T}^{-1}\right) \\
\hline\left(\hat{S}^{-1}-\hat{T}^{-1}\right)^{T} & \left(\hat{C}_{g}^{-1}+\hat{c}^{-1}+\hat{C}_{q}^{-1}\right)
\end{array}\right)  \tag{B.6}\\
\hat{\mathbb{C}}^{-1}=\left(\begin{array}{c|c}
\hat{C}^{-1} & 0 \\
\hline 0 & \hat{c}^{-1}
\end{array}\right) \quad \hat{\mathbb{R}}^{-1}=\left(\begin{array}{cc}
\hat{R}^{-1} & \hat{T}^{-1} \\
\hline 0 & 0
\end{array}\right)
\end{gather*}
$$

Where:

$$
\begin{gathered}
\hat{T}^{-1}=\frac{1}{C_{g}}\left(\begin{array}{ll}
1 & 0 \\
0 & 1 \\
0 & 0
\end{array}\right) \quad \hat{S}^{-1}=\frac{1}{C_{g}}\left(\begin{array}{ll}
0 & 1 \\
0 & 0 \\
1 & 0
\end{array}\right) \quad \hat{c}^{-1}=\frac{1}{\delta z c}\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \quad \hat{C}_{g}^{-1}=\frac{1}{C_{g}}\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \\
\hat{C}^{-1}=\frac{1}{C_{g}}\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{C_{g}}{C_{c}+C_{g}}
\end{array}\right) \quad \hat{R}^{-1}=\frac{1}{C_{g}}\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & \frac{C_{g}}{C_{c}+C_{g}} & 0
\end{array}\right) \quad \hat{C}_{q}^{-1}=\frac{1}{C_{q}}\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \\
\hat{\Upsilon}=\frac{1}{C_{g} C_{J}}\left(\begin{array}{cc}
2 C_{J}+C_{g} & -C_{J} \\
-C_{J} & \frac{\left(C_{c}+C_{g}\right) C_{J}+\left(C_{c}+C_{g}\right) C_{g}+C_{J} C_{g}}{\left(C_{c}+C_{g}\right)} \\
-\frac{C_{g} C_{J}}{\left(C_{c}+C_{g}\right)} & -\frac{C_{g} C_{J}}{\left(C_{c}+C_{g}\right)} \\
-C_{J} & \frac{\left(C_{c}+C_{g}\right) C_{J}+\left(C_{c}+C_{g}\right) C_{g}+C_{J} C_{g}}{\left(C_{c}+C_{g}\right)}
\end{array}\right)
\end{gathered}
$$

Note we re-used some matrix definitions introduced in Appendix A. With these block definitions, we are now ready to separate the vector charges in circulator charges $\vec{Q}_{c i r c}=\left(Q_{A}, Q_{B}, Q_{C}\right)$ and transmission line charges $\vec{Q}_{t l}=\left(Q_{m}, Q_{l}\right)$. The matrix block structure, as we mentioned before, makes
easy to see that the top left $3 \times 3$ block will belong to the circulator charges, the $2 \times 2$ bot right block belongs to the transmission line charges, and the rectangular blocks in the off-diagonal terms will define the interaction between circulator and transmission line. That is not totally true though for $\mathbb{P}$ as we need its inverse. Let us firstly express the first term in the Hamiltonian in (B.3) as follows:

$$
\begin{align*}
& \left(\begin{array}{c|c}
\hat{C}_{\text {circ }}^{-1} & \mathbb{A}^{-1} \\
\hline\left(\mathbb{A}^{T}\right)^{-1} & \hat{C}_{t l}^{-1}
\end{array}\right)=\left[\hat{\mathbb{C}}^{-1}-\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right)\left(\hat{\mathbb{P}}^{-1}\right)^{T}\left(\hat{\mathbb{C}}^{-1}-\hat{\mathbb{R}}^{-1}\right)^{T}\right]= \\
& =\left(\begin{array}{c|c}
\hat{C}^{-1} & 0 \\
\hline 0 & \hat{c}^{-1}
\end{array}\right)-  \tag{B.8}\\
& -\left(\begin{array}{c|c}
\hat{C}^{-1}-\hat{R}^{-1} & -T^{-1} \\
\hline 0 & \hat{c}^{-1}
\end{array}\right)\left(\begin{array}{c|c}
\mathbb{P}_{11}^{-1} & \mathbb{P}_{12}^{-1} \\
\hline \mathbb{P}_{21}^{-1} & \mathbb{P}_{22}^{-1}
\end{array}\right)\left(\begin{array}{cc}
\left(\hat{C}^{-1}-\hat{R}^{-1}\right)^{T} & 0 \\
\hline-\left(T^{-1}\right)^{T} & \left(\hat{c}^{-1}\right)^{T}
\end{array}\right)
\end{align*}
$$

Here the $\mathbb{P}_{i j}^{-1}$ terms are the $i j$ terms of $\mathbb{P}^{-1}$. To calculate its inverse. we can do it using the block matrix definitions.

$$
\begin{align*}
& \mathbb{P}_{11}^{-1}=\hat{\Upsilon}^{-1}+\hat{\Upsilon}^{-1}\left(\hat{S}^{-1}-\hat{T}^{-1}\right) \hat{\mathbb{H}}^{-1}\left(\hat{S}^{-1}-\hat{T}^{-1}\right)^{T} \hat{\Upsilon}^{-1} \\
& \mathbb{P}_{12}^{-1}=\hat{\Upsilon}^{-1}\left(\hat{T}^{-1}-\hat{S}^{-1}\right) \hat{\mathbb{H}}^{-1}  \tag{B.9}\\
& \mathbb{P}_{21}^{-1}=\hat{\mathbb{H}}^{-1}\left(\hat{T}^{-1}-\hat{S}^{-1}\right)^{T} \hat{\Upsilon}^{-1} \\
& \mathbb{P}_{22}^{-1}=\hat{\mathbb{H}}^{-1}
\end{align*}
$$

Where:

$$
\begin{equation*}
\hat{\mathbb{H}}=\left[\left(\hat{C}_{g}^{-1}+\hat{c}^{-1}+\hat{C}_{q}^{-1}\right)-\left(\hat{S}^{-1}-\hat{T}^{-1}\right)^{T} \hat{\Upsilon}^{-1}\left(\hat{S}^{-1}-\hat{T}^{-1}\right)\right] \tag{B.10}
\end{equation*}
$$

Substituting now the $\mathbb{P}^{-1}$ terms from (B.9) back in (B.8), we can re-write the Hamiltonian in (B.3) in the following separate form.

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2} \vec{Q}_{c i r c}^{T} \tilde{C}_{c i r c}^{-1} \vec{Q}_{c i r c}+\frac{1}{2} \vec{Q}_{t l}^{T} \tilde{C}_{t l}^{-1} \vec{Q}_{t l}+\frac{1}{2} \vec{Q}_{c i r c}^{T} \mathbb{A}^{-1} \vec{Q}_{t l}+\frac{1}{2} \vec{Q}_{t l}^{T}\left(\mathbb{A}^{T}\right)^{-1} \vec{Q}_{c i r c}+\mathcal{H}_{t l} \tag{B.11}
\end{equation*}
$$

With:

$$
\left\{\begin{array}{l}
\hat{C}_{\text {circ }}^{-1}=\hat{C}^{-1}-\left(\hat{C}^{-1}-\hat{R}^{-1}\right) \mathbb{P}_{11}^{-1}\left(\hat{C}^{-1}-\hat{R}^{-1}\right)^{T}+  \tag{B.12}\\
\quad+T^{-1} \mathbb{P}_{21}^{-1}\left(\hat{C}^{-1}-\hat{R}^{-1}\right)^{T}+\left(\hat{C}^{-1}-\hat{R}^{-1}\right) \mathbb{P}_{12}^{-1}\left(T^{-1}\right)^{T}-T^{-1} \mathbb{P}_{22}^{-1}\left(T^{-1}\right)^{T} \\
\hat{C}_{t l}^{-1}=\hat{c}^{-1}-\hat{c}^{-1} \mathbb{P}_{22}^{-1} \hat{c}^{-1} \\
\mathbb{A}^{-1}=-\left(\hat{C}^{-1}-\hat{R}^{-1}\right) \mathbb{P}_{12}^{-1} \hat{c}^{-1}+T^{-1} \mathbb{P}_{22}^{-1} \hat{c}^{-1}
\end{array}\right.
$$

Finally, substituting all the values in (B.7) and operating one ends up with the final form of each term in the Hamiltonian as follows:

$$
\hat{C}_{\text {circ }}^{-1}=\left(\begin{array}{ccc}
\alpha & \chi & \zeta  \tag{B.13}\\
\chi & \alpha & \zeta \\
\zeta & \zeta & \sigma
\end{array}\right) \quad \hat{C}_{t l}^{-1}=\left(\begin{array}{cc}
\varrho & \eta \\
\eta & \varrho
\end{array}\right) \quad \mathbb{A}^{-1}=\left(\begin{array}{cc}
\rho & \epsilon \\
\epsilon & \rho \\
\kappa & \kappa
\end{array}\right)
$$

Each term in the matrices corresponds to the following values in (B.14). These can be further simplified thanks to the weak coupling assumption, which is how we proceed in (4.15).
$\alpha=\frac{\delta z c\left(C_{C}\left(C_{g}+2 C_{J}+C_{q}\right)+C_{g}^{2}+4 C_{g} C_{J}+C_{g} C_{q}+3 C_{J}^{2}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+2 C_{J}\right)+C_{g}^{2}+4 C_{g} C_{J}+3 C_{J}^{2}\right)}{\delta z c\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+3 C_{g} C_{J}+C_{g} C_{q}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+C_{J}\right)+C_{g}\left(C_{g}+3 C_{J}\right)\right)} \frac{\left(c+C_{q}\right)}{\delta z c\left(C_{g}+3 C_{J}+C_{q}\right)+C_{q}\left(C_{g}+3 C_{J}\right)}$

$\zeta=\frac{C_{J}}{\delta z c\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+3 C_{g} C_{J}+C_{g} C_{q}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+C_{J}\right)+C_{g}\left(C_{g}+3 C_{J}\right)\right)}\left(c+C_{q}\right)$
$\sigma=\frac{}{\delta z c\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+3 C_{g} C_{J}+C_{g} C_{q}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+C_{J}\right)+C_{g}\left(C_{g}+3 C_{J}\right)\right)}$
$\varrho=\frac{\delta z c\left(C_{g}+3 C_{J}+C_{q}\right)\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+C_{g}\left(3 C_{J}+C_{q}\right)+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}^{2}+C_{g}\left(4 C_{J}+C_{q}\right)+C_{J}\left(3 C_{J}+2 C_{q}\right)\right)+\left(C_{g}+3 C_{J}\right)\left(C_{g}^{2}+C_{g}\left(3 C_{J}+C_{q}\right)+C_{J} C_{q}\right)\right)}{\left(\delta z c\left(C_{g}+3 C_{J}+C_{q}\right)+C_{q}\left(C_{g}+3 C_{J}\right)\right)\left(\delta z c\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+3 C_{g} C_{J}+C_{g} C_{q}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+C_{J}\right)+C_{g}\left(C_{g}+3 C_{J}\right)\right)\right)}$

[^0]
## Appendix C

## 3-CPB approach: Hamiltonian diagonalization

In this appendix we show how to visualize the diagonalization of the circulator Hamiltonian given in the Cooper Pair Box regime in (4.41). Using the number basis ordering given in (4.42), we can write the following matrix form for $\mathcal{H}_{\text {circ }}$.

$$
\mathcal{H}_{\text {circ }}=\left(\begin{array}{cccccccc}
\mathcal{H}_{\text {circ }}^{(0)} & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{C.1}\\
0 & & & & 0 & 0 & 0 & 0 \\
0 & & \mathcal{H}_{c i r c}^{(1)} & & 0 & 0 & 0 & 0 \\
0 & & & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & & & & 0 \\
0 & 0 & 0 & 0 & & \mathcal{H}_{c i r c}^{(2)} & & 0 \\
0 & 0 & 0 & 0 & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{H}_{\text {circ }}^{(3)}
\end{array}\right)
$$

Where $\mathcal{H}_{\text {circ }}^{(0)}$ and $\mathcal{H}_{\text {circ }}^{(3)}$ are $1 \times 1$ matrices (numbers) and $\mathcal{H}_{\text {circ }}^{(1)}$ and $\mathcal{H}_{\text {circ }}^{(2)}$ are $3 \times 3$ matrices. See that we can divide $\mathcal{H}_{\text {circ }}$ in four different sub-spaces depending on the total amount of cooper pairs living in the circulator.

To provide an easy and elegant analytic solution to this Hamiltonian, we will assume the $C_{C}$ capacitance to be small enough so that we can consider $\chi=\zeta$. See that the capacitance terms derived in (4.15) will fulfill this relation as long as $C_{C} \ll C_{g}+3 C_{J}$, or as we have stated, $C_{C} \rightarrow 0$ (3-CPB approach). We also assume that the applied gate voltage is the same for the three different islands in the circulator, i.e., $N_{A, g}=N_{B, g}=N_{C, g}=N_{g}$.

Lets now solve the four different sub-spaces introduced in $(4.42):(0),(1),(2)$ and $(3)$, where the number indicates the total amount of CP charges in the circulator. We will name $\left|\Omega_{j}\right\rangle$ the circulator eigenstates and $E_{j}$ the corresponding eigenenergies, for $j=0-7$.

## C. 1 Sub-spaces (0) and (3)

We will begin looking at the easiest and trivial sub-spaces to solve. The eigenstates and their corresponding eigenenergies will be given by:

$$
\begin{align*}
& \mathcal{H}_{\text {circ }}^{(0)}=4 N_{g}^{2}\left[2 E_{c, \alpha}+E_{c, \sigma}+E_{c, \chi}+2 E_{c, \zeta}\right]=E_{0} \\
& \mathcal{H}_{\text {circ }}^{(3)}=4\left(1-N_{g}\right)^{2}\left[2 E_{c, \alpha}+E_{c, \sigma}+E_{c, \chi}+2 E_{c, \zeta}\right]=E_{7}
\end{align*} \quad \text { for : }\left\{\begin{array}{l}
\left|\Omega_{0}\right\rangle=|000\rangle  \tag{C.2}\\
\left|\Omega_{7}\right\rangle=|111\rangle
\end{array}\right.
$$

Note that the ground state is given by the $|000\rangle$, where the circulator is empty of charges, and the most excited state by $|111\rangle$, where all islands are occupied. This perspective though is lost for $1 / 2 \leq N_{g}$, where then having an empty circulator becomes more unstable than occupying all the islands. Over the sweet spot $\left(N_{g}=1 / 2\right)$, the gate voltage is so big that the charges prefer to stay located in the islands. In the sweet spot though, both eigenstates will have the same eigenenergies, and the system will not mind in staying in one or another (particle-antiparticle symmetry). From now on, we will assume $N_{g}=1 / 2$ to ensure charge stability within the system. Also, we will assume $E_{c, \alpha}=E_{c, \sigma}$ and $E_{c, \chi}=E_{c, \zeta}$, since we are working on the 3-CPB approach.

## C. 2 Sub-space (1)

In this sub-space, the eigenstates will represent the possible configurations of the circulator with one CP charge. The Hamiltonian part that will govern over this sub-space in the sweet spot will be:

$$
\begin{gather*}
\left|N_{A} N_{B} N_{C}\right\rangle=\{|001\rangle,|010\rangle,|100\rangle\} \\
\mathcal{H}_{c i r c}^{(1)}=\left(\begin{array}{ccc}
3 E_{c, \alpha}-E_{c, \chi} & -\frac{E_{J}}{2} e^{-i \varphi} & -\frac{E_{J}}{2} e^{i \varphi} \\
-\frac{E_{J}}{2} e^{i \varphi} & 3 E_{c, \alpha}-E_{c, \chi} & -\frac{E_{J}}{2} e^{-i \varphi} \\
-\frac{E_{J}}{2} e^{-i \varphi} & -\frac{E_{J}}{2} e^{i \varphi} & 3 E_{c, \alpha}-E_{c, \chi}
\end{array}\right) \tag{C.3}
\end{gather*}
$$

Where we already assumed $N_{g}=1 / 2$ and $C_{c} \rightarrow 0$. The eigenstates and their corresponding energies can be directly obtained giving:

$$
\begin{array}{rlrl}
E_{1}= & \frac{E_{J}}{2} \cos (\varphi)-\frac{E_{J}}{2} \sqrt{3} \sin (\varphi)+ & & \\
& +3 E_{c, \alpha}-E_{c, \chi} & \text { for }:\left|\Omega_{1}\right\rangle=\frac{1}{\sqrt{3}}|001\rangle-\frac{e^{i \pi / 3}}{\sqrt{3}}|010\rangle-\frac{e^{-i \pi / 3}}{\sqrt{3}}|100\rangle \\
E_{2}= & \frac{E_{J}}{2} \cos (\varphi)+\frac{E_{J}}{2} \sqrt{3} \sin (\varphi)+ & \text { for }:\left|\Omega_{2}\right\rangle=\frac{1}{\sqrt{3}}|001\rangle-\frac{e^{-i \pi / 3}}{\sqrt{3}}|010\rangle-\frac{e^{i \pi / 3}}{\sqrt{3}}|100\rangle  \tag{C.4}\\
& +3 E_{c, \alpha}-E_{c, \chi} & & \\
E_{3}= & -E_{J} \cos (\varphi)+3 E_{c, \alpha}-E_{c, \chi} & \text { for }:\left|\Omega_{3}\right\rangle=\frac{1}{\sqrt{3}}|001\rangle+\frac{1}{\sqrt{3}}|010\rangle+\frac{1}{\sqrt{3}}|100\rangle
\end{array}
$$

The oscillating character of the energies produces an energy ordering highly depending on the region of $\varphi$ we are operating with. This can be seen in Figure 4.3 (a), where for example for $0<\varphi<\pi / 3$, the ground state is given by $\left|\Omega_{3}\right\rangle$, the first excited state by $\left|\Omega_{1}\right\rangle$ and the second excited state by $\left|\Omega_{2}\right\rangle$. For $\pi / 3<\varphi<2 \pi / 3$ though, the ground state will be given by $\left|\Omega_{1}\right\rangle$, the first excited state by $\left|\Omega_{3}\right\rangle$ and the second excited state still by $\left|\Omega_{2}\right\rangle$.

## C. 3 Sub-space (2)

This subspace is given by the energy configurations brought by having two extra CP charges living in the circulator. The Hamiltonian part $\mathcal{H}_{\text {circ }}^{(2)}$ in the sweet spot will be exactly the same as (C.3). Therefore, the eigenenergies and eigenstates will have exactly the same shape, this time though being:

$$
\begin{array}{rlrl}
E_{4}= & \frac{E_{J}}{2} \cos (\varphi)-\frac{E_{J}}{2} \sqrt{3} \sin (\varphi)+ & \text { for }:\left|\Omega_{4}\right\rangle=\frac{1}{\sqrt{3}}|011\rangle-\frac{e^{i \pi / 3}}{\sqrt{3}}|101\rangle-\frac{e^{-i \pi / 3}}{\sqrt{3}}|110\rangle \\
& +3 E_{c, \alpha}-E_{c, \chi} & & \\
E_{5}= & \frac{E_{J}}{2} \cos (\varphi)+\frac{E_{J}}{2} \sqrt{3} \sin (\varphi)+ & \text { for }:\left|\Omega_{5}\right\rangle=\frac{1}{\sqrt{3}}|011\rangle-\frac{e^{-i \pi / 3}}{\sqrt{3}}|101\rangle-\frac{e^{i \pi / 3}}{\sqrt{3}}|110\rangle  \tag{C.5}\\
& +3 E_{c, \alpha}-E_{c, \chi} & \text { for }:\left|\Omega_{6}\right\rangle=\frac{1}{\sqrt{3}}|011\rangle+\frac{1}{\sqrt{3}}|101\rangle+\frac{1}{\sqrt{3}}|110\rangle
\end{array}
$$

Evaluating $N_{g}$ away from the sweet spot will only bring an energy separation in the eigenenergies from sub-space (1) and (2). This energy splittings can be seen in Figure 4.3.

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[^0]:    $\eta=\frac{C_{J}\left(C_{C}+C_{g}+3 C_{J}\right)}{\delta z c\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+3 C_{g} C_{J}+C_{g} C_{q}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+C_{J}\right)+C_{g}\left(C_{g}+3 C_{J}\right)\right)} \frac{C_{q}^{2}}{\delta z c\left(C_{g}+3 C_{J}+C_{q}\right)+C_{q}\left(C_{g}+3 C_{J}\right)}$
    $\rho=\frac{c\left(C_{C}\left(C_{g}+2 C_{J}+C_{q}\right)+C_{g}^{2}+4 C_{g} C_{J}+C_{g} C_{q}+3 C_{J}^{2}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+2 C_{J}\right)+C_{g}^{2}+4 C_{g} C_{J}+3 C_{J}^{2}\right)}{\delta z c\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+3 C_{g} C_{J}+C_{g} C_{q}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+C_{J}\right)+C_{g}\left(C_{g}+3 C_{J}\right)\right)} \frac{C_{q}}{\delta z c\left(C_{g}+3 C_{J}+C_{q}\right)+C_{q}\left(C_{g}+3 C_{J}\right)}$
    $\epsilon=\frac{C_{J}\left(C_{C}+C_{g}+3 C_{J}\right)}{\left(\delta z c\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+3 C_{g} C_{J}+C_{g} C_{q}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+C_{J}\right)+C_{g}\left(C_{g}+3 C_{J}\right)\right)\right)} \frac{C_{q}\left(c+C_{q}\right)}{\delta z c\left(C_{g}+3 C_{J}+C_{q}\right)+C_{q}\left(C_{g}+3 C_{J}\right)}$
    $\kappa=\frac{C_{J}}{\delta z c\left(C_{C}\left(C_{g}+C_{J}+C_{q}\right)+C_{g}^{2}+3 C_{g} C_{J}+C_{g} C_{q}+2 C_{J} C_{q}\right)+C_{q}\left(C_{C}\left(C_{g}+C_{J}\right)+C_{g}\left(C_{g}+3 C_{J}\right)\right)} C_{q}$

