



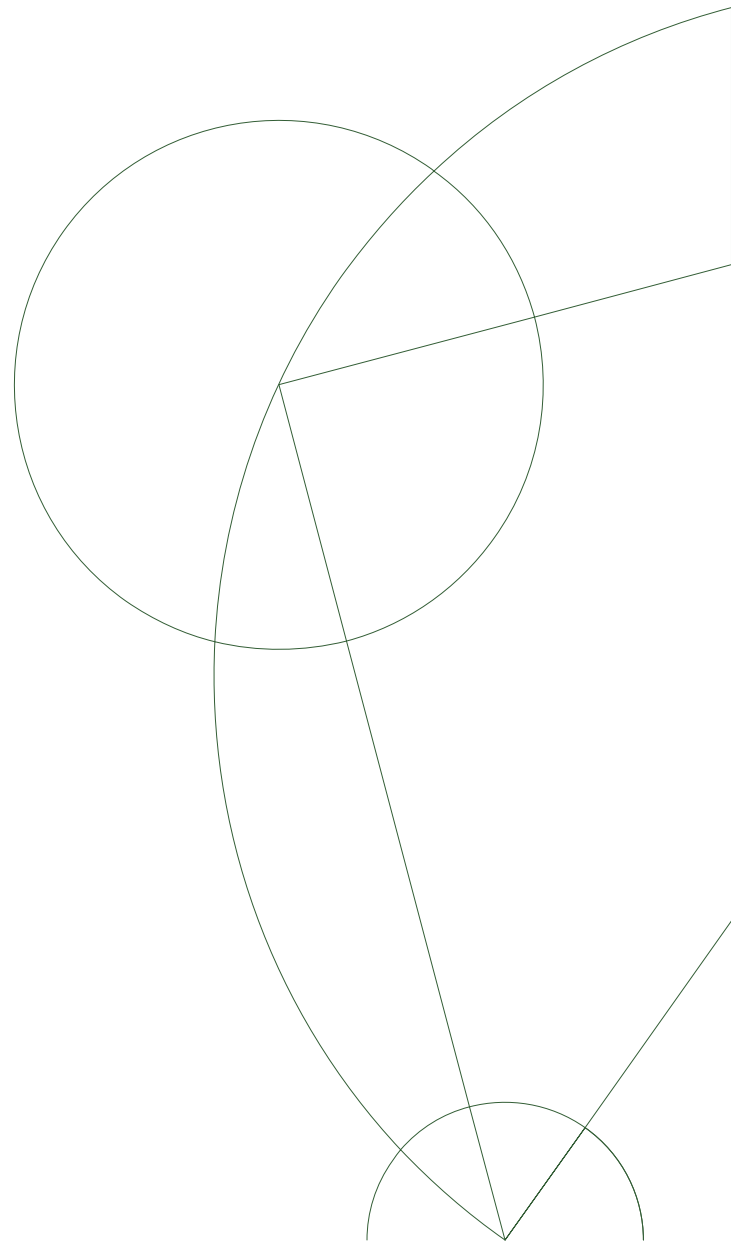
DIPOLE-DIPOLE INTERACTION ANALYSIS IN COUPLING OF LIGHT TO SUPERCONDUCTING QUBITS

Bachelor Thesis

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Abstract

In this thesis the interaction between two dipoles is studied. This is done both in the classical and quantum picture. Explicit expressions for the decay rates, the frequency shifting and the solution of the equations of motion are given. The study of this phenomena is motivated by a photon-qubit entanglement protocol which requires a coupling between a qubit and a dipole-dipole system interacting with incoming photons in a waveguide. By using the model for the dipole-dipole interaction derived in this thesis, we calculate the probability of success of the given protocol for different orientations and distances between the dipoles.

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1 Motivation

One of the big areas of application for modern physics is the area of quantum computation. As opposed to currently available computers, quantum computers operate by qubits, which gives them the upper hand in problems like prime number factorization and simulation of quantum systems [1]. A promising way of creating qubits are the so called superconducting qubits, which interact very little with the environment and therefore preserve coherence (quantum properties) over longer time periods than other forms of qubits [2]. If we in the future make effective quantum computers by superconducting qubits, it is important that they can communicate with each other to exchange information. Current computers communicate with each other by encoding the information of their bits into photons and then sending the photons through waveguides to other computers. If the same is to be done for quantum computers, the information of a qubit has to be encoded into a photon and sent through a waveguide. Because of negligible losses, quantum information is transmitted best by optical photons [3]. Here a problem arises, superconducting qubits require very low temperatures to be super-conductive, but optical photons have very high energies in comparison. Optical photons will therefore destroy the superconducting properties of the qubit and make it lose coherence if they interact directly. To create quantum communication for superconducting qubits, a suitable interface that allows light to interact with qubits indirectly is therefore required.

A protocol for how to create an effective interface was proposed by Das, Elfving, Faez and Sørensen [4]. This protocol proposes to let the superconducting qubit interact with two molecules, and then let photons interact with the molecules instead of the qubit. In their proposal they consider a qubit in the form of a Copper Pair Box (CPB), which has two possible states, $|+\rangle$ and $|-\rangle$. This CPB is put outside the edge of a one dimensional waveguide. Inside the waveguide, two molecules can be found, exchanging energy by dipole-dipole interaction. For simplicity, the molecules can be considered as two leveled systems, then the ground state is denoted by $|g_1g_2\rangle$ and the combined excited states (dressed states) by $|A\rangle$ and $|S\rangle$. The CPB and the molecules interact electrically and can have an influence on the state of each other. As for the photons, they are sent by very short laser pulses, sending very few photons at the time. If the energy difference between the qubit states $|+\rangle$ and $|-\rangle$ is the same as the energy difference between the dressed states $|A\rangle$ and $|S\rangle$, then the incoming photons might undergo a Raman transition, which will encode the state of the qubit into the photon. The process is shown in figure 1.

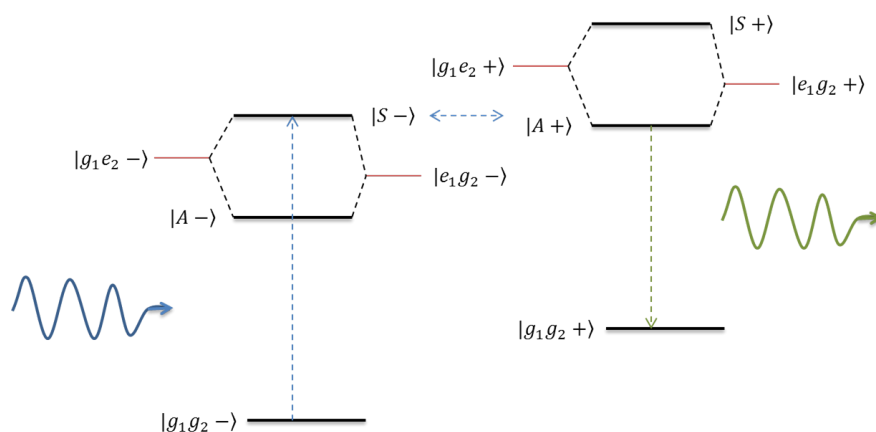


Figure 1: Description of the Raman process. A photon excites the system from the ground state to the $|S-\rangle$ state. The energy of the $|S-\rangle$ is tuned to be equal to the energy of the $|A+\rangle$ state, so that tunneling between those states can occur. After tunneling the system decays to the new ground state, sending a photon with less energy than the incoming one. The state of the photon is then coupled to the state of the qubit.[4]

A crucial requirement for the protocol to work is to have the energy difference between the dressed states of the dipoles to be equal to the energy difference of the qubit. In the article by Das et. al [4] this was phenomenologically taken into account, but a proper understanding of this phenomena requires an accurate dipole-dipole interaction analysis. In this thesis we will therefore give a proper treatment and dive into the dynamics of the dipole-dipole interaction. Because molecules are considered, the interaction is described by quantum mechanics, but as one usually does in physics, before tackling a hard problem, a similar but simpler problem is considered, which usually turns to be helpful. So we begin by solving the problem in classical mechanics.

2 Classical Model

2.1 One dipole

We want to solve the problem of two oscillating dipoles interacting with each other by the electromagnetic field, satisfying Newton's laws. A good place to begin is by looking at one oscillating dipole alone interacting with the electromagnetic field. For the classical problem, we will consider a dipole as being a charged particle attached to a spring. The charge is q , the amplitude of the oscillation is described by x_0 and the spring has a spring constant k . Apart from the dynamics given by the spring, an accelerated charged particle interacts with the electromagnetic field, giving energy in the form of radiation. This results in a radiation reaction force, which is given by [5]

$$\mathbf{F}_{rad} = \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}}. \quad (2.1)$$

Let us assume that the motion is only on the x direction, then adding this term to the equation of motion of the harmonic oscillator we get

$$-\frac{\mu_0 q^2}{6\pi c} \ddot{x} + \ddot{x} + kx = 0, \quad (2.2)$$

We will approach this problem in a way that is very similar to the way it would be in quantum mechanics. We therefore define a classical equivalent to the annihilation operator:

$$\begin{aligned} A &= m\omega x + ip \\ A^* &= m\omega x - ip, \end{aligned} \quad (2.3)$$

where ω denotes the natural spring frequency and p is the momentum of the particle. In the case of the perfect harmonic oscillator without a driving force we have $\dot{x} = p/m$ and from Newton's second law and Hooke's law we get: $\dot{p} = -kx + F_{other}$. Then the derivative of A gives

$$\dot{A} = \omega p - ikx + i\frac{\mu_0 q^2}{6\pi c} \ddot{x} = -i\omega A + i\frac{\mu_0 q^2}{6\pi c} \ddot{x}. \quad (2.4)$$

This is very difficult to solve, so we make our first approximation. We assume that the system oscillates always at the same frequency and over short periods of times the amplitude is constant. So for short time intervals we have

$$A \approx A_0 \exp(i\phi - \omega t). \quad (2.5)$$

We should be aware that by this approximation we have assumed that the system is oscillating with the frequency ω , but in practice this frequency is not equal to the bare spring frequency, because a driving force usually results in a frequency shift for an oscillating system. We will assume that this frequency shift is finite and use ω to denote the shifted oscillation frequency. From equations 2.3 and 2.5 it follows that

$$\ddot{x} \approx -\omega^2 \dot{x} = -\omega^2 \frac{p}{m} = -\frac{\omega^2}{2mi} (A - A^*), \quad (2.6)$$

which we insert into equation 2.4

$$\dot{A} = -i\omega A - \frac{\omega^2 \mu_0 q^2}{12\pi mc} (A - A^*). \quad (2.7)$$

This equation is still difficult to solve. To proceed we move into a rotating frame that follows A . First define the quantity $\tilde{A} = Ae^{i\omega t}$, which is not oscillating periodically. Its derivative is

$$\dot{\tilde{A}} = \dot{A}e^{i\omega t} + i\omega Ae^{i\omega t}. \quad (2.8)$$

Applying this to eq 2.7 we move to the rotating frame:

$$\dot{\tilde{A}} = -\frac{\omega^2 \mu_0 q^2}{12\pi mc} (\tilde{A} - \tilde{A}^* e^{i2\omega t}). \quad (2.9)$$

In this frame we see that the term with A^* oscillates with the frequency $A^* \sim e^{2i\omega t}$ while the other term is not oscillating. For this reason we say that contribution of the A^* term averages to zero and thus we can ignore it. This is the rotating wave approximation. The final equation reads:

$$\dot{\tilde{A}} - \frac{\omega^2 \mu_0 q^2}{12\pi mc} \tilde{A} = -\frac{\gamma}{2} \tilde{A}, \quad (2.10)$$

which we recognize as the differential equation for an exponential decay. Here $\gamma = \frac{\omega^2 \mu_0 q^2}{6\pi mc}$ is the decay rate of the system. The fact that we divide it by two is by convention, since that one normally talks about the decay rate with respect to the energy and not to the amplitude. Since the energy is proportional to the square of the amplitude, its decay rate will be twice as big.

2.2 Two dipoles - Equations of motion

Now we proceed to work with two dipoles that are both oscillating with a frequency ω and are separated at a distance r from each other. The second dipole enters the equation of motion as a driving force:

$$-\frac{\mu_0 q^2}{6\pi c} \ddot{x}_1 + \ddot{x}_1 + kx_1 - F_{2 \rightarrow 1} = 0. \quad (2.11)$$

Again we will try to solve this problem in a way that resembles the quantum mechanical problem. So, similar to eq 2.4 we now have

$$\dot{A}_1 = -i\omega A + i\frac{\mu_0 q^2}{6\pi c} \dot{x}_1 + iF_{2 \rightarrow 1}. \quad (2.12)$$

Next we should find an expression for the interaction force. The general electric field from an oscillating dipole is:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \left[k^2 (\mathbf{n} \times \mathbf{p}) \times \mathbf{n} \frac{e^{ikr}}{r} + (3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}) \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right], \quad (2.13)$$

where \mathbf{n} is the unit normal vector to the position vector \mathbf{r} , \mathbf{k} is the wave vector corresponding to the frequency ω , and \mathbf{p} is the dipole moment vector [6]. A common approximation when working with dipoles is the dipole approximation, where we assume that the field is uniform everywhere where the dipole is oscillating. Within this approximation we can consider the direction of \mathbf{n} to be always the same. In polar coordinates, both dipoles have two independent angles. But because the electric field has azimuthal symmetry, without any loss of generality, the interaction can be described by only three angles, θ_1 , θ_2 and $\Delta\phi$, where the last one is the difference of the azimuthal angles (see figure 2). The electric field radiating from dipole one in the chosen coordinates is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \left[k^2 \sin(\theta_1) p_1 \hat{\mathbf{x}} \frac{e^{ikr}}{r} + (3 \cos(\theta_1) \mathbf{n} - \mathbf{p}_1) \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right]. \quad (2.14)$$

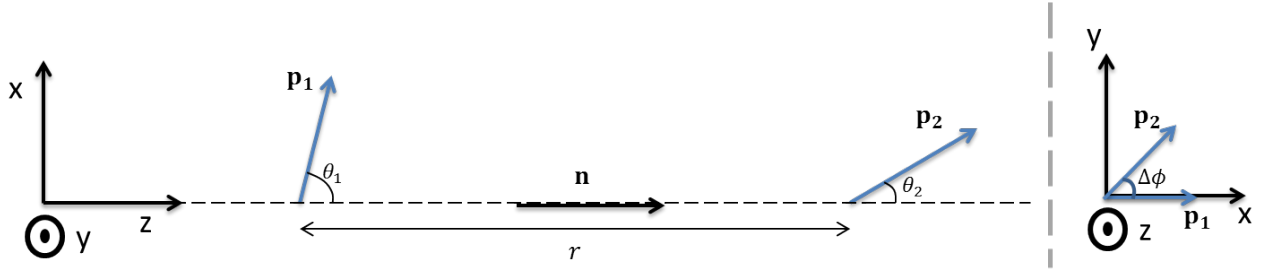


Figure 2: Illustration of our choice of coordinates and angles for the problem of two dipoles interacting with each other. $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$ are vectors pointing in the direction of the dipoles, and (x, y, z) define the basis vectors of the coordinate system that is being used.

We will now make the approximation that the dipoles are only affected by the component of the electric field that is parallel to their dipole moment vector. Then we can write the force acting on the second dipole as $F = q\mathbf{E} \cdot \hat{\mathbf{p}}_2$, where we denote the unit vector in the direction of the second dipole by $\hat{\mathbf{p}}_2$. Plugging into equation 2.14 we get

$$F_{1 \rightarrow 2} = \frac{q}{4\pi\epsilon_0} \left[\sin^2(\theta_1) \sin(\theta_2) \cos(\Delta\phi) \frac{k^2}{r} + (3 \cos(\theta_1) \cos(\theta_2) - \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) \right] p_1(t) e^{ikr}, \quad (2.15)$$

where

$$\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2 = \sin(\theta_1) \sin(\theta_2) \cos(\Delta\phi) + \cos(\theta_1) \cos(\theta_2). \quad (2.16)$$

For later use we will write this result in terms of Green's functions. The electric field of a dipole can be written as [5]

$$\mathbf{E}(\mathbf{r}) = \frac{\omega^2}{\epsilon_0 c^2} \mathbb{G}(\mathbf{r}, \mathbf{r}_0) \cdot \mathbf{p} \quad (2.17)$$

Then the force in eq 2.15 can be expressed as

$$F_{1 \rightarrow 2} = \frac{q\omega^2}{\epsilon_0 c^2} (\mathbb{G}(\mathbf{r}, \mathbf{r}_0; \omega_1) \cdot \mathbf{p}_1) \cdot \hat{\mathbf{p}}_2 = \frac{qp_1(t)\omega^2}{\epsilon_0 c^2} \hat{\mathbf{p}}_2 \cdot (\mathbb{G}(\mathbf{r}, \mathbf{r}_0; \omega_1) \cdot \hat{\mathbf{p}}_1), \quad (2.18)$$

where $p(t)$ is for our problem the same as $qx(t)$, which is oscillating periodically. Thus under the approximation that the system is changing slowly, over short time intervals the dipole moment term behaves as:

$$p(t) \approx qx_0 \cos(\omega t - \phi). \quad (2.19)$$

We note that

$$x_0 \cos(\omega t - \phi) = \frac{A + A^*}{2m\omega}, \quad (2.20)$$

so that we are able to denote the force of the dipoles by using the variables A and A^* , which are the ones we use for our equation of motion. Before proceeding we write equation 2.15 in the following way:

$$F_{1 \rightarrow 2} = \frac{1}{i} C_{1 \rightarrow 2}(r, \theta_1, \theta_2, \Delta\phi) (A + A^*), \quad (2.21)$$

where we have defined

$$\begin{aligned} C_{1 \rightarrow 2}(r, \theta_1, \theta_2, \Delta\phi) &= ie^{ik_1 r} \frac{q^2}{8\pi\epsilon_0 m \omega_1} \left[\sin^2(\theta_1) \sin(\theta_2) \cos(\Delta\phi) \frac{k_1^2}{r} + (3 \cos(\theta_1) \cos(\theta_2) - \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \left(\frac{1}{r^3} - \frac{ik_1}{r^2} \right) \right] \\ &= i \frac{q^2}{2\epsilon_0 m \omega_1} k_1^2 \hat{\mathbf{p}}_2 \cdot (\mathbb{G}(\mathbf{r}, \mathbf{r}_0; \omega_1) \cdot \hat{\mathbf{p}}_1). \end{aligned} \quad (2.22)$$

We consider now two oscillators, denoted by A_1 and A_2 . To begin with we consider them having the same spring frequency. Looking at equation 2.10 we can conclude that in the absence of the second dipole, their individual decay rates would be equal. We now derive the equation of motion for A_1 . Looking back to equation 2.12, we complete the equation of motion by replacing the interaction term from equation 2.21:

$$\dot{A}_1 = -i\omega A_1 - \frac{\gamma}{2}(A_1 - A_1^*) + C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi)(A_2 + A_2^*), \quad (2.23)$$

where we assumed that the system is changing slowly, as we did in the one dipole case. To be able to solve this, we make the rotating wave approximation. To do so we transform into the rotating frame.

$$\dot{\tilde{A}}_1 = -\frac{\gamma}{2}(\tilde{A}_1 - \tilde{A}_1^* e^{i2\omega t}) + C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi)(\tilde{A}_2 + \tilde{A}_2^* e^{i(2\omega t)}). \quad (2.24)$$

By the rotating wave approximation we can ignore the oscillating terms finally getting:

$$\dot{\tilde{A}}_1 = -\frac{\gamma}{2}\tilde{A}_1 + C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi)\tilde{A}_2. \quad (2.25)$$

By a similar procedure we can obtain the equation for A_2

$$\dot{\tilde{A}}_2 = -\frac{\gamma}{2}\tilde{A}_2 + C_{1 \rightarrow 2}(r, \theta_1, \theta_2, \Delta\phi)\tilde{A}_1. \quad (2.26)$$

2.3 Two dipoles - Solution for same frequency case

We write the system of equations in matrix form:

$$\begin{bmatrix} \dot{\tilde{A}}_1 \\ \dot{\tilde{A}}_2 \end{bmatrix} = \begin{bmatrix} -\gamma/2 & C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi) \\ C_{1 \rightarrow 2}(r, \theta_1, \theta_2, \Delta\phi) & -\gamma/2 \end{bmatrix} \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix}. \quad (2.27)$$

We want to solve this system of equations. If we state it simply we get

$$\dot{\tilde{\mathbf{A}}} = \mathbb{M}\tilde{\mathbf{A}}. \quad (2.28)$$

If this was an ordinary differential equation, the solution for $\tilde{\mathbf{A}}$ would be

$$\tilde{\mathbf{A}} = e^{\mathbb{M}t}\tilde{\mathbf{A}}(0), \quad (2.29)$$

which we assume to also apply for a matrix equation. The exponential of a matrix is not easy to calculate in an arbitrary basis, but if we diagonalize the matrix it becomes

$$e^{\mathbb{D}t} = \begin{bmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{bmatrix}, \quad (2.30)$$

where λ_{\pm} are the eigenvalues of the matrix \mathbb{M} . From the diagonal matrix we can recover the original matrix by the equation $\mathbb{M} = R\mathbb{D}R^{-1}$, where R^{-1} is the transformation matrix from standard basis to the diagonalized basis. This matrix is formed by the normed eigenvectors in column form.

By writing $\mathbb{M} = R\mathbb{D}R^{-1}$ and expanding the exponential function we can see that

$$e^{\mathbb{M}t} = R e^{\mathbb{D}t} R^{-1}. \quad (2.31)$$

Taking a look at the matrix \mathbb{M} , we see that $\mathbb{M}\mathbb{M}^* = \mathbb{M}^*\mathbb{M}$, which means that it is a normal matrix. For normal matrices we are sure that the eigenvectors are orthogonal to each other, meaning that R^{-1} is a unitary matrix. This means that $R^{-1} = R^*$.

The eigenvalues of M are

$$\lambda_{\pm} = -\frac{\gamma}{2} \pm C(r, \theta_1, \theta_2, \Delta\phi), \quad (2.32)$$

where we have used that $C_{2 \rightarrow 1} = C_{1 \rightarrow 2} = C$, as both oscillators have the same frequency. The matrix R formed by the eigenvectors is

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (2.33)$$

Combining these results with equation 2.31 we get

$$e^{Mt} = \frac{1}{2} \begin{bmatrix} e^{\lambda_+ t} + e^{\lambda_- t} & e^{\lambda_+ t} - e^{\lambda_- t} \\ e^{\lambda_+ t} - e^{\lambda_- t} & e^{\lambda_+ t} + e^{\lambda_- t} \end{bmatrix}. \quad (2.34)$$

Putting back into equation 2.29 we get the final solution. For oscillator 1 it reads:

$$\tilde{A}_1(t) = \frac{1}{2} \left[(\tilde{A}_1(0) + \tilde{A}_2(0))e^{\lambda_+ t} + (\tilde{A}_1(0) - \tilde{A}_2(0))e^{\lambda_- t} \right]. \quad (2.35)$$

This solution can be divided into a real and an imaginary part. The real part of the solution represents the total decay rate of the system. If the dipoles are on top of each other (their separation distance is zero) and they both start with the same phase, then we should expect the total decay rate of the system to be the double of the decay rate of one dipole. Let us check if this is true. For the condition $A_1(0) = A_2(0)$ equation 2.35 becomes

$$\tilde{A}_1(t) = \tilde{A}_1(0)e^{\lambda_+ t}. \quad (2.36)$$

The decay rate is then given by the real part of the exponential, which we now calculate. For simplicity we will assume that $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $\Delta\phi = 0$. We then get

$$\frac{\Gamma_1}{2} = \text{Re}(\lambda_+) = -\gamma/2 + \text{Re}(C(r, 0, 0, 0)) = -\gamma/2 - \frac{q^2}{8\pi\epsilon_0 m \omega} \left[\left(\frac{k^2}{r} - \frac{1}{r^3} \right) \sin(kr) + \frac{k}{r^2} \cos(kr) \right]. \quad (2.37)$$

We expand the sine and cosine up to third order. The expression inside the square brackets becomes

$$\left(\frac{k^2}{r} - \frac{1}{r^3} \right) \sin(kr) + \frac{k}{r^2} \cos(kr) \approx \left(\frac{k^2}{r} - \frac{1}{r^3} \right) \left(kr - \frac{k^3 r^3}{3!} \right) + \frac{k}{r^2} \left(1 - \frac{k^2 r^2}{2!} \right) = -\frac{k^5 r^2}{3!} + \frac{2k^3}{3}. \quad (2.38)$$

We let $r \rightarrow 0$ and insert this back in equation 2.37

$$\text{Re}(\lambda_+) = -\gamma/2 - \frac{q^2 \omega^2}{12\pi\epsilon_0 m c^3} = -\gamma, \quad (2.39)$$

which is indeed double the decay rate of the single dipole case. A plot of the behavior of the decay rate for oscillator 1 can be seen in figure 3. The decay rate of oscillator 2 is obtained by the real part of the λ_- value, which is very similar to eq 2.37 but the opposite sign after $\gamma/2$. As a consequence, the decay rate of oscillator 2 goes to zero as the distance goes to zero.

The interaction also gives a change in the oscillation frequency for the dipoles, this change is given by the imaginary part of the eigenvalues. The frequency shift for oscillator 1 is:

$$\delta_1 = \text{Im}(\lambda_+) = \text{Im}(C(r, \theta_1, \theta_2, \Delta\phi)), \quad (2.40)$$

which as opposed to the decay rate diverges towards infinity for the distance going to zero. A plot of the behavior of this frequency shift is given in figure 4. For oscillator 2 the behavior of the frequency shift is the same, but with a minus sign, which means that its frequency shift goes towards minus infinity for the distance going to zero.

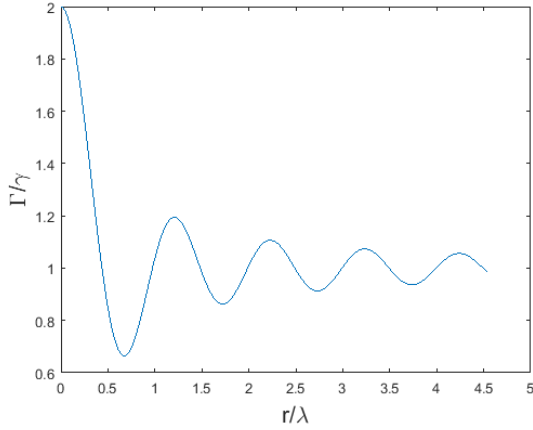


Figure 3: Plot of the decay rate of dipole 1 as a function of the distance between them. The frequency of the spring was assumed to be $\omega = 7.2 \cdot 10^7 \gamma$, $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $\Delta\phi = 0$

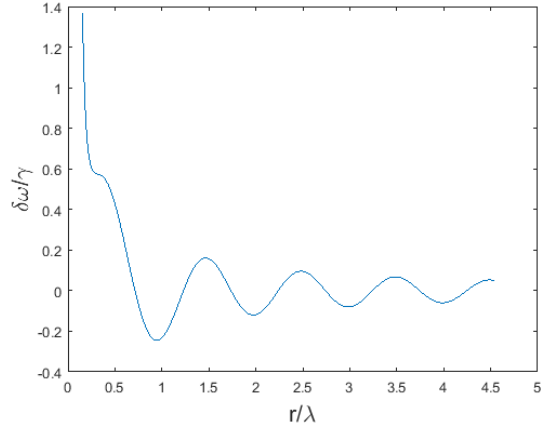


Figure 4: Plot of the frequency shifting for the dipoles due to the dipole dipole interaction as a function of the distance between them. The frequency of the spring was assumed to be $\omega = 7.2 \cdot 10^7 \gamma$, $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $\Delta\phi = 0$

2.4 Two dipoles - Solution for different frequency case

Next we solve this problem for two dipoles oscillating with different spring constants. We begin by looking at equation 2.23:

$$\dot{A}_1 = -i\omega_1 A_1 - \frac{\gamma_1}{2}(A_1 - A_1^*) + C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi)(A_2 + A_2^*).$$

We note that the decay rate is a function of the spring frequency ω and therefore different for the two dipoles. When making the rotating wave approximation, there are now many possible rotating frames to transform to. For this thesis we choose to transform into a frame oscillating with $\omega_s = \frac{\omega_1 + \omega_2}{2}$, so $\tilde{A}_1 = A_1 e^{i\omega_s t}$ and $\tilde{A}_2 = A_2 e^{i\omega_s t}$. Applying this transformation the above equation becomes

$$\dot{\tilde{A}}_1 = i\frac{\Delta\omega}{2}\tilde{A}_1 - \frac{\gamma_1}{2}(\tilde{A}_1 - \tilde{A}_1^* e^{i2\omega_s t}) + C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi)(\tilde{A}_2 + \tilde{A}_2^* e^{i2\omega_s t}), \quad (2.41)$$

where we have defined $\Delta\omega = \omega_2 - \omega_1$. We then apply the rotating wave approximation, where we approximate that the fastly rotating terms contribution averages to zero. The equation of motion becomes

$$\dot{\tilde{A}}_1 = (i\frac{\Delta\omega}{2} - \frac{\gamma_1}{2})\tilde{A}_1 + C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi)\tilde{A}_2 e^{ik_2 r}. \quad (2.42)$$

By a similar procedure the equation of motion for \tilde{A}_2 can be achieved. We will solve this system of equations by the same method as in the same-frequency case. The system in matrix form reads

$$\begin{bmatrix} \dot{\tilde{A}}_1 \\ \dot{\tilde{A}}_2 \end{bmatrix} = \begin{bmatrix} -\gamma_1/2 + i\frac{\Delta\omega}{2} & C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi) \\ C_{1 \rightarrow 2}(r, \theta_1, \theta_2, \Delta\phi) & -\gamma_2/2 - i\frac{\Delta\omega}{2} \end{bmatrix} \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix}, \quad (2.43)$$

which has the eigenvalues

$$\begin{aligned} \lambda_{\pm} &= \frac{-(\gamma_1 + \gamma_2)}{4} \mp \frac{\sqrt{(-i\Delta\omega + \frac{\gamma_1 - \gamma_2}{2})^2 + 4C_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi)C_{1 \rightarrow 2}(r, \theta_1, \theta_2, \Delta\phi)}}{2}, \\ &= \frac{-(\gamma_1 + \gamma_2)}{4} \mp \frac{\sqrt{F(\Delta\omega, \Delta\gamma, r, \theta_1, \theta_2, \Delta\phi)}}{2}, \end{aligned} \quad (2.44)$$

where we defined the function $F(\Delta\omega, \Delta\gamma, r, \theta_1, \theta_2, \Delta\phi)$ to be the function inside the square root to shorten the notation. The naming of the eigenvalues was reverted; the reason will become clear later. The eigenvalues do not have a physical significance as they had in the same-frequency case, but as we will see for the quantum calculations, the difference of the imaginary parts will be an important quantity. It is given by

$$\delta_{total} = |\text{Im}(\lambda_+ - \lambda_-)|, \quad (2.45)$$

a plot of this quantity can be seen in figure 5. We should note that δ_{total} goes towards $\Delta\omega$ for r very large, and it diverges towards infinity for $r \rightarrow 0$.

The matrix that describes the system of equations this time is not normal, which means that we cannot guarantee that the matrix is diagonalizable. It also means that $R^{-1} = R^*$ is no longer a valid assumption. The R matrix is then to be found by inverting the R^{-1} matrix in the old fashion way.

The expression for the eigenvectors is long and ugly, which makes R^{-1} and R even worse. So for the sake of beauty, one more approximation will be made. Assuming $\omega \approx 10^7\gamma$, we have tested some values numerically and found that $\Delta\omega$ is commonly about 6 orders of magnitude bigger than $\Delta\gamma$, so we make the approximation $\gamma_1 = \gamma_2$. The functions $C_{1 \rightarrow 2}$ and $C_{2 \rightarrow 1}$ depend on the value of ω_1 and ω_2 respectively. We will work with values for $\Delta\omega$ that are about 7 orders of magnitude smaller than ω_1 and ω_2 , so the difference in the functions is negligible. We can therefore approximate $C_{1 \rightarrow 2} = C_{2 \rightarrow 1} = C$. With this approximation, the eigenvalues become

$$\lambda_{\pm} = -\frac{\gamma}{2} \mp \frac{\sqrt{C(r, \theta_1, \theta_2, \Delta\phi)^2 - (\frac{\Delta\omega}{2})^2}}{2}. \quad (2.46)$$

We will calculate the solution similarly as we did for the same frequency case, with the only difference that we have to invert R^{-1} without any convenient shortcuts. The solution in matrix form is:

$$\begin{bmatrix} \tilde{A}_1(t) \\ \tilde{A}_2(t) \end{bmatrix} = \frac{1}{B_+ - B_-} \begin{bmatrix} e^{\lambda_- t} B_+ - e^{\lambda_+ t} B_- & \frac{\sqrt{1+|B_-|^2}}{\sqrt{1+|B_+|^2}} B_- (e^{\lambda_- t} - e^{\lambda_+ t}) \\ -\frac{\sqrt{1+|B_+|^2}}{\sqrt{1+|B_-|^2}} B_+ (e^{\lambda_- t} - e^{\lambda_+ t}) & -e^{\lambda_- t} B_- + e^{\lambda_+ t} B_+ \end{bmatrix} \begin{bmatrix} \tilde{A}_1(0) \\ \tilde{A}_2(0) \end{bmatrix}, \quad (2.47)$$

with

$$B_{\pm} = \frac{C(r, \theta_1, \theta_2, \Delta\phi)}{i\frac{\Delta\omega}{2} \pm \sqrt{C(r, \theta_1, \theta_2, \Delta\phi)^2 - (\frac{\Delta\omega}{2})^2}}. \quad (2.48)$$

The change in name for the eigenvalues now becomes clear, as letting $\Delta\omega \rightarrow 0$ we can see that the matrix reduces to the one we had in equation 2.34. But then, do the eigenvalues reduce to the right ones? In this limit, we see that F , the quantity inside the square root becomes $4C^2$, so $\sqrt{F} = \pm 2C$. By comparing the eigenvalues in 2.44 to 2.32, we see that we must choose $\sqrt{F} = -2C$ for them to be equal to each other.

What happens in the more general case where $\Delta\omega \neq 0$? The square root of complex numbers is a multivalued function, but there is only one correct solution, so we must pick the square root that reduces to $\sqrt{F} = -2C$ when $\Delta\omega \rightarrow 0$. Any complex number can be written in polar form by $F = r e^{i\theta}$, and its square root by $\sqrt{F} = r^{\frac{1}{2}} e^{i(\frac{\theta}{2} + n\pi)}$ where n is either 0 or 1. After testing different values numerically, it was found that $n = 0$ gives $\sqrt{F} = -2C$ when $\Delta\omega \rightarrow 0$. So to keep continuity we must choose $\sqrt{F} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ for all values of F .

We have now finished deriving a model for two dipoles interacting classically. This model can give us the motion of two dipoles separated at a given distance, with random orientations and oscillation frequencies that are not too far off from each other. When doing the quantum calculation, if the resulting model resembles the one we derived here, it will be a sign that our calculations might be right.

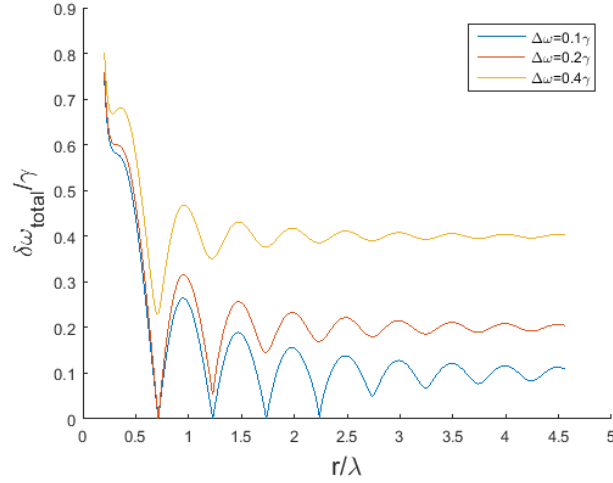


Figure 5: Plot of the absolute value of the difference in frequency shift from two dipoles through different values of $\Delta\omega$. The natural spring frequency was assumed to be $\omega_1 = 2.9 \cdot 10^7 \gamma$.

3 Quantum Model

3.1 One dipole

It is time to turn our attention to quantum mechanics. Now we think of a dipole as an electron orbiting a nucleus, interacting with the quantized electromagnetic field and satisfying the Schrödinger equation. Before we consider the main problem of two dipoles interacting with the electromagnetic field, we work the simpler problem of one dipole alone interacting with the electromagnetic field. This will be done in the Heisenberg picture since it yields a problem similar to the one in the classical picture. The general expression for the electric field operator propagating inside a cubic cavity of volume V in the Heisenberg picture is

$$\hat{E}(\mathbf{r}, t) = i \sum_{\mathbf{k}} \left(\frac{\omega_k \hbar}{\epsilon_0 V} \right)^{\frac{1}{2}} \mathbf{e}_{\mathbf{k}} \left[\hat{a}_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{r}} - \hat{a}_{\mathbf{k}}^\dagger(t) e^{-i\mathbf{k} \cdot \mathbf{r}} \right], \quad (3.1)$$

where $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ are the raising and lowering operators corresponding to the wave vector and polarization \mathbf{k} , and $\mathbf{e}_{\mathbf{k}}$ is a unit vector in the direction of \mathbf{k} [7]. The \mathbf{k} under the sum denotes a sum of all the possible modes inside the volume and the two independent polarizations. This field has the Hamiltonian $H^F = \sum_{\mathbf{k}} \hbar \omega_k \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$.

For simplicity we will now consider a system with only two levels, a ground state $|g\rangle$ and an excited state $|e\rangle$. We let their energy difference be $\hbar \omega_0$ and choose the zero point for the energy to be right between them. The dipole operator for this system is

$$\hat{\mathbf{d}} = \langle g | \hat{\mathbf{d}} | e \rangle |g\rangle \langle e| + \langle e | \hat{\mathbf{d}} | g \rangle |e\rangle \langle g| = \langle e | \hat{\mathbf{d}} | g \rangle (\hat{\sigma}_- + \hat{\sigma}_+), \quad (3.2)$$

where the diagonal terms have been eliminated by parity considerations, the value $\langle e | \hat{\mathbf{d}} | g \rangle$ is assumed to be real and we define the atomic raising and lowering operators as $\hat{\sigma}_+ = |e\rangle \langle g|$ and $\hat{\sigma}_- = |g\rangle \langle e|$ respectively. We will also be using the operator $\hat{\sigma}_z = |e\rangle \langle e| - |g\rangle \langle g|$. These operators fulfill the commutation relations

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z, \quad [\hat{\sigma}_z, \hat{\sigma}_\pm] = 2\hat{\sigma}_\pm. \quad (3.3)$$

The atomic Hamiltonian can be written in terms of these operators by $H^A = \frac{1}{2} \hbar \omega_0 \sigma_z$.

The interaction Hamiltonian between the field and the atom within the dipole approximation can be shown to be $H^I = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}$ [7]. The total Hamiltonian becomes

$$H = H^A + H^F + H^I = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \sum_{\mathbf{k}} \hbar \left[\omega_k \hat{a}_k^\dagger \hat{a}_k + ig_k \hat{a}_k (\hat{\sigma}_+ + \hat{\sigma}_-) - ig_k^* \hat{a}_k^\dagger (\hat{\sigma}_+ + \hat{\sigma}_-) \right] \quad (3.4)$$

with

$$g_k = - \left(\frac{\omega_k}{\hbar\epsilon_0 V} \right)^{\frac{1}{2}} e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{e}_{\mathbf{k}} \cdot \langle e | \hat{\mathbf{d}} | g \rangle. \quad (3.5)$$

We can simplify the Hamiltonian by making the rotating wave approximation. In the free field and free atomic case respectively, the operators have the following time dependence:

$$\hat{a}_k(t) = \hat{a}_k(0)e^{-i\omega_k t}, \quad \hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0)e^{i\omega_k t}, \quad \hat{\sigma}_\pm(t) = \hat{\sigma}_\pm(0)e^{\pm i\omega_0 t}. \quad (3.6)$$

For ω_k close to ω_0 , the terms with $\hat{a}\hat{\sigma}_+$ and $\hat{a}^\dagger\hat{\sigma}_-$ in the Hamiltonian oscillate with frequency $\omega_k + \omega_0$, which is much faster than $\hat{a}\hat{\sigma}_-$ and $\hat{a}^\dagger\hat{\sigma}_+$, which oscillate with frequency $\omega_k - \omega_0$. We then assume that all the relevant physical processes in the one dipole case oscillate with a frequency that is much less than $\omega_k + \omega_0$, so that the contribution of the terms $\hat{a}\hat{\sigma}_+$ and $\hat{a}^\dagger\hat{\sigma}_-$ averages to zero. We should be aware that this only applies for the one dipole case. The Hamiltonian after the approximation reads:

$$H = H^A + H^F + H^I = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \sum_{\mathbf{k}} \hbar \left[\omega_k \hat{a}_k^\dagger \hat{a}_k + ig_k \hat{a}_k \hat{\sigma}_+ - ig_k^* \hat{a}_k^\dagger \hat{\sigma}_- \right]. \quad (3.7)$$

We now derive the Heisenberg equation of motion for the field and the atom. In the case of the field we derive it for the \hat{a} operator:

$$\dot{\hat{a}}_k = \frac{1}{i\hbar} [\hat{a}_k, H] = -i\omega_k \hat{a}_k - g_k^* \hat{\sigma}_-, \quad (3.8)$$

which has the solution

$$\hat{a}_k(t) = -g_k^* \int_{-\infty}^t \hat{\sigma}_-(t') e^{i\omega_k(t'-t)} dt' + \hat{a}_k(0) e^{-i\omega_k t}. \quad (3.9)$$

Similarly we calculate the equation of motion for the $\hat{\sigma}_-$ operator:

$$\dot{\hat{\sigma}}_- = \frac{1}{i\hbar} [\hat{\sigma}_-, H] = -i\omega_0 \hat{\sigma}_- - \sum_{\mathbf{k}} g_k \sigma_z \hat{a}_k, \quad (3.10)$$

where in the last term the normal ordering convention has been used, where the creation operators go before the annihilation operators [8]. We expand equation 3.10 by plugging the expression for $\hat{a}_k(t)$ in 3.9:

$$\dot{\hat{\sigma}}_- = -i\omega_0 \hat{\sigma}_- + \sum_{\mathbf{k}} |g_k|^2 \sigma_z \int_{-\infty}^t \hat{\sigma}_-(t') e^{i\omega_k(t'-t)} dt' + g_k \sigma_z \hat{a}_k(0) e^{-i\omega_k t}. \quad (3.11)$$

We will focus on solving this equation since it contains the dynamics of the dipole. So far the dipole approximation and the rotating wave approximation have been made. To be able to solve the equation we have, one more approximation is required, which is the Markov approximation. This approximation consist of saying that the future state of the coherence operator is not dependent on its past history. This is to be able to take the $\hat{\sigma}_-(t')$ out of the time integral. Mathematically we say $\hat{\sigma}_-(t) \approx \hat{\sigma}_-(0) e^{-i\omega_0 t}$. Then we can write:

$$\int_{-\infty}^t \hat{\sigma}_-(t') e^{-i\omega_k(t-t')} dt' \approx \int_{-\infty}^t \hat{\sigma}_-(0) e^{-i\omega_0 t'} e^{i\omega_k(t'-t)} dt' \approx \hat{\sigma}_-(t) \int_{-\infty}^t e^{i(\omega_0 - \omega_k)(t-t')} dt'. \quad (3.12)$$

Using this, equation 3.11 becomes

$$\dot{\hat{\sigma}}_- = \left[-i\omega_0 - \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \int_0^\infty e^{i(\omega_0 - \omega_{\mathbf{k}})\tau} d\tau \right] \hat{\sigma}_- + g_{\mathbf{k}} \hat{\sigma}_z \hat{a}_{\mathbf{k}}(0) e^{-i\omega_{\mathbf{k}} t}, \quad (3.13)$$

where we defined $\tau = t - t'$. The last term in this equation represents electromagnetic noise, in principle it is always there, but we will assume here that it has very little influence on the dynamics of the system. We wish now to calculate the second term inside the square brackets. By looking at the density of electromagnetic modes in a cavity of volume V , it can be shown that for V very large [7]

$$\sum_{\mathbf{k}} \mathbf{k} \rightarrow \frac{2V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin(\theta) \int_0^\infty dk k^2, \quad (3.14)$$

so the second term inside the brackets in 3.13 becomes

$$\frac{1}{4\pi^2 \epsilon_0 \hbar} \int_0^\pi d\theta \sin(\theta) \int_0^\infty dk k^2 \omega (\mathbf{e} \cdot \langle e | \hat{\mathbf{d}} | g \rangle)^2 \int_0^\infty e^{i(\omega_0 - \omega)\tau} d\tau. \quad (3.15)$$

Now we assume that the expectation value of the dipole moment only has a z component, then we can write $\mathbf{e} \cdot \langle e | \hat{\mathbf{d}} | g \rangle = d \cos \theta$, where d is the value obtained by calculating the expectation value. Then the above expression becomes

$$\frac{d^2}{4\pi^2 \epsilon_0 \hbar c^3} \int_0^\pi d\theta \sin(\theta) \cos^2(\theta) \int_0^\infty d\omega \omega^3 \int_0^\infty e^{i(\omega_0 - \omega)\tau} d\tau. \quad (3.16)$$

The angle integral can be calculated and gives $\frac{2}{3}$. The time integral can be shown to be [9]

$$\int_0^\infty e^{i(\omega_0 - \omega)\tau} d\tau = \pi \delta(\omega_0 - \omega) + iP \left[\frac{1}{\omega_0 - \omega} \right], \quad (3.17)$$

which we plug into equation 3.16 to get

$$\frac{d^2 \omega_0^3}{6\pi \epsilon_0 c^3 \hbar} + i \frac{d^2}{6\pi^2 \epsilon_0 \hbar c^3} P \int_0^\infty \frac{\omega^3}{\omega_0 - \omega} d\omega = \frac{\Gamma}{2} + i\Omega, \quad (3.18)$$

where we now have derived an explicit expression for the decay rate Γ . On the other hand, the frequency shift Ω depends on an integral which value is infinity. This thesis will not cover how to deal with this problem. We will assume that it has a finite value and we will operate with the corrected ω_0 . This is called the lamb shift. The fact that Γ and Ω represent a decay rate and a frequency shift becomes clear when we replace them into eq 3.13:

$$\dot{\hat{\sigma}}_- = - \left[\frac{\Gamma}{2} + i(\omega_0 + \Omega) \right] \hat{\sigma}_- \quad (3.19)$$

3.2 Two dipoles - Raw Equation of motion

Next we derive the equation of motion for two dipoles in quantum electrodynamics. The Hamiltonian will be similar to the one in eq 3.4, with the difference that there will be two contributions for the free atomic Hamiltonian

$$H^A = H^{A(1)} + H^{A(2)} = \sum_{j=1}^2 \frac{1}{2} \hbar \omega_0^{(j)} \hat{\sigma}_z^{(j)}. \quad (3.20)$$

For the electric field we will no longer restrict it to plane waves. From now on we will be using any generalized mode solution for the electromagnetic wave equation, which will be denoted by $\mathbf{F}_{\mathbf{k}}(\mathbf{r})$. The electric field operator then becomes [10]

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i \sum_{\mathbf{k}} \left(\frac{\hbar \omega_{\mathbf{k}}}{2\epsilon_0} \right)^{\frac{1}{2}} \left[\mathbf{F}_{\mathbf{k}}(\mathbf{r}) \hat{a}_{\mathbf{k}}(t) - \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}) \hat{a}_{\mathbf{k}}^\dagger(t) \right], \quad (3.21)$$

so the new interaction Hamiltonian within the dipole approximation becomes

$$H^I = -i \sum_{j=1}^2 \sum_{\mathbf{k}} \left(\frac{\hbar\omega_k}{2\epsilon_0} \right)^{\frac{1}{2}} \mathbf{d}^{(j)} \cdot \mathbf{F}_{\mathbf{k}}(\mathbf{r}_j) \hat{a}_k(t) (\hat{\sigma}_-^{(j)} + \hat{\sigma}_+^{(j)}) - \mathbf{d}^{(j)} \cdot \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_j) \hat{a}_k^\dagger(t) (\hat{\sigma}_-^{(j)} + \hat{\sigma}_+^{(j)}); \quad (3.22)$$

while the field Hamiltonian is unchanged

$$H^F = \sum_{\mathbf{k}} \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k. \quad (3.23)$$

In the one dipole case we did the rotating wave approximation directly in the Hamiltonian; there we assumed that events of the frequency scale $2\omega_0$ were negligible. Now that we have two dipoles, as the distance between them becomes less than the wavelength corresponding to ω_0 , events of this frequency scale can be thought to have an influence on the system, so we will proceed with the Hamiltonian without the rotating wave approximation.

The Heisenberg equations of motion are now derived. We will follow the same procedure as in the one dipole case. The equation of motion for the electric field annihilation operator reads

$$\dot{\hat{a}}_k = \frac{1}{i\hbar} [\hat{a}_k, H] = -i\omega_k \hat{a}_k + \sum_{j=1}^2 \left(\frac{\omega_k}{2\hbar\epsilon_0} \right)^{\frac{1}{2}} \mathbf{d}^{(j)} \cdot \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_j) (\hat{\sigma}_-^{(j)} + \hat{\sigma}_+^{(j)}). \quad (3.24)$$

It has the solution

$$\hat{a}_k(t) = \hat{a}_k(0) e^{-i\omega_k t} + \left(\frac{\omega_k}{2\hbar\epsilon_0} \right)^{\frac{1}{2}} \sum_{j=1}^2 \mathbf{d}^{(j)} \cdot \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_j) \int_{-\infty}^t (\hat{\sigma}_+^{(j)}(t') + \hat{\sigma}_-^{(j)}(t')) e^{i\omega_k(t'-t)} dt'. \quad (3.25)$$

For the coherence operator we will be looking specifically at the one from dipole 1, $\hat{\sigma}_-^{(1)}$. Its equation of motion is

$$\dot{\hat{\sigma}}_-^{(1)} = \frac{1}{i\hbar} [\hat{\sigma}_-^{(1)}, H] = -i\omega_0^{(1)} \hat{\sigma}_-^{(1)} + \sum_{\mathbf{k}} \left(\frac{\omega_k}{2\hbar\epsilon_0} \right)^{\frac{1}{2}} [\mathbf{d}^{(1)} \cdot \mathbf{F}_{\mathbf{k}}(\mathbf{r}_1) \hat{\sigma}_z^{(1)} \hat{a}_k - \mathbf{d}^{(1)} \cdot \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_1) \hat{a}_k^\dagger \hat{\sigma}_z^{(1)}], \quad (3.26)$$

where normal ordering has been used. The next step is to insert the solution for \hat{a}_k from eq. 3.25 and \hat{a}_k^\dagger , which is its hermitian conjugate, to get the full equation of motion:

$$\begin{aligned} \dot{\hat{\sigma}}_-^{(1)} = & -i\omega_0^{(1)} \hat{\sigma}_-^{(1)} + \sum_{\mathbf{k}} \left(\frac{\omega_k}{2\hbar\epsilon_0} \right)^{\frac{1}{2}} \left[\mathbf{d}^{(1)} \cdot \mathbf{F}_{\mathbf{k}}(\mathbf{r}_1) \hat{a}_k(0) e^{-i\omega_k t} - \mathbf{d}^{(1)} \cdot \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_1) \hat{a}_k^\dagger(0) e^{i\omega_k t} \right] \\ & + \sum_{\mathbf{k}} \left(\frac{\omega_k}{2\hbar\epsilon_0} \right) \sum_{j=1}^2 \left[\mathbf{d}^{(1)} \cdot \mathbf{F}_{\mathbf{k}}(\mathbf{r}_1) \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_j) \cdot \mathbf{d}^{(j)} \hat{\sigma}_z^{(1)} \int_{-\infty}^t (\hat{\sigma}_+^{(j)}(t') + \hat{\sigma}_-^{(j)}(t')) e^{i\omega_k(t'-t)} dt' \right. \\ & \left. - \mathbf{d}^{(1)} \cdot \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_1) \mathbf{F}_{\mathbf{k}}(\mathbf{r}_j) \cdot \mathbf{d}^{(j)} \int_{-\infty}^t (\hat{\sigma}_+^{(j)}(t') + \hat{\sigma}_-^{(j)}(t')) e^{-i\omega_k(t'-t)} dt' \hat{\sigma}_z^{(1)} \right], \end{aligned} \quad (3.27)$$

where $\mathbf{F}_{\mathbf{k}}(\mathbf{r}_1) \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_j)$ denotes the tensor product of the mode functions.

3.3 Markov Approximation

To shorten equation 3.27, we will define the noise terms as

$$N = \sum_{\mathbf{k}} \left(\frac{\omega_k}{2\hbar\epsilon_0} \right)^{\frac{1}{2}} \left[\mathbf{d}^{(1)} \cdot \mathbf{F}_{\mathbf{k}}(\mathbf{r}_1) \hat{a}_k(0) e^{-i\omega_k t} - \mathbf{d}^{(1)} \cdot \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}_1) \hat{a}_k^\dagger(0) e^{i\omega_k t} \right], \quad (3.28)$$

and introduce the Green's functions by using the relation [5]

$$\sum_{\mathbf{k}} \omega_{\mathbf{k}} \cdot \mathbf{F}_{\mathbf{k}}(\mathbf{r}') \mathbf{F}_{\mathbf{k}}^*(\mathbf{r}) e^{-i\omega_{\mathbf{k}}(t-t')} = 2 \int d\omega e^{-i\omega(t-t')} \frac{\omega^2}{\pi c^2} \text{Im}\{\mathbb{G}(\mathbf{r}', \mathbf{r}; \omega)\}. \quad (3.29)$$

Reversing the order of the product of the mode functions would bring $\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega)$ instead. In this thesis we will be using the Green's function for free space, for which the order of \mathbf{r} and \mathbf{r}' gives the same function. This property is assumed to simplify the calculations. If a Green's function without this property were to be used, one should be careful of this affects the following calculations.

So far only the dipole approximation has been made, to be able to proceed, we now make the Markov approximation, as we did in equation 3.12. This time we have four different terms:

$$\begin{aligned} \int_{-\infty}^t \hat{\sigma}_-(t') e^{-i\omega_{\mathbf{k}}(t-t')} dt' &\approx \int_{-\infty}^t \hat{\sigma}_-(0) e^{-i\omega_0 t'} e^{i\omega_{\mathbf{k}}(t'-t)} dt' \approx \hat{\sigma}_-(t) \int_{-\infty}^t e^{i(\omega_0 - \omega_{\mathbf{k}})(t-t')} dt', \\ \int_{-\infty}^t \hat{\sigma}_-(t') e^{i\omega_{\mathbf{k}}(t-t')} dt' &\approx \hat{\sigma}_-(t) \int_{-\infty}^t e^{i(\omega_0 + \omega_{\mathbf{k}})(t-t')} dt', \\ \int_{-\infty}^t \hat{\sigma}_+(t') e^{-i\omega_{\mathbf{k}}(t-t')} dt' &\approx \int_{-\infty}^t \hat{\sigma}_+(0) e^{i\omega_0 t'} e^{i\omega_{\mathbf{k}}(t'-t)} dt' \approx \hat{\sigma}_+(t) \int_{-\infty}^t e^{-i(\omega_0 + \omega_{\mathbf{k}})(t-t')} dt', \\ \int_{-\infty}^t \hat{\sigma}_+(t') e^{i\omega_{\mathbf{k}}(t-t')} dt' &\approx \hat{\sigma}_+(t) \int_{-\infty}^t e^{i(-\omega_0 + \omega_{\mathbf{k}})(t-t')} dt'. \end{aligned} \quad (3.30)$$

Applying equations 3.28, 3.29 and 3.30 into 3.27 we get:

$$\begin{aligned} \dot{\hat{\sigma}}_-^{(1)} &= -i\omega_0^{(1)} \hat{\sigma}_-^{(1)} + N + \sum_{j=1}^2 \frac{1}{\hbar\pi\epsilon_0 c^2} \int d\omega \omega^2 \mathbf{d}^{(1)} \cdot \text{Im}\{\mathbb{G}(\mathbf{r}_1, \mathbf{r}_2; \omega)\} \cdot \mathbf{d}^{(j)} \hat{\sigma}_z^{(j)} \\ &\quad \times \left[\hat{\sigma}_z^{(1)} \hat{\sigma}_+^{(j)} \int_0^\infty e^{-i(\omega_0^{(j)} + \omega)\tau} d\tau + \hat{\sigma}_z^{(1)} \hat{\sigma}_-^{(j)} \int_0^\infty e^{i(\omega_0^{(j)} - \omega)\tau} d\tau \right. \\ &\quad \left. - \hat{\sigma}_+^{(j)} \hat{\sigma}_z^{(1)} \int_0^\infty e^{i(-\omega_0^{(j)} + \omega)\tau} d\tau - \hat{\sigma}_-^{(j)} \hat{\sigma}_z^{(1)} \int_0^\infty e^{i(\omega_0^{(j)} + \omega)\tau} d\tau \right]. \end{aligned} \quad (3.31)$$

To proceed we calculate the time integrals by using equation 3.17. Then we define the quantities

$$\Gamma_{\pm}^{(mn)} = \frac{2}{c^2 \hbar \epsilon_0} \int \omega^2 \mathbf{d}^{(m)} \cdot \text{Im}\{\mathbb{G}(\mathbf{r}_m, \mathbf{r}_n; \omega)\} \cdot \mathbf{d}^{(n)} \delta(\omega_0^{(n)} \mp \omega) d\omega, \quad (3.32)$$

$$\Omega_{\pm}^{(mn)} = \frac{1}{\pi c^2 \hbar \epsilon_0} P \int \frac{\omega^2 \mathbf{d}^{(m)} \cdot \text{Im}\{\mathbb{G}(\mathbf{r}_m, \mathbf{r}_n; \omega)\} \cdot \mathbf{d}^{(n)}}{(\omega_0^{(n)} \pm \omega)} d\omega, \quad (3.33)$$

$$\Omega_{\pm}^{(mn)'} = \frac{1}{\pi c^2 \hbar \epsilon_0} P \int \frac{\omega^2 \mathbf{d}^{(m)} \cdot \text{Im}\{\mathbb{G}(\mathbf{r}_m, \mathbf{r}_n; \omega)\} \cdot \mathbf{d}^{(n)}}{-(\omega_0^{(n)} \pm \omega)} d\omega, \quad (3.34)$$

so that equation 3.31 becomes

$$\begin{aligned} \dot{\hat{\sigma}}_-^{(1)} &= -i\omega_0^{(1)} \hat{\sigma}_-^{(1)} + N + \sum_{j=1}^2 \left(\frac{1}{2} \Gamma_-^{(1j)} - i\Omega_+^{(1j)'} \right) \hat{\sigma}_z^{(1)} \hat{\sigma}_+^{(j)} + \left(\frac{1}{2} \Gamma_+^{(1j)} + i\Omega_-^{(1j)} \right) \hat{\sigma}_z^{(1)} \hat{\sigma}_-^{(j)} \\ &\quad - \left(\frac{1}{2} \Gamma_+^{(1j)} + i\Omega_-^{(1j)'} \right) \hat{\sigma}_+^{(j)} \hat{\sigma}_z^{(1)} - \left(\frac{1}{2} \Gamma_-^{(1j)} + i\Omega_+^{(1j)} \right) \hat{\sigma}_-^{(j)} \hat{\sigma}_z^{(1)}. \end{aligned} \quad (3.35)$$

At last we expand the sum over j and apply the product of the atomic operators for the cases where they correspond to the same dipole. For the cases where they correspond to different dipoles we note

that $\hat{\sigma}_-^{(2)}\hat{\sigma}_z^{(1)} = \hat{\sigma}_z^{(1)}\hat{\sigma}_-^{(2)}$. Then equation 3.35 becomes

$$\begin{aligned} \dot{\hat{\sigma}}_-^{(1)} &= -i\omega_0^{(1)}\hat{\sigma}_-^{(1)} + N \\ &+ \left(\frac{1}{2}(\Gamma_-^{(11)} + \Gamma_+^{(11)}) - i(\Omega_+^{(11)'} - \Omega_-^{(11)'})\right)\hat{\sigma}_+^{(1)} + \left(\frac{1}{2}(-\Gamma_+^{(11)} - \Gamma_-^{(11)}) - i(\Omega_-^{(11)} + \Omega_+^{(11)})\right)\hat{\sigma}_-^{(1)} \\ &+ \left(\frac{1}{2}(\Gamma_-^{(12)} - \Gamma_+^{(12)}) - i(\Omega_+^{(12)'} + \Omega_-^{(12)'})\right)\hat{\sigma}_+^{(2)}\hat{\sigma}_z^{(1)} + \left(\frac{1}{2}(\Gamma_+^{(12)} - \Gamma_-^{(12)}) - i(-\Omega_-^{(12)} + \Omega_+^{(12)})\right)\hat{\sigma}_-^{(2)}\hat{\sigma}_z^{(1)} \end{aligned} \quad (3.36)$$

3.4 Rotating wave Approximation

We would like to simplify equation 3.36 even further. We start by calculating equation 3.32. Even though the limits of the integral in equation 3.29 are not specified, we can assume that they are from zero to infinity, since those were the limits for the one dipole case. Then $\Gamma_-^{(nm)} = 0$.

We will from now on use the lamb shifted atomic oscillation frequency $\omega_0^{(1)} = \omega_0^{(1)} + (\Omega_+^{11} + \Omega_-^{11})$. We will also assume that the noise terms have no influence on the motion. Then eq 3.36 becomes

$$\begin{aligned} \dot{\hat{\sigma}}_-^{(1)} &= -i\omega_0^{(1)}\hat{\sigma}_-^{(1)} + \left(\frac{1}{2}\Gamma_+^{(11)} - i(\Omega_+^{(11)'} - \Omega_-^{(11)'})\right)\hat{\sigma}_+^{(1)} - \frac{1}{2}\Gamma_+^{(11)}\hat{\sigma}_-^{(1)} \\ &+ \left(-\frac{1}{2}\Gamma_+^{(12)} - i(\Omega_+^{(12)'} + \Omega_-^{(12)'})\right)\hat{\sigma}_+^{(2)}\hat{\sigma}_z^{(1)} + \left(\frac{1}{2}(\Gamma_+^{(12)} - i(-\Omega_-^{(12)} + \Omega_+^{(12)}))\right)\hat{\sigma}_-^{(2)}\hat{\sigma}_z^{(1)}. \end{aligned} \quad (3.37)$$

To proceed we transform onto the rotating frame, to do so we introduce the quantities $\omega_s = \frac{\omega_0^{(1)} + \omega_0^{(2)}}{2}$, $\tilde{\sigma}_-^{(1)} = \hat{\sigma}_-^{(1)}e^{i\omega_s t}$, $\tilde{\sigma}_-^{(2)} = \hat{\sigma}_-^{(2)}e^{i\omega_s t}$ and $\tilde{\sigma}_+^{(j)} = \tilde{\sigma}_-^{(j)\dagger}$, which oscillate slowly compared to $\hat{\sigma}_p m^{(j)}$. Then substitute into equation 3.37 to get:

$$\begin{aligned} \dot{\tilde{\sigma}}_-^{(1)} &= \left(i\frac{\Delta\omega}{2} - \frac{\Gamma_+^{(11)}}{2}\right)\tilde{\sigma}_-^{(1)} + \left(\frac{\Gamma_+^{(11)}}{2} + i(-\Omega_+^{(12)} + \Omega_-^{(12)})\right)\tilde{\sigma}_-^{(2)}\hat{\sigma}_z^{(1)} \\ &- \left(\frac{\Gamma_+^{(11)}}{2} + i(\Omega_+^{(11)'} - \Omega_-^{(11)'})\right)\tilde{\sigma}_+^{(1)}e^{i2\omega_s t} - \left(-\frac{\Gamma_+^{(12)}}{2} + i(\Omega_+^{(12)'} + \Omega_-^{(12)'})\right)\tilde{\sigma}_+^{(2)}\hat{\sigma}_z^{(1)}e^{i2\omega_s t}, \end{aligned} \quad (3.38)$$

where we make the rotating wave approximation which says that the contribution of the oscillating terms averages to zero. The resulting differential equation is

$$\dot{\tilde{\sigma}}_-^{(1)} = \left(i\frac{\Delta\omega}{2} - \frac{\Gamma_+^{(11)}}{2}\right)\tilde{\sigma}_-^{(1)} + \left(\frac{\Gamma_+^{(12)}}{2} + i(-\Omega_+^{(12)} + \Omega_-^{(12)})\right)\tilde{\sigma}_-^{(2)}\hat{\sigma}_z^{(1)}. \quad (3.39)$$

This equation is very hard to solve because of the $\tilde{\sigma}_-^{(2)}\hat{\sigma}_z^{(1)}$ term, where both operators have a time dependence. This can be solved by changing from the single atom basis to the coupled basis. We do so by multiplying the whole equation by the identity operator for dipole two. In the case of $\hat{\sigma}_-^{(1)}$ we have

$$\hat{\sigma}_-^{(1)} \otimes \mathbb{I}_2 = |g_1\rangle\langle e_1| \otimes (|g_2\rangle\langle g_2| + |e_2\rangle\langle e_2|) = |g_1g_2\rangle\langle e_1g_2| + |g_1e_2\rangle\langle e_1e_2| \approx |g_1g_2\rangle\langle e_1g_2| = \hat{\sigma}_-^{(1)'}, \quad (3.40)$$

where we made the approximation that the double excited state is not present in our system. This approximation is done because in the protocol where we will apply the model, only very few photons are present at a time, so double excitations are negligible [4]. Similarly for $\tilde{\sigma}_-^{(2)}\hat{\sigma}_z^{(1)}$ we get

$$(\tilde{\sigma}_-^{(2)}\hat{\sigma}_z^{(1)}) \otimes \mathbb{I}_2 = \tilde{\sigma}_-^{(2)} \otimes \hat{\sigma}_z^{(1)} = |g_1g_2\rangle\langle g_1e_2| - |e_1g_2\rangle\langle e_1e_2| \approx |g_1g_2\rangle\langle g_1e_2| = -\hat{\sigma}_-^{(2)'} \quad (3.41)$$

Applying this to equation 3.42 we finally get an equation that can be solved:

$$\dot{\tilde{\sigma}}_-^{(1)'} = \left(i \frac{\Delta\omega}{2} - \frac{\Gamma_+^{(11)}}{2} \right) \tilde{\sigma}_-^{(1)'} + \left(\frac{-\Gamma_+^{(12)}}{2} + i(\Omega_+^{(12)} - \Omega_-^{(12)}) \right) \tilde{\sigma}_-^{(2)'}. \quad (3.42)$$

For the second dipole, the same derivation can be made to achieve

$$\dot{\tilde{\sigma}}_-^{(2)'} = \left(-i \frac{\Delta\omega}{2} - \frac{\Gamma_+^{(22)}}{2} \right) \tilde{\sigma}_-^{(2)'} + \left(\frac{-\Gamma_+^{(21)}}{2} + i(\Omega_+^{(21)} - \Omega_-^{(21)}) \right) \tilde{\sigma}_-^{(1)'}. \quad (3.43)$$

We see that this system of equations is equivalent to the one we had in the classical case, and so the solution must be similar.

3.5 Solution

Before stating the solution, we would like to have an expression for all the Ω and Γ constants in the equations. For Γ we have

$$\Gamma_+^{(nm)} = \frac{2\omega_0^{(n)2}}{\hbar\epsilon_0 c^2} \mathbf{d}^{(m)} \cdot \text{Im} \left\{ \mathbb{G}(\mathbf{r}_m, \mathbf{r}_n; \omega_0^{(n)}) \right\} \cdot \mathbf{d}^{(n)}. \quad (3.44)$$

Starting with $\Gamma^{(11)}$ and $\Gamma^{(22)}$, they require the Green's function to be evaluated at the origin, but the Green's function for a point charge has a singularity at the origin. Hence we would rather use another method for calculating them. Luckily we can safely assume that this is the one dipole decay rate, which we calculated in section 3.1. Stating the result we got there we have:

$$\Gamma^{(mm)} = \frac{d^2 \omega_0^3}{3\pi\epsilon_0 c^3 \hbar}. \quad (3.45)$$

As for the non diagonal Γ terms, they can be calculated if the Green's function and the direction of the dipole moments are specified. For this thesis we will be looking at the vacuum Green's function, as we did for the classical case. We will return to the non diagonal Γ terms in shortly. To calculate $\Omega_+^{(12)} - \Omega_-^{(12)}$ we use the Kramers-Kronig relation [6], which for our case becomes:

$$\begin{aligned} \frac{1}{\pi\hbar\epsilon_0 c^2} P \int_0^\infty \left[\frac{\omega^2 \mathbf{d}^{(m)} \cdot \text{Im} \left\{ \mathbb{G}(\mathbf{r}_m, \mathbf{r}_n; \omega) \cdot \mathbf{d}^{(n)} \right\}}{(\omega_0^{(n)} + \omega)} - \frac{\omega^2 \mathbf{d}^{(m)} \cdot \text{Im} \left\{ \mathbb{G}(\mathbf{r}_m, \mathbf{r}_n; \omega) \cdot \mathbf{d}^{(n)} \right\}}{(\omega_0^{(n)} - \omega)} \right] d\omega \\ = \frac{\omega_0^{(n)2}}{\hbar\epsilon_0 c^2} \mathbf{d}^{(m)} \cdot \text{Re} \left\{ \mathbb{G}(\mathbf{r}_m, \mathbf{r}_n; \omega_0^{(n)}) \right\} \cdot \mathbf{d}^{(n)}, \end{aligned} \quad (3.46)$$

which is the expression for $\Omega_+^{(12)} - \Omega_-^{(12)}$. We note that it is the same as 3.44 but with the real part of the Green's function instead. The total non diagonal coefficients are given by the sum of these functions:

$$\begin{aligned} \frac{-\Gamma_+^{(12)}}{2} + i(\Omega_+^{(12)} - \Omega_-^{(12)}) &= \frac{\omega_0^{(2)2}}{\hbar\epsilon_0 c^2} \mathbf{d}^{(1)} \cdot \left(-\text{Im} \left\{ \mathbb{G}(\mathbf{r}_1, \mathbf{r}_2; \omega_0^{(2)}) \right\} + i \text{Re} \left\{ \mathbb{G}(\mathbf{r}_1, \mathbf{r}_2; \omega_0^{(2)}) \right\} \right) \cdot \mathbf{d}^{(2)} \\ &= i \frac{\omega_0^{(2)2}}{\hbar\epsilon_0 c^2} \mathbf{d}^{(1)} \cdot \left(\mathbb{G}(\mathbf{r}_1, \mathbf{r}_2; \omega_0^{(2)}) \right) \cdot \mathbf{d}^{(2)}. \end{aligned} \quad (3.47)$$

We recognize this result as being very similar to the term $C_{2 \rightarrow 1}$ we used for the classical case in equation 2.22. The spatial and angular dependence are the same up to a constant. We define the

quantum equivalent of the function $C_{2 \rightarrow 1}$ as

$$\begin{aligned} Q_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi) &= i \frac{\omega_0^{(2)2}}{\hbar \epsilon_0 c^2} \mathbf{d}^{(1)} \cdot \left(\mathbb{G}(\mathbf{r}_1, \mathbf{r}_2; \omega_0^{(2)}) \cdot \mathbf{d}^{(2)} \right) \\ &= i \frac{e^{ik_2 r} d_1 d_2}{4\pi \epsilon_0 \hbar} \left[\sin^2(\theta_1) \sin(\theta_2) \cos(\Delta\phi) \frac{k_2^2}{r} + (3 \cos(\theta_1) \cos(\theta_2) - \hat{\mathbf{d}}_1 \cdot \hat{\mathbf{d}}_2) \left(\frac{1}{r^3} - \frac{ik_2}{r^2} \right) \right], \end{aligned} \quad (3.48)$$

where

$$\hat{\mathbf{d}}_1 \cdot \hat{\mathbf{d}}_2 = \sin(\theta_1) \sin(\theta_2) \cos(\Delta\phi) + \cos(\theta_1) \cos(\theta_2), \quad (3.49)$$

and we write the system of equations in matrix form:

$$\begin{bmatrix} \dot{\tilde{\sigma}}_-^{(1)} \\ \dot{\tilde{\sigma}}_-^{(2)} \end{bmatrix} = \begin{bmatrix} -\Gamma^{(11)}/2 + i\frac{\Delta\omega}{2} & Q_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi) \\ Q_{1 \rightarrow 2}(r, \theta_1, \theta_2, \Delta\phi) & -\Gamma^{(22)}/2 - i\frac{\Delta\omega}{2} \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_-^{(1)} \\ \tilde{\sigma}_-^{(2)} \end{bmatrix}. \quad (3.50)$$

Now it is evident that we have exactly the same system of equations as in the classical case (equation 2.43). The eigenvalues are obtained by replacing C by Q and γ by Γ in the classical ones (eq 2.44), and the solution is obtained by replacing \hat{A} by $\tilde{\sigma}_-$ in the classical solution (eq 2.47). As the spacial dependence of C and Q is the same, the plots seen in figures 3, 4 and 5 also apply for the quantum mechanical system.

The eigenvalues of the system have a greater physical significance for the quantum system than they have for the classical system. As the equation of motion is diagonalized, we move into the dressed states basis, rendering the equation of motion for the dressed states. Thus the real part of the eigenvalues tells us the decay rate of the dressed states, and the absolute value of the difference of the imaginary parts tells us the energy difference between the dressed states.

$$\begin{aligned} \Gamma_S = \text{Re}(\lambda_+) &= -\frac{\Gamma^{(11)} + \Gamma^{(22)}}{2} + \frac{\text{Re}(\sqrt{F(\Delta\omega, \Delta\gamma, r, \theta_1, \theta_2, \Delta\phi)})}{2}, \\ \Gamma_A = \text{Re}(\lambda_-) &= -\frac{\Gamma^{(11)} + \Gamma^{(22)}}{2} - \frac{\text{Re}(\sqrt{F(\Delta\omega, \Delta\gamma, r, \theta_1, \theta_2, \Delta\phi)})}{2}, \\ \delta_{total} &= |\text{Im}(\lambda_+ - \lambda_-)| = |\text{Im} \sqrt{F(\Delta\omega, \Delta\gamma, r, \theta_1, \theta_2, \Delta\phi)}|, \end{aligned} \quad (3.51)$$

where F is defined as in the classical model:

$$F(\Delta\omega, \Delta\gamma, r, \theta_1, \theta_2, \Delta\phi) = \left(-i\Delta\omega + \frac{\gamma_1 - \gamma_2}{2} \right)^2 + 4Q_{1 \rightarrow 2}(r, \theta_1, \theta_2, \Delta\phi)Q_{2 \rightarrow 1}(r, \theta_1, \theta_2, \Delta\phi). \quad (3.52)$$

We have now successfully developed a model that describes two dipoles interacting with each other in quantum mechanics. We can see that the behavior in the Heisenberg picture of the operators is the same as the behavior of the dipoles in the classical picture. The Heisenberg picture usually renders results that are similar to their classical analogous, which is what we achieved here. Therefore we can be confident that our results are correct.

4 Application to the protocol

Now that we have developed a model that describes the energy difference and decay rate of the dressed states, we would like to see what it can tell us about the photon-qubit entanglement protocol described in the introduction. We now have a superconducting qubit interacting with the system of dipoles. The superconducting qubit at hand has an oscillation frequency of about $5GHz$, and we estimate the

single atom decay rate of the dipoles as being about 20MHz . The atomic oscillation frequency ω_0 has to be in the range of the incoming photons, which are optical photons, so we estimate to be 5THz .

The plots shown in the preceding chapters assumed a ratio between ω_0 and γ in the order of magnitude 10^7 , because for this ratio one can fully appreciate the dynamics of the system. For the problem at hand we have a ratio in the order of 10^5 , and a suitable plot for this case can be seen in figure 6. The resonance condition implies that $\delta_{total} = \omega_q$, where ω_q is the oscillation frequency of the superconducting qubit. In practice, when applying the protocol, the only parameter that can be directly controlled is $\Delta\omega$, so this one must be tuned so that the resonance condition is fulfilled. The value of the energy shifting as a function of distance, angles and $\Delta\omega$ has been calculated in this thesis, and is given by δ_{total} in eq 3.51. In this thesis we will not derive a written expression for the required $\Delta\omega$ value to achieve resonance. Instead we have done numerical calculations for which value of $\Delta\omega$ is required for given values of $r, \theta_1, \theta_2, \Delta\phi$ to achieve resonance. The result for $\theta_1 = \theta_2 = \frac{\pi}{2}, \Delta\phi = 0$ can be seen in figure 7.

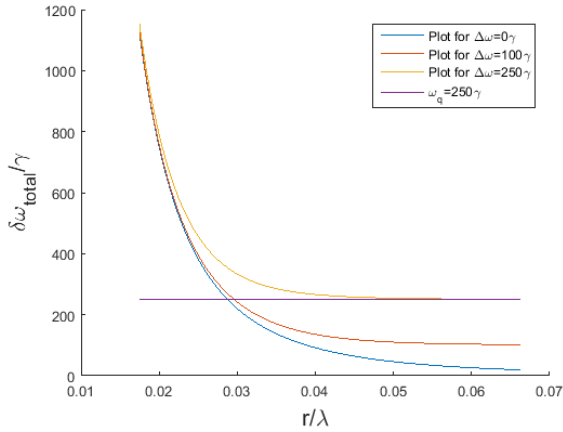


Figure 6: Energy difference of the dressed states as a function of the distance between the molecules. We use the assumptions mentioned in the main text, $\theta_1 = \theta_2 = \pi/2$ and $\Delta\phi = 0$

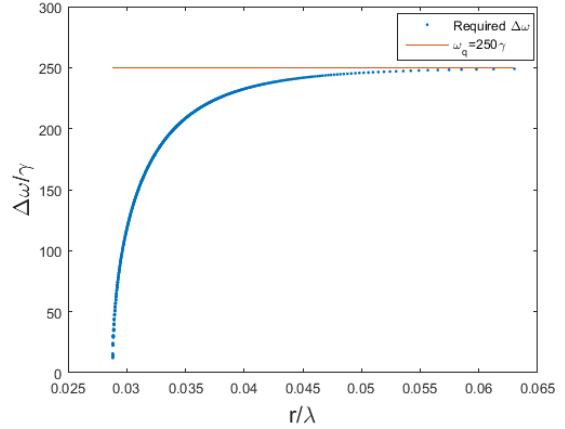


Figure 7: Plot of the required $\Delta\omega$ for the energy difference of the dressed states to be equal to the qubit energy $\theta_1 = \theta_2 = \pi/2$ and $\Delta\phi = 0$

After achieving resonance, we would like to know the probability for the Raman process. This is given by the equation:

$$P = \left(\frac{\gamma_{1D}}{\gamma} \right)^2 \frac{\Gamma_A \Gamma_S}{\gamma^2} \frac{G^2}{\frac{\Gamma_A^2 \Gamma_S^2}{\gamma^2} + G^2}, \quad (4.1)$$

which was obtained by private conversation with the authors of the protocol, its derivation will not be covered in this thesis. This equation assumes that the qubit and the dressed states are in resonance, which means $\delta_{total} = \omega_q$. We can calculate the decay rates Γ_S and Γ_A by using equation 3.51. The quantity $\left(\frac{\gamma_{1D}}{\gamma} \right)^2$ is a correction factor to the fact that the scattering happens in a one dimensional waveguide, not in three dimensions as we calculated for this thesis. The term G is the transition rates between the state $|S-\rangle$ to $|A+\rangle$. We will now find an expression for this transition rate. So far we have only looked at the problem of the two dipoles ignoring the superconducting qubit. The interaction Hamiltonian of the qubit and the molecules is given by

$$H_{mq}^I = \sum_{j=1}^2 \frac{\hbar g_j}{4} \hat{\tau}_x \otimes (\hat{\sigma}_z^{(j)} + \mathbb{I}), \quad (4.2)$$

where g_j is the coupling constant for the qubit with the dipoles [11], its value will not be calculated in this thesis. For simplicity we will approximate $g_2 = 0$, assuming that the qubit only couples to one of the molecules. If this term is added to the original Hamiltonian, the new system of Heisenberg equations of motion becomes

$$\begin{bmatrix} \dot{\tilde{\sigma}}_-^{(1)'} \\ \dot{\tilde{\sigma}}_-^{(2)'} \end{bmatrix} = \begin{bmatrix} i\frac{\Delta\omega}{2} - \frac{\Gamma^{(11)}}{2} & \frac{\Gamma^{(12)}}{2} + i\Omega^{(12)} \\ \frac{\Gamma^{(21)}}{2} + i\Omega^{(21)} & -i\frac{\Delta\omega}{2} - \frac{\Gamma^{(22)}}{2} \end{bmatrix} - i\hat{\tau}_x \begin{bmatrix} g_{c1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_-^{(1)'} \\ \tilde{\sigma}_-^{(2)'} \end{bmatrix}, \quad (4.3)$$

where we defined $\Omega^{(nm)} = \Omega_+^{(nm)} - \Omega_-^{(nm)}$. The expression for this quantity can be seen in equation 3.46. For later use, we rewrite equation 4.3 in the following way:

$$\dot{\sigma} = \left(\begin{bmatrix} -\frac{\Gamma^{(11)}}{2} & \frac{\Gamma^{(12)}}{2} \\ -\frac{\Gamma^{(21)}}{2} & -\frac{\Gamma^{(22)}}{2} \end{bmatrix} + i \begin{bmatrix} \frac{\Delta\omega}{2} & \Omega^{(12)} \\ \Omega^{(21)} & -\frac{\Delta\omega}{2} \end{bmatrix} + i\hat{\tau}_x \begin{bmatrix} g_{c1} & 0 \\ 0 & 0 \end{bmatrix} \right) \sigma. \quad (4.4)$$

We wish to find the transition rate from the dressed state $|S-\rangle$ to $|A+\rangle$. By bringing equation 4.3 into the eigenbasis of its first matrix (which we will denote as M matrix), we get the dynamics of the dressed states. In this basis the first matrix becomes a diagonal matrix, and the second becomes a matrix with four different terms that couple the dressed states by interacting with the qubit. The diagonalization of the M matrix can be seen in section 2.4. A problem arises though because the M matrix is not hermitian, so when moving to its eigenbasis, the terms that couple $|S-\rangle$ to $|A+\rangle$ are unclear. As we soon will see, this problem does not arise if we move to the eigenbasis of a hermitian matrix. Therefore we make the following approximation: instead of transforming to eigenbasis of the M matrix, we transform into the eigenbasis of the second matrix in 4.4 (which we denote by Ω matrix), which is hermitian under the approximation $\Omega^{(12)} = \Omega^{(21)} = \Omega$. Bringing the system into the eigenbasis of the Ω matrix we write

$$U^\dagger \dot{\sigma} = i \left(\begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} - U^\dagger \begin{bmatrix} -\frac{\Gamma^{(11)}}{2} & \frac{\Gamma^{(12)}}{2} \\ -\frac{\Gamma^{(21)}}{2} & -\frac{\Gamma^{(22)}}{2} \end{bmatrix} U - \hat{\tau}_x U^\dagger \begin{bmatrix} g_{c1} & 0 \\ 0 & 0 \end{bmatrix} U \right) U^\dagger \sigma, \quad (4.5)$$

where U and U^\dagger are the unitary transformation matrices. The coupling terms we are interested in is then given by the cross entrances of $U^\dagger \bar{g} U$, which we now calculate. First we wish to calculate the eigenvalues and the eigenvectors of the Ω matrix. It has the eigenvalues

$$\lambda_{\pm} = \pm \sqrt{\frac{\Delta\omega^2}{4} + \Omega^2}. \quad (4.6)$$

Defining the function $L_{\pm} = \frac{\Omega}{\lambda_{\pm} - \frac{\Delta\omega}{2}}$ the normed eigenvectors become

$$V_+ = \frac{1}{\sqrt{L_+^2 + 1}} \begin{bmatrix} L_+ \\ 1 \end{bmatrix}, V_- = \frac{1}{\sqrt{L_-^2 + 1}} \begin{bmatrix} L_- \\ 1 \end{bmatrix} \quad (4.7)$$

Which compose the unitary matrix U^\dagger . The g matrix in the eigenbasis becomes

$$U^\dagger \bar{g} U = g_{c1} \begin{bmatrix} \frac{L_+}{L_+^2 + 1} & \frac{L_+}{L_+^2 + 1} \\ \frac{L_-}{L_-^2 + 1} & \frac{1}{L_-^2 + 1} \end{bmatrix}. \quad (4.8)$$

The non diagonal entries of this matrix couple the dressed states $|A\rangle$ and $|S\rangle$ through an interaction with the qubit. It is still unclear which one specifically couples $|S-\rangle$ to $|A+\rangle$, but because both non diagonal terms are the same, by symmetry the coupling term must be:

$$G = g_{c1} \frac{L_+}{L_+^2 + 1}. \quad (4.9)$$

This quantity tells us the rate for the transition between the states $|S-\rangle$ and $|A+\rangle$. Now we have all the functions that describe the Raman transition probability in equation 4.1. In figures 8 and 9 we show plots of of this probability for different angle configurations and different distances between the dipoles. For every set of angles and distances it is assumed that the quantity $\Delta\omega$ is tuned so that the dipoles are in resonance with the qubit.

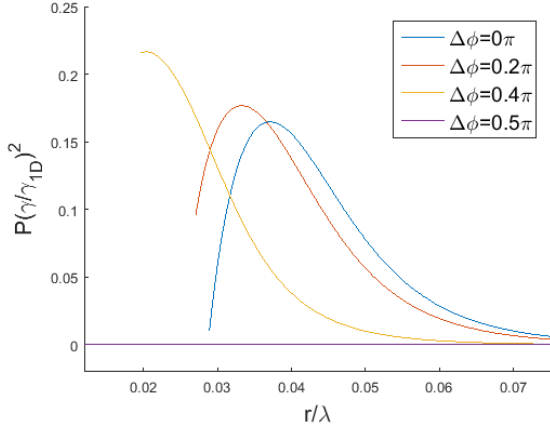


Figure 8: Scattering probability for the Raman process for different values of the difference in azimuthal angles $\Delta\phi$, the polar angles are $\theta_1 = \theta_2 = \pi/2$. We assumed the qubit coupling to be $g_1 = 4\gamma$. The regions in the left before the lines begin are regions where resonance cannot be achieved due to a too strong dipole-dipole interaction.

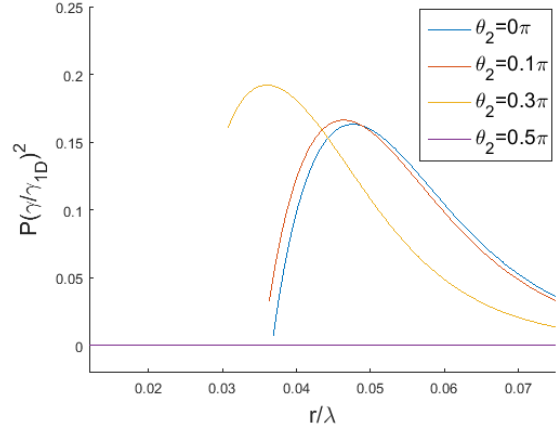


Figure 9: Scattering probability for the Raman process for different values of the polar angle θ_2 . The other angles are $\Delta\phi = 0$ and $\theta_1 = 0$. We assumed the qubit coupling to be $g_1 = 4\gamma$. The regions in the left before the lines begin are regions where resonance cannot be achieved due to a too strong dipole-dipole interaction.

We will finish with a brief discussion of the results presented in figures 8 and 9. We can see that for every set of angles there is an optimum value of r that maximizes the probability. The maximum probability is about 20% and falls quickly around both sides of the peak. There is also a minimum value of r required for the transition to be possible. Every distance below this minimum, results in a dipole-dipole interaction that is too strong, making resonance impossible.

We can see that in the angle configurations where there is no dipole interaction, ie. $(\theta_1, \theta_2, \Delta\phi) = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ and $(\theta_1, \theta_2, \Delta\phi) = (0, \frac{\pi}{2}, 0)$, the scattering probability is zero, which can be expected, as the coupling parameter becomes zero. On the other hand, the best probabilities are not achieved for the configurations where the dipole-dipole interaction is strongest. The reason is that even though they have the biggest coupling parameter, they also have the fastest decay rates, lowering the probabilities.

5 Conclusion

In this thesis we have seen a possible protocol that allows coupling of light to superconducting qubits without direct interaction, and we achieved a better understanding of the physics in the problem. We have derived a suitable model of the dipole-dipole interaction present in the protocol, and we have calculated its probability of success under different conditions.

We derived explicit expressions for the decay rates, frequency shifting and equations of motion of two oscillating dipoles interacting with each other by the electromagnetic field. We found that the decay rates of the two dipoles go towards 2 and 0 times the single dipole decay rate respectively as the distance between them goes to zero. On the other hand, as their distance goes towards infinity, the decay rate goes towards the original decay rate for both dipoles. They also create a shift in their oscillation frequency, which goes towards infinity for the distance going to zero, and towards their frequency difference as the distance goes to infinity. We found that the dynamics obtained by looking at the classical dynamics of two dipoles are very similar to the dynamics obtained by doing the same calculations with quantum mechanics when looking at the Heisenberg picture.

We applied the results to the given photon-qubit entanglement protocol. Using our model we could calculate numerically what frequency difference for the oscillation of the dipoles was necessary to achieve resonance condition. We found that Raman scattering is possible if the distance between the dipoles is in the range from 0.2λ to about 0.7λ , depending on the orientation of the dipoles. We also calculated the success probability once the resonance condition is achieved. Against expectations we found that angle configurations where the dipoles do not interact strongly give better probability curves than configurations with strong interactions. We also found that for an optimal configuration, approximately one of every five photons will undergo a Raman scattering that entangles them with the qubit. We can therefore conclude that from a theoretical perspective, this protocol is a feasible way to entangle photons and superconducting qubits, which might in the future open the possibility of communication between quantum computers.

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