THE EFFECT OF DEPHASING IN WAVEGUIDE QED

Application to a two-level single emitter

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Abstract

Recent experimental advances in nanophotonics has improved the ability of quantum emitters strongly coupled to waveguides to create strong light-matter interactions, which is of immense interest in fields like quantum computation and quantum communication. The effects of dephasing on the photon scattering from such interfaces are calculated for a two-level system, and an expression for the dephasing-induced noise function in the transmittance is derived. Dephasing is found to weaken the light-matter interactions significantly in this case. Finally, the first steps are taken towards extension to a quantum emitter with an arbitrary number of excited states.

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Introduction

Light-matter interactions, involving emission and absorption of photons at an atomic scale, is the cornerstone of quantum optics, and for this to be applicable, we need to be able to describe and control different types of light-matter interfaces.

In the field of quantum computing, the goal is to create quantum optical elements analogously to the electrical components of classical computers, and for this to be realized, there is a demand for good light-matter interfaces. A huge challenge here is that photons have a reluctancy to interact internally; such an interaction usually requires a huge number of photons. However, we are interested in strong interactions at the few-photon level for the system to exhibit quantum features. Single-photon interactions are desirable since they enable logic operations for quantum information technology. Many optical devices rely on this, e.g. photon-photon gates, single-photon transistors and switches, which can be used for optical quantum information processing [1].

When a quantum emitter is coupled to an optical waveguide which guides the direction of propagation and slow down the photons, it opens up for enhancement of the interactions between the field and the emitter [2], which can be a huge advantage for applications within quantum networks and communication, among other fields. Emitters coupled to optical waveguides are of great interest for quantum communication over long distances, since this can only be accomplished through optical media [3]. Photons emitted into a waveguide also has a possible application as propagating ("flying") qubits which can connect stationary qubits in a quantum network [4]. In the past, cavity QED was usually implemented in quantum networks, but waveguides have the advantage that they can provide efficient out-coupling of light. Furthermore, they have shown to be promising as a key ingredient for scaling up the devices involved [5] while maintaining the quantum properties, which is a fundamental challenge for the implementation of quantum networks.

Strong nonlinear interactions at the single-photon level can be obtained by sending low-intensity light through a nonlinear medium. Different ways of obtaining the strongest possible confinement of the field has been explored in this case, including plasmons propagating on a conducting nanowire [1] and graphene plasmons [6].

Another application is transferring of the quantum state, which is essential for quantum information processing and can be used for quantum simulation of many-body physics [7].

The field of quantum information processing has emerged over the past few years, for which emitters coupled to waveguides have shown to be promising; this is a very big area of research in modern quantum optics. The given light-matter interface is probed by sending light through the waveguide and studying how it scatters as a response, so it's crucial to be able to calculate the transmission and reflection of the field.

A formalism for describing the photon scattering from emitters coupled to a 1D-waveguide was developed recently [8], which can be applied to an arbitrary number of emitters with different level structures, providing solutions for low-intensity input fields. Dissipative dynamics plays a big role in quantum optics, and particularly spontaneous emission, which gives all excited states a finite lifetime, needs to be taken into consideration. Another common problem for physical implementations of waveguide QED is dephasing (decoherence). Many systems are dependent on coherence, for example quantum processors, where the current solution is to isolate and cool the system to the order of mK. The formalism in [8] accounts for decay of the excited states via a non-Hermitian term in the Hamiltonian, but it has not included the effect of dephasing.

This project will be concerned with how to include dephasing in such a formalism, and how dephasing affects the scattering of light in a one-dimensional waveguide coupled to an emitter. Dephasing will be included for a single two-level atom in a 1D-waveguide, and the first steps towards an extension to a multilevel emitter will be taken.

1 Theory

1.1 Atomic operators and their time evolution

In the basic theory of quantum mechanics, isolated systems are described by state vectors $|\Psi\rangle = \sum_i c_i |i\rangle$ satisfying Schrödingers Equation together with $\sum_i |c_i|^2 = 1$ and $\langle i|j\rangle = \delta_{ij}$ [9]. This is the description that contains the most information about the system. However, if the system is open, interacting with its environment, the exact state of the system cannot be known. This is exactly the case in quantum optics, which is much concerned with the interactions of light with matter. For the case of weak interactions, it is of course possible to apply pertubation theory, but if one is interested in various coupling strengths, a common alternative is to use the density operator, which for a pure state is defined as

$$\hat{\rho} = |\Psi\rangle \langle\Psi| = \sum_{i,j} \rho_{ij} |i\rangle \langle j| , \quad \rho_{ij} = c_i c_j^* \tag{1.1}$$

and has the special properties

$$\mathrm{Tr}(\hat{\rho}) = 1 \qquad\qquad \langle\hat{A}\rangle = \mathrm{Tr}(\hat{A}\hat{\rho}) \tag{1.2}$$

where $\hat{A}$ is an arbitrary operator [10]. Since the trace is cyclic, the relation above is very powerful, since it allows us to evaluate the expectation value independently of choice of basis.

Now, let the system be an atom, the matrix elements of the density operator will contain the state population probabilities in the diagonal and the coherences in the off-diagonal. The systems we will be concerned with during the following chapters will be single atoms with a ground state and one or multiple excited states, for which we will find the equations of motion. For an atom with ground state $|g\rangle$ and excited state $|e\rangle$ we can define the atomic transition operators $\hat{\sigma}_{ij} \equiv |i\rangle \langle j|$, whose expectation values are the density matrix elements, $\langle\hat{\sigma}_{ij}\rangle = \rho_{ji}$. From now on the more intuitive notation $\hat{\sigma}_{(-)+} = |(g)e\rangle \langle(e)g|$ will be used for describing (de)excitations of the atom.

Note that the time evolution of the density operator is

$$\frac{\partial\hat{\rho}}{\partial t} = \left(\frac{\partial}{\partial t}|\Psi\rangle\right)\langle\Psi| + |\Psi\rangle\left(\frac{\partial}{\partial t}\langle\Psi|\right) = -\frac{i}{\hbar}\left(\hat{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\hat{H}\right) = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] \tag{1.3}$$

while the time evolution of the transition operators (along with any arbitrary operator $\hat{A}$ in general) are governed by the Heisenberg equation

$$\frac{\partial\hat{\sigma}_\pm}{\partial t} = \frac{i}{\hbar}[\hat{H},\hat{\sigma}_\pm] \tag{1.4}$$

The sign reversal arises from the density operators describing time evolution in the state vectors, propagating with the total Hamiltonian, because of which we are necessarily in the Schrödinger picture.

1.2 Light-matter interactions and the quantization of light

If an atom sits in free space, it can be treated as an isolated system with the total Hamiltonian containing only the terms of the states with their respective energies. When a radiation field is present, the atom couples to the electric and magnetic field components of the light via its electric dipole moment and its spin magnetic moment, respectively. However, the strength of the magnetic field is $1/c$ of the electric field strength, which results in the former to be safely neglected throughout the following work. Thus, the field results in a coupling of the atom to the electric field, which is described via the potential $\hat{H}^I = -\hat{\mathbf{d}} \cdot \mathbf{E}$, where $\hat{\mathbf{d}}$ is the dipole moment operator $\hat{\mathbf{d}} = d_0(\hat{\sigma}_+ + \hat{\sigma}_-)$, and $\mathbf{E}$ is the electric field [11].

Let's start defining a classical electromagnetic field which satisfies Maxwell's equations without sources. Because of the convenience of boundary conditions, we consider a one-dimensional cavity extending from $z = -L/2$ to $z = L/2$. For a single-mode EM-field propagating with wavevector $\mathbf{k}{=}k\hat{\mathbf{z}}$ inside this cavity, we have a possible solution for the E-field, satisfying $\nabla \cdot \mathbf{E} = 0$,

$$\mathbf{E}(z,t) = C\dot{A}(t)\text{cos}(kz)\hat{\mathbf{x}}, \tag{1.5}$$

where $\dot{A}(t)$ is a term collecting the time-dependency of the E-field, $k = (2n+1)\pi/L$ and C is some constant. Using Faraday's equation, we obtain

$$\dot{\mathbf{B}} = -\nabla \times \mathbf{E} = Ck\dot{A}(t)\text{sin}(kz)\hat{\mathbf{y}} \Rightarrow \mathbf{B}(z,t) = CkA(t)\text{sin}(kz)\hat{\mathbf{y}} \tag{1.6}$$

$$\dot{\mathbf{E}} = c^2\nabla \times \mathbf{B} = -c^2k^2CA(t)\text{cos}(kz) \Rightarrow A(t) = A_0\text{sin}(\pm\omega t), \tag{1.7}$$

where we have used the vacuum dispersion relation $\omega = ck$. As a step towards the quantization of the fields, we evaluate the energy W of the field inside the cavity, which we can relate to the Hamiltonian of a quantized field. We integrate the energy density of the field over the volume of the cavity

$$W = \frac{V}{2L}\int_{-L/2}^{L/2} dz \left(\epsilon_0\mathbf{E}^2 + \mu_0^{-1}\mathbf{B}^2\right) = \frac{V}{2L}C^2\int_{-L/2}^{L/2} dz \left(\epsilon_0\dot{A}^2(t)\text{cos}^2(kz) + \frac{k^2}{\mu_0}A^2(t)\text{sin}^2(kz)\right), \tag{1.8}$$

which leaves us with

$$W = \frac{\epsilon_0 V}{4}C^2\left(\dot{A}^2(t) + \omega^2A^2(t)\right) \tag{1.9}$$

We can try to relate this to the Hamiltonian for a harmonic oscillator with position q and momentum $p = m\dot{q}$,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 = \frac{m}{2}\left(\dot{\hat{q}}^2 + \omega^2\hat{q}^2\right) \tag{1.10}$$

Defining $m = \epsilon_0 VC^2/2$, we end up with the exact same equations $W \sim \hat{H}$, where $A(t) \sim \hat{q}$, apart from $\hat{H}$ being an operator equation. This tells us that a classical field mode can be associated with a quantum harmonic oscillator and motivates us to quantize the the classical electromagnetic field by canonical quantization, introducing $[\hat{q},\hat{p}] = i\hbar$.

We introduce the ladder operators known from the solution to the quantum harmonic oscillator,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i\hat{p}}{m\omega}\right) \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i\hat{p}}{m\omega}\right) \tag{1.11}$$

with the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$.

If we look again at equations 1.5 and 1.6, now with $A \rightarrow \hat{q}$, we can substitute the canonical conjugate quantities $\hat{q}$ and $\hat{p}$ by $\hat{a}$ and $\hat{a}^\dagger$;

$$\hat{a} - \hat{a}^\dagger = i\sqrt{\frac{2m}{\hbar\omega}}\dot{\hat{q}} \Rightarrow \hat{\mathbf{E}} = -iC\sqrt{\frac{\hbar\omega}{2m}}(\hat{a} - \hat{a}^\dagger)\text{cos}(kz)\hat{\mathbf{x}} = -i\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}(\hat{a} - \hat{a}^\dagger)\text{cos}(kz)\hat{\mathbf{x}} \tag{1.12}$$

$$\hat{a} + \hat{a}^\dagger = \sqrt{\frac{2m\omega}{\hbar}}\hat{q} \Rightarrow \hat{\mathbf{B}} = \frac{1}{c}\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}(\hat{a} + \hat{a}^\dagger)\text{sin}(kz)\hat{\mathbf{y}} \tag{1.13}$$

Evaluating the product of the ladder operators

$$\hat{a}^\dagger\hat{a} = \frac{1}{2\hbar}\left(\frac{\hat{p}^2}{m\omega} + m\omega\hat{q}^2\right) + \frac{i}{2\hbar}[\hat{q},\hat{p}] = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}, \tag{1.14}$$

we can express the Hamiltonian of the quantized field as $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + 1/2\right)$.

Defining an eigenstate $|n\rangle$ of the field Hamiltonian with eigenvalue n, such that $\hat{H}|n\rangle = E_n|n\rangle$, we can show that

$$\hat{a}^\dagger\hat{H}|n\rangle = E_n\hat{a}^\dagger|n\rangle \Rightarrow \hat{H}\hat{a}^\dagger|n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger|n\rangle \tag{1.15}$$

We see that the eigenvalue of state $\hat{a}^\dagger|n\rangle$ is $(E_n + \hbar\omega)$, which shows us that $\hat{a}^\dagger$ "creates" an energy quantum $\hbar\omega$ and consequently appropiately referred to a the creation operator, while on the other hand $\hat{a}$ is the annihilation operator. Because $|n\rangle$ are eigenstates of $\hat{H}$, we're motivated to define the product of the creation/annihilation operators as the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$, for which the number states $|n\rangle$ are eigenstates, where n represents the number of photons in the given state.

The derivations above were based on a single-mode field, and this will also be the only case of interest in the following chapters. Both the semiclassical and the fully quantized model will be considered interacting with an emitter in a one-dimensional waveguide which, first of all, restricts the field to one dimensional propagation. The waveguide can be designed so that it supports only a single field mode of the field, and can be treated as a single-mode field when calculating the dynamics. Similarily, the waveguide has to be able to support the given polarization of the emitted photons.

A quantized electromagnetic field propagating in the z-direction couples to the emitter in the same way as the classical field; for an E-field with the form of eq. 1.12,

$$\hat{H}^I = -\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} = -g(\hat{a} - \hat{a}^\dagger)\hat{\mathbf{d}}\cdot\hat{\mathbf{x}}, \tag{1.16}$$

where $g = i\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\cos(kz)$ is the coupling constant. Compared to the case of a classical field, the resulting Hamiltonian now has an additional contribution $\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}} + \hat{H}_{\text{interaction}} = \hat{H}_{\text{atom}} + \hbar\omega\hat{a}^\dagger\hat{a} + d_0 g(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)$.

The theory of quantized electromagnetic fields has the important difference from the semiclassical theory that it can explain transitions in between the states of an atom in free space. If we look at the wavefunction for an isolated two-level system, the ground- and excited states don't overlap and hence a transition is only possible when coupling a quantized field to the atom, which leads us into the field of quantum electrodynamics (QED).

In practise, it is not possible to isolate the system completely from quantized field modes. Even in vacuum, the ground state of the number state, the energy is $\hbar\omega/2$, just as the ground state of the harmonic oscillator - which is known as the zero-point energy. Because of this, the variance of the quantized E-field is non-zero as well; these are the so-called vacuum fluctuations which give rise to several effects, of which the most important might be spontaneous emission, where the atom decays without necessarily an applied field present, but by interaction with surrounding field modes of the environment.

The magnitude of the vacuum field oscillations at frequencies close to the transition frequency of the atom is on the order of 10^6 smaller than the strength of the electric binding of the electron to the nucleus. For the vacuum field oscillations to have an impact, it then has to act for around a million cycles, which matches a typical lifetime of the excited state ($\sim$ ns) [12].

1.3 Dephasing and the master equation

A pure state $|\Psi\rangle = \sum_i c_i|i\rangle$ is a coherent superposition of basis states $|i\rangle$; there exists a definite phase relation between different states. However, this coherence is fragile. If the system is not completely isolated, it will be subject to interactions with the environment. As mentioned above, the interaction of an atom with photons in the environment causes spontaneous emission where the atom emits a photon and decays, thus affecting the populations. The coupling to the environment leads to generation of entanglement between the system and environment, which results in an exchange of information and will cause decoherence so that the pure state ultimately transforms into a classical statistical mixture. The information about the phase of the superposition is lost to the environment, which is what is known as dephasing. In terms of the density matrix, dephasing corresponds to a decay of the off-diagonal elements, while spontaneous emission corresponds to a change in the populations for the excited to the ground state. The sources can be e.g. nuclear spins, magnetic fields, electical or mechanical vibrations (phonons), which can kick the excited state out of its equilibrium, causing the energy level to fluctuate. These effects are weaker than the photon interactions, but still has a significant impact for systems which are dependent on the coherence. An example of such a system are qubits. The state of a two-level system, defined as $|\Psi\rangle = c_0 |0\rangle + c_1 |1\rangle$, is what forms a qubit, and in order for quantum computation to be realized it is crucial to preserve its coherence. Hence, quantum computation is a research area where it is of great interest to understand the effect of dephasing in the system.

Dephasing occurs because the system is not completely isolated, but on the other hand, a system in complete isolation is not desirable, since we can gain no information about the system itself. Therefore, dephasing will always be a challenge for any quantum system of interest.

Weisskopf and Wigner proved that the interaction of matter with quantized field modes gives rise to an irreversible exponential decay rate of the excited state population [13]. Now we're interested in including these dissipative dynamics of the system to the total dynamics of our density matrix. We can use the quantum jump description, where spontaneous emission leads to a sudden energy "jump" to the ground state. The decay is then described by adding a jump operator, better known as a Lindblad operator, $\mathscr{L} = \sqrt{\gamma}\hat{\sigma}_-$ to equation 1.3. Similarily, dephasing can be included as $\mathscr{L}_{\not{d}} = \sqrt{\gamma_d}\hat{\sigma}_{ee}$. In the same way, all the contributions to the non-Hermiticity of the Hamiltonian can be contained in Lindblad operators, and the new equation of motion reads

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{1}{2}\sum_m \{\mathscr{L}_m^\dagger \mathscr{L}_m, \hat{\rho}\} + \sum_m \mathscr{L}_m \hat{\rho} \mathscr{L}_m^\dagger, \tag{1.17}$$

which is known as the *master equation* [13], and the non-Hermitian terms are frequently referred to as the relaxation superoperator. The master equation approach is widely used in quantum optics, since it makes a good approximation to a linear equation of motion for conditions often fulfilled by quantum optical systems. When applying the relevant terms to the master equation and evaluating it, it yields Langevin-Bloch equations which are the equations of motion for the density matrix elements.

2 Dynamics of the semiclassical model

For the sake of increasing complexity and to illustrate the differences and similarities between the dynamics of the semiclassical and the QED model, we start finding and solving the Langevin-Bloch equations semiclassically. In subchapter 2.2 we find the time-dependent solutions to these equations and use them to watch the Rabi oscillations arising for a sufficiently strong field.

2.1 Langevin-Bloch equations and their steady state solutions

We consider the simplest case possible; a two-level atom with a ground state $|g\rangle$ and an excited state $|e\rangle$ in a classical radiation field. We can view the Hamiltonian as the sum of the Hamiltonian of the atom in isolation and the Hamiltonian arising from the interaction of the atom with the field, such that $\hat{H} = \hat{H}_{\text{atom}} + \hat{H}^I$. The transition frequency between the levels is $\omega_{eg} = \omega_e - \omega_g$, so we define $\hat{H}_{\text{atom}} = \hbar\omega_{eg} |e\rangle \langle e|$.

Confined to the z-direction by a 1D-waveguide, we can combine equations 1.5 and 1.7 to form $\mathbf{E}(z,t) = \pm\omega C A_0 \cos(kz \pm \omega t)\hat{\mathbf{x}} = \mathbf{E_0}\cos(kz \pm \omega t)$. Defining the frequency of light $\omega = \omega_0$, the E-field moving to the right is given by $\mathbf{E}(\mathbf{z},t) = \mathbf{E_0}\cos(kz - \omega_0 t)$, where $k = \frac{2\pi}{\lambda}$, specifies the mode with wavelength λ supported by the waveguide. Since the length scale of the atom is much smaller than the wavelength of optical radiation, $kz \ll 1$ in the area of interest, and we can approximate the electric field to be constant over the extent of the atom at a given time, which is known as the dipole approximation [11]. Thus we make the approximation for the E-field appearing in the system Hamiltonian $\mathbf{E}(z,t) \approx \mathbf{E}(t) = \mathbf{E_0}\cos(\omega_0 t)$.

Now, for the interaction we get the term

$$\hat{H}^I = -\hat{\mathbf{d}} \cdot \mathbf{E} = -\frac{E_0 d_0}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})(\hat{\sigma}_+ + \hat{\sigma}_-), \tag{2.1}$$

which contains two terms describing the situations "*excitation* by *emitting* a photon" and vice versa. Motivated by energy conservation, we discard these two terms and end up with the total Hamiltonian

$$\hat{H} = \hbar\omega_{eg}\hat{\sigma}_{ee} - \frac{E_0 d_0}{2}(e^{-i\omega_0 t}\hat{\sigma}_+ + e^{i\omega_0 t}\hat{\sigma}_-) \tag{2.2}$$

We apply the Lindblad description of decay and dephasing by including the two Lindblad terms $\mathcal{L} = \sqrt{\gamma}\hat{\sigma}_-$ for decay. For the case of dephasing, the excited state interacts with the environment and "jumps" back to the excited state, which results in its respective Lindblad operator $\mathcal{L}_{\mathscr{d}} = \sqrt{\gamma_d}\hat{\sigma}_{ee}$.

Using the master equation (eq. 1.17), this leads to the equations of motion for the density operator; the so-called Langevin-Bloch equations

$$\dot{\rho}_{ee} = -\gamma\rho_{ee} + \frac{i}{2}\Omega(e^{-i\omega_0 t}\rho_{ge} - e^{i\omega_0 t}\rho_{eg}) \tag{2.3}$$

$$\dot{\rho}_{gg} = \gamma\rho_{ee} - \frac{i}{2}\Omega(e^{-i\omega_0 t}\rho_{ge} - e^{i\omega_0 t}\rho_{eg}) \tag{2.4}$$

$$\dot{\rho}_{eg} = [-i\omega_{eg} - \frac{1}{2}(\gamma + \gamma_d)]\rho_{eg} + \frac{i}{2}\Omega e^{-i\omega_0 t}(\rho_{ee} - \rho_{gg}) \tag{2.5}$$

$$\dot{\rho}_{ge} = [i\omega_{eg} - \frac{1}{2}(\gamma + \gamma_d)]\rho_{ge} - \frac{i}{2}\Omega e^{i\omega_0 t}(\rho_{ee} - \rho_{gg}) \tag{2.6}$$

where $\Omega = E_0 d_0/\hbar$ is the standard definition of the Rabi frequency arising from the interaction.

In order to have more neatly appearing equations we transform to a frame rotating with the field frequency ω_0. We do not need to make a unitary transformation, but can simply transform the density matrix terms individually. Getting rid of the exponential terms also has a practical purpose, since the field typically oscillates so rapidly that it's at least hard to measure in the laboratory. In our new frame, $\rho'_{eg} = \rho_{eg}e^{i\omega_0 t}$, so $\dot{\rho}_{eg} = \dot{\rho}'_{eg}e^{-i\omega_0 t} - i\omega_0\rho'_{eg}e^{-i\omega_0 t}$. Substituting this relation into the above equations and finally dropping the primes, we get

$$\dot{\rho}_{ee} = -\gamma\rho_{ee} + \frac{i}{2}\Omega(\rho_{ge} - \rho_{eg}) \tag{2.7}$$

$$\dot{\rho}_{gg} = \gamma\rho_{ee} - \frac{i}{2}\Omega(\rho_{ge} - \rho_{eg}) \tag{2.8}$$

$$\dot{\rho}_{eg} = [-i\Delta - \frac{1}{2}(\gamma + \gamma_d)]\rho_{eg} - \frac{i}{2}\Omega(\rho_{ee} - \rho_{gg}) \tag{2.9}$$

$$\dot{\rho}_{ge} = [i\Delta - \frac{1}{2}(\gamma + \gamma_d)]\rho_{ge} + \frac{i}{2}\Omega(\rho_{ee} - \rho_{gg}) \tag{2.10}$$

where $\Delta \equiv \omega_{eg} - \omega_0$ is the detuning.

Exploiting that $\mathrm{Tr}(\hat{\rho}) = 1$, the four coupled Langevin-Bloch equations can be reduced to three variables since $\rho_{gg} = 1 - \rho_{ee}$ and $\dot{\rho}_{gg} = -\dot{\rho}_{ee}$. Solving for steady state, we can show

$$\rho_{eg} = \frac{\gamma\Omega(2\Delta + i(\gamma+\gamma_d))}{\gamma(\gamma+\gamma_d)^2 + 2\Omega^2(\gamma+\gamma_d) + 4\gamma\Delta^2} \tag{2.11}$$

If we assume $\gamma_d \ll \gamma$, we arrive at the steady state solutions, which we will need in chapter 3

$$\rho_{ee}^{s.s.} = \frac{\Omega^2}{2\Omega^2 + \gamma(\gamma+\gamma_d) + 4\Delta^2} \tag{2.12}$$

$$\rho_{gg}^{s.s.} = \frac{\Omega^2 + \gamma(\gamma+\gamma_d) + 4\Delta^2}{2\Omega^2 + \gamma(\gamma+\gamma_d) + 4\Delta^2} \tag{2.13}$$

$$\rho_{eg}^{s.s.} = \frac{\Omega(2\Delta + i\gamma)}{2\Omega^2 + \gamma(\gamma+\gamma_d) + 4\Delta^2} \tag{2.14}$$

We now find the steady-state solutions in the weak driving limit, since this will prove useful when we reach chapter 4. With a weak input field, we can approximate 2.11 using $\Omega^2 \approx 0$ and obtain the equations,

$$\rho_{eg,wd}^{s.s.} = \frac{i\Omega}{\gamma + \gamma_d + 2i\Delta} \tag{2.15}$$

$$\rho_{ee,wd}^{s.s.} = \frac{i\Omega}{2\gamma}(\rho_{ge,wd}^{s.s.} - \rho_{eg,wd}^{s.s.}) = \left(1 + \frac{\gamma_d}{\gamma}\right)\frac{\Omega^2}{4\Delta^2 + (\gamma+\gamma_d)^2} \tag{2.16}$$

Note that we have the relation $\rho_{ee} = (1 + \gamma_d/\gamma)\rho_{eg}\rho_{ge}$ for steady state and weak driving. Say the system starts in the ground state, $\rho_{gg}(0) = 1$, for a weak input field the populations can be approximated as constant so that $\rho_{gg}(t) \approx 1$ and $\rho_{ee}(t) \approx 0$, which also implies $\rho_{ee} \approx \rho_{eg}\rho_{ge}$. However, as we found above, dephasing leads to an additional term $(\gamma_d/\gamma)\rho_{eg}\rho_{ge}$.

2.2 Time-dependent solutions and Rabi oscillations

In order to find the time-dependent solutions to equations 2.7-2.10, it is convenient to change variables to $v = i(\rho_{eg} - \rho_{ge})$ and $w = \rho_{ee} - \rho_{gg}$, which is valid for $\Delta = 0$ [14]. The Langevin-Bloch equations yields, in terms of the new variables, the second-order differential equation

$$\ddot{w}(t) + \gamma\dot{w} + \Omega^2 w = -\frac{\Omega}{2}(\gamma+\gamma_d)v(t) - \gamma, \tag{2.17}$$

which leads to the time-dependent solutions at exact resonance for a field turned on at $t = 0$,

$$\rho_{gg}(t) = -\frac{\Omega^2}{2\Omega^2 + \gamma(\gamma+\gamma_d)}\left[-\frac{1}{2}\left(\frac{\gamma+\gamma_d}{\Omega} + 1\right)\left(e^{A^+t} + e^{A^-t}\right) + \frac{\gamma+\gamma_d}{2\Omega}\left(e^{B^+t} + e^{B^-t}\right) - \frac{\gamma(\gamma+\gamma_d)}{\Omega^2} - 1\right] \tag{2.18}$$

$$\rho_{ee}(t) = \frac{\Omega^2}{2\Omega^2 + \gamma(\gamma+\gamma_d)}\left[-\frac{1}{2}\left(\frac{\gamma+\gamma_d}{\Omega} + 1\right)\left(e^{A^+t} + e^{A^-t}\right) + \frac{\gamma+\gamma_d}{2\Omega}\left(e^{B^+t} + e^{B^-t}\right) + 1\right] \tag{2.19}$$

$$\rho_{eg}(t) = -\frac{-i\Omega^2}{2\Omega^2 + \gamma(\gamma+\gamma_d)}\left[\frac{\gamma}{2\Omega}\left(\frac{\gamma+\gamma_d}{\Omega} + 1\right)\left(e^{A^+t} + e^{A^-t}\right) + e^{B^+t} + e^{B^-t} + \frac{\gamma^2 - \gamma\Omega^2}{\Omega^3}\right] \tag{2.20}$$

$$A^{\pm} = \frac{1}{2}\left(-\gamma \pm \sqrt{-4\Omega^2 + \gamma^2}\right) \qquad B^{\pm} = \frac{1}{4}\left(-(\gamma+\gamma_d) \pm \sqrt{-16\Omega^2 + (\gamma+\gamma_d)^2}\right)$$

where $-\frac{1}{2}((\gamma+\gamma_d)/\Omega + 1)$ is a normalization factor.

For $\Omega > \gamma/2$ and $\gamma_d < \gamma$, the populations exhibit sinosoidal behavior and we see what is known as Rabi oscillations (fig. 1). Here, the time-dependent field acts periodically as a source or sink of energy for the two-level atom, while the system undergoes cycles of absorption followed by emission as it absorbs energy from the field to make transitions to the excited state until the process is reversed at energy excess. At resonance the *entire* population is undergoing a transition at each half-cycle, and the transistion probability oscillates between 0 and 1. Since this requires a high-frequent field, this is a semiclassical concept. There does exist a quantum parallel to this behavior in cavity QED, where the power can grow large enough for Rabi oscillations to occur even for an emitter in vacuum, but here we will not deal with cavity QED. If we could isolate this system completely, these oscillations would go on forever, but with decay included, we observe the envelope of the population oscillations to decay exponentially at rate γ. Thus, the excited state population decays with γ while the ground state population grows at a corresponding rate, until the populations find an equilibrium approximately equally devided between the two states.

We can think of the field hitting an ensemble of atoms, where the pattern can be seen as the interference of the populations of each of the atoms. Since spontaneous emission happens at random points in time, the populations get out of phase and we end up observing only the average population.

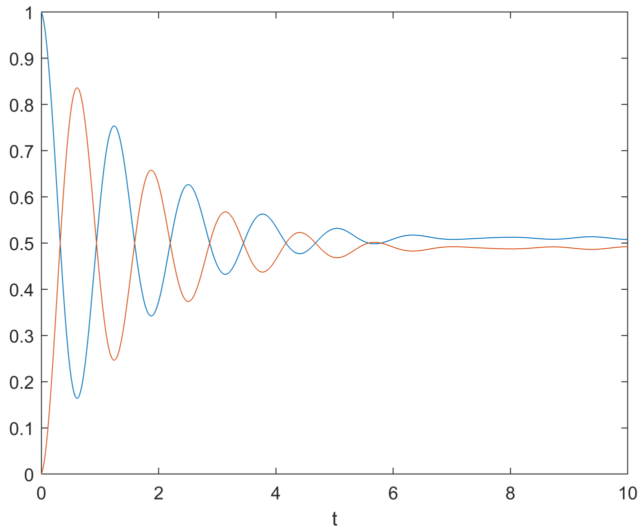

Figure 1: The solutions for $\rho_{gg}(t)$ (blue) and $\rho_{ee}(t)$ (red) with $\Omega = 5$, $\gamma = 1$ and $\gamma_d = 0.1$ for a two-level atom with decay and dephasing in the semiclassical model.

3 Solving the scattering problem in a fully quantized model

We now apply the results to a fully quantum mechanical model as the one described in [1]. In this article, the transmittance and reflectance is calculated for plasmons propagating on a nanowire, but the method applies equally well for two-level atom in a waveguide. The article doesn't include dephasing, so in this chapter we will go through the same calculations and find the dephasing correction to the model. The model is described by the Hamiltonian

$$\hat{H} = \hbar(\omega_{eg} - i\Gamma'/2)\hat{\sigma}_{ee} + \int dk\hbar c|k|\hat{a}_k^\dagger \hat{a}_k - \hbar g \int dk(\hat{\sigma}_+ \hat{a}_k e^{ikz_a} + \hat{\sigma}_- \hat{a}_k^\dagger e^{-ikz_a}), \tag{3.1}$$

where the first term is a non-hermitian term with Γ' representing decay outside the waveguide, g is the plasmon-emitter coupling, and z_a is the position of the emitter along the z-axis.

Considering near-resonance situations only, the E-field can be separated into two independent fields, since there is a near-unit probability for the atom to absorb an incoming photon which is reemitted to either the left or right (if not to unsupported modes). With this assumption, the field is defined as

$$\hat{E}_{L(R)}(z) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikz_a} \hat{a}_{L(R),k}, \tag{3.2}$$

where subscipt $L(R)$ represents the left(right)-going field, and the electric field operators satisfy the commutation relations

$$[\hat{E}_R(z), \hat{E}_R^\dagger(z_a)] = [\hat{E}_L(z), \hat{E}_L^\dagger(z_a)] = \delta(z - z_a) \tag{3.3}$$

$$[\hat{E}_R(z), \hat{E}_L^\dagger(z_a)] = [\hat{E}_L(z), \hat{E}_R^\dagger(z_a)] = 0 \tag{3.4}$$

The input field is a coherent state, defined as the displacement operator D acting on vacuum at initial time, where the atom is in the ground state,

$$|\tilde{\psi}(t \to -\infty)\rangle = D(\{\alpha_k e^{-i\nu_k t}\}) |vac\rangle |g\rangle \tag{3.5}$$

Now the following transformation is made

$$|\tilde{\psi}\rangle = D(\{\alpha_k e^{-i\nu_k t}\}) |\psi\rangle \Rightarrow |\psi(t \to -\infty)\rangle = |vac\rangle |g\rangle \tag{3.6}$$

so, with an input field coming from the left, the right-going E-field transforms as $\hat{E}_R \to \hat{E}_R + \mathcal{E}_c$, where $\mathcal{E}_c$ is the external field amplitude

$$\mathcal{E}_c = \frac{1}{\sqrt{2\pi}} \int dk \alpha_k e^{ikz_a} \tag{3.7}$$

Note that $\mathcal{E}_c$ is an eigenvalue for $\hat{E}$ acting on the coherent state, but enhanced with a factor 2 after the transformation, i.e.

$$\hat{E}_R |\alpha\rangle = \mathcal{E}_c |\alpha\rangle \qquad\qquad \hat{\hat{E}}_R |\alpha\rangle = (\hat{E}_R + \mathcal{E}_c) |\alpha\rangle = 2\mathcal{E}_c |\alpha\rangle \tag{3.8}$$

If we look at the interaction Hamiltonian only, i.e. the last term of eq. 3.1, this can be rewritten as

$$\hat{H}^I = -\hbar\sqrt{2\pi} g \hat{E}(\hat{\sigma}_+ + \hat{\sigma}_-) = -\hbar \frac{\sqrt{2\pi} g}{d_0} \hat{d} \cdot \hat{E} \tag{3.9}$$

$$\hat{H}^I |\alpha\rangle = -\hbar\sqrt{2\pi} g \mathcal{E}_c (\hat{\sigma}_+ + \hat{\sigma}_-) |\alpha\rangle \tag{3.10}$$

as defined before the transformation of the E-field. Comparing this to the interaction term of the Hamiltonian in equation 2.2, which can be rewritten as $\hat{H}^I = -\hbar\frac{\Omega}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)$ in the Schrödinger picture, we see that the interaction terms of the Hamiltonians in the two models are very alike. It comes naturally to define the quantum Rabi frequency $\Omega_c = \sqrt{2\pi} g \mathcal{E}_c$ and obtaining the relation $\Omega = 2\Omega_c$, in which case the interaction terms are equal for the two models. We will use Ω without the subscipt as classical Rabi frequency.

After this, the Heisenberg equations of motion can be applied, and we obtain the wave equation

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial z}\right) E(z) = i\sqrt{2\pi} g \delta(z - z_a)\hat{\sigma}_-, \tag{3.11}$$

which has the solution for the right-going E-field

$$\hat{E}_R(z,t) = \hat{E}_{R,free}(z - ct) + i\sqrt{2\pi}\frac{g}{c}\sigma_-(t - (z - z_a)/c)\theta(z - z_a), \tag{3.12}$$

where $\hat{E}_{R,free}$ is the field at initial time (before reaching the atom) and $\theta(z)$ is a step function representing measurement to the right of the atom.

3.1 Steady-state transmittance

Now we will use first-order correlation functions [15] for $\hat{E}_{L(R)}$ in order to find the steady-state transmittance (reflectance) at resonance ($\Delta = 0$).

For the right-going field, the first-order correlation function is

$$G^1 = \left\langle (\hat{E}_R^\dagger + \mathcal{E}_c^*)(\hat{E}_R + \mathcal{E}_c) \right\rangle \tag{3.13}$$

Since $\hat{E}_{R,free}$ is the field at initial time and the initial state was transformed to vacuum (cf. eq. 3.6), it will have no effect. Thus, all terms including $\hat{E}_{R,free}$ will vanish, and the remaining expression is

$$\begin{aligned} G^1 &= |C|^2\rho_{ee} + C^*\mathcal{E}_c\rho_{ge} + C\mathcal{E}_c^*\rho_{eg} + |\mathcal{E}_c|^2 = |C|^2\rho_{ee} + 2Re(C^*\mathcal{E}_c\rho_{ge}) + |\mathcal{E}_c|^2 \\ &= |C|^2\rho_{ee} + 2Re(-\frac{i}{c}\Omega_c\rho_{ge}) + |\mathcal{E}_c|^2, \end{aligned} \tag{3.14}$$

where $C = i\frac{\sqrt{2\pi g}}{c}$ and we have used $\Omega_c = \sqrt{2\pi}g\mathcal{E}_c$

From this, after inserting the steady-state density matrix elements from eq. 2.12-2.14, we find the steady state transmittance

$$\mathcal{T} = \left[\left(\frac{\Gamma_{1D}}{\gamma}\right)^2 - \frac{2\Gamma_{1D}}{\gamma} + 1 + 8\left(\frac{\Omega_c}{\gamma}\right)^2\right]\frac{1}{1+\gamma_d/\gamma+(\Omega_c/\gamma)^2} = \frac{1+8(1+P)^2(\Omega_c/\gamma)^2}{(1+P)^2(1+\gamma_d/\gamma+8(\Omega_c/\gamma)^2)}, \tag{3.15}$$

where the Purcell factor $P \equiv \Gamma_{1D}/\Gamma'$ is the relative fraction of decays into the waveguide modes $\Gamma_{1D} = 4\pi g^2/c$ and into all other modes Γ', and the total decay rate is $\gamma = \Gamma_{1D} + \Gamma'$.

Doing the same calculation for the left-travelling field

$$G^1 = \langle \hat{E}_L{}^\dagger \hat{E}_L \rangle = |C|^2\rho_{ee}^{s.s.} = \frac{2\pi g^2}{c^2}\frac{\Omega^2}{2\Omega^2 + \gamma^2 + \gamma\gamma_d} \tag{3.16}$$

we arrive at the expression for the steady-state reflectance

$$\mathcal{R} = \left(1 + \frac{1}{P}\right)^{-2}\left(1 + \frac{\gamma_d}{\gamma} + 8\left(\frac{\Omega_c}{\gamma}\right)^2\right)^{-1} \tag{3.17}$$

Setting $\gamma_d = 0$ in the found $\mathcal{T}$ and $\mathcal{R}$, they reduce to the results in [1] as expected.

3.2 Transmitted field

We now look at the average transmitted field in the weak driving limit; the limit where the intensity of the input field is so low that it can hardly cause transitions. For a system initially in the ground state, the field will consequently obey $\rho_{gg} \approx 1, \rho_{ee} \approx 0$.

$$\langle \hat{E}_{out} \rangle = \langle \hat{E}_R + \mathcal{E}_c \rangle = i\frac{\sqrt{2\pi}g}{c}\langle \hat{\sigma}_{ge} \rangle + \mathcal{E}_c = \frac{i}{2\sqrt{2\pi}g}\Gamma_{1D}\rho_{eg} + \mathcal{E}_c \tag{3.18}$$

Inserting the expression for ρ_{eg} in steady state and with weak driving (eq. 2.15), we obtain

$$\langle \hat{E}_{out} \rangle = \left(1 - \frac{\Gamma_{1D}}{\gamma + \gamma_d + 2i\Delta}\right)\langle \hat{E}_{in} \rangle \;\Leftrightarrow\; T = \left(1 - \frac{\Gamma_{1D}}{\gamma + \gamma_d + 2i\Delta}\right) \tag{3.19}$$

for the average transmitted field.

In the recent article by Das et al. [8], which was mentioned in the introduction, a formalism is developed for the transmission through a 1D-waveguide. The formalism is very general; it can be applied to a system of an arbitrary number of emitters with both a ground- and an excited state manifold and can include different types of effects. This is a powerful relation, completely trivializing the calculation of the transmission through a single two-level emitter. As an example, we show how to use it to arrive at the same conclusion as above.

First, the non-Hermitian Hamiltonian is defined as $H_{nh} = \big(\Delta - \frac{i}{2}(\gamma + \gamma_d)\big)$. The scattering amplitude is defined by $\mathscr{A}^+[H_{nh}]_{ee}^{-1}\mathscr{A}$ where $\mathscr{A} = \sqrt{\Gamma_{1D}/2}$, so

$$\mathscr{A}^+[H_{nh}]_{ee}^{-1}\mathscr{A} = \sqrt{\Gamma_{1D}/2}\,\langle e|\, H_{nh}\,|e\rangle^{-1}\sqrt{\Gamma_{1D}/2} = \frac{\Gamma_{1D}}{2}\,\langle e|\, H_{nh}\,|e\rangle^{-1} \tag{3.20}$$

Having this, the transmission can be calculated directly from

$$\langle E_{out}\rangle = \big(1 + i\mathscr{A}^+[H_{nh}]_{ee}^{-1}\mathscr{A}\big)\,\langle E_{in}\rangle = \left(1 - \frac{\Gamma_{1D}}{\gamma + \gamma_d + 2i\Delta}\right)\langle \hat{E}_{in}\rangle \tag{3.21}$$

which is the same as we found in eq. 3.19.

4 The scattering problem with stochastic dephasing and a weak driving field

Until this point, dephasing has been described as an effect acting, in the same way as decay, as a Lindblad operator in the master equation. They are both caused from interaction with field modes, but in stead of $\mathscr{L}$ which causes decay to the ground state directly, $\mathscr{L}_{\mathscr{d}}$ affects the coherence terms only. This description was sufficient to solve the scattering problem for the two-level atom, but in order to get a more accurate description of dephasing we now introduce it as term included in the Hamiltonian, $\hat{H^d} = \frac{\hbar}{2}f(t)\hat{\sigma_3}$ [13] in stead of the Lindblad operator $\sqrt{\gamma_d}\hat{\sigma}_{ee}$. This will allow us to evaluate multitime correlations between atomic operators.

4.1 New equations of motion

We consider the new dephasing term in the Hamiltonian, $\hat{H^d} = \frac{\hbar}{2}f(t)\hat{\sigma_3}$. The function $f(t)$ is a Gaussian stochastic function, for which we define the properties

$$\big\langle f(t)\big\rangle = 0 \qquad\qquad \big\langle f(t)f(t')\big\rangle = \gamma_d\delta(t-t') \tag{4.1}$$

i.e. it is delta correlated with itself (white noise). This is the Markovian approximation, meaning the interactions with the enviroment are instantaneous compared to the time scale of our dynamics such that the system has no memory.

We know that dephasing causes decay of the density matrix coherence terms, so now we investigate what happens with the dynamics of the coherences due to this new description with the Hamiltonian

$$\hat{H} = \frac{\hbar}{2}(\omega_{eg} + f(t))\hat{\sigma}_3 - \frac{E_0 d_0}{2}(e^{-i\omega_0 t}\hat{\sigma}_+ + e^{i\omega_0 t}\hat{\sigma}_-) \tag{4.2}$$

Evaluating the Heisenberg equation, we get

$$\dot{\hat{\sigma}}_+ = \frac{i}{\hbar}[\hat{H}, \hat{\sigma}_+] = i(\omega_{eg} + f(t))\hat{\sigma}_+ + \frac{i}{2}\Omega e^{i\omega_0 t}\hat{\sigma}_3, \tag{4.3}$$

where Ω is the usual Rabi frequency.

From the decay Lindblad operator, we have one additional term $\dot{\hat{\sigma}}_+ = -\frac{\gamma}{2}\hat{\sigma}_+$, as we found in the derivation of eq. 2.6.

In the weak driving limit we can use the approximation $\hat{\sigma}_3 \approx -1$, which results in uncoupling of the first order differential equations in terms of $\hat{\sigma}_\pm$ that can be solved directly analytically. This is the limit where we go to the quantum regime, and where the interaction probability is maximized, since a too strong field can cause saturation effects, resulting in the field carrying no signature from the emitter by the time it reaches the detector. Note that in ch. 3 we did not use weak driving since the input field was quantized. At the few-photon level the field is necessarily too weak to cause Rabi behavior.

Transforming the equations to a frame rotating with the field frequency ω_0 via $\hat{\sigma}'_+ = \hat{\sigma}_+ e^{-i\omega_0 t}$, we finally obtain the equation

$$\dot{\hat{\sigma}}_+ = i\left(\Delta + \frac{i\gamma}{2} + f(t)\right)\hat{\sigma}_+ - \frac{i}{2}\Omega \tag{4.4}$$

whose solution can be written as

$$\hat{\sigma}_+(t) = e^{i\int_0^t(\Delta + i\frac{\gamma}{2} + f(t))dt}\hat{\sigma}_+(0) - \frac{i}{2}\Omega\int_0^t e^{i\int_{t_1}^t(\Delta + i\frac{\gamma}{2} + f(t_2))dt_2}dt_1 \tag{4.5}$$

Now

$$\rho_{ge}(t) = \langle\hat{\sigma}_+(t)\rangle = \left\langle e^{i\int_0^t(\Delta + i\frac{\gamma}{2} + f(t))dt}\right\rangle\rho_{ge}(0) - \frac{i}{2}\Omega\int_0^t dt_1\left\langle e^{i\int_{t_1}^t(\Delta + i\frac{\gamma}{2} + f(t_2))dt_2}\right\rangle \tag{4.6}$$

An important property of the Gaussian stochastic function is that when summing up a large number of the random uncorrelated events, the sum will go towards a Gaussian distribution with expectation value $\langle f(t)\rangle$ according to the Central Limit Theorem [13]. This allows us to evaluate the expectation values of the stochastic terms as follows,

$$\langle e^{i\int_0^t f(t)dt}\rangle = e^{-\int_0^t dt\langle f(t)\rangle^2/2} = e^{-\gamma_d t/2} \tag{4.7}$$

After applying this property to equation 4.6, we end up with

$$\rho_{ge}(t) = e^{i\Delta t}e^{-\frac{1}{2}(\gamma+\gamma_d))t}\rho_{ge}(0) - \frac{i}{2}\Omega\int_0^t dt_1 e^{i(\Delta + \frac{i}{2}(\gamma+\gamma_d))(t-t_1)} \tag{4.8}$$

Now we want to find the steady state solution in order to calculate the transmission through the emitter. Taking the time derivative of ρ_{ge}, we get

$$\dot{\rho}_{ge}(t) = \left(i\Delta - \frac{1}{2}(\gamma+\gamma_d)\right)\rho_{ge}(0)e^{i\Delta t}e^{-\frac{1}{2}(\gamma+\gamma_d)t} + \frac{i}{2}\Omega(1 - e^{i\Delta t}e^{-\frac{1}{2}(\gamma+\gamma_d)t}) \tag{4.9}$$

Over a long time scale compared to the lifetime of the coherences, the decay terms in the first term will vanish and we will be left with the condition for steady state $\dot{\rho}_{ge}(t) = \frac{i}{2}\Omega \approx 0$, since we are in the weak driving limit. Changing to this time scale, eq. 4.8 becomes

$$\rho_{ge}(t) \approx -\frac{i}{2}\Omega\int_{-\infty}^t dt_1 e^{i(\Delta + \frac{i}{2}(\gamma+\gamma_d))(t-t_1)} \tag{4.10}$$

We use this approximation to find the expression for steady-state $\rho_{ge}(t)$ with weak driving

$$\rho_{ge,wd}^{s.s.}(t) \approx -\frac{i}{2}\Omega\int_{-\infty}^t dt_1 e^{i(\Delta + \frac{i}{2}(\gamma+\gamma_d))(t-t_1)} = -\frac{\Omega}{2\Delta + i(\gamma+\gamma_d)} = \frac{-i\Omega}{\gamma+\gamma_d - 2i\Delta} \tag{4.11}$$

which is exactly the same result as we got for the steady state solutions in the weak driving limit (eq. 2.15) when solving the density matrix dynamics in the semiclassical model (using $\rho_{ge} = \rho_{eg}^*$).

The corresponding steady state solutions for $\hat{\sigma}_\pm$, which we will be using in our further calculations, are

$$\hat{\sigma}^{s.s.}_{+,wd}(t) = -\frac{i}{2}\Omega \int_{-\infty}^{t} e^{i\int_{t_1}^{t}(\Delta+i\frac{\gamma}{2}+f(t_2))dt_2} dt_1 \tag{4.12}$$

$$\hat{\sigma}^{s.s.}_{-,wd}(t) = \frac{i}{2}\Omega \int_{-\infty}^{t} e^{-i\int_{t_1}^{t}(\Delta-i\frac{\gamma}{2}+f(t_2))dt_2} dt_1 \tag{4.13}$$

4.2 The noise term in the transmittance

Our goal is to repeat the calculations from chapter 3, but with the transition operators we found in the previous subchapter (eq. 4.12 and 4.13), where we used a stochastic model for dephasing. The difference is that this time we have used the weak driving approximation on a semiclassical model and included detuning.

The expression for the first-order correlation function is

$$\begin{aligned} G^1 = \langle \hat{E}^\dagger_{out}(t)\hat{E}_{out}(t')\rangle &= \left\langle \frac{2\pi g^2}{c^2}\hat{\sigma}_+(t)\hat{\sigma}_-(t') - i\frac{\sqrt{2\pi}g}{c}\hat{\sigma}_+(t)\mathcal{E}_c + i\frac{\sqrt{2\pi}g}{c}\hat{\sigma}_-(t')\mathcal{E}_c^* + |\mathcal{E}_c|^2 \right\rangle \\ &= \frac{\Gamma_{1D}^2}{8\pi g^2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t')\rangle + \frac{i\Gamma_{1D}}{2\sqrt{2\pi}g}\left(\langle\hat{\sigma}_-(t')\rangle\mathcal{E}_c^* - \langle\hat{\sigma}_+(t)\rangle\mathcal{E}_c\right) + \langle\hat{E}^\dagger_{in}(t)\hat{E}_{in}(t')\rangle \end{aligned} \tag{4.14}$$

where, again, we have used that the decay into the waveguide is $\Gamma_{1D} = 4\pi g^2/c$. Now the interesting part is the correlation term. If we set $t' = t$, we expect simply to get $\langle\hat{\sigma}_+(t)\sigma_-(t)\rangle = \langle\hat{\sigma}_{ee}(t)\rangle = \rho_{ee}$, but in this description we're also able to allow for the interactions with the environment to change the coherence functions by evaluating the more general expression for transmittance when $t' \neq t$.

The last three terms of equation 4.14 are straightforward;

$$\frac{i\Gamma_{1D}}{2\sqrt{2\pi}g}\left(\langle\hat{\sigma}_-(t')\rangle\mathcal{E}_c^* - \langle\hat{\sigma}_+(t)\rangle\mathcal{E}_c\right) + \langle\hat{E}^\dagger_{in}\hat{E}_{in}\rangle = \left(1 - \frac{2\Gamma_{1D}(\gamma+\gamma_d)}{4\Delta^2+(\gamma+\gamma_d)^2}\right)\langle\hat{E}^\dagger_{in}\hat{E}_{in}\rangle \tag{4.15}$$

Now to the first term. We start by evaluating it with $t' = t$

$$\begin{aligned} \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle &= \frac{\Omega^2}{4}\left\langle \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_1' e^{i\int_{t_1}^{t}(\Delta+i\frac{\gamma}{2}+f(t_2))dt_2} e^{-i\int_{t_1'}^{t}(\Delta-i\frac{\gamma}{2}+f(t_2))dt_2} \right\rangle \\ &= \frac{\Omega^2}{4}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_1' e^{(i\Delta-\frac{\gamma}{2})(t-t_1)} e^{(-i\Delta-\frac{\gamma}{2})(t-t_1')} \left\langle e^{i\int_{t_1}^{t} f(t_2)dt_2} e^{-i\int_{t_1'}^{t} f(t_2)dt_2} \right\rangle \\ &= \frac{\Omega^2}{4}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_1' e^{(i\Delta-\frac{\gamma}{2})(t-t_1)} e^{(-i\Delta-\frac{\gamma}{2})(t-t_1')} e^{-\frac{\gamma_d}{2}|t_1'-t_1|} \\ &= \frac{\Omega^2}{4}\int_{-\infty}^{t} dt_1 e^{(i\Delta-\frac{\gamma}{2})(t-t_1)} \left(\int_{-\infty}^{t_1} dt_1' e^{(-i\Delta-\frac{\gamma}{2})(t-t_1')} e^{-\frac{\gamma_d}{2}(t_1-t_1')} + \int_{t_1}^{t} dt_1' e^{(-i\Delta-\frac{\gamma}{2})(t-t_1')} e^{-\frac{\gamma_d}{2}(t_1'-t_1)}\right) \\ &= \left(1+\frac{\gamma_d}{\gamma}\right)\frac{\Omega^2}{4\Delta^2+(\gamma+\gamma_d)^2} \end{aligned} \tag{4.16}$$

Comparing with our result from the semiclassical model, equation 2.16, we see that we got exactly the same expression for ρ_{ee} as expected.

By combining equation 4.15 and 4.16, we find the transmission coefficent for $t' = t$,

$$\mathcal{T}_{t'=t} = \left(1+\frac{\gamma_d}{\gamma}\right)\frac{\Gamma_{1D}^2}{4\Delta^2+(\gamma+\gamma_d)^2} - \frac{2\Gamma_{1D}(\gamma+\gamma_d)}{4\Delta^2+(\gamma+\gamma_d)^2} + 1 \tag{4.17}$$

This result is plotted as a function of time in fig. 2. The waveguide is usually designed so that Γ_{1D} is often up to several orders of magnitude larger than Γ' as long as there is a strong field-atom coupling; for this plot the relation $\Gamma_{1D} = 0.95\gamma$ was chosen. The situation $\gamma_d = \gamma = 1$ is not very realistic, but is included to illustrate the extreme case where reflection and transmission are equally probable at resonance.

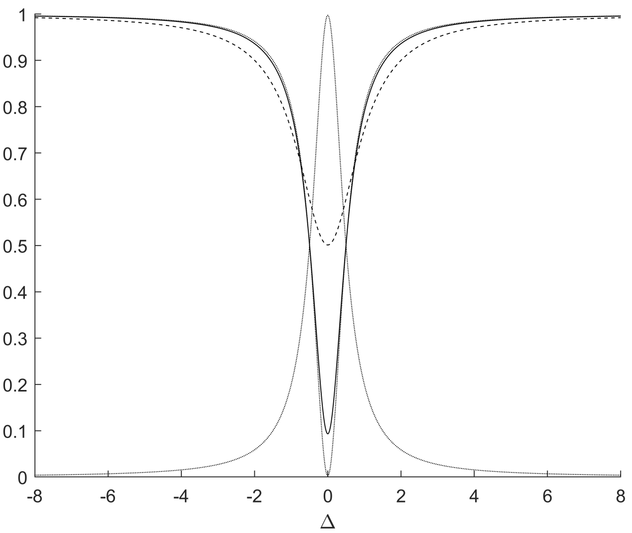

Figure 2: Transmittance as a function of Δ with $\gamma = 1$, $\Gamma_{1D} = 0.95$ and various values of γ_d; $\gamma_d = 0$ plotted together with its corresponding reflectance (dotted lines), $\gamma_d = 1$ (dashed line) and $\gamma_d = 0.1$ (solid line)

Until now, we extracted no new information, apart from a pleasing confirmation of the calculations so far. The interesting part is when we look at the multitime correlation of the output field, because this is something that we couldn't achieve with the density matrix approach in chapter 3. When the times are equal, we see the interactions with the environment as instantaneous, but now we will allow for the interactions to have a finite duration and change the transition operators. The expectation is that the stochastic description of dephasing results in a noise term added to equation 3.19 in the following manner (as predicted in [8]),

$$\langle E_{out} \rangle = \left(1 - \frac{\Gamma_{1D}}{\gamma + \gamma_d + 2i\Delta}\right)\langle E_{in} \rangle + \mathcal{F}(t) \tag{4.18}$$

We start by evaluating the correlation term

$$\big\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t')\big\rangle = \frac{\Omega^2}{4}\Big\langle \int_{-\infty}^{t} e^{i\int_{t_1}^{t}(\Delta + i\frac{\gamma}{2} + f(t_2))dt_2} dt_1 \int_{-\infty}^{t'} e^{-i\int_{t'_1}^{t'}(\Delta - i\frac{\gamma}{2} + f(t'_2))dt'_2} dt'_1 \Big\rangle \tag{4.19}$$

We proceed by assuming $t > t'$, which allows us to split up the first integral as

$\int_{-\infty}^{t} \to \int_{-\infty}^{t'} + \int_{t'}^{t}$. This gives us

$$\begin{aligned}
\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t')\rangle &= \frac{\Omega^2}{4}\Bigg[\int_{-\infty}^{t'} dt_1 \int_{-\infty}^{t'} dt_1' e^{(i\Delta-\frac{\gamma}{2})(t-t_1)} e^{-(i\Delta+\frac{\gamma}{2})(t'-t_1')} \left\langle e^{i\int_{t_1}^{t} f(t_2)dt_2} e^{-i\int_{t_1'}^{t'} f(t_2')dt_2'}\right\rangle \\
&+ \int_{t'}^{t} dt_1 e^{i\Delta-\frac{\gamma}{2}(t-t_1)} \left\langle e^{i\int_{t_1}^{t} f(t_2)dt_2}\right\rangle \int_{-\infty}^{t'} dt_1' e^{(-i\Delta-\frac{\gamma}{2})(t'-t_1')} \left\langle e^{-i\int_{t_1'}^{t'} f(t_2')dt_2'}\right\rangle \Bigg] \\
&= \frac{\Omega^2}{4}\Bigg[\int_{-\infty}^{t'} dt_1 \int_{-\infty}^{t'} dt_1' e^{(i\Delta-\frac{\gamma}{2})(t-t_1)} e^{(-i\Delta-\frac{\gamma}{2})(t'-t_1')} e^{-\frac{\gamma_d}{2}(t-t')} e^{-\frac{\gamma_d}{2}|t_1'-t_1|} \\
&+ \int_{t'}^{t} dt_1 \int_{-\infty}^{t'} dt_1' e^{(i\Delta-\frac{1}{2}(\gamma+\gamma_d))(t-t_1)} e^{(i\Delta+\frac{1}{2}(\gamma+\gamma_d))(t_1'-t')}\Bigg] \\
&= \frac{\gamma+\gamma_d}{\gamma}\frac{\Omega^2}{4\Delta^2+(\gamma+\gamma_d)^2} e^{(i\Delta-\frac{1}{2}(\gamma+\gamma_d))(t-t')} + \frac{\Omega^2}{4\Delta^2+(\gamma+\gamma_d)^2}\left(1-e^{(i\Delta-\frac{1}{2}(\gamma+\gamma_d))(t-t')}\right) \\
&= \left(1+\frac{\gamma_d}{\gamma} e^{(i\Delta-\frac{1}{2}(\gamma+\gamma_d))(t-t')}\right)\frac{\Omega^2}{4\Delta^2+(\gamma+\gamma_d)^2} \qquad (4.20)
\end{aligned}$$

Expressing this in terms of the coherences, we now have

$$\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t')\rangle = \left(1+\frac{\gamma_d}{\gamma} e^{(i\Delta-\frac{1}{2}(\gamma+\gamma_d))(t-t')}\right)\rho_{eg}\rho_{ge} \qquad (4.21)$$

Evaluating the correlation between the transition operators at different times thus leads to an exponential factor proportional to $\exp(t-t')$ in the decay term. Recall that the derivation was based on the assumption that $t > t'$, which could as well has been the opposite and given an $\exp(t'-t)$ term in stead. The general expression for the transmittance is therefore

$$\mathcal{T}_{t'\neq t} = \left(1+\frac{\gamma_d}{\gamma} e^{(i\Delta-\frac{1}{2}(\gamma+\gamma_d))|t-t'|}\right)\frac{\Gamma_{1D}^2}{4\Delta^2+(\gamma+\gamma_d)^2} - \frac{2\Gamma_{1D}(\gamma+\gamma_d)}{4\Delta^2+(\gamma+\gamma_d)^2} + 1 \qquad (4.22)$$

If we look at the transmission as a function of the time difference we can see that the transmission probability is maximal at $t' = t$. For large time differences we reach the limit when $\mathcal{T}_{t'\neq t} \to \frac{\Gamma_{1D}^2 - 2\Gamma_{1D}(\gamma+\gamma_d)}{4\Delta^2+(\gamma+\gamma_d)^2} + 1$, which corresponds to the first term reducing to $\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t')\rangle = \rho_{eg}\rho_{ge}$, i.e. the transition operators become uncorrelated and dephasing has no other effect on the transmission than enhancing the decay rate. For $\gamma_d/\gamma = 0.1$ and for a excited state lifetime around 1 ns, the correlation lasts around 0.1 ns.

The last step is to figure out the noise term $\mathcal{F}(t)$ in equation 4.18. Taking the modulo square of this entire equation, we get

$$\frac{\langle E_{out}^\dagger E_{out}\rangle}{\langle E_{in}^\dagger E_{in}\rangle} = \mathcal{T}_{t'\neq t} = |T|^2 + \langle \mathcal{F}(t)^*\mathcal{F}(t')\rangle \qquad (4.23)$$

Since we have

$$|T|^2 = \frac{\Gamma_{1D}^2 - 2\Gamma_{1D}(\gamma+\gamma_d)}{4\Delta^2+(\gamma+\gamma_d)^2} + 1, \qquad (4.24)$$

what we are left with is

$$\langle \mathcal{F}(t)^*\mathcal{F}(t')\rangle = \frac{\gamma_d}{\gamma}\frac{\Gamma_{1D}^2}{4\Delta^2+(\gamma+\gamma_d)^2} e^{(i\Delta-\frac{1}{2}(\gamma+\gamma_d))|t-t'|} \qquad (4.25)$$

Now we want to make sure the system has no memory in order to stay within the Markovian regime. Looking at a long time scale compared to the time difference between the events

$|t - t'|$, the only contribution will be from $t \approx t'$ and we get we can approximate 4.25 as a delta function $\kappa\delta(t-t')$, where $\kappa = \int_{-\infty}^{t} dt' \langle \mathcal{F}(t)^* \mathcal{F}(t') \rangle$, so

$$\langle \mathcal{F}(t)^* \mathcal{F}(t') \rangle \simeq \frac{\Gamma_{1D}^2}{4\Delta^2 + (\gamma+\gamma_d)^2} \frac{2\gamma_d}{\gamma(\gamma+\gamma_d)} \delta(t-t') \tag{4.26}$$

This is the amplitude of the noise arising from dephasing in the transmission of a weak driving field through our two-level emitter. Inserting this in equation 4.23, we have that the steady-state transmittance at time t is

$$\mathcal{T} = 1 + \frac{\Gamma_{1D}^2 - 2\Gamma_{1D}(\gamma+\gamma_d)}{4\Delta^2 + (\gamma+\gamma_d)^2} + \frac{2\Gamma_{1D}^2 \gamma_d}{\gamma(\gamma+\gamma_d)(4\Delta^2 + (\gamma+\gamma_d)^2)} \tag{4.27}$$

At resonance and with $\Gamma_{1D}/\gamma = 0.95$, the transmittance is $2.5 \cdot 10^{-3}$ for $\gamma_d = 0$, which increases by two orders of magnitude to 0.25 for $\gamma_d/\gamma = 0.2$.

5 Generalization to a multilevel emitter

The dynamics of a two-level system is interesting theoretically and gives us the opportunity to try out complicated calculations on a simple system, which can be seen as the foundation of all other possible level structures. However, since its structure is so specific and simple, its applications are very limited. More flexible results are needed, and we will therefore intent to generalize the calculations to a multilevel emitter.

In the following, we will still consider a single emitter in a 1D-waveguide, but now it is composed of a ground state and an arbitrary number of excited states. The distinction is made based on the energy difference in between the states, so in this case it's only the transition frequency between ground and the excited states that is optical, which allows us to treat this level structure as a ground state with an excited state manifold. Meanwhile, the presence of multiple excited states will still change the dynamics.

5.1 Density matrix with multiple excited states

For the density matrix, the crucial difference in changing to a multilevel emitter is that we now have the possibility for coherences $\rho_{e_i e_j}$ between the i^{th} and j^{th} excited states.

In the treatment of the two-level system, dephasing happened between the excited state and the environment, which we saw as loss of coherence.

For the case of multiple excited states, dephasing can occur between all possible combinations of the excited states and each differential equation $\rho_{e_i g}$ will contain the information of where dephasing happened before decaying from the i^{th} state. The dephasing operator for each state must sum up all possible excited state paths into an effective decay rate for the last state where dephasing ocurred.

This means our dephasing operator for the i^{th} state is now $\mathcal{L}_{\mathcal{d},i} = \sum_{l,m} \sqrt{\gamma_{d,lm}^i} |e_l\rangle \langle e_m|$, which describes all the possible paths of dephasing between the excited states $|e_l\rangle$ and $|e_m\rangle$ before decaying from the i^{th} state, while the decay operator will simply generalize to
$\mathcal{L}_i = \sqrt{\gamma_i} |g\rangle \langle e_i|$.

The dipole moments within the excited state manifold are negligible, so that for N excited states the dipole moment operator $\hat{d} = (d_{ge_1}\hat{\sigma}_{ge_1} + d_{ge_2}\hat{\sigma}_{ge_2} + ... + d_{ge_N}\hat{\sigma}_{ge_N} + h.c.)$.

The total Hamiltonian in a frame rotating with ω_0 becomes

$$\hat{H} = \hbar \sum_i \left(\Delta_i |e_i\rangle \langle e_i| - \frac{\Omega_i}{2} (|g\rangle \langle e_i| + |e_i\rangle \langle g|) \right), \tag{5.1}$$

where $\Delta_i = \omega_{e_i} - \omega_g - \omega_0$ and $\Omega_i = E_0 d_{ge_i} \hbar^{-1}$.

We will now use the master equation again to obtain the Langevin-Bloch equations for the multilevel emitter. It will provide clarity to set up the matrices involved;

$$\hat{H} = \hbar \begin{pmatrix} 0 & -\frac{\Omega_1}{2} & -\frac{\Omega_2}{2} & \dots & -\frac{\Omega_N}{2} \\ -\frac{\Omega_1}{2} & \Delta_1 & 0 & \dots & 0 \\ -\frac{\Omega_2}{2} & 0 & \Delta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\Omega_N}{2} & 0 & \dots & & \Delta_N \end{pmatrix} \tag{5.2}$$

$$\mathcal{L} = \begin{pmatrix} 0 & \sqrt{\gamma_1} & \sqrt{\gamma_2} & \dots & \sqrt{\gamma_N} \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{L}_{\not{d}} = \sum_i \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \sqrt{\gamma^i_{d,11}} & \sqrt{\gamma^i_{d,12}} & \dots & \sqrt{\gamma^i_{d,1N}} \\ 0 & \sqrt{\gamma^i_{d,21}} & \sqrt{\gamma^i_{d,22}} & \dots & \sqrt{\gamma^i_{d,2N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sqrt{\gamma^i_{d,N1}} & \sqrt{\gamma^i_{d,N2}} & \dots & \sqrt{\gamma^i_{d,NN}} \end{pmatrix} \tag{5.3}$$

$$\sum_m \mathcal{L}_m{}^\dagger \mathcal{L}_m = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & \gamma_1 + \sum_{i,m} \gamma^i_{d,m1} & \sqrt{\gamma_1\gamma_2} + \sum_{i,m} \sqrt{\gamma^i_{d,m1}\gamma^i_{d,m2}} & \dots \\ 0 & \sqrt{\gamma_1\gamma_2} + \sum_{i,m} \sqrt{\gamma^i_{d,m1}\gamma^i_{d,m2}} & \gamma_2 + \sum_i \gamma^i_{d,m2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.4}$$

$\mathcal{L}\hat{\rho}\mathcal{L}^\dagger$ has a contribution to ρ_{gg} only, while $\mathcal{L}_{\not{d}}\hat{\rho}\mathcal{L}_{\not{d}}{}^\dagger$ gives contributions to the $\rho_{e_ie_j}$-terms.

This gives us the Langevin-Bloch equations for the population terms,

$$\dot{\rho}_{gg} = \sqrt{\gamma_i}\sqrt{\gamma_j}\rho_{e_ie_j} - \frac{i}{2}\Omega_i(\rho_{ge_i} - \rho_{e_ig}) \tag{5.5}$$

$$\dot{\rho}_{e_ie_j} = \frac{i}{2}\left(\Omega_i\rho_{ge_j} - \Omega_j\rho_{e_ig}\right) - \frac{1}{2}\left[\left(\sqrt{\gamma_j\gamma_n} + \sqrt{\gamma^\alpha_{d,mj}\gamma^\alpha_{d,mn}}\right)\rho_{e_ie_n} + \left(\sqrt{\gamma_i\gamma_n} + \sqrt{\gamma^\alpha_{d,mi}\gamma^\alpha_{d,mn}}\right)\rho_{e_ne_j}\right] + \sqrt{\gamma^\alpha_{d,jm}\gamma^\alpha_{d,in}}\rho_{e_me_n} \tag{5.6}$$

where the Einstein summation convention is used.

For the coherence terms, we obtain

$$\dot{\rho}_{e_ig} = \left[-i\Delta_i\rho_{e_ig} - \frac{1}{2}\left(\sqrt{\gamma_i}\sum_m \sqrt{\gamma_m}\rho_{e_mg} + \sum_{\alpha,l,m} \rho_{e_mg}\sqrt{\gamma^\alpha_{d,lm}}\sqrt{\gamma^\alpha_{d,li}}\right)\right] - \frac{i}{2}\left(\sum_j \Omega_j\rho_{e_ie_j} - \Omega_i\rho_{gg}\right) \tag{5.7}$$

$$\dot{\rho}_{ge_i} = \left[i\Delta_i\rho_{ge_i} - \frac{1}{2}\left(\sqrt{\gamma_i}\sum_m \sqrt{\gamma_m}\rho_{ge_m} + \sum_{\alpha,l,m} \rho_{ge_m}\sqrt{\gamma^\alpha_{d,lm}}\sqrt{\gamma^\alpha_{d,li}}\right)\right] + \frac{i}{2}\left(\sum_j \Omega_j\rho_{e_je_i} - \Omega_i\rho_{gg}\right) \tag{5.8}$$

It's easy to check that these equations reduce to equations 2.7-2.10 for the case $i = j = 1$. Now for the ρ_{gg} and the coherence terms, nothing new happens physically except that the dissipative terms come from multiple levels, and there is a unique dipole value for each of the excited states' coherences with the ground state, which gives N Rabi frequencies for N

excited states. The crucial difference lies in the $\rho_{e_ie_j}$-terms which now contain contributions from dephasing as well.

Now the big question is whether we can manage to solve these equation and get something nice out of it. We start with the simpler task; solving for the coherences to find an expression for the transmitted field $\langle \hat{E}_{out} \rangle$.

5.2 Transmitted field

In order to solve for the transmitted field, recall the relation from eq. 3.18, where the coupling constant g now represents the sum of the coupling constants g_i for each of the i^{th} states, so that

$$\langle \hat{E}_{out} \rangle = \mathcal{E}_c + \frac{i}{2} \sum_i \frac{\Gamma^i_{1D}}{2\sqrt{2\pi} g_i} \rho^{s.s.}_{e_ig,wd} = \left(1 + i \sum_i \frac{\Gamma^i_{1D}}{\Omega_i} \rho^{s.s.}_{e_ig,wd} \right) \langle \hat{E}_{in} \rangle. \tag{5.9}$$

We need only to find the weak driving steady state solution for ρ_{eg} in order to be able to calculate the transmitted field.

We can reformulate the master equation by defining the non-hermitian Hamiltonian

$$\hat{H}_{nh} = \sum_i \Delta_i \left| e_i \right\rangle \left\langle e_i \right| - \frac{i}{2} \sum_m \mathcal{L}^\dagger_m \mathcal{L}_m \tag{5.10}$$

Then the master equation becomes

$$\dot{\hat{\rho}} = -i[\hat{H}_{nh}, \hat{\rho}] + \sum_m \mathcal{L}_m \hat{\rho} \mathcal{L}_m{}^\dagger + [\hat{V}_-, \hat{\rho}] + [\hat{V}_+, \hat{\rho}], \tag{5.11}$$

where $\hat{V}_\pm = \frac{i}{2} \sum_i \Omega_i \hat{\sigma}^i_\pm$

The notation is convenient for this purpose since $\hat{H}_{nh}$ is defined in excited-state space only. For ρ_{eg}, the equation reduces to

$$\dot{\rho}_{eg} = -i H_{nh} \rho_{eg} + \frac{i}{2} \Omega (\rho_{gg} - \rho_{e_ie_j}) \tag{5.12}$$

In the weak driving limit, the steady-state equation for ρ_{eg} becomes

$$\rho_{eg} = \frac{1}{2} [H_{nh}]^{-1} \Omega \tag{5.13}$$

Expanding the matrices, we have the vector equation for ρ_{eg}

$$\rho^{s.s.}_{eg,wd} = \frac{1}{2} \begin{pmatrix} \Delta_1 - \frac{i}{2}\Gamma_{11} & -\frac{i}{2}\Gamma_{12} & \dots & -\frac{i}{2}\Gamma_{1N} \\ -\frac{i}{2}\Gamma_{21} & \Delta_2 - \frac{i}{2}\Gamma_{22} & \dots & -\frac{i}{2}\Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{i}{2}\Gamma_{N1} & -\frac{i}{2}\Gamma_{N2} & \dots & \Delta_N - \frac{i}{2}\Gamma_{NN} \end{pmatrix}^{-1} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_N \end{pmatrix} \tag{5.14}$$

where $\Gamma_{ij} = \sqrt{\gamma_i \gamma_j} + \sqrt{\gamma^j_{d,mi} \gamma^j_{d,mj}}$

The model is compatible with the theory for the two-level system, since for $N = 1$, the matrix equation reduces to a scalar $\rho_{eg} = \frac{1}{2} \left(\Delta - \frac{i}{2}(\gamma + \gamma_d) \right)^{-1} \Omega$, equally to eq. 2.15.

We can try to apply this result to a 3-level system with one ground state ($N = 2$). In this case,

$$\rho_{eg} = \frac{1}{2} \left((\Delta_1 - \frac{i}{2}\Gamma_{11})(\Delta_2 - \frac{i}{2}\Gamma_{22}) + \frac{1}{4}\Gamma_{12}\Gamma_{21} \right)^{-1} \begin{pmatrix} \Omega_1(\Delta_2 - \frac{i}{2}\Gamma_{22}) + \frac{i}{2}\Omega_2\Gamma_{12} \\ \frac{i}{2}\Omega_1\Gamma_{21} + \Omega_2(\Delta_1 - \frac{i}{2}\Gamma_{11}) \end{pmatrix} \tag{5.15}$$

and the transmission, using the notation $d_{ge_i} = d_i$, is

$$\langle \hat{E}_{out} \rangle = \left(1 + i \sum_i \frac{\Gamma^i_{1D}}{\Omega_i} \rho^{s.s.}_{e_ig,wd} \right) \langle \hat{E}_{in} \rangle = \left(1 + \frac{i}{2\det(H_{nh})} \begin{pmatrix} \Gamma^1_{1D}(\Delta_2 - \frac{i}{2}\Gamma_{22} + \frac{id_2}{2d_1}\Gamma_{12}) \\ \Gamma^2_{1D}(\Delta_1 - \frac{i}{2}\Gamma_{11} + \frac{id_1}{2d_2}\Gamma_{21}) \end{pmatrix} \right) \langle \hat{E}_{in} \rangle \tag{5.16}$$

6 Conclusion and outlook

This thesis has shown how dephasing affects the transmission of light through a two-level system in a 1D-waveguide.

The steady-state transmittance at resonance for a quantized model was calculated, where dephasing was found to contribute with a term $(1+P)^2\gamma_d/\gamma$ in the denominator to give

$$\mathcal{T} = \frac{1+8(1+P)^2(\Omega_c/\gamma)^2}{(1+P)^2(1+\gamma_d/\gamma+8(\Omega_c/\gamma)^2)}$$

The scattering problem was solved for a semiclassical model in the weak driving limit, where the average power of the transmitted field was found to be

$$\langle \hat{E}_{out}\rangle = \left(1-\frac{\Gamma_{1D}}{\gamma+\gamma_d+2i\Delta}\right)\langle \hat{E}_{in}\rangle \ \Leftrightarrow \ T = \left(1-\frac{\Gamma_{1D}}{\gamma+\gamma_d+2i\Delta}\right)$$

in which case the effect of dephasing was essentially to increase the decay rate.

Dephasing was included as a stochastic function, which made it possible to evaluate multitime correlations between the transition operators when the weak driving limit was applied. This treatment allowed us to evaluate the dephasing-induced noise term $\mathcal{F}(t)$ in the transmittance, which was found to have the amplitude

$$\left\langle \mathcal{F}(t)^*\mathcal{F}(t')\right\rangle \simeq \frac{\Gamma_{1D}^2}{4\Delta^2+(\gamma+\gamma_d)^2}\frac{2\gamma_d}{\gamma(\gamma+\gamma_d)}\delta(t-t')$$

for a system without memory. The steady-state transmittance at time t is then

$$\mathcal{T} = 1+\frac{\Gamma_{1D}^2-2\Gamma_{1D}(\gamma+\gamma_d)}{4\Delta^2+(\gamma+\gamma_d)^2}+\frac{2\Gamma_{1D}^2\gamma_d}{\gamma(\gamma+\gamma_d)(4\Delta^2+(\gamma+\gamma_d)^2)}.$$

This result shows that the transmittance at near-resonance increases very rapidly with dephasing. At resonance and for $\Gamma_{1D}/\gamma = 0.95$, the transmittance without dephasing is essentially zero, while when $\gamma_d/\gamma = 0.2$, as much as 25 % of the light is transmitted. Dephasing thus has a significant impact on the scattering in the system, reducing the light-matter interactions.

In the case of a single ground state and N excited states, the ρ_{eg}, which is now a N-dimensional vector, was found to be

$$\rho_{eg,wd}^{s.s.} = \frac{1}{2}\begin{pmatrix} \Delta_1-\frac{i}{2}\Gamma_{11} & -\frac{i}{2}\Gamma_{12} & \dots & -\frac{i}{2}\Gamma_{1N} \\ -\frac{i}{2}\Gamma_{21} & \Delta_2-\frac{i}{2}\Gamma_{22} & \dots & -\frac{i}{2}\Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{i}{2}\Gamma_{N1} & -\frac{i}{2}\Gamma_{N2} & \dots & \Delta_N-\frac{i}{2}\Gamma_{NN} \end{pmatrix}^{-1}\begin{pmatrix}\Omega_1\\ \Omega_2\\ \vdots \\ \Omega_N\end{pmatrix}$$

where $\Gamma_{ij} = \sqrt{\gamma_i\gamma_j}+\sqrt{\gamma_{d,mi}^j\gamma_{d,mj}^j}$

This can be used to calculate the transmission directly for a single emitter with an arbitrary number of excited states via the relation

$$T = \left(1+i\sum_i \frac{\Gamma_{1D}^i}{\Omega_i}\rho_{eig,wd}^{s.s.}\right)$$

The expression gets increaily complex rapidly as N increases but an example was provided for the simplest extension; an emitter with one ground- and two excited states, for which we found

$$T = \left(1+\frac{i}{2\det(H_{nh})}\begin{pmatrix}\Gamma_{1D}^1(\Delta_2-\frac{i}{2}\Gamma_{22}+\frac{id_2}{2d_1}\Gamma_{12})\\ \Gamma_{1D}^2(\Delta_1-\frac{i}{2}\Gamma_{11}+\frac{id_1}{2d_2}\Gamma_{21})\end{pmatrix}\right)$$

The next steps towards a complete formalism for dephasing in emitters with a single ground state would obviously be to find the transmittance $\langle \hat{E}^{\dagger}_{out}\hat{E}_{out}\rangle / \langle \hat{E}^{\dagger}_{in}\hat{E}_{in}\rangle$ of the field generalized for multiple excited states. This is a much more complicated calculation than for the transmitted power, since it requires an expression for $\rho_{e_i e_j}$; the coherence between two arbitrary excited states, which is now a tensor equation. Deriving this is a harder task than ρ_{eg}, since all eight terms in the master equation in the form of 5.11 now contribute (where we could neglect five of the terms for the case of ρ_{eg}). It would be of great interest to find the dephasing-induced noise term in the transmittance, as it was done for a two-level system, but for this, as we have seen, the density matrix approch does not suffice. There isn't a straightforward way to generalize the method with stochastic dephasing, so a different approach to handle this is yet to be developed. Other extensions could be to solve the scattering problem for multiple emitters and for other types of level structures.

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