Quantum memory in 2-level systems

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1 Resume

1.1 English

In this thesis we consider usage of two-level systems as memory for photons. The technique called controlled and reversible inhomogeneous broadening (CRIB) is treated analytically where it's possible and perturbatively when it's not. We combine ideas from earlier implementations of CRIB with a new idea of only turning on broadening when the whole light pulse is absorbed. Thus we introduce stages that have no broadening before and after stages that have broadening.

For the non-broadened stages we take equations that were originally developed for a three-level system and adopt it for our needs. We show that there're specific modes of electric field that have better storage efficiency than others even before any broadening has taken place. For the broadened stages we take the same initial transformation of the equations as in the non-broadened case but since an exact analytic solution to the resulting equations cannot be found we make a perturbative approximation. We only look at a specific form of broadening distribution – a Gaussian – since it's convenient to work with while still being non-trivial.

In the end we arrive at expression of efficiency and show that loss in the broadening stages becomes smaller as broadening is increased. Thus if we assume that we store one of the optimal modes in our memory it works practically with 100 % efficiency. This efficiency goes down when we introduce one particular source of loss – addition of non-reversible broadening.

1.2 Dansk

Emnet af dette bachelorprojekt er kvantehukommelse for fotoner i to-niveau systemer. En teknik kaldet "controlled and reversible inhomogeneous broadening" (CRIB) er behandlet analytisk, hvor det er muligt, og perturbativt, hvor det er ikke muligt. Vi kombinerer ideer fra tidligere realiseringer af CRIB med en ny ide, som går ud på at slå forbredningen til efter hele lyspulsen er blevet absorberet. Derfor introducerer vi trin, som ikke har nogen forbredning, før og efter trin, som har forbredning.

For de ikke forbredede trin tager vi ligninger, som blev oprindeligt brugt til at beskrive et tre-niveau system og modificerer, så det passer med vores system. Vi viser, at der er bestemte modes af lys, der har en bedre lagringseffektivitet end andre, og den forskel indtræder før enhver forbredning har sket. For de forbredede trin tager vi den samme transformation af ligninger som i det ikke forbredede tilfælde, men siden den eksakte analytiske løsning kunne ikke blive fundet, blev vi nødt til at lave en pertubativ approksimation. Vi kigger kun på en bestemt form af fordelingsfunktionen for forbredning, en Gaussfordeling, siden den er nem at håndtere og samtidig en af de ikke trivielle.

Til sidst kommer vi frem til et udtryk for effektivitet og viser, at tab i trin med forbredning bliver mindre jo mere forbredt bliver de. På den måde, hvis vi antager, at vi gemmer en af de optimale modes i hukommelsen, så virker den praktisk talt med 100 % effektivitet. Denne effektivitet bliver dog mindre, hvis vi tager med påvirkning af den ikke reversible forbredning i modellen.

2 Introduction

To make quantum computing practically realisable it's of great importance to be able to store state of photons for extended periods of time. Many different practical implementations of quantum memory have been proposed. A lot of challenges here stem from the fact that there're two conceptually different things that have to accomplished at once. First of all light has to be able to easily leave its "footprint" in the memory. When talking about atoms that requires them to transition easily between states. However, this also means that those atoms will just as easily transition back to their original state whereby loosing stored information. Hence the other component of a good memory is a way to fix the aquired state for as long as possible and then release it on demand.

One way to do it is to have a Λ-type system with two levels that couple strongly to incoming light and a third level that has much longer lifetime than the excited state (metastable). If one manages to transfer superposition of ground and excited state to superposition of ground and the metastable state on storage and reverse this process on retrieval then the memory has effectively the desired properties. Analysis of that approach can be found in [1].

The main motivation for the technique that is analysed in this thesis is a wish for a simpler system. We want to only have 2 levels instead of 3. The mechanism that makes sure that state of the atoms stays the way it was is called controlled and reversible inhomogeneous broadening (CRIB) as can be found in [3]. It may be viewed as a destructive interference effect. If those two-level atoms were left on their own they would radiate their excitation back as soon as possible. But if we broaden the transition of different atoms by different amount (hence "inhomogeneous") we effectively archieve partial destructive interference between the radiated waves and thus prevent any significant intensity from escaping the ensemble. At the same time atoms become more and more dephased. To retrieve the stored information they need to be rephased first, and this is done by reversing the broadening.

The question now is when to turn on inhomogeneous broadening. In [3] it is done at in the very beginning of the storage process. Then in the middle of desired storage time interval the broadening is reversed. (See Fig. 1 (a)) With this setup the maximal efficiency of forward retrieval reached 54 %. Qualitatively the difficulty of reaching full efficiency can be understood as a consequence of broadening being the very mechanism whose purpose was *prevent* atoms from radiating and here we have it on at all times. Also without any additional sources of loss it can be shown that broadening introduces effective attenuation of field intensity – introducing non-zero optical depth into the model. And in fact the quoted maximal efficiency occurs at very specific value of optical depth while at optical depths 0 and ∞ the efficiency is 0.

The solution seems to be to have broadening turned off on retrieval. This naturally means that broadening on storage should also be turned on after some time and not at the very beginning. (See Fig. 1 (b)) What we set out to do in this thesis is to take a hybrid approach combining ideas from [1] and [3]. What was taken from [1] is description of the system with broadening turned off. Even though it describes a 3 level system the equations could be adopted quite easily to our 2 level case. While not much besides a general idea could be taken from [3] and original approach had to be developed.

This thesis is structured as follows. In section 3 the model is introduced. It may be argued that a lot of the careful quantum optics considerations in there are unnecessary for the goal of the thesis and serve only as motivation for writing down final equations of motion (10), (11) for non-broadened case and (13), (14) when broadening is on. However, in that section one can also find conventions and definitions of rescaled variables that are used in all subsequent sections. Section 4 contains a high-level overview of operation of memory and that is elaborated upon in sections 5 and 6. Finally in section 7 a particular source of loss is considered.

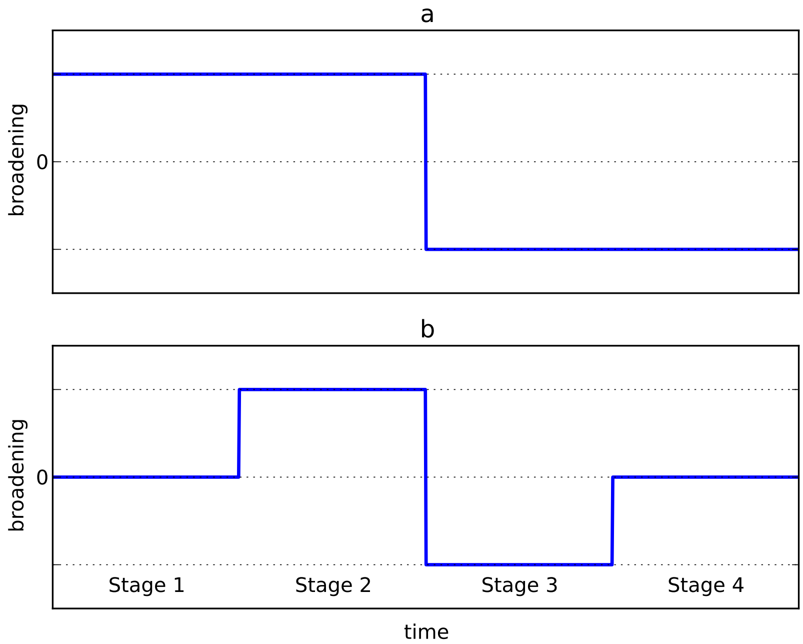


Figure 1: This is a diagram of CRIB based memory operation. Diagram "a" depicts what can be found in [3] while diagram "b" is the approach used in this thesis. (A more detailed description of different stages can be found in Section 4.)

3 Model

Our system consists of N two-level atoms coupled to a quantised electric field. We treat everything in one dimension so the electric field is only dependent on the spatial coordinate z, and atoms are assumed to be confined in a finite length L along the z-axis. The transverse extent of the ensemble is assumed to be the same as the cross-section of the incoming light pulse.

The two levels of the atom with index n are denoted as $|g\rangle_n$ and $|e\rangle_n$ for ground and excited state. The ground state energy is taken to be 0. The frequency of the atomic transition is initially assumed to be the same for all atoms and is called ω_{eg}.

Keeping ourselves to the Schrödinger picture for now will make operators time-independent. As usual we split the electric field in two parts:

$$\hat{\mathcal{E}}_S(z) = \hat{\mathcal{E}}_S^{(+)}(z) + \hat{\mathcal{E}}_S^{(-)}(z)$$

where $\hat{\mathcal{E}}_S^{(-)} = \left(\hat{\mathcal{E}}_S^{(+)}\right)^\dagger$. Assuming that the field describes a pulse propagating in the forward direction and has a narrow bandwidth around central frequency ω_L we express $\hat{\mathcal{E}}_S^{(+)}(z)$ as integral over continuum of creation operators $\hat{a}(\omega)$

$$\hat{\mathcal{E}}_S^{(+)}(z) = i\int_0^\infty \mathcal{E}_0(\omega)e^{i\omega z/c}\hat{a}(\omega)\,\mathrm{d}\omega \approx i\mathcal{E}_0(\omega_L)\int_0^\infty e^{i\omega z/c}\hat{a}(\omega)\,\mathrm{d}\omega$$

Where $\mathcal{E}_0(\omega) = \sqrt{\hbar\omega/(4\pi c\epsilon_0 A)}$ and creation operators have commutation relation

$$\left[\hat{a}(\omega), \hat{a}^\dagger(\omega')\right] = \delta(\omega - \omega').$$

Our system is then described by Hamiltonian

$$\hat{H} = \hat{H}_L + \hat{H}_A + \hat{H}_{\text{int}}$$

where

$$\begin{aligned}
\hat{H}_L &= \hbar \int_0^\infty \omega \hat{a}^\dagger(\omega)\hat{a}(\omega)\, \mathrm{d}\omega, \\
\hat{H}_A &= \hbar\omega_{eg} \sum_{n=1}^{N} |e\rangle_n \langle e|_n, \\
\hat{H}_{\text{int}} &= -\wp \sum_{n=1}^{N} \left(\hat{\mathcal{E}}_S^{(+)}(z) + \hat{\mathcal{E}}_S^{(-)}(z)\right) \left(|e\rangle_n\langle g|_n + |g\rangle_n\langle e|_n\right).
\end{aligned}$$

Here $\wp$ is the dipole strength.

Now we transform into the interaction picture using

$$\hat{H}_0 = \hat{H}_L + \hbar\omega_L \sum_{n=1}^{N} |e\rangle_n\langle e|_n.$$

From this we define unitary operator $\hat{U}_0 = \exp(-iH_0 t/\hbar)$ and seek solutions of the Schrödinger equation in the form $|\psi'\rangle = \hat{U}_0^\dagger|\psi\rangle$. Using the fact that the Schrödinger equation is satisfied for $|\psi\rangle$ one can show that for $|\psi'\rangle$ the effective Hamiltonian is

$$\hat{H}' = \hat{U}_0^\dagger \hat{H} \hat{U}_0 - i\hbar \hat{U}_0^\dagger \frac{\mathrm{d}\hat{U}_0}{\mathrm{d}t} = \hat{U}_0^\dagger \hat{H}\hat{U}_0 - \hat{H}_L - \hbar\omega_L \sum_{n=1}^{N} |e\rangle_n\langle e|_n.$$

Since $\hat{H}_A$ and $\hat{H}_L$ commute with $\hat{U}_0$ we need only consider how $\hat{U}_0$ acts on $\hat{H}_{\text{int}}$. For any operator $\hat{A}$ we have that $\hat{U}_0^\dagger \hat{A}\hat{U}_0$ is the solution of the Heisenberg equation $\frac{\mathrm{d}}{\mathrm{d}t}\hat{A} = \frac{i}{\hbar}[\hat{H}_0, \hat{A}]$ so to calculate $\hat{U}_0^\dagger \hat{H}\hat{U}_0$ we consider commutators

$$\begin{aligned}
\frac{i}{\hbar}\left[\hat{H}_L, \hat{\mathcal{E}}_S^{(+)}(z)\right] &= \frac{i}{\hbar}\left[\hbar\int_0^\infty \omega'\hat{a}^\dagger(\omega')\hat{a}(\omega')\,\mathrm{d}\omega',\ i\mathcal{E}_0(\omega_L)\int_0^\infty e^{i\omega z/c}\hat{a}(\omega)\,\mathrm{d}\omega\right] \\
&= -\mathcal{E}_0(\omega_L)\int_0^\infty\int_0^\infty \omega' e^{i\omega z/c}\left[\hat{a}(\omega')^\dagger,\ \hat{a}(\omega)\right]\hat{a}(\omega')\,\mathrm{d}\omega'\,\mathrm{d}\omega \\
= \mathcal{E}_0(\omega_L)\int_0^\infty \omega e^{i\omega z/c}\hat{a}(\omega)\,\mathrm{d}\omega &\approx \mathcal{E}_0(\omega_L)\omega_L\int_0^\infty e^{i\omega z/c}\hat{a}(\omega)\,\mathrm{d}\omega = -i\omega_L\hat{\mathcal{E}}_S^{(+)}(z)
\end{aligned}$$

(here we used the assumption of narrow bandwidth of the field again) and

$$\begin{aligned}
\frac{i}{\hbar}\left[\hbar\omega_L\sum_{m=1}^{N}|e\rangle_m\langle e|_m,\ |e\rangle_n\langle g|_n\right] &= i\omega_L\sum_{m=1}^{N}\left(|e\rangle_m\langle e|_m|e\rangle_n\langle g|_n - |e\rangle_n\langle g|_n|e\rangle_m\langle e|_m\right) \\
&= i\omega_L|e\rangle_n\langle g|_n.
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{\mathcal{E}}_I^{(+)}(z,t) &= \hat{\mathcal{E}}_S^{(+)}(z)e^{-i\omega_L t}, \\
|e\rangle_n\langle g|_n(t) &= |e\rangle_n\langle g|_n e^{i\omega_L t}.
\end{aligned}$$

We can get time evolution of $\hat{\mathcal{E}}_I^{(-)}(z,t)$ and $|g\rangle_n\langle e|_n$ by Hermitian conjugation. If we now expand the two parethenses in $\hat{H}_{\text{int}}$ then we end up with terms either having no time dependence (when exponentials cancel) or those having $\exp(\pm 2\omega_L t)$ time dependence. Making the rotating wave approximation we drop the latter ones. Our final effective Hamiltonian which is approximately time-independent then becomes

$$\hat{H}' = \hbar(\omega_{eg} - \omega_L)\sum_{n=1}^{N}|e\rangle_n\langle e|_n - \wp\sum_{n=1}^{N}\left(\hat{\mathcal{E}}_S^{(+)}(z)|e\rangle_n\langle g|_n + \hat{\mathcal{E}}_S^{(-)}(z)|g\rangle_n\langle e|_n\right). \tag{1}$$

If we let $\delta z = L/K$ where K is a positive integer then we can define operators that only act on atoms in the interval $[k(\delta z),(k+1)(\delta z)]$:

$$\hat{\mathcal{P}}_{k,ij} = \frac{\sqrt{N}}{N_k}\sum_{n=1}^{N_k}|i\rangle_n\langle j|_n$$

where both i and j can be either g or e and if we assume that atoms are distributed uniformly throughout the whole length L we have that $N_k = N(\delta z)/L = N/K$.

Commutation relation is then

$$\begin{aligned}\left[\hat{\mathcal{P}}_{k,ij},\hat{\mathcal{P}}_{k',i'j'}\right] &= \frac{N}{N_k N_{k'}}\sum_{m=1}^{N_k}\sum_{n=1}^{N_{k'}}\left[|i\rangle_m\langle j|_m,|i'\rangle_n\langle j'|_n\right]\\ &= N\frac{\delta_{kk'}}{N_k^2}\sum_{n=1}^{N_k}\left(\delta_{ji'}|i\rangle_n\langle j'|_n - \delta_{j'i}|i'\rangle_n\langle j|_n\right) = \sqrt{N}\frac{\delta_{kk'}}{N_k}\left(\delta_{ji'}\hat{\mathcal{P}}_{k,ij'} - \delta_{j'i}\hat{\mathcal{P}}_{k,i'j}\right).\end{aligned} \tag{2}$$

Using these operators and defining $\Delta_0 = \omega_{eg} - \omega_L$ Hamiltonian (1) can be written as

$$\hat{H}' = \Delta_0\frac{\sqrt{N}}{L}(\delta z)\sum_{k=0}^{K-1}\hat{\mathcal{P}}_{k,ee} + \wp\frac{\sqrt{N}}{L}(\delta z)\sum_{k=0}^{K-1}\left(\hat{\mathcal{E}}_S^{(+)}(z)\hat{\mathcal{P}}_{k,eg} + \hat{\mathcal{E}}_S^{(-)}(z)\hat{\mathcal{P}}_{k,ge}\right). \tag{3}$$

If K is chosen sufficiently big such that $\hat{\mathcal{P}}_{k,ij}$ can be considered a function $\hat{\mathcal{P}}_{ij}(z)$ then $\sqrt{N}(\delta_{kk'}/N_k)$ in the commutator (2) will get replaced by $(L/\sqrt{N})\delta(z-z')$ and Hamiltonian (3) becomes

$$\hat{H}' = \frac{\sqrt{N}}{L}\int_0^L\left(\Delta_0\hat{\mathcal{P}}_{ee}(z) - \wp\left(\hat{\mathcal{E}}_S^{(+)}(z)\hat{\mathcal{P}}_{eg}(z) + \hat{\mathcal{E}}_S^{(-)}(z)\hat{\mathcal{P}}_{ge}(z)\right)\right)\mathrm{d}z. \tag{4}$$

We are interested in removing fast oscillations and thus rather work with a slowly varying (both in time and space) version of $\mathcal{E}_I^{(+)}(z,t)$. We also want to work with dimensionless quantities and thus using $g_0 = \sqrt{\hbar\omega_L/(2\epsilon_0 V)}$ (where V is product of length of the ensemble L and quantisation area A) define

$$\hat{E}(z,t) = \frac{1}{g_0}\exp(i\omega_L t - i\omega_L z/c)\hat{\mathcal{E}}_I^{(+)}(z,t) = \sqrt{\frac{L}{2\pi c}}\int_0^\infty \hat{a}(\omega)e^{i(\omega-\omega_L)z/c}e^{i(\omega-\omega_L)t}\,\mathrm{d}\omega \tag{5}$$

$$\approx \frac{1}{g_0}\exp(-\omega_L z/c)\hat{\mathcal{E}}_S^{(+)}(z,t) \tag{6}$$

Since we assumed that our field had a small bandwidth around ω_L and was propagating in forward direction the states corresponding to creation and destruction operators with negative frequencies can be assumed to be in vacuum state. This allows us to extend lower range of integration in (5) to $-\infty$ and write $\hat{E}$ in terms of Fourier transform of $\hat{a}(\omega)$:

$$\hat{a}(t) = \int_{-\infty}^{\infty} \hat{a}(\omega) e^{-i\omega t}\, \mathrm{d}\omega$$

that has commutator

$$\left[\hat{a}(t), \hat{a}^{\dagger}(t')\right] = \int_{-\infty}^{\infty} e^{-i\omega(t-t')}\, \mathrm{d}\omega = 2\pi\delta(t-t').$$

And so

$$\hat{E}(z,t) = \sqrt{\frac{L}{2\pi c}}\hat{a}(t - z/c) \tag{7}$$

so it has commutator

$$\left[\hat{E}(z,t), \hat{E}^{\dagger}(z',t)\right] = \frac{L}{2\pi c} 2\pi\delta\left((z-z')/c\right) = L\delta(z-z'). \tag{8}$$

With such a definition the expectation value $\langle \hat{E}^{\dagger}(z,t)\hat{E}(z,t)\rangle$ is number of photons present in the field up to a dimensionless constant.

To find dynamics of operator (5) we first consider how it entered the interation picture. To that end we look at

$$\begin{aligned}
\frac{i}{\hbar}\left[\hat{H}_L, \hat{\mathcal{E}}_S(z)e^{-i\omega_L z/c}\right] &= \mathcal{E}_0(\omega_L)\int_0^{\infty} \omega e^{i(\omega-\omega_L)z/c}\hat{a}(\omega)\, \mathrm{d}\omega \\
= -i\mathcal{E}_0(\omega_L)\Bigg(c\int_0^{\infty} i\frac{\omega-\omega_L}{c}&e^{i(\omega-\omega_L)z/c}\hat{a}(\omega)\, \mathrm{d}\omega + i\omega_L \int_0^{\infty} e^{i(\omega-\omega_L)z/c}\hat{a}(\omega)\, \mathrm{d}\omega\Bigg) \\
&= -c\frac{\partial}{\partial z}\left(\hat{\mathcal{E}}_S^{(+)}(z)e^{-i\omega_L z/c}\right) - i\omega_L\left(\hat{\mathcal{E}}_S^{(+)}(z)e^{-i\omega_L z/c}\right).
\end{aligned}$$

Now adding contribution due to $\hat{H}'$ as given in (4) results in equation

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\hat{\mathcal{E}}_I^{(+)}(z,t)e^{-i\omega_L z/c}\right) = -c\frac{\partial}{\partial z}&\left(\hat{\mathcal{E}}_I^{(+)}(z,t)e^{-i\omega_L z/c}\right) - i\omega_L\left(\hat{\mathcal{E}}_I^{(+)}(z,t)e^{-i\omega_L z/c}\right) \\
&+\frac{i}{\hbar}\left[\hat{H}', \hat{\mathcal{E}}_I^{(+)}(z,t)e^{-i\omega_L z/c}\right]
\end{aligned}$$

which upon rearranging and using definition (5) becomes

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)\hat{E}(z,t) = \frac{i}{\hbar}\left[\hat{H}', \hat{E}(z,t)\right]. \tag{9}$$

Using approximation (6) and commutator (8) the right hand side of (9) is

$$\frac{i}{\hbar}\left[\hat{H}', \hat{E}(z,t)\right] = i\frac{\sqrt{N}g_0}{L\hbar}\int_0^L \wp\left[\hat{E}(z,t), \hat{E}^{\dagger}(z,t)\right] e^{-i\omega_L z'/c}\hat{\mathcal{P}}_{ge}(z)\, \mathrm{d}z' = i\beta\hat{P}(z)$$

where we defined $\hat{P}(z) = e^{-i\omega_L z/c}\hat{\mathcal{P}}_{ge}(z)$ and $\beta = \sqrt{N}g_0\wp/\hbar$. Equation (9) can now be written as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)\hat{E}(z,t) = i\beta\hat{P}(z,t). \tag{10}$$

Assuming most of the population is in the ground state such that in our normalisation it means that $\mathcal{P}_{ee}(z) = \sqrt{N}$ and $\mathcal{P}_{gg}(z) = 0$ time dependence of $\hat{P}$ is given by

$$
\begin{aligned}
&\frac{i}{\hbar}\left[\hat{H}', \hat{\mathcal{P}}_{ge}(z)\right] \\
&= \frac{i}{\hbar}\frac{\sqrt{N}}{L}\int_0^L \left(i\hbar\Delta_0\left[\hat{\mathcal{P}}_{ee}(z'), \hat{\mathcal{P}}_{ge}(z)\right] - \wp\hat{\mathcal{E}}_S^{(+)}(z')\left[\hat{\mathcal{P}}_{eg}(z'), \hat{\mathcal{P}}_{ge}(z)\right]\right)\mathrm{d}z' \\
&= i\int_0^L \delta(z-z')\left(-\Delta_0\hat{\mathcal{P}}_{ge}(z) - \frac{\wp}{\hbar}\hat{\mathcal{E}}_S^{(+)}(z')\left(\hat{\mathcal{P}}_{ee}(z') - \hat{\mathcal{P}}_{gg}(z')\right)\right)\mathrm{d}z' \\
&= -i\Delta_0\hat{\mathcal{P}}_{ge}(z) - i\frac{\wp}{\hbar}\hat{\mathcal{E}}_S^{(+)}(z)\left(\hat{\mathcal{P}}_{ee}(z) - \hat{\mathcal{P}}_{gg}(z)\right) \\
&\approx -i\Delta_0\hat{\mathcal{P}}_{ge}(z) + i\frac{\wp\sqrt{N}}{\hbar}\hat{\mathcal{E}}_S^{(+)}(z)
\end{aligned}
$$

which implies

$$\frac{\partial}{\partial t}\hat{P}(z,t) = -i\Delta_0\hat{P}(z,t) + i\beta\hat{E}(z,t). \tag{11}$$

We shall also need to deal with the situation when atoms are inhomogeneously broadened with distribution which we call G. In anticipation of making our equations of motion dimensionless we note that $1/\mu$ where $\mu = L\beta^2/c$ turns out to be the natural time scale of our system and thus we want to work with detunings $\tilde{\Delta} = \Delta/\mu$ and $\tilde{\Delta}_0 = \Delta_0/\mu$. Our G then satisfies $\int_{-\infty}^{\infty} G(\tilde{\Delta}')\,\mathrm{d}\tilde{\Delta}' = 1$ and thus is dimensionless.

Generalisation to this broadened case is then straightforward. We introduce operators that only act on spatial interval $[k(\delta z), (k+1)(\delta z)]$ and detuning interval $[\tilde{\Delta}_0 + m(\delta\tilde{\Delta}), \tilde{\Delta}_0 + (m+1)(\delta\tilde{\Delta})]$:

$$\hat{\mathcal{S}}_{km,ij} = \frac{\sqrt{N}}{N_{km}}\sum_{n=1}^{N_{km}} |i\rangle_n\langle j|_n$$

where $N_{km} = NG\left(m(\delta\tilde{\Delta})\right)(\delta\tilde{\Delta})(\delta z)/L$. Again we make $(\delta\tilde{\Delta})$ and (δz) small to consider $\mathcal{S}$ to be a function of $\tilde{\Delta}$ and z. With these operators the analog to Hamiltonian (4) becomes

$$
\begin{aligned}
\hat{H}' = \int_{-\infty}^{\infty}\frac{\sqrt{N}G(\tilde{\Delta})}{L}\int_0^L &\Big(\hbar\mu(\tilde{\Delta}_0 + \tilde{\Delta})\hat{\mathcal{S}}_{ee}(z) \\
&-\wp\left(\hat{\mathcal{E}}_S^{(+)}(z)\hat{\mathcal{S}}_{eg}(z,\tilde{\Delta}) + \hat{\mathcal{E}}_S^{(-)}(z)\hat{\mathcal{S}}_{ge}(z,\tilde{\Delta})\right)\Big)\,\mathrm{d}z\,\mathrm{d}\tilde{\Delta}.
\end{aligned} \tag{12}
$$

Defining $\hat{\sigma}(z,t,\tilde{\Delta}) = e^{-i\omega_L z/c}\hat{\mathcal{S}}_{ge}(z,t,\tilde{\Delta})$ equations of motion (10) and (11) become

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)\hat{E}(z,t) = i\beta\int_{-\infty}^{\infty} G(\tilde{\Delta}')\hat{\sigma}(z,t,\tilde{\Delta}')\,\mathrm{d}\tilde{\Delta}' \tag{13}$$

and

$$\frac{\partial}{\partial t}\hat{\sigma}(z,t,\tilde{\Delta}) = -i\mu(\tilde{\Delta}_0 + \tilde{\Delta})\hat{\sigma}(z,t,\tilde{\Delta}) + i\beta\hat{E}(z,t). \tag{14}$$

To describe reversal of detunings we can replace $\tilde{\Delta}$ by $-\tilde{\Delta}$ in (13) and (14).

We want to use all the derived equations of motion to calculate efficiencies, i.e. $\eta_E = \langle\hat{E}^\dagger(z,t)\hat{E}(z,t)\rangle$ or $\eta_P = \langle\hat{P}^\dagger(z,t)\hat{P}(z,t)\rangle$. We assume that a photon with mode function Φ

comes in the atomic ensemble and excites polarisation in the atomic ensemble with mode function Θ. Formally we can define creation operators for these modes as

$$\hat{c}_\Phi = \int_0^\infty \Phi^*(t)\hat{E}(z=z_0,t)\,\mathrm{d}t$$
$$\hat{c}_\Theta = \int_0^L \Theta^*(z,t)\hat{P}(z,t)\,\mathrm{d}z$$

If we assume that the state in expression for η_E is $|1\rangle = \hat{c}_\Phi^\dagger|0\rangle$ then $\eta_E = \int_0^\infty \left|\Phi(t)\right|^2 \mathrm{d}t$ and simliarly for P and σ. Thus we can consider equations (10), (11), (13) and (14) as equations of these mode functions and simply drop hats. We'll also assume that mode functions are normalised, i.e. $\int_0^L |P(z)|^2\,\mathrm{d}z = 1$ and $\int_0^\infty |E(z=z_0,t)|^2\,\mathrm{d}t = 1$. When after manipulations (say after storage and retrieval) with these mode functions we calculate these integrals and find that they are not normalised it's to be interpreted as probability leaking into other modes. We shall also continue calling P the "polarisation operator" occasionally even though it's not an operator anymore. Also we shall talk about "light pulses" and "photons" interchangibly when referring to electric field as the derived equations can just as well describe classical and quantum light. Provided classical light is weak so assumption of most of the population being in the ground state is satisfied we can just take expectation values of equations of motion with a coherent state and get equations for functions instead.

4 Principle of operation

In stage 1 light comes in the atomic ensemble and excites atoms. We shall assume that stage 1 is just long enough for light to be absorbed by the atomic ensemble. In stage 2 inhomogeneous broadening is introduced to prevent atoms from radiating their excitation as electric field again. In stage 3 inhomogeneous broadening is reversed to make the atomic ensemble evolve back to its state in the beginning of stage 2. In stage 4 inhomogeneous broadening is turned off and light is radiated in forward direction.

Significant part of description of stage 1 and 4 can be treated in analytic way as was shown in [1]. In this thesis some elaboration is given on the equations compared to the mentioned article. Stages 2 and 3 are made more complicated by the having inhomogeneous broadening and thus only perturbative treatment of those is given.

We assume that all the stages don't have any decay. At later point a particular decay mechanism in stage 2 and 3 will be considered.

5 Stage 1 and 4

We consider (10), (11) as equations of mode functions and change variables to $\tilde{z} = z/L$ and $\tau = \mu(t - z/c)$. Since

$$c\,\frac{\partial}{\partial z}\,E(\tilde{z},\tau) = c\left(\frac{\partial E}{\partial \tilde{z}}\,\frac{\partial \tilde{z}}{\partial z} + \frac{\partial E}{\partial \tau}\,\frac{\partial \tau}{\partial z}\right) = \frac{c}{L}\,\frac{\partial E}{\partial \tilde{z}} - \mu\,\frac{\partial E}{\partial \tau}$$
$$\frac{\partial}{\partial t}\,E(\tilde{z},\tau) = \frac{\partial E}{\partial \tilde{z}}\,\frac{\partial \tilde{z}}{\partial t} + \frac{\partial E}{\partial \tau}\,\frac{\partial \tau}{\partial t} = \mu\,\frac{\partial E}{\partial \tau}$$
$$\frac{\partial}{\partial t}\,P(\tilde{z},\tau) = \frac{\partial P}{\partial \tilde{z}}\,\frac{\partial \tilde{z}}{\partial t} + \frac{\partial P}{\partial \tau}\,\frac{\partial \tau}{\partial t} = \mu\,\frac{\partial P}{\partial \tau}$$

these equations become

$$\frac{\partial}{\partial \tilde{z}}\,E(\tilde{z},\tau) = i\frac{\mu}{\beta}P(\tilde{z},\tau) \tag{15}$$

and

$$\frac{\mu}{\beta}\frac{\partial}{\partial\tau}P(\tilde{z},\tau)=-i\frac{\Delta_0}{\mu}\frac{\mu}{\beta}P(\tilde{z},\tau)+iE(\tilde{z},\tau). \tag{16}$$

Absorbing μ/β into definition of P we end up with completely dimensionless equations

$$\frac{\partial}{\partial\tilde{z}}E(\tilde{z},\tau)=iP(\tilde{z},\tau) \tag{17}$$

and

$$\frac{\partial}{\partial\tau}P(\tilde{z},\tau)=-i\tilde{\Delta}_0P(\tilde{z},\tau)+iE(\tilde{z},\tau). \tag{18}$$

Here it'd be apropriate to discuss how stages 2 and 3 are related to 1 and 4. We've just changed time variable to combination of time and spatial variable. This means that when we say that stage 1 – for instance – had duration of τ_1 then this actually means that stage 2 began at different real times in the ensemble of atoms and broadening was turned on gradually starting from $z=0$ and propagating with speed of light towards $z=L$. Effects of turning broadening in some other way was not investigated in this thesis since the one mentioned was the most natural. In the sequel τ_n will denote duration of stage n understood in the sense above.

If we define $E_{\text{in}}(\tau)=E(\tilde{z}=0,\tau)$ and Laplace transform in space ($\tilde{z}\to u$) equations (17), (18) we get

$$-E_{\text{in}}(\tau)+u\bar{E}(u,\tau)=i\bar{P}(u,\tau) \tag{19}$$

and

$$\frac{\partial}{\partial\tau}\bar{P}(u,\tau)=-i\tilde{\Delta}_0\bar{P}(u,\tau)+i\bar{E}(u,\tau). \tag{20}$$

Expressing $\bar{E}(u,\tau)$ in terms of $\bar{P}(u,\tau)$ using (19) and substituting into (20) we get

$$\frac{\partial}{\partial t}\bar{P}(u,\tau)+\left(i\tilde{\Delta}_0+\frac{1}{u}\right)\bar{P}(u,\tau)=i\frac{E_{\text{in}}(\tau)}{u} \tag{21}$$

In stage 1 we assume that there're no excitations of atoms, i.e. $P(z,t=z/c)=P(\tilde{z},\tau=0)=0$. This expresses the fact that at time $t=z/c$ the light pulse has just reached atoms at the position z. Solving (21) under this initial condition gives

$$\bar{P}(u,\tau)=i\int_0^\tau\exp\left(-\left(i\tilde{\Delta}_0+\frac{1}{u}\right)(\tau-\tau')\right)\frac{E_{\text{in}}(\tau')}{u}\,\mathrm{d}\tau'.$$

Taking inverse Laplace transform ($u\to\tilde{z}$) we arrive at

$$P(\tilde{z},\tau)=i\int_0^\tau\exp\left(-i\tilde{\Delta}_0(\tau-\tau')\right)J_0\left(2\sqrt{(\tau-\tau')\tilde{z}}\right)E_{\text{in}}(\tau')\,\mathrm{d}\tau' \tag{22}$$

where J_0 is Bessel function of the first kind of order 0.

This is the final polarisation operator for stage 1. Turning now to stage 4 we need to solve equations (19) and (20) under assumption that $E_{\text{in}}(\tau)=0$ and $P(\tilde{z},\tau=0)=P_3(\tilde{z})$. Equation (21) now has solution

$$\bar{P}(u,\tau)=\exp\left(-\left(i\tilde{\Delta}_0+\frac{1}{u}\right)\tau\right)\bar{P}_3(u)$$

and inserting this into (19) we get

$$\bar{E}(u,\tau) = i\exp(-i\tilde{\Delta}_0\tau)\frac{1}{u}\exp\left(-\frac{1}{u}\tau\right)\bar{P}_3(u).$$

We take inverse Laplace transform ($u \to \tilde{z}$) using convolution theorem. Defining $E_{\text{out}} = E(\tilde{z} = 1, \tau)$ we arrive at

$$E_{\text{out}}(\tau) = ie^{-i\tilde{\Delta}_0\tau}\int_0^1 J_0\left(2\sqrt{\tau z'}\right)P_3(1-z')\,\mathrm{d}z'. \tag{23}$$

The question now is when and in what sense equations (22) and (23) can be considered inverses of each other. First of all one useful property of (23) is that it preserves inner products. For any functions f and g defined on $(0,1)$ their inner product in L^2 Hilbert space sense is

$$\langle f|g\rangle = \int_0^1 f^*(\tilde{z})g(\tilde{z})\,\mathrm{d}\tilde{z}.$$

We can define retrieval operator by

$$R(f)(\tau) = ie^{-i\tilde{\Delta}_0\tau}\int_0^1 J_0\left(2\sqrt{\tau(1-z')}\right)f(z')\,\mathrm{d}z'.$$

such that (23) can be written $E_{\text{out}}(\tau) = R(P_3)(\tau)$ with suitable substitution. Then property of R can be stated symbolically as

$$\langle R(f)|R(g)\rangle = \langle f|g\rangle \tag{24}$$

where inner product on the left hand side is to be understood as integration from 0 to ∞. This can be proved by using (48) from Appendix A. We have

$$\begin{aligned}
\langle R(f)|R(g)\rangle &= \int_0^\infty R(f)^*(\tau)R(g)(\tau)\,\mathrm{d}\tau \\
&= \int_0^\infty \int_0^1 J_0\left(2\sqrt{\tau(1-z')}\right)f^*(z')\,\mathrm{d}z' \int_0^1 J_0\left(2\sqrt{\tau(1-z'')}\right)g(z'')\,\mathrm{d}z''\,\mathrm{d}\tau \\
&= \int_0^1\int_0^1 \delta\left((1-z')-(1-z'')\right)f(z')g(z'')\,\mathrm{d}z''\,\mathrm{d}z' = \int_0^1 f^*(z')g(z')\,\mathrm{d}z' \\
&= \langle f|g\rangle.
\end{aligned}$$

In particular it means that efficiency of the retrieved electric field is the same as efficiency of the polarisation operator:

$$\eta_E = \langle R(P)|R(P)\rangle = \langle P|P\rangle = \eta_P.$$

The significance of this result it two-fold. First of all it means that stage 4 has no loss and if stage 2 and 3 were not there then all losses if any would be caused by stage 1. Second is that when we begin to consider effects of stage 2 and 3 it's only nessesary to calculate efficiency of polarisation operator in the end of stage 3 since we know that efficiency of the retrieved field would be exactly the same.

We need to stress the fact that even though the result just derived is very useful in simplifying arguments in the current model we must remember that it only holds for the case without decay. The way decay was included in the model of [1] would in our model result in addition of factors of $e^{-\tilde{\Gamma}\tau}$ (for some rescaled decay rate $\tilde{\Gamma} = \Gamma/\mu$) in (22) and (23). Then instead of the expression (48) from Appendix A we would need to consider integral

$$\int_0^\infty e^{-\tilde{\Gamma}t} J_0(2\sqrt{z'\tau}) J_0(2\sqrt{z''\tau})\,\mathrm{d}\tau$$

which doesn't seem to be as easily tractable. Numerical optimisation of the process of storage and forward retrieval as can be found in [1] then showed that the best mode of P (i.e. the one that has maximal efficiency) converged towards $P(\tilde{z}) = \sqrt{30}\tilde{z}(1-\tilde{z})$ as decay rate went to 0. Since in our model we effectively work in this limit we should be able to reproduce this. In fact it seems that any polarisation operator mode can be retrieved and then stored optimally.

To show that we consider process of storage. We see that here the concept of time reversal comes naturally from the equations. When we shall say in the following that we store a particular input field E_{in} it needs to be understood that we store a time reversed copy of it. In terms of our equations it will mean that we integrate $E_{\mathrm{in}}(\tau - \tau')$ instead of $E_{\mathrm{in}}(\tau')$ in expression (22). Then making substitution $x = \tau - \tau'$ that equation will become

$$P(\tilde{z},\tau) = i\int_0^\tau \exp\left(i\tilde{\Delta}_0 x\right) J_0\left(2\sqrt{\tilde{z}x}\right) E_{\mathrm{in}}(x)\,\mathrm{d}x \tag{25}$$

We also need to consider what happens in the limit when $\tau \to \infty$ and call this process storage with infinite storage time. Making upper limit ∞ is rather unphysical and would be made impossible by having decay included in the model but it's nevertheless a useful tool to explore boundaries of the current model.

Assume that stage 2 and 3 are not there and assume further that in the beginning of stage 4 we have some polarisation operator P_0. If we retrieve it and then store it on resonance ($\tilde{\Delta}_0 = 0$) with infinite storage time we get (again using (48) from Appendix A)

$$P(\tilde{z}) = -\int_0^\infty J_0\left(2\sqrt{x\tilde{z}}\right)\int_0^1 J_0\left(2\sqrt{xz'}\right) P_0(1-z')\,\mathrm{d}z'\,\mathrm{d}x = -P_0(1-\tilde{z})$$

This means that efficiency $\eta = \int_0^1 |P(\tilde{z})|^2\,\mathrm{d}\tilde{z}$ of this process is equal to 1.

Now we know that if we take any mode that was already stored in the memory, read it out and then read the resulting field back in again – we end up with practically the same thing that we started with – up to mirroring about $\tilde{z}$-axis or around point $\tilde{z} = 1/2$. What is left to be determined is whether the opposite process will work without any losses. Namely if we start with arbitrary field, store it and then retrieve it again. Our intuition suggests that given that the fields of travelling waves could not be satisfactorily described with infinite sums of discrete creation operators and needed continuum wave formalism instead their space is much too large to be mapped onto an ensemble of atoms with a finite spatial extent. What we're after is to find a convincing counterexample.

Consider $E_{\mathrm{in}} = \sqrt{2}e^{-\tau}$. Storing it with infinite storage time and with $\tilde{\Delta}_0 = 0$ we get

$$P(\tilde{z}) = \sqrt{2}i\int_0^\infty J_0\left(2\sqrt{\tilde{z}x}\right) E_{\mathrm{in}}(x)\,\mathrm{d}x = \sqrt{2}ie^{-z}.$$

Already here we see that efficiency of storage $\int_0^1 |P(\tilde{z})|^2\,\mathrm{d}\tilde{z} = 1 - e^{-2} < 1$ and that means that this input field couldn't be mapped onto state of atoms in its entirety even in the limit of infinite storage time. For completeness we can calculate electric field that would have been retrieved from this mode using (23). Plot of it can be found in Fig. 2.

The conclusion of this section is that there exist particular modes of electric field that can be stored (in stage 1) with 100 % efficiency. They can all be generated by taking some mode of polarisation operator and then finding output field. On the other hand not all input fields can be stored and efficiency of storage has to be determined on case by case basis.

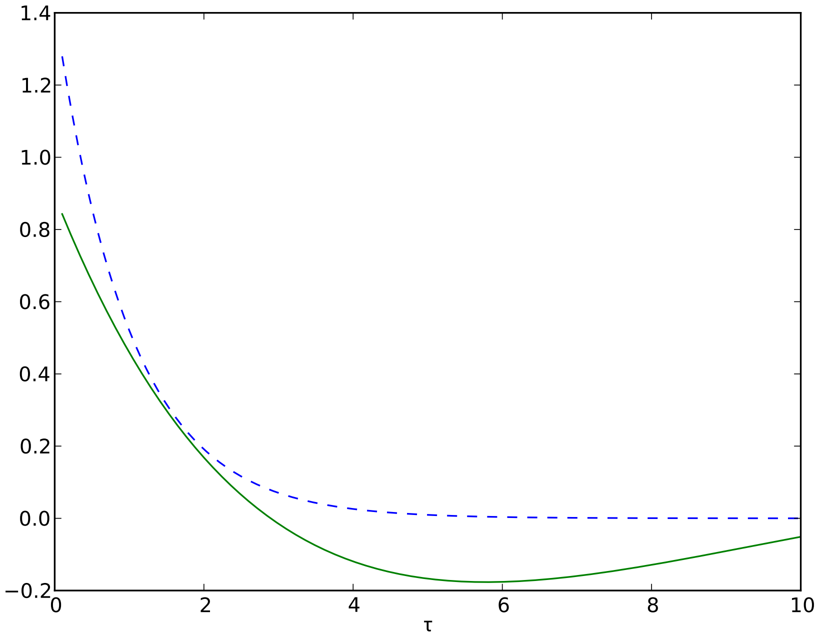


Figure 2: Plot of exponential input mode $E_{\text{in}}(\tau) = \sqrt{2}e^{-\tau}$ (dashed) and corresponding output mode (solid).

6 Stage 2 and 3

We look at the equations (13) and (14). Making the same change of variables as in Section 5 and absorbing a factor of μ/β into σ our equations become

$$\frac{\partial}{\partial \tilde{z}} E(\tilde{z}, \tau) = i \int_{-\infty}^{\infty} G(\tilde{\Delta}')\sigma(\tilde{z}, \tau, \tilde{\Delta}') \, \mathrm{d}\tilde{\Delta}' \tag{26}$$

and

$$\frac{\partial}{\partial \tau} \sigma(\tilde{z}, \tau, \tilde{\Delta}) = -i(\tilde{\Delta}_0 + \tilde{\Delta})\sigma(\tilde{z}, \tau, \tilde{\Delta}) + iE(\tilde{z}, \tau). \tag{27}$$

We assume that the electric field got completely absorbed in stage 1 and so $E_{\text{in}}(\tau) = 0$ for stage 2 and 3. Even under this assumption some field will be radiated in stage 2 due to imperfections of our memory so this assumption is less true for stage 3 but for simplicity of having only to consider evolution of σ in these two stages we assume that anyway.

Taking the Laplace transform ($\tilde{z} \to u$) of (26) and (27) we get

$$u\bar{E}(u, \tau) = i \int_{-\infty}^{\infty} G(\tilde{\Delta}')\bar{\sigma}(u, \tau, \Delta') \, \mathrm{d}\Delta' \tag{28}$$

and

$$\frac{\partial}{\partial \tau} \bar{\sigma}(u, \tau, \tilde{\Delta}) = -i(\tilde{\Delta}_0 + \tilde{\Delta})\bar{\sigma}(u, \tau, \tilde{\Delta}) + i\bar{E}(u, \tau). \tag{29}$$

When we insert $\bar{E}(u,\tau)$ from (28) into (29) we arrive at the equation

$$\frac{\partial}{\partial\tau}\,\bar{\sigma}(u,\tau,\tilde{\Delta}) = -i(\tilde{\Delta}_0+\tilde{\Delta})\bar{\sigma}(u,\tau,\tilde{\Delta}) - \frac{1}{u}\int_{-\infty}^{\infty} G(\tilde{\Delta}')\bar{\sigma}(u,\tau,\tilde{\Delta}')\,\mathrm{d}\tilde{\Delta}'. \tag{30}$$

This is the analog of equation (21) in case of broadening. We can now control contribution of the term containing $1/u$ of the right hand side by integrating it with distribution G. We also see that in a sense we have a system of infinitely many coupled differential equations (one for each $\tilde{\Delta}$) and this is unfortunate if we want an exact solution. We note, however, that it's us who brought this problem upon our heads when we made operator $\hat{\sigma}$ depend on a real variable $\tilde{\Delta}$ instead of an index of an interval of detunings. If we didn't do that step we'd end up with at most N equations – one for each atom. In that form it would be much more suited for finding numerical solution.

Now we turn to the initial conditions. For σ in stage 2 the initial condition is that $\bar{\sigma}(u,\tau=0,\tilde{\Delta})$ should be proportional to $\bar{P}(u,\tau=\tau_1)$. Since they both have the same normalisation they are actually equal. For stage 4 the initial condition is

$$\bar{P}(u,\tau=0) = \int_{-\infty}^{\infty} G(\tilde{\Delta}')\bar{\sigma}(u,\tau=\tau_3,-\tilde{\Delta}')\,\mathrm{d}\tilde{\Delta}'.$$

(The $-\tilde{\Delta}$ in the argument is used to remind that σ in stage 3 has reversed broadening compared to stage 2.)

To get any further analysis we assume a specific form of G. Defining $\tilde{\gamma}=\gamma/\mu$ we choose

$$G(\tilde{\Delta}) = \frac{1}{\sqrt{2\pi\tilde{\gamma}^2}}\exp\left(-\frac{\tilde{\Delta}^2}{2\tilde{\gamma}^2}\right), \tag{31}$$

i.e. a Gaussian and try to find an approximative solution for equation (30). First of all to get some intuitive feeling for it suppose for a moment that the second term on right hand side of (30) is equal to 0. Then time dependence of $\bar{\sigma}(u,\tau,\tilde{\Delta})$ is a simple complex exponential one. If we assume (30) to describe evolution in stage 2 then evolution in stage 3 is given by the same equation with $\tilde{\Delta}$ replaced by $-\tilde{\Delta}$. After these two stages the initial value of $\bar{\sigma}$, i.e. $\bar{\sigma}(u,\tau=0,\tilde{\Delta})$ will get multiplied by a factor of $\exp\Big(-i\tilde{\Delta}_0(\tau_2+\tau_3)-i\tilde{\Delta}(\tau_2-\tau_3)\Big)$. We don't want to have dependence on $\tilde{\Delta}$ in the final value since the initial condition for stage 4 is

$$P(\tilde{z},\tau=0) \propto \int_{-\infty}^{\infty} G(\tilde{\Delta}')\exp(i\tilde{\Delta}'(\tau_2-\tau_3))\,\mathrm{d}\tilde{\Delta}' = \exp\left(-\frac{1}{2}(\tau_2-\tau_3)^2\gamma^2\right)$$

and will be less than 1 unless $\tau_2=\tau_3$. So in order for the first term in equation (30) to not incur any losses we need stage 2 and stage 3 to have the same duration.

We want to consider effect of the second term on the right hand side of (30) as a perturbation. Introducing

$$\sigma_S(\tilde{z},\tau,\tilde{\Delta}) = e^{i(\tilde{\Delta}_0+\tilde{\Delta})\tau}\sigma(\tilde{z},\tau,\tilde{\Delta}) \tag{32}$$

equation (30) can be rewritten as

$$\frac{\partial}{\partial\tau}\,\bar{\sigma}_S(u,\tau,\tilde{\Delta}) = -\frac{1}{u}e^{i\tilde{\Delta}\tau}\int_{-\infty}^{\infty} G(\tilde{\Delta}')\bar{\sigma}_S(u,\tau,\tilde{\Delta}')e^{-i\tilde{\Delta}'\tau}\,\mathrm{d}\tilde{\Delta}'. \tag{33}$$

Doing perturbative approximation we replace σ_S in the integrand by its value in $\tau=0$:

$$\frac{\partial}{\partial\tau}\,\bar{\sigma}_S(u,\tau,\tilde{\Delta}) \approx -\frac{1}{u}e^{i\tilde{\Delta}\tau}\int_{-\infty}^{\infty} G(\tilde{\Delta}')\bar{\sigma}_S(u,\tau=0,\tilde{\Delta}')e^{-i\tilde{\Delta}'\tau}\,\mathrm{d}\tilde{\Delta}'. \tag{34}$$

For the case of stage 2 we have that $\bar{\sigma}_S(u, \tau = 0, \tilde{\Delta}) = \bar{\sigma}(u, \tau = 0, \tilde{\Delta}) = \bar{P}(u, \tau = \tau_1)$ is actually independent of $\tilde{\Delta}$. Hence we take $\bar{\sigma}_S(u, \tau = 0, \tilde{\Delta})$ out of the integral and defining $P_1(\tilde{z}) = \sigma_S(\tilde{z}, \tau = 0, \tilde{\Delta})$ we can write solution to (34) as

$$\bar{\sigma}_S(u, \tau, \tilde{\Delta}) = \left(1 - \frac{1}{u}\int_0^\tau e^{i\tilde{\Delta}\tau'} \int_{-\infty}^{\infty} G(\tilde{\Delta}')e^{-i\tilde{\Delta}'\tau'} \,\mathrm{d}\tilde{\Delta}' \,\mathrm{d}\tau'\right) \bar{P}_1(u)$$

Using the specific form of our G from equation (31) and definition of σ_S in (32) we arrive at

$$\bar{\sigma}(u, \tau, \tilde{\Delta}) = e^{-i(\tilde{\Delta}_0+\tilde{\Delta})\tau} \left(1 - \frac{1}{u}\int_0^\tau \exp\left(-\frac{1}{2}\gamma^2\tau'^2 + i\tilde{\Delta}\tau'\right) \mathrm{d}\tau'\right) \bar{P}_1(u) \tag{35}$$

At this point it's best to take a shortcut. We could do a full perturbative treatment of stage 3 by obtaining equation (33) for $\bar{\sigma}_S(u, \tau, -\tilde{\Delta})$ and solving that. This is done in Appendix C in full detail. It turns out that in the limit where τ_2 (and consequently also τ_3) is big the result is the same as if we completely disregarded $1/u$ term in the equation (30). I.e. making changes for reversed broadening we assume that stage 3 is described by equation

$$\frac{\partial}{\partial\tau}\,\bar{\sigma}(u, \tau, -\tilde{\Delta}) = -i(\tilde{\Delta}_0 - \tilde{\Delta})\bar{\sigma}(u, \tau, -\tilde{\Delta}) \tag{36}$$

which solving under initial condition $\bar{\sigma}(u, \tau = 0, -\tilde{\Delta}) = \bar{\sigma}(u, \tau = \tau_2, \tilde{\Delta})$, assuming that $\tau_3 = \tau_2$ and using expression (35) gives in the end of stage 3 the following expression:

$$\bar{\sigma}(u, \tau = \tau_2, -\tilde{\Delta}) = e^{-i2\tilde{\Delta}_0\tau_2} \left(1 - \frac{1}{u}\int_0^{\tau_2} \exp\left(-\frac{1}{2}\gamma^2\tau'^2 + i\tilde{\Delta}\tau'\right) \mathrm{d}\tau'\right) \bar{P}_1(u). \tag{37}$$

From (37) we can find $\bar{P}(u, \tau = \tau_2) = \int_{-\infty}^{\infty} G(\tilde{\Delta}')\bar{\sigma}(u, \tau = \tau_2, -\tilde{\Delta}') \,\mathrm{d}\tilde{\Delta}'$ and take inverse Laplace transform using convolution theorem. Then expression for P in the end of stage 3 in terms of what is was in the end of stage 1 is given by

$$P_3(\tilde{z}) = e^{-i2\tilde{\Delta}_0\tau_2} P_1(\tilde{z}) - e^{-i2\tilde{\Delta}_0\tau_2} \int_0^{\tilde{z}} P_1(z') \,\mathrm{d}z' \int_0^{\tau_2} e^{-\tilde{\gamma}^2\tau'^2} \,\mathrm{d}\tau'. \tag{38}$$

Compared to (51) in Appendix C it only has two last terms. But when we take limit $\tau_2 \to \infty$ they both give the same:

$$\left|P(\tilde{z}, \tau = \tau_2)\right|^2 \to \left|P_1(\tilde{z}) - \frac{\sqrt{\pi}}{2\tilde{\gamma}}\int_0^{\tilde{z}} P_1(z') \,\mathrm{d}z'\right|^2.$$

Assuming that stage 1 didn't have any losses the efficiency of storage is given by

$$\begin{aligned} \eta &= \int_0^1 \left|P(\tilde{z}, \tau = \tau_2)\right|^2 \mathrm{d}\tilde{z} \\ &= 1 - \frac{\sqrt{\pi}}{\tilde{\gamma}}\int_0^1 P_1(\tilde{z}) \int_0^{\tilde{z}} P_1(z') \,\mathrm{d}z' \,\mathrm{d}\tilde{z} + \frac{\pi}{4\tilde{\gamma}^2}\int_0^1 \left(\int_0^{\tilde{z}} P_1(z') \,\mathrm{d}z'\right)^2 \mathrm{d}\tilde{z}. \end{aligned} \tag{39}$$

It's clear that this result is not valid for $\tilde{\gamma} \to 0$. In that limit we effectively have no broadening and thus a more apropriate description would've been using equation (22) for P of stage 4 for which perturbative assumption may be not satisfied. On the other hand when $\tilde{\gamma} \to \infty$ the efficiency goes to 1 as we have hoped.

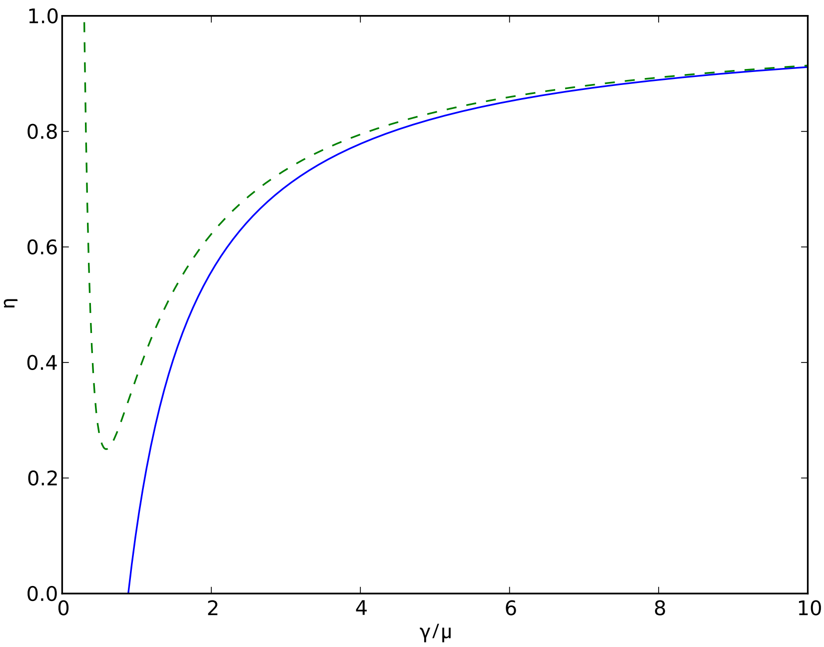


Figure 3: This is a plot of storage efficiency η as function of $\tilde{\gamma} = \gamma/\mu$ for $P(\tilde{z}) = 1$. Solid line has only linear contribution while dashed line also has quadratic contribution. Both curves are clearly not valid as expressions for efficiency for $\tilde{\gamma}$ below 1.

As shown in Appendix B term linear in $1/\tilde{\gamma}$ can be maximised (so that η is minimised) by taking P constant, i.e. $P(\tilde{z}) = 1$ and then

$$\int_0^1 P_1(\tilde{z}) \int_0^{\tilde{z}} P_1(z')\,\mathrm{d}z'\,\mathrm{d}\tilde{z} = \int_0^1 \tilde{z}\,\mathrm{d}\tilde{z} = \frac{1}{2}.$$

Strictly speaking what we show there is that it has a stationary point for this form. But we can see that this term will be smaller for any other form of P. Moreover by taking $P_1(\tilde{z}) = \sqrt{2}\cos(\pi\tilde{z})$ we get

$$\int_0^1 P_1(\tilde{z}) \int_0^{\tilde{z}} P_1(z')\,\mathrm{d}z'\,\mathrm{d}\tilde{z} = \frac{2}{\pi}\int_0^1 \cos(\pi\tilde{z})\sin(\pi\tilde{z})\,\mathrm{d}\tilde{z} = 0$$

while

$$\int_0^1 \left(\int_0^{\tilde{z}} P_1(z')\,\mathrm{d}z'\right)^2 \mathrm{d}\tilde{z} = \frac{2}{\pi^2}\int_0^1 \sin^2(\pi\tilde{z})\,\mathrm{d}\tilde{z} = \frac{1}{\pi^2}$$

and thus term linear in $1/\tilde{\gamma}$ vanishes but the term quadratic in $1/\tilde{\gamma}$ does not and we end up with efficiency greater than 1 for all values of $\tilde{\gamma}$. This serves as a motivation to drop the quadratic term as it's not valid for at least some P's. Impact of this decision for constant P can be judged from Figure 3.

We conclude this section by noting that by discussion in Section 5 electric field after retrieval has the same efficiency as the efficiency of polarisation operator.

7 Additional sources of loss

The only reason that our memory works practically with 100 % efficiency when broadening is chosen large enough is because we don't really have any sources of loss. One obvious generalisation is to have some part of broadening that is not reversed. If this kind of broadening is present and cannot be neglected as we did in all the previous derivations then it's caused by intrinsic properties of the atomic ensemble and will affect all four stages. However, in the framework we've developed it's hard to find an analytic solution to equations of motion when broadening is present in the model. This is why we needed to resort to perturbation theory for stage 2 and 3. This approach wouldn't work for stage 1 and 4 – at least in the current form – because there P and E do change a lot contrary to the pertubative assumption.

Thus we imagine that this non-reversible broadening is turned on in the beginning of stage 2 and is turned off as soon as stage 3 ends. It effectively means that $\tilde{\Delta}_0$ in equations (26) and (27) is now distributed with some density function G_0. To keep things concrete let's assume that it's a Gaussian of the same form as G that we looked at (cf. (31)) with width $\tilde{\gamma}_0 = \gamma_0/\mu$. Hence stages 2 and 3 are now described by equations

$$\frac{\partial}{\partial\tilde{z}}\,E(\tilde{z},\tau) = i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_0(\tilde{\Delta}_0')G(\tilde{\Delta}')\sigma(\tilde{z},\tau,\tilde{\Delta}_0',\tilde{\Delta}')\,\mathrm{d}\tilde{\Delta}_0'\,\mathrm{d}\tilde{\Delta}' \tag{40}$$

and

$$\frac{\partial}{\partial\tau}\,\sigma(\tilde{z},\tau,\tilde{\Delta}) = -i(\tilde{\Delta}_0+\tilde{\Delta})\sigma(\tilde{z},\tau,\tilde{\Delta}) + iE(\tilde{z},\tau). \tag{41}$$

Going through the same calculations as in equations (30) through (37) for this case then calculating $P(u,\tau_2) = \int_0^\infty\int_0^\infty G_0(\tilde{\Delta}_0')G(\tilde{\Delta}')\sigma(u,\tau,\tilde{\Delta}_0',\tilde{\Delta}')\,\mathrm{d}\tilde{\Delta}_0'\,\mathrm{d}\tilde{\Delta}'$ and taking inverse Laplace transform we get

$$P(\tilde{z},\tau_2) = e^{-2\tilde{\gamma}_0^2\tau_2^2}P_1(\tilde{z}) - \int_0^{\tilde{z}} P_1(z')\,\mathrm{d}z'\int_0^{\tau_2} e^{-\frac{1}{2}\tilde{\gamma}_0^2(\tau'-2\tau_2)^2}e^{-\frac{1}{2}\tilde{\gamma}^2\tau'^2}e^{-\frac{1}{2}(\tilde{\gamma}_0^2+\tilde{\gamma}^2)\tau'^2}\,\mathrm{d}\tau'. \tag{42}$$

We see that setting $\tilde{\gamma}_0 = 0$ we recover (38) (with $\tilde{\Delta}_0 = 0$). It also almost as if we actually took (38) and integrated it with distribution G_0 which would have simply added a factor of $e^{-2\tilde{\gamma}_0^2\tau_2^2}$ there. We do have some cross terms in the exponents in (42) and this is not strictly true, but the general picture is clear though: $|P(\tilde{z},\tau_2)| \to 0$ as $\tau_2 \to 0$.

It's possible to write the integral in the correction part in terms of error functions but this doesn't make expression (42) clearer and it's easier to judge its behaviour from Fig. 4 (b) and compare it to the case with no intrinsic broadening on Fig. 4 (a).

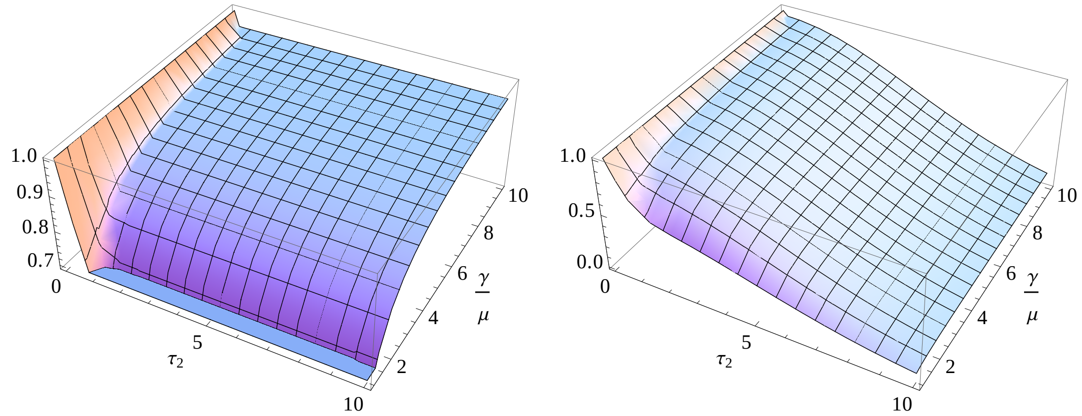
Now we want to show that distribution G_0 is in fact related to optical depth of the atomic ensemble. The physical process that is to be modeled is simply a light pulse propagating through the atomic ensemble without any extra broadening and reversing of it ever taking place. For this we consider equations that look a lot like equations (26) and (27):

$$\frac{\partial}{\partial\tilde{z}}\,E(\tilde{z},\tau) = i\int_{-\infty}^{\infty} G_0(\tilde{\Delta}_0')\sigma(\tilde{z},\tau,\tilde{\Delta}_0')\,\mathrm{d}\tilde{\Delta}_0', \tag{43}$$

$$\frac{\partial}{\partial\tau}\,\sigma(\tilde{z},\tau,\tilde{\Delta}_0) = -i\tilde{\Delta}_0\sigma(\tilde{z},\tau,\tilde{\Delta}_0) + iE(\tilde{z},\tau). \tag{44}$$

Now σ describes atoms that were broadened by the intrinsic broadening from 0 to some detuning $\tilde{\Delta}_0$. If we formally integrate (44) under initial condition $\sigma(\tilde{z},\tau=-\infty,\tilde{\Delta}_0) = 0$ we get

$$\begin{aligned}\sigma(\tilde{z},\tau,\tilde{\Delta}_0) &= i\int_{-\infty}^{\tau}\exp(-i\Delta_0(\tau-\tau'))E(\tilde{z},\tau)\,\mathrm{d}\tau' \\ &= i\int_{-\infty}^{\infty}\Theta(\tau-\tau')\exp(-i\Delta_0(\tau-\tau'))E(\tilde{z},\tau)\,\mathrm{d}\tau'\end{aligned} \tag{45}$$



(a) $\tilde{\gamma}_0 = 0$ (b) $\tilde{\gamma}_0 = 0.1$

Figure 4: Storage efficiency with non-reversible broadening $\tilde{\gamma}_0$ for $\hat{P}(\tilde{z}) = 1$

where Θ is Heaviside step function. Taking Fourier transform ($\tau \to \omega$) using convolution theorem (45) becomes

$$\sigma(\tilde{z}, \omega, \tilde{\Delta}_0) = i\tilde{E}(\tilde{z}, \omega)\left(\sqrt{\frac{\pi}{2}}\delta(\omega - \tilde{\Delta}_0) + \frac{1}{\sqrt{2\pi}}\frac{i}{\omega - \tilde{\Delta}_0}\right). \tag{46}$$

Inserting (46) into Fourier transform of (43) results in

$$\begin{aligned}\frac{\partial}{\partial \tilde{z}}\,\tilde{E}(\tilde{z}, \omega) &= -\tilde{E}(\tilde{z}, \omega)\int_{-\infty}^{\infty} G_0(\tilde{\Delta}_0')\left(\sqrt{\frac{\pi}{2}}\delta(\omega - \tilde{\Delta}_0') + \frac{1}{\sqrt{2\pi}}\frac{i}{\omega - \tilde{\Delta}_0'}\right)\mathrm{d}\tilde{\Delta}_0' \\ &= -\tilde{E}(\tilde{z}, \omega)\left(\sqrt{\frac{\pi}{2}}G_0(\omega) - \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G_0(\tilde{\Delta}_0')\frac{1}{\omega - \tilde{\Delta}_0'}\,\mathrm{d}\tilde{\Delta}_0'\right).\end{aligned}$$

From this we see that

$$\left|\tilde{E}(\tilde{z} = 1, \omega)\right|^2 = \exp\left(-\sqrt{2\pi}G_0(\omega)\right)\left|\tilde{E}(\tilde{z} = 0, \omega)\right|^2.$$

Defining optical depth on resonance from this expression as $d_0 = \frac{1}{2}\sqrt{2\pi}G_0(0)$ we have for Gaussian G_0 that

$$d_0 = \frac{1}{2\tilde{\gamma}_0} = \frac{\mu}{2\gamma_0}.$$

Going back to the expression (42) we can rewrite and simplify it somewhat. First of all we see that the effect of G_0 on the correction term is to make it smaller. However the impact on the leading term is more significant. So at cost of pushing efficiency somewhat lower than it actually is we can neglect all factors containing γ_0 in the correction term and get

$$P(\tilde{z}, \tau_2) = e^{-2\tilde{\gamma}_0^2\tau_2^2}P_1(\tilde{z}) - \int_0^{\tilde{z}} P_1(z')\,\mathrm{d}z'\int_0^{\tau_2} e^{-\tilde{\gamma}^2\tau'^2}\,\mathrm{d}\tau'.$$

which basically makes it the same as (38) with leading term multiplied by decaying factor. Keeping only linear term in efficiency as before and expanding the exponential in the leading term we get for $P_1(\tilde{z}) = 1$ that

$$\eta = 1 - 4\tilde{\gamma}_0^2\tau_2^2 - \int_0^{\tau_2} e^{-\tilde{\gamma}^2\tau'^2}\,\mathrm{d}\tau'.$$

Expressing the second term via optical depth and taking worst case estimate for the third term we get

$$\eta = 1 - \frac{\tau_2^2}{d_0^2} - \frac{\sqrt{\pi}}{2\tilde{\gamma}}. \tag{47}$$

On the other hand storage duration τ_2 which is of order of $\frac{1}{\tilde{\gamma}_0}$ and pulse length (duration of stage 1) τ_1 which is of order 1 (both in the rescaled variables wrt μ) have a ratio of order

$$\frac{1}{\tilde{\gamma}_0} = 2d_0$$

and should be much greater than 1 or else the memory is useless. Thus we see that if we make efficiency close to 1 by taking big optical depth then we cannot store our pulse for very long time and vice versa. This is the same conclusion that was reached in [3] when analysing effects of non-reversible broadening – that there's a trade off between storage duration and efficiency.

8 Discussion and outlook

There're plenty possibilities for further analysis. The biggest of them is considering effects of non-reversible broadening during stage 1 and 4. Alternatively we can consider effects of decay as found in [1]. It's not clear whether any new analytic results can be obtained in these cases or whether one would have to resort to numerical calculations.

Other interesting thing to consider is finite speed of broadening reversal. In this thesis we assume that broadening switches between $\tilde{\Delta}$ and $-\tilde{\Delta}$ immediately. In reality broadening would be around 0 for some time and there the system will radiate light causing efficiency to be lower. Quantifying this effect would then give answer to whether several reversals of broadening instead of one is of any advantage. One reason for doing several reversals is that it will effectively divide time spent in stage 2 and 3 by some integer and hence make the perturbative treatment of stage 2 and 3 as presented in this thesis valid for lower values of $\tilde{\gamma}$.

Finally even though analytic approach of [3] wasn't used here they seem to claim that their obtained efficiency of 54 % is the same independent of the mode of the incoming photon. We found that in our model some difference is present and it would be insightful to track down difference in assumptions that causes difference in the result.

9 Conclusion

We started with the idea of only applying inhomogeneous broadening for a finite time intervals and only after the field has already excited the atoms. With this setup we can show that this CRIB based memory works with efficiency approaching 100 % as broadening is increased. Compared to the previous result of [3] which claimed 54 % this constitutes a clear improvement. We note that in the course of the derivation of this main result we had to make several idealisations which could impede practical realisation of this principle. These include usage of specific mode of the incoming light in the limit of infinite storage time and no other sources of loss than light escaping the atomic ensemble due to inability of the atomic ensemble to contain all the information about it in the initial stage. The only other source of loss that we considered was introduction of non-reversible broadening in the atomic ensemble and it was shown to make efficiency do down. It should be noted, however, that those cited 54 % in [3] were also obtained in the case of absent non-reversible broadening. Just as in [3] we concluded that in case of non-reversible broadening there exists a tradeoff between efficiency and storage duration.

References

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A Bessel functions

We want to prove the identity

$$\int_0^\infty J_0(2\sqrt{ux})J_0(2\sqrt{vx})\,\mathrm{d}x = \delta(u-v). \tag{48}$$

It looks a lot like a known result [2]

$$\int_0^\infty xJ_0(ux)J_0(vx)\,\mathrm{d}x = \frac{1}{u}\delta(u-v).$$

Making substitution in (48): $t = 2\sqrt{x}$, $x = \frac{1}{4}t^2$, $\mathrm{d}x = \frac{1}{2}t\,\mathrm{d}t$ we get

$$\int_0^\infty \frac{1}{2}tJ_0(\sqrt{u}t)J_0(\sqrt{v}t)\,\mathrm{d}t = \frac{1}{2\sqrt{u}}\delta(\sqrt{u}-\sqrt{v}).$$

Using substitution $y = \sqrt{u}$, $\mathrm{d}y = \frac{1}{2\sqrt{u}}\,\mathrm{d}u$ we have for some function f and interval $[c,d]$ that contains both $\sqrt{v}$ and v that

$$\int_c^d \frac{\delta(\sqrt{u}-\sqrt{v})}{2\sqrt{u}}f(u)\,\mathrm{d}u = \int_c^d \delta(y-\sqrt{v})f(y^2)\,\mathrm{d}y = f(v) = \int_c^d \delta(y-v)f(y)\,\mathrm{d}y$$

and hence that

$$\int_0^\infty J_0(2\sqrt{ux})J_0(2\sqrt{vx})\,\mathrm{d}x = \frac{1}{2\sqrt{u}}\delta(\sqrt{u}-\sqrt{v}) = \delta(u-v).$$

B Stationary points for storage efficiency

We want to find P that extremised term linear in $1/\tilde{\gamma}$ in (39). Defining

$$f(\tilde{z}) = \int_0^{\tilde{z}} P(z')\,\mathrm{d}z'$$

and assuming that P is real-valued we can recast this problem into standard Euler-Lagrange language. The functional whose stationary points we're interested in is

$$I = \int_0^1 F(f, f', \tilde{z})\,\mathrm{d}\tilde{z} = \int_0^1 f'(\tilde{z})f(\tilde{z})\,\mathrm{d}\tilde{z}$$

under constraint that

$$J = \int_0^1 G(f, f', \tilde{z}) = \int_0^1 f'(\tilde{z})^2\,\mathrm{d}\tilde{z} = 1$$

Euler-Lagrange equation is

$$\begin{aligned}&\frac{\partial F}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}\tilde{z}}\left(\frac{\partial F}{\partial f'}\right) + \lambda\left(\frac{\partial G}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}\tilde{z}}\left(\frac{\partial G}{\partial f'}\right)\right) = 0\\ &= f'(\tilde{z}) - \frac{\mathrm{d}}{\mathrm{d}\tilde{z}}\big(f(\tilde{z})\big) - \lambda\frac{\mathrm{d}}{\mathrm{d}\tilde{z}}\left(2f'(\tilde{z})\right)\\ &= -2\lambda f''(\tilde{z})\end{aligned}$$

and this implies that

$$P(\tilde{z}) = f'(\tilde{z}) = 1.$$

C Perturbative treatment of stage 3

As was already mentioned stage 3 can be described by equation (30) with $\tilde{\Delta}$ replaced by $-\tilde{\Delta}$. Doing the same substitution in definition of σ_S in (32) we get the equation

$$\frac{\partial}{\partial \tau}\,\bar{\sigma}_S(u,\tau,-\tilde{\Delta}) = -\frac{1}{u}e^{-i\tilde{\Delta}\tau}\int_{-\infty}^{\infty} G(\tilde{\Delta}')\sigma_S(u,\tau,-\tilde{\Delta}')e^{i\tilde{\Delta}'\tau}\,\mathrm{d}\tilde{\Delta}' \tag{49}$$

to which doing the perturbative approximation we can write the formal solution as

$$\begin{aligned}\bar{\sigma}_S(u,\tau,-\tilde{\Delta}) = &-\frac{1}{u}\int_0^{\tau} e^{-i\Delta\tau'}\int_{-\infty}^{\infty} G(\tilde{\Delta}')\sigma_S(u,\tau=0,-\tilde{\Delta}')e^{i\tilde{\Delta}'\tau'}\,\mathrm{d}\tilde{\Delta}'\,\mathrm{d}\tau' \\ &+\bar{\sigma}_S(u,\tau=0,-\tilde{\Delta})\end{aligned} \tag{50}$$

Since broadening got reversed between stage 2 and stage 3 we have that the initial condition is given by

$$\bar{\sigma}_S(u,\tau=0,-\tilde{\Delta}) = \bar{\sigma}(u,\tau=0,-\tilde{\Delta}) = \bar{\sigma}(u,\tau=\tau_2,\tilde{\Delta}).$$

Hence using the solution we obtained in (35) and inserting it into (50) we get

$$\begin{aligned}&\bar{\sigma}_S(u,\tau,-\tilde{\Delta}) \\ &= -\frac{1}{u}\bar{P}_1(u)\int_0^{\tau} e^{-i\tilde{\Delta}\tau'}\int_{-\infty}^{\infty} G(\tilde{\Delta}')e^{-i\tilde{\Delta}_0\tau_2}e^{i\tilde{\Delta}'(\tau'-\tau_2)}\,\mathrm{d}\tilde{\Delta}'\,\mathrm{d}\tau' \\ &+\frac{1}{u^2}\bar{P}_1(u)\int_0^{\tau} e^{-i\tilde{\Delta}\tau'}\int_{-\infty}^{\infty} G(\tilde{\Delta}')\int_0^{\tau_2} e^{-i\tilde{\Delta}_0\tau_2}e^{-\frac{1}{2}\tilde{\gamma}^2\tau''^2+i\tilde{\Delta}'(\tau''+\tau'-\tau_2)}\,\mathrm{d}\tau''\,\mathrm{d}\tilde{\Delta}'\,\mathrm{d}\tau' \\ &+\bar{P}_1(u)e^{-i(\tilde{\Delta}_0+\tilde{\Delta})\tau_2} \\ &-\frac{1}{u}\bar{P}_1(u)e^{-i(\tilde{\Delta}_0+\tilde{\Delta})\tau_2}\int_0^{\tau_2} e^{-\frac{1}{2}\tilde{\gamma}^2\tau'^2+i\tilde{\Delta}\tau'}\,\mathrm{d}\tau'.\end{aligned}$$

Our plan of action from now on is to evaluate all the $\tilde{\Delta}$ integrals first as those are easy. Now we can write expression for $\bar{\sigma}(u,\tau,-\tilde{\Delta}) = e^{-i(\tilde{\Delta}_0-\tilde{\Delta})\tau}\bar{\sigma}_S(u,\tau,-\tilde{\Delta})$ and assuming that $\tau_3=\tau_2$ we get in the end of stage 3

$$\begin{aligned}&\bar{\sigma}(u,\tau=\tau_2,-\tilde{\Delta}) \\ &= -\frac{1}{u}\bar{P}_1(u)e^{-i2\tilde{\Delta}_0\tau_2}\int_0^{\tau_2} e^{-i\tilde{\Delta}(\tau'-\tau_2)}e^{-\frac{1}{2}\tilde{\gamma}^2(\tau'-\tau_2)^2}\,\mathrm{d}\tau' \\ &+\frac{1}{u^2}\bar{P}_1(u)e^{-i2\tilde{\Delta}_0\tau_2}\int_0^{\tau_2} e^{-i\tilde{\Delta}(\tau'-\tau_2)}\int_0^{\tau_2} e^{-\frac{1}{2}\tilde{\gamma}^2\tau''^2}e^{-\frac{1}{2}\tilde{\gamma}^2(\tau''+\tau'-\tau_2)^2}\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\ &+\bar{P}_1(u)e^{-i2\tilde{\Delta}_0\tau_2} \\ &-\frac{1}{u}\bar{P}_1(u)e^{-i2\tilde{\Delta}_0\tau_2}\int_0^{\tau_2} e^{-\frac{1}{2}\tilde{\gamma}^2\tau'^2+i\tilde{\Delta}\tau'}\,\mathrm{d}\tau'.\end{aligned}$$

Now we find $\bar{P}(u, \tau = \tau_2) = \int_{-\infty}^{\infty} G(\tilde{\Delta}')\bar{\sigma}(u, \tau = \tau_2, -\tilde{\Delta}')\,\mathrm{d}\tilde{\Delta}'$ and take inverse Laplace transform using convolution theorem:

$$\begin{aligned}
&P(\tilde{z}, \tau = \tau_2) \\
&= -e^{-i2\tilde{\Delta}_0\tau_2} \int_0^{\tilde{z}} P_1(z')\,\mathrm{d}z' \int_0^{\tau_2} e^{-\tilde{\gamma}^2(\tau'-\tau_2)^2}\,\mathrm{d}\tau' \\
&+ e^{-i2\tilde{\Delta}_0\tau_2} \int_0^{\tilde{z}} (1-z')P_1(z')\,\mathrm{d}z' \int_0^{\tau_2}\int_0^{\tau_2} e^{-\frac{1}{2}\tilde{\gamma}^2\left((\tau'-\tau_2)^2+\tau''^2+(\tau''+\tau'-\tau_2)^2\right)}\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
&+ e^{-i2\tilde{\Delta}_0\tau_2} P_1(\tilde{z}) \\
&- e^{-i2\tilde{\Delta}_0\tau_2} \int_0^{\tilde{z}} P_1(z')\,\mathrm{d}z' \int_0^{\tau_2} e^{-\tilde{\gamma}^2\tau'^2}\,\mathrm{d}\tau'.
\end{aligned} \tag{51}$$