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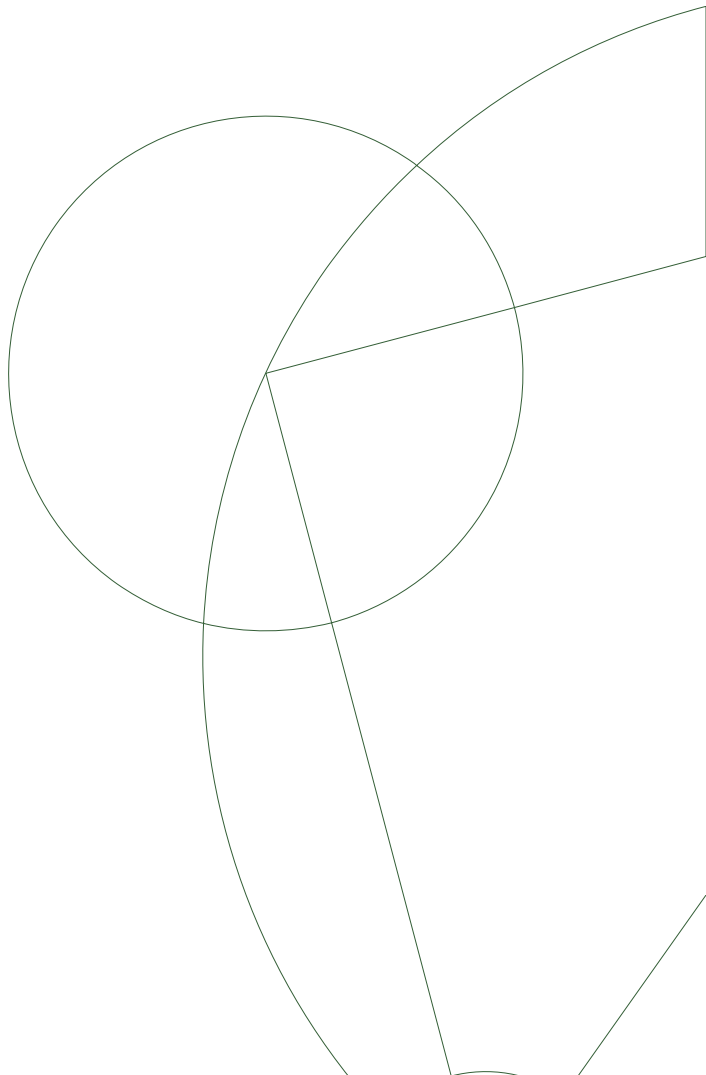
MICROWAVE SPECTROSCOPY OF QUANTUM DOT SYSTEMS

WRITTEN BY:

PETER DANIEL JOHANNSSEN

UNDER SUPERVISION OF:

JENS PAASKE



Abstract

This thesis describes the classical circuitry that is used for microwave reflectometry of single and double quantum dots. The theory is built up from a simple capacitive coupling between the coaxial cable and the dots to a more complicated circuit which takes impedance mismatches into account. The full circuit allows for measurement of the tunneling resistance/capacitance between a single dot and ground or between two dots, which in principle allows us to create a charge stability diagram. Tunneling between leads and dots is not included.

For a double dot with one energy level per dot the possible number of electrons is $n \in \{0, 1, 2, 3, 4\}$. The simplest non-trivial case is $n = 1$ and $n = 3$, for which the problem is solved to linear order using linear response theory, this gives us a way of calculating the interdot current as a convolution of the applied voltage and a periodic function.

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1 INTRODUCTION

A semiconductor quantum dot (QDot) consists of two semiconducting layers (Hanson et al. 2007) (for instance GaAs and AlGaAs), doped in such a way that a two-dimensional electron gas accumulates at the interface between the materials. In the case of AlGaAs one can dope with Si (Hanson et al. 2007) to introduce the electrons. The fact that the electrons are constrained to the interface causes them to have some interesting properties that are not seen in metals (three-dimensional electron gases). We can add metallic “wires” to the surface of AlGaAs and by applying voltages to the wires we can affect the energy levels of the dots. See Appendix B.1 for an image, taken from (Laucht et al. 2021). The dots can be thought of as artificial atoms – they can even have different energy levels and can under the right conditions receive and release electrons.

Usually QDot devices have at least two leads from which electrons can enter or leave the dot; this is shown in Figure 1.1. The electrons enter and leave the dot through a process called *tunneling* at a rate Γ , the so-called tunneling rate which can be calculated using Fermi’s Golden Rule, as we will see in Section 1.1. By changing the electrochemical potential in the leads one can get the electron to hop in and out of the dot.

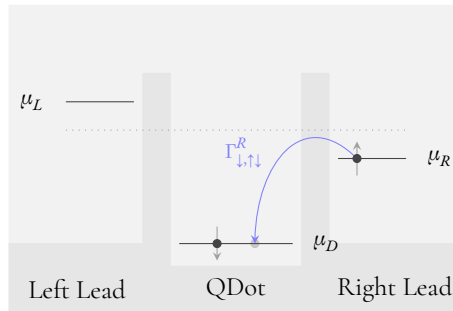


FIGURE 1.1: The left, right and dot electrochemical potentials. In this figure an electron with spin up is tunneling from the right lead onto the dot, which already has an electron with spin down. The tunneling rate is $\Gamma_{\downarrow,\uparrow}^R$.

In Figure 1.1 we see a tunneling process, where an up electron tunnels from the right lead into the dot, which is already occupied by a down electron. The tunneling rate, $\Gamma_{\downarrow,\uparrow}$, depends on the chemical potential of the relevant lead and the dot, as well as the occupancy of the dot, due to the electrostatic interaction. Note that generally $\Gamma = \Gamma^L + \Gamma^R$ so the total rate is the rate from the left lead and from the right lead onto the dot.

1.1 COULOMB BLOCKADE

This section is based on (Bruus and Flensberg 2016, Chapter 10).

Consider Figure 1.1 where a QDot with a single level is located between a left and a right lead. Electrons

can tunnel from the leads into the dot and vice versa. The following transitions are possible

$$|0\rangle \xrightarrow{\Gamma_{0,\uparrow}} |\uparrow\rangle \quad (\text{I.Ia})$$

$$|0\rangle \xrightarrow{\Gamma_{0,\downarrow}} |\downarrow\rangle \quad (\text{I.Ib})$$

$$|\uparrow\rangle \xrightarrow{\Gamma_{\uparrow,\downarrow}} |\uparrow\downarrow\rangle \quad (\text{I.Ic})$$

$$|\downarrow\rangle \xrightarrow{\Gamma_{\downarrow,\uparrow}} |\uparrow\downarrow\rangle \quad (\text{I.Id})$$

as well as the reverse transitions, where we flip the indices of the rates. The master Equation becomes

$$\frac{d}{dt} \begin{pmatrix} P_0 \\ P_\uparrow \\ P_\downarrow \\ P_{\uparrow\downarrow} \end{pmatrix} = \begin{pmatrix} -(\Gamma_{0,\uparrow} + \Gamma_{0,\downarrow}) & \Gamma_{\uparrow,0} & \Gamma_{\downarrow,0} & 0 \\ \Gamma_{0,\uparrow} & -(\Gamma_{\uparrow,0} + \Gamma_{\uparrow,\downarrow}) & 0 & \Gamma_{\uparrow\downarrow,\uparrow} \\ \Gamma_{0,\downarrow} & 0 & -(\Gamma_{\downarrow,0} + \Gamma_{\downarrow,\uparrow}) & \Gamma_{\uparrow\downarrow,\downarrow} \\ 0 & \Gamma_{\uparrow,\downarrow} & \Gamma_{\downarrow,\uparrow} & -(\Gamma_{\uparrow\downarrow,\uparrow} + \Gamma_{\uparrow\downarrow,\downarrow}) \end{pmatrix} \begin{pmatrix} P_0 \\ P_\uparrow \\ P_\downarrow \\ P_{\uparrow\downarrow} \end{pmatrix} = 0 \quad (\text{I.2})$$

where the final equality only holds in steady state. Let us begin by simplifying the problem and assuming that there is spin-degeneracy, i.e we cannot tell the difference between $|\uparrow\rangle$ and $|\downarrow\rangle$. Additionally we assume that the probabilities are normalised: $P_0 + 2P_1 + P_2 = 1$, this gives us the following expression for the probabilities:

$$P_0 = \frac{\Gamma_{1,0}\Gamma_{2,1}}{\Gamma_{1,0}\Gamma_{2,1} + \Gamma_{0,1}(\Gamma_{1,2} + 2\Gamma_{2,1})}, \quad P_1 = \frac{2\Gamma_{0,1}\Gamma_{2,1}}{\Gamma_{1,0}\Gamma_{2,1} + \Gamma_{0,1}(\Gamma_{1,2} + 2\Gamma_{2,1})}, \quad P_2 = \frac{\Gamma_{0,1}\Gamma_{1,2}}{\Gamma_{1,0}\Gamma_{2,1} + \Gamma_{0,1}(\Gamma_{1,2} + 2\Gamma_{2,1})} \quad (\text{I.3})$$

The current through the dot is the number of electrons that enter the dot from the left (right) lead per unit time:

$$I = -e(\Gamma_{0,1}^L P_0 + (\Gamma_{1,2}^L - \Gamma_{1,0}^L) P_1 - \Gamma_{2,1}^L P_2) = -e \left(\frac{\Gamma_{2,1}(\Gamma_{0,1}^L \Gamma_{1,0} - 2\Gamma_{1,0}^L \Gamma_{0,1}) + \Gamma_{0,1}(2\Gamma_{1,2}^L \Gamma_{2,1} - \Gamma_{2,1}^L \Gamma_{1,2})}{\Gamma_{1,0}\Gamma_{2,1} + \Gamma_{0,1}(\Gamma_{1,2} + 2\Gamma_{2,1})} \right) \quad (\text{I.4})$$

The Coulomb repulsion between electrons creates what is referred to as *charging energy*, which is a minimal energy that is required in order for us to force two electrons to be in the same space at the same time. Additionally the electrons have an energy that is associated with the gate voltage:

$$E_n = E_C n^2 - eV_G n \quad (\text{I.5})$$

Assuming the leads are reservoirs in thermal equilibrium, in which case they follow a Fermi-Dirac distribution, the transition rates are given by Fermi's golden rule:

$$\Gamma_{n,n+1}^s = \Gamma^s n_F(E_{n+1} - E_n - \mu_s) \quad (\text{I.6a})$$

$$\Gamma_{n,n-1}^s = \Gamma^s (1 - n_F(E_n - E_{n-1} - \mu_s)) \quad (\text{I.6b})$$

where s can either be the left or the right lead and μ^s is the electrochemical potential at the s -th dot (left or right), which is set by the source-drain voltage, V_{SD} . The temperature dependence of the Fermi-Dirac distribution implies that in order to perform precision measurements one needs to decrease the setup's temperature. For instance, the centre for Quantum Devices at the Niels Bohr Institute performs their measurements at $\approx 20\text{mK}$ (Berritta 2021). We now have everything we need in order to calculate the current through the dot. In the following figure we show the differential conductivity, $G \equiv \frac{\partial I}{\partial V_{SD}}$:

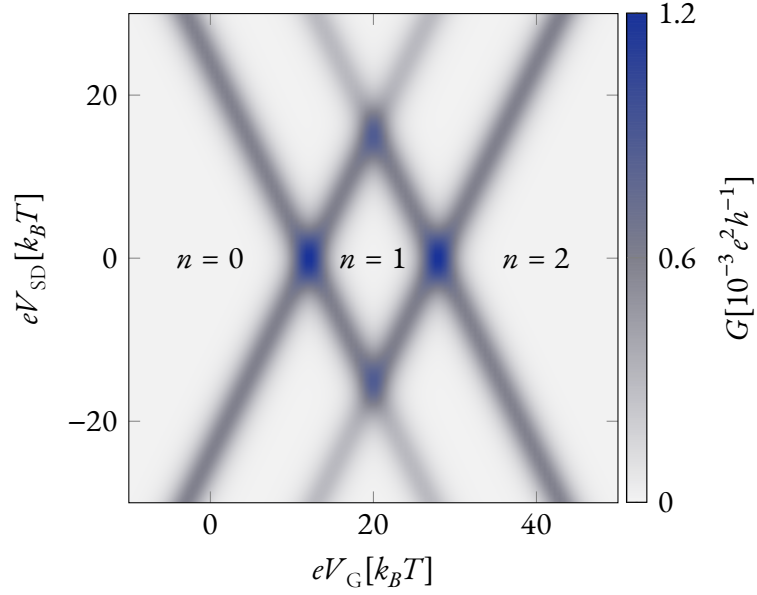


FIGURE 1.2: Differential conductivity as a function of gate- and source-drain voltages. Parameters: $E_C = 14k_B T$, $\hbar\Gamma^s = 0.1k_B T$

The inverse of G tells us about the effective resistance through the dot, which we in the next section will refer to as the tunneling resistance, R_T . However, it should be clear that this is not a linear resistor, because R_T is dependent on the source-drain voltage as well as the gate voltages: we will treat it as linear though. The tunneling resistance is a good measure for the “phase-transition”: when the resistance drops we are in the areas where the dots can conduct electricity, which is the region where the number of electrons on the dot can change. The width and height of the Coulomb Diamond above is set by the charging energy, which arises due to the Coulomb force: the Coulomb force *blocks* electrons from moving in or out of the dot in the central diamond, which is why it is called the Coulomb Blockade.

1.1.1 DOUBLE DOT

The master Equation for the double dot can be set up in a similar fashion. However, what is interesting is that in the regime where the number of collective electrons on the double dot system is conserved ($n_L + n_R = \text{const.}$) we cannot use the method from Figure 1.2 to establish whether the gate voltages are such that the number of electrons on each of the dots changes, because the overall resistance of the dot will be infinite: using DC-voltages we can only measure the regions where a net current can pass *through* the QDot system. Therefore, in gate-voltage-configuration where the number of electrons is constant we cannot yield any information about what is happening inside the QDot system. This is *not* the case if we use reflectometric methods: we can for instance see the transition between $(2, 0)$ and $(1, 1)$, where (n, m) represents the configuration where there are n and m electrons in the left and right dot respectively. In a transition like this we cannot see anything using DC-methods, because there will never be a current through the system.

1.2 REFLECTOMETRY

We will now introduce the coaxial cable, which will be modelled classically. The coaxial cable serves as a transmission line through which we send signals and receive the reflected wave created by the system we

are investigating.

A coaxial cable consists of an outer pipe-like conductors and an inner conductor that are separated by a dielectric medium. The dielectric medium between the conductors can be polarised, therefore the coaxial cable has as a capacitance per unit length, and additionally the conductors themselves have an inductance per unit length as depicted in the following figure (Sørensen 2009, pp. 86):

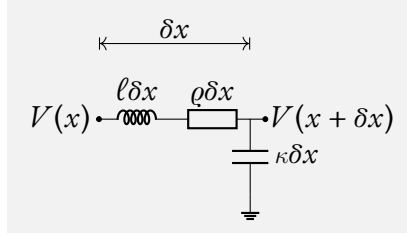


FIGURE 1.3: Model of an infinitesimal part of the coaxial cable, with inductance per unit length, ℓ , capacitance per unit length, κ and resistance per unit length, ρ .

The voltage drop over the length δx is (Clerk et al. 2010)

$$V(x + \delta x) - V(x) = -\ell \delta x \frac{\partial I}{\partial t} - \rho I \delta x, \quad \rightsquigarrow \quad \frac{\partial V}{\partial x} = -\ell \frac{\partial I}{\partial t} - \rho I \quad (1.7)$$

Similarly, the current leakage is

$$I(x + \delta x) - I(x) = -\kappa \delta x \frac{\partial V}{\partial t}, \quad \rightsquigarrow \quad \frac{\partial I}{\partial x} = -\kappa \frac{\partial V}{\partial t} \quad (1.8)$$

Combining the two, we get:

$$\frac{\partial^2 V}{\partial x^2} = \kappa \ell \frac{\partial^2 V}{\partial t^2} + \rho \kappa \frac{\partial V}{\partial t} \quad (1.9)$$

If we assume that the resistance is negligible, this becomes a one dimensional D'Alembert equation, with $v = (\kappa \ell)^{-1/2}$, whose solution consists of a left and right propagating wave: the incident wave and the reflected wave. Assuming our cable has length, L (Müller 2019):

$$\tilde{V}(x, t) = \begin{cases} \tilde{V}_{\text{in}} e^{i(\omega t - kx)} + \tilde{V}_{\text{out}} e^{i(\omega t + kx)} & 0 < x < L \\ \tilde{V}_{\text{tra}} e^{i(\omega t - kx)} & x > L \end{cases} \quad (1.10)$$

where $\tilde{V}_{\text{in}}$, $\tilde{V}_{\text{out}}$ and $\tilde{V}_{\text{tra}}$ are the incident, reflected and transmitted voltages respectively. The wavenumber, k , is defined as $k \equiv \frac{\omega}{v}$. The incident and reflected wave propagate on the same part of the cable, whereas the transmitted wave passes through the sample and goes to ground from there. Therefore the relevant voltage is not the voltage that we apply ($\tilde{V}_{\text{in}}$), but rather the superposition of incident and reflected voltages.

From Equation 1.8 we have $\partial_x I = -\kappa \partial_t V$, which gives us a way of calculating the impedance of the load $Z_L = \frac{\tilde{V}}{\tilde{I}}$ (Müller 2019):

$$Z_L = \frac{k (\tilde{V}_{\text{in}} e^{-ikx} + \tilde{V}_{\text{out}} e^{ikx}) e^{i\omega t}}{\kappa \omega (\tilde{V}_{\text{in}} e^{-ikx} - \tilde{V}_{\text{out}} e^{ikx}) e^{i\omega t}} = \frac{k (1 + r e^{2ikx})}{\kappa \omega (1 - r e^{2ikx})}, \quad r \equiv \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \left(\frac{Z_L - Z_c}{Z_L + Z_c} \right) e^{-2ikx}, \quad Z_c \equiv \sqrt{\frac{\ell}{\kappa}} \quad (1.11)$$

We see that the coefficient of reflection, r , has a phase factor which depends on where you measure the reflection, this is because the phase of the wave in Equation 1.10 depends on both time and position. We will henceforth ignore this phase factor, because we can assume that we perform the measurement point at $x = 0$ where there is no phase.

Typically experimentalists work with 50Ω transmission lines, which implies that the load impedance would need to have a similar value in order for r to differ from either one or minus one. When this is the case we can send a signal through the coaxial cable, measure the reflected signal, and calculate the impedance of the load, using the equation above. For instance if the load is shorted, then we can plug $Z \rightarrow 0$ into the expression for the reflection coefficient, which gives us $r = -1$, this means that the incident is flipped. However, if we have an open load, we let $Z_L \rightarrow \infty$ and get $r = 1$, which means that the wave is reflected as is.

The incident and reflected waves are related through the following relation (Clerk et al. 2010, App. C, p. 61), which is derived in Appendix A.1.1:

$$\tilde{V}_{\text{out}}(L, t) = \tilde{V}_{\text{in}}(L, t) + Z_L \tilde{I}(L, t) \quad (1.12)$$

where $\tilde{I}(L, t)$ is the current that *enters* the load.

2 REFLECTOMETRY

2.1 A SHORT NOTE ON CAPACITANCES

Whenever charges in our circuit interact with each other, we model them as two capacitors. For instance, the charges in the leads interact with charges in the dots, through the Coulomb force, therefore there is a capacitive coupling. Therefore, in the following sections there will be many capacitances that are introduced. Not only do the capacitors model the interactions between dots, but also the interactions between our macroscopic apparatuses and the dots.

We will use many values for the capacitances throughout this thesis; at first we will use $C \sim \text{pF}$, because this is the *required* order of magnitude to get a reflection coefficient that differs from one. However, if we assume that the capacitance between a gate and a dot can be treated as a plate capacitor the capacitance is $C = \frac{\epsilon_0 \epsilon_r A}{d}$ where A is the area of the plates and d is the distance between them; for $A \sim (100\text{nm})^2$ and $d \sim 100\text{nm}$ we get $C \sim 1\text{aF}$ and if we consider the fact that the electrons are inside of a material this value should increase with the relative permittivity. In the case of GaAs we have that $\epsilon_r \approx 10$ (Gallium Arsenide 2021). Therefore, from Section 2.2.3, which is when we achieve impedance matching and we can choose the capacitances more freely we will use $C \sim 10\text{aF}$, which is approximately in correspondence with (Bruus and Flensberg 2016, p. 157, Fig 10.2) and (Scholze 1969, p. 112).

We will look at frequencies in the interval $[0, 10]\text{GHz}$.

2.2 CLASSICAL CALCULATION

Let us build our classical description of a double dot up step by step, simply beginning with a capacitor as the load impedance. The charges on a single dot interact with the charges in the coaxial cable through the Coulomb-interaction, which is exactly what a capacitor is.

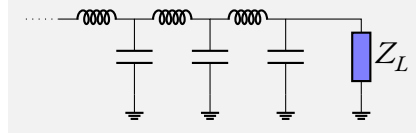


FIGURE 2.1: The coaxial cable (the parallel inductors and capacitors) is terminated by a load impedance, which we will describe using inductive, capacitive and resistive components.

2.2.1 SINGLE DOT

In the case where the load is a simple capacitor, the impedance is $Z = \frac{1}{i\omega C}$, giving us a reflection coefficient of:

$$r = \frac{\omega^2 C^2 Z_c^2 - 1 + 2i\omega C Z_c}{1 + \omega^2 C^2 Z_c^2} \quad (2.1)$$

The absolute value of which is equal to one: the entire signal is reflected, however there is a phase shift between the V_{in} and V_{out} , given by $\varphi = \arctan\left(\frac{2\omega C Z_c}{\omega^2 C^2 Z_c^2 - 1}\right)$.

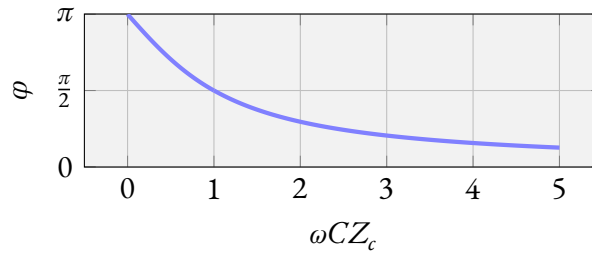


FIGURE 2.2: Phase of r as a function of $\omega C Z_c$, there is clearly resonance when $\omega = \frac{1}{Z_c C}$

However, in reality the QDot will also interact with the lead going to ground capacitively, which in our naïve model would correspond to an additional capacitor. Additionally we can change the energy level of the dot by applying a voltage to the ground gate:

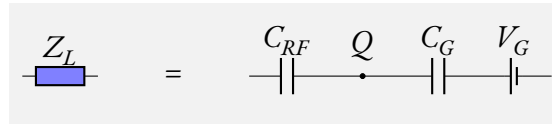


FIGURE 2.3: A naïve description of a QDot. C_{RF} is the capacitive coupling between the coaxial cable and the dot, and C_G is the coupling between the dot and the gate.

Classically we can just describe this as two capacitors in series, hence we can use our previous expression and replace $C \rightsquigarrow \left(\frac{1}{C_{RF}} + \frac{1}{C_G}\right)^{-1}$: we still only see the effect of the dot as a phase shift between V_{in} and V_{out} .

It is worth noting that the value of V_G does not affect the oscillatory behaviour of the charges at C_G^* , it merely sets the zero-point voltage, which cannot be seen in the reflected signal: for a QDot described quantum mechanically this is not the case, as the gate voltages determine the phase[†] (number of charges on the dot).

By expressing the charges on the capacitors in terms of the applied, gate and dot voltages, we can derive an expression for the energy[‡] required to put Q charges on this simple dot:

$$E = \frac{1}{C_G + C_{RF}} \left(\frac{Q^2}{2} + (C_{RF}V(t) + C_G V_G) Q \right) \quad (2.2)$$

which can be obtained by expressing the total charge, Q , only in terms of the voltages.

The next step towards the double dot is to allow the charge on the dot to change, we can do by connecting the island to ground through a tunnel junction, which we model as a resistor, R_T , and capacitor, C_T , in parallel[§] (Wiel et al. 2003, p. 3, Fig 1),:

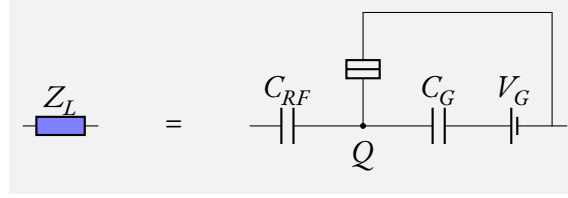


FIGURE 2.4: Classical single dot with a tunnel junction to ground, which means that Q no longer is fixed.

In this case the load impedance is given by:

$$Z_L = \frac{1}{i\omega C_{RF}} \frac{1 + i\omega Z_{\text{eff}}(C_G + C_{RF})}{1 + i\omega C_G Z_{\text{eff}}} \quad (2.3)$$

where $Z_{\text{eff}} = \frac{R_T(1-i\omega R_T C_T)}{1+\omega^2 R_T^2 C_T^2}$, is the effective impedance of the tunnel junction. The derivation of Z_L can be found in Appendix A.1.2. The tunneling process involves a dissipative part, because energy is sent to the environment when an electron transitions to a lower energy level. The capacitive part on the one hand is the Coulomb-interaction between electrons on the respective dots but is also a quantum mechanical quantity, given as $C \propto -e^2 \frac{\partial^2 E}{\partial V_{\text{SD}}^2}$ (Petersson et al. 2010). The value of the tunneling-capacitance provides an issue, because it is not a physical capacitor, but rather a way of modelling a quantum mechanical system in electrical circuits. However, for the double dot we can use $C_T \approx 10\text{fF}$ (Petersson et al. 2010), and because we lack an estimate for the value for the single dot I will use 10fF in Figure 2.5[¶].

We can now calculate the reflection coefficient using Equations 1.11 and 2.3:

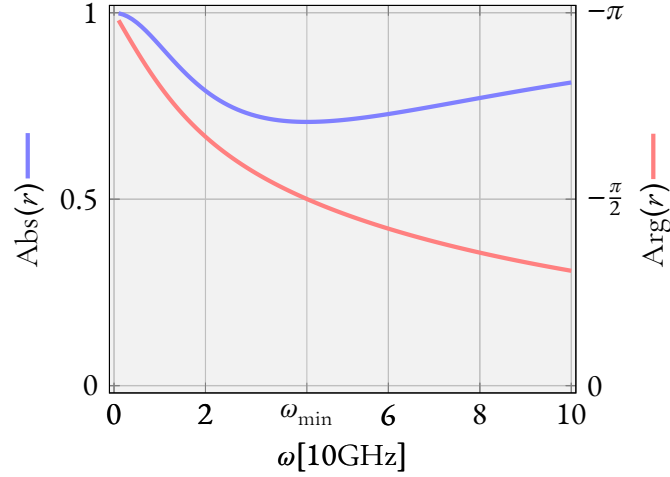
* this is a general point: DC voltages do not appear in the AC Kirchhoff Laws

† phase in the sense of phase-transitions

‡ note similarity to the expression we had for the energy on a dot in the Coulomb Blockade section: the first term is the charging energy and the second term is the energy associated with the height of the potential

§ this extra capacitor would modify the energy on the dot to $E = \frac{1}{C_G + C_{RF} + C_T} \left(\frac{Q^2}{2} + (C_{RF}V(t) + C_G V_G) Q \right)$

¶ Strictly speaking this is not valid, because the one is a tunneling capacitor between two dots and the other between a dot and a lead.

FIGURE 2.5: $C_G = C_{RF} = 6.7\text{pF}$; $C_T = 10\text{fF}$; $R_T = 25\Omega$;

The minimum lies at:

$$\omega_{\min} = \frac{1}{\sqrt{C_G C_{RF} R_T Z_c + C_{RF} C_T R_T Z_c}} \quad (2.4)$$

Using this we can calculate the R_T which will give us the best (largest) absorption:

$$R_T = Z_c \left(\frac{C_{RF}}{C_{RF} + C_G + C_T} \right) \quad (2.5)$$

This tells us that the classical single dot has an optimal R_T smaller than or equal to Z_c , however R_T represents dissipation during a tunneling process, and should be $R_T \geq \frac{\hbar}{4e^2} \sim 10\text{k}\Omega$ (Scholze 1969). For the classical circuit, however, impedance matching is not achieved for this value of R_T and the entire signal will be reflected.

Using Equations 2.4 and 2.5 we can simplify $\text{Abs}(r)$ assuming that we can choose R_T and C_T such that $\omega_{\min} = 2\text{GHz}$ and such that we have the smallest possible $r_{\min}$ we get that the minimum of $\text{Abs}(r)$ is:

$$\text{Abs}(r)_{\text{optimal}} = \frac{1}{\omega_{\min} C_{RF} Z_c} \quad (2.6)$$

For $\omega_{\min} = 2\text{GHz}$ and $Z_c = 50\Omega$ this is only valid for $C_{RF} \geq \frac{1}{\omega_{\min} Z_c} = 10\text{pF}$ (otherwise we are looking at a maximum in $\text{Abs}(r)$, not a minimum). Therefore, even in the most optimal case possible, if we hold the requirement for our minimum to be in the GHz region, we will need an $C_{RF} \geq 10\text{pF}$ **. In reality though we cannot set R_T nor C_T because they are emergent properties of a quantum mechanical system.

This implies that when performing reflectometry on single dots using the circuit above it will always be the case that there is an impedance mismatch. However, there is a way of adjusting the impedance, as we will see shortly.

2.2.2 DOUBLE DOT

Let us now add a second dot to the system and connect the dots with a tunnel junction:

** note that if we want $\omega_{\min}$ to be in the MHz region we need a $C_{RF} \geq 10\text{nF}$.

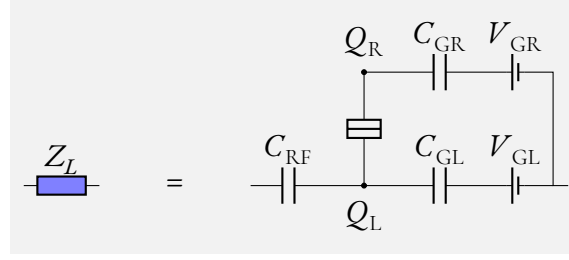


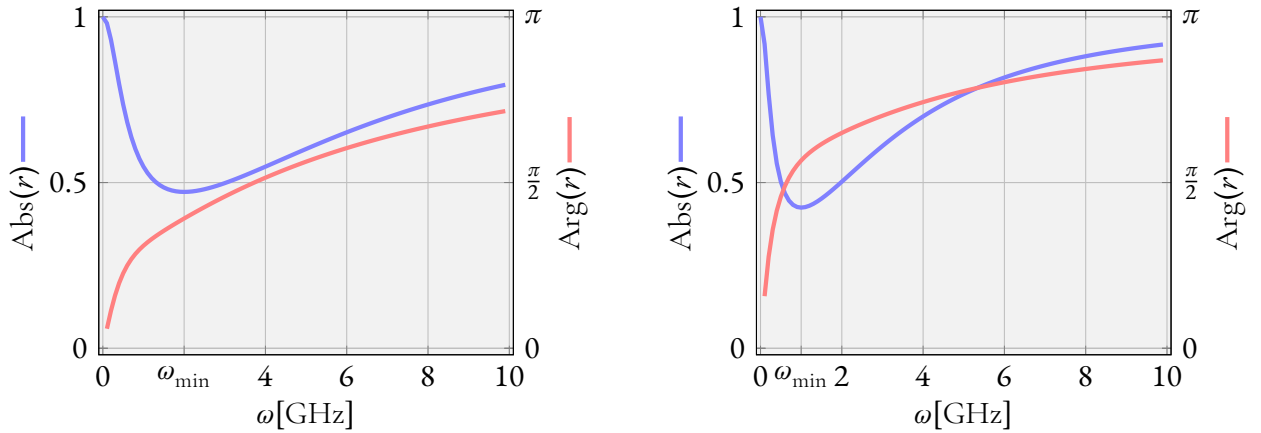
FIGURE 2.6: Classical double dot with a tunneling junction connecting the dots.

We can use Kirchhoff's AC laws to find an expression for the effective impedance:

$$\begin{aligned}
 Z_L &= \frac{\omega C_{GR} Z_{\text{eff}} (C_{RF} + C_{GL}) - i (C_{GL} + C_{GR} + C_{RF})}{\omega C_{RF} (C_{GR} + C_{GL} (1 + i\omega C_{GR} Z_{\text{eff}}))} \\
 &= - \frac{i\omega R_T (C_T (C_{GL} + C_{GR} + C_{RF}) + C_{GR} (C_{GL} + C_{RF})) + C_{GL} + C_{GR} + C_{RF}}{\omega C_{RF} (\omega R_T (C_{GL} (C_{GR} + C_T) + C_{GR} C_T) - i (C_{GL} + C_{GR}))}
 \end{aligned} \tag{2.7}$$

The derivation can be found in Appendix A.1.3. We must assume that all components are linear, in order to use Kirchhoff's AC laws, which also implies that Q_L will oscillate at the same frequency as the incident wave.

This, once again allows us to calculate the reflective coefficient:

FIGURE 2.7: Left: $C_{GL} = 4\text{pF}$, $C_{GR} = 20\text{pF}$, $C_{RF} = 90\text{pF}$, $C_T = 10\text{fF}$, $R_T = 78.9\Omega$, $Z_c = 50\Omega$, $\omega_{\min} = 2\text{GHz}$.

Right: $C_{GL} = 10\text{pF}$, $C_{GR} = 10\text{pF}$, $C_{RF} = 10\text{pF}$, $C_T = 10\text{fF}$, $R_T = 41.9\Omega$, $Z_c = 50\Omega$, $\omega_{\min} = 1\text{GHz}$

The minimum of which is at:

$$\omega_{\min} = \sqrt{\frac{C_{GL} + C_{GR} + C_{RF}}{R_T Z_c C_{RF} (C_{GR} C_T + C_{GL} (C_{GR} + C_T))}} \xrightarrow{C=C_{RF}=C_{GL}=C_{GR}} \omega_{\min} = \sqrt{\frac{3}{R_T Z_c C^2 (1 + 2\frac{C_T}{C})}} \tag{2.8}$$

However, when one uses values for C_{GR} , C_{GL} and C_{RF} that even approach the values cited in the literature (Bruus and Flensberg 2016, p. 157, Fig 10.2) and (Scholze 1969, p. 112), the dip in $\text{Abs}(r)$ becomes smaller, and the frequency at which the dip happens increases outside of the interval $\omega \in [0, 10]\text{GHz}$ (see Figures B.2 and B.3). This implies that there is once again an impedance *mismatch* when we use small capacitances.

2.2.3 ADDING THE INDUCTOR AND PARASITIC CAPACITOR

We would like to match the impedances, so that we can measure a reflection coefficient different from one even though we use capacitances closer to those cited in the literature. Therefore we would like to introduce components which allow us to (approximately) set the resonant frequency. Adding these two components appears to be common practice, see for example (Y. Y. Liu et al. 2020; Mizokuchi et al. 2021; Schoelkopf et al. 1998).

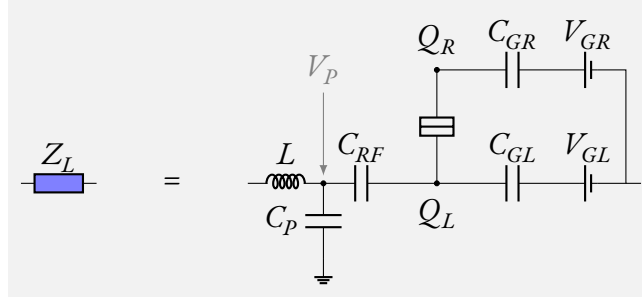


FIGURE 2.8: Classical double dot with a tunnel junction connecting the dots. An inductor and a parasitic capacitor have been added to achieve impedance matching. The value of the voltage between C_{RF} and L will be referred to as V_P .

The total impedance of which is:

$$Z_L = \frac{-C_{GR}LZ_{\text{eff}}\omega^3(C_{GL}(C_P + C_{RF}) + C_PC_{RF})}{\omega(iC_{GR}Z_{\text{eff}}\omega(C_{GL}(C_P + C_{RF}) + C_PC_{RF}) + C_{RF}(C_{GL} + C_{GR} + C_P) + C_P(C_{GL} + C_{GR}))} \quad (2.9)$$

$$+ i \frac{(L\omega^2(C_{RF}(C_{GL} + C_{GR} + C_P) + C_P(C_{GL} + C_{GR})) - (C_{GL} + C_{GR} + C_{RF})) + C_{GR}Z_{\text{eff}}\omega(C_{GL} + C_{RF})}{\omega(iC_{GR}Z_{\text{eff}}\omega(C_{GL}(C_P + C_{RF}) + C_PC_{RF}) + C_{RF}(C_{GL} + C_{GR} + C_P) + C_P(C_{GL} + C_{GR}))}$$

The derivation can be found in Appendix A.I.4

Note that by isolating the current between the dots, $\tilde{I}_{GR}^{\dagger\dagger}$, and integrating, we can write $\tilde{Q}_L = \tilde{Q}_L^0 + \delta\tilde{Q}_L$, where $\tilde{Q}_L^0$ is the integration constant, and $\delta\tilde{Q}_L$ is the oscillatory part, which will be proportional to $\tilde{V}(t)$. Therefore, for future use, we can in fact write $\delta\tilde{Q}_L(t) = \tilde{\chi} \int_0^t dt' \delta(t-t')\tilde{V}(t')$, where $\tilde{\chi}$ is the proportionality constant we get from Kirchhoff.

The addition of the two components couples an LC resonator to the double dot system. When $C_{RF} \rightarrow 0$ this gives us “resonance” ($\text{Arg}(r) = \frac{\pi}{2}$) at $\omega = \frac{1}{\sqrt{LC_P}}$ ^{††}. However, when $C_{RF} \neq 0$ the double dot system will influence the frequency at which we have resonance, as well as the size of the dip in $\text{Abs}(r)$ near resonance, due to the tunneling resistor^{§§}.

^{††}the current that goes from the node denoted by Q_L through Q_R and down to ground.

^{††}Note that $\text{Abs}(r)$ is one when $C_{RF} \rightarrow 0$ because the impedance is purely reactive.

^{§§}values of C_P and L were chosen such that resonance is inside the interval $\omega \in [0, 10]\text{GHz}$ and such that $\text{Abs}(r)$ differs significantly from 1 at resonance.

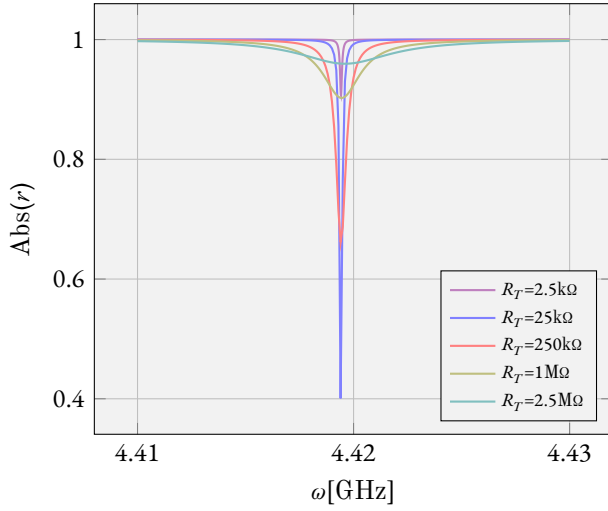


FIGURE 3.9a: Behaviour of the resonance dip for different values of R_T . $C_{GL} = 10\text{aF}$, $C_{GR} = 10\text{aF}$, $C_{RF} = 10\text{aF}$, $C_p = 100\text{aF}$, $L = 480\mu\text{H}$, $Z_c = 50\Omega$. Here C_T is set to zero.

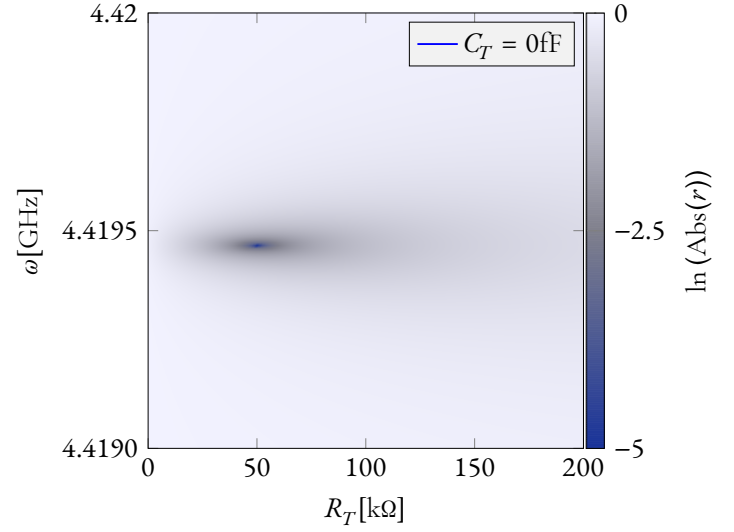


FIGURE 3.9b: $\ln(\text{Abs}(r))$ as a function of R_T and ω . $C_{GL} = 10\text{aF}$, $C_{GR} = 10\text{aF}$, $C_{RF} = 10\text{aF}$, $C_p = 100\text{aF}$, $L = 480\mu\text{H}$, $Z_c = 50\Omega$. Here C_T is set to zero

The tunneling capacitor is set to 0 in Figures 3.9a and 3.9b because quantum dot systems are occasionally modelled as a non-linear resistor (Persson et al. 2010), and because the result is most simple in this case. However, we have included C_T in our previous calculations, therefore we will add it once again. Adding C_T again moves the resonance point and makes the system less sensitive to changes in R_T :

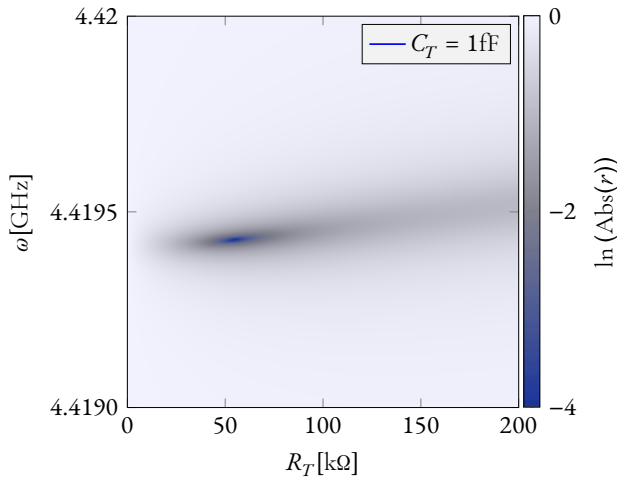


FIGURE 3.10a: $\ln(\text{Abs}(r))$ as a function of R_T and ω . $C_{GL} = 10\text{aF}$, $C_{GR} = 10\text{aF}$, $C_{RF} = 10\text{aF}$, $C_p = 100\text{aF}$, $L = 480\mu\text{H}$, $Z_c = 50\Omega$ and $C_T = 1\text{fF}$

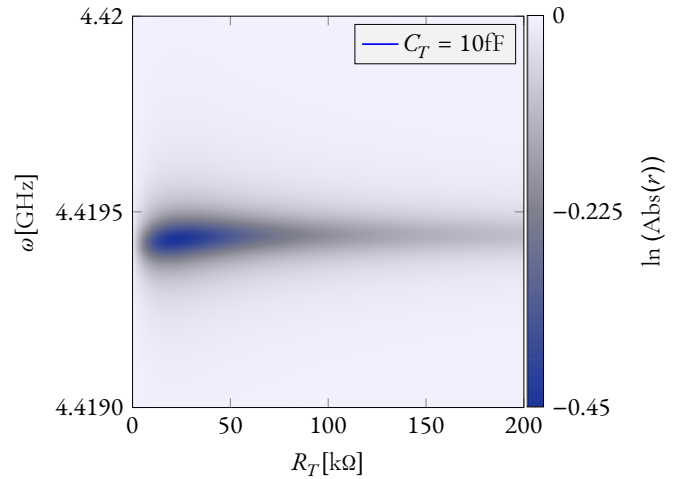


FIGURE 3.9b: $\ln(\text{Abs}(r))$ as a function of R_T and ω . $C_{GL} = 10\text{aF}$, $C_{GR} = 10\text{aF}$, $C_{RF} = 10\text{aF}$, $C_p = 100\text{aF}$, $L = 480\mu\text{H}$, $Z_c = 50\Omega$ and $C_T = 10\text{fF}$

It should be noted that the figures above are inspired by (Y. Y. Liu et al. 2020, Figure 2 (e) and (f)), though they describe a single dot and here we describe the double dot system. Additionally, they use frequencies in the MHz region, which accounts for the difference in values chosen for L and C_p .

Additionally for very large R_T ($> 100\text{M}\Omega$) the absorption dip appears once again, however, at a different frequency:

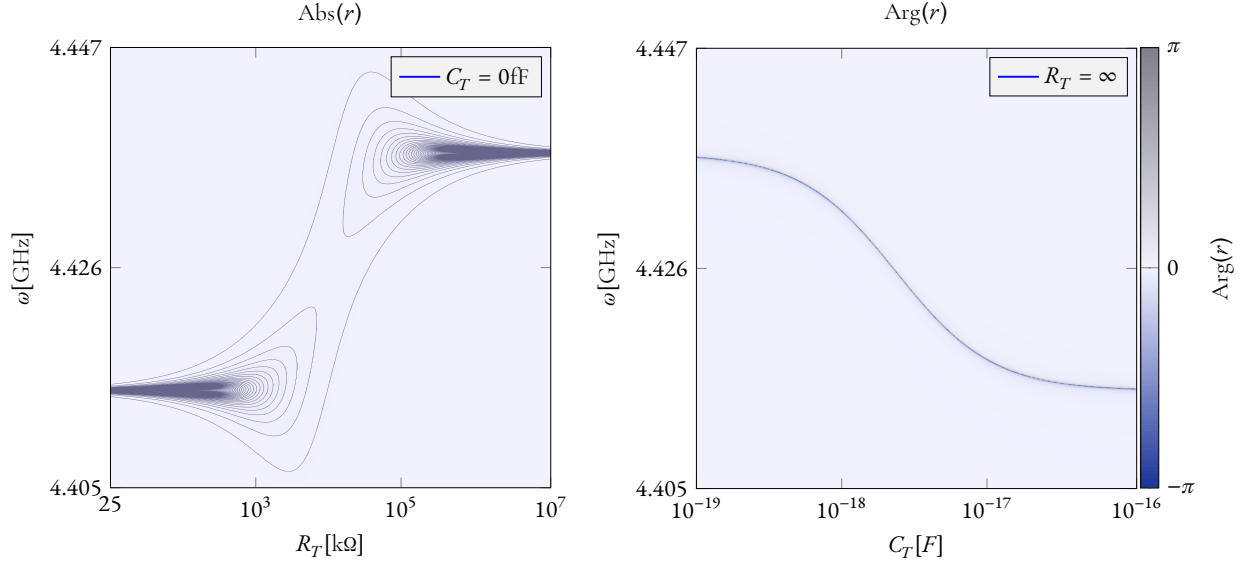


FIGURE 2.10: Left: $\text{Abs}(r)$ as a function of R_T and ω for $C_T = 0$. $C_{GL} = 10\text{aF}$, $C_{GR} = 10\text{aF}$, $C_{RF} = 10\text{aF}$, $C_P = 100\text{aF}$, $L = 480\mu\text{H}$ and $Z_c = 50\Omega$. The lines show “equipotential” lines for $\text{Abs}(r)^2$ in order to show the differences close to $\text{Abs}(r) = 1$ more clearly. The minimum of is $\text{Abs}(r) = 0.2$. Right: $\text{Arg}(r)$ as a function of C_T and ω for $R_T = \infty$. Same parameters as left.

Unfortunately a general expression for the resonant frequency was not obtained, however, it is possible to find it in the two limits, where $R_T \rightarrow 0$ and $R_T \rightarrow \infty$:

$$\omega_{\min}^{R_T \rightarrow 0} = \frac{\sqrt{C_{GL} + C_{GR} + C_{RF}}}{\sqrt{LC_P (C_{GL} + C_{GR} + C_{RF}) + LC_{RF} (C_{GL} + C_{GR})}} \quad (2.10a)$$

$$\omega_{\min}^{R_T \rightarrow \infty} = \frac{\sqrt{C_{GR} (C_{GL} + C_{RF}) + C_T (C_{GR} + C_{RF} + C_{GL})}}{\sqrt{LC_P (C_{GR} (C_{GL} + C_{RF}) + C_T (C_{GR} + C_{RF} + C_{GL})) + LC_{RF} (C_{GL}C_{GR} + C_{GL}C_T + C_{GR}C_T)}} \quad (2.10b)$$

It becomes apparent that the difference in resonant frequency in the limits is because of a play-off between the parasitic capacitor and the double-dot system. In the $R_T \rightarrow 0$ limit the two dots behave as one large dot, because they are connected. In the opposite limit the two dots behave as two independent dots that are coupled capacitively.

In principle this implies that by using the circuit in Figure 2.8 and performing reflectometry on the double dot system with frequencies in $\omega \in [0, 10]\text{GHz}$ one can create a charge stability diagram for the double dot. This is because in regions where the number of electrons on each dot is stable we have that $R_T \rightarrow \infty$, and in the regions where electrons can move around $R_T \approx \frac{h}{4e^2} \sim 10\text{k}\Omega$ (Scholze 1969). This implies that there will be values of the left and right gate voltages where R_T takes a value where the reflection coefficient has a value different from one and we know that in these regions the system is approaching a state where the electrons can tunnel. Similarly we can look at $\text{Arg}(r)$ and determine where capacitance changes significantly, which will also be near the transition, because this is where the dots can ‘feel’ each other most.

Unfortunately a general expression for the resonance condition was not obtained for $R_T \in \mathbb{R} \setminus \{0\}$. Even

if there is no closed form expression for the resonance condition, one could rely on numerical calculations to find the minimum.

In Figure 2.10 on the right we see a similar plot, however here we plot $\text{Arg}(r)$ as a function of C_T and ω . We see a similar shift in resonance frequency, which *has* be solved for. The behaviour around resonance though is quite different, because the transition from non-resonant to resonant behaviour is much sharper when we plot $\text{Arg}(r)$, however, when $R_T = \infty$ there is no dip in $\text{Abs}(r)$ because the impedance is purely imaginary.

Note that these results also hold for the single dot with a tunnel junction to ground, as this corresponds to the limit where $C_{GR} \rightarrow \infty$.

2.3 CLASSICAL ENERGY LEVELS: DOUBLE DOT

We are now interested in an expression for the energies on the two dots as a function of the number of electrons on each dot, which we will need this for the quantum mechanical description. We begin by calculating the charges on each dot, given all the voltages (Wiel et al. 2003):

$$Q_L = C_{RF} (V_{\text{dot,L}} - V(t)) + C_{GL} (V_{\text{dot,L}} - V_{GL}) + C_T (V_{\text{dot,L}} - V_{\text{dot,R}}) \quad (2.11)$$

$$Q_R = C_{GR} (V_{\text{dot,R}} - V_{GR}) + C_T (V_{\text{dot,R}} - V_{\text{dot,L}}) \quad (2.12)$$

By expressing this as a matrix equation and defining the sum of capacitances as C_L and C_R :

$$\begin{pmatrix} Q_L + C_{RF}V(t) + C_{GL}V_{GL} \\ Q_R + C_{GR}V_{GR} \end{pmatrix} = \begin{pmatrix} C_L & -C_T \\ -C_T & C_R \end{pmatrix} \begin{pmatrix} V_{\text{dot,L}} \\ V_{\text{dot,R}} \end{pmatrix}, \quad \begin{pmatrix} C_L \\ C_R \end{pmatrix} \equiv \begin{pmatrix} C_{RF} + C_{GL} + C_T \\ C_{GR} + C_T \end{pmatrix} \quad (2.13)$$

we solve for the voltages on each of the dots, merely by inverting the matrix:

$$\begin{pmatrix} V_{\text{dot,L}} \\ V_{\text{dot,R}} \end{pmatrix} = \frac{1}{C_L C_R - C_T^2} \begin{pmatrix} C_R & C_T \\ C_T & C_L \end{pmatrix} \begin{pmatrix} Q_L + C_{RF}V(t) + C_{GL}V_{GL} \\ Q_R + C_{GR}V_{GR} \end{pmatrix} = \frac{1}{C_L C_R - C_T^2} \begin{pmatrix} C_R & C_T \\ C_T & C_L \end{pmatrix} \begin{pmatrix} Q_L + Q_{EL} \\ Q_R + Q_{ER} \end{pmatrix} \quad (2.14)$$

The charges Q_{EL} and Q_{ER} aren't physical charges, but relate the dot voltages to the gate and rf-voltages; only Q_{EL} (and Q_L and Q_R) is time-dependent. The value $C_L C_R - C_T^2$ is never equal to zero, because C_L and C_R are equal to C_T plus some positive constant[¶].

We now take the line integral of the dot voltage vector with respect to the dot charge vector, giving us a scalar quantity: the energy of the system:

$$E = \frac{E_{CL}}{2} n_L^2 + \frac{E_{CR}}{2} n_R^2 + E_{CT} n_L n_R + E_L n_L + E_R n_R \quad (2.15)$$

where we've defined

$$E_{CL} \equiv \frac{e^2 C_R}{C_L C_R - C_T^2}, \quad E_{CR} \equiv \frac{e^2 C_L}{C_L C_R - C_T^2}, \quad E_{CT} \equiv \frac{2e^2 C_T}{C_L C_R - C_T^2} \quad (2.16a)$$

$$E_L \equiv -e \left(\frac{C_R Q_{EL} + C_T Q_{ER}}{C_L C_R - C_T^2} \right), \quad E_R \equiv -e \left(\frac{C_T Q_{EL} + C_L Q_{ER}}{C_L C_R - C_T^2} \right) \quad (2.16b)$$

[¶]or rather the value is only zero when $C_{RF} = C_{GL} = C_{GR} = 0$, in which case the circuit is fully disconnected.

In the case that we are using the LC resonator circuit, we need to replace $V(t)$ with $V_P(t)$, which is the voltage to the left of C_{RF} which is derived in Appendix A.1.5:

$$V_P(t) = V_{\text{out}}(t) \left(1 + \frac{i\omega L}{Z_c}\right) + V_{\text{in}}(t) \left(1 - \frac{i\omega L}{Z_c}\right) \quad (2.17)$$

This only affects the term Q_{EL} , which would be modified to

$$Q_{EL} = C_{GL}V_{GL} + C_{RF} \left(V_{\text{out}}(t) \left(1 + \frac{i\omega L}{Z_c}\right) + V_{\text{in}}(t) \left(1 - \frac{i\omega L}{Z_c}\right) \right) \quad (2.18)$$

2.4 LINEAR RESPONSE THEORY

Let us now focus on a different approach to describing the double dot, which requires a bit of quantum mechanics. We will think of the load as the classical circuit, where we have replaced the two dots and the tunnel junction with a quantum mechanical two-level system (hence we are removing R_T and C_T from the circuit):

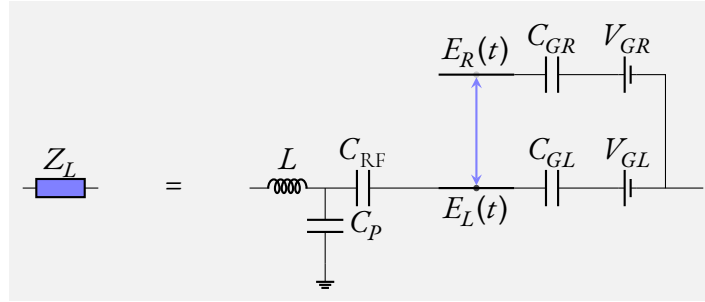


FIGURE 2.11: Quantum mechanical double dot modelled as a two-level system whose energy levels are set by the circuit above.

The number of charges on the double dot system is conserved^{***}, therefore we can treat the separate cases individually. The system consists of two levels, therefore the occupancy can be anywhere between $n = 0$ and $n = 4$. Luckily we do not need to look at each of the cases, because there are only 3 distinct cases, one of which (when $n = 0$ and $n = 4$) is trivial.

When $n = 0$ and $n = 4$ there is no dynamics, because there are either no electrons, or all the electrons are frozen due to the Pauli principle.

For $n = 1$ the electron will follow a Hamiltonian evolution, tunneling from the one dot to the other. The case with $n = 3$ is equivalent to $n = 1$ because the Pauli exclusion principle locks two of the charges in place, so there is still only one electron moving.

Finally, for $n = 2$ we have two distinct behaviours. Firstly if the two spins are parallel the system is inert, just like $n = 0$ and $n = 4$. Secondly if the spins are opposite the system evolves following Hamiltonian evolution, whose matrix representation is a 4×4 matrix.

The following figure we see the dynamical classes: the static class where there is no dynamics, the simple class where only one electron can move and the double class where two electrons are mobile:

^{***}Due to our constraint that the double dot system is only couple capacitively to its environment

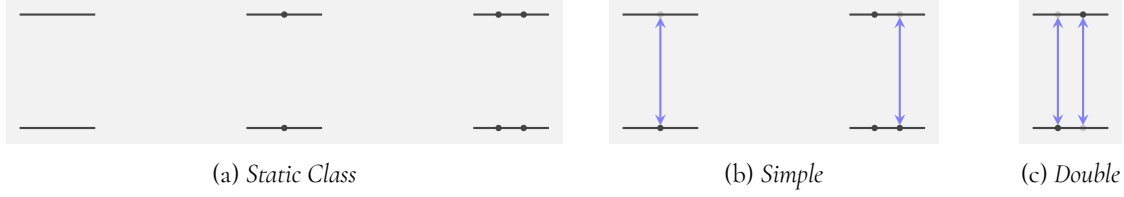


FIGURE 2.12: (a) Static class where no electrons are mobile due to the Pauli exclusion principle. (b) one electron is mobile, either because there only is one electron, or because two of the three electrons are frozen in place. (c) Two mobile electrons, only possible if they have opposite spin.

See Table B.1 for a visualisation for the different Hilbert-subspaces. Due to our assumption that tunneling events cannot flip spin and that the number of electrons is conserved we can treat the three classes separately. We will focus on the case where we only have one electron (hole) tunneling.

2.4.1 SINGLE OCCUPANCY: LINEAR RESPONSE THEORY

Let us now use the expression for the classical energy to create a second quantisation Hamiltonian. When $n = 1$ (3) there cannot be an electron (hole) on the left and right dot at the same time, therefore $n_L n_R = 0$, additionally there cannot be more than one electron (hole) at each site, which implies that $n_a^2 = n_a$. Therefore the Hamiltonian simplifies when we constrict ourselves to the Hilbert subspace, which consists of states that have one electron (hole) in them:

$$H = \left(\frac{E_{CL}}{2} + E_L \right) c_L^\dagger c_L + \left(\frac{E_{CR}}{2} + E_R \right) c_R^\dagger c_R - \tau c_L^\dagger c_R - \tau^* c_R^\dagger c_L \quad (2.19)$$

let us rewrite this in such a way, that it is easier to determine which coefficients are time dependent and which are not^{†††}:

$$H_0 = \varepsilon_L c_L^\dagger c_L + \varepsilon_R c_R^\dagger c_R - \tau c_L^\dagger c_R - \tau^* c_R^\dagger c_L, \quad H'(t) = \Theta(t) \delta_L(t) c_L^\dagger c_L \quad (2.20)$$

where the total Hamiltonian is $H = H_0 + H'$. The Heavyside function, $\Theta(t)$, tells us that we turn on the perturbation at $t = 0$, before which the system evolved “freely”. Because we are modelling the tunnel junction as a quantum mechanical system, we will set $C_T \rightarrow 0$ and $R_T \rightarrow \infty$. Let us assume that $\delta_L(t) = \delta_L \cos(\omega t + \varphi_0)$. The phase shift is essentially an additional parametre that tells us about the phase shift between our drive and the systems H_0 evolution. We intend to treat $H'(t)$ as a perturbation, which is why we split the time-dependent part from the time-independent part. The energies above were taken from Equation 2.16, in the limit where $C_T \rightarrow 0$:

$$\varepsilon_L \equiv \frac{e}{C_{GL} + C_{RF}} \left(\frac{e}{2} - C_{GL} V_{GL} \right), \quad \delta_L(t) \equiv -\frac{e C_{RF}}{C_{GL} + C_{RF}} V(t) \quad (2.21a)$$

$$\varepsilon_R \equiv \frac{e}{C_{GR}} \left(\frac{e}{2} - C_{GR} V_{GR} \right), \quad \delta_R \equiv 0 \quad (2.21b)$$

where $V(t) = V_{\text{in}}(t) + V_{\text{out}}(t)$, or when we use the LC resonator we would use $V_p(t)$, which still is a linear combination of V_{in} and V_{out} , as discussed previously, see Equation 2.17. It should be noted that $V_{\text{in}}(t)$ and $V_{\text{out}}(t)$ do not in general oscillate at the same frequency. For convenience let us define $\varepsilon \equiv \frac{\varepsilon_L + \varepsilon_R}{2}$ and $\Delta \equiv \frac{\varepsilon_L - \varepsilon_R}{2}$, so that

$$H_0 = (\varepsilon + \Delta) c_L^\dagger c_L + (\varepsilon - \Delta) c_R^\dagger c_R + \tau c_L^\dagger c_R + \tau^* c_R^\dagger c_L \quad (2.22)$$

^{†††} $\delta_L(t)$ is zero because we are setting $C_T = 0$

which we can diagonalise with a unitary transformation using

$$U \equiv \begin{pmatrix} u & v \\ -v & u \end{pmatrix}, \quad u \equiv \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\Delta}{\sqrt{\Delta^2 + \tau^2}}}, \quad v \equiv \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\Delta}{\sqrt{\Delta^2 + \tau^2}}} \quad (2.23)$$

We wish to use linear response theory to calculate the system's response to our perturbation up to linear order in the perturbation; for us to do that we need to use the Kubo Formula (Bruus and Flensberg 2016, p. 93):

$$\langle A \rangle(t) \approx \langle \psi_I(0) | A_I(t) | \psi_I(0) \rangle + \frac{i}{\hbar} \int_0^t dt' \langle \psi_I(0) | [H'_I(t'), A_I(t)] | \psi_I(0) \rangle \quad (2.24)$$

for some operator A . The index I tells us to use the interaction picture, where the phase factors $e^{\pm \frac{i}{\hbar} H_0 t}$ have been attached to the states and operators. The first term is the expectation value before we have switched our perturbation on, and the second term is the *response* to the perturbation. The operator whose expectation value we are interested in is n_L , because $-e\dot{n}_L$ is the current between the dots which we could, in principle use to calculate the reflected signal.

Let us simplify things by assuming that we start the system in an eigenstate to the Hamiltonian, say $|\psi_+\rangle$. In this case we get (for the full calculation see A.2.1)

$$\langle n_L \rangle(t) = \langle n_L \rangle_0 + \langle \delta n_L \rangle(t) \quad (2.25)$$

where

$$\langle n_L \rangle_0 = u^2 = \frac{1}{2} \left(1 + \frac{\Delta^2}{\Delta^2 + \tau^2} \right), \quad \langle \delta n_L \rangle(t) = \frac{\tau^2}{2\hbar(\Delta^2 + \tau^2)} \int_0^t dt' H_S(t') \sin(\Omega(t - t')) \quad (2.26)$$

where $\hbar\Omega = 2\sqrt{\Delta^2 + \tau^2}$. The quantum capacitance is given by $C^Q = -e^2 \frac{\partial^2 E_{\pm}}{\partial \Delta^2}$ (Petersson et al. 2010), which for the two level system is $\mp \frac{8e^2\tau^2}{\hbar^3\Omega^3}$. In the following figure we see the quantum capacitance as a function of Δ , beside the two energy levels.

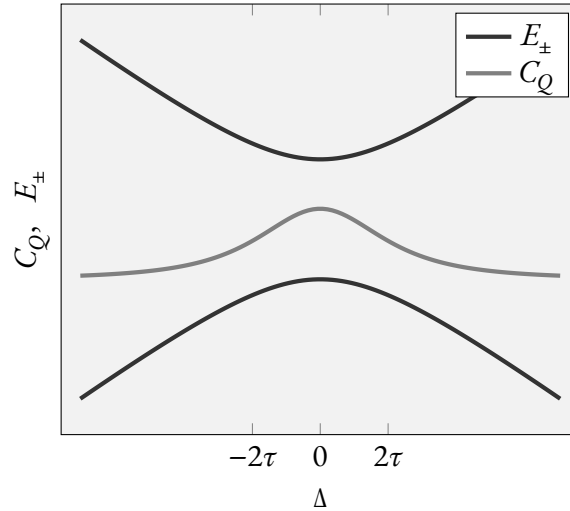


FIGURE 2.13: The quantum capacitance, C_Q , and the energy levels of the two-level system as a function of Δ . The capacitance has a peak where the energy levels have the greatest curvature.

By instead calculating the charge on the left dot $\langle Q_L \rangle(t) = -e \langle n_L \rangle(t)$ and by using that $\delta_L(t') = -eV_D(t')$ we get

$$\langle Q_L \rangle(t) = \frac{1}{2} \Omega C_Q \int_0^t dt' V_D(t') \sin(\Omega(t - t')) \quad (2.27)$$

where $V_D = \frac{C_{RF}}{C_{GL} + C_{RF}} V_P(t)$. Hence we can define the susceptibility (Clerk et al. 2010, p. 21, Eq. 4.2) of the system as

$$\chi(t - t') \equiv \frac{1}{2} \Omega C_Q \sin(\Omega(t - t')) \quad (2.28)$$

Which for the example above, when $V_P(t)$ is monochromatic we would get:

$$\langle \delta Q_L \rangle(t) = \frac{\Omega C_Q V_D}{2} \int_0^t dt' \cos(\omega t' + \varphi_0) \sin(\Omega(t - t')) \quad (2.29)$$

$$= \frac{C_Q V_D}{2} \left(\cos(\omega t + \varphi_0) + \frac{1}{2} (\cos((\omega + \Omega)t + \varphi_0) + \cos((\omega - \Omega)t + \varphi_0)) \right) \quad (2.30)$$

However, if we use non-eigenstate initial conditions the expressions become more complicated, due to the Rabi oscillation, for example when $|\psi(0)\rangle = |\psi_L\rangle$, see A.2.I.

Not only does our perturbation cause the (otherwise stationary) system to oscillate at the driving frequency, but also at $\omega \pm \Omega$, this is shown in the following figure:

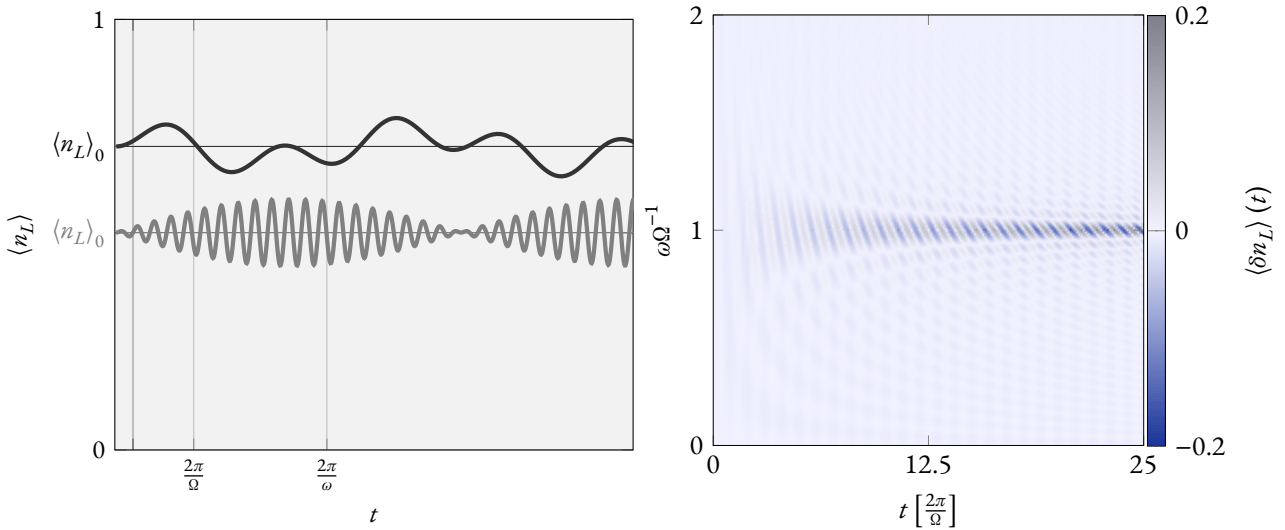


FIGURE 2.14: Left: $\langle n_L \rangle_0$ for different set of parameters. Initial condition: $|\psi(0)\rangle = |\psi_+\rangle$. Black: $\tau = 1.2\Delta$, $\omega = \frac{\Omega}{\epsilon}$, $\delta_L = 0.02\Delta$, $\varphi_0 = 0$. Grey: $\tau = 10\Delta$, $\omega = 1.05\Omega$, $\delta_L = 0.01\Delta$ and $\varphi_0 = 0$. Right: $\langle \delta n_L \rangle$ as a function of ω and t . $\Delta = 1.707\tau$ and $\delta_L = 0.0025\Delta$.

We see that the system responds best when our driving frequency matches the natural frequency of the system, and that the response is not monochromatic.

The fact that our linear response theory tells us that the system will oscillate at frequencies other than the incident frequency implies that we no longer can use the ratio $r = \frac{V_{\text{out}}}{V_{\text{in}}}$ to completely describe our reflected wave: V_{out} will have higher harmonics, just like $\dot{Q}_L$, as we can see in the following equation, which relates V_{out} (unknown) to V_{in} (known) and Q_L (unknown/known)^{†††}. It is derived in Appendix A.I.6:

$$\left(\frac{1}{Z_c} - C_{\text{all}} \left(1 - \frac{L}{Z_c} \frac{d}{dt} \right) \right) V_{\text{out}} = \left(\frac{1}{Z_c} - C_{\text{all}} \left(1 + \frac{L}{Z_c} \frac{d}{dt} \right) \right) V_{\text{in}} - \frac{C_{RF}}{C_{GL} + C_{RF}} \dot{Q}_L \quad (2.31)$$

^{†††}we've calculated Q_L for monochromatic driving, but in reality we need to calculate what happens if the perturbation is not monochromatic.

where

$$C_{\text{all}} = C_P + C_{RF} \left(1 - \frac{C_{RF}}{C_{GL} + C_{RF}} \right) \quad (2.32)$$

What this also implies is that our initial assumption, that $H'_S(t) \sim \cos(\omega t + \varphi_0)$ is not valid. The voltage that sets the energy levels of the two-level system is $V_P(t)$, which is a linear combination of V_{in} and V_{out} , the latter of which is not monochromatic, as we have just seen. Therefore, we have to use a more general form, such as

$$\langle \delta Q_L \rangle(t) = \frac{1}{2} \Omega C_Q \int_0^t dt' V_D(t') \sin(\Omega(t - t')) = \int_0^t dt' \chi(t - t') V_D(t') \quad (2.33)$$

where our new $V'_D(t)$ (and hence $H'_S(t')$) has to include higher harmonics. However, even this is an issue: in order to know what $H'_S(t)$ looks like, we need to know what the result, $\dot{Q}_L$, is so that we know what voltage we are applying to the system. Therefore, we would need to find a general way of solving this type of a problem: perhaps through Fourier Analysis, where we leave $H'_S(t)$'s Fourier coefficients as unknown and later match them to the coefficients we find for $\dot{Q}_L$ and V_{out} .

However, as we can see in Equation 2.33 we can write $\langle Q_L \rangle(t)$ as a convolution of $V_D(t')$ and a ‘susceptibility’, χ , which in the case above is simple because we have assumed we start the system in an eigenstate to H_0 . This is useful, because it means that the Fourier transform is a product: $\langle \delta Q_L \rangle(\omega) = \chi(\omega) V'_D(\omega)$. However in the case where we do not start the system in an eigenstate, the integrand in Equation 2.33 has terms that also depend purely on t and t' ; these are only important in the transient phase. Once the system has reached a steady state, when the behaviour cannot depend on initial conditions, we can write the integrand as $\chi(t - t') V_D(t')$ ^{§§§}.

3 DISCUSSION & CONCLUSION

Our description of classical dots began with an over-simplified circuit, which did not include a dissipative part. The coefficient of reflection had magnitude 1, however, the phase depended on ω . Once we added the tunneling resistor, R_T , we saw that there were frequencies where the absorption was significant: frequency intervals where we deliver energy into the system. However, in physical quantum dots, there is also the possibility that electrons tunnel from the leads into (and out of) the dot and between the dots. This implies that a more accurate model of quantum dots would include tunneling resistors parallel to C_{RF} and C_G .

The small scale of quantum dot systems presented the next problem: using 50Ω transmission lines requires the gate capacitors to be $\sim$ pF, whereas the relevant capacitances in QDot systems is closer to $\sim$ aF, therefore, we included an LC resonator in our circuit, to solve the problem of impedance mismatching.

Once this was done, we had designed a circuit that can be used to measure the tunneling resistance or capacitance for a double dot (and also a single dot). However, in this thesis we did not find* a general expression for the frequency at which resonance occurs (for $0 < R_T < \infty$), which means that we could not determine what the magnitude of absorption or the location of resonance depends on. This implies that we could not derive a method of determining R_T from an $\text{Abs}(r)$ as a function of ω . However,

^{§§§}this statement requires the system include dissipation.

*The expression became too large for even Mathematica to work with.

even if this expression does not exist, we could rely on numerical analyses to obtain a result that can be used in experiments. If we had an analytic expression for the location and depth of the minimum of $\text{Abs}(r)$ we could use the Fermi Golden Rule description of double (and single) dots to predict the exact shape and form of a Charge Stability (and Coulomb Diamond) Diagram, which can be measured by ‘merely’ attaching a quantum dot into a classical reflectometry device. Another issue is that the size of the absorption depends greatly on the value of C_T , for instance for $C_T = 100\text{fF}$ the absorption becomes almost negligible, see Figure B.4. However, we can expect the value of C_T to be 10fF , as per (Petersson et al. 2010).

In electrodynamic courses one usually assumes that two capacitors in a circuit can be described as independent of one another, however quantum dots are so small that the charges on capacitors C_i and C_j can feel each others Coulomb forces, and therefore cannot be treated as independent: we need to include mutual capacitances. However, this implies that there are additional fitting parametres, as it is difficult to determine their size.

An additional assumption that was made throughout this thesis, is that we can assume that R_T is independent of the voltage applied to it, i.e that it is a linear resistor. However, we know that R_T is not a physical resistor, but just a tool we use to model an inherently non-linear effect, therefore we cannot treat R_T as time-independent when solving Kirchhoff’s Laws, as $R_T(V(t))$ is time-dependent. By looking into how R_T is dependent of source-drain voltage, we can determine a more accurate form of Kirchhoff’s laws, which inevitably will include higher harmonic terms. The description used in this thesis only applies to the regions where R_T is approximately independent of V_{SD} , which is in the regions where the number of charges per dot is an integer (far away from the transitions).

The effective impedance between the dots was treated as a parallel R_T and C_T throughout the classical description, therefore we attempted to model the tunnel junction as a quantum mechanical system. However, due to the complicated interaction in linear response theory the reflected wave was not monochromatic, which implied that we could not use the usual tools to find a reflection coefficient nor an impedance, because these tools require monochromatic waves. However an expression was obtained that gave the system’s response, $\langle \delta n_L \rangle(t)$ as a convolution of the perturbation and a periodic function, which in the simplest case was proportional to $\sin(\Omega(t - t'))$.

The Hamiltonian in Equation 2.20 only involved hopping between the levels, but it did not include spontaneous decay. Therefore, the only energy loss from the coaxial cable is energy that is absorbed into the system. However, due to spontaneous decay a double quantum dot system can emit energy into the environment. Therefore the next step in describing quantum double dot systems would be to include spontaneous decay in the description. Unfortunately, due to the fact that V_{out} no longer is monochromatic, we cannot derive an expression for the effective impedance of the double dot system. However, we can note that in the classical case we had that $\tilde{Q}_L = \tilde{Q}_L^0 + \int_0^t dt' \tilde{\chi} \delta(t - t') \tilde{V}(t')$, and in the linear response theory case, we had that $\langle Q_L \rangle = \langle Q_L \rangle_0 + \int_0^t dt' \chi(t - t') V_D(t')$. Therefore, given more time I would look further into the input-output theory description quantum dot systems, where the susceptibilities play a central role.

BIBLIOGRAPHY

Hanson, R. et al. (2007). ‘Spins in few-electron quantum dots’. In: *Rev. Mod. Phys.* 79 (4), 1217–1265. DOI: 10.1103/RevModPhys.79.1217. URL: <https://link.aps.org/doi/10.1103/RevModPhys.79.1217>.

- Laucht, Arne et al. (Feb. 2021). ‘Roadmap on quantum nanotechnologies’. In: *Nanotechnology* 32.16, p. 162003. DOI: 10.1088/1361-6528/abb333. URL: <https://doi.org/10.1088/1361-6528/abb333>.
- Bruus, Henrik and Karsten Flensberg (2016). *Many-Body Quantum Theory in Condensed Matter Physics. An Introduction*. Oxford University Press. ISBN: 9780198566335.
- Berritta, Fabrizio (2021). *Visiting QDev’s laboratory*.
- Sørensen, Anders (2009). *Noter til Fysik 6 – Elektrodynamik og Bølger*. Accessed: 2020–12-22. URL: https://www.astro.ku.dk/~hansen/EM2/Bog_master.pdf.
- Clerk, A. A et al. (2010). ‘Introduction to quantum noise, measurement, and amplification’. In: *Rev. Mod. Phys.* 82, 1155 (2010) 82.2, 1155–1208.
- Müller, Jörg (2019). *EM 2: Laboratory Exercises*. Absalon. Accessed: 2020–12-22.
- Gallium Arsenide* (2021). Accessed: 2021–06-04. URL: <https://www.microwaves101.com/encyclopedias/gallium-arsenide>.
- Scholze, A. (1969). ‘Simulation of Single-Electron Devices’. PhD thesis. Swiss Federal Institute Of Technology Zürich.
- Wiel, W. G. van der et al. (2003). ‘Electron Transport Through Double Quantum Dots’. In: *Reviews of Modern Physics* 75, 1–22.
- Petersson, K. D. et al. (2010). ‘Charge and Spin State Readout of a Double Quantum Dot Coupled to a Resonator’. In: *NANO LETTERS* 10, 2789–2793.
- Liu, Y. Y. et al. (2020). ‘Radio frequency reflectometry in silicon-based quantum dots’. unpublished.
- Mizokuchi, R. et al. (2021). ‘Radio-frequency single electron transistors in physically defined silicon quantum dots with a sensitive phase response’. In: *Nature*.
- Schoelkopf, R. J. et al. (1998). ‘The Radio-Frequency Single-Electron Transistor (RF-SET): A Fast and Ultrasensitive Electrometer’. In: *Science* 280.
- Persson, F. et al. (2010). ‘Excess Dissipation in a Single-Electron Box: The Sisyphus Resistance’. In: *NANO LETTERS* 10, 953–957.
- Spiegel, Murray R., Seymour Lipschutz and John Liu (2013). *Mathematical Handbook of Formulas and Tables*. McGraw Hill. ISBN: 9780071795371.

A CALCULATIONS

A.1 CLASSICAL CIRCUIT

A.1.1 V_{out} AS A FUNCTION OF V_{in}

In Section 1.2 we introduced the total voltage at position x as $V(x, t) = V_{\text{in}}(x, t) + V_{\text{out}}(x, t)$. Additionally we derived that $\partial_x I = -\kappa \partial_t V$ (Clerk et al. 2010, App. C, p. 61):

$$I(x, t) = -\frac{1}{Z_c} (V_{\text{in}}(x, t) - V_{\text{out}}(x, t)) \quad (\text{A.1})$$

where we've used that $\frac{\omega}{k} = v = (\ell\kappa)^{-\frac{1}{2}}$, and that $\frac{\kappa}{\sqrt{\ell\kappa}} = \sqrt{\frac{\kappa}{\ell}} = \frac{1}{Z_c}$. Now, by solving for $V_{\text{out}}(x, t)$:

$$V_{\text{out}}(x, t) = V_{\text{in}}(x, t) + Z_c I(x, t) \quad (\text{A.2})$$

specifically, this holds at the border between the coaxial cable and the load:

$$V_{\text{out}}(L, t) = V_{\text{in}}(L, t) + Z_c I(L, t) \quad (\text{A.3})$$

in which case $I(L, t)$ is the current that enters the load.

A.1.2 SINGLE DOT WITH DRAIN

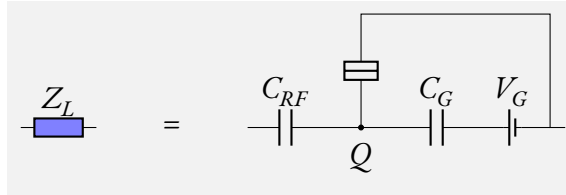


FIGURE A.1: Classical single dot with a tunnel junction to ground, which means that Q no longer is fixed.

In the case with a single dot that is connected to ground with a tunnel junction, Kirchhoff's Laws look as follows:

$$V(t) = -\frac{\tilde{Q}_{RF}}{C_{RF}} + \frac{\tilde{Q}_G}{C_G} + V_G \quad (\text{A.4})$$

$$V(t) = -\frac{\tilde{Q}_{RF}}{C_{RF}} + Z_{\text{eff}} \tilde{I}_T \quad (\text{A.5})$$

where $Z_{\text{eff}} = \frac{R_T(1-iR_T\omega C_T)}{1+R_T^2\omega^2 C_T^2}$ is the effective impedance of the tunnel junction. Additionally we have that

$$\tilde{Q} = \tilde{Q}_{RF} + \tilde{Q}_G \quad (\text{A.6})$$

Differentiating

$$\tilde{V}(t) = -\frac{\tilde{I}_{RF}}{i\omega C_{RF}} + \frac{\tilde{I}_G}{i\omega C_G} \quad (\text{A.7})$$

$$\tilde{V}(t) = -\frac{\tilde{I}_{RF}}{i\omega C_{RF}} + Z_{\text{eff}} \tilde{I}_T \quad (\text{A.8})$$

$$-\tilde{I}_{RF} = \tilde{I}_G + \tilde{I}_T = \tilde{I}_G - \dot{\tilde{Q}} \quad (\text{A.9})$$

Note that we have assumed that $I_T \sim e^{i\omega t}$. We define the charge on the dot as $\tilde{Q} = \tilde{Q}_{RF} + \tilde{Q}_G$. This charge can tunnel through the tunnel junction.

$$\begin{pmatrix} \tilde{V}(t) \\ \tilde{V}(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{i\omega C_{RF}} & \frac{1}{i\omega C_G} & 0 \\ -\frac{1}{i\omega C_{RF}} & 0 & Z_{\text{eff}} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{I}_{RF} \\ \tilde{I}_G \\ \tilde{I}_T \end{pmatrix} \quad (\text{A.10})$$

This corresponds exactly to the previous example, but where we let $C_{GR} \rightarrow \infty$.

$$\begin{pmatrix} \tilde{I}_{RF} \\ \tilde{I}_G \\ \tilde{I}_T \end{pmatrix} = \frac{1}{\omega Z_{\text{eff}} (C_G + C_{RF}) - i} \begin{pmatrix} -i\omega^2 C_G C_{RF} Z_{\text{eff}} & -\omega C_{RF} & \omega C_{RF} Z_{\text{eff}} \\ \omega C_G (1 + i\omega C_{RF} Z_{\text{eff}}) & -\omega C_G & \omega C_G Z_{\text{eff}} \\ -\omega C_G & \omega (C_G + C_{RF}) & -i \end{pmatrix} \begin{pmatrix} \tilde{V}(t) \\ \tilde{V}(t) \\ 0 \end{pmatrix} \quad (\text{A.11})$$

and hence

$$\tilde{I}_{RF} = \frac{\tilde{V}(t) \omega C_{RF} (1 + i\omega C_G Z_{\text{eff}})}{i - \omega Z_{\text{eff}} (C_G + C_{RF})} \quad (\text{A.12})$$

Using that $Z_L = -\frac{\tilde{V}(t)}{\tilde{I}_{RF}}$ The load impedance is

$$Z_L = \frac{1}{i\omega C_{RF}} \frac{1 + i\omega Z_{\text{eff}} (C_G + C_{RF})}{1 + i\omega C_G Z_{\text{eff}}} \quad (\text{A.13})$$

A.1.3 DOUBLE DOT IMPEDANCE MISMATCH

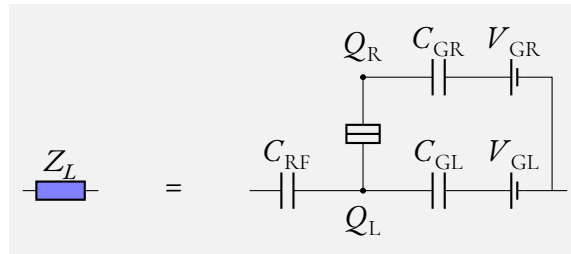


FIGURE A.2: Classical double dot with a tunnel junction connecting the dots.

We can obtain Kirchhoff's AC laws by treating the tunnel junction as an effective impedance, once again:

$$\tilde{V}(t) = -\frac{\tilde{I}_{RF}}{i\omega C_{RF}} + \frac{\tilde{I}_{GL}}{i\omega C_{GL}} \quad (\text{A.14})$$

$$\tilde{V}(t) = -\frac{\tilde{I}_{RF}}{i\omega C_{RF}} + \frac{\tilde{I}_{GR}}{i\omega C_{GR}} + Z_{eff}\tilde{I}_{LR} \quad (\text{A.15})$$

$$0 = \tilde{I}_{GL} + \tilde{I}_{GR} + \tilde{I}_{RF} \quad (\text{A.16})$$

$$\tilde{I}_{LR} = \tilde{I}_{GR} \quad (\text{A.17})$$

and so

$$\begin{pmatrix} \tilde{V}(t) \\ \tilde{V}(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{i\omega C_{RF}} & \frac{1}{i\omega C_{GL}} & 0 \\ -\frac{1}{i\omega C_{RF}} & 0 & Z_{eff} + \frac{1}{i\omega C_{GR}} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{I}_{RF} \\ \tilde{I}_{GL} \\ \tilde{I}_{LR} \end{pmatrix} \quad (\text{A.18})$$

By inverting this we can solve for $\tilde{I}_{RF}$:

$$-\tilde{I}_{RF} = \frac{V(t)\omega C_{RF} (C_{GR} + C_{GL} (1 + i\omega C_{GR} Z_{eff}))}{\omega C_{GR} Z_{eff} (C_{RF} + C_{GL}) - i (C_{GL} + C_{GR} + C_{RF})} \quad (\text{A.19})$$

hence

$$\begin{aligned} Z_L &= \frac{\omega C_{GR} Z_{eff} (C_{RF} + C_{GL}) - i (C_{GL} + C_{GR} + C_{RF})}{\omega C_{RF} (C_{GR} + C_{GL} (1 + i\omega C_{GR} Z_{eff}))} \\ &= -\frac{i\omega R_T (C_T (C_{GL} + C_{GR} + C_{RF}) + C_{GR} (C_{GL} + C_{RF})) + C_{GL} + C_{GR} + C_{RF}}{\omega C_{RF} (\omega R_T (C_{GL} (C_{GR} + C_T) + C_{GR} C_T) - i (C_{GL} + C_{GR}))} \end{aligned} \quad (\text{A.20})$$

A.1.4 DOUBLE DOT IMPEDANCE MATCHING

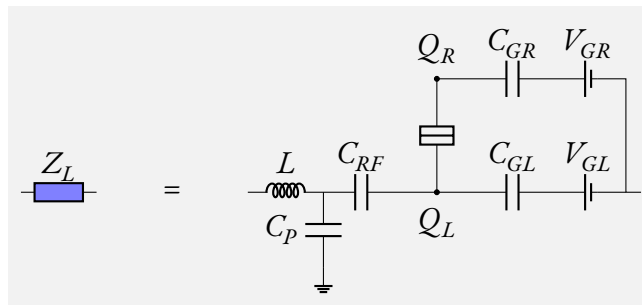


FIGURE A.3: Classical double dot with a tunnel junction connecting the dots. An inductor and a parasitic capacitor have been added to achieve impedance matching

Once again we set up Kirchhoff's AC laws:

$$V(t) = i\omega L_p \tilde{I}_{LP} + \frac{\tilde{I}_{RF}}{i\omega C_{RF}} + \frac{\tilde{I}_{GL}}{i\omega C_{GL}} \quad (\text{A.2Ia})$$

$$V(t) = i\omega L_p \tilde{I}_{LP} + \frac{\tilde{I}_{RF}}{i\omega C_{RF}} + Z_{\text{eff}} \tilde{I}_{GR} + \frac{\tilde{I}_{GR}}{i\omega C_{GR}} \quad (\text{A.2Ib})$$

$$V(t) = i\omega L_p \tilde{I}_{LP} + \frac{\tilde{I}_{CP}}{i\omega C_p} \quad (\text{A.2Ic})$$

$$\tilde{I}_{RF} = \tilde{I}_{GR} + \tilde{I}_{GL} \quad (\text{A.2Id})$$

$$\tilde{I}_{LP} = \tilde{I}_{RF} + \tilde{I}_{CP} \quad (\text{A.2Ie})$$

Which we just need to write in matrix form and then invert. Now it is $\tilde{I}_{LP}$ that is the relevant current, because that is the current *from* the coaxial cable.

$$Z_L = \frac{\tilde{V}(t)}{\tilde{I}_{LP}} = \frac{-C_{GR} L Z_{\text{eff}} \omega^3 (C_{GL}(C_p + C_{RF}) + C_p C_{RF})}{\omega(i C_{GR} Z_{\text{eff}} \omega (C_{GL}(C_p + C_{RF}) + C_p C_{RF}) + C_{RF}(C_{GL} + C_{GR} + C_p) + C_p(C_{GL} + C_{GR}))} \quad (\text{A.22})$$

$$+ i \frac{(L\omega^2(C_{RF}(C_{GL} + C_{GR} + C_p) + C_p(C_{GL} + C_{GR})) - (C_{GL} + C_{GR} + C_{RF})) + C_{GR} Z_{\text{eff}} \omega (C_{GL} + C_{RF})}{\omega(i C_{GR} Z_{\text{eff}} \omega (C_{GL}(C_p + C_{RF}) + C_p C_{RF}) + C_{RF}(C_{GL} + C_{GR} + C_p) + C_p(C_{GL} + C_{GR}))}$$

By letting $R_T \rightarrow \infty$ we get the following expression:

$$Z_L = i\omega L + \frac{(C_T(C_{GL} + C_{GR} + C_{RF}) + C_{GR}(C_{GL} + C_{RF}))}{i\omega(C_{GL}(C_{GR} + C_T)(C_p + C_{RF}) + C_{GR} C_T(C_p + C_{RF}) + C_{GR} C_p C_{RF} + C_p C_{RF} C_T)} \quad (\text{A.23})$$

INTERDOT CURRENT

From the inversion of Kirchhoff's laws, we can also calculate the interdot current. Integrating this gives us $Q_L(t) = Q_L^0 + \delta Q_L(t)$, where Q_L^0 is the integration constant:

$$\delta \tilde{Q}_L(t) = - \frac{C_{GR} C_{RF} \omega}{-i L \omega^2 (C_{RF}(C_{GL} + C_{GR} + C_p) + C_p(C_{GL} + C_{GR})) + C_{GR} L \omega^3 Z_{\text{eff}} (C_{GL}(C_p + C_{RF}) + C_p C_{RF}) - C_{GR} \omega Z_{\text{eff}} (C_{GL} + C_{RF}) + i(C_{GL} + C_{GR} + C_{RF})} \frac{\tilde{V}(t)}{i\omega} \quad (\text{A.24})$$

We can write this as $\delta \tilde{Q}_L(t) \int_0^t dt' \tilde{\chi} \delta(t - t') \tilde{V}(t')$, which will be used so that we can compare it to the result from linear response theory.

A.1.5 VOLTAGE RELATIONS: MONOCHROMATIC

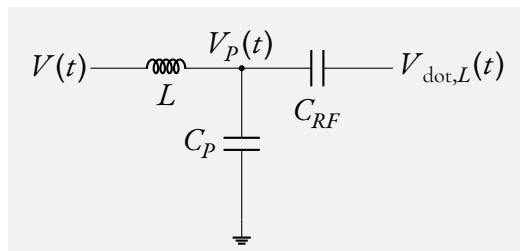


FIGURE A.4: First part of the circuit

The current through the components is

$$\tilde{I}_L = \frac{1}{i\omega L} (\tilde{V}_P(t) - \tilde{V}(t)) \quad (\text{A.25a})$$

$$\tilde{I}_{RF} = i\omega C_{RF} (\tilde{V}_P(t) - \tilde{V}_{\text{dot},L}) \quad (\text{A.25b})$$

$$\tilde{I}_P = i\omega C_P \tilde{V}_P(t) \quad (\text{A.25c})$$

$$0 = \tilde{I}_L + \tilde{I}_{RF} + \tilde{I}_P \quad (\text{A.25d})$$

Additionally we have that

$$\tilde{V}_{\text{out}}(t) = \tilde{V}_{\text{in}}(t) - Z_c \tilde{I}_L \quad (\text{A.26})$$

and so

$$\tilde{V}_{\text{out}} - \tilde{V}_{\text{in}} = -\frac{Z_c}{i\omega L} (\tilde{V}_P - \tilde{V}_{\text{in}} - \tilde{V}_{\text{out}}) \quad (\text{A.27})$$

Giving us

$$\tilde{V}_P = \tilde{V}_{\text{in}} \left(1 + \frac{i\omega L}{Z_c}\right) + \tilde{V}_{\text{out}} \left(1 - \frac{i\omega L}{Z_c}\right) \quad (\text{A.28})$$

This means that

$$\tilde{V}_{\text{out}}(t) = \tilde{V}_{\text{in}}(t) + Z_c (\tilde{I}_{RF} + \tilde{I}_P) \quad (\text{A.29})$$

$$= \tilde{V}_{\text{in}}(t) + Z_c (i\omega (C_P + C_{RF}) \tilde{V}_P(t) - i\omega C_{RF} \tilde{V}_{\text{dot},L}) \quad (\text{A.30})$$

The voltage on the dot:

$$\tilde{V}_{\text{dot},L} = \frac{C_R}{C_L C_R - C_T^2} (\tilde{Q}_L + C_{RF} \tilde{V}_P(t)) \quad (\text{A.31})$$

putting it all together

$$\begin{aligned} \frac{\tilde{V}_{\text{out}}(t) - \tilde{V}_{\text{in}}(t)}{Z_c} &= i\omega (C_P + C_{RF}) \tilde{V}_P - i\omega C_{RF} \left(\frac{C_R}{C_L C_R - C_T^2} (\tilde{Q}_L + C_{RF} \tilde{V}_P(t)) \right) \\ &= \left(i\omega (C_P + C_{RF}) - \frac{i\omega C_{RF}^2 C_R}{C_L C_R - C_T^2} \right) \left(\tilde{V}_{\text{in}} \left(1 + \frac{i\omega L}{Z_c}\right) + \tilde{V}_{\text{out}} \left(1 - \frac{i\omega L}{Z_c}\right) \right) - \frac{i\omega C_{RF} C_R}{C_L C_R - C_T^2} \tilde{Q}_L \end{aligned} \quad (\text{A.32})$$

(A.33)

Isolating

$$\begin{aligned} \tilde{V}_{\text{out}}(t) \left(1 - (Z_c - i\omega L) \left(i\omega (C_P + C_{RF}) - \frac{i\omega C_{RF}^2 C_R}{C_L C_R - C_T^2} \right) \right) &= \tilde{V}_{\text{in}}(t) \left(1 + (Z_c + i\omega L) \left(i\omega (C_P + C_{RF}) - \frac{i\omega C_{RF}^2 C_R}{C_L C_R - C_T^2} \right) \right) \\ &\quad - \frac{Z_c C_R C_{RF} \dot{\tilde{Q}}_L}{C_L C_R - C_T^2} \end{aligned} \quad (\text{A.34})$$

Note that when $\dot{\tilde{Q}}_L = 0$ we can once again obtain the expression for Z_L with the entire circuit in the limit where $R_T \rightarrow \infty$, which is the limit where the charges can't oscillate between the dots.

A.1.6 VOLTAGE RELATIONS: POLYCHROMATIC

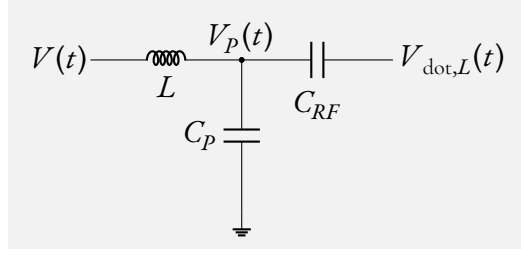


FIGURE A.5: First part of the circuit

The voltage drops are

$$\dot{I}_L = \frac{1}{L} (V_P(t) - V(t)) \quad (\text{A.35a})$$

$$Q_{RF} = C_{RF} (V_P(t) - V_{\dot{L}}(t)) \quad (\text{A.35b})$$

$$Q_P = C_P V_P(t) \quad (\text{A.35c})$$

$$0 = I_L + I_{RF} + I_P \quad (\text{A.35d})$$

We will assume that $C_T = 0$, because this is only relevant for the quantum mechanical description.

But we can take time derivatives until we have $\dot{I}$ everywhere:

$$\dot{I}_L = \frac{1}{L} (V_P(t) - V(t)) \quad (\text{A.36})$$

$$\dot{I}_{RF} = C_{RF} (\ddot{V}_P(t) - \ddot{V}_{\dot{L}}(t)) \quad (\text{A.37})$$

$$\dot{I}_P = C_P \ddot{V}_P(t) \quad (\text{A.38})$$

$$0 = \dot{I}_L + \dot{I}_{RF} + \dot{I}_P \quad (\text{A.39})$$

Additionally we have that

$$V_{\text{out}}(t) = V_{\text{in}}(t) - Z_c I_L \quad \rightarrow \quad \dot{V}_{\text{out}}(t) = \dot{V}_{\text{in}}(t) - Z_c \dot{I}_L \quad (\text{A.40})$$

and so

$$\dot{V}_{\text{out}} - \dot{V}_{\text{in}} = -\frac{Z_c}{L} (\ddot{V}_P - \ddot{V}_{\text{in}} - \ddot{V}_{\text{out}}) \quad (\text{A.41})$$

Giving us

$$\ddot{V}_P = \left(1 + \frac{L}{Z_c} \frac{d}{dt}\right) \dot{V}_{\text{in}} + \left(1 - \frac{L}{Z_c} \frac{d}{dt}\right) \dot{V}_{\text{out}} \quad (\text{A.42})$$

This means that

$$\dot{V}_{\text{out}}(t) = \dot{V}_{\text{in}}(t) + Z_c (\dot{I}_{RF} + \dot{I}_P) \quad (\text{A.43})$$

$$= \dot{V}_{\text{in}}(t) + Z_c ((C_P + C_{RF}) \ddot{V}_P(t) - C_{RF} \ddot{V}_{\dot{L}}(t)) \quad (\text{A.44})$$

The voltage on the dot:

$$\ddot{V}_{\dot{L}} = \frac{1}{C_{GL} + C_{RF}} (\ddot{Q}_L + C_{RF} \ddot{V}_P(t)) \quad (\text{A.45})$$

putting it all together

$$\frac{\dot{V}_{\text{out}} - \dot{V}_{\text{in}}}{Z_c} = (C_P + C_{RF}) \ddot{V}_P - \frac{C_{RF}}{C_{GL} + C_{RF}} (\ddot{Q}_L + C_{RF} \ddot{V}_P) \quad (\text{A.46})$$

$$= \left(C_P + C_{RF} \left(1 - \frac{C_{RF}}{C_{GL} + C_{RF}} \right) \right) \ddot{V}_P - \frac{C_{RF}}{C_{GL} + C_{RF}} \ddot{Q}_L \quad (\text{A.47})$$

plugging in for $\ddot{V}_P(t)$:

$$\frac{\dot{V}_{\text{out}} - \dot{V}_{\text{in}}}{Z_c} = \underbrace{\left(C_P + C_{RF} \left(1 - \frac{C_{RF}}{C_{GL} + C_{RF}} \right) \right)}_{C_{\text{all}}} \left(\left(1 + \frac{L}{Z_c} \frac{d}{dt} \right) \dot{V}_{\text{in}} + \left(1 - \frac{L}{Z_c} \frac{d}{dt} \right) \dot{V}_{\text{out}} \right) - \frac{C_{RF}}{C_{GL} + C_{RF}} \ddot{Q}_L \quad (\text{A.48})$$

Integrating and setting the integration constant to zero, because it is not important for the current analysis, and isolating V_{out} dependent terms on the left:

$$\left(\frac{1}{Z_c} - C_{\text{all}} \left(1 - \frac{L}{Z_c} \frac{d}{dt} \right) \right) V_{\text{out}} = \left(\frac{1}{Z_c} - C_{\text{all}} \left(1 + \frac{L}{Z_c} \frac{d}{dt} \right) \right) V_{\text{in}} - \frac{C_{RF}}{C_{GL} + C_{RF}} \dot{Q}_L \quad (\text{A.49})$$

A.2 QUANTUM MECHANICS

A.2.1 LINEAR RESPONSE THEORY

Let us begin by expressing H_0 in terms of the average energy $\varepsilon \equiv \frac{\varepsilon_L + \varepsilon_R}{2}$ and the difference in energy $\Delta \equiv \frac{\varepsilon_L - \varepsilon_R}{2}$:

$$H_0 = (\varepsilon + \Delta) c_L^\dagger c_L + (\varepsilon - \Delta) c_R^\dagger c_R + \tau c_L^\dagger c_R + \tau c_R^\dagger c_L \quad (\text{A.50})$$

The eigenvalues are $E_{\pm} = \varepsilon \pm \sqrt{\Delta^2 + \tau^2}$. But let us ignore the average energy, because the dynamics is independent of the average energies, and only dependent on the energy differences: the average energy is on the diagonal of H_0 , so it can be written as $\varepsilon \mathbb{1}$, which is unchanged by the upcoming unitary transformation. We can transform H_0 into a diagonal matrix using the following unitary tranformation:

$$U \equiv \begin{pmatrix} u & v \\ -v & u \end{pmatrix}, \quad u \equiv \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\Delta}{\sqrt{\Delta^2 + \tau^2}}}, \quad v \equiv \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\Delta}{\sqrt{\Delta^2 + \tau^2}}} \quad (\text{A.51})$$

We can show that $UH_0U^\dagger$ is diagonal, by using the following relations:

$$u^2 + v^2 = 1, \quad u^2 - v^2 = \frac{\Delta}{\sqrt{\Delta^2 + \tau^2}}, \quad 2uv = \frac{\tau}{\sqrt{\Delta^2 + \tau^2}} \quad (\text{A.52})$$

This tells us that

$$i\hbar (U |\dot{\psi}\rangle) = UH_0U^\dagger (U |\psi\rangle) \quad (\text{A.53})$$

Hence we are changing basis from the left-right basis to the $\{|\psi_+\rangle, |\psi_-\rangle\}$ basis:

$$\begin{pmatrix} |\psi_+\rangle \\ |\psi_-\rangle \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} |\psi_L\rangle \\ |\psi_R\rangle \end{pmatrix} \quad (\text{A.54})$$

We can also change the basis of our Fermion operators:

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} c_L \\ c_R \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} c_L \\ c_R \end{pmatrix} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \quad (\text{A.55})$$

note that the determinant is one. Now, in order to use the Kubo formula for linear response theory, we need to use the interaction picture, where both states and operators are time-dependent (Bruus and Flensberg 2016, p. 82)

$$|\psi_I(t)\rangle \equiv e^{\frac{i}{\hbar}H_0 t} |\psi_S(t)\rangle, \quad A_I(t) \equiv e^{\frac{i}{\hbar}H_0 t} A_S(t) e^{-\frac{i}{\hbar}H_0 t} \quad (\text{A.56})$$

The indices I and S denote that these are wave functions and operators in the interaction and Schrödinger picture respectively. $A_S(t)$ can both be time-dependent and independent of time, the only difference being that in the former case the $A_I(t)$ time-dependent is both due to H_0 and due its inherent time-dependence.

In a way this removes the “trivial” time dependence of the wave functions due to H_0 so that we can focus on the time-dependence due to the perturbation. In fact, in this picture the Schrödinger Equation becomes

$$i\hbar\partial_t |\psi_I(t)\rangle = H'_I(t) |\psi_I(t)\rangle \quad (\text{A.57})$$

note that the perturbation does not necessarily commute with H_0 and therefore we need to use the interaction operator, however H_0 commutes with itself, therefore there is no difference between H_0 in the Schrödinger and the interaction picture. Unfortunately the full form of the unitary operator $\hat{U}(t, 0)$ that evolves a state from $|\psi_I(0)\rangle$ to $|\psi_I(t)\rangle$ is quite complicated, and for the purpose of this thesis it is sufficient to look at it to linear order:

$$U_I(t, 0) \approx 1 - \frac{i}{\hbar} \int_0^t dt' H'_I(t') \quad (\text{A.58})$$

This is the case because we are viewing $H'_I(t)$ as a perturbation to H_0 and hence are assuming that the energy scales of $H'_I(t)$ are small compared to H_0

To linear order the expectation value of an operator $\hat{A}(t)$ is

$$\langle A \rangle(t) = \langle \psi_I(t) | A_I(t) | \psi_I(t) \rangle \approx \left\langle \psi_I(0) \left| \left(1 + \frac{i}{\hbar} \int_0^t dt' H'_I(t') \right) A_I(t) \left(1 - \frac{i}{\hbar} \int_0^t dt'' H'_I(t'') \right) \right| \psi_I(0) \right\rangle \quad (\text{A.59})$$

to linear order in H' this becomes

$$\langle A \rangle(t) \approx \langle \psi_I(0) | A_I(t) | \psi_I(0) \rangle + \frac{i}{\hbar} \int_0^t dt' \langle \psi_I(0) | [H'_I(t'), A_I(t)] | \psi_I(0) \rangle \quad (\text{A.60})$$

In our case we are interested in $n_L = c_L^\dagger c_L$ and the perturbation is $H'(t) = \delta_L(t) c_L^\dagger c_L + \delta_R(t) c_R^\dagger c_R$. Let us look at the second term more closely:

$$\frac{i}{\hbar} \int_0^t dt' \langle \psi_S(0) | e^{\frac{i}{\hbar}H_0 t'} H'_S(t') e^{\frac{i}{\hbar}H_0(t-t')} n_L e^{-\frac{i}{\hbar}H_0 t} | \psi_S(0) \rangle - \langle \psi_S(0) | e^{\frac{i}{\hbar}H_0(t)} n_L e^{\frac{i}{\hbar}H_0(t'-t)} H'_S(t') e^{-\frac{i}{\hbar}H_0 t'} | \psi_S(0) \rangle \quad (\text{A.61})$$

The phase factors pose an issue, therefore it makes sense to change basis into $\{|\psi_+\rangle, |\psi_-\rangle\}$ because this is the eigenbasis to H_0 . Suppose that $|\psi_S(0)\rangle = \alpha_+ |\psi_{S,+}\rangle + \alpha_- |\psi_{S,-}\rangle$, where we can find $\beta_\pm^0$ from initial conditions using Equation A.54. This also means that we have to rewrite n_L and n_R in the eigenbasis:

$$n_L = c_L^\dagger c_L = (uc_+^\dagger - vc_-^\dagger)(uc_+ - vc_-) = u^2 c_+^\dagger c_+ + v^2 c_-^\dagger c_- - uv(c_+^\dagger c_- + c_-^\dagger c_+) \quad (\text{A.62})$$

$$n_R = v^2 c_+^\dagger c_+ + u^2 c_-^\dagger c_- + uv(c_+^\dagger c_- + c_-^\dagger c_+) \quad (\text{A.63})$$

This allows us to calculate $n_L |\psi_S(0)\rangle$, which we assume is a linear combination of the eigenbasis $|\psi_S(0)\rangle = \alpha_+ |\psi_+\rangle + \alpha_- |\psi_-\rangle$, where the constants $\alpha_\pm$ are determined by initial conditions.

$$n_L |\psi_S(0)\rangle = u^2 \alpha_+ |\psi_+\rangle + v^2 \alpha_- |\psi_-\rangle - uv (\alpha_- |\psi_+\rangle + \alpha_+ |\psi_-\rangle) \quad (\text{A.64})$$

$$n_R |\psi_S(0)\rangle = v^2 \alpha_+ |\psi_+\rangle + u^2 \alpha_- |\psi_-\rangle + uv (\alpha_- |\psi_+\rangle + \alpha_+ |\psi_-\rangle) \quad (\text{A.65})$$

And in this basis the exponential operators just become phase factors, $e^{-\frac{i}{\hbar} H_0 t} |\psi_\pm\rangle = e^{-\frac{i}{\hbar} E_\pm t} |\psi_\pm\rangle$:

$$n_L e^{-\frac{i}{\hbar} H_0 t} |\psi_S(0)\rangle = u^2 \alpha_+ e^{-i\omega_+ t} |\psi_+\rangle + v^2 \alpha_- e^{-i\omega_- t} |\psi_-\rangle - uv (\alpha_- e^{-i\omega_- t} |\psi_+\rangle + \alpha_+ e^{-i\omega_+ t} |\psi_-\rangle) \quad (\text{A.66})$$

$$n_R e^{-\frac{i}{\hbar} H_0 t} |\psi_S(0)\rangle = v^2 \alpha_+ e^{-i\omega_+ t} |\psi_+\rangle + u^2 \alpha_- e^{-i\omega_- t} |\psi_-\rangle + uv (\alpha_- e^{-i\omega_- t} |\psi_+\rangle + \alpha_+ e^{-i\omega_+ t} |\psi_-\rangle) \quad (\text{A.67})$$

where we've defined $\hbar\omega_\pm = E_\pm$. And hence

$$e^{\frac{i}{\hbar} H_0 t} n_L e^{-\frac{i}{\hbar} H_0 t} |\psi_S(0)\rangle = u^2 \alpha_+ |\psi_+\rangle + v^2 \alpha_- |\psi_-\rangle - uv (\alpha_- e^{i(\omega_+ - \omega_-)t} |\psi_+\rangle + \alpha_+ e^{-i(\omega_+ - \omega_-)t} |\psi_-\rangle) \quad (\text{A.68})$$

$$e^{\frac{i}{\hbar} H_0 t} n_R e^{-\frac{i}{\hbar} H_0 t} |\psi_S(0)\rangle = v^2 \alpha_+ |\psi_+\rangle + u^2 \alpha_- |\psi_-\rangle + uv (\alpha_- e^{i(\omega_+ - \omega_-)t} |\psi_+\rangle + \alpha_+ e^{-i(\omega_+ - \omega_-)t} |\psi_-\rangle) \quad (\text{A.69})$$

note that $\omega_+ - \omega_- = 2\sqrt{\Delta^2 + \tau^2} = \hbar\Omega$. So let us explicitly write n_L

$$e^{\frac{i}{\hbar} H_0 t} n_L e^{-\frac{i}{\hbar} H_0 t} |\psi_S(0)\rangle = (u^2 \alpha_+ - uv \alpha_- e^{i\Omega t}) |\psi_+\rangle + (v^2 \alpha_- - uv \alpha_+ e^{-i\Omega t}) |\psi_-\rangle \quad (\text{A.70})$$

We can similarly calculate

$$e^{\frac{i}{\hbar} H_0 t'} H_S(t') e^{-\frac{i}{\hbar} H_0 t'} |\psi_S(0)\rangle = \delta_L \cos(\omega t' + \varphi_0) e^{\frac{i}{\hbar} H_0 t'} n_L e^{-\frac{i}{\hbar} H_0 t'} |\psi_S(0)\rangle \quad (\text{A.71})$$

letting H_0 operate on the ket and letting n_L operate on the result

$$\begin{aligned} e^{\frac{i}{\hbar} H_0 t'} H_S(t') e^{-\frac{i}{\hbar} H_0 t'} |\psi_S(0)\rangle &= \delta_L \cos(\omega t' + \varphi_0) e^{\frac{i}{\hbar} H_0 t'} n_L e^{-\frac{i}{\hbar} H_0 t'} |\psi_S(0)\rangle \\ &= \delta_L \cos(\omega t' + \varphi_0) (u^2 \alpha_+ |\psi_+\rangle + v^2 \alpha_- |\psi_-\rangle - uv (\alpha_- e^{i\Omega t'} |\psi_+\rangle + \alpha_+ e^{-i\Omega t'} |\psi_-\rangle)) \\ &= \delta_L \cos(\omega t' + \varphi_0) ((u^2 \alpha_+ - uv \alpha_- e^{i\Omega t'}) |\psi_+\rangle + (v^2 \alpha_- - uv \alpha_+ e^{-i\Omega t'}) |\psi_-\rangle) \end{aligned} \quad (\text{A.72})$$

This tells us that

$$\begin{aligned} &\langle \psi_S(0) | e^{\frac{i}{\hbar} H_0 t'} H_S'(t') e^{\frac{i}{\hbar} H_0 (t-t')} n_L e^{-\frac{i}{\hbar} H_0 t} | \psi_S(0) \rangle \\ &= \delta_L \cos(\omega t' + \varphi_0) (u^2 \alpha_+ - uv \alpha_- e^{-i\Omega t'}) (u^2 \alpha_+ - uv \alpha_- e^{i\Omega t}) \\ &\quad + \delta_L \cos(\omega t' + \varphi_0) (v^2 \alpha_- - uv \alpha_+ e^{i\Omega t'}) (v^2 \alpha_- - uv \alpha_+ e^{-i\Omega t}) \end{aligned} \quad (\text{A.73})$$

multiplying out

$$\begin{aligned} &\langle \psi_S(0) | e^{\frac{i}{\hbar} H_0 t'} H_S'(t') e^{\frac{i}{\hbar} H_0 (t-t')} n_L e^{-\frac{i}{\hbar} H_0 t} | \psi_S(0) \rangle \\ &= \delta_L \cos(\omega t' + \varphi_0) (u^2 \alpha_+ (u^2 \alpha_+ - uv \alpha_- e^{i\Omega t}) - uv \alpha_- (u^2 \alpha_+ e^{-i\Omega t'} - uv \alpha_- e^{i\Omega(t-t')})) \\ &\quad + \delta_L \cos(\omega t' + \varphi_0) (v^2 \alpha_- (v^2 \alpha_- - uv \alpha_+ e^{-i\Omega t}) - uv \alpha_+ (v^2 \alpha_- e^{i\Omega t'} - uv \alpha_+ e^{i\Omega(t-t')})) \end{aligned} \quad (\text{A.74})$$

writing it in this way allows us to calculate the total integrand, which is $2i$ times the imaginary part of the value above.

$$\begin{aligned} f(t, t') &= \langle \psi_S(0) | e^{\frac{i}{\hbar} H_0 t'} H_S'(t') e^{\frac{i}{\hbar} H_0 (t-t')} n_L e^{-\frac{i}{\hbar} H_0 t} | \psi_S(0) \rangle - \langle \psi_S(0) | e^{\frac{i}{\hbar} H_0 t} n_L e^{\frac{i}{\hbar} H_0 (t'-t)} H_S'(t') e^{-\frac{i}{\hbar} H_0 t'} | \psi_S(0) \rangle \\ &= 2i \delta_L \cos(\omega t' + \varphi_0) (-u^3 v \alpha_- \alpha_+ \sin(\Omega t) + u^3 v \alpha_- \alpha_+ \sin(\Omega t') + u^2 v^2 (\alpha_-)^2 \sin(\Omega(t-t'))) \\ &\quad + 2i \delta_L \cos(\omega t' + \varphi_0) (uv^3 \alpha_- \alpha_+ \sin(\Omega t) - uv^3 \alpha_- \alpha_+ \sin(\Omega t') - u^2 v^2 (\alpha_+)^2 \sin(\Omega(t-t'))) \end{aligned} \quad (\text{A.75})$$

we have that $u^2 - v^2 = \frac{\Delta}{\sqrt{\Delta^2 + \tau^2}}$ and $2uv = \frac{\tau}{\sqrt{\Delta^2 + \tau^2}}$

$$f(t, t') = i \delta_L \cos(\omega t' + \varphi_0) \frac{\tau}{\Delta^2 + \tau^2} (\alpha_- \alpha_+ \Delta (-\sin(\Omega t) + \sin(\Omega t'))) + \frac{\tau}{2} ((\alpha_-)^2 - (\alpha_+)^2) \sin(\Omega(t-t'))$$

STARTING IN THE LEFT DOT

Consider the case where we start the system in $|\psi(0)\rangle = |\psi_L\rangle = u |\psi_+\rangle - v |\psi_-\rangle$: $\alpha_+ = u$ and $\alpha_- = -v$

$$f(t, t') = -i\delta_L \cos(\omega t' + \varphi_0) \frac{\tau^2 \Delta}{2(\Delta^2 + \tau^2)^{\frac{3}{2}}} (\sin(\Omega(t - t')) - \sin(\Omega t) + \sin(\Omega t')) \quad (\text{A.76})$$

Integrating

$$\delta n_L(t) \equiv \frac{i}{\hbar} \int_0^t dt' f(t, t') = \frac{\delta_L \tau^2 \Delta}{2\hbar (\Delta^2 + \tau^2)^{\frac{3}{2}}} \int_0^t dt' \cos(\omega t' + \varphi_0) (\sin(\Omega(t - t')) - \sin(\Omega t) + \sin(\Omega t')) \quad (\text{A.77})$$

Note that $H'_S(t') = \delta_L \cos(\omega t' + \varphi_0)$ has been taken along throughout the entire calculation, unchanged. This implies that for a general $H'_S(t')$ we would use:

$$\delta n_L(t) \equiv \frac{i}{\hbar} \int_0^t dt' f(t, t') = \frac{\tau^2 \Delta}{2\hbar (\Delta^2 + \tau^2)^{\frac{3}{2}}} \int_0^t dt' H'_S(t') (\sin(\Omega(t - t')) - \sin(\Omega t) + \sin(\Omega t')) \quad (\text{A.78})$$

For the monoharmonic perturbation I solved the integral using Mathematica:

$$\frac{i}{\hbar} \int_0^t dt' f(t, t') = \frac{\delta_L \tau^2 \Delta}{2\hbar (\Delta^2 + \tau^2)^{\frac{3}{2}}} \frac{4\Omega \sin^2\left(\frac{\Omega t}{2}\right) \cos\left(\frac{\omega t}{2} + \varphi_0\right) \left(\Omega \sin\left(\frac{\omega t}{2}\right) \cot\left(\frac{\Omega t}{2}\right) - \omega \cos\left(\frac{\omega t}{2}\right)\right)}{\omega(\omega - \Omega)(\omega + \Omega)} \quad (\text{A.79})$$

$$= \frac{4\delta_L \tau^2 \Delta}{\hbar^2 (\Delta^2 + \tau^2) \omega(\omega - \Omega)(\omega + \Omega)} \left(\underbrace{\Omega \cos\left(\frac{\omega t}{2} + \varphi_0\right) \sin\left(\frac{\omega t}{2}\right)}_{\frac{1}{2}(\sin \varphi_0 + \sin(\omega t + \varphi_0))} \underbrace{\cos\left(\frac{\Omega t}{2}\right) \sin\left(\frac{\Omega t}{2}\right)}_{\frac{1}{2} \sin \Omega t} - \underbrace{\omega \cos\left(\frac{\omega t}{2} + \varphi_0\right) \cos\left(\frac{\omega t}{2}\right)}_{\frac{1}{2}(\cos \varphi_0 + \cos(\omega t + \varphi_0))} \underbrace{\sin^2\left(\frac{\Omega t}{2}\right)}_{\frac{1}{2}(1 - \cos \Omega t)} \right) \quad (\text{A.80})$$

$$= \frac{\delta_L \tau^2 \Delta}{\hbar^2 (\Delta^2 + \tau^2) \omega(\omega - \Omega)(\omega + \Omega)} \left(\Omega \left(\underbrace{\sin \varphi_0 \sin \Omega t}_{\frac{1}{2}(\cos(\Omega t - \varphi_0) - \cos(\Omega t + \varphi_0))} + \underbrace{\sin(\omega t + \varphi_0) \sin \Omega t}_{\frac{1}{2}(\cos((\omega - \Omega)t + \varphi_0) - \cos((\omega + \Omega)t + \varphi_0))} \right) - \right. \\ \left. \omega \left(\cos \varphi_0 + \cos(\omega t + \varphi_0) - \underbrace{\cos \varphi_0 \cos \Omega t}_{\frac{1}{2}(\cos(\Omega t - \varphi_0) + \cos(\Omega t + \varphi_0))} - \underbrace{\cos(\omega t + \varphi_0) \cos \Omega t}_{\frac{1}{2}(\cos((\omega - \Omega)t + \varphi_0) + \cos((\omega + \Omega)t + \varphi_0))} \right) \right) \quad (\text{A.81})$$

The sin and cos product rules were obtained from (Spiegel, Lipschutz and J. Liu 2013) We see that there is a term that oscillates at the same frequency (and same) phase as the incident voltage. However, we also get terms with frequencies Ω and $\Omega \pm \omega$.

We also are interested in $\langle n_L \rangle_0$:

$$\langle n_L \rangle_0 = \langle \psi_S(0) | e^{\frac{i}{\hbar} H_0 t} n_L e^{-\frac{i}{\hbar} H_0 t} | \psi_S(0) \rangle = \alpha_+ (u^2 \alpha_+ - uv \alpha_- e^{i\Omega t}) + \alpha_- (v^2 \alpha_- - uv \alpha_+ e^{-i\Omega t}) \quad (\text{A.82})$$

if we again assume we start the system in the left dot then $\alpha_+ = u$ and $\alpha_- = -v$:

$$\langle n_L \rangle_0 = u^4 + v^4 + u^2 v^2 (e^{i\Omega t} + e^{-i\Omega t}) = 1 - 2u^2 v^2 (1 - \cos(\Omega t)) = 1 - \frac{\tau^2}{2(\tau^2 + \Delta^2)} (1 - \cos(\Omega t)) \quad (\text{A.83})$$

for $t = 0$ this is equal to one, and otherwise is smaller or equal to one.

STARTING IN AN EIGENSTATE

Alternatively, if we begin the system in an eigenstate, say $|\psi_+\rangle$ we get that $\alpha_+ = 1$ and $\alpha_- = 0$, therefore

$$\langle n_L \rangle_0 = u^2 = \frac{1}{2} \left(1 + \frac{\Delta^2}{\Delta^2 + \tau^2} \right) \quad (\text{A.84})$$

and

$$\langle \delta n_L \rangle(t) = \frac{\tau^2}{2\hbar(\Delta^2 + \tau^2)} \int_0^t dt' H_S(t') \sin(\Omega(t - t')) \quad (\text{A.85})$$

Which for the example above would be

$$\langle \delta n_L \rangle(t) = \frac{\tau^2 \delta_L}{2\hbar(\Delta^2 + \tau^2)} \int_0^t dt' \cos(\omega t' + \varphi_0) \sin(\Omega(t - t')) \quad (\text{A.86})$$

$$= \frac{\tau^2 \delta_L}{2\hbar(\Delta^2 + \tau^2)^{\frac{3}{2}}} \left(\cos(\omega t + \varphi_0) + \frac{1}{2} (\cos((\omega + \Omega)t + \varphi_0) + \cos((\omega - \Omega)t + \varphi_0)) \right) \quad (\text{A.87})$$

B PLOTS, TABLES & IMAGES

B.1 QUANTUM DOT DEVICE, GAAS

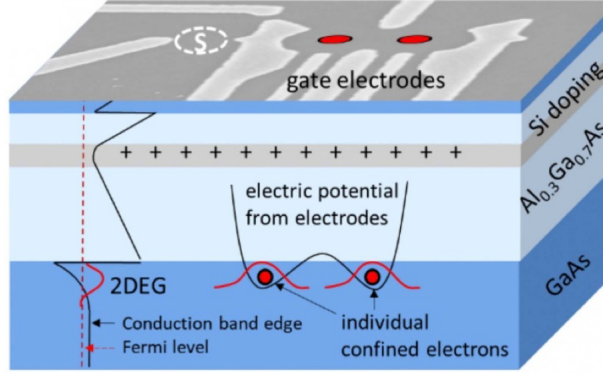


FIGURE B.1: Depiction of a quantum dot device, obtained from (Laucht et al. 2021)

B.2 CLASSICAL DOT

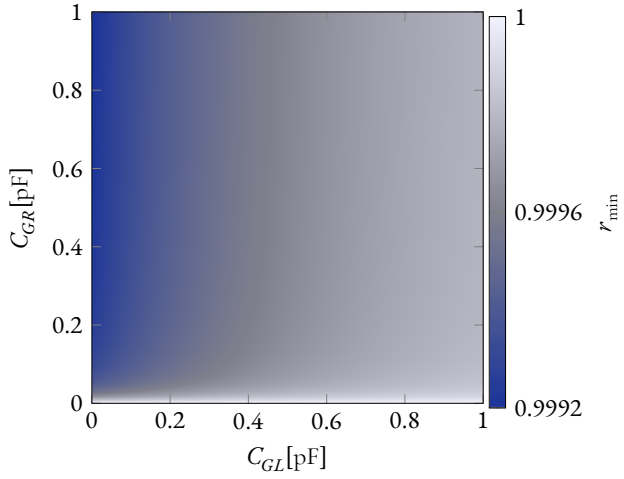


FIGURE B.2: Minimum of $\text{Abs}(r)$ as a function of C_{GL} and C_{GR} with $C_{RF} = 0.1\text{pF}$, $C_T = 10\text{fF}$, $R_T = 100\text{k}\Omega$, $Z_c = 50\Omega$.

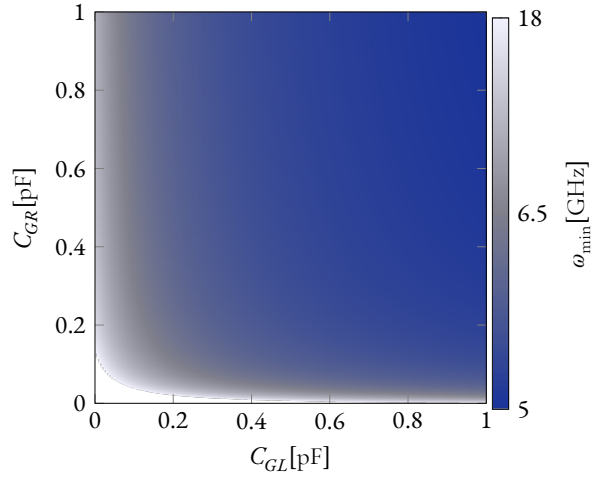


FIGURE B.3: $\omega_{\min}$ as a function of C_{GL} and C_{GR} with $C_{RF} = 0.1\text{pF}$, $C_T = 10\text{fF}$, $R_T = 100\text{k}\Omega$, $Z_c = 50\Omega$.

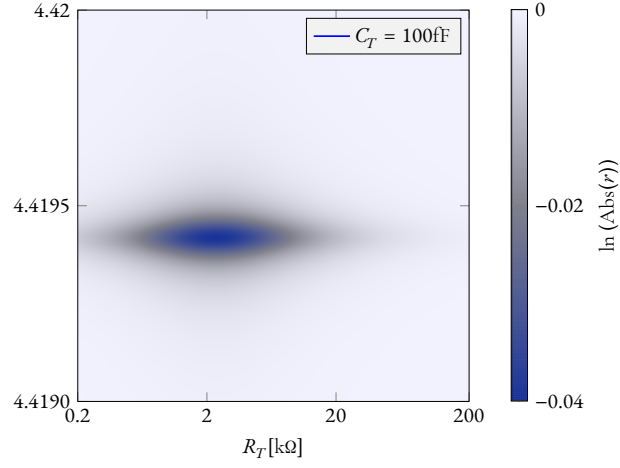


FIGURE B.4: $\ln(\text{Abs}(r))$ as a function of R_T and ω . $C_{GL} = 10\text{aF}$, $C_{GR} = 10\text{aF}$, $C_{RF} = 100\text{aF}$, $C_P = 100\text{aF}$, $L = 480\mu\text{H}$, $Z_c = 50\Omega$ and $C_T = 100\text{fF}$

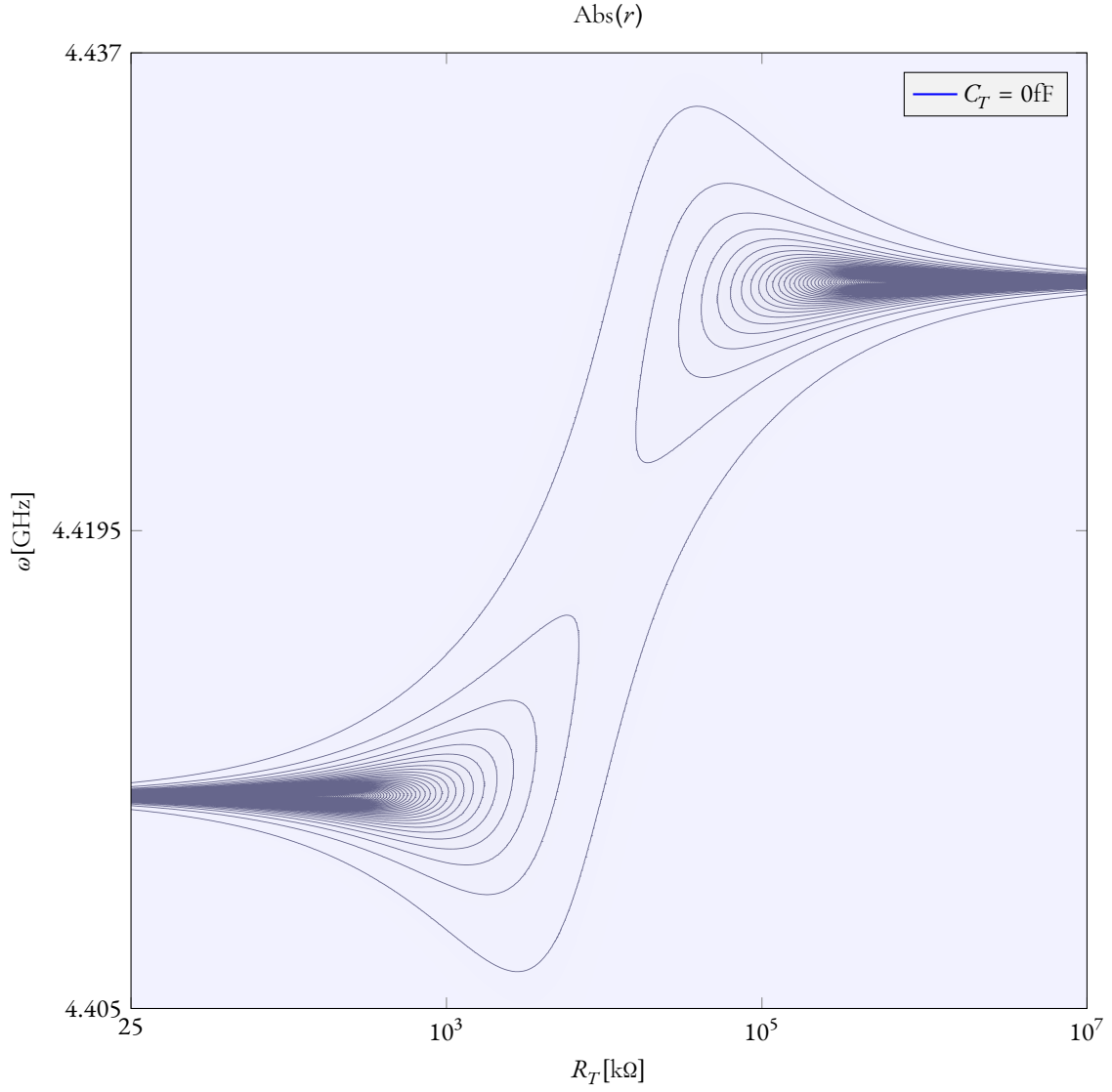


FIGURE B.5: $Abs(r)$ as a function of R_T and ω . $C_{GL} = 10aF$, $C_{GR} = 10aF$, $C_{RF} = 100aF$, $C_p = 100aF$, $L = 480\mu H$, $Z_c = 50\Omega$ and $C_T = 0fF$. The smallest value of $Abs(r) = 0.2$ and the greatest is one. The minimum is $Abs(r) = 0.2$ and the lines are changes in $Abs(r)^2$ such that it is easier to see differences close to $Abs(r) = 1$. There are 150 lines

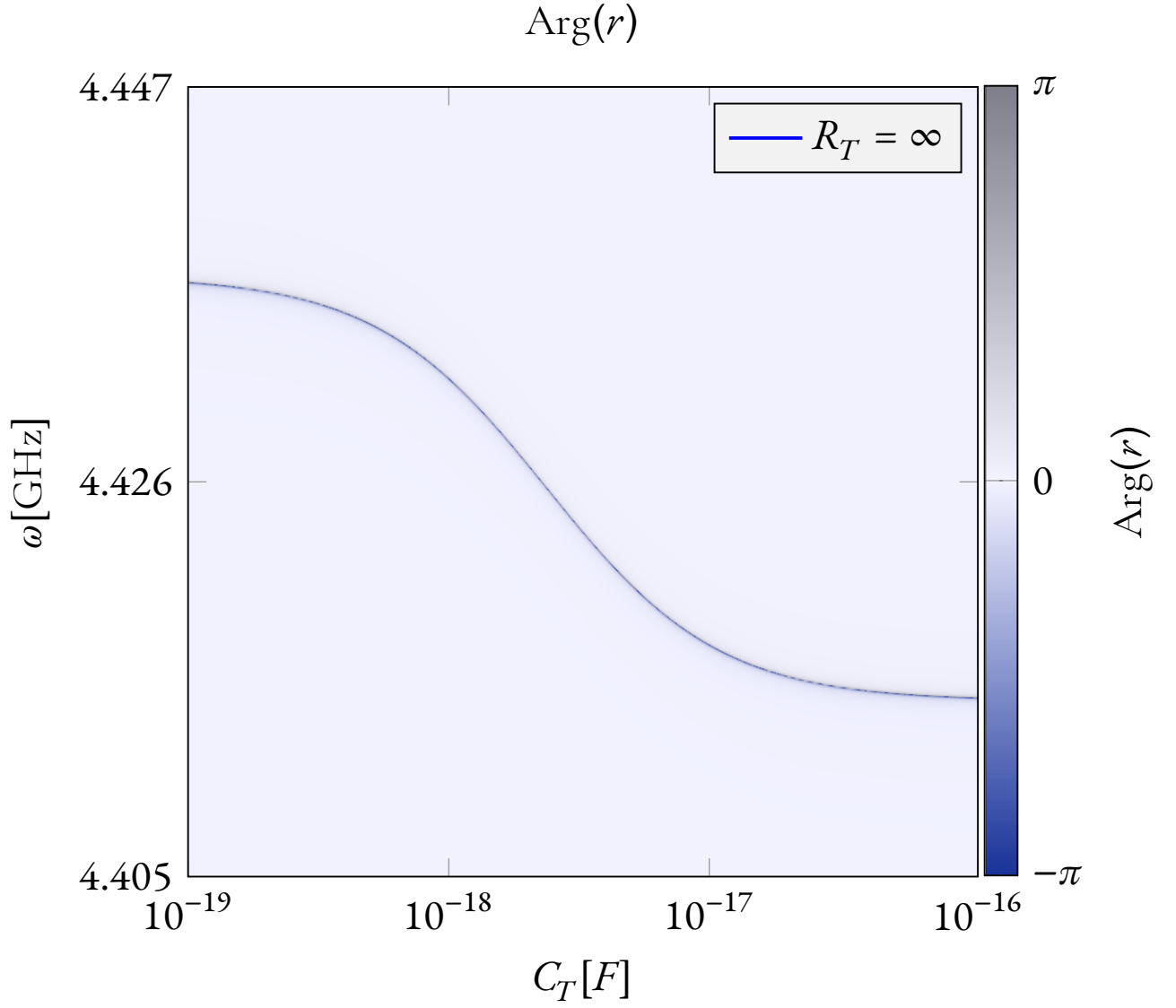


FIGURE B.6: $\text{Arg}(r)$ as a function of C_T and ω . $C_{GL} = 10\text{aF}$, $C_{GR} = 10\text{aF}$, $C_{RF} = 100\text{aF}$, $C_P = 100\text{aF}$, $L = 480\mu\text{H}$, $Z_c = 50\Omega$ and $R_T = \infty$.

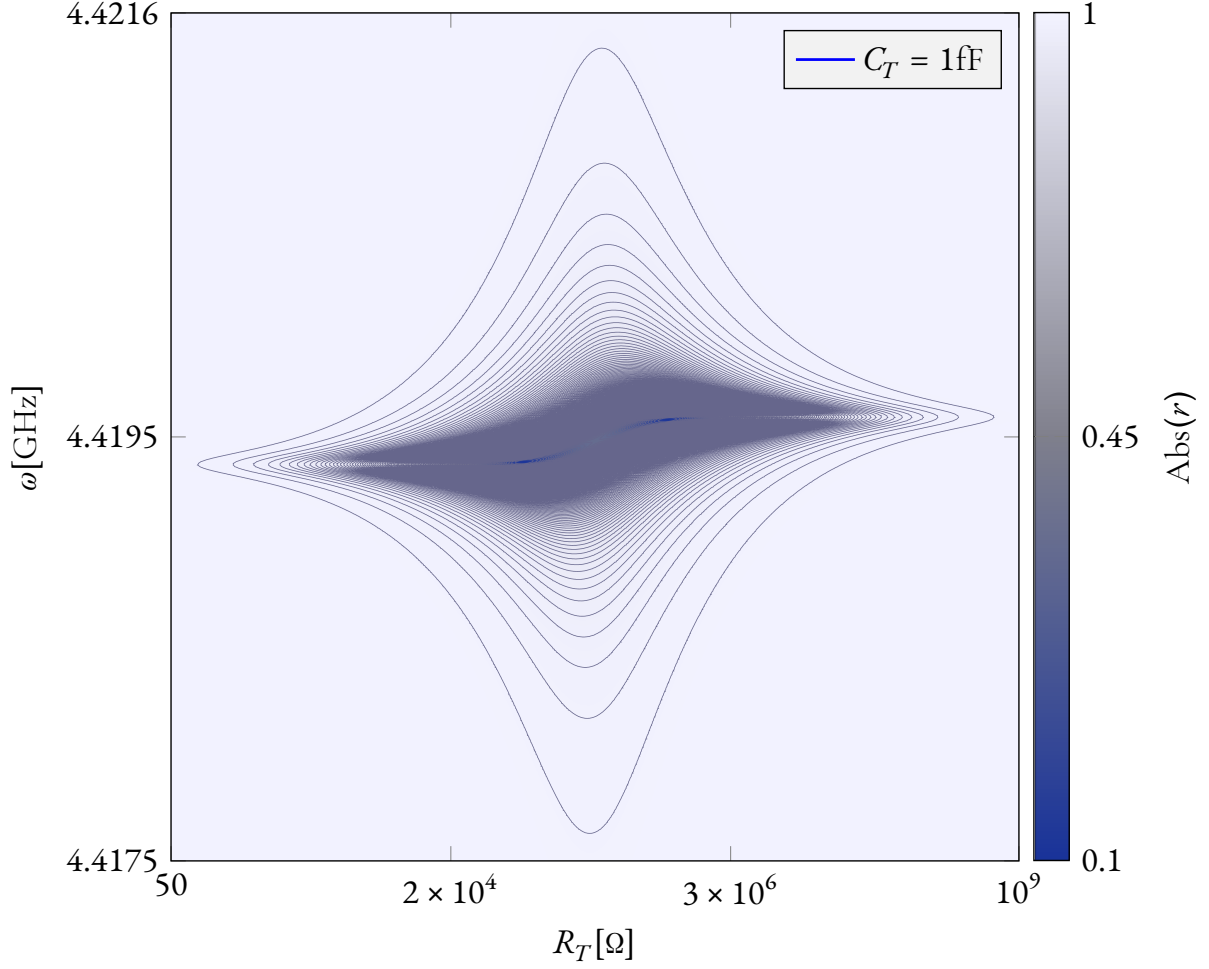


FIGURE B.7: $\text{Abs}(r)$ as a function of R_T and ω . $C_{GL} = 10\text{aF}$, $C_{GR} = 10\text{aF}$, $C_{RF} = 100\text{aF}$, $C_p = 100\text{aF}$, $L = 480\mu\text{H}$, $Z_c = 50\Omega$ and $C_T = 1\text{fF}$. The lines are changes in $\text{Abs}(r)^2$ so as to see changes close to $\text{Abs}(r) = 1$. There are 150 lines.

B.3 QUANTUM MECHANICAL DESCRIPTION

In the following Figure I show the entire Hilbert space, which decouples into separate Hilbert subspaces:

	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
Static	$ 0\rangle$		$ \uparrow\rangle \otimes \uparrow\rangle$ $ \downarrow\rangle \otimes \downarrow\rangle$		$ \uparrow\downarrow\rangle \otimes \uparrow\downarrow\rangle$
Dynamic		$ \uparrow\rangle \otimes 0\rangle$ $ 0\rangle \otimes \uparrow\rangle$ $ \downarrow\rangle \otimes 0\rangle$ $ 0\rangle \otimes \downarrow\rangle$	$ \uparrow\downarrow\rangle \otimes \downarrow\rangle$ $ \uparrow\downarrow\rangle \otimes 0\rangle$ $ \downarrow\rangle \otimes \uparrow\rangle$ $ 0\rangle \otimes \uparrow\downarrow\rangle$	$ \uparrow\rangle \otimes \uparrow\downarrow\rangle$ $ \uparrow\downarrow\rangle \otimes \uparrow\rangle$ $ \downarrow\rangle \otimes \uparrow\downarrow\rangle$ $ \uparrow\downarrow\rangle \otimes \downarrow\rangle$	

TABLE B.1: Hilbert subspaces in the case where we ignore spin flips. The states in the same box can evolve into one another.