



Driven Problems in Quantum and Classical Mechanics with Floquet Theory

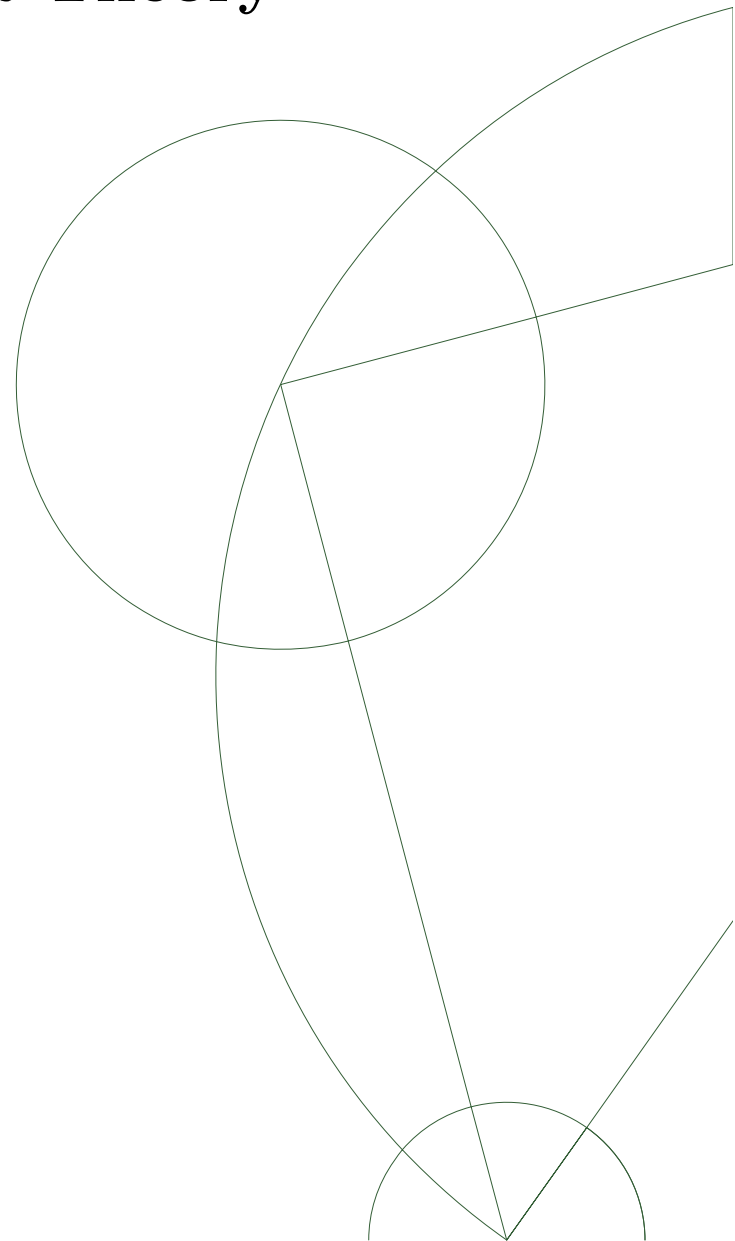
Mads Anders Jørgensen

Bachelor's project in Physics

Supervisor: Jens Paaske

Niels Bohr Institute
University of Copenhagen

Signature:
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Abstract

In this paper general Floquet theory is introduced and applied to various time dependent problems in both quantum and classical mechanics. Analytical solutions are found to the harmonically driven oscillator and the inverted harmonically driven oscillator and their stability has been studied. The transition probability to the n 'th state of the harmonic oscillator is found for the harmonically driven oscillator using two methods. The stability of the abruptly driven classical oscillator, the harmonically driven classical oscillator and the quantum oscillator with time-dependent frequency have been assessed and stable and unstable solutions have been found.

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1 Introduction

Driven systems appear in various forms in both classical mechanics and quantum mechanics. Some simple forms allow for analytical solutions, which can then be analyzed and studied in detail. For those that do not, there are tools to make qualitative statements about their general behaviour. In this paper, various problems, both with and without analytical solutions, in both classical and quantum mechanics, will be examined and their general behaviours identified with the help of Floquet theory. Floquet multipliers will be used to show stability of a classical system via Poincaré sections. The stability of various driven systems with different initial conditions have been assessed through identification of the quasi-energy of the system. Stable and unstable solutions have been found to the harmonic quantum oscillator. The transition probabilities of the 0th state of the driven harmonic oscillator to the n 'th state of the harmonic oscillator have been found and analyzed and resonance has been discussed.

2 Floquet theory

2.1 Floquet's theorem

Floquet's theorem [1] states that the solution to a problem of the form $\dot{x} = A(t)x$, where $A(t+T) = A(t)$ is some periodic function with period T , is given by

$$\Phi = P(t)e^{\pm i\mu t}, \quad (2.1)$$

where $P(t+T)=P(t)$ is a periodic function with the the same period T . Second order problems, $\ddot{x} = A(t)x$ can be rewritten as

$$\frac{d}{dt} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ A(t) & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \quad (2.2)$$

so that their solution is on Floquet form.

If $H = A(t) - \partial_t^2$ is the operator for which $H\Phi = 0$, then we have that

$$HP(t) = i\mu P(t). \quad (2.3)$$

This can be seen by inserting the proposed solution into Φ and carrying out the differentiations. In quantum mechanics, we usually say that

$$\mu = \frac{\epsilon_n}{\hbar}, \quad (2.4)$$

where ϵ_n is called the quasi-energy. The quantity μt is a measure of the phase acquired after a time t , for real values of μ . It can also be used to make a qualitative assessment of the stability of a given system. For real μ , the solution is normalizable and the motion is bounded. Imaginary (or complex) values of μ allow for diverging or decaying solutions. For example, consider the case in which $\mu = -ia$, where $a > 0$. This would make the solution tend to diverge as $t \rightarrow \infty$, and is thus not normalizable.

If \hat{M} is the operator which propagates time by one period when applied to a state, such that $\hat{M}\Phi(t) = \Phi(t+T)$, then the eigenvalues, ρ , of the operator \hat{M} are called the Floquet multipliers, and are given by

$$\rho = e^{i\mu T}. \quad (2.5)$$

If the Floquet multiplier $|\rho| \leq 1$, then the system is stable and will either be periodic or decay. If $|\rho| > 1$, then the solution will grow exponentially with time.

Just like ρ , the operator \hat{M} can be used to make a qualitative statement about the stability of a given system, even of ones with no analytical solution. Given a set of initial conditions, one can make a Poincaré section by applying the operator multiple times. This can reveal the periodicity, decay, or divergence of the solution, without having to solve the problem analytically.

2.2 Time evolution operator of a driven pendulum

An easily accessible example of using this method to determine stability is the case of the abruptly driven pendulum [2].

Given the equation

$$\begin{aligned}\ddot{\theta} + (1+r)\theta &= 0, \text{ for } 0 \leq t < \pi x \\ \ddot{\theta} + (1-r)\theta &= 0, \text{ for } \pi x \leq t < 2\pi x,\end{aligned}$$

the solution is of the form

$$\theta(t) = \theta(0) \cos(\omega t) + \dot{\theta}(0) \frac{\sin(\omega t)}{\omega},$$

where $\omega = \sqrt{1 \pm r} = \Gamma_{\pm}/(\pi x)$, + or - depends on t.

The time evolution operator M, which fulfills $M\theta(t) = \theta(t+T)$, can be split up into two matrices, each responsible for half a period, $\frac{T}{2}$. For the first half-period, we use $\omega = \sqrt{1+r}$.

$$\begin{aligned}\theta(\pi x) &= \theta(0) \cos(\Gamma_+) + \dot{\theta}(0) \frac{\sin(\Gamma_+)}{\omega} \text{ and} \\ \dot{\theta}(\pi x) &= -\theta(0) \sqrt{1+r} \sin(\Gamma_+) + \dot{\theta}(0) \cos(\Gamma_+).\end{aligned}$$

In matrix notation, this gives us

$$\begin{pmatrix} \theta(\pi x) \\ \dot{\theta}(\pi x) \end{pmatrix} = \begin{pmatrix} \cos(\Gamma_+) & \frac{\sin(\Gamma_+)}{\sqrt{1+r}} \\ -\frac{\sin(\Gamma_+)}{\sqrt{1+r}} & \cos(\Gamma_+) \end{pmatrix} \begin{pmatrix} \theta(0) \\ \dot{\theta}(0) \end{pmatrix}. \quad (2.6)$$

To find the matrix that takes us the remaining halfway, we just assume that πx is our new starting point and use the exact same argument. Note that this time $\omega = \sqrt{1-r}$, since we are in the second half of the period.

$$\begin{aligned}\theta(2\pi x) &= \theta(\pi x) \cos(\Gamma_-) + \dot{\theta}(\pi x) \frac{\sin(\Gamma_-)}{\sqrt{1-r}} \text{ and} \\ \dot{\theta}(2\pi x) &= -\theta(\pi x) \sqrt{1-r} \sin(\Gamma_-) + \dot{\theta}(\pi x) \cos(\Gamma_-),\end{aligned}$$

which in matrix notation gives

$$\begin{pmatrix} \theta(2\pi x) \\ \dot{\theta}(2\pi x) \end{pmatrix} = \begin{pmatrix} \cos(\Gamma_-) & \frac{\sin(\Gamma_-)}{\sqrt{1-r}} \\ -\frac{\sin(\Gamma_-)}{\sqrt{1-r}} & \cos(\Gamma_-) \end{pmatrix} \begin{pmatrix} \theta(\pi x) \\ \dot{\theta}(\pi x) \end{pmatrix}. \quad (2.7)$$

In total, we have that $\theta(2\pi x) = M\theta(0)$, where

$$M = \begin{pmatrix} \cos(\Gamma_-) & \frac{\sin(\Gamma_-)}{\sqrt{1-r}} \\ \frac{-\sin(\Gamma_-)}{\sqrt{1-r}} & \cos(\Gamma_-) \end{pmatrix} \begin{pmatrix} \cos(\Gamma_+) & \frac{\sin(\Gamma_+)}{\sqrt{1+r}} \\ \frac{-\sin(\Gamma_+)}{\sqrt{1+r}} & \cos(\Gamma_+) \end{pmatrix}. \quad (2.8)$$

From here, we can find the eigenvalues of the \hat{M} , which are the Floquet multipliers, ρ . From those, we can find the values of μ , since $\mu = \pm \frac{\ln(\rho)}{iT}$. For example, if we set $x = 0.9$ and $r = 0.7$, we find that $\mu = \pm 0.139$, and therefore expect the system to be stable, given these values. This can be verified by examining the Poincaré section, which we can make by applying \hat{M} multiple times to a set of initial conditions. Given the initial condition

$$\begin{pmatrix} \theta(0) \\ \dot{\theta}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (2.9)$$

the resulting phase space diagram can be seen in figure 1(a). As expected, the system is stable and just keeps going around in phase space in the outlined shape.

If, instead, we were to study the case in which $x = 0.96$ and $r = 0.7$, then $\mu = \pm 0.517i$ suddenly becomes purely imaginary, and we would expect the solution to diverge. Using the exact same method for the Poincaré section, we arrive at figure 1(b). This solution keeps growing exponentially, even if it is slow a first.

If you are more harmonically inclined, and would prefer a harmonically changing frequency rather than the abrupt one studied here, the equation of motion takes the form of the Mathieu equation, whose solution is also analyzable from a Floquet perspective and will be studied in chapter 5.2.

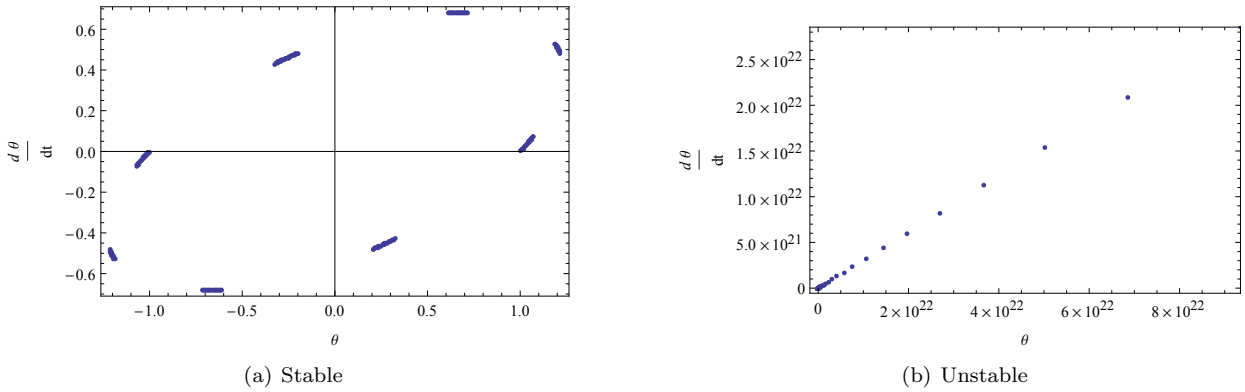


Figure 1: Poincaré sections of the two systems, with the first 200 periods imaged. (a) Stable solution with initial conditions $x = 0.9$ and $r = 0.7$, and $\mu = \pm 0.139$. (b) Unstable solution with initial conditions $x = 0.96$ and $r = 0.7$, and $\mu = \pm 0.517i$

3 Driven Quantum Oscillator

3.1 The general solution

In this chapter¹ we will study the behaviour of the the harmonic oscillator when exposed to a driving perturbation, $F(t)$. The Schrödinger equation for such a system is given by

$$i\hbar\dot{\Psi}(x, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2 - xF(t) \right] \Psi(x, t). \quad (3.1)$$

This can be solved by bringing it on a form that resembles the unperturbed harmonic oscillator, and hiding the perturbation behind transformations made along the way [5]². To get there, we will start by making a change of variables,

$$x \rightarrow y = x - \zeta(t), \quad (3.2)$$

so that we can write the Schrödinger equation in the new coordinate y ,

$$i\hbar\dot{\Psi}(y, t) = \left[i\hbar\dot{\zeta}(t) \frac{\partial}{\partial y} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_0^2(y + \zeta(t))^2 - (y + \zeta(t))F(t) \right] \Psi(y, t). \quad (3.3)$$

From here, it is useful to perform the following unitary transformation,

$$\Psi(y, t) = e^{im\dot{\zeta}y/\hbar} \phi(y, t), \quad (3.4)$$

where $\zeta(t)$ obeys the newtonian equation of motion

$$m\ddot{\zeta} + m\omega_0^2\zeta = F(t). \quad (3.5)$$

Inserting this into the above and calculating the LHS and RHS separately, we get for the LHS:

$$i\hbar\dot{\Psi}(y, t) = e^{im\dot{\zeta}y/\hbar} \left[i\hbar\dot{\phi} - my\ddot{\zeta}\phi \right]. \quad (3.6)$$

Using (3.5), we eliminate the double time derivative and get the following expression for the LHS

$$i\hbar\dot{\Psi}(y, t) = e^{im\dot{\zeta}y/\hbar} \left[i\hbar\dot{\phi} - y\phi F(t) + y\phi m\omega_0^2\zeta \right]. \quad (3.7)$$

For the RHS, we get

$$\begin{aligned} & \left[i\hbar\dot{\zeta}(t) \frac{\partial}{\partial y} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_0^2(y + \zeta(t))^2 - (y + \zeta(t))F(t) \right] \Psi(y, t) \\ &= e^{im\dot{\zeta}y/\hbar} \left[i\hbar\dot{\zeta} \frac{\partial \phi}{\partial y} - m\dot{\zeta}^2\phi - \frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial y^2} - i\hbar\dot{\zeta} \frac{\partial \phi}{\partial y} + \frac{m}{2}\dot{\zeta}^2\phi + \frac{m}{2}\omega_0^2 y^2\phi + \frac{1}{2}m\omega_0^2\zeta^2\phi + m\omega_0^2\zeta y\phi - yF(t)\phi - \zeta F(t)\phi \right] \end{aligned} \quad (3.8)$$

Equating the RHS and LHS and reducing, we see that

$$i\hbar\dot{\phi} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{m}{2}\omega_0^2 y^2 - L(\zeta, \dot{\zeta}, t) \right] \phi, \quad (3.9)$$

where $L = \frac{m}{2}\dot{\zeta}^2 - \frac{1}{2}m\omega_0^2\zeta^2 + \zeta F(t)$ is the Lagrangian for a driven harmonic oscillator.

¹Chapters 3.1 and 3.2 have been written in collaboration with Jon Brogaard.

²Note that there are multiple mistakes in this part of [5] which we have corrected.

Introducing another unitary transformation specifically to get the lagrangian term to cancel out,

$$\phi(y, t) = e^{\frac{i}{\hbar} \int_0^t dt' L(\zeta, \dot{\zeta}, t')} \chi(y, t), \quad (3.10)$$

we thereby end up with a form whose solution is known,

This transformation gives us the form of a normal, unperturbed harmonic oscillator

$$i\hbar \dot{\chi}(y, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m \omega_0^2 y^2 \right] \chi(y, t). \quad (3.11)$$

The stationary states of the harmonic oscillator are

$$\varphi_n(y) = \left(\frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-\frac{m\omega_0}{2\hbar} y^2}, \quad (3.12)$$

where $H_n(x)$ ³ are the Hermite polynomials [6]. The energies of the harmonic oscillator are known to be $E_n = \hbar\omega_0(n + \frac{1}{2})$. Combining this with the stationary states, φ_n , of the harmonic oscillator, we get that

$$\chi_n(y, t) = \left(\frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-\frac{m\omega_0}{2\hbar} y^2 - \frac{i}{\hbar} E_n t}. \quad (3.13)$$

Inserting this expression into our expression for ϕ , and then that expression into the one for ψ , we get

$$\phi_n(y, t) = \left(\frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-\frac{m\omega_0}{2\hbar} y^2 - \frac{i}{\hbar} [E_n t - \int_0^t dt' L]}, \quad (3.14)$$

$$\psi_n(y, t) = \left(\frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-\frac{m\omega_0}{2\hbar} y^2 + \frac{i}{\hbar} [m\dot{\zeta}(t)y - E_n t + \int_0^t dt' L]}. \quad (3.15)$$

Now, substituting our variable from y back to x

$$\psi_n(x, t) = \left(\frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(x - \zeta(t)) e^{-\frac{m\omega_0}{2\hbar} (x - \zeta(t))^2 + \frac{i}{\hbar} [m\dot{\zeta}(t)(x - \zeta(t)) - E_n t + \int_0^t dt' L]}. \quad (3.16)$$

Now we have our general wavefunction for an arbitrary $F(t)$, which appears in the lagrange function in the exponent. This is just a harmonic oscillator with a shifted coordinate and a complex phase in exponential factor.

3.2 Periodic monochromatic driving force

Let us now try to work with a simple harmonic driving force by setting

$$F(t) = A \sin(\omega t + \theta). \quad (3.17)$$

Classically, this would be like placing a harmonic oscillator (with frequency ω_0) on a platform performing harmonic movement itself, with frequency ω . With this force, a solution to 3.5 is

$$\zeta(t) = \frac{A \sin(\omega t + \theta)}{m(\omega_0^2 - \omega^2)}, \quad (3.18)$$

where $\omega \neq \omega_0$.

In order to calculate the action, we need to first calculate the time derivative of the moving coordinate, $\dot{\zeta}$,

$$\dot{\zeta} = \frac{A\omega \cos(\omega t + \theta)}{m(\omega_0^2 - \omega^2)}. \quad (3.19)$$

³For ease of notation throughout this paper, it is implicit that the argument of the Hermite polynomial $H_n(x)$ is $\sqrt{\frac{m\omega_0}{\hbar}} x$.

We now calculate the action as it appears in our expression for ψ ,

$$\int_0^t dt' L(\zeta, \dot{\zeta}, t') = \int_0^t dt' \left[\frac{A^2 \omega^2 \cos^2(\omega t' + \theta)}{2m(\omega_0^2 - \omega^2)^2} - \frac{m\omega_0^2 \left(\frac{A \sin(\omega t' + \theta)}{m(\omega_0^2 - \omega^2)} \right)^2}{2} + \frac{A^2 \sin^2(\omega t' + \theta)}{m(\omega_0^2 - \omega^2)} \right] = \frac{A^2 [\omega t(\omega_0^2 - \omega^2) + (3\omega^2 - \omega_0^2) \cos(2\theta + \omega t) \sin(\omega t)]}{4m\omega(\omega_0^2 - \omega^2)^2}. \quad (3.20)$$

Inserting this into 3.16 we find the wavefunctions

$$\psi_n(x, t) = \left(\frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(x - \zeta(t)) e^{-\frac{m\omega_0}{2\hbar}(x - \zeta(t))^2 + \frac{i}{\hbar} \left[m\dot{\zeta}(t)(x - \zeta(t)) - E_n t + \frac{A^2 [\omega t(\omega_0^2 - \omega^2) + (3\omega^2 - \omega_0^2) \cos(2\theta + \omega t) \sin(\omega t)]}{4m\omega(\omega_0^2 - \omega^2)^2} \right]} \quad (3.21)$$

3.3 Floquet

Since ζ and $\dot{\zeta}$ are both periodic in t , then so are the Hermite polynomials. If we, in the exponent of (3.21), both add and subtract the expression

$$\frac{t}{T} \int_0^T dt' L(\zeta, \dot{\zeta}, t') = \frac{A^2}{4m(\omega_0^2 - \omega^2)} t, \quad (3.22)$$

we can then identify the quasi-energies as being

$$\epsilon_n = \hbar\omega_0 \left(n + \frac{1}{2} \right) - \frac{A^2}{4m(\omega_0^2 - \omega^2)}, \quad (3.23)$$

which then leaves us with the floquet modes of the form

$$\Phi_n = \left(\frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(x - \zeta(t)) e^{-\frac{m\omega}{2\hbar}(x - \zeta(t))^2 + \frac{i}{\hbar} \left[m\dot{\zeta}(t)(x - \zeta(t)) + \frac{A^2 (3\omega^2 - \omega_0^2) \cos(2\theta + \omega t) \sin(\omega t)}{4m\omega(\omega_0^2 - \omega^2)^2} \right]}. \quad (3.24)$$

This quasi-energy is strictly real, and so the system is stable, since the Floquet multiplier $|\rho| = |e^{\frac{i}{\hbar}\epsilon_n T}| = 1$. It is worth noting that as $\omega \rightarrow \omega_0$, the last term in equation 3.23 starts to dominate, such that the quasi-energies lie closer and closer (as long as $A \neq 0$), and at resonance the quasi-energies form a continuous spectrum, as seen in figure 2. If we keep increasing ω past ω_0 , then the quaienergies will flip back around to go from very negative to very positive. This just means that the phase of the solution (the μ -part from equation 2.1) changes direction in phase space.

3.3.1 The inverted harmonic potential

It is interesting to consider what happens if we turn the potential of the harmonic oscillator upside-down. If we do this, then equation 3.5 takes the form

$$m\ddot{\zeta} - m\omega_0^2\zeta = F(t), \quad (3.25)$$

and the Lagrange function becomes

$$L = \frac{m}{2} \dot{\zeta}^2 + \frac{1}{2} m\omega_0^2 \zeta^2 + \zeta F(t). \quad (3.26)$$

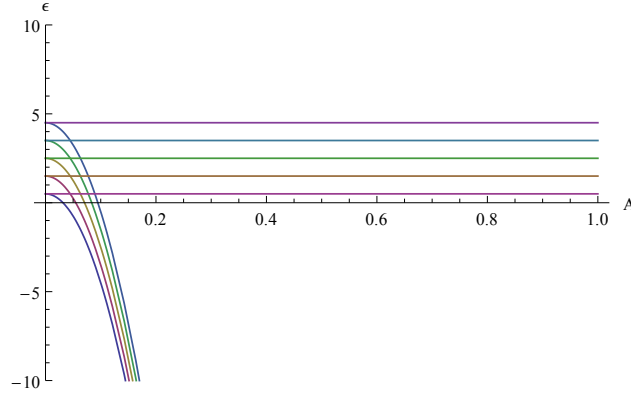


Figure 2: Parametric plot of the quasi-energies in (3.23) at $\frac{\omega}{\omega_0} = 0.999$, plotted together with the eigen-energies of the harmonic oscillator. Both are in units of $\hbar\omega_0$.

Now, we can follow the derivation of chapter 3.1 all the way down to (3.11), but with a negative sign in front of the $\frac{1}{2}m\omega_0^2 y^2$ -term. The solutions to the inverted oscillator potential [7] are given by

$$\chi(y, t) = \exp \left[\frac{1}{2} \left(\frac{i}{\hbar} m \omega_0 y^2 - \omega_0 t \right) + i k y e^{-\omega_0 t} + \frac{i k^2}{4 \omega_0 m} e^{-2 \omega_0 t} \right]. \quad (3.27)$$

The derivation of equation 3.27 can be found in [7] or in appendix A. To get the full solution, we have to transform back to $\psi(x, t)$, just like we did for the harmonic oscillator. Doing so we get

$$\psi(x, t) = \exp \left[\frac{1}{2} \left(\frac{i}{\hbar} m \omega_0 (x - \zeta(t))^2 - \omega_0 t \right) + i k (x - \zeta(t)) e^{-\omega_0 t} + \frac{i k^2}{4 \omega_0 m} e^{-2 \omega_0 t} + \frac{i}{\hbar} \left(m \dot{\zeta}(t) (x - \zeta(t)) + S(t) \right) \right]. \quad (3.28)$$

However, since normalization of equation 3.27 is far from trivial, we will comment mostly on what is readily available through the solution to (3.25) and the corresponding action.

The most general solution to equation 3.25 is

$$\begin{aligned} \zeta(t) &= -\frac{A \sin(\omega t)}{m(\omega_0^2 + \omega^2)} + \alpha e^{\omega_0 t} + \beta e^{-\omega_0 t} \\ \dot{\zeta}(t) &= -\frac{A \omega \cos(\omega t)}{m(\omega_0^2 + \omega^2)} + \alpha \omega_0 e^{\omega_0 t} - \beta \omega_0 e^{-\omega_0 t}. \end{aligned}$$

If we choose to start the system in $\zeta(t) = 0$, with a finite initial velocity $\dot{\zeta}(0) = -\frac{A \omega}{m(\omega_0^2 + \omega^2)}$, then we must have that $\alpha = \beta = 0$, such that

$$\begin{aligned} \zeta(t) &= -\frac{A \sin(\omega t)}{m(\omega_0^2 + \omega^2)} \\ \dot{\zeta}(t) &= -\frac{A \omega \cos(\omega t)}{m(\omega_0^2 + \omega^2)}. \end{aligned}$$

These expressions we can plug into the lagrangian to ultimately calculate the action integral, v . If we do that, we get that

$$S = \frac{A^2 ((3\omega^2 + \omega_0^2) \sin(2\omega t) - 2t\omega (\omega^2 + \omega_0^2))}{8m\omega (\omega^2 + \omega_0^2)^2}. \quad (3.29)$$

This is a combination of a harmonic term and a term linear in t . The term linear in t contributes to the quasi-energies in the same way that it did for the potential of the upright harmonic oscillator. The remainder oscillates harmonically with the same period as ζ and $F(t)$. Since $\zeta(t)$ is periodic, we can write this on Floquet form as a quasi-energy

$$\epsilon_n = \alpha + \frac{A^2}{4m(\omega_0^2 + \omega^2)}, \quad (3.30)$$

where α is the contribution to the quasi energy from the general solution, and a Floquet mode which is periodic with period $t = \frac{2\pi}{\omega}$. The quasi-energies of this potential look a lot like the ones of equation 3.23, but we see that there is no resonance phenomenon in this case. If α is real, the quasi-energy is real, and so the system is stable, regardless of the strength of the driving force, A . We can rationalize this by noticing that the initial velocity is dependent on the driving force, and so a stronger driving force will have a faster initial velocity to stabilize. This velocity makes up for the initial "push" that would otherwise unbalance the system.

If we try again with the same potential, but different initial conditions, we stumble upon a curiosity. Say we want our particle to start out at rest at the top of the potential barrier, so that the initial velocity is 0, just like the initial position, then we see that

$$\zeta(0) = 0 \Leftrightarrow \alpha = -\beta \quad (3.31)$$

and

$$\dot{\zeta}(0) = 0 \Leftrightarrow \alpha = \frac{A \frac{\omega}{\omega_0}}{2m(\omega_0^2 + \omega^2)}. \quad (3.32)$$

Using this, we finally get an expression for $\zeta(t)$ and $\dot{\zeta}(t)$, namely

$$\begin{aligned} \zeta(t) &= \frac{A}{m(\omega_0^2 + \omega^2)} \left(\frac{\omega}{\omega_0} \sinh(\omega_0 t) - \sin(\omega t) \right) \\ \dot{\zeta}(t) &= \frac{A\omega}{m(\omega_0^2 + \omega^2)} (\cosh(\omega_0 t) - \cos(\omega t)). \end{aligned}$$

With this equation of motion, the action becomes

$$S = -\frac{A^2 (2t\omega^3\omega_0 - 2\omega^3 \sinh(2\omega_0 t) + 8\omega^3 \cos(\omega t) \sinh(\omega_0 t) + 2t\omega\omega_0^3 - (3\omega^2\omega_0 + \omega_0^3) \sin(2\omega t))}{8m\omega\omega_0 (\omega^2 + \omega_0^2)^2}, \quad (3.33)$$

which, contrary to the prior actions calculated, is *not* made up of only terms linear and harmonic in t , but now contains sinh-terms. For this reason, it is not immediately apparent that this expression can be brought on Floquet form. In fact, this can be seen from the fact that $\zeta(t)$ grows as the time goes, and is therefore not periodic. The shift of equation (3.28) grows with time and the solution therefore seems unstable. As a result, it is not possible to balance a particle on top of a potential hill of this form, with these initial conditions, unless the monotonically increasing shift exactly cancels out with something else in (3.28) to make the floquet mode periodic. This instability is in agreement with what we would expect classically; if you push a ball

away from an unstable equilibrium, you would have to push even harder to get it back to the equilibrium position since you would now also have to push against the potential.

We would expect to be able to write the solution on Floquet form, though, because even though we have changed the initial conditions, the potential is still periodic and so is compatible with the requirements of Floquet's theorem.

It can be shown that the quasi-energies of the upright driven harmonic oscillator, with initial conditions $\zeta(0) = 0$ and $\dot{\zeta}(0) = 0$, are the same as the quasi-energies as those of the harmonic oscillator with the initial conditions considered above, in equation 3.23.

3.4 Overlaps of the driven oscillator with the harmonic oscillator

If we assume that our upright (regular) harmonic oscillator is in the ground state when we apply our driving perturbation, we can try to find the overlap of the n 'th state of the unperturbed oscillator with the 0 'th state of the driven oscillator. This result we can use to find the probability of finding the system in an excited state of the harmonic oscillator. Using equations 3.12 and 3.21

$$P_n(t) = |\langle \varphi_n | \psi_0 | \varphi_n | \psi_0 \rangle|^2, \quad (3.34)$$

where the usual inner product is used, such that

$$\langle \varphi_n | \psi_0 | \varphi_n | \psi_0 \rangle = \int_{-\infty}^{\infty} dx \varphi_n^*(x, t) \psi_0(x, t). \quad (3.35)$$

This can be evaluated for different values of n . As an example, the overlap with the groundstate ($n=0$) is given by

$$\begin{aligned} \langle \varphi_n | \psi_0 | \varphi_n | \psi_0 \rangle = \exp & \left[\frac{1}{4\hbar} \left(-\frac{A^2 \omega_0 \sin^2(\omega t + \phi)}{m(\omega_0^2 - \omega^2)^2} - \frac{A^2 \omega^2 \cos^2(\omega t + \phi)}{m\omega_0(\omega_0^2 - \omega^2)^2} \right. \right. \\ & \left. \left. - 2i \left(\frac{A^2 (t\omega(\omega_0^2 - \omega^2) + (3\omega^2 - \omega_0^2) \sin(\omega t) \cos(\omega t + 2\phi))}{2m\omega(\omega_0^2 - \omega^2)^2} + \frac{A^2 \omega \sin(\omega t + \phi) \cos(\omega t + \phi)}{m(\omega_0^2 - \omega^2)^2} \right) \right) \right]. \end{aligned} \quad (3.36)$$

If we examine the relation between consecutive overlaps (here we have examined overlaps upto $n = 4$), we find a recurring factor (up to a factor dependent on n),

$$\frac{\langle \varphi_{n+1} | \psi_0 | \varphi_{n+1} | \psi_0 \rangle}{\langle \varphi_n | \psi_0 | \varphi_n | \psi_0 \rangle} = \frac{A(i\omega \cos(\omega t) + \omega_0 \sin(\omega t))}{\sqrt{2(n+1)m\omega_0\hbar(\omega_0^2 - \omega^2)}}. \quad (3.37)$$

In the general expression for $\langle \varphi_n | \psi_0 | \varphi_n | \psi_0 \rangle$, we can combine equations 3.12, 3.16, 3.18 and 3.19, and take all terms that are independent of both x and n outside of the integration in the inner product. This factor must be the same for all n , regardless of the overlap examined. This leads us to guess an expression for the overlap of the 0 'th state of the driven oscillator and the n 'th state of the harmonic oscillator.

$$\begin{aligned} \langle \varphi_n | \psi_0 | \varphi_n | \psi_0 \rangle_{guess} = & \frac{\left(\frac{A(\omega_0 \sin(\omega t + \phi) + i\omega \cos(\omega t + \phi))}{\sqrt{m\omega_0\hbar(\omega_0^2 - \omega^2)}} \right)^n}{\sqrt{2^n n!}} \exp \left[\frac{1}{4\hbar} \left(-\frac{A^2 \omega_0 \sin^2(\omega t + \phi)}{m(\omega_0^2 - \omega^2)^2} - \frac{A^2 \omega^2 \cos^2(\omega t + \phi)}{m\omega_0(\omega_0^2 - \omega^2)^2} \right. \right. \\ & \left. \left. - 2i \left(\frac{A^2 (t\omega(\omega_0^2 - \omega^2) + (3\omega^2 - \omega_0^2) \sin(\omega t) \cos(\omega t + 2\phi))}{2m\omega(\omega_0^2 - \omega^2)^2} + \frac{A^2 \omega \sin(\omega t + \phi) \cos(\omega t + \phi)}{m(\omega_0^2 - \omega^2)^2} \right) \right) \right]. \end{aligned} \quad (3.38)$$

When we want to calculate the transition probability, the expression will simplify greatly, since the complex phase will vanish, to yield

$$P_{n,guess}(\omega, t) = |\langle \varphi_n | \psi_0 \rangle_{guess}|^2 = e^{-\frac{1}{2} \left[\frac{A^2}{m\omega_0\hbar} \left(\frac{\omega_0^2 \sin^2(\omega t) - \omega^2 \cos^2(\omega t)}{(\omega_0^2 - \omega^2)^2} \right) \right]} \left[\frac{A^2}{2m\omega_0\hbar} \left(\frac{\omega_0^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)}{(\omega_0^2 - \omega^2)^2} \right) \right]^n \frac{1}{n!}. \quad (3.39)$$

4 The Heisenberg Picture

4.1 Analytical solution in Heisenberg picture

To justify the form of equation 3.39, it is educational to look at the problem in the Heisenberg picture, where the wave functions are all stationary, and it is the operators which are considered to have the time dependence. The following derivation follows closely along the lines of reference [3].

The hamiltonian of a general driven and damped oscillator is given by

$$H(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - xF(t) - pG(t), \quad (4.1)$$

where $F(t)$ is the driving term and $G(t)$ is the damping term. Expressing this in terms of the raising and lowering operators of the unperturbed harmonic oscillator, $a^\dagger(t)$ and $a(t)$ respectively, we get

$$H = \hbar\omega_0 \left(a^\dagger a + \frac{1}{2} \right) + f(t)a + f^* a^\dagger, \quad (4.2)$$

where

$$a = \sqrt{\frac{m\omega_0}{2\hbar}} \left(x + i \frac{p}{m\omega_0} \right) \text{ and } f(t) = -\sqrt{\frac{\hbar}{2m\omega_0}} F(t) + i\sqrt{\frac{\hbar m\omega_0}{2}} G(t). \quad (4.3)$$

In the Heisenberg picture, it is the operators that have time dependence, while the wavefunctions are stationary. Using the known commutation relation for $a(t)$ and $a^\dagger(t)$ (which can be derived using the commutation relation of x and p , $[x, p] = i\hbar$),

$$[a(t), a^\dagger(t)] = I, \quad (4.4)$$

we can find the equation of motion in for the operator $a(t)$.

The equation of motion in the Heisenberg picture is given by (here the explicit time dependence has been suppressed for ease of reading)

$$\begin{aligned} i\hbar \frac{da(t)}{dt} &= [a(t), H(t)] \\ &= \hbar\omega_0 \left(a a^\dagger a + \frac{1}{2} a \right) + a f a + a f^* a^\dagger - \hbar\omega_0 \left(a^\dagger a^2 + \frac{1}{2} a \right) - f a a - f^* a^\dagger a \\ &= \hbar\omega_0 a a^\dagger a + f^* a a^\dagger - \hbar\omega_0 a^\dagger a a - f^* a^\dagger a \\ &= \hbar\omega_0 (a a^\dagger - a^\dagger a) a + f^* (a a^\dagger - a^\dagger a) \\ &= \hbar\omega_0 a + f^* \\ &\Downarrow \\ \frac{da(t)}{dt} + i\omega_0 a(t) &= -\frac{i}{\hbar} f^*(t). \end{aligned} \quad (4.5)$$

The solution to this differential equation can be found by construction of a Green's function. The solution then becomes

$$a(t) = -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt' G(t-t') f^*(t'), \quad (4.6)$$

where $G(t-t')$ is the Green's function which satisfies the relation

$$\frac{dG(t-t')}{dt} + i\omega_0 G(t-t') = \delta(t-t'), \quad (4.7)$$

since

$$\begin{aligned} \frac{da(t)}{dt} &= -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt' \frac{dG(t-t')}{dt} f^*(t') = -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt' (\delta(t-t') - i\omega_0 G(t-t')) f^*(t') \\ &= -\frac{i}{\hbar} f^*(t) - i\omega_0 a(t). \end{aligned} \quad (4.8)$$

The Green's function used is, of course, discontinuous at the time $t = t'$, but the solution to the homogeneous equation 4.7 has an exponential time dependence. If we use the Green's functions that are a combination of this exponential time dependence and the step-function, η , which is also discontinuous at t' , we have a two functions. One that goes backwards in time (advanced) and another that goes forward in time (retarded),

$$\begin{aligned} G_R(t-t') &= \eta(t-t') e^{-i\omega_0(t-t')} \\ G_A(t-t') &= -\eta(t'-t) e^{-i\omega_0(t-t')}. \end{aligned}$$

Additionally, if we call the solutions to the homogeneous eq. 4.5 ($f^*(t') = 0$) a_{in} and a_{out} , then the solution to the inhomogeneous differential equation will be

$$\begin{aligned} a(t) &= a_{in}(t) - \frac{i}{\hbar} \int_{-\infty}^{\infty} dt' G_R(t-t') f^*(t') = a_{in}(t) - \frac{i}{\hbar} \int_{-\infty}^t dt' e^{-i\omega_0(t-t')} f^*(t') \\ a(t) &= a_{out}(t) - \frac{i}{\hbar} \int_{-\infty}^{\infty} dt' G_A(t-t') f^*(t') = a_{in}(t) + \frac{i}{\hbar} \int_t^{\infty} dt' e^{-i\omega_0(t-t')} f^*(t'). \end{aligned}$$

a_{in} and a_{out} describe the system respectively before and after we apply the perturbation. Equating the two expressions for $a(t)$ and combining the integrations into one, we end up with following relation

$$a_{out}(t) = a_{in}(t) - \frac{i}{\hbar} \int_{-\infty}^{\infty} dt' e^{-i\omega_0(t-t')} f^*(t'). \quad (4.9)$$

Since a_{in} and a_{out} apply to the system before the perturbation is started and after the perturbation ends, respectively, they are the solutions to 4.5 with the right-hand side set to 0, which means that they have an exponential time dependence, such that

$$\begin{aligned} a_{in}(t) &= a_{in} e^{-i\omega_0 t} \\ a_{out}(t) &= a_{out} e^{-i\omega_0 t}, \end{aligned}$$

and so we can cancel out the $e^{-i\omega_0 t}$ terms in 4.9 to get

$$a_{out} = a_{in} - \frac{i}{\hbar} g^*(\omega_0), \quad (4.10)$$

where $g(\omega_0)$ is the fourier transform of $f(t)$,

$$g(\omega_0) = \int_{-\infty}^{\infty} dt' e^{-i\omega_0 t'} f(t'). \quad (4.11)$$

We can write a relation between a_{out} and a_{in} with a unitary S as

$$a_{out} = S^\dagger a_{in} S. \quad (4.12)$$

The hamiltonian before the perturbation is applied is $H_{in} = \hbar\omega_0(a_{in}^\dagger a_{in} + \frac{1}{2})$ and the hamiltonian after the perturbation is stopped is $H_{out} = \hbar\omega_0(a_{out}^\dagger a_{out} + \frac{1}{2})$. Imagine that we stop the perturbation right as we measure the overlaps. Then H_{out} will be the hamiltonian for the system as we measure it, and it will behave as a continuously driven system. The operators $a_{in}^\dagger a_{in}$ and $a_{out}^\dagger a_{out}$ from the hamiltonians have the eigenvalues $n = 0, 1, 2, 3, \dots$, and the eigenvector corresponding to the eigenvalue n , we call $|n\rangle_{in}$ or $|n\rangle_{out}$, respectively.

To find a relationship between $|n\rangle_{out}$ and $|n\rangle_{in}$, we consider that

$$\begin{aligned} a_{out}^\dagger a_{out} |n\rangle_{out} &= n |n\rangle_{out}, \text{ but it must also be true that} \\ a_{out}^\dagger a_{out} |n\rangle_{out} &= S^\dagger a_{in}^\dagger a_{in} S |n\rangle_{out}. \end{aligned}$$

For this to be true, we must have that

$$|n\rangle_{out} = S^\dagger |n\rangle_{in}, \quad (4.13)$$

such that the last term is

$$S^\dagger a_{in}^\dagger a_{in} S |n\rangle_{out} = S^\dagger a_{in}^\dagger a_{in} S S^\dagger |n\rangle_{in} = n S^\dagger |n\rangle_{in}. \quad (4.14)$$

Together with the operator relation between an operator a and a complex number α

$$e^{-\alpha a^\dagger + \alpha^* a} a e^{\alpha a^\dagger - \alpha^* a} = a + \alpha, \quad (4.15)$$

we have all that is needed to determine an expression for S . From equation 4.10,

$$a_{out} = S^\dagger a_{in} S = a_{in} - \frac{i}{\hbar} g^*(\omega_0), \quad (4.16)$$

we can identify

$$\begin{aligned} a &= a_{in} \\ \alpha &= -\frac{i}{\hbar} g^*(\omega_0) \end{aligned}$$

and write

$$S = e^{(-\frac{i}{\hbar} g^*(\omega_0) a_{in}^\dagger - \frac{i}{\hbar} g(\omega_0) a)}. \quad (4.17)$$

Now, we want to know the probability of the driven harmonic oscillator being in an excited state of the unperturbed oscillator. In other words, assuming that the oscillator is in the ground state before the perturbation, we want to calculate

$${}_{out} \langle n|0|n|0 \rangle_{in} = {}_{out} \langle n|S|0 \rangle_{out} = {}_{in} \langle n|S|0 \rangle_{in}. \quad (4.18)$$

Using the relation, with a and b being operators,

$$e^a e^b = e^{a+b+\frac{1}{2}[a,b]}, \quad (4.19)$$

we can find, using the commutation relation between a^\dagger and a established earlier, that

$$S|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} \left(1 - \alpha^* a + \frac{1}{2}(\alpha^* a)^2 + \dots\right) |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle, \quad (4.20)$$

because using the lowering operator on a groundstate results in 0. Now we can just include the bra on from the left and utilize the orthogonality of the eigenstates of the harmonic oscillator to get

$$\begin{aligned} \langle n|S|0\rangle &= e^{-\frac{1}{2}|\alpha|^2} \langle n|e^{\alpha a^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \langle n|\left(1 + \alpha a^\dagger + \frac{1}{2}(\alpha a^\dagger)^2 + \dots + \frac{(\alpha a^\dagger)^2}{n!} + \dots\right)|0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}. \end{aligned} \quad (4.21)$$

Inserting α and taking the square of the absolute value, we find the transition probability P_n to be

$$P_n(\omega_0) = |{}_{out} \langle n|S|0 \rangle_{in}|^2 = e^{-\frac{|g(\omega_0)|^2}{\hbar^2}} \left|\frac{g(\omega_0)}{\hbar}\right|^{2n} \frac{1}{n!} \quad (4.22)$$

4.2 The driven harmonic oscillator in the Heisenberg picture

If we now study the case of a driven harmonic oscillator, with driving force

$$F(t) = A \sin(\omega t), \quad (4.23)$$

and no damping, we see that we can use the Heisenberg picture to calculate the transition probabilities. We want to know the chance of a driven harmonic oscillator transitioning to a state reminiscent of an excited state n of the harmonic oscillator at time t . The driving force, $F(t)$, is the same $F(t)$ that appears in equation 4.1, with the damping term $G(t) = 0$.

We can now, with ease, calculate $f(t)$. Since $G(t) = 0$,

$$f(t) = -\sqrt{\frac{\hbar}{2m\omega_0}} F(t) = -\sqrt{\frac{\hbar}{2m\omega_0}} A \sin(\omega t). \quad (4.24)$$

From here, it is our goal to calculate the fourier transform, $g(\omega_0, t)$, where t is the time at which we measure the transition probability.

$$g(\omega_0, t) = -\sqrt{\frac{\hbar}{2m\omega_0}} A \int_{-\infty}^t dt' e^{-i\omega_0 t'} \sin(\omega t'). \quad (4.25)$$

To evaluate this integral in the endpoints, it is prudent to have the perturbation be applied adiabatically to the oscillator. This is done by adding an infinitesimal positive term, O_+ , that is linear in time, in the exponent. If we at the same time rewrite the $\sin(\omega t')$ to exponential form, the integral becomes (disregarding the constants in front of the integral for now)

$$\begin{aligned}
\int_{-\infty}^t dt' e^{-i(\omega_0 + O_+)t'} \sin(\omega t') &= \frac{1}{2i} \int_{-\infty}^t dt' e^{-i(\omega_0 + iO_+)t'} \left(e^{i\omega t'} - e^{-i\omega t'} \right) \\
&= \frac{1}{2i} \int_{-\infty}^t dt' \left(e^{-i(\omega_0 + iO_+ - \omega)t'} - e^{-i(\omega_0 + iO_+ + \omega)t'} \right) \\
&= \frac{1}{2i} \left[\left(\frac{e^{-i(\omega_0 + iO_+ - \omega)t'}}{-i(\omega_0 + iO_+ - \omega)} + \frac{e^{-i(\omega_0 + iO_+ + \omega)t'}}{i(\omega_0 + iO_+ + \omega)} \right) \right]_{-\infty}^t
\end{aligned} \tag{4.26}$$

Here, the purpose of the term O_+ is to make the integral go to 0 as $t' \rightarrow 0$, while it is negligible at time $t' = t$ and the terms in the denominator are virtually unaffected by the presence of the term. Hence, we get that

$$\begin{aligned}
\frac{1}{2i} \left[\left(\frac{e^{-i(\omega_0 + iO_+ - \omega)t'}}{-i(\omega_0 + iO_+ - \omega)} + \frac{e^{-i(\omega_0 + iO_+ + \omega)t'}}{i(\omega_0 + iO_+ + \omega)} \right) \right]_{-\infty}^t &\approx \frac{1}{2i} \left(\frac{e^{-i(\omega_0 t - \omega t)}}{-i(\omega_0 - \omega)} + \frac{e^{-i(\omega_0 t + \omega t)}}{i(\omega_0 + \omega)} \right) \\
&= \frac{e^{-i(\omega_0 t - \omega t)}}{2(\omega_0 - \omega)} - \frac{e^{-i(\omega_0 t + \omega t)}}{2(\omega_0 + \omega)} = e^{-i\omega_0 t} \left[\frac{e^{i\omega t}}{2(\omega_0 - \omega)} - \frac{e^{-i\omega t}}{2(\omega_0 + \omega)} \right] \\
&= e^{-i\omega_0 t} \left[\frac{(\omega_0 + \omega)e^{i\omega t} - (\omega_0 - \omega)e^{-i\omega t}}{2(\omega_0^2 - \omega^2)} \right] = e^{-i\omega_0 t} \left[\frac{i\omega_0 \sin(\omega t) + \omega \cos(\omega t)}{(\omega_0^2 - \omega^2)} \right].
\end{aligned} \tag{4.27}$$

In the end, we get that

$$g(\omega_0, t) = -\sqrt{\frac{\hbar}{2m\omega_0}} A e^{-i\omega_0 t} \left[\frac{i\omega_0 \sin(\omega t) + \omega \cos(\omega t)}{(\omega_0^2 - \omega^2)} \right]. \tag{4.28}$$

The absolute square of this expression is easily found, since it is of the form $g = x + iy$. Suppressing the dependence on ω_0 ,

$$|g(t)|^2 = \frac{\hbar}{2m\omega_0} A^2 \left[\frac{\omega_0^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)}{(\omega_0^2 - \omega^2)^2} \right]. \tag{4.29}$$

This we insert into equation 4.22 to get

$$P_n(t) = e^{-\left[\frac{1}{2m\omega_0 \hbar} A^2 \left(\frac{\omega_0^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)}{(\omega_0^2 - \omega^2)^2} \right) \right]} \left[\frac{A^2}{2m\omega_0 \hbar} \left(\frac{\omega_0^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)}{(\omega_0^2 - \omega^2)^2} \right) \right]^n \frac{1}{n!}. \tag{4.30}$$

Comparing this with the absolute square of equation 3.38 (note that a big part of the exponent will cancel out),

$$|\langle \varphi_n | \psi_0 \rangle_{guess}|^2 = e^{-\frac{1}{2} \left[\frac{A^2}{m\omega_0 \hbar} \left(\frac{\omega_0^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)}{(\omega_0^2 - \omega^2)^2} \right) \right]} \left[\frac{A^2}{2m\omega_0 \hbar} \left(\frac{\omega_0^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)}{(\omega_0^2 - \omega^2)^2} \right) \right]^n \frac{1}{n!}, \tag{4.31}$$

we see that these expressions are identical. Below are plotted the transition probabilities of (4.30) and (4.31) close to, and far from, resonance.

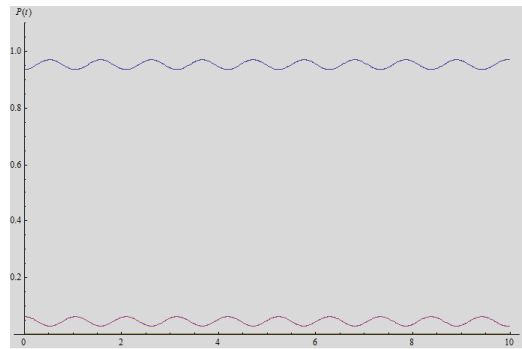


Figure 3: Parametric plot of the transition probabilities to the first 15 states, at ω relatively far from resonance, at $\frac{\omega}{\omega_0} = 1.5$.

Note that only the transitions to the ground state (blue) and the first excited state (red) are clearly visible; the rest are negligibly small in this case.

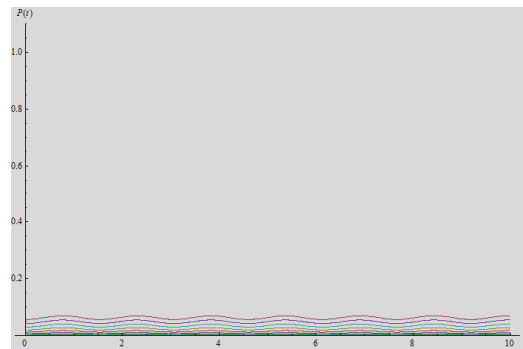


Figure 4: Parametric plot of the transition probabilities of the first 15 states close to resonance, at $\frac{\omega}{\omega_0} = 1.025$.

From figures 3 and 4 we see that as we approach resonance, the transition probabilities tend toward 0. This is because the $\frac{1}{\omega_0^2 - \omega^2}$ -term in equation 4.30 starts to dominate the $\frac{1}{n!}$ -term, for all n . As we approach resonance, more and more transitions become plausible, and so in the limit $\omega \rightarrow \omega_0$ all transitions will be equally probable and since there is no upper limit for n , the probabilities will tend toward 0. From equation 3.23, we see that as $\omega \rightarrow \omega_0$, the quasi-energies tend to lie closer and closer, and finally becoming a continuous spectrum at resonance. Far from resonance, all transitions but those to the groundstate and first excited state of the harmonic oscillator are negligible.

5 Harmonically varying frequency and the Mathieu equation

5.1 Time-dependent frequency

In the situation in which we have a quantum harmonic oscillator which oscillates with a time dependent frequency, $\omega(t)$, the schrödinger equation takes the form

$$i\hbar\dot{\Psi}(x, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega(t)^2 x^2 \right] \Psi(x, t). \quad (5.1)$$

According to [4], the solution to this equation is of the form

$$\psi(x, t) = \frac{1}{\sqrt{r(t)}} e^{-i\Phi(x, t)} \chi(y, \tau), \quad (5.2)$$

where $y = \frac{x}{r(t)}$ and $\tau = \frac{\gamma(t)}{\omega_0}$, ω_0 is the natural frequency of the oscillator before we perturb it by making the frequency time-dependent at $t \rightarrow -\infty$, $\chi(y, \tau)$ is the solution to the unperturbed oscillator (with $\omega(t) = \omega_0$), and

$$\zeta(t) = r(t)e^{i\gamma(t)}, \text{ where } r(t) = |\zeta(t)| \quad (5.3)$$

is, just as in chapter 3.1, the solution to the classical equation of motion

$$\zeta(t) + \omega^2(t)\ddot{\zeta}(t) = 0. \quad (5.4)$$

ζ obeys the initial conditions

$$\zeta(t) \rightarrow e^{i\omega_0 t} \text{ as } t \rightarrow -\infty, \quad (5.5)$$

which means that we "started" the perturbation infinitely long ago.

To check the legitimacy of this solution, we will calculate the LHS and the RHS independently. In doing so, we get that

$$i\hbar\dot{\Psi}(x, t) = \left[-\frac{i}{2}\hbar r^{-\frac{3}{2}}\dot{r}\chi + \hbar r^{-\frac{1}{2}}\chi \frac{\partial\Phi}{\partial t} - i\hbar r^{-\frac{1}{2}}\frac{\partial\chi}{\partial y} \frac{x}{r^2}\dot{r} + i\hbar r^{-\frac{1}{2}}\frac{\partial\chi}{\partial\tau} \frac{\dot{\gamma}}{\omega_0} \right] e^{-i\Phi} \quad (5.6)$$

and

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega(t)^2 x^2 \right] \Psi(x, t) = \frac{\hbar^2}{2m} r^{-\frac{1}{2}} e^{-i\Phi} \left[i \frac{\partial^2\Phi}{\partial x^2} \chi + 2i \frac{\partial\Phi}{\partial x} \frac{\partial\chi}{\partial y} \frac{1}{r} + \left(\frac{\partial\Phi}{\partial x} \right)^2 \chi - \frac{\partial^2\chi}{\partial y^2} \frac{1}{r^2} \right] + \frac{1}{2}m\omega(t)^2 x^2 \chi. \quad (5.7)$$

Forcing the terms that are linear in $\frac{\partial\chi}{\partial y}$ to cancel out, we find that $\Phi = -\frac{m\dot{r}}{2\hbar r} x^2$. If we insert this relation into equations 5.6 and 5.7, and then insert 5.6 and 5.7 into equation 5.1, we get

$$i\hbar \frac{\partial\chi}{\partial\tau} \frac{\dot{\gamma}}{\omega_0} = \frac{1}{2} \left[m \left(\frac{\dot{r}}{r} \right)^2 + m \frac{d}{dt} \left(\frac{\dot{r}}{r} \right) + m\omega(t)^2 \right] x^2 \chi - \frac{\hbar^2}{2mr^2} \frac{\partial^2\chi}{\partial y^2}. \quad (5.8)$$

To get from here and to an expression reminiscent of the harmonic oscillator, we can consider the complex number

$$a(t) = a_1 + ia_2 = -i \frac{\dot{\zeta}}{\zeta} = \dot{\gamma}(t) - i \frac{\dot{r}}{r}. \quad (5.9)$$

Rewriting ζ as $\zeta = \alpha + i\beta$ and inserting this in the expression for $a(t)$, we find that

$$\begin{aligned} a_1 &= \text{Re} \left[-i \frac{\dot{\zeta}}{\zeta} \right] = \frac{\alpha\dot{\beta} - \beta\dot{\alpha}}{\alpha^2 + \beta^2} \equiv -i \frac{\Delta(\zeta, \zeta^*)}{2|\zeta|^2} = \frac{\omega_0}{r^2} \\ a_2 &= \text{Im} \left[-i \frac{\dot{\zeta}}{\zeta} \right] = -\frac{\alpha\dot{\alpha} + \beta\dot{\beta}}{\alpha^2 + \beta^2} = -\frac{\dot{\zeta}\zeta^* + \zeta\dot{\zeta}^*}{2|\zeta|^2} = -\frac{\dot{r}}{r}, \end{aligned} \quad (5.10)$$

where $\Delta(\zeta, \zeta^*) = \dot{\zeta}\zeta^* - \dot{\zeta}^*\zeta = 2i\omega_0$ is the Wronskian. Comparing this result to equation 5.9, we see that $\dot{\gamma} = \frac{\omega_0}{r^2}$.

Finally, we can use this to show the fact that

$$\omega^2(t) = -\frac{\ddot{\zeta}}{\zeta} = -\left[\frac{d}{dt}\left(\frac{\dot{\zeta}}{\zeta}\right) + \left(\frac{\dot{\zeta}}{\zeta}\right)^2\right] = -i\dot{\gamma} - \frac{d}{dt}\left(\frac{\dot{r}}{r}\right) - \left(\frac{\dot{r}}{r}\right)^2 + \dot{\gamma}^2 - 2i\dot{\gamma}\frac{\dot{r}}{r} = -\frac{\ddot{r}}{r} + \frac{\omega_0^2}{r^4} \Leftrightarrow \frac{\ddot{r}}{r} + \omega(t)^2 = \frac{\omega_0^2}{r^4}, \quad (5.11)$$

and use this along with $\dot{\gamma} = \frac{\omega_0}{r^2}$ to reduce equation 5.8 to the harmonic oscillator,

$$i\hbar\frac{\partial\chi}{\partial\tau}(y, \tau) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_0^2 y^2\right]\chi(y, \tau), \quad (5.12)$$

the solution to which is, of course,

$$\chi_n(y, \tau) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-\frac{m\omega_0}{2\hbar}y^2 - \frac{i}{\hbar}E_n\tau}. \quad (5.13)$$

So equation 5.2 is indeed the solution to equation 5.1. All that remains to be done is to solve equation 5.4 and insert the solution into equation 5.2 to get the general solution.

This is no easy task since few forms of $\omega(t)$ give analytically solvable equations of motion. If, however, $\omega(t)$ is periodic with some period T , such that $\omega(t+T) = \omega(t)$, then the equation is called the Mathieu equation, and the solutions are called Mathieu functions. The Mathieu functions are infinite series, and therefore inconvenient for our purposes. However, since the frequency is periodic, Floquet's theorem states that the solution must be of the form of equation 2.1, such that $r(t)$ is periodic, assuming that the quasi-energies are strictly real. Since $r(t)$ is the scaling of the oscillator length of equations 5.2 and 5.13 through the relation $y = \frac{x}{r(t)}$, this means that the y -coordinate is periodically stretching and contracting. A way to imagine this would be picturing the ground state of the harmonic oscillator. It has a gaussian shape, which would alternate between a slim, tall curve and a flat, wide one.

If, however, the quasi-energies are complex, then $r(t) = |\zeta|$ will either grow or decay exponentially in time, depending on the sign of $\text{Im}[\mu]$. If this happens, then one of two scenarios will happen:

If $|\zeta| \rightarrow \infty$ as $t \rightarrow \infty$, then $\psi(x, t)$ would be infinitely flat and infinitely wide. There would be a uniform chance (of 0) to find the particle anywhere in the universe. One could call this an unstable solution⁴, but in reality, we just know less and less about the position of the particle as time passes.

If $|\zeta| \rightarrow 0$ as $t \rightarrow \infty$, then $\psi(x, t)$ becomes reminiscent of a delta function. This means that we know the position of the particle to be $x = 0$ with absolute certainty, but would cause us to know nothing about its momentum. This solution⁵ then therefore localizes the particle at $x = 0$. This is the same effect that we would expect to happen to an unperturbed hamonic oscillator as $\omega_0 \rightarrow \infty$.

For this reason, we will differentiate the solutions that give rise to oscillating solutions ($\mu \in \mathbb{R}$) from those that give rise to "unstable" solutions ($\mu \in \mathbb{C}$).

The Mathieu equation is generally written as

$$\ddot{\zeta} + (a - 2q \cos(2\Gamma))\zeta = 0, \quad (5.14)$$

where this Γ is dimensionless, time-dependent and unrelated to the Γ in section 2.2. It is possible to find the values of μ by numerically integrating (5.14) to find the trace of the time propagation operator, \hat{M} . The

⁴We will refer to it as such, anyhow.

⁵It is not an unstable solution, per se, but I will categorize it as such.

trace of \hat{M} is the sum of the eigenvalues, ρ , and the product of the eigenvalues is the determinant of the operator, which is 1⁶. This leads to an expression for ρ ,

$$\begin{aligned}\rho_+ + \rho_- &= \text{Tr}(\hat{M}) \text{ and} \\ \rho_+ \rho_- &= \text{Det}(\hat{M}) = 1 \Leftrightarrow \\ \rho_{\pm} &= \frac{\text{Tr}(\hat{M}) \pm \sqrt{(\text{Tr}(\hat{M}))^2 - 4}}{2}.\end{aligned}\tag{5.15}$$

Since the only stable solutions are the ones where μ is real or, alternatively, $|\rho| = 1$, we can make a plot showing combinations of a and q that result in stable solutions. Figure 5 shows the regions of stability for the Mathieu equation as a function of a and q .

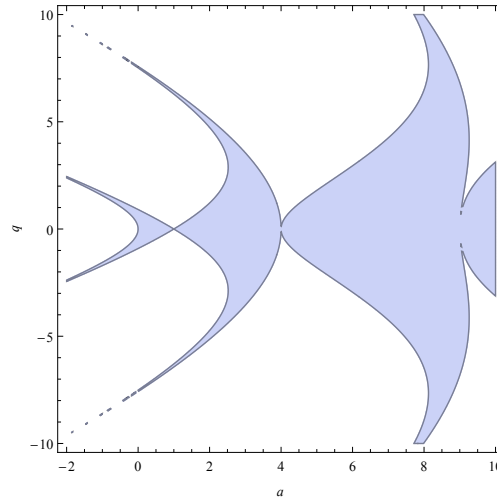


Figure 5: Plot of combinations of a and q that produce respectively stable and "unstable" solutions for equation 5.2. The grey area is the region for which $|\rho| = 1$, which produces stable solutions. The white area is the region for which $|\rho| \neq 1$, which results in the special "unstable" solutions to (5.2).

5.2 The harmonically driven pendulum

One example of (5.4) could be the (relatively slowly) vertically driven pendulum doing small amplitude oscillations [2]. If the pendulum (of length l) is fastened at a point which moves in time as $Y(t) = Y_0 \cos(\Omega t)$, then the coordinates of motion of the pendulum are

$$\begin{aligned}x &= l \sin(\theta) \approx l\theta \\ y &= Y(t) + l(1 - \cos(\theta)) \approx Y(t) + \frac{1}{2}l\theta^2.\end{aligned}\tag{5.16}$$

Because we have small amplitude oscillation, we are going to neglect terms that are of the order $(\theta\dot{\theta})^2$ and \dot{Y}^2 . In doing so, we can write the Lagrangian of the system as

⁶page 203 of [2]

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy = \frac{1}{2}m\left((l\dot{\theta})^2 + 2\dot{Y}l\dot{\theta}\right) - mg\left(Y + \frac{l\theta^2}{2}\right). \quad (5.17)$$

The Lagrange equation then becomes

$$\begin{aligned} \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= 0 \Leftrightarrow \\ m\dot{Y}l\dot{\theta} - mgl\theta - m\frac{d}{dt}\left(l^2\dot{\theta} + \dot{Y}l\theta\right) &= m\left[\dot{Y}l\dot{\theta} - gl\theta - l^2\ddot{\theta} - \ddot{Y}l\theta - \dot{Y}l\dot{\theta}\right] = 0, \end{aligned} \quad (5.18)$$

which then provides us with the equation of motion for θ ,

$$\ddot{\theta} + \left(\frac{g + \ddot{Y}}{l}\right)\theta = 0. \quad (5.19)$$

Identifying θ as ζ from equation 5.4, then we see that $\omega(t)^2 = \left(\frac{g + \ddot{Y}}{l}\right)$, which is periodic, since $\ddot{Y} = -Y_0\Omega^2 \cos(\Omega t)$. To bring it on the form of equation 5.14, we make the substitution $\Gamma = \frac{1}{2}\Omega t$ so that $dt = \frac{2}{\Omega}d\Gamma$ which leaves us with

$$\frac{\Omega^2}{4}\frac{\partial^2 \theta}{\partial \Gamma^2} + \left(\frac{g - Y_0\Omega^2 \cos(2\Gamma)}{l}\right)\theta, \quad (5.20)$$

which then finally gives us

$$\frac{\partial^2 \theta}{\partial \Gamma^2} + \left(\left(\frac{2\omega_0}{\Omega}\right)^2 - \frac{4Y_0 \cos(2\Gamma)}{l}\right)\theta = 0, \quad (5.21)$$

where we have identified the natural frequency of the undriven pendulum, $\omega_0^2 = \frac{g}{l}$. Comparing (5.21) and (5.14), we can identify

$$a = \left(\frac{2\omega_0}{\Omega}\right)^2 \quad \text{and} \quad q = \frac{2Y_0}{l}. \quad (5.22)$$

From figure 5 we see that for $a = 1$, there is only the a very small band of q that results in solutions with $|\rho| = 1$. This happens when $\Omega = 2\omega_0$ for almost all ratios $\frac{Y_0}{l}$, save for a select spectrum. This could be a sort of resonance that causes unstable solutions.

The area in which $a < 0$ corresponds to the inverted pendulum ($\omega_0^2 \rightarrow -\omega_0^2$) and we can see that it, too, has stable solutions.

6 Conclusion

Floquet theory is a useful tool to make qualitative assessments about various periodic system. It is useful in quantum mechanics, since any periodic hamiltonian will put the Schrödinger equation on Floquet form. It is a good ansatz for a solution to periodic problems, but other tools are needed to get an analytical solution. For systems that have no analytical solution, Floquet theory can be a powerful asset in numerical calculations, as seen in chapter 2.2.

A Appendix

To get to equation 3.27, we first consider the Schrödinger equation of the system,

$$i\hbar \frac{\partial \psi(y, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} - \frac{1}{2} m \omega_0^2 y^2 \right] \psi(y, t). \quad (\text{A.1})$$

If we make the transformation

$$\psi(y, t) = e^{\frac{1}{2} \left(\frac{i}{\hbar} m \omega_0 y^2 - \omega_0 t \right)} \Phi(y, t), \quad (\text{A.2})$$

and insert it in (A.1), then the Schrödinger equation takes the form

$$i\hbar \frac{\partial \Phi(y, t)}{\partial t} = -i\hbar \omega_0 y \frac{\partial \Phi(y, t)}{\partial y} - \frac{\hbar^2}{2m} \frac{\partial^2 \Phi(y, t)}{\partial y^2}. \quad (\text{A.3})$$

If we then make a scaling of the variable y , such that $y = q(t)z$, and $q(t)$ is the solution to the classical system with lagrangian

$$L = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega_0^2 q^2. \quad (\text{A.4})$$

Lagrange's equation then leaves us with an equation of motion for q ,

$$\ddot{q} = \omega_0^2 q \Leftrightarrow q(t) = e^{\omega_0 t}. \quad (\text{A.5})$$

Now, the transformation changes the time-derivative

$$\frac{\partial \Phi(y, t)}{\partial t} = \frac{\partial \tilde{\Phi}(z, t)}{\partial t} - \frac{\dot{q}y}{q^2} \frac{\partial \tilde{\Phi}(z, t)}{\partial t} = \frac{\partial \tilde{\Phi}(z, t)}{\partial t} - y \omega_0 \frac{\partial \tilde{\Phi}(z, t)}{\partial t}, \quad (\text{A.6})$$

where tilde indicates that it is a function of a different variable. Inserting this into (A.3), the terms with $\frac{\partial \tilde{\Phi}}{\partial t}$ cancel and we end up with

$$i\hbar \frac{\partial \tilde{\Phi}(z, t)}{\partial t} = -\frac{\hbar^2}{2m} e^{-2\omega_0 t} \frac{\partial^2 \tilde{\Phi}(z, t)}{\partial z^2}, \quad (\text{A.7})$$

where the exponential function comes from the change of variables.

This can be solved by separation of variables, such that we get

$$-\frac{\partial^2 \tilde{\Phi}(z, t)}{\partial z^2} = \frac{2im}{\hbar} e^{2\omega_0 t} \frac{\partial \tilde{\Phi}(z, t)}{\partial t} = k^2. \quad (\text{A.8})$$

The time-part gives

$$\frac{2im}{\hbar} e^{2\omega_0 t} \frac{\partial \tilde{\Phi}(z, t)}{\partial t} = k^2 \Leftrightarrow \tilde{\Phi}(z, t) = \exp\left(\frac{ik^2 \hbar}{4m\omega_0} e^{-2\omega_0 t}\right) \Phi(z), \quad (\text{A.9})$$

the position part gives the equation of a free particle. The solution to the free particle is

$$\Phi(z) = a e^{ikz}, \quad (\text{A.10})$$

where we will omit the scalar factor a , since the free particle is non-normalizable. All in all, the total solution, where we have transformed back to $\psi(x, t)$, is

$$\psi(y, t) = \exp\left[\frac{1}{2} \left(\frac{i}{\hbar} m \omega_0 y^2 - \omega_0 t \right) + ik y e^{-\omega_0 t} + \frac{ik^2}{4\omega_0 m} e^{-2\omega_0 t}\right]. \quad (\text{A.11})$$

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