Shapiro steps in Josephson Junctions

Peder Heiselberg

Niels Bohr Institute

University of Copenhagen

12/6-13

Abstract

We use Ginzburg-Landau theory to describe the phase transition to the superconducting state and the current through a superconductor. Applying quantum mechanics to the Josephson junction following Feynmann's method the Josephson relations for AC and DC current through the junctions is obtained. The resistively shunted Josephson junction and the resistively and capacitively shunted Josephson junction are analyzed. When subject to an alternating voltage driven circuit Shapiro spikes occur in the I vs. V characteristics of the circuit. If the circuit is driven by an AC current Shapiro steps appear with jumps when the average voltage is an integer multiplum of the frequency times the Josephson constant.

Vi anvender Ginzburg-Landau teori til at beskrive faseovergangen til det superledende fase, og strømmen gennem en superleder. Ved brug af kvantemekanik på en Josephson junction findes ud fra Feynmann's metode Josephson relationerne for AC of DC strømme gennem junctionen. Josephson junctionen med modstand og Josephson junctionen med modstand of capacitans analyseres. Når kredsløbene bliver drevet med en vekselstrøm opstår Shapiro spikes i I vs. V karakteristikken. Hvis kredsløbet drives med en vekselstrøm fremkommer Shapiro steps med hop når then gennemsnitlige spænding er et integer multiplum af frekvensen multipliceret med Josephson konstanten.

Contents

1 Introduction

In this Bachelor thesis we aim to investigate the nature of the Shapiro step phenomena involved with the Josephson junction. From the basic concepts of the superconductor we will slowly but surely deduce the relevant physics needed to understand why and how they occur. Starting with the Ginzburg-Landau theory (and a bit of microscopic theory) we seek to understand the superconductor. Then we look at the Josephson Junctions, which is two superconductors with a (thin) insulator between that couples the superconductors, and derive the relevant current equations governing the junction, i.e. the DC and AC Josephson equations. With this background information at hand we then enter the curcuit realm apply what we learned to circuits with Josephson junctions, resistive and capacitive shunt components. Subsequently we study circuits driven by alternating voltages from microwave radiation and calculate the Shapiro spikes. Finally we explain the fundamentals governing the step structure known as Shapiro steps when the circuit is driven by alternating currents.

2 Ginzburg-Landau theory

In order to describe the transition to the superconducting state V. L. Ginzburg and L. D. Landau assumed a free energy functional defined in the vicinity of the phase transition point. The functional is a function of an order parameter, that is small near the transition point allowing for expansion in powers.

Assuming the order parameter is linked to the wavefunction of superconducting electrons, and therefore a complex quantity, Ginzburg and Landau described the condensate with free energy functional of a single one-particle wave function $\Psi(r)$ as the complex order parameter. The functional is real, therefore only the absolute value of the wave function

appears. Expanding the free energy functional F in powers of the order parameter they obtained [1]

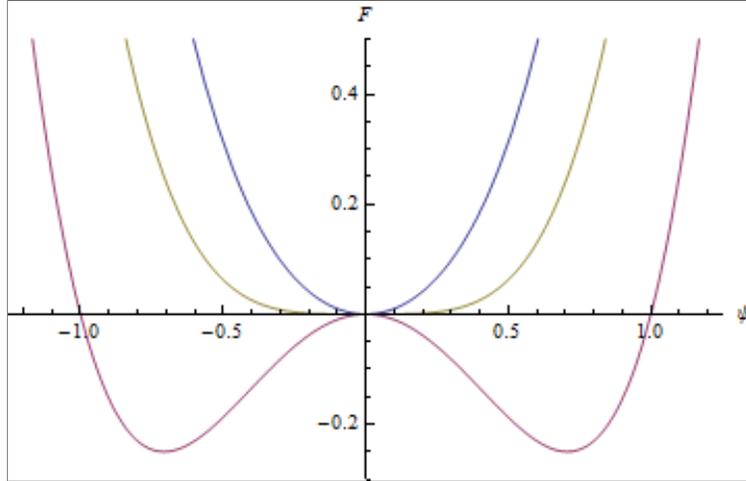$$F(\psi) = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \mathcal{O}(|\psi|^6)$$

Figure 1: A schematic of the free energy functional for $T > T_c$ (Blue curve), $T < T_c$ (Red) and $T = T_c$(Yellow). The red curve resembles a "mexican hat" in three dimensions, i.e. plotted vs. both the real and imaginary part of ψ.

It is clear that the potential is dominated by the fourth order term invoking $\beta > 0$ therefore we can neglect higher order terms since $F(\psi)$ is bounded from below [2]. If $\alpha \geq 0$ then $F(\psi)$ has a single minimum at $\psi = 0$, since ψ is related to the number of superconducting electrons in the condensate this is a normal metal. If $\alpha \leq 0$ then a spontaneous symmetry braking occurs and $F(\psi)$ has an extrema at

$$\frac{\partial F}{\partial|\psi|} = 2\alpha|\psi| + 2\beta|\psi|^3 = 0 \quad \Rightarrow \quad |\psi| = 0 \quad \text{or} \quad |\psi| = \sqrt{-\frac{\alpha}{\beta}} \quad \text{(the minimum)} \quad (1)$$

as shown in the "mexican hat" potential of Fig(1). The phase transitions happens exactly when $\alpha(T) = 0$ or when the temperature drops to the critical temperature T_c for superconducting transition. Assuming $\alpha(T)$ is smooth we can expand it in T around T_c.

$$\alpha(T) = \alpha_0(T_c - T) \quad . \quad (2)$$

The order parameter below T_c then satisfies

$$|\psi| = \sqrt{\frac{\alpha_0}{\beta}}\sqrt{T_c - T} \quad . \quad (3)$$

---

[1]The functional can not depend on the global phase of Ψ since the global phase of quantum states is not observable.

[2]The expansion of F to fourth order is only valid when ψ is small.

The scaling of ψ also shown in Fig.(2) is characteristic of mean-field theories as the Landau theory. This also implies that the order parameter is considered constant in time and space.
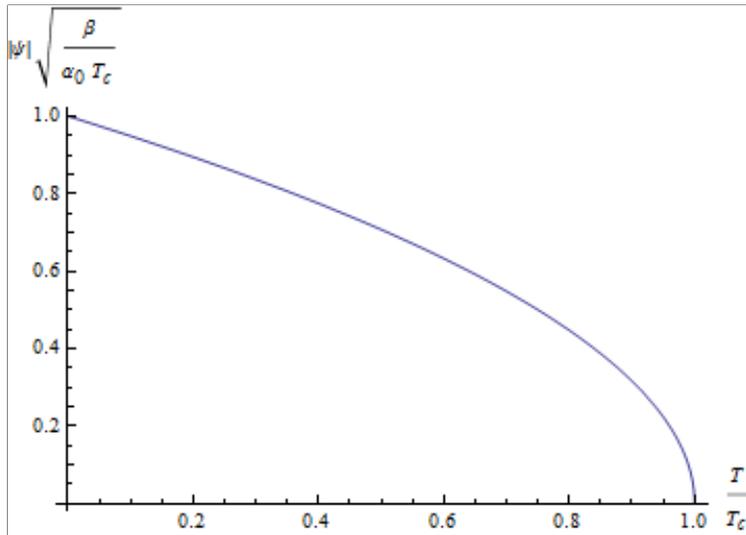

Figure 2: A stretch of $|\psi|$

So what are the underlying physical of this phase transition? To answer this question we have to look at the microscopic theory. $\alpha < 0$ requires an attraction between electrons which may seem strange as electrons repel each other. However, the lattice of positively charged atoms cancel this repulsion on average. It can even distort such that a small net attractive interaction results between pairs. Such lattice vibrations are called phonons. Cooper found that this can lead to pairing of opposite spin electrons with momenta near the Fermi surface. Schrieffer found the many electron wavefunction in what is now referred to as the Bardeen-Cooper-Schrieffer (BCS) theory. Unfortunately the phonon attraction is so weak that thermal fluctuations destroys the pairs at a few degrees Kelvin depending on the metal corresponding to that α increase to zero at this critical temperature.

Until now we have only described a phase transition so how do we apply it to the superconductor? To describe superconductors we assume the charge q and mass m^* for the particles forming the condensate[3] and then add two additional terms to the energy functional.

$$F[\psi, A] = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}|(\frac{\hbar}{i}\nabla - \frac{q}{c}A)\psi|^2 + \frac{|B|^2}{2\mu_0} \qquad (4)$$

Here $A = \nabla \times B$ is the vector potential required by gauge invariance and the last term is the energy density of the magnetic field. The equilibrium value of the order parameter can be determined from the minimum of the functional, whereas the actual value of the free energy is given by the value of the functional at the equilibrium order parameter.

---

[3]The free energy is gauge invariant only if a universal value is taken for the charge. Ginzburg and Landau argued that there was no reason to choose it to be different from the the electron charge. It is now known that $q = 2e$ should be chosen a universal value for all superconductors and $m^* = 2m = 2m_e$.

Minimizing the functional with respect to the order parameter ψ the first Ginzburg-Landau equation takes the form

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}(\frac{\hbar}{i}\nabla - \frac{q}{c}A)^2\psi = 0 \tag{5}$$

This equation shares some similarities with the time-independent Schrödinger equation but is different due to a nonlinear term that determines the order parameter. With Ampères law we find the second Ginzburg-Landau equation

$$j = \frac{c}{4\pi}\nabla \times B = i\frac{q\hbar}{2m^*}(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{q^2}{m^*c}|\psi|^2A \tag{6}$$

In the limit of uniform $\psi(r)$

$$j = -\frac{q^2|\psi|^2}{m^*c}A \tag{7}$$

this should reproduce the London equation

$$j = -\frac{e^2 n_s}{mc}A \tag{8}$$

Which require

$$n_s = 2|\psi|^2 \tag{9}$$

One would think that the phase of the wavefunction, which is a typical microscopic quantum mechanical quantity, cannot be measured, and is therefore of limited importance. This is indeed so for an isolated superconductor. However, when there is a weak contact between two superconductors, which prevents the establishment of thermodynamic equilibrium but allows the transfer of electrons from one superconductor to the other, their phase difference can lead to interesting phenomena.

3   DC and AC Josephson effects

Let us turn our attention to the Josephson junction. A Josephson junction consists of two superconductors, 1 and 2, separated by an insulating barrier which electrons can tunnel through. Each superconductor can be described by a quantum mechanical wave function.

We shall derive the Josephson equations in two different ways - both illustrative. First from the Ginzburg-Landau equations and second by Feynmann's method.

When magnetic fields are absent we obtain from the first Ginzburg-Landau Eq. (5)

$$-\frac{\hbar^2}{m^*}\nabla^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0. \tag{10}$$

The order parameter in bulk was $|\psi_0|^2 = -\alpha/\beta$ below T_c and defining the Ginzburg-Landau coherence length $\xi = \hbar/\sqrt{-2m^*\alpha}$ we obtain

$$\xi^2 \nabla^2 \psi = -\psi(1 - \frac{|\psi|^2}{|\psi_0|^2}).$$

(11)

If we place the insulating layer at $x = 0$ and assume that its thickness d is much smaller than the Ginzburg-Landau coherence length, then $d^2\psi/dx^2 \simeq 0$ because it is multiplied by the large number ξ^2/d^2 in the above equation. Therefore the order parameter must be a linear function in the insulator, $-d/2 < x < d/2$. Furthermore requiring that it is continuous and matches the order parameters in both superfluids, it must have the form

$$\psi(x) = \psi_0[(\frac{1}{2} - x/d)e^{i\chi_1} + (\frac{1}{2} + x/d)e^{i\chi_2}]$$

(12)

where χ_1 and χ_2 are the phases in the first and second superfluid. Note that we assume the size of the order parameter is $|\psi_0|$ is the same in both superfluids.

Inserting the above order parameter in the second Ginzburg-Landau Eq. (6) we obtain the superfluid current density

$$
\begin{aligned}
j &= i\frac{q\hbar}{2m^*}[\psi\nabla\psi^* - \psi^*\nabla\psi] \\
&= \frac{2e\hbar}{m^*d}|\psi_0|^2 \sin(\chi_2 - \chi_1).
\end{aligned}
$$

(13)

(14)

This is the DC Josephson equation. The current through the Josephson junction is proportional the sine of the phase difference between the two superfluid phases. We now turn to Feynmann's derivation of the Josephson equations

The wave function for the superconducting electrons can be writen as a sum

$$\Psi = \sum_\alpha C_\alpha(t)\psi_\alpha \qquad \alpha = 1, 2$$

of the states ψ_1 and ψ_2 in superconductor 1 and 2 respectively. The wave functions for the uncoupled superconductors obeys the Schrödinger equation.

$$i\hbar\frac{d}{dt}\psi_\alpha = E_\alpha\psi_\alpha$$

(15)

If the wave functions are normalized such that

$$\int \psi_\beta^* \psi_\alpha dV = \delta_{\alpha\beta}$$

(16)

The tunneling through the barrier couples the two conductors.

$$i\hbar\frac{d}{dt}\Psi = H\Psi$$

Inserting we obtain

$$i\hbar \frac{d}{dt} \sum_{\alpha} C_{\alpha}\psi_{\alpha} = \sum_{\alpha} C_{\alpha}H\psi_{\alpha}$$

$$i\hbar \sum_{\alpha} \dot{C}_{\alpha}\psi_{\alpha} + C_{\alpha}\dot{\psi}_{\alpha} = \sum_{\alpha} C_{\alpha}H\psi_{\alpha}$$

Multiplying by ψ_{β}^{*}

$$i\hbar \sum_{\alpha} \psi_{\beta}^{*}\dot{C}_{\alpha}\psi_{\alpha} + \psi_{\beta}^{*}C_{\alpha}\dot{\psi}_{\alpha} = \sum_{\alpha} C_{\alpha}\psi_{\beta}^{*}H\psi_{\alpha} \qquad (17)$$

Integrating over the space

$$\int i\hbar \sum_{\alpha} \psi_{\beta}^{*}\dot{C}_{\alpha}\psi_{\alpha} + \psi_{\beta}^{*}C_{\alpha}\dot{\psi}_{\alpha}dV = \int \sum_{\alpha} C_{\alpha}\psi_{\beta}^{*}H\psi_{\alpha}dV$$

and using Eq. (16) and Eq. (15) we obtain

$$\sum_{\alpha} i\hbar \dot{C}_{\alpha}\delta_{\alpha\beta} + \int \psi_{\beta}^{*}C_{\alpha}E_{\alpha}\psi_{\alpha}dV = \sum_{\alpha} C_{\alpha}H_{\beta\alpha}$$

$$\sum_{\alpha} i\hbar \dot{C}_{\alpha}\delta_{\alpha\beta} = \sum_{\alpha} [H_{\beta\alpha} - E_{\alpha}\delta_{\alpha\beta}]C_{\alpha}$$

Because of the delta function within the sum only the terms in which $\alpha = \beta$ are nonzero.

$$i\hbar \frac{d}{dt}C_{\beta} = \sum_{\alpha} [H_{\beta\alpha} - E_{\alpha}\delta_{\alpha\beta}]C_{\alpha}$$

Here

$$H_{\beta\alpha} = \int \psi_{\beta}^{*}H\psi_{\alpha}dV$$

are the matrix elements in the Hamilton matrix.

Assume that a voltage V is applied between the two superconductors. If zero potential is assumed to occur in the middle of the barrier between the two superconductors, superconductor 1 will be at potential $-1/2V$ with Cooper pair potential energy $+eV$, while superconductor 2 will be at potential $+1/2V$ with Cooper pair potential energy $-eV$[8].
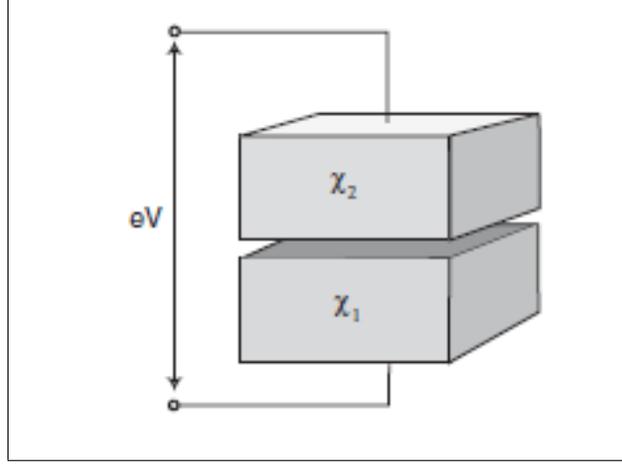
Figure 3: The Josephson junction

The diagonal elements

$$H_{11} = E_1 + e^*V/2 = E_1 + eV \qquad H_{22} = E_2 - e^*V/2 = E_2 - eV$$

correspond to the energy levels of state 1 and 2 and $e^* = 2e$ is the charge of each cooper pair. Off diagonal elements is a coupling constant which describes the transistions between states 1 and 2.

$$H_{12} = H_{21} = -K$$

We can now write the coupled equations.

$$i\hbar \frac{d}{dt} C_1 = eV C_1(t) - K C_2(t) \tag{18}$$

$$i\hbar \frac{d}{dt} C_2 = -eV C_2(t) - K C_1(t) \tag{19}$$

The coefficients are normalized such that

$$|C_1|^2 = N_1 \quad C_1 = \sqrt{N_1} e^{i\chi_1} \tag{20}$$

$$|C_2|^2 = N_2 \quad C_2 = \sqrt{N_2} e^{i\chi_2} \tag{21}$$

here $N_{1,2}$ is the number of superconducting electrons in each electrode.

Inserting Eq. (20) into Eq. (18) we obtain

$$i\hbar \frac{d}{dt} \sqrt{N_1} e^{i\chi_1} = eV \sqrt{N_1} e^{i\chi_1} - K \sqrt{N_2} e^{i\chi_2}$$

Multiplying with the complex conjugate of C_1 we get

$$i\hbar [\frac{d}{dt} N_1 + i2N_1 \frac{d}{dt} \chi_1] = 2eV N_1 - 2K \sqrt{N_1 N_2} e^{i(\chi_2 - \chi_1)} \tag{22}$$

8

Following the same calculations with Eq. (21) and Eq. (19) and multiplying with the complex conjugate of C_2 we obtain

$$i\hbar[\frac{d}{dt}N_2 + i2N_2\frac{d}{dt}\chi_2] = -2eVN_2 - 2K\sqrt{N_1N_2}e^{i(\chi_1-\chi_2)} \tag{23}$$

We define $\phi = \chi_2 - \chi_1$ as the phase difference. Splitting Eq. (22) and Eq. (23) into its real and imaginary parts we obtain four equations Imaginary parts:

$$\hbar\frac{d}{dt}N_1 = -2K\sqrt{N_1N_2}\sin(\phi) \tag{24}$$

$$\hbar\frac{d}{dt}N_2 = +2K\sqrt{N_1N_2}\sin(\phi) \tag{25}$$

Real parts:

$$\hbar N_1\frac{d}{dt}\chi_1 = -eVN_1 + K\sqrt{N_1N_2}\cos(\phi) \tag{26}$$

$$\hbar N_2\frac{d}{dt}\chi_2 = eVN_2 + K\sqrt{N_1N_2}\cos(\phi) \tag{27}$$

By adding Eq. (24) and Eq. (25)

$$\hbar\frac{d}{dt}N_1 + \hbar\frac{d}{dt}N_2 = -2K\sqrt{N_1N_2}sin(\phi) + 2K\sqrt{N_1N_2}sin(\phi)$$

$$\hbar\frac{d}{dt}N_1 + \hbar\frac{d}{dt}N_2 = 0$$

$$\hbar\frac{d}{dt}N_1 = -\hbar\frac{d}{dt}N_2$$

Therefore

$$N_1 + N_2 = constant \tag{28}$$

corresponding to the conservation of charge. By subtracting Eq. (24) and Eq. (25)

$$\hbar\frac{d}{dt}N_1 - \hbar\frac{d}{dt}N_2 = -2K\sqrt{N_1N_2}\sin(\phi) - 2K\sqrt{N_1N_2}\sin(\phi)$$

$$\hbar\frac{d}{dt}[N_1 - N_2] = -4K\sqrt{N_1N_2}\sin(\phi)$$

$$\frac{d}{dt}[N_2 - N_1] = \frac{4K\sqrt{N_1N_2}}{\hbar}\sin(\phi)$$

Using Eq. (28) and multiplying with the electron charge we get

$$2e\frac{d}{dt}[N_2] = \frac{4eK\sqrt{N_1N_2}}{\hbar}\sin(\phi) \quad . \tag{29}$$

9

We see that the left side is exactly the supercurrent I_s since the charge of a Cooper pair is 2e. This equation is the same as Eq. (14) found from Ginzburg-Landau theory since the current density is just the current per area. The constants d and K are thus related. Eq(29) can then be written as

$$I_s = I_c \sin(\phi) \tag{30}$$

with the critical Josephson current given by

$$I_c = \frac{4eK\sqrt{N_1 N_2}}{\hbar}$$

Equation (30) describes the DC Josephson effect. The current can penetrate the insulating barrier as long as there is an interaction ($K > 0$) between the superconducting regions.

By dividing Eq. (26) with N_1 and Eq. (27) by N_2 and subtracting them we obtain

$$\hbar \frac{d}{dt}\phi = 2eV - K\frac{N_2 - N_1}{\sqrt{N_1 N_2}}\cos(\phi)$$

The last term is usually ignored because it is related to the overall phase $\chi_1 + \chi_2$ of the device which plays no role. However, in principle either the charge difference or the coupling must be small. In this case we arrive at

$$\hbar \frac{d}{dt}\phi = 2eV \tag{31}$$

Equation (31) describes the AC Josephson effect. The phase difference increases with time if there is a voltage between the two superconductors.

In the following chapters we will solve the Josephson equations for gradually more complicated circuits driven by DC and AC voltages and currents eventually arriving at the Shapiro steps.

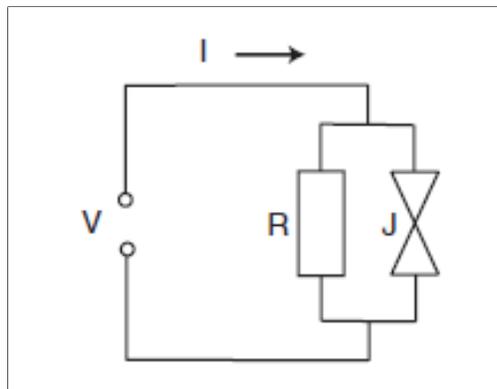# 4 Resistively shunted Josephson junction (RSJ)

Figure 4: The resistively shunted Josephson junction circuit.

Consider first the AC Josephson effect in a circuit with a Josephson junction in parallel with a resistance R, which therefore acts as a shunt. See Fig.(4) The total current which is the sum from Kirchhoffs laws takes familiar form

$$I = \frac{V}{R} + I_c sin(\phi) \tag{32}$$

Here, V is the voltage and I the total current. Using Eq. (31) we get the full current equation

$$I = \frac{\hbar}{2eR}\frac{\partial \phi}{\partial t} + I_c sin(\phi) \tag{33}$$

which is a first order nonlinear ordinary differential equation. It can easily be solved assuming a constant current I.

If we look at the case $I_c \geq I$ the phase does not change with time and is therefore stationary

$$\phi = \arcsin(I/I_c) \tag{34}$$

and the voltage is zero. The phase difference reaches $\pi/2$ for $I = I_c$

If $I > I_c$ the phase grows with time and we get a nonvanishing voltage. Rewriting Eq. (33)

$$\frac{\hbar}{2eR}\frac{d\phi}{I - I_c \sin(\phi)} = dt \tag{35}$$

Let t_0 be the time it takes for the phase to increase by 2π. Integrating Eq. (35) for such a period of time we obtain

$$t_0 = \frac{\hbar}{2eR}\frac{2\pi}{\sqrt{I^2 - I_c^2}} \tag{36}$$

The voltage varies over this period but we can find its average $\langle V \rangle$ from Eq. (31)

$$\langle V \rangle = \frac{1}{t_0}\int_0^{t_0} V dt = \frac{\hbar}{2e}\frac{1}{t_0}\int_0^{t_0}\frac{d\phi}{dt}dt = \frac{2\pi\hbar}{2e}\frac{1}{t_0} \tag{37}$$

Inserting Eq. (36) into Eq. (37) the current-voltage relation takes the form

$$\langle V \rangle = R\sqrt{I^2 - I_c^2} \tag{38}$$

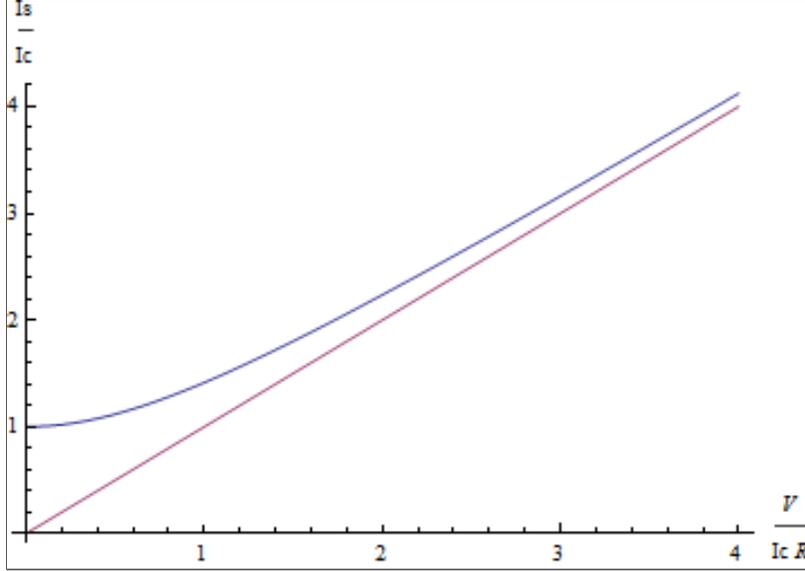This $I(V)$ curve is shown in Fig. (5), it almost has a step at $V = 0$.

Figure 5: The I/V characteristics of the RSJ

5 Resistively and capacitively shunted Josephson junction (RCSJ)

Next we add a capacitor to our circuit in parallel. A capacitor represent the case in which the current is'nt solely carried by tunneling electron pairs or leakage currents through the insulator barrier(shunt), but also takes into account that charge can build up at the interface layers between the Josephson junction and causes displacement currents.
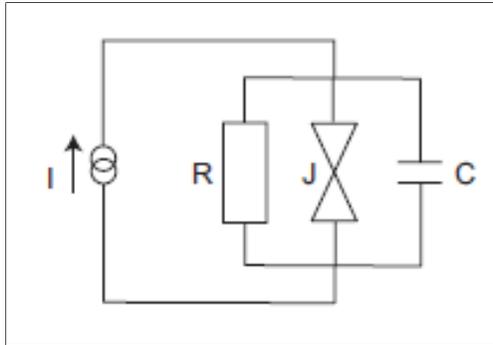

Figure 6: The resistively and capacitively shunted Josephson junction circuit driven by a current I.

The current through the capacitor

$$I_C = C\frac{\partial V}{\partial t} = \frac{\hbar C}{2e}\frac{\partial^2 \phi}{\partial t^2} \tag{39}$$

12

also has to be added to the supercurrent of Eq. (31) besides the shunted current as follows from Kirchhoffs laws, giving the total current

$$I = \frac{\hbar C}{2e}\frac{\partial^2 \phi}{\partial t^2} + \frac{\hbar}{2eR}\frac{\partial \phi}{\partial t} + I_c \sin(\phi) \tag{40}$$

This can be written as an equation that is a mechanical analogue to a forced pendulum

$$\frac{\hbar C}{2e}\frac{\partial^2 \phi}{\partial t^2} + \frac{\hbar}{2eR}\frac{\partial \phi}{\partial t} = I - I_c \sin(\phi). \tag{41}$$

Here the first term is the acceleration of the phase, the second is a dissipation, and the right hand side is the force from a potential given as

$$U = -\int (I - I_c \sin(\phi))d\phi = -I\phi - I_c \cos(\phi) + k. \tag{42}$$

Choosing arbitrarily the integration constant as $k = I_c$ we arrive at

$$U = I_c[1 - \cos(\phi)] - I\phi = I_c\left[1 - \cos(\phi) - \phi\frac{I}{I_c}\right] \tag{43}$$

The potential of Eq(43) is called a tilted washboard potential. It is evident from Fig.(7) that when $I \ll I_c$ the particle is trapped in a minimum with constant phase just as in the previous section for the RSJ. When $I \gg I_c$ the particle rolls down rapidly with almost linearly increasing phase and $I = V/R$ again as for the RSJ.
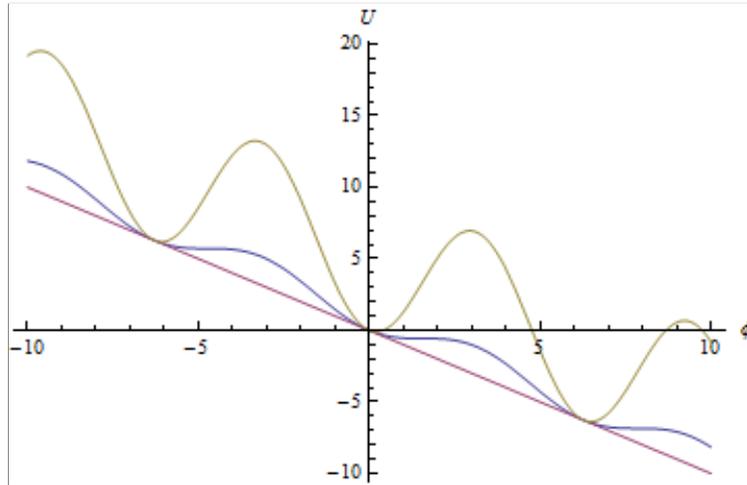

Figure 7: The tilted washboard potential illustrated with $I \ll I_c$ (yellow) and $I > I_c$(blue).

The interesting case occurs when $I < I_c$ and for finite capacitance. Here there are in fact two solutions! It can either be trapped with zero voltage as for the RSJ, or it can slowly roll such that the dissipation cancels the acceleration. This double valued solution is a hysteresis

effect, i.e. when the current is increased the voltage takes the value given by forward arrow in Fig. (8), whereas when the current is decreased it takes the other value.
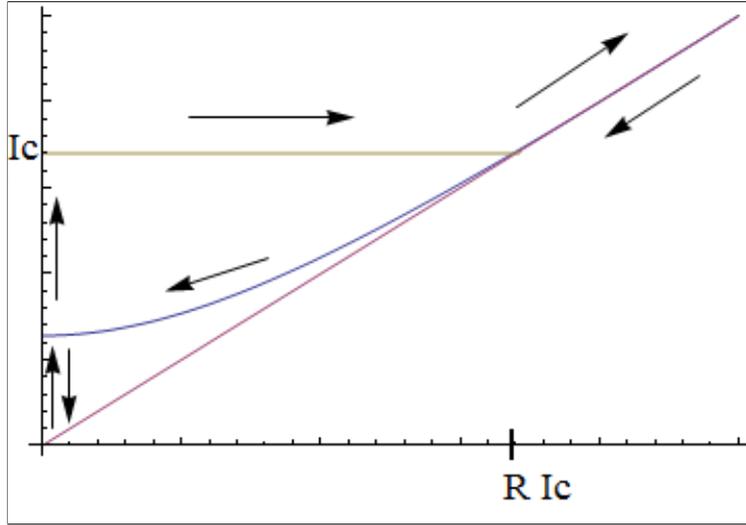

Figure 8: Hysterisis effect in the RCSJ, see text.

It is important to notice the flat plateau when the current is I_c and the voltage jumps. This is the first example of a Shapiro step.

6   Shapiro Spikes

Above we studied circuits driven by constant currents. We now turn to AC circuits. It is mathematically easier to analyse the case where the circuit is driven by an AC voltage. The practical way to implement such a modulation is to bathe the Josephson junction in microwave radiation.

Let us therefore apply both a DC and an AC voltage across the junction.

$$V = V_0 + V_1 cos(\omega t)$$

The phase difference follows from Eq. (31)

$$
\begin{aligned}
\phi &= \int \frac{2eV}{\hbar} dt = \frac{2e}{\hbar} \int [V_0 + V_1 cos(\omega t)] dt \\
&= \phi_0 + \frac{2e}{\hbar} V_0 t + \frac{2e}{\hbar\omega} V_1 sin(\omega t)
\end{aligned}
$$

The supercurrent follows from Eq. (30)

$$I_s = I_c sin(\phi) = I_c \operatorname{Im}[\exp(i(\phi_0 + \frac{2e}{\hbar} V_0 t + \frac{2eV_1}{\hbar\omega} \sin(\omega t)))]. \tag{44}$$

14

where
Substituting $\omega_j = \frac{2e}{\hbar}V_0$, $z = \frac{2eV_1}{\hbar\omega}$ and $\alpha = \omega t$ in Eq. (44)

$$
\begin{aligned}
I_s &= I_c \operatorname{Im}[\exp(i(\phi_0 + \omega_j t + z\sin(\alpha)))] \\
&= I_c \operatorname{Im}[\exp(i(\phi_0 + \omega_j t))\exp(iz\sin(\alpha))]
\end{aligned}
$$

Next use the expansion[9]

$$
\begin{aligned}
\exp(iz\sin(\alpha)) &= J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos(2k\alpha) + 2i\sum_{k=0}^{\infty} J_{2k+1}(z)\sin((2k+1)\alpha) \\
&= \sum_{k=-\infty}^{\infty} J_k(z)\cos(k\alpha) + i\sum_{k=-\infty}^{\infty} J_k(z)\sin(k\alpha)
\end{aligned}
$$

Due to the parity $J_k(z) = (-1)^k J_{-k}(z)$ of the Bessel functions, the components with odd k drop out from the first sum, while the components with even k drop out from the second sum.

$$
\begin{aligned}
I_s &= I_c \operatorname{Im}[\exp(i(\phi_0 + \omega_j t))(\sum_{k=-\infty}^{\infty} J_k(z)\cos(k\alpha) + i\sum_{k=-\infty}^{\infty} J_k(z)\sin(k\alpha))] \\
&= I_c \operatorname{Im}[\exp(i(\phi_0 + \omega_j t))\sum_{k=-\infty}^{\infty} (-1)^k J_k(z)\exp(-ik\alpha)] \\
&= I_c \operatorname{Im}[\sum_{k=-\infty}^{\infty} (-1)^k J_k(z)\exp(i(\phi_0 + \omega_j t))\exp(-ik\alpha)] \\
&= I_c \sum_{k=-\infty}^{\infty} (-1)^k J_k(z)\sin(\phi_0 + \omega_j t - k\alpha)
\end{aligned}
$$

Substituting back and adding the shunt current V_0/R the total current takes the form

$$
I = I_s + \frac{V_0}{R} = I_c \sum_{k=-\infty}^{\infty} (-1)^k J_k(\frac{2eV_1}{\hbar\omega})\sin(\phi_0 + \frac{2e}{\hbar}V_0 t - k\omega t) + \frac{V_0}{R} \tag{45}
$$

also known as the inverse AC Josephson effect. The DC part of the current is V_0/R unless

$$
V_0 = \frac{k\hbar\omega}{2e}, \quad k = 0, \pm1, \pm2, ... \tag{46}
$$

Then the supercurrent has the DC component

$$
I_s = I_c(-1)^k J_k(\frac{2eV_1}{\hbar\omega})\sin(\phi_0) \tag{47}
$$

If the potential is not precisely given by Eq. (46) the supercurrent will oscillate slowly with an amplitude given by Eq. (47) with $\phi_0 = \pi/2$, i.e. given by the Bessel function value.

The resulting DC current therefore increase linear as V_0/R except when the potential is an integer multipla of $\hbar\omega/2e$ as given by Eq.(46) where the DC supercurrent suddenly jumps to the value given by Eq. (47). These are called Shapiro spikes and are shown in Fig(9)
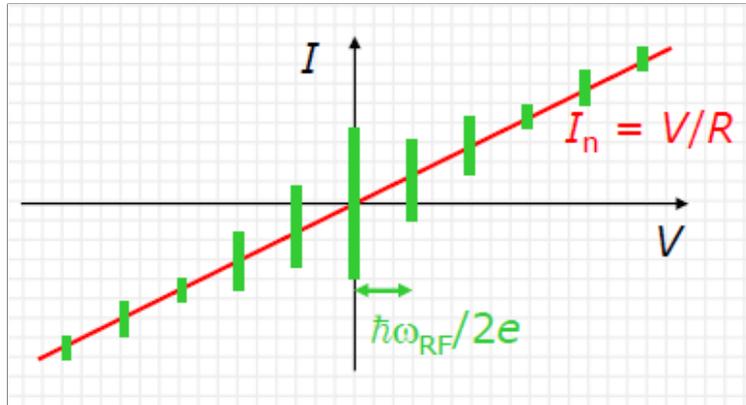

Figure 9: Shapiro spikes of width $\hbar\omega/2e$.

7   Shapiro Steps

Realistic circuits are usually driven by a current. Applying an AC current we thus need to solve the nonlinear second order differential Eq. (40). This is very complicated and can only be done numerically. The behavior of the phase is quite intricate, however, in terms of the DC current versus the average voltage a simple ladder behavior appears as shown in Fig(10). These are called Shapiro steps.
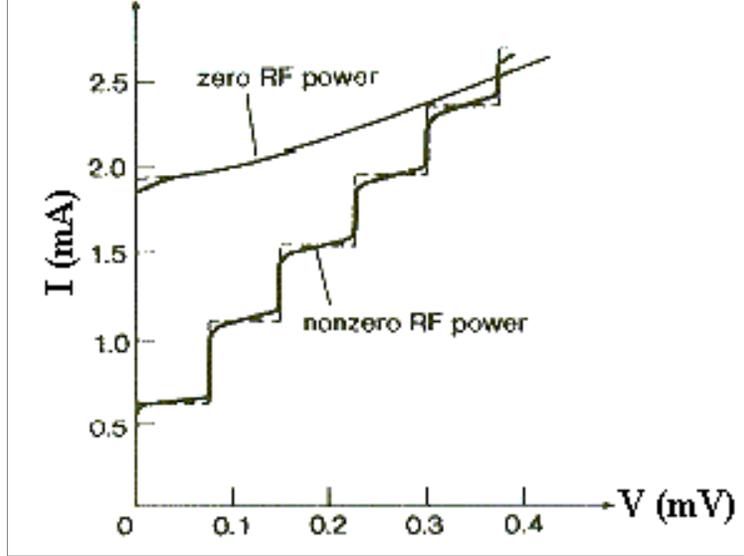
Figure 10: Current-voltage characteristics of a point-contact Josephson junction with applied rf power at 35 GHz. Solid lines show experimental results. Broken lines gives results of calculations without noise.

The Shapiro steps follow naturally from the results in the previous sections. The jumps occur precisely when the average voltage match $\langle V \rangle = k\hbar\omega/2e$, $k = 0, \pm 1, \pm 2, ...$, as described for the Shapiro spikes. The plateaus occur as in the RCSJ. Note that the Shapiro spikes occurring for voltage driven circuits leads to double valued voltages for the same current. This does not occur for the current driven circuits due to the hysteresis effect as described for the RCSJ. Instead the plateau appears and a Shapiro step occurs when the voltage match an integer value of $\hbar\omega/2e$. The currents jump at the Shapiro step by an amount given by the Bessel function expression above.
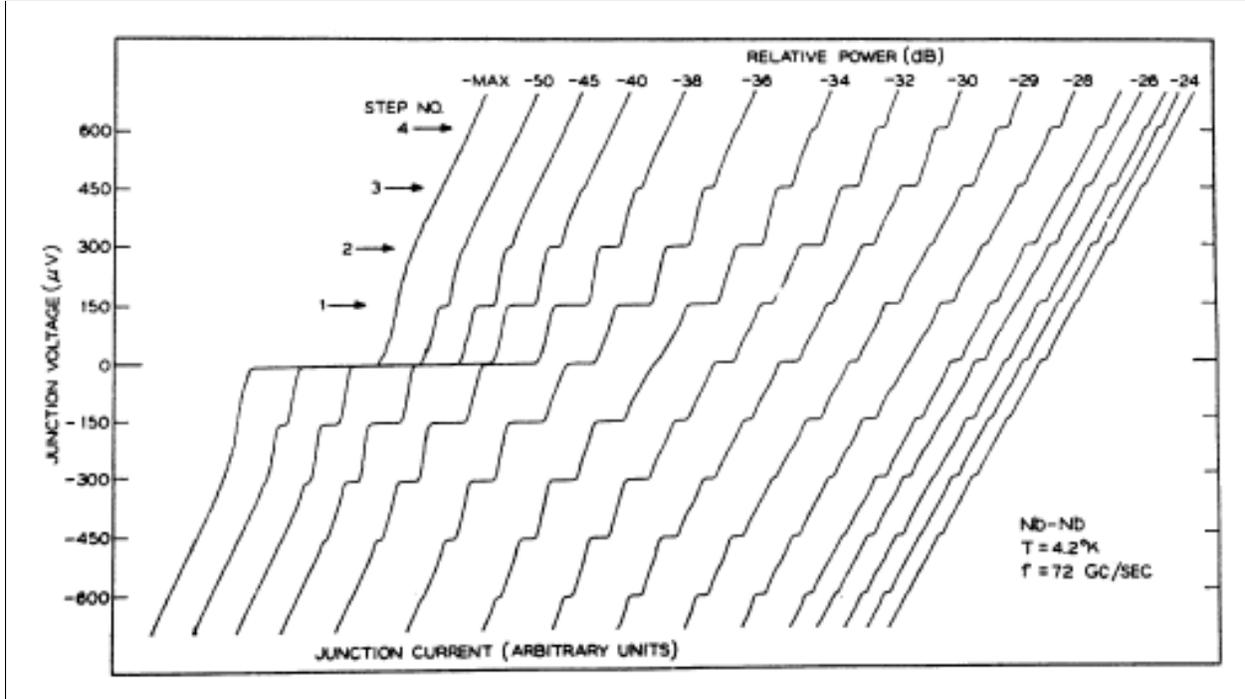
Figure 11: The current-voltage characteristics for a current-driven Josephson junction placed in microwave fields of different power, according to the measurements of C. C. Grimes and S. Shapiro [1].

8   Summary and conclusion

Starting from the Ginzburg-Landau description of superfluidity we have derived the AC and DC equations for the Josephson junction. We have solved these for circuits with Josephson junction with shunt resistance and capacitance. Driving these circuits with an AC voltage could also be solved yielding the Shapiro spikes. Driving them with an AC voltage led to the Shapiro steps where the current jump when the average voltage is an integer multiple of the AC frequency divided by the Josephson constant $2e/h = 483597,011 GHz/Volt$. The current jumps by the Josephson current times a Bessel function of order k.

Since the voltage makes integer steps in terms of the frequency and fundamental constants like the electron charge and Planck's constant, the Shapiro steps and the AC Josephson effect provide the currently most accurate physical standards for the volt using the Caesium frequency. The most precise measurement of the electron charge is made from the Josephson constant $h/2e$ and the von Klitzing constant $\hbar/e^2$ measured in the quantum Hall effect.

Mode locking occurs in many physical systems. Two pendulum clocks on the wall can become synchronous due to the weak coupling between them. Pacemaker cells syncronize to make your heart pump. The Shapiro spikes and steps can be interpreted as mode locking in the Josephson junction. When the average voltage of the source is close to an integer multiple of the frequency of the source, $2e\langle V \rangle = \hbar\omega$, the Josephson junction tunes in with

18

the source.

Shapiro steps have recently been observed in nanotube Josephson junctions [5], superconducting nanowires [6] and High-Temperature superconductors with additional interesting effects. Fractional Shapiro steps have been observed in circuits with several Josephson junctions although all fractions can not be explained.

This fundamental understanding of the phases and currents in Josephson junctions are important in applications as SQUIDS mentioned in the introduction, which can measure the magnetic flux and fields precisely. Josephson junctions are further exploited to make flux Qubits that are essential for superconducting quantum computers. These and many more applications are outside the scope of this bachelor thesis.

References

[1] S. Shapiro, "Josephson currents in superconducting tunneling: The effect of microwaves and other observations" Physical Review Letters 11, 80 (1963). C. C. Grimes and S. Shapiro, Physical Review 169, 397 (1968).

[2] Josephson, B. D., "Possible new effects in superconductive tunnelling," Physics Letters 1, 251 (1962)

[3] V.L. Ginzburg and L.D. Landau, Zh. Eksp. Teor. Fiz. 20, 1064 (1950).

[4] Jenó Sólyom, "Fundamentals of the Physics of Solids", Springer-Verlag Berlin Heidelberg 2009.

[5] J. P.Cleuziou et al. , PRL 99, 117001 (2007).

[6] R. C. Dinsmore III et al. Applied physics letters 93 (2008).

[7] N. B Kopnin, "Introduction to The Theory of Superconductivity", Cryocourse 2009.

[8] C. P. Poole Jr., H. A. Farach, R. J. Creswick, "Superconductivity", Academic Press, London 1995.

[9] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals: 8.511 3.