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Master thesis

Symbol and Coaction of l-loop Banana Integrals

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1 Abstract

We construct a symbol and coaction operator on l-loop equal-mass banana integrals. Our work is based on a general construction of a coaction on unipotent qunatities due to Brown. In this construction we are able to define a coaction through the symbol operator. The symbol operator itself is defined on a pair of unipotent periods $[\xi_i, \xi_j]$. Each period belong to a vector that satisfy a unipotent differential equation. The symbol expression is determined by the matrix that solves the unipotent differential equation. The equal-mass banana integrals, do not satisfy such a unipotent differential equation. Instead we consider a vector that contain all the master integrals and $l-1$ numbers of $\tau_i(z)$'s for $0 \leq i \leq l-1$. The $\tau_i(z)$'s are defined as the ratio between two periods in the Frobenius basis. This vector satisfy a unipotent differential equation. We provide a general expression for the unipotent differential equation at arbitrary loop order.

We give explicit examples of the symbol and coaction for equal-mass master integrals at $l = 2, 4$. We find that for $l = 2$ the symbol length is two and for $l \geq 3$ the symbol length is three. We comment on the meaning of these results and compare our findings for $l = 2$ with earlier works by Broedel et al. [arXiv:1803.10256].

Finally we comment on further improvements and discuss how to generalise to generic mass.

Contents

2 Introduction

High precision calculations of physical observables in QFT usually require evaluating many Feynman integrals up to a certain loop order. Most notoriously they are needed in precision calculation of cross sections for high-energy collider experiments [1]. Feynman integrals beyond one-loop order with different masses and momenta are notoriously difficult to evaluate analytically [2]. For instance, the Higgs production with generic masses has only been evaluated at two-loop orders [3]. The computation of multi-loop integrals is still a bottleneck in precision calculations. The main reason for this is that the functions involved are in general not elementary functions (for instance rational or algebraic). The functions will have a complicated branch cut structure dictated by unitarity and intermediate virtual particles going on shell [4]. In recent years there has been a lot of investigation into the mathematical construction of Feynman integrals. This investigation has showed that a large class of phenomenologically important integrals can be expressed in terms of functions called *multiple polylogarithms* (MPLs) [5]. There exist several numerical techniques for their computations. The success of MPLs is owed to the fact that their mathematical and algebraic properties are well understood [6]. A particular powerful tool for computing MPLs is the symbol operator. It maps a MPL to a tensor whose entries are algebraic functions. The advantage of the symbol is that it makes complicated relations between MPLs trivial. Although the symbol of MPLs is easy to work with, a lot of information is lost as it maps all constants to zero. Some of this information can be recovered by using the more general coaction. The symbol can be seen as the maximally iterated coaction [7].

While many classes of Feynman integrals can be represented in terms of MPLs, there are still many cases in which this is not possible. The simplest class of Feynman integrals that cannot be computed in terms of MPLs, is the two-loop sunrise integral with non-zero masses [8]. This integral describes a two-loop process involving 3 masses and external momentum p^2 [9]. The sunrise integral is particularly relevant in electro-weak physics, where non-zero masses naturally occur. It also appears as sub topologies in many higher-order calculations, like the two-loop corrections to top pair production or in the computation of higher-point functions in massless theories. It has been shown that in order to evaluate the sunrise integrals, one has to generalize to elliptic functions. As such there have over the years been an interest in investigating a new class of functions called *multiple elliptic polylogarithms* (eMPLs) [10].

In [11], the symbol and coaction on the equal-mass two loop sunrise integral was calculated. The construction was based on work from Brown [12]. The construction allows one to build a symbol and coaction on arbitrary periods. In this thesis we use the work of Brown to

construct a symbol and coaction on a class of Feynman integrals called banana integrals. These integrals describe a l-loop processes, depicted in Figure 1, involving $(l+1)$ masses and external momentum p^2. The goal is to provide a general construction of a symbol and coaction operator on equal-mass banana integrals for arbitrary loop-order. We give explicit examples in the case of 2 and 4 loops and compare our findings with the results of Broedel et al. [11], [13].

This thesis is organized as follows. In section 3 we review the master integrals related to the banana integrals. We also give a brief overview of the maximal cuts related to the master integrals. In section 4 we review MPLs, coaction and symbols. We introduce the symbol on MPLs in a slighty unconventional way, as it will prepare us for the construction of symbols on arbitrary periods. Finally we review the work of [11] on how to build a coaction on arbitrary periods. In section 5 we review the differential equations that are satisfied by the banana integrals. This includes a special type of differential equation known as the Picard-Fuchs differential equation. We go in more detail about maximal cuts. Finally we review a type of relations between maximal cuts called Griffiths transversality. In section 6, we apply the construction of symbols and coaction on the equal-mass master integrals defined in section 3. We compare our symbol and coaction with the results of Broedel et al. [11], [13]. In section 7, we attempt to generalize our symbol and coaction to generic-mass master integrals. This involves generalizing the Griffiths transversality to a generic-mass case. In section 8 we state our conclusions.

3 Feynman Integrals

In this section we introduce the Feynman diagram of an l-loop banana diagram. The section is a review of an article by Bönisch et. al [6]. We will in this section introduce the master integrals in dimensional regularization, associated with the l-loop Feynman diagram. We also give a brief overview on the topic of maximal cuts.

3.1 Sunrise Diagram

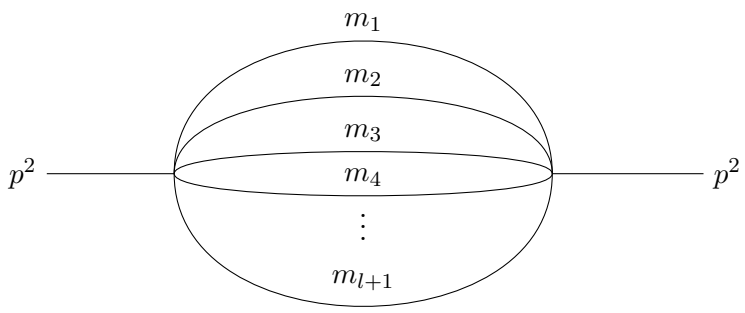

Figure 1: l-loop banana diagram with external momentum p and internal masses $m_1, ..., m_{l+1}$

The banana diagram, which will be our main focus in this thesis, describes an l-loop process with external momentum p^2 and $l+1$ internal masses $m_1, ..., m_{l+1}$. The banana Feynman diagram can be viewed in figure 1. To calculate the amplitude of such a diagram one is required to evaluate nested integrals of the type

$$I_{\underline{\nu}}(\underline{x}; D) = \int \left(\prod_{r=1}^{l} \frac{d^D k_r}{(i\pi)^{D/2}} \right) \left(\prod_{j=1}^{n} \frac{1}{P_j^{\nu_j}} \right) \tag{3.1}$$

, with $P_j = q_j^2 - m_j^2 + i\varepsilon$ and the exponents $\underline{\nu}$ is a set of $\mathbb{Z}^p$. The momentum k_j is the momenta flowing through the propagator, made out of linear combinations of loop momenta k_j and external momenta p. D the space-time dimension. In particular we will work in $D = 2 - 2\varepsilon$ dimensions. By working with the integrals in $D = 2 - 2\varepsilon$ dimensions, there are less undetermined momenta k_r to integrate, thus reducing the complexity of the integral. We are able to relate $D = 2 - 2\varepsilon$ dimensional integrals to $D = 4 - 4\varepsilon$ dimensional integrals, by using dimensional recurrence relations [14]. There will be at most $n = l + 1$ propagators

of the form

$$\begin{aligned} P_j &= k_j^2 - m_j^2,\ 1 \leq j \leq l, \\ P_{l+1} &= (k_1 + k_2 + ... + k_l - p)^2 - m_{l+1}^2. \end{aligned} \tag{3.2}$$

It is evident that the integral will only depend on the masses and the dot products between external momentum. In other words, the equation is Lorentz invariant. The Lorentz invariant quantities will be referred to as scales $\underline{x}$. Furthermore it is possible to conclude on the basis of dimensional analysis, that the only dependence of these scales is through the dimensionless ratio $z_j = x_{k+1}/x_1$ for $1 \leq k \leq N$.
The scales are

$$\begin{aligned} \underline{x} &= \left(p^2, m_1^2, ..., m_{l+1}^2\right), \\ \underline{z} &= \left(m_1^2/p^2, ..., m_{l+1}^2/p^2\right). \end{aligned} \tag{3.3}$$

3.2 Master Integrals

The family of integrals described by eq. 3.1 are by no means independent. In fact they can be represented by a linear combinations of a basis of integrals called *master integrals*. The basis is always finite and can be derived via. integration-by-part (IBP) identities.
It will later prove useful for us to group the family of integrals into sectors that share the same denominators in the integrand in eq. 3.1. We say that two integrals $I_{\underline{\nu}}(\underline{x}; D)$ and $I_{\underline{\nu}'}(\underline{x}; D)$ belong to the same sector if the "zero" placements in $\underline{\nu}$ have the same placements in $\underline{\nu}'$ and vice versa.
Furthermore we also define the notion of ordered sectors, where the lowest order contains the most zeros in $\underline{\nu}$ and the higest sector contains the least zeros. In total there will be $2^{l+1} - 1$ master integrals, distributed among $l + 2$ sectors.
There are $l + 1$ sectors of the form $\underline{\nu} = \{1, ..., 1, 0, 1, ..., 1\}$, all proportional to an l-loop tadpole integral of the form

$$J_{l,i}(\underline{z}; \varepsilon) = \frac{(-1)^{l+1}}{\Gamma(1 + l\varepsilon)} (p^2)^{l\varepsilon} \varepsilon^l I_{1,...,1,0,1,...,1}(\underline{x}; 2 - 2\varepsilon) = -\frac{\Gamma(1 + \epsilon)}{\Gamma(1 + l\varepsilon)} \prod_{j=1, j\neq i}^{l+1} z_j^{-\varepsilon}, \tag{3.4}$$

with the i determining which index has the "zero". The final sector contains $2^{l+1} - l - 2$ master integrals for each $k \in \{0, 1\}^{l+1}$ with $1 \leq |\underline{k}| \leq l - 1$.

$$\begin{aligned} J_{l,\underline{0}}(\underline{z}; \varepsilon) &= \frac{(-1)^{l+1}}{\Gamma(1 + l\varepsilon)} (p^2)^{1+l\varepsilon} I_{1,...,1}(\underline{x}; 2 - 2\varepsilon), \\ J_{l,\underline{k}}(\underline{z}; \varepsilon) &= (1 + 2\varepsilon)...(1 + k\varepsilon) \partial_{\underline{z}}^{\underline{k}} J_{l,0}(\underline{z}; \varepsilon), \end{aligned} \tag{3.5}$$

with $\partial_{\underline{z}}^{k} = \prod_{i=1}^{l+1} \partial_{z_i}^{k_i}$.
It was checked in [6], for the first few loop orders, that these integrals form a basis of master integrals.
In the limit where some scales disappear or become equal, the numbers of master integrals are reduced. In the equal mass case, $m_i^2 = m^2$, the numbers of master integrals are reduced to $l+1$. With a given normalization these can be written as

$$\begin{aligned} J_{l,0}(z;\varepsilon) &= \frac{(-1)^{l+1}}{\Gamma(1+l\varepsilon)}(m^2)^{l\varepsilon}\varepsilon^l I_{1,\ldots,1,0}(\underline{x}; 2-2\varepsilon) = -\frac{\Gamma(1+\epsilon)}{\Gamma(1+l\varepsilon)}, \\ J_{l,1}(z;\varepsilon) &= \frac{(-1)^{l+1}}{\Gamma(1+l\varepsilon)}(m^2)^{1+l\varepsilon} I_{1,\ldots,1}(\underline{x}; 2-2\varepsilon), \\ J_{l,k}(z;\varepsilon) &= (1+2\varepsilon)\ldots(1+k\varepsilon)\partial_z^k J_{l,1}(z;\varepsilon). \end{aligned} \tag{3.6}$$

It is worth mentioning that in the beginning we make no assumption about which basis of master integrals are being used. However we will later on, restrict ourselves to the equal-mass case, as this vastly simplifies our calculations. In section 5.6, we will make the extra simplification of going to the limit $\epsilon \to 0$, due to the fact that the relations we consider in that section are not defined in dimensional regularization.

3.3 Maximal Cuts

In the sections to come we will need to consider the maximal cut of eq. (3.6), which means that we set all propagators in eq. (3.1) on shell. It is worth briefly mentioning them, on the topic of Feynman diagrams. Closed expressions for the equal-mass maximal cuts can be found for $l \leq 3$ in the large momentum region $0 \leq z \leq 1/(l+1)^2$. At one loop this is simply an algebraic function

$$\varpi_{1,0}(z) = \frac{z}{\sqrt{1-4z}}. \tag{3.7}$$

At two loop order, the maximal cuts can be described in terms of complete elliptic integrals of the first kind. We define

$$\lambda = \frac{(\sqrt{z}+1)(3\sqrt{z}-1)^3}{(\sqrt{z}-1)(3\sqrt{z}+1)^3}, \tag{3.8}$$

which is related to the branch points of the elliptic curve. The maximal cuts can be expressed as

$$\begin{aligned} \varpi_{2,0}(z) &= \frac{2z}{\pi\sqrt{1-27z^2+18z+8\sqrt{z}}}\mathrm{K}(1-\lambda), \\ \varpi_{2,1}(z) &= -\frac{4z}{\sqrt{1-27z^2+18z+8\sqrt{z}}}\mathrm{K}(\lambda), \end{aligned} \tag{3.9}$$

where $K(\lambda)$ is the complete elliptic integral of the first kind

$$\mathrm{K}(z) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}. \tag{3.10}$$

The expressions in other regions can be found by analytical continuation cf. refs. [15], [16] and [17].
At three loop orders the maximal cuts can be written as products of elliptic integrals of the first kind [18]. We define

$$\begin{aligned}
\lambda_1 &= \frac{32z^{3/2}}{i\sqrt{1-16z} - 8(i\sqrt{1-4z} - 2\sqrt{z})z + i\sqrt{1-4z}}, \\
\lambda_2 &= -\frac{32z^{3/2}}{i\sqrt{1-16z} + 8(i\sqrt{1-4z} - 2\sqrt{z})z - i\sqrt{1-4z}},
\end{aligned} \tag{3.11}$$

that are again related to the branch points. The maximal cuts can be written as

$$\begin{aligned}
\varpi_{3,0}(z) &= \frac{\sqrt{\lambda_1\lambda_2}}{4\pi^2}\mathrm{K}(\lambda_1)\mathrm{K}(\lambda_2), \\
\varpi_{3,1}(z) &= -\frac{\sqrt{\lambda_1\lambda_2}}{2\pi}\mathrm{K}(\lambda_1)\mathrm{K}(1-\lambda_2) - \frac{\sqrt{\lambda_1\lambda_2}}{4\pi}i\mathrm{K}(\lambda_1)\mathrm{K}(\lambda_2), \\
\varpi_{3,2}(z) &= \frac{\sqrt{\lambda_1\lambda_2}}{6}\mathrm{K}(1-\lambda_1)\mathrm{K}(1-\lambda_2) + \frac{i\sqrt{\lambda_1\lambda_2}}{6}\mathrm{K}(\lambda_1)\mathrm{K}(1-\lambda_2) \\
&\quad - \frac{\sqrt{\lambda_1\lambda_2}}{8}\mathrm{K}(\lambda_1)\mathrm{K}(\lambda_2).
\end{aligned} \tag{3.12}$$

Although no closed form solution exists for higher loop orders, it is possible to obtain a power series type expression with increasing powers of $\log(z)$. It is only possible to construct such a series around points known as *Maximal Unipotent Monodromy* (MUM)-points. These will be covered in section 5.2 and 5.3. By increasing powers of $\log(z)$, we mean for instance that the maximal cut of $J_{l,0}(z)$ is a power series $\varpi_{l,0}(z)$. The maximal cut of $J_{l,1}(z)$ contain $\log(z)$. The maximal cut of $J_{l,2}(z)$ contain $\log^2(z)$ and so on up to a function that contain $\log^{l-1}(z)$ for $J_{l,l-1}(z)$. The basis associated with the maximal cuts $\varpi_{l,0}(z), \varpi_{l,1}(z), ..., \varpi_{l,l-1}(z)$ is called the Frobenius basis.

4 Symbols and Coaction

The goal of this section is to define the concept of a symbol and coaction operator on arbitrary (motivic) periods. To do this we start out by reviewing [4] and [7], which covers the topic of *Multiple Polylogarithms* (MPLs). The symbol and coactions of MPLs have long been known and are in general well understood. By first introducing symbols and coaction on MPLs we are better equipped to deal with symbol and coaction on arbitrary periods. Finally we will review [11], which describes how to construct a coaction on arbitrary (motivic) periods.

4.1 Multiple polylogarithms

MPLs are defined recursively for $n \geq 0$ with the iterated integral [4]

$$G(a_1, ..., a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, ..., a_n; t), \tag{4.1}$$

where the start of the recursion is defined as $G(; z) = 1$. We also define for the special case, in which all the a_i's are zeros, to be

$$G(0, ..., 0; z) = \frac{1}{n!} \log^n z. \tag{4.2}$$

The number of a_i's, or equivalently the number of integrations, is called the weight of the MPL.
There exists also a series representation of eq. 4.1

$$\mathrm{Li}_{m_1,...,m_k}(z_1, ..., z_k) = \sum_{0<n_1<n_2...<n_k} \frac{z_1^{n_1} z_2^{n_2} ... z_k^{n_k}}{n_1^{m_1} n_2^{m_2} ... n_k^{m_k}}, \tag{4.3}$$

which is only defined for a convergent series, i.e $|z_i| < 1$. There exists special cases of MPLs where z's or a's take special values. The most notorious, which is worth mentioning are *multiple zeta-values* (MZV's)

$$\zeta_n = \mathrm{Li}_{m_1,...,m_n}(1, ..., 1). \tag{4.4}$$

The iterated integral representation and the series representation of MPLs cover the same space of functions, however each representation comes with a different, although very similar, algebra. The itererated integral representation of MPLs is equipped with a *shuffle multiplication* ⧢

$$\begin{aligned} G(a_1, ..., a_{n_1;z}) G(a_{n_1+1}, ..., a_{n_1+n_2}) &= G\left((a_1, ..., a_{n_1;z}) \sqcup\!\sqcup (a_{n_1+1}, ..., a_{n_1+n_2}); z\right) \\ &= \sum_{\sigma \in \Sigma(n_1, n_2)} G(a_{\sigma(1)}, ..., a_{\sigma(n_1+n_2)}; z), \end{aligned} \tag{4.5}$$

where $\Sigma(n_1, n_2)$ is the subset of the symmetric group $S_{n_1+n_2}$, which is the set of shuffles of $n_1 + n_2$ elements given by

$$\Sigma(n_1, n_2) = \{\sigma \in S_{n_1+n_2} | \sigma^{-1}(1) < ... < \sigma^{-1}(n_1), \sigma^{-1}(n_1+1) < ... < \sigma^{-1}(n_1+n_2). \quad (4.6)$$

In other words, σ contains all symmetric permutations of $s_1 = (a_1, ..., a_{n_1})$ into $s_2 = (a_{n_1+1}, ..., a_{n_1+n_2})$, where the original order of s_1 and s_2 remain the same.
This property turns MPLs into a *shuffle algebra* $\mathcal{A}_s$.
On the other hand, the series representation of MPLs have a *quasi-shuffle algebra*, sometimes refered to as a stuffle algebra, which is equipped with a quasi-shuffle multiplication $\sqcup\!\sqcup_q$. The quasi-shuffle multiplication is defined recursively as [7]

$$\begin{aligned}\mathrm{Li}(l_1, ..., l_k)\mathrm{Li}(l_{k+1}, ..., l_{k_1+k_2}) &= \mathrm{Li}(l_1 l_2 ... l_k \sqcup\!\sqcup_q l_{k_1} ... l_{k_1+k_2})) \\ &= \mathrm{Li}(l_1(l_2 ... l_k \sqcup\!\sqcup_q l_{k+1} ... l_{k_1+k_2})) + \mathrm{Li}(l_{k+1}(l_1 l_2 ... l_k \sqcup\!\sqcup_q l_{k_2} ... l_{k_1+k_2})) \\ &+ \mathrm{Li}((l_1 \circ l_{k+1})(l_2 ... l_k \sqcup\!\sqcup_q l_{l_k+2} ... l_{k_1+k_2})),\end{aligned} \quad (4.7)$$

where each l_i, called the letters of the MPL, are defined as $l_i = (m_i, x_i)$ and $\mathrm{Li}(l_1, ..., l_n) = \mathrm{Li}_{m_1,...,m_n}(x_1, ..., x_n)$. Furthermore, the operator $\circ$ acts on letters as $l_i \circ l_j = (m_i, x_i) \circ (m_j, x_j) = (m_i + m_j, x_i x_j) = l_{ij}$.
Using both shuffle and quasi-shuffle multiplication, one can find relations between MPLs. For instance, consider the multiple zeta value ζ_2^2. Using shuffle multiplication we find

$$\zeta_2^2 = [-G(0, 1; 1)]^2 = 2G(0, 1, 0, 1; 1) + 4G(0, 0, 1, 1, ; 1) = 2\zeta_{22} + 4\zeta_{31} \quad (4.8)$$

and the quasi-shuffle relation

$$\zeta_2^2 = [\mathrm{Li}_2(1)]^2 = 2\mathrm{Li}_{22}(1, 1) + \mathrm{Li}_4(1) = 2\zeta_{22} + \zeta_4, \quad (4.9)$$

we find the relation $\zeta_4 = 4\zeta_{31}$.

We conclude this section with a note. When working with MPLs we may either choose to work in the iterated integral representation and thus in a shuffle algebra or the series representation and work in the quasi-shuffle algebra. The duality enables us to find relations between MPLs. We can also find relations, via. linearity, change of variable and stokes theorem [7]. In principle there could in theory exist additional relations, but it is difficult to prove their existence. The fact that we cannot for sure now all the relations is slightly problematic when constructing a coaction operator. We will be explain this in more detail in section 4.3. For now we will only hint at its solution. The solution is defining objects called *motivic multiple polylogarithms*. These objects have exactly all the relations we know of and nothing more.

4.1.1 Hopf Algebra

MPLs have both shuffle and quasi-shuffle algebra, however only one is manifest, depending on which representation one chooses to work with. Another type of algebra that is of interest is the *Hopf Algebra*. The properties of a Hopf algebra, are directly linked to the idea of a coaction. Strictly speaking MPLs do not have a Hopf algebra, but many of the properties of an Hopf algebra are also shared by MPLs. A Hopf algebra is a combination of an algebra called a bialgebra and an extra algebraic structure called the antipode. We will not be using the antipode and therefore make no further distinction between bialgebra and Hopf algebra. A Hopf algebra is an algebra $\mathcal{A}$ that has unit element and is associative and distributive. The algebra contains a map $\cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, which essentially works as multiplication, taking a pair of elements $(a, b) \in \mathcal{A}$ to their product $ab \in \mathcal{A}$. The pair of elements will be denoted as $a \otimes b$, which behave according to the bilinearity conditions

$$\begin{aligned} (a+b)\otimes c = a\otimes c + b\otimes c, \qquad & a\otimes(b+c) = a\otimes b + a\otimes c, \\ (ka)\otimes b = a\otimes(kb) = k(a\otimes b), \quad & (a\otimes b)\cdot(c\otimes d) = (ac)\otimes(bd), \end{aligned} \tag{4.10}$$

$\forall a, b, c \in \mathcal{A}$ and $k \in \mathbb{Q}$. To form a Hopf algebra we need one additional algebraic structure called the *coalgebra*. The coalgebra is an algebra $\mathcal{A}$ equipped with a linear map

$$\begin{aligned} \Delta : \mathcal{A} &\to \mathcal{A}\otimes\mathcal{A} \\ a \mapsto \Delta(a) &= \sum_i a_i^1 \otimes a_i^2, \end{aligned} \tag{4.11}$$

called a comultiplication. Comultiplication assigns to each element $a \in \mathcal{A}$ its coproduct $\Delta(a) \in \mathcal{A} \otimes \mathcal{A}$, a linear combination of pairs of elements. Comultiplication is also required to be *coassociative* $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$.
We may apply comultiplcation once again on eq. 4.11, however we may proceed in two ways: Either we act with the comultiplcation on a_i^1 or act with it on a_i^2

$$\begin{aligned} \Delta(\Delta(a)) &= \sum_i \Delta(a_i^1)\otimes a_i^2 = \sum_{i,j} a_{i,j}^{(1,1)} \otimes a_{i,j}^{(1,2)} \otimes a_i^2, \\ \Delta(\Delta(a)) &= \sum_i a_i^1 \otimes \Delta(a_i^2) = \sum_{i,j} a_i^1 \otimes a_{i,j}^{(2,1)} \otimes a_{i,j}^{2,2}, \end{aligned} \tag{4.12}$$

Coassociativity states that these two expressions are equal, which means that there is a unique way of iterating the coproduct.
The Hopfalgebra has both a multiplication $\cdot$ and comultiplication Δ and we require these two operators to satisfy

$$\Delta(ab) = \Delta(a)\cdot\Delta(b), \tag{4.13}$$

Finally we will define how comultiplication acts. Consider a set of letters $\{a_1, ..., a_n\}$ and a vector space $\mathcal{A}$ that is spanned by all linear combinations of words, with rational coefficients in these letters. We define the comultiplication to act on letters as

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad x \in \{a_1, ..., a_n\}, \tag{4.14}$$

in which case we also say that the comultiplication acts trivially on x. For words of length $n \geq 2$ we use the property of eq. 4.13.
If we consider the $G(a_0, ..., a_n; z)$'s in the same fashion as we do for words $a_0...a_n \in \mathcal{A}$, we get the result of a comultiplication acting on a MPL

$$\Delta(G(\vec{a}) \sum_{\vec{b} \subseteq \vec{a}} G(\vec{b}; z) \otimes G_{\vec{b}}(\vec{a}; z), \tag{4.15}$$

where we have introduced the shorthand notation $\vec{a} = (a_0, ..., a_n)$ and $\vec{b} = (b_0, ..., b_r)$ for $r \leq n$. The sum runs over all order preserving subsets $\vec{b}$ of $\vec{a}$, including the empty set. The function $G_{\vec{b}}(\vec{a}; z)$ is the iterated integral, with the same integrands as in $G(\vec{a})$. The difference lies in the integration contour γ_b, which only encircles all the singularities at the points $z_i = a_i \in \vec{b}$.

4.2 Symbol and co-actions of (motivic) MPLs

4.2.1 Motivic MPLs

We now have all the mathematical tools we need to construct a coaction on MPLs. However, as we mentioned at the end of section 4.1, when we work with MPLs we can find relations between them using shuffle or quasi-shuffle algebra. As it turns out however, it is very difficult to prove that there does not exist more relations. For coactions we want to work within a structure that contains all relations we know of and nothing more. For this reason we construct a set of objects called *motivic multiple polylogarithms*, which exactly contain only the relations we know of, such as shuffle and quasi-shuffle, but also relations relating to linearity, change of variables and stokes. We denote such objects as $G^m(a_1, ..., a_n; z)$. There exists a map called the period map

$$\begin{gathered} \text{per} : \mathcal{P}^m_{\text{MPL}} \to \mathbb{C} \\ G^m(a_1, ..., a_n; z) = G(a_1, ..., a_n; z), \end{gathered} \tag{4.16}$$

which maps a motivic MPL to its non-motivic form. The objects in $\mathcal{P}^m_{\text{MPL}}$ contain any relation, which can be derived for non-motivic MPLs using shuffle, quasi-shuffle, linearity,

change of variable and stokes thoerem. As a result, there exists a motivic counterpart of eq. 4.5, where all the $G(\vec{a})$ have been replaced with $G^m(\vec{a})$.
The set of motivic MPLs are denoted as $\mathcal{P}^m_{\text{MPL}}$. Motivic MPLs do not quite have a Hopf algebra, much like their non-motivic counterpart. Instead they are a comodule of another set of objects called *de Rham multiple polylogarithms*. For this project it is not necessary to understand comodules in detail, only that motivic MPLs are comodules.

4.2.2 de Rham multiple polylogarithms

We may think of de Rham MPLs as motivic MPLs modulo its discontinuities i.e $i\pi$. We denote this algebra as $\mathcal{P}^{\partial\mathcal{R}}_{\text{MPL}}$ by factoring out $\mathbb{Q}(i\pi)$

$$\mathcal{P}^{\partial\mathcal{R}}_{\text{MPL}} = \mathcal{P}^m_{\text{MPL}}/\mathbb{Q}(i\pi^m). \tag{4.17}$$

Elements in this algebra are denoted as $G^{\partial\mathcal{R}}(a_1, ..., a_n; z)$. There exists no period map for de Rham MPLs as these would be ambiguous by terms proportional to $i\pi$. As opposed to motivic MPLs $\mathcal{P}^{\partial\mathcal{R}}_{\text{P}}$ is actually a Hopf algebra, with an associated comultiplication (eq. 4.15). This is also what we ment when we said that the algebra $\mathcal{A}$ we described in section 4.1.1 is not truly a Hopf algebra. By factoring out powers of $i\pi$ of our algebra, we can write $\mathcal{A} = \mathbb{Q}(i\pi) \otimes \mathcal{H}$, where $\mathcal{H}$ is a Hopf algebra.

4.2.3 Coaction on MPLs

We are now ready to define the concept of a coaction. We start from the general construction by considering the the linear map

$$\begin{gathered} \Delta : M \to M \otimes C, \\ v \mapsto \Delta(v) = \textstyle\sum_i v_i^{(1)} \otimes a_i^{(2)}, \\ \text{with} \\ v, v_i^{(1)} \in M, \quad a_i^{(2)} \in C, \end{gathered} \tag{4.18}$$

where C is a coassociative coalgebra (the bialgebra is an example of such a coalgebra see eq. 4.11) and M is a comodule. We require the linear map to satisfy

$$(\text{id} \otimes \Delta)\Delta(v) = (\Delta \otimes \text{id})\Delta(v). \tag{4.19}$$

A linear map of the type in eq. 4.18 that also satisfy eq. 4.19 defines a coaction of C on M. From our above discussion on motivic and de Rham MPLs, we mentioned that $\mathcal{P}^m_{\text{MPL}}$

is a comodule and $\mathcal{P}^{\partial\mathcal{R}}_{\text{MPL}}$ is a Hopf algebra (bialgebra) and therefore also a coalgebra. Thus we can apply the definition in eq. 4.18 to our (motivic) MPLs

$$\Delta_{\text{MPL}}(G^m(\vec{a}))\sum_{\vec{b}\subseteq\vec{a}} G^m(\vec{b};z)\otimes G^{\partial\mathcal{R}}_{\vec{b}}(\vec{a};z). \tag{4.20}$$

Note that this equation is very similar to eq. 4.15. Indeed the sum runs over the same set and $G^{\partial\mathcal{R}}_{\vec{b}}(\vec{a};z)$ has the same interpretation as $G_{\vec{b}}(\vec{a};z)$. The only difference is that the first tensor slot belongs to motivic MPLs $\mathcal{P}^m_{\text{MPL}}$ and the second belongs to de Rham MPLs $\mathcal{P}^{\partial\mathcal{R}}_{\text{MPL}}$.

To understand the importance of the tensorial splitting of motivic -and de Rham periods, we look at an example.

Consider the MZV $\zeta_2^2 \in H$, where H is a Hopf algebra. From the shuffle and quasi-shuffle algebra one can find the relation

$$\zeta_2^2 = \frac{5}{4}\zeta_4. \tag{4.21}$$

Using this relation we can write the coproduct of ζ_2^2 as

$$\Delta(\zeta_2^2) = \frac{5}{4}\Delta(\zeta_4) = \frac{5}{4}\left[\zeta_4\otimes 1 + 1\otimes\zeta_4\right] = \zeta_2^2\otimes 1 + 1\otimes\zeta_2^2. \tag{4.22}$$

We may also calculate the coproduct using the compatibility between multiplication and comultiplication in eq. 4.13

$$\begin{aligned}\Delta(\zeta_2\cdot\zeta_2) &= [\zeta_2\otimes 1 + 1\otimes\zeta_2]\cdot[\zeta_2\otimes 1 + 1\otimes\zeta] \\ &= \zeta_2^2\otimes 1 + 2\zeta_2\otimes\zeta_2 + 1\otimes\zeta_2.\end{aligned} \tag{4.23}$$

Immediately we see a contradiction between the two results and as such a coproduct on ζ_2^2 is prohibited. In fact this turns out to be true for all ζ_n with positive even n. The trouble lies in the fact that for MZVs with even n, you can find relations like the one in eq. 4.21, which leads to a contradiction between the coproduct and compatibility statement. The coaction in a way fixes this issue. Acting with the coaction in eq. 4.20 on an arbitrary ζ_n we obtain

$$\Delta_{\text{MPL}}(\zeta_n^m) = \zeta_n^m\otimes 1 + 1\otimes\zeta_n^{\partial\mathcal{R}}. \tag{4.24}$$

It can be shown that all ζ_n (still positive even n) are rational numbers times a positive power of $i\pi$. From the definition of $\mathcal{P}^{\partial\mathcal{R}}_{\text{MPL}}$ it follows then that $\zeta_n^{\partial\mathcal{R}}$ is zero. Thus for all even positive n the coaction on ζ_n reduces to

$$\Delta(\zeta_n) = \zeta_n^m\otimes 1, \qquad n = 2,4,6,8... \tag{4.25}$$

This also means that the coaction acts trivially on $(i\pi)^m$

$$\Delta_{\text{MPL}}((i\pi)^m) = (i\pi)^m \otimes 1. \tag{4.26}$$

In fact de Rham MPLs $\mathcal{P}^{\partial\mathcal{R}}_{\text{MPL}}$ are defined exactly to mod out all discontinuities, defined across a branch cut

$$\text{Disc}_z f(z) = f(z + i\varepsilon) - f(z - i\varepsilon), \tag{4.27}$$

which for MPLs are powers of $i\pi$. As a consequence, all discontinuities only act on the first tensor slot

$$\Delta\text{Disc} = (\text{Disc} \otimes \text{id})\Delta. \tag{4.28}$$

We note that this is a general result for a coaction, which we emphasize by leaving out the usual "MPL" subscript.

4.2.4 Symbol operator

Another useful operator on MPLs is the symbol operator. The symbol is defined recursively for a transcendental function F_w with trancendental weight w. Any differential of a MPL F_w can be written as

$$dF_w = \sum_i F_{w-1,i} dR_i, \tag{4.29}$$

where the $F_{w-1,i}$ are trancendental functions of weight $w-1$ and R_i are algebraic functions. The symbol on F_w is then defined as

$$\mathcal{S}(F_w) = \sum_i \mathcal{S}(F_{w-1,i}) \otimes R_i, \tag{4.30}$$

with the definition that $\mathcal{S}(F_0) = F_0$. The recursive nature of eq. 4.29 make strict claims about which functions the symbol can and can not be defined for. Any function that, for instance, satisfy an inhomogeneous version of the differential equation in eq. 4.29, can not have a symbol defined for them, due to the simple fact that the recursiveness would never terminate in such a case.
MPLs do satisfy a homogeneous equation of the type in eq. 4.29, which explicitly has the form

$$dG(a_1, ..., a_n; z) = \sum_{i=1}^{n} G(a_1, ..., \hat{a}_i, ..., a_n; z) d\log \frac{a_{i-1} - a_i}{a_{i+1} - a_i}, \tag{4.31}$$

where $\hat{a}_i$ indicates the argument that is absent. We also set $a_{n+1} = 0$ and $a_0 = z$. It can easily be seen that the symbol of an MPL has the form

$$\mathcal{S}(G(a_1, ..., a_n; z)) = \sum_{i=0}^{n} \mathcal{S}(G(a_1, ..., \hat{a}_i, ..., a_n; z)) \otimes \frac{a_{i-1} - a_i}{a_{i+1} - a_i}, \tag{4.32}$$

where the recursion stops at $\mathcal{S}(G(;z)) = \mathcal{S}(1) = 1$. Let us consider the explicit example, in which we have the differential equation $dI = AI$, with $I = (G(a_1, a_2; z), G(a_1; z), G(a_2; z), G(;z))$, and by convention we set $G(;z) = 1$. This vector satisfy the differential equation with

$$A = \begin{pmatrix} 0 & 0 & \mathrm{d}\log\frac{a_1-z}{a_1-a_2} & 0 \\ 0 & 0 & 0 & \mathrm{d}\log\frac{a_1-z}{a_1} \\ 0 & 0 & 0 & \mathrm{d}\log\frac{a_2-z}{a_1} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.33}$$

Using this matrix, we can very quickly calculate $\mathcal{S}(G(a_1, a_2; z))$, simply by reading off the entries in A

$$\begin{aligned} \mathcal{S}(I_1) = \mathcal{S}(G(a_1, a_2; z)) &= \sum_i \mathcal{S}(I_i) \otimes A_{i,1} \\ &= \sum_i \left(\sum_j \mathcal{S}(I_j) \otimes A_{j,i} \right) \otimes A_{i,1} \\ &= \sum_i \left(\sum_j A_{j,i} \right) \otimes A_{i,1} \\ &= \mathrm{d}\log\frac{a_2 - z}{a_2} \otimes \mathrm{d}\log\frac{a_1 - z}{a_1 - a_2}. \end{aligned} \tag{4.34}$$

In the second equality we simply inserted the definition of the symbol in a recursive manner. In a more general manner, one would need to keep inserting the symbol, until the recursion ends. This provides one with n sums $\mathcal{S}(G(a_1, ..., a_n; z))$. In the third equality we used the fact that, we had reached the end of our symbol i.e, $\mathcal{S}(G(;z)) = \mathcal{S}(1) = 1$. In a more general case for $G(\vec{a}; z)$, the vector I has the form

$$I = \left(I_{\vec{b}}\right)_{\vec{b}\subseteq\vec{a}} = \left(\int_0^z \omega_{\vec{b}} \right)_{\vec{b}\subseteq\vec{a}}, \tag{4.35}$$

where $\vec{b}$ is once again all preserving subsets of $\vec{a}$.
The symbol has various properties. In particular it is linear and maps the product of MPLs to the shuffle product

$$\mathcal{S}(a \cdot b) = \mathcal{S}(a) \sqcup\!\sqcup \mathcal{S}(b). \tag{4.36}$$

Both the symbol and coaction gives relations between MPLs. The symbol can however be seen as a more coarse alternative to the coaction, as it also maps all constants to zero.

4.3 Constructing a generalized co-action

Constructing a coaction for more generalized periods than MPLs, begins at the definition of the *motivic coaction*. The motivic coaction provides a general framework, where it is possible to define a coaction on arbitrary periods. We define the motivic coaction Δ^m as

$$\Delta^m([\gamma,\omega]) = \sum_i [\gamma,\omega_i]\otimes[\omega_i,\omega]. \tag{4.37}$$

The pair $[\gamma,\omega]$ should be thought of a motivic period, where γ is an integration contour and ω is a differential form that can be integrated over γ. Thus, the pair is not itself an integral, but rather the *motivic avatar* of the integral $\int_\gamma \omega$. All the relations that this integral satisfy, such as linearity in both contour and integrand also apply for its motivic counterpart. It is possible to define an injective[1] map called the *period map* in the same fashion as we did for MPLs

$$per : [\gamma,\omega] \mapsto \int_\gamma \omega. \tag{4.38}$$

Since the map is injective, no information is lost and we will therefore make no distinction between the motivic period $[\gamma,\omega]$ and the integral $\int_\gamma \omega$.
The second pair of forms $[\omega_i,\omega]$ in 4.37 are a *de Rham period* of differential forms. Unlike the motivic period, a de Rham period can not strictly be interpreted as an integral. The reason being that the integrals we consider are usually made up of multivalued functions and so the integral will have branch cuts. Suddenly you will get different answers depending on the number of times you cross the branch cuts and in which direction. We want the integrals we consider to be independent of the contour one integrates over. Given the way the coaction acts on discontinuities (see eq. 4.28) the second tensor slot should be invariant under such deformations of the contour. Thus, it is only possible to interpret it as an ordinary integral up to its branch cuts.
In the MPL case we saw that we could simply define $\mathcal{P}^{\partial\mathcal{R}}_{\text{MPL}}$ as being the motivic periods modulo their discontinuities ($i\pi$). In more general cases, not all discontinuities will be proportional to powers of $i\pi$. Specifically for elliptical integrals their discontinuities are more complicated

$$\text{Disc}_\lambda K(\lambda) = K(\lambda+i\varepsilon) - K(\lambda-i\epsilon) = \theta(\lambda-1)\frac{2}{\sqrt{\lambda}}K(1-1/\lambda), \tag{4.39}$$

where $K(\lambda)$ is the complete elliptic integral of the first kind and $\theta(\lambda-1)$ is the heavyside step function. It is therefore not immediately clear how one can make explicitly calculations

[1] A folklore conjecture that we assume to be true

using eq. 4.37.
To get around this problem, we are inspired by the construction of Broedel et. al [12], in which symbols are associated to de Rham periods. The idea is then, that you can calculate the symbols and use these to write the coaction.
We saw in section 4.3 how it is only possible to define symbols on functions that satisfy a first order differential equation with no homogeneous part. We will therefore restrict ourselves to a set of integrals $\underline{I} = \left(\int_\gamma \xi_1, ..., \int_\gamma \xi_n\right)$, that satisfy a differential equation of the type

$$d\underline{I}(x) = A\underline{I}(x), \tag{4.40}$$

where A is a $n \times n$ matrix containing one-forms. The differential equation has non-trivial solutions only in the case where $dA = A \wedge A$, which we will assume is satisfied going forward. This differential equation is also special, when we assume that A is a nilpotent matrix. A nilpotent matrix A is defined as a matrix, which for some positive integer n called the index, has the property that $A^n = 0$. If this condition is satisfied, it is possible to find a basis, in which A is strictly upper triangular. We will assume that we are working in such a basis in eq. 4.40. A differential equation with a nilpotent matrix is called a unipotent differential equation and so are the basis elements of $\underline{I}$ called unipotent. Note, that it is due to the nilpotent properties, that we are able to create something that has recursive properties, as we saw the symbol should have in eq. 4.30. Finally, we also define the concept of a unipotent matrix as the sum of the identity matrix and a nilpotent matrix.
We now show how it is possible to define the notion of a symbol on pairs of differential forms $[\xi_i, \xi_j]$ that enter the basis I. We consider the unipotent matrix

$$T_A = 1 + [A]^R + [A|A]^R + [A|A|A]^R + ..., \tag{4.41}$$

which, due to the nilpotent properties of A, converge at some final matrix. The matrix multiplication corresponds to ordinary multiplication combined with the concatenation of words formed out of the one-forms that enter into A. The superscript R denotes the operation that reverses words:

$$[\omega_0|...|\omega_k]^R = [\omega_k|...|\omega_0]. \tag{4.42}$$

This way of writing words is called the bar construction, which we denote [12]

$$[\omega_0|\omega_1|...|\omega_k] = \omega_0 \otimes \omega_1 \otimes ... \otimes \omega_k. \tag{4.43}$$

The bar construction comes with an extra set of instructions on how to integrate over each form $\omega_i = dtf_i(t))$

$$\int_\gamma [\omega_0|\omega_1|...|\omega_n] = \int_{0\leq t_1\leq t_2,...,\leq t_n\leq 1} dt_0 f_1(t_0)...dt_n f_n(t_n), \tag{4.44}$$

where $t \in [0,1]$ is a local coordinate parametrizing the path γ. Finally we define the symbol of a pair of periods $[\xi_i, \xi_j]$ as

$$\mathcal{S}([\xi_i, \xi_j]) = \langle \xi_i | T_A | \xi_j \rangle = (T_A)_{ji}. \tag{4.45}$$

To see how all this plays out, an example is in order.
Consider the nilpotent matrix of one-forms ω_0, ω_1

$$A = \begin{pmatrix} 0 & \omega_0 & 0 \\ 0 & 0 & \omega_1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.46}$$

That solves the unipotent differential equation with $I = (\xi_1, \xi_2, \xi_3)$. Then we have

$$[A|A]^R = \begin{pmatrix} 0 & 0 & [\omega_1|\omega_0] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [A|A|A]^R = 0, \tag{4.47}$$

so that

$$T_A = \begin{pmatrix} 1 & [\omega_0] & [\omega_1|\omega_0] \\ 0 & 1 & [\omega_1] \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.48}$$

We now find that the symbol on the pair $[\xi_3, \xi_1]$ is

$$\mathcal{S}([\xi_3, \xi_1]) = \langle \xi_3 | T_A | \xi_1 \rangle = [\omega_1|\omega_0]. \tag{4.49}$$

From the definition of the symbol in eq. 4.45, it appears as though the symbol strongly depends on the specific choice of basis. Since the symbol gives relations between functions we would expect the symbol to be invariant towards such differences in basis choice. This indeed turns out to be true under the circumstances that the words satisfy certain relations among one another. Given that the forms ω_i's are closed, meaning that $\mathrm{d}\omega_i = 0$, these relations read

$$\begin{aligned} [\omega_0|...|\omega_k|df|\omega_{k+1}|...|\omega_n] &= [\omega_0|...|\omega_k|f\omega_{k+1}|...|\omega_n] - [\omega_0|...|\omega_k f|\omega_{k+1}|...|\omega_n] \\ [df|\omega_0|...|\omega_n] &= [f\omega_0...|\omega_n] - f[\omega_0...|\omega_n] \\ [\omega_0...|\omega_n|df] &= f[\omega_0...|\omega_n] - [f\omega_0...|\omega_n f]. \end{aligned} \tag{4.50}$$

These relations are simply integration by parts relations at the level of the symbol.
To illustrate this, an example is once again in order. Consider the same matrix A as before, but now we make the basis change

$$I' = MI, \qquad M = \begin{pmatrix} 1 & f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.51}$$

for some algebraic function f. In this basis we get the differential equation

$$dI' = A'I', \qquad M = \begin{pmatrix} 0 & \omega_0 + df & f\omega_1 \\ 0 & 0 & \omega_1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.52}$$

I' is still satisfy an unipotent differential equation and as such we calculate eq. 4.41

$$T_{A'} = \begin{pmatrix} 1 & [\omega_0 + df] & [\omega_1|\omega_0 + df] + [f\omega_1] \\ 0 & 1 & [\omega_1] \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.53}$$

We then obtain

$$\begin{aligned} \mathcal{S}([\xi_3, \xi_1]) &= \langle \xi_3 | T_{A'} | \xi_1 \rangle \\ &= \langle \xi_3' | T_{A'} | \xi_1' \rangle - f \langle \xi_3 | T_{A'} | \xi_2' \rangle \\ &= [\omega_1|\omega_0] + [\omega_1|df] + [f\omega_1] - f[\omega_1]. \end{aligned} \tag{4.54}$$

Comparing this with what we found in eq. 4.49 we see that the two expressions agree when they satisfy the relations eq. 4.50.

$$[f\omega_1] - f[\omega_1] = -[\omega_1|df]. \tag{4.55}$$

We have now solved our initial problem of working with pairs of forms $[\omega_i, \omega]$. The symbol operator $\mathcal{S}$ that takes the pair of forms and turn them into something with which we can make explicit calculations with. We are now ready to define a map Δ by composing the motivic coaction with the symbol operator

$$\Delta = (\mathrm{id} \otimes \mathcal{S})\Delta^m. \tag{4.56}$$

Letting it act on an integral $I_a = \int_\gamma \omega_a$ which is part of a larger unipotent basis $I_a \in \underline{I} = (\int_\gamma \omega_0, ..., \int_\gamma \omega_n)$, we achieve the explicit result

$$\Delta(I_a) = \sum_c I_c \otimes \mathcal{S}([\omega_c, \omega_a]). \tag{4.57}$$

One can check that such a map satisfies all the axioms of a coaction. It is worth noting that, since the symbol map is only defined for unipotent quantities, so is the coaction in eq. 4.56. The motivic coaction in eq. 4.37, is however defined for arbitrary motivic periods. MPLs are unipotent quantities and as a result, our coaction in eq. 4.56 can recover the coaction used on MPLs.

One can show that in general, any motivic period can be decomposed into a linear combination of products of a *unipotent* u_I and *semi-simple* period s_i

$$x = \sum_i s_i u_i. \tag{4.58}$$

The semi-simple part is the part we want the coaction to not care about. We want the coaction to treat these objects as constants. As such we define the coaction Δ to act trivially on semi-simple objects

$$\Delta(s_i) = s_i \otimes 1. \tag{4.59}$$

The unipotent part contain the quantities that we do care about and we want the coaction to act non-trivially on these elements. The coaction will act on unipotent elements as

$$\Delta(u_i) = u_i \otimes 1 + 1 \otimes [du_i]. \tag{4.60}$$

The coaction preserves all the properties that we know from MPLs, such as compatibility between multiplication and coaction and discontinuities

$$\begin{aligned} \Delta(x_1 x_2) &= \Delta(x_1)\Delta(x_2) \\ \Delta(\text{Disc } x) &= (\text{Disc } x \otimes 1)\Delta(x). \end{aligned} \tag{4.61}$$

The coaction acting on a derivative can also be constructed by decomposing an element into semi-simple and unipotent quantities

$$\Delta(\partial x) = \sum_i \left(\Delta(u_i)\partial_z s_i + s_i(id \otimes \partial_z)\Delta(u_i)\right). \tag{4.62}$$

We have seen that the key to building a symbol and coaction operator is finding a set of master integrals that satisfy a unipotent differential equation as in eq. 4.40. The next step will therefore be to explore the differential equations that are satisfied by our master integrals of interest in eq. 3.6.

5 Differential Systems

In this section we review [6], which covers the topic of differential equations that are satisfied by our master integrals. As we saw in the previous section, to build a symbol and coaction operator, we need our master integrals to satisfy a unipotent differential equation. We will introduce the Wronskian, which can help us express our master integrals in a particularly beneficial form. On the topic of differential equations, we also explore a particular type of differential equations called the *Picard-Fuchs* differential equations. These differential equations are satisfied by the maximal cuts. The notion of maximal cuts was briefly covered in section 3, but in this section we will explain it in more detail. Finally we will cover a set of bilinear relations between the maximal cuts, known as Griffiths Transversality conditions

5.1 Gauss-Manin Differential equation

We collect all the master integrals into a vector $\underline{I}(\underline{z};\varepsilon) = (I_1(\underline{z};\varepsilon), ..., I_M(\underline{z};\varepsilon))$, which satisfy a linear differential equation of the type

$$\mathrm{d}\underline{I}(\underline{z};\varepsilon) = \mathbf{A}(\underline{z};\varepsilon)\underline{I}(\underline{z};\varepsilon), \tag{5.1}$$

where $d = \sum_{i=1}^{N} dz_i \partial_{z_i}$ is the total differential and $\mathbf{A}(\underline{z};\varepsilon)$ is a matrix of one-forms. This type of differential equation is called a *Gauss-Manin* differential equation. The matrix $\mathbf{A}(\underline{z})$ is called the *Gauss-Manin connection* made up of linear combinations of one-forms.
The basis $\underline{I}(\underline{z};\varepsilon)$ is by no means unique. In fact, for any invertible matrix $\mathbf{M}(\underline{z},\varepsilon)$, a change of basis of the type

$$\underline{I}(\underline{z},\varepsilon) = \mathbf{M}(\underline{z},\varepsilon)\underline{J}(\underline{z},\varepsilon), \tag{5.2}$$

will also be a solution to eq. (5.1).
It can be shown that it is always possible to change basis to a ε-regular basis defined as a basis of master integrals $\underline{J}(\underline{z};\varepsilon)$ that are finite and non-zero as $\varepsilon \to 0$.
If we arrange our master integrals such that $\underline{J}(\underline{z};\varepsilon) = \left(\underline{J}_1(\underline{z};\varepsilon)^T, ..., \underline{J}_s(\underline{z};\varepsilon)^T\right)$, where the elements of $\underline{J_r}(\underline{z};\varepsilon)$ are of the same sector. The master integrals in each sector instead satisfy an inhomogeneous differential equation

$$\mathrm{d}\underline{J}_r(\underline{z};\varepsilon) = \mathbf{B}_r(\underline{z};\varepsilon)\underline{J}_r(\underline{z};\varepsilon) + \underline{N}_r(\underline{z};\varepsilon), \tag{5.3}$$

where the inhomogeneity collects terms from lower orders. The homogeneous version of eq. (5.3), is the differential equation satisfied by the maximal cuts of $\underline{J}_r(\underline{z};\varepsilon)$, found by setting $\underline{N}_r(\underline{z};\varepsilon) = 0$.

For the the equal-mass case, the inhomogeneity in sector $(1, ..., 1)$ has the form

$$\underline{N}_l(z;\varepsilon) = \left(0, ..., 0, (-1)^{l+1}(l+1)! \frac{z}{z^l \prod_{k\in\Delta^{(l)}}(1-kz)} \frac{\Gamma(1+\varepsilon)^l}{\Gamma(1+l\varepsilon)}\right)^T, \tag{5.4}$$

with

$$\Delta^{(l)} = \bigcup_{j=0}^{\lceil \frac{l-1}{2} \rceil} \left[(l+1-2j)^2\right]. \tag{5.5}$$

The inhomogeneity is regular at $\varepsilon \to 0$. We will in section 5.4 show how to obtain the solution matrix $\mathbf{B}_r(z;\varepsilon)$ and the inhomogenuity $N_l(z;\varepsilon)$ in the equal-mass case and in section 7.1 how to obtain them for generic-mass.

We will spend the remainder of this section talking about the homogenous verison of eq. 5.3. This homogenous differential equation is satisfied by the maximal cuts of $\underline{J}_r(\underline{z};\varepsilon)$. In the sections to come, we will characterize the maximal cuts in more details, for now it is sufficient to know that a general solution to the differential equation can be cast in the form of an $M_r \times M_r$ matrix $\mathbf{W}_r(\underline{z})$ called the Wronskian

$$d\mathbf{W}_r(\underline{z}) = \mathbf{B}_r(\underline{z})\mathbf{W}_r(\underline{z}), \tag{5.6}$$

where $\mathbf{B}_r(\underline{z}) = \lim_{\varepsilon\to 0} \mathbf{B}_r(\underline{z};\varepsilon)$. We can use the wronskian to simplify our differential equation by making the change of basis

$$L_r(\underline{z};\varepsilon) = \mathbf{W}_r(z) J_r(\underline{z};\varepsilon), \tag{5.7}$$

which leads us to the important equation

$$dL_r(z;\varepsilon) = \mathbf{W}_r^{-1}(\underline{z}) \left[\mathbf{B}_r(\underline{z};\varepsilon) - \mathbf{B}_r(\underline{z})\right] \mathbf{W}_r(\underline{z}) + \mathbf{W}_r^{-1}(\underline{z}) N_r(\underline{z};\varepsilon). \tag{5.8}$$

At first glance, one may think that we have merely made things more complicated, but since everything is ε-regular the limit $\lim_{\varepsilon\to 0} \mathbf{B}_r(\underline{z};\varepsilon) = \mathbf{B}_r(\underline{z})$ provides us with a much simpler result

$$dL_r(\underline{z}) = \mathbf{W}_r^{-1}(\underline{z}) N_r(\underline{z}). \tag{5.9}$$

The simplification works especially well in the equal-mass case. The vector $N_r(z)$ in eq. 5.4 has only one non-zero element. This means that $dL_r(z)$ will only be the rightmost column of $\mathbf{W}_r^{-1}$ times the non-zero value in $N_r(z)$.

5.2 Picard Fuchs Differential equations

Another type of differential equation that we will make frequent use of, is the *Picard-Fuchs differential equation*. This is a higher order differential equation satisfied by each master integral.

$$\mathcal{L}_{k,\varepsilon} I_k(\underline{z}; D) = R_k(\underline{z}; \varepsilon). \tag{5.10}$$

Where the inhomogenity $R_k(\underline{z}; \varepsilon)$ is related to master integrals of lower sectors. The differential operator $\mathcal{L}_{k,\varepsilon}$ has the general form

$$\mathcal{L}_{k,\varepsilon} = \sum_{j_1,...,j_m \geq 0} Q_{k,j_1...j_m}(\underline{z}; \varepsilon) \partial_{z_1}^{j_1} ... \partial_{z_m}^{j_m}, \tag{5.11}$$

where $Q_{k,j_1...j_m}(\underline{z}; \varepsilon)$ are polynomials in both $\underline{z}$ and ε. The differential operator is defined such that it will annihilate the maximal cuts in the limit $\varepsilon \to 0$, which we will assume exists. The one parameter Picard-Fuchs differential equation of the form

$$\mathcal{L} f(z) = 0 \text{ with } \mathcal{L} = B_n(z)\partial_z^n + B_{n-1}(z)\partial_z^{n-1} + ...B_0(z), \tag{5.12}$$

has n independent solutions f_i for $1 \leq i < n$ that spans the solution space $\mathrm{Sol}(\mathcal{L})$. Around the singularities of the solution space, a variety of series expansions are allowed depending on the type of singularity. We distinguish between three types of points; *Ordinary*, *regular singular* and *irregular singular points*. The differential equation has an ordinary point $z = z_0$ if its coefficients $p_i(z) = B_i(z)/B_n(z)$ are analytic in a neighborhood of z_0 for all $0 \leq i < n$. A point z_0 is called regular singular if $(z - z_0)^{n-i} p_i(z)$ are analytic in a neighbourhood of z_0. Finally we define a point z_0 that is neither ordinary nor regular singular as an irregular singular point.

Feynman integrals in general are expected to only have regular singularities. Moreover the singularities $z_0 \neq \infty$ are zeroes of the discriminant of $\mathcal{L}_{k,\varepsilon}$, $\mathrm{Disc}(\mathcal{L})|_{z=z_0} = B_n(z_0)$. For $z_0 = \infty$ one makes the substitution $t = 1/z$ and makes the analysis around $t = 0$.

To find the n independent local solutions to eq. (5.12), one starts at the indicial equation

$$B_n(0)\alpha^n + B_{n-1}(0)\alpha^{n-1} + ... + B_0(0)\alpha = 0. \tag{5.13}$$

Note that we have assumed that $z_0 = 0$, which can be done without loss of generality as there is always the option to perform an appropriate substitutions of the type $z \to z' = z - z_0$ or $z' = 1/z$. The structure of the solutions space will depend on the type of singularity z_0. If z_0 is an ordinary point, there will be n different solutions $\alpha_1, ..., \alpha_n$ to eq. (5.13). The solution will be a power series type solution around z_0:

$$z^{\alpha_i} \Sigma_{i,0}(z) = z^{\alpha_i} \sum_{k=0}^{\infty} a_{i,k} z^k, \tag{5.14}$$

where $\Sigma_{i,0}(z)$ has a non-vanishing radius of convergence and is normalized as $\Sigma_{i,0}(0)$=1. The coefficients $a_{i,k}$ can be calculated by letting $\mathcal{L}$ act on eq. 5.14, as $\mathcal{L}$ should annihilate any element of the solution space.
If z_0 is a regular-singular point, one again finds that there are n numbers of independent solutions, however this time some of them may appear with multiplicities. We sort them as $(\alpha_1, ..., \alpha_1, \alpha_2, ..., \alpha_2, ..., \alpha_m, ..., \alpha_m)$. If a given indicial α_i has the property of $\alpha_i - \alpha_j \notin \mathbb{Z}$, for distinct i and j the solution space is spanned by a power series type solution as in eq. 5.14. Let there be r of such solutions. The remaining $n - r$ solutions, appear with multiplicity s. The solution space in this case is spanned by $s - 1$ logarithmic solutions containing up to s powers of $\log(z)$. These solutions are spanned by

$$z^{\alpha_i} \sum_{j=0}^{k} \frac{1}{(k-j)!} \log^{k-j}(z) \Sigma_{i,j}(z), \tag{5.15}$$

where $\Sigma_{i,j}(z)$ is again a power series type solution that is convergent up to the nearest singularity and normalized as $\Sigma_{i,j}(z) = \delta_{j,0} + \mathcal{O}(z)$ for $j \geq 0$.

In some special cases you find that there exists a singular point in which all indices are equal. In this case the solution space is characterized by an increasingly higher order of logarithmic solutions as in eq. 5.15. Such a point is called a *point of maximally unipotent monodromy* or 'MUM'-point for short and the associated basis that spans the solution space is called the Frobenius basis $\varpi_0(z), ...\varpi_{n-1}(z)$

5.3 Maximal Cuts series expansion for equal-mass banana integral

As mentioned in section 3.3, there exists only closed-form expressions for the maximal cuts for $l \leq 3$. It is, however, possible to develope an infinite series expression for the remaining higher loop orders, which will be necessary if they are to be investigated.
We start out by looking at the Picard-Fuchs differential operator in the equal-mass case. This operator is supposed to annihilate the maximal cuts of $J_{l,0}(z;0)$ (eq. 3.6). The differential operator $\mathcal{L}_l$ has regular singular points at

$$z \in \{0, \infty\} \cup \bigcup_{j=0}^{\lceil \frac{l-1}{2} \rceil} \left\{ \frac{1}{(l+1-2j)^2} \right\}. \tag{5.16}$$

For all loop orders l, the equal-mass Picard-Fuchs operators has a single MUM-point at $z = 0$ with indicials $\alpha_i = 1$. The nearest singularity to $z = 0$ is at $z = 1/(l+1)^2$, which is

one of the thresholds where the propagators become on-shell.
The solution space of the $z = 0$ MUM-point is spanned by the Frobenius basis given by

$$\varpi_{l,k}(z) = \sum_{j=0}^{k} \frac{1}{(k-j)!} \log^{k-j}(z) \Sigma_{l,j}(z), \tag{5.17}$$

where $\Sigma_{l,k}(z)$ is a power series, normalized such that $\Sigma_{l,k}(z) = \delta_{k,0} z + \mathcal{O}(z^2)$. For $k = 0$ we have

$$\varpi_{l,0} = \Sigma_{l,0}(z) = \sum_{k_1,\ldots,k_{l+1} \geq 0} \left(\frac{|k|!}{k_1! k_2! \ldots k_{l+1}!} \right)^2 z^{|k|+1}, \tag{5.18}$$

where $|k| = k_1 + \ldots + k_{l+1}$. In the following sections it will be handy to consider the vector of elements of the Frobenius basis $\Pi(z) = (\varpi_0(z), \ldots, \varpi_{n-1}(z))$. To get explicit expressions for the remaining $k > 0$ solutions, one makes use of the normalization of $\Sigma_{l,k}(z)$ and the fact that the Picard-Fuchs operator annihilates every element in the Frobenius basis $\mathcal{L}\varpi_{l,k}(z) = 0$. Doing so, one is able to determine the coefficient of $\Sigma_{l,k}(z)$ order by order.

5.4 Connection between Picard-Fuchs and Gauss-Manin differential equation

To solve differential systems satisfied by master integrals, it is usually easier to first consider the maximal cuts and then later extend this to the uncut version. The reason being that the maximal cuts are well understood and it can be shown that they always satisfy the differential equation

$$d\mathbf{W}_l(z) = \mathbf{B}_l(z) \mathbf{W}_l(z), \tag{5.19}$$

where $\mathbf{W}_l(z)$ is an $l \times l$-matrix called the wronskian, which can be chosen to be

$$\mathbf{W}_l(z) = \begin{pmatrix} \varpi_{l,0}(z) & \varpi_{l,1}(z) & \ldots & \varpi_{l,l-1}(z) \\ \partial_z \varpi_{l,0}(z) & \partial_z \varpi_{l,1}(z) & \ldots & \partial_z \varpi_{l,l-1}(z) \\ \vdots & \vdots & \vdots & \vdots \\ \partial_z^{l-1} \varpi_{l,0}(z) & \partial_z^{l-1} \varpi_{l,1}(z) & \ldots & \partial_z^{l-1} \varpi_{l,l-1}(z) \end{pmatrix}. \tag{5.20}$$

Due to the increasing order of derivatives, the first $l-1$ rows of $\mathbf{B}_l(z)$ are easily found as they merely need to relate a $\partial_z^{n+1} \varpi_{l,i}(z)$ to a $\partial_z^n \varpi_{l,i}(z)$. This is achieved simply by having the first $l-1$ rows have a diagonal line of ones offset by one column to the right. To get the entries of the final row, one has to find expressions for an l-order differential acting on each basis element of the Frobenius basis. To find these expressions one makes use of the fact that the Picard-Fuchs operator annihilates the Frobenius basis $\mathcal{L}_l \varpi_{l,k} = 0$, for all

$0 \leq k \leq l-1$. Writing the differential operator as $\mathcal{L}_l = \sum_{i=0}^{l} B_{l,i}\partial_z^i$, one can solve for the highest order differential, from which you get the expression

$$\partial_z^l \varpi_{l,k}(z) = -\frac{B_{l,l-1}(z)}{B_{l,l}(z)}\partial_z^{l-1}\varpi_{l,k}(z) - \frac{B_{l,l-2}(z)}{B_{l,l}(z)}\partial_z^{l-2}\varpi_{l,k}(z) - ... - \frac{B_{l,0}(z)}{B_{l,l}(z)}\varpi_{l,k}(z). \quad (5.21)$$

These are then the desired coefficients that enter the final row and we can write the solution matrix as

$$\mathbf{B}_l(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{B_{l,0}(z)}{B_{l,l}(z)} & -\frac{B_{l,1}(z)}{B_{l,l}(z)} & -\frac{B_{l,2}(z)}{B_{l,l}(z)} & \cdots & -\frac{B_{l,l-1}(z)}{B_{l,l}(z)} \end{pmatrix}. \quad (5.22)$$

As stated before, once the differential equation that satisfies the maximal cuts has been found, it is easier to generalize to the uncut case. To do so we notice that in the uncut case, the Picard-Fuchs operator turns into an inhomogeneous differential equation as in eq. 5.10; $\mathcal{L}_l J_{l,1}(z) = S_l(z)$. Following the same analysis as in the cut case, it is then clear that the inhomogeneous part would become

$$N_l(z) = \left(0, ..., 0, \frac{S_l(z)}{B_{l,l}(z)}\right). \quad (5.23)$$

5.5 Multivariable Picard-Fuchs differential equation

It is possible to extend the one parameter Picard-Fuchs differential equation to a multivariable version. In this case we consider a set of differential operators $\mathcal{D} = \{\mathcal{L}_1, ..., \mathcal{L}_s\}$ and we are looking for functions $f(\underline{z})$ that are simultaneously annihilated by all the differential operators in $\mathcal{D}$. We refer to the set of linearly independent solutions as the solution space of $\mathcal{D}$

$$\text{Sol}(\mathcal{D}) = \{f(\underline{z}) | \mathcal{L}_i f(\underline{z}) = 0 \text{ for all } \mathcal{L}_i \in \{\mathcal{D}\}. \quad (5.24)$$

The choice of $\mathcal{D}$ is not necessarily unique, in the sense that two different sets of operators $\mathcal{D} = \{\mathcal{L}_1, ..., \mathcal{L}_r\}$ and $\mathcal{D}' = \{\mathcal{L}'_1, ..., \mathcal{L}'_{r'}\}$ might generate the exact same solution space. Even the lengths and order of differential between the two sets can differ, while still having the same solutions space. Sometimes a single, well constructed, differential operator can be enough to find all solutions. We will use this freedom in choosing a set of operators, when we derive a symbol and coaction on the generic-mass master integrals.

In the multivariable case, it is again possible to describe a basis of local solutions. Close

to an ordinary point, the solution space is expressed in terms of a generalized power series solution

$$\left(\prod_{i=1}^{m} z_i^{\alpha_i}\right) \sum_{j_1,...,j_m \geq 0} a_{j_1,...,j_m} z_1^{j_1} ... z_m^{j_m}, \tag{5.25}$$

with indicials $(\alpha, ..., \alpha_m)$.
Close to singular points, the solution space can contain multivariable logarithmic solution

$$\left(\prod_{i=1}^{m} z_i^{\alpha_i}\right) \sum_{\substack{j_1,...,j_m \geq 0 \\ k_1,...,k_m \geq 0}} a_{j_1,...,j_m,k_1,...,k_m} \log^{k_1}(z_1) ... \log^{k_m}(z_m) z_1^{j_1} ... z_m^{j_m}, \tag{5.26}$$

where $k_1, ..., k_m$ depends on the multiplicity and differences of the local indicials.
For the multivariable case there again exists a Frobenius basis around a MUM-point where all the indicials are equal. One such MUM-point is located at $\underline{z}_0$ by which $\underline{z}_{0,i} = 0$ for all i. At this point there exists one power series type solution[1]

$$\varpi_0(\underline{z}) = \sum_{j_1,...,j_{l+1} \geq 0} \left(\frac{|j|!}{j_1! ... j_{l+1}!}\right)^2 \prod_{n=1}^{l+1} z_n^{j_n}. \tag{5.27}$$

It is possible to define $l+1$ differential operators $\mathcal{D} = \{\mathcal{L}_1, .., \mathcal{L}_{l+1}\}$ that simultaneously annihilate eq. 5.27. The rest of the solutions have, much like the equal-mass case, an increase in orders of logarithms. There is however some degeneracy in the order of logarithms. For instance $l \geq 2$ there exists $l+1$ single logarithmic solutions, which can be normalized as

$$\varpi_1^k(\underline{z}) = \log(z_k) + \mathcal{O}(\underline{z}), \tag{5.28}$$

for each $k = 1, ..., l+1$. Solutions of higher logarithmic degree $r \geq 2$ are normalized as

$$\varpi_r^k(\underline{z}) = \prod_{i=1}^{r} \log(z_{j_{k,i}}) + \mathcal{O}(\underline{z}) \quad \text{for } \{j_{k,1}, ..., j_{k,r}\} \in T_r^{(l+1)}, \tag{5.29}$$

where $k = 1, ..., \binom{l+1}{l-(r-1)}$. The set $\underline{J}_{k,r} = \{j_{k,1}, ..., j_{k,r}\}$ is a subset of the object $T_r^{(l+1)}$ that contain all the subsets of $\{1, ..., l+1\}$ of length r. More precisely $\underline{J}_{k,r}$ is the k'th element of $T_r^{(l+1)}$.
By considering that $\mathcal{L}_i \varpi_r^k(\underline{z}) = 0$, it should be possible to derive the coefficients for arbitrary orders of $\underline{z}$, much the same way as it was explained in the equal mass Frobenius basis.

5.6 Griffiths Relations

We will spend the following section investigating a type of bilinear relation between maximal cuts known as Griffiths transversality. These relations will be especially important to us, due

to the fact that they can greatly reduce the polynomial complexity of the inverse Wronskian $\mathbf{W}_l(z)$ used in eq. 5.9. In general the inverse Wronskian of size $M_l \times M_l$ is going to have polynomials of order $l-1$ in elements of $\mathbf{W}_l(z)$, but as we will see, this number can be greatly reduced. Griffiths transversality will give us a series of relations between maximal cuts and we will refere to these collectively as the Griffiths relations. The maximal cuts $\underline{\Pi}_l(z)$ in the equal-mass case satisfy

$$\underline{\Pi}_l(z)\Sigma_l\partial_z^k\underline{\Pi}_l(z) = \begin{cases} 0, & k < l-1 \\ C_l(z), & k = l-1 \end{cases}, \tag{5.30}$$

where, Σ_l is the intersection matrix

$$\Sigma_l = \begin{pmatrix} 0 & ... & 0 & 1 \\ 0 & ... & -1 & 0 \\ & & & \\ 0 & .^{.^{.}} & 0 & 0 \\ (-1)^{l+1} & ... & 0 & 0 \end{pmatrix} \tag{5.31}$$

and $C_l(z)$ is the Yukawa-coupling

$$C_l(z) = \frac{1}{z^{l-3}\prod_{k\in\Delta^{(l)}}(1-kz)}. \tag{5.32}$$

Griffiths transversality will provide us with relations between $\underline{\Pi}(z)$ its derivatives. We will collectively refere to these relations as the Griffiths relations. We now focus on deriving a differential equation satisfied by eq. 5.32, as this will provide us with two additional bonuses that will come in handy. The first one being that the derivation can be used as a guideline for generalizing the result to generic-mass. We will also see that we can derive more Griffiths relations, that can be used to express the inverse wronskian.
We start out by using the product rule, to derive relations between first orders of derivatives with respect to the Frobenius basis

$$C_l(z) = \partial_z(\underline{\Pi}_l(z)^T\Sigma_l\partial_z^{l-2}\underline{\Pi}(z)) - \partial_z\underline{\Pi}_l(z)^T\Sigma_l\partial_l^{l-2}\underline{\Pi}_l(z). \tag{5.33}$$

The first term is zero, due to eq. 5.30. We can continue using the product rule this way and find a more generalized version of eq. 5.30

$$C_l(z) = (-1)^k\partial_z^k\underline{\Pi}_l(z)^T\Sigma_l\partial_z^{l-1-k}\underline{\Pi}_l(z). \tag{5.34}$$

If one then differentiates the Griffiths relation 5.30 and use the product rule successively, one gets

$$\begin{aligned} \underline{\Pi}_l(z)^T\Sigma_l\partial_z^l\underline{\Pi}_l(z) &= \partial_z C_l(z) - \partial_z\underline{\Pi}_l(z)^T\Sigma_l\partial_z^{l-1}\underline{\Pi}_l(z) \\ &= l\partial_z C_l(z) + (-1)^l\partial_z^l\underline{\Pi}_l(z)^T\Sigma_l\underline{\Pi}_l(z). \end{aligned} \tag{5.35}$$

Due to the $(-1)^{l+1}$ symmetry of Σ_l, we can reduce this to

$$\underline{\Pi}_l(z)^T \Sigma_l \partial_z^l \underline{\Pi}_l(z) = \frac{l}{2} \partial_z C_l(z). \tag{5.36}$$

So far we have only developed relations between different degrees of differential orders of the Frobenius basis. To find the differential equation, whose solution is the Yukawa-coupling we make use of the fact that the Picard-Fuchs operator is an l-order differential equation that annihilates the Frobenius basis:

$$\begin{aligned} 0 &= \underline{\Pi}_l(z)^T \Sigma_l \mathcal{L}_l \underline{\Pi}_l(z) \\ &= B_{l,l-1}(z) \underline{\Pi}_l(z)^T \Sigma_l \partial_z^{l-1} \underline{\Pi}_l(z) + B_{l,l}(z) \underline{\Pi}_l(z)^T \Sigma_l \partial_z^l \underline{\Pi}_l(z). \end{aligned} \tag{5.37}$$

We can now use the relations we have found above, namely eq. 5.30 and 5.35, to find a differential equation that is satisfied by $C_l(z)$

$$\partial_z C_l(z) + \frac{2}{l} \frac{B_{l,l-1}}{B_{l,l}} C_l(z) = 0. \tag{5.38}$$

We can also use this result in eq. 5.35 to generate a series of relations through differentiation. To start out with, we see that we can rewrite eq. 5.35, by using the Picard Fuchs operator

$$\begin{aligned} \partial_z C_l(z) &= \underline{\Pi}_l(z)^T \Sigma_l \partial_z^l \underline{\Pi}_l(z) + \partial_z \underline{\Pi}_l(z)^T \Sigma_l \partial_z^{l-1} \underline{\Pi}_l(z) \\ &= -\sum_{j=1}^{l-1} \frac{B_{l,j}(z)}{B_{l,l}(z)} \underline{\Pi}_l(z)^T \Sigma_l \partial_z^j \underline{\Pi}_l(z) + \partial_z \underline{\Pi}_l(z)^T \Sigma_l \partial_z^{l-1} \underline{\Pi}_l(z). \end{aligned} \tag{5.39}$$

Due to Griffiths relations, we realize that the only non-vanishing term in the sum is $j = l-1$. We can then use the differential equation 5.38 to collect terms. This gives us even more relations between the Frobenius basis

$$\left(1 - \frac{l}{2}\right) \partial_z C_l(z) = \partial_z \underline{\Pi}_l(z)^T \Sigma_l \partial_z^{l-1} \underline{\Pi}_l(z). \tag{5.40}$$

We can proceed in this fashion i.e taking the derivative of eq. 5.40 and using the Picard-Fuchs operator to re-write the l-order differential. Eventually we will have enough relations to calculate all the entries in the matrix

$$\mathbf{Z}_l(z) = \begin{pmatrix} \underline{\Pi}(z)^T \Sigma_l \underline{\Pi}(z) & \dots & \underline{\Pi}(z) \Sigma_l \partial_z^{l-1} \underline{\Pi}(z) \\ \vdots & \ddots & \vdots \\ \partial_z^{l-1} \underline{\Pi}(z) \Sigma_l \underline{\Pi}(z) & \dots & \partial_z^{l-1} \underline{\Pi}(z) \Sigma_l \partial_z^{l-1} \underline{\Pi}(z) \end{pmatrix}. \tag{5.41}$$

In terms of $\mathbf{Z}_l(z)$, it is possible to express the inverse Wronskian as

$$\mathbf{W}_l(z)^{-1} = \Sigma_l \mathbf{W}_l(z)^T \mathbf{Z}_l(z)^{-1}. \tag{5.42}$$

We explicitly show the result for the l'th column of the inverse wronskian, with $1 \leq k \leq l$, as this will be of great use for us later on,

$$W(z)^{-1}_{k,l} = \frac{(-1)^{l+k}\varpi_{l,l-k}(z)}{C_l(z)}. \tag{5.43}$$

We once again emphasise the importance of the Griffiths relations. We see that our results will be linear in elements of the Frobenius basis and its derivatives. Had we instead inverted the Wronskian as if it was a generic matrix, we would be left with $(l-1)$ polynomials of elements of the Wronskian.

6 Symbol and coaction on l-loop banana diagram

At this point, we have all the tools we need to construct our own Symbol and Coaction operators on the l-loop banana integrals. In this section we will use the material covered thus far, to build a symbol and coaction on periods of the equal-mass banana master integrals. We saw in section 4.3, that the essential part of building a Symbol, is to find a Unipotent differential equation that is satisfied by our master integrals. We will compute the symbol and coaction on master integrals related to the loop orders $l = 2, 4$. We will compare these results with what has previously been found by Broedel et. al [11] and [13].

6.1 Unipotent System

It was shown (eq. 5.9) that it is possible to do a change of basis, to achieve a differential equation written in terms of the inverse Wronskian and an inhomogeneous term arising from the inhomogeneous differential equation eq. 5.3 where the inhomogenous term is eq. 5.4. If we also make use of the Griffiths relations relations, we are able to express the differential equation for $1 \leq k \leq l$ as

$$dL_{l,k}(z) = (-1)^{k+1} \frac{(l+1)!}{z^2} \varpi_{l,k}(z) dz, \tag{6.1}$$

With inspiration from eq. 4.58, we decompose the equation into a semi-simple and unipotent quantity. We define the unipotent quantity $\tau_{k-1}(z)$ to be

$$\tau_j(z) = \frac{\varpi_{l,j}(z)}{\varpi_{l,l-1}(z)} \qquad 0 \leq j \leq l-1. \tag{6.2}$$

The choice of τ is arbitrary in the sense that we could have chosen any ratio between two elements of the Frobenius basis or any other linear combination of Frobenius elements. We have chosen this particular choice of τ, as it leads to fewer logarithmic singularities. To see this, note that $\varpi_{l,l-1}(z)$ will contain the most powers of $\log(z)$ when you make the expansion following eq. 5.17. Thus when you take the high momentum limit $z \to 0$ you will have $\tau_j(z) \to 0$ is finite. The differential equation is then expressed in terms of semi-simple and unipotent qunaitites

$$dL_{l,k}(z) = (-1)^{k+1} \frac{(l+1)!}{z^2} \varpi_{l,l-1}(z) \tau_k(z) dz. \tag{6.3}$$

We now consider the differential equation eq. 4.40, that our master integrals must satisfy in order to build a symbol and coaction on them. It is clear from the discussion in section

5.1, that in order to build a co-action and symbol operator, we need to have a Gauss-manin type differential equation with a nilpotent connection matrix $\mathbf{N}(\underline{z})$

$$d\underline{I}_l(\underline{z}) = \mathbf{N}(\underline{z})\underline{I}_l(\underline{z}), \tag{6.4}$$

to gain a unipotent differential equation. When the connection is nilpotent we say that the master integrals and differential equation are unipotent. For the differential equation to not have a trivial solution, the connection has to satisfy the so called *integrability condition* $d\mathbf{N}(\underline{z}) = \mathbf{N}(\underline{z}) \wedge \mathbf{N}(\underline{z})$. Note that for the equal-mass case, this condition is trivially satisfied. To see this, note that in the equal-mass case, the matrix $\mathbf{N}_l(z)$ will only contain one-forms of the type αdz. Since the wedge operator $\wedge$ is an anti-symmetric operator $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$, where ω_1 and ω_2 are one forms, the right hand side the equation must be zero. The differential operator acts on an r-form ω as $\mathrm{d}_r\omega = \frac{1}{r!}\left(\partial/\partial_{x^\nu}\omega\right) dx^\nu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_r}$[19]. In the case of one-forms in a one-dimensional space we have $\mathrm{d}_z\omega_1(z) = \partial_z\omega(z)dz \wedge dz$, which is zero due to the anti-symmetry of $\wedge$ and the integrability conditions holds. It is evident that the master integrals in eq. 5.9 do not satisfy a differential equation like eq. 6.4. It is however possible to "force" them into satisfying one, by expanding the basis of integrals. We define the expanded L-basis as

$$\underline{T}_l(z) = (L_{l,1}(z), ..., L_{l,l}(z), \tau_0(z), ..., \tau_{l-2}(z), 1). \tag{6.5}$$

This basis will satisfy the differential equation 6.4 and produce a nilpotent connection. We can for instance look at the $l = 2$ example in which we have the basis

$$\underline{T}_l(z) = (L_{l,1}(z), L_{l,2}(z), \tau_0(z), 1), \tag{6.6}$$

which satisfies the unipotent differential equation

$$d\begin{pmatrix} L_{2,1}(z) \\ L_{2,2}(z) \\ \tau_0(z) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{3!}{z^2}\varpi_{2,1}(z)dz \\ 0 & 0 & -\frac{3!}{z^2}\varpi_{2,1}(z)dz & 0 \\ 0 & 0 & 0 & d\tau_0(z) \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_{2,1}(z) \\ L_{2,2}(z) \\ \tau_0(z) \\ 1 \end{pmatrix}, \tag{6.7}$$

where we have used eq. 6.3 to find the entries of the matrix.
For a general loop order l, $\mathbf{N}_l(z)$ will be a $2l \times 2l$ matrix with entries

$$\mathbf{N}_l(z)_{i,j} = \begin{cases} (-1)^{j+1}\frac{(l+1)!}{z^2}\varpi_{l,l-1}(z)dz, & \text{for } i+j = 2l+1, \quad 1 \leq j \leq l, \quad l < i \leq 2l \\ d\tau_{j-l-1}(z), & \text{for } i = 2l, \quad l < j < 2l \\ 0 & \text{otherwise} \end{cases}. \tag{6.8}$$

It can be shown that for any loop order l, $\mathbf{N}_l(z)$ has index 3 i.e, $\underline{\boldsymbol{N}}_l(z)^3 = 0$. Symbols and coactions for banana diagrams at $l = 2$ and $l = 3$ are known results. The interesting results are for $l = 4$ as this is where the Feynman integrals can no longer be expressed in terms of elliptic curves. We will therefore present the nilpotent matrix for $l = 4$, for reference sake

$$\mathbf{N}_4(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5!}{z^2}\varpi_{4,2}(z)dz \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5!}{z^2}\varpi_{4,2}(z)dz & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5!}{z^2}\varpi_{4,3}(z)dz & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5!}{z^2}\varpi_{4,3}(z)dz & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d\tau_0(z) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d\tau_1(z) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d\tau_2(z) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.9}$$

6.2 Constructing a symbol operator

We will now use our unipotent differential system in eq. 6.4 to build a symbol operator following the instructions in section 4.3. Our first step is to define define the unipotent matrix T_A 4.41 for our differential equation

$$T_N = 1 + [\mathbf{N}_l]^R + [\mathbf{N}_l|\mathbf{N}_l]^R + [\mathbf{N}_l|\mathbf{N}_l|\mathbf{N}_l]^R + ..., \tag{6.10}$$

where we have suppressed the z-dependence. Using eq. 6.8, we can derive a general construction of $[\mathbf{N}_l|\mathbf{N}_l]^R$ and $[\mathbf{N}_l|\mathbf{N}_l|\mathbf{N}_l]^R$.

$$\begin{aligned} [\mathbf{N}_l|\mathbf{N}_l]^R_{ij} &= \begin{cases} \left[d\tau_j|(-1)^{j+1}\frac{4!}{z^2}\varpi_{l,l-j}(z)dz\right], & \text{for } 2 \le j \le l, \quad i = 2l \\ 0, & \text{otherwise} \end{cases}. \\ [\mathbf{N}_l|\mathbf{N}_l|\mathbf{N}_l]^R_{ij} &= 0 \end{aligned} \tag{6.11}$$

We now do as instructed and define the symbol on a pair of periods $[\xi_i, \xi_j]$, where $\xi_i, \xi_j \in \underline{T}_l(z)$, as

$$\mathcal{S}\left([\xi_i, \xi_j]\right) = (T_N)_{ji}\,. \tag{6.12}$$

We can write a program in MATHEMATICA, that calculates the symbol on all combinations of pairs $[\xi_i, \xi_j]$ in $\underline{T}_l(z)$. For starters we calculate the symbol on a $l = 2$ basis. We list all

non-zero answers:

$$\begin{aligned}
&\mathcal{S}([\xi_i, \xi_i]) = 1, \\
&\mathcal{S}([\tau_0, L_{r,1}(z)]) = \left[-\frac{2!}{z^2}\varpi_{2,1}(z)dz\right], \\
&\mathcal{S}([1, L_{2,1}(z)]) = \left[-\frac{2!}{z^2}\varpi_{2,1}(z)dz\right], \\
&\mathcal{S}([1, L_{2,2}(z)]) = \left[d\tau_0|\frac{2!}{z^2}\varpi_{2,1}(z)dz\right], \\
&\mathcal{S}([1, \tau_0(z)]) = d\tau_0.
\end{aligned} \tag{6.13}$$

Let us take a few seconds to understand what these results mean. The symbols of the type $\mathcal{S}([1, \xi_j])$ are what we would usually refere to as $\mathcal{S}(\xi_j)$. The other types of symbol, namely $\mathcal{S}([\tau_0(z), L_{2,1}(z)])$ have a different interpretation. The pair can in some sense be seen as the integral $L_{2,1}(z)$ integrated over a contour that encircles all of $\tau_0(z)$'s singularities. We stress that this is only an analogy, as a de Rham pair does not permit an integral representation. Once we know how the symbol operator acts on elements of $\underline{T}_l(z)$, we can calculate the symbol in our original $\underline{J}_l(z)$ basis. To see this, we note that the transformation we need to do is $\underline{J}_l(z) = \mathbf{W}_l(z)\underline{L}_l(z)$. This means that we can express our $\underline{J}_l(z)$ in terms of elements of $\underline{L}_l(z)$ multiplied with elements of $\mathbf{W}_l(z)$, the latter of which we can reexpress in terms of $\tau_k(z)$'s. Finally, we can use the shuffle property of the symbol in eq. 4.36 to find the symbol of elements in $\underline{J}_l(z)$. For instance, in a two loop example, we find that

$$\begin{aligned}
&\underline{J}_{2,1}(z) = L_1(z)\varpi_1(z)\tau_0(z) + L_2(z)\varpi_1(z), \\
&\underline{J}_{2,2}(z) = L_1(z)\left[\varpi_1(z)\partial_z\tau_0(z) + \tau_0(z)\partial_z\varpi_1(z)\right] + L_2\partial_z\varpi_1(z),
\end{aligned} \tag{6.14}$$

has the symbols

$$\begin{aligned}
&\mathcal{S}([1, \underline{J}_{2,1}(z)]) = \varpi_{2,1}(z)\left[d\tau_0|\frac{3!}{z^2}\varpi_{2,1}(z)dz\right], \\
&\mathcal{S}([1, \underline{J}_{2,2}(z)]) = \varpi_1\partial_z\tau_0(z)\left[\frac{3!}{z^2}\varpi_1(z)dz\right] + \partial_z\varpi_1(z)\left[d\tau_0|\frac{3!}{z^2}\varpi_1(z)dz\right],
\end{aligned} \tag{6.15}$$

where we have used eq. 4.50 to simplify our expressions. Let us now compare our results with what has previous been found by Broedel et. al [11]. They derived the symbol on the full Feynman integral

$$\begin{aligned}
S(p^2, m^2) &= \int \frac{d^dk_1 d^dk_2}{(k_1^2 - m_1^2)(k_2^2 - m_2^2)((k_1 + k_2 - p)^2 - m^2)} \\
&= \frac{2\omega_1}{(s + m^2)\sqrt{a_{12}a_{43}}}J(\tau) + \mathcal{O}(\epsilon),
\end{aligned} \tag{6.16}$$

where $\omega_1 = K(\lambda)$ and $\omega_2 = 2iK(1-\lambda)$ are the two periods of the elliptic curve and they define $\tau = \frac{\omega_2}{\omega_1}$. The function $J(\tau)$ is a linear combination of *Elliptical Multiple Polylogarithms* or eMPLs for short. These are functions defined as

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ z_1 & \cdots & z_k \end{smallmatrix}; z, \tau\right) = \int_0^z dz' g^{(n_1)}(z' - z_1, \tau)\tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ z_2 & \cdots & z_k \end{smallmatrix}; z, \tau\right). \tag{6.17}$$

These functions also form a shuffle-algebra, similar to ordinary MPLs. In the article they find that the symbol of eq. 6.16 is

$$S(J(\tau)) = \frac{2}{\pi^2}\left[\mathrm{d}\tau\, h^{(3)}_{6,0,1}|\mathrm{d}\tau\right] + \frac{2}{\pi^2}\left[\mathrm{d}\tau\, h^{(3)}_{6,3,4}|\mathrm{d}\tau\right] + \frac{5}{\pi}\left[\mathrm{d}\tau\, h^{(3)}_{6,3,1}|\mathrm{d}\tau\right], \tag{6.18}$$

where $h^{(n)}_{N,r,s}$ are Eisenstein series, represented by

$$h^{(n)}_{N,r,s} = -\sum_{\substack{(\alpha,\beta)\in\mathbb{Z} \\ (\alpha,\beta)\neq(0,0)}} \frac{e^{2\pi i(s\alpha - r\beta)/N}}{(\alpha+\beta\tau)^{2n}}. \tag{6.19}$$

Since eq. 6.18 is the symbol for the full Feynman integral and eq. 6.15 are only symbols for master integrals, to properly compare the two we would need to consider the sum of the symbols in eq. 6.15. It is however evident that both eq. 6.15 and 6.18 have a maximal symbol length of 2. We also note that the our symbols have a $d\tau_0$ in the leftmost spot, whereas τ is located in the rightmost spot of eq. 6.18. This is not an issue as we can use the relations in eq. 4.50 to shift the position of the $d\tau_0$ in exchange of a sign flip. We have attempted to check if the two results agree numerically. The bar constructions is an object that can be integrated over and is not itself permitted to be evaluated. By integrating a bar construction, one can map it to the complex numbers. Unfortunately this was not possible. There is no direct way of expressing dz in terms of $d\tau_k(z)$, we must instead express all $d\tau_k(z)$'s in terms of dz's. The resulting expressions are complicated and could not be integrated to a finite expression.

We are now interested in calculating the symbol for higher loop orders. We once again consider the example $l = 4$ and calculate all the non-zero symbol values

$$
\begin{aligned}
&\mathcal{S}([\xi_i,\xi_i]) = 1\\
&\mathcal{S}([\tau_0(z),L_{4,4}(z)]) = \left[\frac{5!}{z^2}\varpi_{4,3}(z)\right]\\
&\mathcal{S}([\tau_1(z),L_{4,3}(z)]) = \left[-\frac{5!}{z^2}\varpi_{4,3}(z)\right]\\
&\mathcal{S}([\tau_2(z),L_{4,2}(z)]) = \left[\frac{5!}{z^2}\varpi_{4,3}(z)\right]\\
&\mathcal{S}([1,L_{4,1}(z)]) = \left[-\frac{5!}{z^2}\varpi_{4,3}(z)\right]\\
&\mathcal{S}([1,L_{4,2}(z)]) = \left[d\tau_2|\frac{5!}{z^2}\varpi_{4,3}(z)\right]\\
&\mathcal{S}([1,L_{4,3}(z)]) = \left[d\tau_1|-\frac{5!}{z^2}\varpi_{4,3}(z)\right]\\
&\mathcal{S}([1,L_{4,4}(z)]) = \left[d\tau_0|\frac{5!}{z^2}\varpi_{4,3}(z)\right]\\
&\mathcal{S}([1,\tau_0(z)]) = d\tau_0\\
&\mathcal{S}([1,\tau_1(z)]) = d\tau_1\\
&\mathcal{S}([1,\tau_2(z)]) = d\tau_2.
\end{aligned}
\tag{6.20}
$$

By first inspection these all appear to be of length 2. However if we calculate the symbol for $\underline{J}_4(z)$ we will see that the length of the symbols increase. We only list the $\mathcal{S}(\underline{J}_{4,1}(z))$ calculations as the symbol on the other 3 calculations are only more complicated, but provide us with the same information as the symbol of $\underline{J}_{4,1}(z)$.

$$
\mathcal{S}([1,J_{4,1}(z)] = \varpi_{4,3}(z)\left(\left[-\frac{5!}{z^2}\varpi_{4,3}(z)dz|\tau_0\right] + \left[d\tau_2|\frac{5!}{z^2}\varpi_{4,3}(z)dz|\tau_1\right] - \left[\tau_1|\frac{5!}{z^2}\varpi_{4,3}(z)dz\tau_2\right]\right). \tag{6.21}
$$

In this case the second and third term are length 3 symbols. The symbol on the remaining three elements of $\underline{J}_4(z)$ are also of length three. With our methods, one can actually show that all integrals $\underline{J}_l(z)$ where $l \geq 3$ have a symbol length of 3. In [13] it was found that the master integrals of the $l = 3$ banana diagram could be expressed as

$$
\begin{aligned}
\tilde{M}_1(x(\tau)) &= -\frac{4\zeta_3}{\pi^3} - \frac{i}{3\pi^3}\left(81\mathcal{I}\left(\begin{smallmatrix}0&0&4\\0&0&0\end{smallmatrix};\tau\right) - 90\mathcal{I}\left(\begin{smallmatrix}0&0&4\\0&0&1\end{smallmatrix};\tau\right) + 10\mathcal{I}\left(\begin{smallmatrix}0&0&4\\0&0&3\end{smallmatrix};\tau\right) - \mathcal{I}\left(\begin{smallmatrix}0&0&4\\0&0&4\end{smallmatrix};\tau\right)\right)\\
\tilde{M}_2(x(\tau)) &= \frac{i}{3\pi^3}\left(81\mathcal{I}\left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix};\tau\right) - 90\mathcal{I}\left(\begin{smallmatrix}0&4\\0&1\end{smallmatrix};\tau\right) + 10\mathcal{I}\left(\begin{smallmatrix}0&4\\0&3\end{smallmatrix};\tau\right) - \mathcal{I}\left(\begin{smallmatrix}0&4\\0&4\end{smallmatrix};\tau\right)\right)\\
\tilde{M}_3(x(\tau)) &= \frac{i}{18\pi^3}\left(81\mathcal{I}\left(\begin{smallmatrix}0\\4\end{smallmatrix};\tau\right) - 90\mathcal{I}\left(\begin{smallmatrix}4\\1\end{smallmatrix};\tau\right) + 10\mathcal{I}\left(\begin{smallmatrix}4\\3\end{smallmatrix};\tau\right) - \mathcal{I}\left(\begin{smallmatrix}4\\4\end{smallmatrix};\tau\right)\right)
\end{aligned}
\tag{6.22}
$$

where $\mathcal{I}\left(\begin{smallmatrix}n_1,\dots n_k\\ p_1,\dots,p_k\end{smallmatrix};\tau\right)$ are iterated integrals of modular forms. The symbol length of these iterated integrals are equal to the numbers of iterated integrals, which is given by the numbers of colomns in $\left(\begin{smallmatrix}n_1,\dots n_k\\ p_1,\dots,p_k\end{smallmatrix}\right)$. Thus we see that $\tilde{M}_1(x(\tau))$ will have a symbol length of three, which is in accordance with what we find.

6.3 Constructing a coaction

Now that we have defined a symbol on various master integrals we can use

$$\Delta(T_{l,k}) = \sum_{i=1}^{2l} \left(T_{l,i}(z) \otimes \mathcal{S}([\xi_i, \xi_k])\right), \tag{6.23}$$

to calculate the coaction the same master integrals. We start by calculating the coaction on $l = 2$ master integrals

$$\begin{aligned}
&\Delta(L_{l,1}(z)) = L_{l,1}(z) \otimes 1 - 1 \otimes \left[\frac{3!}{z^2}\varpi_{2,1}dz\right] \\
&\Delta(L_{l,2}(z)) = L_{l,2}(z) \otimes 1 + 1 \otimes \left[d\tau_0|\frac{3!}{z^2}\varpi_{2,1}(z)dz\right] + \tau_0(z) \otimes \left[\frac{3!}{z^2}\varpi_{2,1}(z)dz\right] \\
&\Delta(\tau_0(z)) = \tau_0 \otimes 1 + 1 \otimes [d\tau_0]
\end{aligned} \tag{6.24}$$

First of all we note that the coaction acts non-trivally on all unipotent quantities. This is in accordance with what was found in section 4.3, when we discussed the decomposition of semi-simple and unipotent quantities. As we want to work out the coaction on our original $\underline{J}_l(z)$-basis, we use the compatability between multiplication and coaction to write

$$\Delta\left(T_{l,k}(z)T_{l,j}(z)\right) = \Delta(T_{l,k}(z)) \cdot \Delta(T_{l,j}(z)). \tag{6.25}$$

The basis $\underline{J}_2(z)$ is given in eq. 6.14 in terms of $\underline{L}_2(z)$ and $\tau_k(z)$. We have to be careful when we apply the coaction on the elements of $\underline{J}_{l,2}(z)$ as it contain terms with both unipotent and semi-simple quantities. We therefore have to use the compatebility between multiplication and cocation, in such a way that we in the end act with the coaction on quantities that are either unipotent or semi-simple but never a mix of the two. For instance the first term in 6.14 can be split as

$$\begin{aligned}
\Delta(L_{2,1}(z)(\varpi_{2,1}(z)\tau_0(z))) &= \Delta(L_{2,1}(z))\Delta(\varpi_{2,1}(z)\tau_0(z)) \\
&= \Delta(L_{2,1}(z))\Delta(\varpi_{2,1}(z))\Delta(\tau_0(z)).
\end{aligned} \tag{6.26}$$

We are now only operating with the coaction on two unipotent quantities $\Delta(L_{2,1}(z))$ and $\Delta(\tau_0)$ and a single semi-simple quantity $\Delta(\varpi_{2,1}(z))$. We know how the coaction acts on

the unipotent quantities and we know that it should act trivially on semi-simple quantities. We therefore find the symbol becomes

$$\begin{aligned}\Delta(L_{2,1}(z)(\varpi_{2,1}(z)\tau_0(z))) = \tau_0(z)\varpi_{2,1}(z)\left[\frac{3!}{z^2}\varpi_{2,1}(z)dz\right]\otimes 1 - \tau_0(z)\varpi_{2,1}(z)\otimes\left[\frac{3!}{z^2}\varpi_{2,1}(z)dz\right]\\ + \varpi_{2,1}(z)L_{2,1}(z)\otimes[d\tau_0] - \varpi_{2,1}(z)\otimes\left[\frac{3!}{z^2}\varpi_{2,1}(z)dz\right][d\tau_0],\end{aligned} \tag{6.27}$$

where we have used the fact that $(a_1\otimes b_1)(a_2\otimes b_2) = (a_1a_2\otimes b_1b_2)$. Calculating the symbol for the second term in $\underline{J}_{2,1}(z)$, we get the full coaction on $\underline{J}_2(z)$

$$\begin{aligned}\Delta(\underline{J}_{2,1}(z)) = \varpi_{2,1}(z)\Bigg(L_{2,1}(z)\otimes 1 + L_{2,1}(z)\otimes[d\tau_0] + \tau_0(z)L_{2,1}\otimes 1\\ - 1\otimes[d\tau_0]\left[\frac{3!}{z^2}\varpi_{2,1}(z)dz\right] + 1\otimes\left[d\tau_0|\frac{3!}{z^2}\varpi_{2,1}(z)dz\right]\Bigg).\end{aligned} \tag{6.28}$$

To calculate the coaction on the second element of $\underline{J}_2(z)$, namely $\underline{J}_{2,2}(z)$, we have to also make use of eq. 4.62 that describes how the coaction acts on derivatives. Doing so we find that the coaction becomes

$$\begin{aligned}\Delta(\underline{J}_{2,2}(z)) = \partial_z\varpi_{2,1}(z)\Bigg(L_{2,2}(z)\otimes 1 + L_{2,1}(z)\otimes[d\tau_0] + \tau_0(z)L_{2,1}(z)\otimes 1\\ - 1\otimes[d\tau_0]\left[\frac{3!}{z^2}\varpi_{2,1}(z)dz\right] + 1\otimes\left[d\tau_0|\frac{3!}{z^2}\varpi_{2,1}(z)dz\right]\Bigg).\end{aligned} \tag{6.29}$$

Once again we compare with the results by Broedel et. al [11]. In their paper, they find that the coaction acting on $J(\tau)$ becomes

$$\Delta(J(\tau)) = J(\tau)\otimes 1 + 1\otimes X_2 + X_{11}\otimes[d\tau_0], \tag{6.30}$$

where

$$\begin{aligned}X_2 = \frac{4}{\pi^2}\left(\left[d\tau h^{(3)}_{12,0,1}|d\tau\right] - \left[d\tau h^{(3)}_{12,0,5}|d\tau\right] + \left[d\tau h^{(3)}_{12,5,1}|d\tau\right] - \left[d\tau h^{(3)}_{12,6,5}|d\tau\right]\right)\\ + \frac{4}{\pi^2}\left(\left[d\tau h^{(3)}_{12,3,4}|d\tau\right] - \left[d\tau h^{(3)}_{12,3,2}|d\tau\right] + \left[d\tau h^{(3)}_{12,9,4}|d\tau\right] - \left[d\tau h^{(3)}_{12,9,2}|d\tau\right]\right)\\ + \frac{10}{\pi^2}\left(\left[d\tau h^{(3)}_{12,3,1}|d\tau\right] - \left[d\tau h^{(3)}_{12,9,5}|d\tau\right] + \left[d\tau h^{(3)}_{12,3,5}|d\tau\right] - \left[d\tau h^{(3)}_{12,9,1}|d\tau\right]\right)\end{aligned} \tag{6.31}$$

and X_{11} contain linear combinations of eMPLs. Once again $\Delta(J(\tau))$ is the coaction acting on a whole Feynman integral, so in order to properly compare the two, we should consider

the coaction acting on a linear combination of $\underline{J}_{2,1}(z)$ and $\underline{J}_{2,2}(z)$. If we write up our coaction as

$$\begin{aligned}\Delta(\alpha\underline{J}_{2,1}+\beta\underline{J}_{2,2}) &= \mathcal{K}(z)\otimes 1 + 1\otimes Y_1 + Y_2\otimes[d\tau] \\ &\quad -(\alpha\varpi_{2,1}(z)+\beta\partial\varpi_{2,1}(z))\left(1\otimes[d\tau_0]\left[\frac{3!}{z^2}\varpi_{2,1}(z)dz\right]\right),\end{aligned} \tag{6.32}$$

where α and β are $\mathbb{C}$ coefficients and

$$\begin{aligned}\mathcal{K}(z) &= \alpha\varpi_{2,1}(z)(L_{2,1}(z)+\tau_0(z)L_{2,1}(z))+\beta\partial\varpi_{2,1}(z)(L_{2,2}(z)+\tau_0(z)L_{2,1}(z)),\\ Y_1 &= 2(\alpha\varpi_{2,1}(z)+\beta\partial\varpi_{2,1}(z))\left[d\tau_0|\frac{3!}{z^2}\varpi_{2,1}(z)dz\right],\\ Y_2 &= (\alpha\varpi_{2,1}(z)+\beta\partial_z\varpi_{2,1}(z))L_{2,1}(z).\end{aligned} \tag{6.33}$$

If we now compare the overall structure of 6.30 and 6.32, we see that the two equations do not agree in terms of their tensor structure. In our calculations, we find that there is an extra term with two bar constructions multiplied together in the righthand tensor slot. It has not been possible to verify, whether or not the two equations agree numerically for the same reasons as in the symbol case. We now move on to calculate the coaction on master integrals of higher loop order. We consider the case $l=4$ and once again use eq. 6.23 to calculate the coactions

$$\begin{aligned}&\Delta(\tau_k(z)) = \tau_k\otimes 1+1\otimes d\tau_k,\\ &\Delta(L_{4,1}(z)) = L_{4,1}(z)\otimes 1 - 1\otimes\left[\frac{5!}{z^2}\varpi_{4,3}(z)dz\right]\\ &\Delta(L_{4,2}(z)) = L_{4,2}(z)\otimes 1+1\otimes\left[d\tau_2|\frac{5!}{z^2}\varpi_{4,3}(z)dz\right]+\tau_2(z)\otimes\left[\frac{5!}{z^2}\varpi_{4,3}dz\right],\\ &\Delta(L_{4,3}(z)) = L_{4,3}(z)\otimes 1-1\otimes\left[d\tau_1|\frac{5!}{z^2}\varpi_{4,3}(z)dz\right]-\tau_1(z)\otimes\left[\frac{5!}{z^2}\varpi_{4,3}dz\right],\\ &\Delta(L_{4,4}(z)) = L_{4,4}(z)\otimes 1+1\otimes\left[d\tau_0|\frac{5!}{z^2}\varpi_{4,3}(z)dz\right]+\tau_0(z)\otimes\left[\frac{5!}{z^2}\varpi_{4,3}dz\right],\\ &\Delta(1) = 1\otimes 1.\end{aligned} \tag{6.34}$$

Again we see that the overall structure is the same as in the $l=2$ case.

We will conclude this section with a comment. We have a freedom in choosing, which parts to consider unipotent and which parts to consider semi-simple. For instance, it is not necessary to expand the $\underline{L}_l(z)$ basis with $l-1$ elements $\tau_k(z)$ to produce a unipotent differential equation. We could just as well have expanded it with a simple "1". Had we

done so, we would have declared that all of $\underline{L_l}(z)$ is semi-simple. This would have given us a symbol of length one and we would have gained no new information.
Due to the arbitrariness of choosing a semi-simple and unipotent quantity, it is possible that there exists another choice, in which the symbol length is even longer than what we've found. It is therefore also important to note that even though we find that the symbol length is 3 for $l > 2$, it may be an artifact of choosing a poor decomposition of semi-simple and unipotentcy.

7 Extending the symbol and coaction to the generic-mass case

In this section we will consider how to build a symbol and coaction operator on the generic mass master integrals described in eq. 3.4. The sector $(1, ..., 1)$ contains $2^{l+1} - l - 2$ master integrals and as a result, the matrices we will consider, quickly become computationally heavy. Following our methods in the equal mass case, we are interested in deriving a generic mass version of eq. 5.9. To do this we require two things: A vector containing the inhomogeneous parts of the differential equation described in eq. 5.3 and a generic mass version of the inverse Wronskian $W(\underline{z})^{-1}$ described in terms of generic-mass Griffiths relations.

7.1 Generic mass master integrals and their differential equations

We start out by finding a vector containing the inhomogeneities. Due to the fact that we are now in a multivariable system and the differential operator acts as $\mathrm{d} = \sum_i \mathrm{d}z_i \partial_{z_i}$, the differential equation in eq. 5.3, will essentially give us the sum of $l+1$ differential equations

$$\begin{aligned} \mathrm{d}\underline{J}_l(\underline{z}) &= \sum_i^{l+1} \mathrm{d}z_i \partial_{z_i} \underline{J}_l(\underline{z}) \\ &= \sum_i^{l+1} \mathrm{d}z_i \mathbf{B}_{l,i}(\underline{z}) \underline{J}_r + \sum_i^{l+1} \mathrm{d}z_i \underline{N}_{l,i}(\underline{z}). \end{aligned} \tag{7.1}$$

We can therefore work out $\underline{N}_{l,i}$ for each i independent of each other and in the end sum them together to get the full inhomogeneous term $\underline{N}_l(\underline{z})$. In section 5.4, we saw how we could at first study the maximal cut version of the differential equation. This would lead to most of the rows of $\mathbf{B}_l(z)$, in eq. 5.22, being trivial and would therefore not contribute to the inhomogeneous term. The entries of the non-trivial row could be found by applying the Picard-Fuchs operator to the Frobenius basis and solving for the highest order of derivative. In the uncut case we then knew that the only row, which could contain an inhomogeneous part would be the non-trivial one. We can do the same type of analysis we did for the equal-mass case, by first solving the Wronskian system $\mathrm{d}\mathbf{W}_l(\underline{z}) = \mathbf{B}_l(\underline{z})\mathbf{W}_l(\underline{z})$, where

$$\mathbf{W}(\underline{z}) = \begin{pmatrix} \varpi_{l,\underline{s}_1}(\underline{z}) & \varpi_{l,\underline{s}_2}(\underline{z}) & ... & \varpi_{l,\underline{s}_\lambda}(\underline{z}) \\ \partial_{z_1}\varpi_{l,\underline{s}_1}(\underline{z}) & \partial_{z_1}\varpi_{l,\underline{s}_2}(\underline{z}) & ... & \partial_{z_1}\varpi_{l,\underline{s}_\lambda}(\underline{z}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{\underline{z}}^{\underline{s}_\lambda}\varpi_{l,\underline{s}_1}(\underline{z}) & \partial_{\underline{z}}^{\underline{s}_\lambda}\varpi_{l,\underline{s}_2}(\underline{z}) & ... & \partial_{\underline{z}}^{\underline{s}_\lambda}\varpi_{l,\underline{s}_\lambda}(\underline{z}) \end{pmatrix}, \tag{7.2}$$

with $\lambda = 2^{l+1} - l - 2$ and $\underline{s} = \left\{\underline{k} \in \{0,1\}^{l+1} \mid |\underline{s}_1| \leq |\underline{s}_2| \leq ... \leq |\underline{s}_\lambda|\right.$ is the set of all $\underline{k}$ with a special ordering.

We will start out, by considering the Wronskian system for $l = 2$. By first analysing $\mathbf{B}_{l=2}(z)$, we will be able to generalize it to arbitrary l. Consider for example the generic mass $l = 2$ differential equation acting on the Frobenius basis $\Pi_2(\underline{z}) = (\varpi_0(\underline{z}), \varpi_{100}(\underline{z}), \varpi_{010}(\underline{z}), \varpi_{001}(\underline{z}))$

$$\mathrm{d}\begin{pmatrix} \varpi_0(\underline{z}) \\ \varpi_{100}(\underline{z}) \\ \varpi_{010}(\underline{z}) \\ \varpi_{001}(\underline{z}) \end{pmatrix} = \begin{pmatrix} \varpi_{100}(\underline{z}) \\ \varpi_{200}(\underline{z}) \\ \varpi_{110}(\underline{z}) \\ \varpi_{101}(\underline{z}) \end{pmatrix} \mathrm{d}z_1 + \begin{pmatrix} \varpi_{010}(\underline{z}) \\ \varpi_{110}(\underline{z}) \\ \varpi_{020}(\underline{z}) \\ \varpi_{011}(\underline{z}) \end{pmatrix} \mathrm{d}z_2 + \begin{pmatrix} \varpi_{001}(\underline{z}) \\ \varpi_{101}(\underline{z}) \\ \varpi_{011}(\underline{z}) \\ \varpi_{102}(\underline{z}) \end{pmatrix} \mathrm{d}z_3, \tag{7.3}$$

where we used the fact that $\partial_{z_i} \varpi_{\underline{k}}$ increases the i'th element of $\underline{k}$ by one. It is evident that, it is only the top row, that has a trivial linear dependence of $\Pi_2(\underline{z})$. The other rows will have to be derived from multivariable Pichard-Fuchs differential equations. In the two loop example, there are 6 elements that need to be solved for, namely all the elements $\varpi_{\underline{p}}(\underline{z}) = \partial_{\underline{z}}^{\underline{p}} \varpi_0(\underline{z})$ in which $|p| = 2$. We will refer to the set of all $\varpi_{\underline{p}}(\underline{z})$, that needs to be solved for as $\underline{\Omega}_l(\underline{z})$, i.e $\underline{\Omega}_2(\underline{z}) = \{\varpi_{200}(\underline{z}), \varpi_{020}(\underline{z}), \varpi_{002}(\underline{z}), \varpi_{110}(\underline{z}), \varpi_{101}(\underline{z}), \varpi_{011}(\underline{z})\}$.

Due to the fact that the multivariable Picard-Fuchs operator may contain multiple terms, whose derivatives are of order $n = 2$, it is evident that one needs to set up a system of equations and solve all elements of $\underline{\Omega}_l(\underline{z})$, simultaneously.

This puts some constraints on the set of Picard-Fuchs operators we should consider. As we mentioned in section 5.5, two different sets of operators $\mathcal{D}$ and $\mathcal{D}'$, might generate the exact same solution space. In this case it is necessary to use a set of differential operators, that are of order, at least $n \geq l$. It is also important that the numbers of terms in the Picard-Fuchs operators, that are of order $n \geq l$, are the same as the numbers of Picard-Fuchs operators available, as it will otherwise mean that the system of equations is either over -or under determined and there will not exist a unique solutions for $\varpi_{\underline{p}}(\underline{z})$ in terms of the Frobenius basis. We state this requirement as

$$\#\text{terms with } n \geq l \;=\; \#\text{Picard-Fuchs operators.} \tag{7.4}$$

It is therefore preferable to consider a set of operators that have an exact order of $n = l$, as this would reduce the amount of elements one needs to solve for.

Thus, in the two loop example, we need to find a set of 6 Picard-Fuchs operators of order two $\mathcal{D}_{l=2}^{(n=2)} = \{\mathcal{L}_1, ..., \mathcal{L}_6\}$ to completely describe all elements of $\underline{\Omega}_l(\underline{z})$ in terms of elements in the Frobenius basis. The two loop example is a special case in which all the elements that needs to be solved for have $|p| = l$ and in general this is not true. The indices of the

Frobenius basis are generated by $\underline{s}$, thus any $\varpi_{\underline{i}}(\underline{z})$, in which $\underline{i}$ contains a number that is greater than 1, yet has $|i| \leq l$, is not a part of the Frobenius basis. As they are not part of the Frobenius basis, they are by definition going to be a part of $\underline{\Omega}_l(\underline{z})$. The explicit index of $\varpi_{\underline{p}}(\underline{z}) \in \underline{\Omega}_l(\underline{z})$ can be found by considering the set containing all possible ways to fill $l+1$ slots, while keeping the sum at l, modular $\underline{s}$. In other words we consider the set $\underline{\omega} = \left\{ \underline{p} \in (j_1, j_2, ...j_{l+1}) \pmod{\underline{s}} \mid \sum_i^{l+1} j_i \leq l \ , \ |\underline{\omega}_1| \leq |\underline{\omega}_2| \leq ... \leq |\underline{\omega}_m| \right.$.
In general we will need to consider a set of operators $\mathcal{D}_l^{(n=l)} = \{\mathcal{L}_1, ..., \mathcal{L}_m\}$, where m corresponds to the numbers of elements in $\underline{\Omega}_l(\underline{z})$, which one can calculate to be

$$m = l + \sum_{n=1}^{l} n(l-n) \binom{l+1}{l-(n-1)}. \tag{7.5}$$

The operators themselves, can in general be written as

$$\mathcal{L}_{l,i} = \sum_{\underline{p} \in \underline{\omega}} \beta_{i,\underline{p}}^{(l)}(\underline{z}) \partial_{\underline{z}}^{\underline{p}} + \sum_{\underline{k} \in \underline{s}} \beta_{i,\underline{k}}^{(l)}(\underline{z}) \partial_{\underline{z}}^{\underline{k}} \quad \text{for } 1 \leq i \leq m, \tag{7.6}$$

where $\beta_{i,j}^{(l)}(\underline{z})$ are polynomials in $\underline{z}$. The system of equations, one has to write up to find all the undetermined $\varpi_{\underline{p}}(\underline{z})$ takes the general form of solving a $m \times m$ matrix

$$\begin{gathered} \boldsymbol{\beta}_l(\underline{z}) \cdot \underline{\Omega}(\underline{z}) = \underline{b}(\underline{z}) \\ = \\ \begin{pmatrix} \beta_{1,\underline{\omega}_1}^{(l)}(\underline{z}) & ... & \beta_{1,\underline{\omega}_m}^{(l)}(\underline{z}) \\ \vdots & \ddots & \vdots \\ \beta_{s,\underline{\omega}_1}^{(l)}(\underline{z}) & ... & \beta_{s,\underline{\omega}_m}^{(l)}(\underline{z}) \end{pmatrix} \begin{pmatrix} \varpi_{\underline{\omega}_1}(\underline{z}) \\ \vdots \\ \varpi_{\underline{\omega}_m}(\underline{z}) \end{pmatrix} = \begin{pmatrix} -\sum_{\underline{k} \in \underline{s}} \beta_{1,\underline{k}}^{(l)}(\underline{z}) \partial_{\underline{z}}^{\underline{k}} \\ \vdots \\ -\sum_{\underline{k} \in \underline{s}} \beta_{m,\underline{k}}^{(l)}(\underline{z}) \partial_{\underline{z}}^{\underline{k}} \end{pmatrix}. \end{gathered} \tag{7.7}$$

This equation can easily be solved computationally by use of Gaussian Elimination, however the end result will introduce long and complicated $\underline{z}$ polynomials.
Finally we are ready to find the inhomogeneous terms in eq. 7.1. To do this we only need to realize that each Picard-Fuchs operator acts on $\underline{J}_l(\underline{z})$ with an inhomogenous term $\mathcal{L}_{l,i}\underline{J}_l(\underline{z}) = S_{l,i}(\underline{z})$ with $1 \leq i \leq s$. As such, we need to add $S_{l,i}(\underline{z})$ on the i'th row of eq. 7.7. Again we define a vector $\underline{\mathcal{J}}_l(\underline{z})$, that contain all elements $J_{l,\underline{p}}(\underline{z})$ that needs to be solved for and write the system of equations

$$\boldsymbol{\beta}_l(\underline{z}) \cdot \underline{\mathcal{J}}_l(\underline{z}) = \underline{j}_l(\underline{z}) + \underline{S}_l, \tag{7.8}$$

where we have defined $\underline{j}_l(\underline{z}) = \left\{ -\sum_{\underline{k} \in \underline{s}} \beta_{i,\underline{k}}(\underline{z}) J_{l,k}(\underline{z}) \ \mid \ 1 \leq i \leq m \right\}$ and $\underline{S}_l(\underline{z}) = (S_{l,1}(z), ..., S_{l,m}(z))$.
Once one has solved for all $\underline{\mathcal{J}}_l(\underline{z})$, they will in general dependent on a linear combination of

elements in $\underline{J}_l(\underline{z})$ as well as a linear combination of the inhomogeneities $\underline{S}_l(\underline{z})$. The latter will be contained in the inhomogenous vector $\underline{N}_l(\underline{z})$

$$\underline{N}_l(\underline{z}) = \begin{pmatrix} 0 \\ c_{1,1}(\underline{z})S_{l,1}(\underline{z}) + ... + c_{1,m}(\underline{z})S_{l,m}(\underline{z}) \\ \vdots \\ c_{l-1,1}(\underline{z})S_{l,1}(\underline{z}) + ... + c_{l-1,m}(\underline{z})S_{l,m}(\underline{z}) \end{pmatrix}, \tag{7.9}$$

where the coefficients $c_{i,j}(\underline{z})$ are one-forms of the type $c_{i,j} = \sum_i t_i dz_i$.

7.2 Generic mass Griffiths relations

We now shift focus in an attempt to generalize the Griffiths relations in section 5.6 to generic-mass. The Griffiths relation in eq. 5.30 generalizes to generic-mass as

$$\underline{\Pi}_l(\underline{z})\Sigma_l\partial_{\underline{z}}^{\underline{k}}\underline{\Pi}(\underline{z}) = \begin{cases} 0 & \text{for } 0 \leq |\underline{k}| < l-1 \\ C_{\underline{k}}(\underline{z}) & \text{for } |\underline{k}| = l-1 \end{cases}, \tag{7.10}$$

where we once again use the shorthand notation $\partial_{\underline{z}}^{\underline{k}} = \prod_i^{l+1} \partial_{z_i}^{\underline{k}_i}$. In this case $\underline{\Pi}(\underline{z})$ is the multivalued Frobenius basis described in section 5.5. The intersection matrix Σ_l can in general be quite different than in the equal-mass case. As of writing this thesis, the explicit form of Σ_l in the generic-mass case it is not known, only that it is anti-symmetric for even l and symmetric for uneven l. From now on we will assume that $|\underline{k}| = l-1$ and consider the relations we gain from that. Just as in the equal-mass case, we can gain more relations by using the product rule. However, since $\underline{k}$ is now a vector, we have a choice of which derivative to "pull out" of $\partial_{z_i}^{\underline{k}}$. We can keep it general by pulling out ∂_{z_i}, we just have to be sure that $\underline{k}_i \geq 1$ when doing so. By pulling out one ∂_{z_i} we get the equation

$$C_{\underline{k}}(\underline{z}) = \partial_{z_i}\left(\underline{\Pi}(\underline{z})\Sigma_l\partial_{\underline{z}}^{\underline{k}-1_i}\underline{\Pi}(\underline{z})\right) - \partial_{z_i}\underline{\Pi}(\underline{z})\Sigma_l\partial_{\underline{z}}^{\underline{k}-1_i}\underline{\Pi}(\underline{z}), \tag{7.11}$$

where $\underline{k} - 1_i$, means that we are subtracting one from the i'th entry in $\underline{k}$. Equivalently one can think of $1_i = (0, ..., 0, 1, 0, ..., 0)$ as an $l+1$ vector, where the i indicates the entry that has the 1. The first term on the right hand side is zero due to the fact that $|k - 1_i| < l-1$. We can once again pull out a new derivative ∂_{z_j} from the second term in eq. 7.11, where j need not be different from i

$$C_{\underline{k}}(\underline{z}) = -\partial_{z_j}\partial_{z_i}\left(\partial_{z_i}\underline{\Pi}(\underline{z})\Sigma_l\partial_{\underline{z}}^{\underline{k}-1_i-1_j}\underline{\Pi}(\underline{z})\right) + \partial_{z_j}\partial_{z_i}\underline{\Pi}(\underline{z})\Sigma_l\partial_{\underline{z}}^{\underline{k}-1_i-1_j}\underline{\Pi}(\underline{z}). \tag{7.12}$$

It is once again possible to show that the first term on the right hand side vanishes due to eq. 7.10. We can keep going in this way, and in general we obtain

$$C_{\underline{k}}(\underline{z}) = \partial_{\underline{z}}^{\underline{s}} \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}-\underline{s}} \underline{\Pi}(\underline{z}), \tag{7.13}$$

where $|s| \leq |k|$ and $k_i - s_i \geq 0$ for all $1, \leq i \leq l+1$.
We will now derive the next set of Griffiths relations, by taking the i'th derivative of eq. 7.10

$$\begin{aligned} \partial_{z_i} \left(\underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}} \underline{\Pi}(\underline{z}) \right) &= \partial_{z_i} C_{\underline{k}}(\underline{z}) \\ \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}+1_i} \underline{\Pi}(\underline{z}) &= \partial_{z_i} C_{\underline{k}}(\underline{z}) - \partial_{z_i} \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}} \underline{\Pi}(\underline{z}). \end{aligned} \tag{7.14}$$

If we now pull out an ∂_{z_j} from $\partial_{\underline{z}}^{\underline{k}}$ in the second term, we find

$$\begin{aligned} \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}+1_i} \underline{\Pi}(\underline{z}) &= \partial_{z_i} C_{\underline{k}}(\underline{z}) - \partial_{z_j} \left(\partial_{z_i} \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}-1_j} \underline{\Pi}(\underline{z}) \right) + \partial_{z_j} \partial_{z_i} \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}-1_j} \underline{\Pi}(\underline{z}) \\ &= \partial_{z_i} C_{\underline{k}}(\underline{z}) - \partial_{z_j} C_{\underline{k}+1_i-1_j}(\underline{z}) + \partial_{z_j} \partial_{z_i} \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}-1_j} \underline{\Pi}(\underline{z}). \end{aligned} \tag{7.15}$$

Of course we still have to keep in mind whether or not this exists, in other words whether or not $\underline{k}_j - 1_j \geq 0$. Using this successively, we get

$$\underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}+1_i} \underline{\Pi}(\underline{z}) = \partial_{z_i} C_{\underline{k}}(\underline{z}) + \sum_{n=1}^{l+1} (-1)^{\underline{k}_n+1} \underline{k}_n \partial_{z_n} C_{|\underline{k}+1_i-1_n|}(\underline{z}) + (-1)^{|\underline{k}+1_i|} \partial_{\underline{z}}^{\underline{k}} \underline{\Pi}(\underline{z}) \Sigma_l \underline{\Pi}(\underline{z}). \tag{7.16}$$

Note that we no longer have to keep in mind whether or not $C_{\underline{k}+1_i-1_n}$ exists. If it does it must mean that $\underline{k}_n = 0$ and the term wont contribute anyway. Due to the $(-1)^{l+1}$ symmetry of Σ_l we can write 7.16 as

$$\underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}+1_i} \underline{\Pi}(\underline{z}) = \frac{1}{2} \partial_{z_i} C_{\underline{k}}(\underline{z}) - \sum_{n=1}^{l+1} (-1)^{\underline{k}_n} \frac{\underline{k}_n}{2} \partial_{z_n} C_{\underline{k}+1_i-1_n}(\underline{z}). \tag{7.17}$$

We can insert eq. 7.17 back into eq. 7.14 to find yet more relations

$$\partial_{z_i} \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}} \underline{\Pi}(\underline{z}) = \frac{1}{2} \partial_{z_i} C_{\underline{k}}(\underline{z}) + \sum_{n=1}^{l+1} (-1)^{\underline{k}_n} \frac{\underline{k}_n}{2} \partial_{z_n} C_{\underline{k}+1_i-1_n}(\underline{z}). \tag{7.18}$$

Finally we can obtain more relations by using the fact that the all Picard-Fuchs operators $\mathcal{L}_{l,i}$ annihilate the maximal cuts,

$$\begin{aligned} \partial_{z_i} \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}} \underline{\Pi}(\underline{z}) &= \partial_{z_i} C_{\underline{k}}(\underline{z}) - \underline{\Pi}(\underline{z}) \Sigma_l \partial_{\underline{z}}^{\underline{k}+1_i} \underline{\Pi}(\underline{z}) \\ &= \partial_{z_i} C_{\underline{k}}(\underline{z}) - \underline{\Pi}(\underline{z}) \Sigma_l \sum_{\underline{\eta} \in \sigma^{(l)}/(\underline{k}+1_i)} \frac{\beta_{i,\underline{\eta}}^{(l)}(\underline{z})}{\beta_{i,\underline{k}+1_i}^{(l)}(\underline{z})} \partial_{\underline{z}}^{\underline{\eta}} \underline{\Pi}(\underline{z}), \end{aligned} \tag{7.19}$$

where $\sigma^{(l)}$ is the set that contains all ways of filling $l+1$ slots, such that their sum is $\leq l$. This can also be seen as the collection $\sigma^{(l)} = \underline{s} \cup \underline{\omega}$ where $\underline{s}$ and $\underline{\omega}$ are defined as in the previous section.

We can take the derivative of eq. 7.14 once more and use the Picard-Fuchs operators to gain new relations. It is worth noting once again, that choosing a particular set of Picard-Fuchs operators may be more suited for gaining new Griffiths relations. When acting with $\partial_{\underline{z}}^{\eta}(\underline{z})$ on $\underline{\Pi}(\underline{z})$, some of the elements can be described in terms of $\Pi(\underline{z})$, whereas other elements will be a part of $\underline{\Omega}(\underline{z})$, containing all the $\varpi_{\underline{p}}(\underline{z})$ that we needed to solve for in the previous section. In the sum of eq. 7.19, we have removed one of the l-order differentials and as such the sum still contains $s-1$ elements of $\underline{\Omega}(\underline{z})$, where s is defined as eq. 7.5. Solving the remaining $s-1$ elements of $\underline{\Omega}(\underline{z})$, the sum will only contain differentials of order $l-1$ or lower. As such we are able to use our Griffiths relations and simplify the computations. To do this we once again need to find a set $\mathcal{D}_l^{(n=2)} = \{\mathcal{L}_1, ..., \mathcal{L}_m\}$, whose solution space is spanned by $\underline{\Pi}(\underline{z})$.

We can keep taking derivatives of eq. 7.14 and using the Picard-Fuchs operator to gain more and more relations. Eventually we will have enough relations to construct a generic-mass version of the matrix

$$\mathbf{Z}_l(\underline{z}) = \begin{pmatrix} \underline{\Pi}(\underline{z})\Sigma_l\underline{\Pi}(\underline{z}) & ... & \underline{\Pi}(\underline{z})\Sigma_l\partial_{\underline{z}}^{\underline{s}\lambda}\underline{\Pi}(\underline{z}) \\ \vdots & \ddots & \vdots \\ \partial_{\underline{z}}^{\underline{s}\lambda}\underline{\Pi}(\underline{z})\Sigma_l\underline{\Pi}(\underline{z}) & ... & \partial_{\underline{z}}^{\underline{s}\lambda}\underline{\Pi}(\underline{z})\Sigma_l\partial_{\underline{z}}^{\underline{s}\lambda}\underline{\Pi}(\underline{z}) \end{pmatrix}. \tag{7.20}$$

Note that due to the $(-1)^{l+1}$-symmetry of Σ_l, $\mathbf{Z}_l(\underline{z})$ has the same symmetric; $\mathbf{Z}_l(\underline{z}) = (-1)^{l+1}\mathbf{Z}(\underline{z})^T$. One can show that in the generic-mass case with Σ_l the inverse Wronskian can still be expressed as

$$\mathbf{W}(\underline{z})^{-1} = \Sigma_l \mathbf{W}(\underline{z})^T \mathbf{Z}(\underline{z})^{-1}. \tag{7.21}$$

In the generic-mass case, we are then once again able to describe the inverse Wronskian linearly in terms of entries in $\mathbf{W}_l(\underline{z})$.

7.3 Symbol and coaction on generic-mass master integrals for even l

We now have all the ingredients we need in order to build a Symbol and coaction in the generic-mass case. First of all we make the basis change in eq. 5.7 and obtain the differential equation

$$\mathrm{d}\underline{L}_l(\underline{z}) = \mathbf{W}_l(\underline{z})^{-1}\underline{N}_l(\underline{z}), \tag{7.22}$$

in the $\epsilon \to 0$ limit. We are not able to write $\mathrm{d}\underline{L}_l(z)$ explicitly due to the fact that we do not know how to construct Σ_l in the generic-mass case. Depending on where Σ_l has entries,

$\mathrm{d}L_l(\underline{z})$ may be linear in just one element or linear in many elements of the Frobenius basis. We can however make comments on which quantities we want to be unipotent and semi-simple. We choose to define the unipotent quantities as the ratio

$$\tau_{\underline{k}}(\underline{z}) = \frac{\varpi_{\underline{k}}(\underline{z})}{\varpi_{\underline{s}_\lambda}(\underline{z})}. \tag{7.23}$$

Our argumentation is the same as in the equal-mass case: $\varpi_{\underline{s}_\lambda}(\underline{z})$ will contain more $\log(\underline{z})$ divergences than most other elements in $\underline{\Pi}(\underline{z})$. It is however possible that, in the generic-mass case there can be a degeneracy in numbers of $\log(\underline{z})$ divergences and as such there can exist elements in $\underline{\Pi}(\underline{z})$ with as many, but never more, $\log(\underline{z})$ divergences.
The Wronskian contains derivatives up to $l-1$ order. Therefore $\mathrm{d}\underline{L}_l(z)$ will contain elements that are proportional to both $\tau_{\underline{k}}(z)$ and $\partial^{\underline{k}}_{\underline{z}}\tau_{\underline{k}}(z)$. W can write each element of $\mathrm{d}\underline{L}_l(z)$ as being some function

$$f(\varpi_0(z), ..., \varpi_{\underline{s}_\lambda}(z), \underline{z}) = X(\underline{z}) + D(\underline{z}), \tag{7.24}$$

where

$$\begin{aligned} X(\underline{z}) &= \sum_i^\lambda \sum_{\underline{s}} \alpha_{\underline{s}_i}(\underline{z}) \partial^{\underline{s}_i}_{\underline{z}} \varpi_\lambda(\underline{z}) \tau_{\underline{s}_i}(\underline{z}), \\ D(\underline{z}) &= \sum_i^\lambda \sum_{\underline{s}} \sum_{\substack{m,n \\ n\neq 0 \\ \underline{s}_i = m+n}} \gamma_{\underline{s}_i}(\underline{z}) \partial^{\underline{m}}_{\underline{z}} \varpi_\lambda(\underline{z}) \partial^{\underline{n}}_{\underline{z}} \tau_{\underline{k}}(\underline{z}). \end{aligned} \tag{7.25}$$

The coefficients $\alpha_{\underline{s}_i}(\underline{z})$ and $\gamma_{\underline{s}_i}(\underline{z})$ are linear combinations of one-forms of the form $\sum_i \omega_i(\underline{z}) dz_i$. Strictly speaking the elements $\tau_{\underline{k}}(\underline{z})$ are not unipotent quantities, since they do not obey a unipotent differential equation. To make them unipotent quantities, we should do the same trick as we did in the equal-mass case, which is to expand the basis $\underline{L}_l(\underline{z})$ with all $\tau_{\underline{k}}(\underline{z})$

$$\underline{T}_l(\underline{z}) = \left(L_1(\underline{z}), L_2(\underline{z}), ..., L_\lambda(\underline{z}), \tau_0(\underline{z}), ..., \tau_{s_{\lambda-1}}(\underline{z}), 1\right). \tag{7.26}$$

This once again allows us to write a unipotent differential equation

$$d\underline{T}_l(\underline{z}) = \mathbf{N}_l(\underline{z}) \underline{T}(\underline{z}), \tag{7.27}$$

where $\mathbf{N}_l(\underline{z})$ is a $2\lambda \times 2\lambda$ nilpotent matrix. Unfortunately we are unable to go any further in our derivations. Without an explicit form of Σ_l it is not possible to explicitly give $\mathbf{N}_l(\underline{z})$. As such we will not be able to calculate its symbol nor can we make any comments on the length of the symbol in the generic-mass case. Working out the form of Σ_l in the generic-mass case would be very interesting in future work.

8 Conclusion

In this thesis we have presented an explicit construction of a symbol and coaction operator on l-loop, equal-mass banana integrals. We are able to express our master integrals as functions that depend on elements in the Frobenius basis. Furthermore, we show this dependance can be made linear, due to Griffiths transversality. We decompose our master integrals into a semi-simple and unipotent part. We choose to define our unipotent quantities as the ratio between two elements in the Frobenius basis. We construct a unipotent differential equation, for arbitrary loop order, that is satisfied by a vector containing all our master integrals and the unipotent elements. The unipotent differential equation can be used to construct a symbol operator on pairs of periods in the extended basis. The symbol operator allow us to define a coaction. We explicitly provide the expressions of the symbol and coaction on $l = 2, 4$ master integrals. We find that the symbol length of our master integrals is two at two loop orders and three for all loop orders higher than two. This agrees with what has previously been found by Broedel et al. [11]. Right now, we are only able to compare the two results in their tensorial structure. It would in the future be interesting to verify that the two results also agree numerically. We compare our coaction results with that of Broedel et al. [11]. In this case, we find one extra term that does not match what they find the coaction to be. A resolution to this disagreement has not been found. Finally we attempt to generalize our symbol and coaction operator to generic-mass master integrals. We are able to derive relations for generic-mass maximal cuts using a generalized version of Griffiths transversality. As the explicit form of the intersection matrix Σ_l in the generic-mass case is unknown, we are unable to derive any symbol or coaction for generic-mass master integrals. Our shortcomings suggest an interesting direction of future research. It would be interesting to find an explicit form of the intersection matrix in the generic-mass case. Once the intersection matrix is known, it is easy to define a symbol and coaction on the generic-mass master integrals using the methods we described in section 7. The symbols and coactions of generic-mass master integrals are particularly interesting. At the pseudo thresholds $p^2 = (m_1 + m_2 - m_3 - 2m_i)$ with $i = 1, 2, 3$, the sunrise diagram ($l = 2$) can be expressed in terms of MPLs [20]. It would be interesting to apply our definition of the symbol and coaction at the pseudo-threshold. By doing so, we should obtain the usual symbol and coaction of MPLs. Another direction that is worth exploring is the choices of semi-simple and unipotent quantities. We have before commented that the choice of semi-simple and unipotent is arbitrary, but some choices will endow the symbol with more information than others. It would be interesting to do more research into variations of semi-simple and unipotent quantities.

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