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Carroll Geometry and Ultra-Relativistic Gravity

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Abstract

The Carroll group, which emerges as the ultra-relativistic limit of the Poincaré group, and its local realization in terms of Carroll geometry have recently received renewed interest due to their connection to flat space holography. The study of Carroll geometry, Carroll gravity and Carrollian field theories is thus currently emerging as a new direction of research. It is the intention of this thesis to further develop the subject of Carroll geometry and its application to the study of gravity. The main goal of this thesis is to derive and analyze a theory of Carrollian gravity through a novel small c expansion of the Einstein-Hilbert action. To attain this objective we review and further develop basic notions of Carroll geometry. In particular, we will review the construction of Carroll geometry through a gauging procedure and its appearance as the natural geometry on null hypersurfaces in Lorentzian geometry. We shall develop basic aspects of Carrollian field theories by considering the construction of energy-momentum tensors. Furthermore, the degenerate metric structure of Carroll geometry does not naturally single out a distinguished connection like the Levi-Civita connection of Lorentzian geometry. Thus, we will explore what obstructions to such a natural connection exist and what choices need to be made to single out a Carrollian analog of the Levi-Civita connection. These preliminary considerations allow us to perform an expansion of the Einstein-Hilbert action in powers of c^2. We will consider this expansion up to next-to-leading order. As the leading-order theory resembles the 3+1 decomposition of general relativity, we review this framework in order to adapt the methods thereof to the Carrollian theory. We then examine the leading-order theory in detail and develop new methods for obtaining solutions and computing boundary charges. Finally, we present several examples of solutions to the leading-order theory using the methods developed in this thesis.

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Contents

Chapter 1

Introduction

The most successful theories of nature all realize relativistic spacetime symmetries, whether it be the global Poincaré symmetry of Quantum Field Theory (QFT) or the local Lorentz symmetry of General Relativity (GR). Still, during the past few years there has been a renewed interest in so-called non-Lorentzian geometries building on the Galilei and Carroll groups. These alternate spacetime symmetry groups can respectively be obtained as either the infinite or vanishing speed of light limit (c) of the Poincaré group. Exploring these limits may seem like an academic exercise, but there is good reason to consider both.

The Galilei group has historically gained the most attention because it naturally occurs as the spacetime symmetry group of effective descriptions of systems moving much slower than the speed of light. The geometric study of non-relativistic (NR) spacetimes goes back to the work of Cartan [1, 2] who geometrized Newtonian gravity in what is today known as Newton-Cartan (NC) geometry. One natural application of this NC geometric framework is to approximate GR at small speeds and weak fields in a post-Newtonian approximation [3]. More recently, NR theories have been applied to address problems of modern physics either as predictive physical models or as toy models to gain insights that could help understand their relativistic counterparts. The NC framework presents itself as the natural covariant formulation of many NR phenomena: the NC setup has for example found use in biophysics. The authors of [4] study NC submanifolds and find it to be a natural framework for describing fluids moving on curved membranes. Another example is the application of a relaxed variation of NC, known as Aristotelian geometry, to generalizing hydrodynamics to non-boost invariant fluids [5]. Finally, NC geometry has also been applied to condensed matter systems where maybe most notably Son [6], guided in part by NR covariance, constructed an effective field theory modeling quantum Hall states. Another research direction is attempting to develop a theory of quantum gravity in the NR realm and to use that as a guide to construct a relativistic theory. A proposal for a theory of quantum gravity is Hořava–Lifshitz gravity [7], which has been shown to be realizable in terms of dynamical NC geometry [8]. Related to these efforts, work has also been done to understand non-relativistic string theory and holography [9–11] to obtain a more tractable NR limit of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. Many of the recent developments within NR holography were spurred by the discovery that NC geometry emerges naturally when trying to extend the AdS/CFT correspondence to a non-relativistic setting. Specifically, it was shown that the relevant boundary geometry for holography on spacetimes exhibiting asymptotic Lifshitz scaling symmetry is the so-called torsional Newton-Cartan geometry [12–15].

The converse limit $c \to 0$ of the Poincaré group results in the Carroll group, which despite it being less well-known, also has relevance for profound problems in theoretical physics. In particular, Carroll geometry turns out to be intimately related to some of the building blocks of flat space holography. To appreciate the importance of this, it is instructive to consider the more developed cousin of flat space holography: Anti-de Sitter space holography. The ideas of the AdS/CFT correspondence [16] and the holographic principle have been some of the most influential paradigms in 21$^{\text{st}}$ century theoretical physics and have provided a new angle to study quantum gravity and strongly coupled field theories. However, applying the holographic principle only to asymptotically AdS spacetimes is unsatisfactory for multiple reasons: Firstly, the area law of black hole entropy [17] dictates that the gravitational degrees of freedom grow as the area rather than the volume. This suggests that the gravity theory should have a holographic description living in a lower dimensional space. As black holes are a generic feature of GR and not only asymptotically AdS spacetmes, one would expect a holographic principle to extend to asymptotically flat spacetimes as well. Secondly, from a practical point of view, physical systems are often well-approximated by asymptotically flat spacetimes rather than asymptotically AdS spacetimes. Consequently, holography would be a more useful tool if it could be extended to flat space.

Hence, an important research direction is to understand flat space holography. A first indication that Carroll geometry is related to this program is the fact that part of the conformal boundary of asymptotically flat spacetimes, i.e. light-like infinity, is a null hypersurface. In chapter 2 we will show that Carroll geometry is naturally induced on null hypersurfaces in Lorentzian theories [18]. Thus, boundary field theories in flat space holography would presumably couple to a Carrollian geometry. In addition to this preliminary observation, one can also take inspiration from one of the first clues of the AdS/CFT correspondence: the asymptotic symmetry group of AdS. Specifically, Brown and Henneaux [19] found that the asymptotic symmetry algebra of AdS_3 is two commuting copies of the Virasoro algebra. This algebra corresponds to the symmetry algebra realized by the dual 2D conformal field theory. Following this idea, one can consider the asymptotic symmetry group of asymptotically flat spacetimes known as the BMS group [20, 21]. The structure of Carroll geometry also arises naturally in considering BMS symmetry because the BMS group has been shown to be isomorphic to the conformal Carroll group [22–24]. Some work has been done exploring candidate boundary field theories for flat space holography e.g. from both the BMS [25] and conformal Carroll [26] perspectives. Hence a major motivation for understanding how Carrollian field theories can be constructed and how they behave is that Carroll geometry may present itself as the natural framework for understanding flat space holography. Other approaches to flat space holography also exist such as celestial CFTs [27], where one maps the scattering amplitudes of a QFT in an asymptotically flat spacetime to correlators of a CFT living on a so-called celestial sphere at null infinity.

Another null hypersurface where Carroll geometry has found use is the event horizon of a black hole. In [28] the authors consider the membrane paradigm, which is an effective description of black hole dynamics as a fluid living on the event horizon. Specifically, they show that the equations governing the black hole dynamics can be understood as Carrollian conservation laws akin to those we will derive in chapter 2.

In a recent paper [29], the authors extended the methods of [30] and performed a fully covariant large c expansion of the Einstein-Hilbert (EH) action to obtain an action principle for Newtonian gravity in the NC framework. The methods developed in this paper can be equally applied in an ultra-relativistic limit of the EH action to derive a Carrollian theory of gravity. Some efforts have already been made to study Carroll gravity from an ultra-relativistic limit [31–33] as well as from an effective field theory approach [18]. However, no prior works have considered a systematic ultra-relativistic expansion of the EH action, and hence it presents a natural opportunity to study a Carrollian theory of gravity.

The goal of this thesis is to develop the understanding of Carroll geometry including Carrollian gravity as well as field theories coupling to a Carrollian background. We will focus on classical aspects of Carroll geometry and in particular explore how it emerges from both the Carroll symmetry algebra and as an induced geometry in Lorentzian theories. We further aim to address some of the natural questions of classical field theory such as gauge and diffeomorphism covariance, the construction of energy-momentum tensors and the existence of conserved symmetry charges. To answer these questions in a covariant manner the formulae call for a choice of connection. However, Carroll geometry does not have an established connection like the Levi-Civita connection in Riemannian geometry. Hence, we will also investigate to what extent Carroll structures can be endowed with a natural connection. These tools all become valuable in an ultra-relativistic expansion of the Einstein-Hilbert action, where we repurpose the methods of [29] to obtain an action principle for Carroll gravity order by order in c^2. Furthermore, we review the 3+1 formalism in order to pursue a new connection between the ultra-relativistic expansion and the 3+1 decomposition. Finally, having a Carrollian theory of gravity at hand we investigate what kind of dynamics is possible in the ultra-relativistic limit.

1.1 Structure of the thesis

The thesis is roughly organized in two parts: Chapters 2 and 3 review and develop Carroll geometry, while chapters 4, 5 and 6 are concerned with the expansion of the EH action and subsequent analysis of the theory.

We start in chapter 2 by considering the emergence of the Carroll algebra from an ultra-relativistic limit and the construction of the Carroll geometry through a gauging procedure. The second half of chapter 2 presents the connection to null hypersurfaces and explores different aspects of general field theories coupled to a Carroll background. In chapter 3 we address the question of natural connections for Carroll geometries. In particular, we review the notion of intrinsic torsion and its consequences. Furthermore, we develop a procedure that singles out an analog of the Levi-Civita connection for Carroll geometry.

Chapter 4 follows the methods of [29] and primes the EH action for an ultra-relativistic expansion. Further, the leading-order (LO) and next-to-leading-order (NLO) action as well as equations of motion (EOM) are derived (only partially for NLO). In chapter 5 we take an intermezzo and review the basics of the 3+1 decomposition of general relativity in anticipation of its use in the LO theory. Furthermore, the 3+1 formalism provides us with an alternate approach to expanding the EH action. Chapter 6 is devoted to exploring the LO theory in vacuum and possible solutions along with a characterization through boundary charges. Finally in chapter 7, we summarize the results of the thesis and discuss ideas for future work.

Chapter 2

Carroll Group and Geometry

The Carroll group is best understood as an ultra-relativistic limit i.e. $c \to 0$ of the Poincaré group as it was first done by Lévy-Leblond [34]. Taking the limit of vanishing speed of light corresponds to the light cone collapsing to a line, rendering all spatially separated points causally disconnected. This is in contrast to the more well-known Galilean limit $c \to \infty$ which flattens the light-cone and consequently, information propagates instantaneously. The name Carroll is due to the rather peculiar causal structure and hence is a reference to Lewis Carroll's stories about Alice in Wonderland. The Carrollian causal structure also implies that particles do not move and cannot be boosted to do so [35].

To contrast these converse limits in c and appreciate the duality between them [18], we can schematically consider the scaling of the components of a Lorentz transformation in c

$$\Lambda^{\mu}{}_{\nu} = \left[\begin{array}{c|c} \mathcal{O}(1) & \mathcal{O}(c^{-1}) \\ \hline \mathcal{O}(c) & \mathcal{O}(1) \end{array}\right], \tag{2.1}$$

where we used the coordinates $x^\mu = (t, x^1, \ldots, x^d)$ on Minkowski space. From (2.1) it is clear that in the limit $c \to 0$ the row dominates, while for $c \to \infty$ the column survives. Hence we see the two opposite limits of c give rise to similar structures as they are related by a transpose. Their embedding in the larger group $\mathrm{GL}(d+1, V)$ is however different and sets the two limiting groups apart. One way to see this duality is that the Galilean limit singles out a co-dimension 1 spatial subspace of the vector space V it is acting on, while the Carrollian limit distinguishes a co-dimension 1 subspace of V^*, the dual vector space.

In this chapter, we will review in section 2.1 how the Carroll algebra emerges as the limit of Poincaré and in section 2.2 how to gauge the algebra to obtain the corresponding geometry. In section 2.3 we will present the connection between null hypersurfaces and Carroll geometry, and finally in section 2.4 we explore field theories coupled to a Carrollian background.

2.1 The Carroll algebra

As stated, we can derive the Carroll algebra as an Inönü-Wigner contraction [36] of the Poincaré algebra. If we consider a $d+1$ dimensional spacetime then the Poincaré algebra takes the form

$$[J_{AB}, P_C] = 2\eta_{C[A}P_{B]}, \tag{2.2a}$$

$$[J_{AB}, J_{CD}] = 4\eta_{[A[D}J_{C]B]},, \tag{2.2b}$$

with $\eta_{AB} = \mathrm{diag}(-1, 1, \ldots, 1)$ being the Minkowski metric, J_{AB} anti-symmetric and upper case indices $A, B, C, \ldots = 0, 1, \ldots, d$. To obtain the Carroll limit, we explicitly split space and time by redefining generators and reintroducing factors of the speed of light through the contraction parameter $\epsilon \sim c$

$$P_0 = \epsilon^{-1} H, \qquad J_{0a} = \epsilon^{-1} C_a, \tag{2.3}$$

where for lower case indices $a, b, c, \ldots = 1, 2, \ldots, d$. Due to the signature of the flat metric η_{AB} we simply raise and lower the indices $a, b, \ldots$ with the Kronecker-δ. To obtain the Carroll algebra as the contraction of (2.2a)-(2.2b), we need to consider the implications of (2.3) when taking the limit $\epsilon \to 0$. We can consider a few examples of this contraction

$$[P_a, J_{0b}] = \delta_{ab} P_0 \quad \Rightarrow \quad \epsilon^{-1}[P_a, C_b] = \epsilon^{-1}\delta_{ab} H \quad \xrightarrow{\epsilon \to 0} \quad [P_a, C_b] = \delta_{ab} H, \tag{2.4a}$$

$$[P_0, J_{0a}] = P_a \quad \Rightarrow \quad \epsilon^{-2}[H, C_a] = P_a \quad \xrightarrow{\epsilon \to 0} \quad [H, C_a] = 0. \tag{2.4b}$$

Repeating this process for all possible combinations of generators yields the following non-zero Lie brackets for the Carroll algebra

$$[J_{ab}, P_c] = 2\delta_{c[a} P_{b]}, \tag{2.5a}$$

$$[J_{ab}, C_c] = 2\delta_{c[a} C_{b]}, \tag{2.5b}$$

$$[J_{ab}, J_{cd}] = 4\delta_{[a[d} J_{c]b]}, \tag{2.5c}$$

$$[P_a, C_b] = \delta_{ab} H. \tag{2.5d}$$

On a group-theoretic level, one can also show that the Carroll algebra can be obtained as the semi-direct sum of $\mathfrak{so}(d)$ (realized by J_{ab}) acting on the Heisenberg algebra (for P_a and C_a) with H as its central element. Another way of splitting the algebra, that will have relevance for the gauging procedure, is to consider the Carroll algebra as the semi-direct sum of rotations and boosts $\mathfrak{g} = \langle J_{ab}, C_a \rangle$ with the abelian ideal of translations $\mathfrak{t} = \langle H, P_a \rangle$.

2.2 Gauging kinematical algebras

There exists several equivalent ways of approaching the gauging of the Carroll algebra (2.5a)-(2.5d), which ultimately boils down to how the vielbein and spin connection transform under and act according to the underlying Lie algebra. One approach [8] uses so-called $\bar{\delta}$-transformations constructed such that they suggest the generator (H, P_a) as the vielbeine, and one uses these to obtain the geometric data from the Lie algebra structure. We will follow a slightly different path using the theory of affine connections on a principal bundle [37], from which one reduces down to the usual frame bundle objects. We will outline the mathematical background in section 2.2.1 and subsequently work through the example of the Poincaré algebra in section 2.2.2 before repeating the procedure for Carroll algebra in section 2.2.3.

2.2.1 Principal bundles and affine connections

We start by noting that both the Poincaré group and the Carroll group (and other kinematical groups) have a semi-direct product structure i.e. they can be written as

$$A = G \ltimes T, \tag{2.6}$$

where A is the affine group (e.g. Poincaré or Carroll), $T = \mathbb{R}^{d+1}$ and $G \leq \mathrm{GL}(d+1)$ (the Lorentz group in the example of Poincaré). This structure also appears at the level of the algebra

$$\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{t}, \tag{2.7}$$

with $\mathfrak{a}$, $\mathfrak{g}$ and $\mathfrak{t}$ being the corresponding Lie algebras.

We can then consider a principal A-bundle $\hat{P} \xrightarrow{\hat{\pi}} M$ with the spacetime $(d+1)$-manifold M as the base space. We can further equip $\hat{P}$ with an Ehresmann connection $\hat{\omega} \in \Omega^1(\hat{P}, \mathfrak{a})$ i.e. a 1-form taking values in the Lie algebra of A and transforming in the adjoint representation. The Ehresmann connection encodes how nearby affine frames relate and will thus, with the Lie algebra split (2.7) in mind, give rise to both the vielbeine valued in $\mathfrak{t}$ and the spin connection valued in $\mathfrak{g}$. Specifically, it satisfies two important features: It annihilates horizontal vectors and, when acting on a fundamental vector field, it returns the Lie algebra element generating the vector field. The next step is to reduce the affine bundle to a frame bundle which amounts to reducing the structure group A down to G and reinterpreting the pullback connection. Technically, the reduction of structure group is possible if we can supply a section $\hat{\sigma}$ of the associated bundle $\hat{P} \times_A (A/G)$ [37]. Choosing a section of this associated bundle intuitively corresponds to choosing an origin in each of the affine frames. Thus, we reduce our principal A-bundle $\hat{P}$ to a principle G-bundle P. Let $\gamma : P \to \hat{P}$ be the inclusion map of this G-restriction. We can then consider the pullback of the connection $\hat{\omega}$ to the G-bundle P

$$\gamma^* \hat{\omega} = \omega + \theta, \tag{2.8}$$

where we utilized the natural split of the Lie algebra i.e. $\omega \in \Omega^1(P, \mathfrak{g})$ and $\theta \in \Omega^1(P, \mathfrak{t})$. As the structure group of P is G, an Ehresmann connection on P should be valued in $\mathfrak{g}$. Further, the pullback turns the components of $\hat{\omega}$ that give rise to the projective property on $\mathfrak{t}$ into horizontal components making up θ. Thus, we interpret ω to be an Ehresmann connection on P and θ as the solder form i.e. the 1-form that relates vectors in the tangent bundle TM with vectors in the frame bundle. With these identifications, we have reduced the affine bundle $\hat{P}$ into a frame bundle P with solder form θ and connection ω.

In the following sections 2.2.2 and 2.2.3 we will not be dealing with the full objects ω and θ, but rather their pullbacks to the base manifold M through a section σ of the frame bundle P. The section σ then has the interpretation as a choice of moving frames and the pullback of the connection and solder form realize the spin connection and vielbeine with respect to that choice of frame. We will abuse notation and still denote $\sigma^*\omega$ and $\sigma^*\theta$ as ω and θ, respectively. The construction can be visualized by the following diagram

$$\begin{tikzcd}
\text{Affine-bundle} \arrow[dr, bend left] & & \text{Frame-bundle} \arrow[d] & & (d+1)\text{-manifold} \arrow[dl, bend right] \\
& \hat{P} \arrow[r, "\hat{\pi}"] & P \arrow[l, bend left, "\gamma"] \arrow[r, "\pi"] & M \arrow[l, bend left, "\sigma"] & \\
& \text{Choice of origin} & & \text{Choice of frame} &
\end{tikzcd} \tag{2.9}$$

What one in physics usually thinks of as gauge transformations of the pulled-back forms or local representatives ω and θ, corresponds in this language to changing the section σ i.e. the choice of frame. It can be shown [38] that if two choices of moving frames are related by a gauge transformation[1] $g : M \to G$, then the local representatives transform as

$$\theta' = g^{-1}\theta g, \tag{2.10a}$$
$$\omega' = g^{-1}\omega g + g^{-1}dg. \tag{2.10b}$$

[1]Here and in the rest of the thesis we are glossing over the fact that for non-trivial frame bundles the local representatives and transformation rules only exist on local trivializations.

The transformations (2.10a)-(2.10b) are familiar from gauge theories. The co-frame (2.10a) transforms covariantly in the adjoint representation like e.g. the field strength of Yang-Mills theory, while the connection transforms non-covariantly (2.10b) with respect to the gauge group.

Importantly, the connection ω gives rise to a notion of parallel transport and thus a covariant derivative. Technically, this can be implemented by considering the induced connection on associated bundles with typical fiber being tensor powers of $\mathfrak{t}$ and $\mathfrak{t}^*$ which both have a natural $\mathfrak{g}$-action in terms of the adjoint and co-adjoint action, respectively. This covariant derivative is sometimes called an adapted connection as tensors that are left invariant at the level of group action induce tensor fields in the tangent bundle which are automatically parallel.

2.2.2 Gauging the Poincaré algebra

As a perhaps more familiar example, we will warm up by going through the gauging procedure for the Poincaré algebra (2.2a)-(2.2b). The Poincaré algebra splits according to (2.7) as $\mathfrak{g} = \langle J_{AB} \rangle$ and $\mathfrak{t} = \langle P_a \rangle$. The starting point for the gauging procedure is defining the connection ω and the co-frame θ valued in $\mathfrak{g}$ and $\mathfrak{t}$, respectively, as

$$\theta = e^A \otimes P_A, \tag{2.11a}$$

$$\omega = \frac{1}{2}\Omega^{AB} \otimes J_{AB}, \tag{2.11b}$$

with $e^A, \Omega^{AB} \in \Omega^1(M)$ and $\Omega^{(AB)} = 0$. We can then consider how θ and ω transform under a infinitesimal $\mathfrak{g}$-valued gauge transformation

$$\Sigma = \frac{1}{2}\Lambda^{AB} \otimes J_{AB}, \tag{2.12}$$

with Λ^{AB} being some anti-symmetric parameter and $\Sigma \in \Omega^0(M, \mathfrak{g})$. Under Σ, the co-frame θ transforms according to the infinitesimal adjoint action corresponding to (2.10a)

$$\delta\theta = [\theta, \Sigma] = \frac{1}{2}\Lambda^{AB} e^C \otimes [P_C, J_{AB}] = \Lambda^A{}_B e^B \otimes P_A, \tag{2.13}$$

where we used the bracket (2.2a) and lowered the index with η_{AB}. For the connection, the infinitesimal transformation rule follows from the finite transformation (2.10b) as

$$\delta\omega = d\Sigma + [\omega, \Sigma] = \frac{1}{2} d\Lambda^{AB} \otimes J_{AB} + \frac{1}{4}\Lambda^{AB}\Omega^{CD} \otimes [J_{CD}, J_{AB}] \tag{2.14}$$

$$= \frac{1}{2}\left[d\Lambda^{AB} + 2\Lambda_C{}^{[A}\Omega^{B]C}\right] \otimes J_{AB}, \tag{2.15}$$

where we used (2.2b). These results can be written out in component form

$$\delta e^A_\mu = \Lambda^A{}_B e^B_\mu, \tag{2.16a}$$

$$\delta\Omega_\mu{}^{AB} = \partial_\mu\Lambda^{AB} + 2\Lambda_C{}^{[A}\Omega_\mu{}^{B]C}, \tag{2.16b}$$

which we recognize as the usual laws for local Lorentz transformations. The next natural thing to compute is the associated torsion and curvature 2-forms, which follow from the Cartan structure equations. In particular, we can compute the torsion T as

$$\begin{aligned} T = D\theta = d\theta + [\omega \wedge \theta] &= de^A \otimes P_A + \frac{1}{2}\Omega^{AB} \wedge e^C \otimes [J_{AB}, P_C] \\ &= \left[de^A - \Omega^A{}_B \wedge e^B\right] \otimes P_A, \end{aligned} \tag{2.17}$$

where D denotes the covariant exterior derivative and $[\cdot \wedge \cdot]$ is the wedge product of Lie algebra valued differential forms defined by $[(\omega \otimes A) \wedge (\eta \otimes B)] = \omega \wedge \eta \otimes [A, B]$. The curvature 2-form R follows in a similar fashion

$$\begin{aligned} R = D\omega = d\omega + \frac{1}{2}[\omega \wedge \omega] &= \frac{1}{2} d\Omega^{AB} \otimes J_{AB} + \frac{1}{8}\Omega^{AB} \wedge \Omega^{CD}[J_{AB}, J_{CD}] \\ &= \frac{1}{2}\left[d\Omega^{AB} + \Omega^{A}{}_{C} \wedge \Omega^{BC}\right] \otimes J_{AB}. \end{aligned} \tag{2.18}$$

The factor of $\frac{1}{2}$ in the curvature structure equations is due to the fact that ω does not transform covariantly. If we further define

$$T = \frac{1}{2} T_{\mu\nu}{}^{A} dx^{\mu} \wedge dx^{\nu} \otimes P_A, \tag{2.19a}$$

$$R = \frac{1}{4} R_{\mu\nu}{}^{AB} dx^{\mu} \wedge dx^{\nu} \otimes J_{AB}, \tag{2.19b}$$

then we can write out the torsion and curvature 2-forms in components to obtain the familiar expressions [39]

$$T_{\mu\nu}{}^{A} = 2\partial_{[\mu} e^{A}_{\nu]} - 2\Omega_{[\mu}{}^{AB} e_{\nu]B} \tag{2.20a}$$

$$R_{\mu\nu}{}^{AB} = 2\partial_{[\mu} \Omega_{\nu]}{}^{AB} + 2\Omega_{[\mu}{}^{AC} \Omega_{\nu]}{}^{B}{}_{C}. \tag{2.20b}$$

As an example of how this connection is adapted to the Lorentzian structure of the frame bundle, we can compute the covariant derivative of the metric. As the metric is a co-variant tensor, we need to introduce a basis of $\mathfrak{t}^*$ which we define by

$$\gamma^{A}(P_B) = \delta^{A}_{B}. \tag{2.21}$$

The metric can then be written as $\eta = \eta_{AB}\, \gamma^{A} \otimes \gamma^{B}$, which has a naturally induced transformation in terms of the co-adjoint action. We can then directly compute the covariant derivative of the metric

$$\begin{aligned} D\eta &= D(\eta_{AB})\, \gamma^{A} \otimes \gamma^{B} + \eta_{AB}\, D\gamma^{A} \otimes \gamma^{B} + \eta_{AB}\, \gamma^{A} \otimes D\gamma^{B} \\ &= d(\eta_{AB})\, \gamma^{A} \otimes \gamma^{B} + \eta_{AB}\, (\mathrm{ad}^{*}_{\omega}\, \gamma^{A}) \otimes \gamma^{B} + \eta_{AB}\, \gamma^{A} \otimes (\mathrm{ad}^{*}_{\omega}\, \gamma^{B}) \\ &= -\eta_{AB} \Omega_{C}{}^{A}\, \gamma^{C} \otimes \gamma^{B} - \eta_{AB} \Omega_{C}{}^{B}\, \gamma^{A} \otimes \gamma^{C} = -(\Omega_{AB} + \Omega_{BA})\, \gamma^{A} \otimes \gamma^{B} = 0, \end{aligned} \tag{2.22}$$

where ad^{*}_{ω} is the co-adjoint action of ω and the last expression vanishes by the anti-symmetry of Ω_{AB}. From the calculation (2.22) it is clear that the metric is conserved by the covariant derivative due to it being annihilated by the action of the Lorentz algebra $\mathrm{ad}^{*}_{J_{AB}}(\eta_{AB}\, \gamma^{A} \otimes \gamma^{B}) = 0$. This property of the covariant derivative is often contributed to the fact that Ω_{AB} is anti-symmetric, which is correct from an operational point of view, but in our language follows as a direct consequence of the algebra (2.2a)-(2.2b).

Finally, we can relate the connection ω on the frame bundle to an affine connection with coefficients $\Gamma^{\rho}_{\mu\nu}$. To do this we consider the co-frame θ as an identity map between sections of the tangent bundle and the frame bundle. We then impose the so-called vielbein postulate, which is the assertion that θ is parallel in the connection ω. Thus for any vector field X^{μ} we can compute

$$\begin{aligned} 0 &\overset{!}{=} \mathcal{D}_X \theta \\ &= \mathcal{D}_X(e^{A}_{\mu} dx^{\mu} \otimes P_A) = \mathcal{D}_X(e^{A}_{\mu}) dx^{\mu} \otimes P_A + e^{A}_{\mu} \mathcal{D}_X(dx^{\mu}) \otimes P_A + e^{A}_{\mu} dx^{\mu} \otimes \mathcal{D}_X(P_A) \\ &= X^{\rho}\left[\partial_{\rho} e^{A}_{\mu} - \Gamma^{\sigma}_{\rho\mu} e^{A}_{\sigma} - \Omega_{\rho}{}^{A}{}_{B} e^{B}_{\mu}\right] dx^{\mu} \otimes P_A, \end{aligned} \tag{2.23}$$

which in component form reads

$$\partial_\mu e^A_\nu - \Gamma^\rho_{\mu\nu} e^A_\rho - \Omega_\mu{}^A{}_B e^B_\nu = 0. \tag{2.24}$$

Analogously to how co-bases (2.21) are canonically defined, we can also define a frame related to the co-frame θ i.e. a map that relates co-vectors in the tangent bundle to co-vectors in the frame bundle. In components, the frame e^μ_A corresponding the co-frame e^A_μ can be defined by

$$e^A_\mu e^\nu_A = \delta^\nu_\mu, \qquad e^A_\mu e^\mu_B = \delta^A_B. \tag{2.25}$$

The e^μ_A provides the necessary inverses for us to solve (2.24) for the affine connection coefficients

$$\Gamma^\rho_{\mu\nu} = e^\rho_A \partial_\mu e^A_\nu - e^\rho_A \Omega_\mu{}^A{}_B e^B_\nu. \tag{2.26}$$

Hence, we see that we can obtain all the geometrical objects of pseudo-Riemannian geometry by gauging the Poincaré algebra (2.2a)-(2.2b). A further step one can take is to impose vanishing torsion on the connection, which allows us to solve (2.17) for the Levi-Civita connection and thereby specialize to the usual setting of GR.

2.2.3 Gauging the Carroll algebra

The results of this and the next two subsections reproduce those of [18], but we use the slightly different gauging approach as described in sections 2.2.1 and 2.2.2. We now repeat and slightly expand the gauging procedure carried out for the Poincaré algebra in section 2.2.2 for the Carroll algebra. The split of the generators into a semi-direct sum is, in the case of Carroll, $\mathfrak{g} = \langle J_{ab}, C_a \rangle$ and $\mathfrak{t} = \langle H, P_a \rangle$. We again only need the pullback of the solder form and Ehresmann connection, which we now define using the Lie algebra generators of $\mathfrak{t}$ and $\mathfrak{g}$, respectively

$$\theta = \tau \otimes H + e^a \otimes P_a, \tag{2.27a}$$

$$\omega = \Omega^a \otimes C_a + \frac{1}{2}\Omega^{ab} \otimes J_{ab}, \tag{2.27b}$$

where $\tau, e^a, \Omega^a, \Omega^{ab} \in \Omega^1(M)$ and $\Omega^{(ab)} = 0$. Having established a co-frame and a connection, we can consider their response to a gauge transformation of the form

$$\Sigma = \lambda^a C_a + \frac{1}{2}\lambda^{ab} J_{ab}, \tag{2.28}$$

with λ^a, λ^{ab} being real coefficients and λ^{ab} anti-symmetric. The co-frame transforms covariantly

$$\begin{aligned}\delta\theta = [\theta, \Sigma] &= \lambda^a \tau \otimes [H, C_a] + \lambda^a e^b \otimes [P_a, C_a] + \lambda^{ab} \tau \otimes [H, J_{ab}] + \frac{1}{2}\lambda^{ab} e^c \otimes [P_c, J_{ab}] \\ &= \lambda_a e^a \otimes H + \lambda^a{}_b e^b \otimes P_a,\end{aligned} \tag{2.29}$$

while the gauge connection transforms according to the infinitesimal form of (2.10b)

$$\begin{aligned}\delta\omega &= d\Sigma + [\omega, \Sigma] \\ &= d\lambda^a \otimes C_a + \frac{1}{2} d\lambda^{ab} \otimes J_{ab} + \frac{1}{2}\lambda^a \Omega^{bc} \otimes [J_{bc}, C_a] + \frac{1}{2}\lambda^{ab}\Omega^c \otimes [C_c, J_{ab}] + \frac{1}{4}\lambda^{ab}\Omega^{cd} \otimes [J_{cd}, J_{ab}] \\ &= (d\lambda^a + \lambda^a{}_b \Omega^b - \lambda_b \Omega^{ab}) \otimes C_a + \frac{1}{2}(d\lambda^{ab} + \lambda^a{}_c \Omega^{cb} - \lambda^b{}_c \Omega^{ca}) \otimes J_{ab},\end{aligned} \tag{2.30}$$

where we left out zero brackets in the second line. Alternatively, we can write out the same transformation rules in component form

$$\delta\tau_\mu = \lambda_a e^a_\mu, \tag{2.31a}$$
$$\delta e^a_\mu = \lambda^a{}_b e^b_\mu, \tag{2.31b}$$
$$\delta\Omega_\mu{}^a = \partial_\mu\lambda^a + \lambda^a{}_b\Omega_\mu{}^b - \lambda_b\Omega_\mu{}^{ab}, \tag{2.31c}$$
$$\delta\Omega_\mu{}^{ab} = \partial_\mu\lambda^{ab} + \lambda^a{}_c\Omega_\mu{}^{cb} - \lambda^b{}_c\Omega_\mu{}^{ca}. \tag{2.31d}$$

The choice of connection can be further characterized by its torsion and curvature as computed through Cartan structure equations. The torsion 2-form T is given by

$$T = D\theta = d\theta + [\omega\wedge\theta] = (d\tau + e_a\wedge\Omega^a)\otimes H + (de^a - e_b\wedge\Omega^{ba})\otimes P_a. \tag{2.32}$$

Similarly we can calculate the curvature 2-form as

$$\begin{aligned} R &= D\omega = d\omega + \frac{1}{2}[\omega\wedge\omega] \\ &= d\Omega^a\otimes C_a + \frac{1}{2}d\Omega^{ab}\otimes J_{ab} + \frac{1}{4}2\Omega^a\wedge\Omega^{bc}\otimes[C_a, J_{bc}] + \frac{1}{8}\Omega^{ab}\wedge\Omega^{cd}\otimes[J_{ab}, J_{cd}] \\ &= (d\Omega^a - \Omega^{ab}\wedge\Omega_b)\otimes C_a + \frac{1}{2}(d\Omega^{ab} - \Omega^{ca}\wedge\Omega^b{}_c)\otimes J_{ab}, \end{aligned} \tag{2.33}$$

where the indices $a, b, c, \ldots$ are raised and lowered using δ_{ab}. We can equivalently represent this in component form by defining

$$T = \frac{1}{2}T_{\mu\nu}(H)dx^\mu\wedge dx^\nu\otimes H + \frac{1}{2}T_{\mu\nu}{}^a(P)dx^\mu\wedge dx^\nu\otimes P_a, \tag{2.34a}$$
$$R = \frac{1}{2}R_{\mu\nu}{}^a(C)dx^\mu\wedge dx^\nu\otimes C_a + \frac{1}{4}R_{\mu\nu}{}^{ab}(J)dx^\mu\wedge dx^\nu\otimes J_{ab}, \tag{2.34b}$$

with respect to which we find

$$T_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]} + 2e^a_{[\mu}\Omega_{\nu]a}, \tag{2.35a}$$
$$T_{\mu\nu}{}^a(P) = 2\partial_{[\mu}e^a_{\nu]} - 2e^b_{[\mu}\Omega_{\nu]b}{}^a, \tag{2.35b}$$
$$R_{\mu\nu}{}^a(C) = 2\partial_{[\mu}\Omega_{\nu]}{}^a - 2\Omega^{ab}_{[\mu}\Omega_{\nu]b}, \tag{2.35c}$$
$$R_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu}\Omega^{ab}_{\nu]} - 2\Omega_{[\mu}{}^{ca}\Omega_{\nu]}{}^b{}_c. \tag{2.35d}$$

The connection ω of course corresponds to an affine connection through the vielbein postulate. We will, however, postpone this to section 2.2.5.

It will also prove useful to introduce a frame (v^μ, e^μ_a) corresponding to the co-frame (τ_μ, e^a_μ), which we do by the defining relations

$$v^\mu\tau_\nu = -1, \qquad v^\mu e^a_\mu = 0, \qquad \tau_\mu e^\mu_a = 0, \qquad e^a_\mu e^\mu_b = \delta^b_a. \tag{2.36}$$

We will also occasionally use $\vartheta_a \equiv e^\mu_a\partial_\mu$ when dealing with abstract tensors. From (2.36) the last of the possible contractions of the vielbeine follows

$$e^a_\mu e^\nu_a = \delta^\nu_\mu + \tau_\mu v^\nu. \tag{2.37}$$

The transformation law (2.31a) and (2.31b) together with the relations (2.36) further imply the transformations of the frame

$$\delta v^\mu = 0, \tag{2.38a}$$
$$\delta e^\mu_a = v^\mu\lambda_a + \lambda_a{}^b e^\mu_b. \tag{2.38b}$$

Finally, we also have a covariant derivative associated with the connection ω that acts in the adjoint representation on a vector field $X = X^0 H + X^a P_b$ as

$$DX = dX + [\omega, X] = (dX^0 - X^a \Omega_a) \otimes H + (dX^a + X^b \Omega_b{}^a) \otimes P_a. \tag{2.39}$$

There is likewise a natural covariant derivative on co-vectors in the frame bundle for which we need to introduce a co-basis (η, γ^a) defined by

$$\eta(H) = -1, \qquad \eta(P_a) = 0, \qquad \gamma^a(H) = 0, \qquad \gamma^a(P_b) = \delta^a_b. \tag{2.40}$$

Co-vectors in $\mathfrak{t}^*$ naturally transform in the co-adjoint representation. Thus, the covariant derivative for a co-vector $\Theta = \Theta_0 \eta + \Theta_a \gamma^a$ is

$$D\Theta = d\Theta + \operatorname{ad}^*_\omega \Theta = d\Theta_0 \otimes \eta + (d\Theta_a - \Omega_a \Theta_0 - \Omega_a{}^b \Theta_b) \otimes \gamma^a. \tag{2.41}$$

These associated derivatives are, as mentioned, adapted to the Carroll structure and one example of this can be observed by considering the covariant derivative of H which follows from (2.39)

$$DH = 0. \tag{2.42}$$

The vanishing of this derivative can be traced back to the fact that H is central, or in other words, H is annihilated by all elements of $\mathfrak{g}$ (in fact also $\mathfrak{a}$). One can also, by a computation analogous to (2.22), show that $\delta_{ab}\, \gamma^a \otimes \gamma^b$ is conserved by the covariant derivative. We will return to these adapted properties of the covariant derivative in section 2.2.5.

2.2.4 Carroll invariants

It is clear that only the vector field v^μ of the vielbeine is a tensorial quantity as it is invariant under local tangent space transformations (2.38a). However, the co-frame only transforms under rotations (2.31b) and hence we can define an invariant degenerate metric

$$h_{\mu\nu} \equiv \delta_{ab} e^a_\mu e^b_\nu, \tag{2.43}$$

whose invariance is easily seen by

$$\delta h_{\mu\nu} \overset{(2.38a)}{=} \delta_{ab}(\lambda^a{}_c e^c_\mu e^b_\nu + e^a_\mu \lambda^b{}_c e^c_\nu) = 2 e^a_\mu e^b_\nu \lambda_{(ab)} = 0. \tag{2.44}$$

From the definitions (2.43) and (2.36), we also see that the kernel of $h_{\mu\nu}$ is spanned by v^μ, that is

$$v^\mu h_{\mu\nu} = 0. \tag{2.45}$$

Carroll structures are often defined as a spacetime manifold equipped with a nowhere-vanishing vector field v^μ and a spatial metric $h_{\mu\nu}$ whose kernel is spanned by v^μ [22]. As v^μ is gauge invariant, it is natural to consider its integral curves which fiber the manifold. These considerations give rise to another construction of Carroll structures as a fiber bundle $M \xrightarrow{\pi} S$ with typical fiber $\mathbb{R}$ over a spatial base manifold S [24, 40]. The fiber bundle construction naturally possesses the subspace spanned by v^μ as the kernel of the projection push-forward π_*. Further, a choice of spatial subspace in T^*M can be seen as equipping the fiber bundle with an Ehresmann connection.

The degeneracy of $h_{\mu\nu}$ entails that it is non-invertible, but we can still define a projective inverse

$$h^{\mu\nu} \equiv \delta^{ab} e^\mu_a e^\nu_b, \tag{2.46}$$

which by the defining relations (2.36) satisfies

$$v^\mu \tau_\mu = -1, \qquad v^\mu h_{\mu\nu} = 0, \qquad \tau_\mu h^{\mu\nu} = 0, \qquad h_{\mu\rho} h^{\rho\nu} - \tau_\mu v^\nu = \delta^\nu_\mu. \tag{2.47}$$

Carrying out a calculation similar to (2.44), one can determine the transformation of $h^{\mu\nu}$ and we state the full set of transformations

$$\delta v^\mu = 0, \tag{2.48a}$$
$$\delta h_{\mu\nu} = 0, \tag{2.48b}$$
$$\delta \tau_\mu = \lambda_\mu, \tag{2.48c}$$
$$\delta h^{\mu\nu} = 2 v^{(\mu} \lambda^{\nu)}, \tag{2.48d}$$

with $\lambda_\mu = \lambda_a e^a_\mu$ and $\lambda^\mu = \lambda_a \delta^{ab} e^\mu_b$.

The relations (2.47) give rise to a set of projection operators

$$-v^\mu \tau_\nu, \qquad \text{and} \qquad h^\mu_\nu \equiv h^{\mu\rho} h_{\rho\nu}, \tag{2.49}$$

called temporal and spatial, respectively. The two projectors span the entire tangent space due to the completeness relations (2.47). It is important to note that these projectors are not boost invariant due to (2.48a)-(2.48d) and consequently the induced split of the tangent space is not a true Carrollian notion. However, we can always complete the basis by choosing a spatial subspace and construct these boost non-invariant projectors. This split suggests defining spatial and temporal indices as being annihilated by $-v^\mu \tau_\nu$ and h^μ_ν, respectively. The fact that we can raise and lower the flat indices with the Kronecker-δ then carries over to the curved indices in the sense that spatial indices can be raised and lowered with $h^{\mu\nu}$ and $h_{\mu\nu}$, respectively.

A third and perhaps less obvious Carroll invariant is the tensor density e of weight -1 given by

$$e = \sqrt{\det(\tau_\mu \tau_\nu + h_{\mu\nu})}. \tag{2.50}$$

The density e is not manifestly boost invariant as it depends on τ_μ which transforms according to (2.48c) under boosts. However, a short calculation shows

$$\delta e = \frac{1}{2} e (v^\mu v^\nu + h^{\mu\nu}) \delta(\tau_\mu \tau_\nu + h_{\mu\nu}) = \frac{1}{2} e (v^\mu v^\nu + h^{\mu\nu}) 2 \tau_{(\mu} \lambda_{\nu)} = 0, \tag{2.51}$$

that e is indeed invariant. Having determined a density associated with the geometry, we can define a covariant measure $e\, d^{d+1}x$ necessary to write down Carrollian field theory actions. These three invariant quantities could all have been anticipated from the group action as v^μ, $h_{\mu\nu}$ and e in the frame bundle correspond to tensors left invariant by the action of the Carroll group. This is analogous to how one can show that the metric and the Levi-Civita symbol are the only invariants of $\mathrm{SO}(d)$.

A one-derivative object of interest is the so-called extrinsic curvature

$$K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_v h_{\mu\nu}, \tag{2.52}$$

whose invariance under the group action is manifest. A short computation further shows that $K_{\mu\nu}$ is spatial

$$v^\mu K_{\mu\nu} = -\frac{1}{2} v^\mu \mathcal{L}_v h_{\mu\nu} = \frac{1}{2} h_{\mu\nu} \mathcal{L}_v v^\mu = \frac{1}{2} h_{\mu\nu} [v, v]^\mu = 0, \tag{2.53}$$

and thus we can raise and lower its indices

$$K^{\mu}{}_{\nu} \equiv h^{\mu\rho} K_{\rho\nu}, \qquad K^{\mu\nu} \equiv h^{\mu\rho} h^{\nu\sigma} K_{\rho\sigma}. \tag{2.54}$$

Utilizing this property of the extrinsic curvature, we can construct scalars by repeated contractions

$$K \equiv h^{\mu\nu} K_{\mu\nu}, \qquad K_{\mu\nu} K^{\mu\nu}, \qquad \ldots, \tag{2.55}$$

which despite the gauge non-invariant $h^{\mu\nu}$ are all boost invariant because all potential transformations are projected out as in (2.51).

2.2.5 The affine connection

As with gauging of the Poincaré algebra in section (2.2.2), the frame bundle connection (2.27b) also corresponds to an affine connection $\Gamma^{\rho}_{\mu\nu}$ and the relation between them can be derived with the vielbein postulate. In particular, if we let X^{μ} be any vector field, we find

$$\begin{aligned} 0 \overset{!}{=} \mathcal{D}_X \theta &= \mathcal{D}_X (\tau_\mu dx^\mu \otimes H + e^a_\mu dx^\mu \otimes P_a) \\ &= X^\mu (\partial_\mu \tau_\nu - \Gamma^\rho_{\mu\nu} \tau_\rho - \Omega_{\mu a} e^a_\nu) dx^\nu \otimes H + X^\mu (\partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho - \Omega_\mu{}^a{}_b e^b_\nu) dx^\nu \otimes P_a. \end{aligned} \tag{2.56}$$

Equivalently, one can write this in component form

$$\partial_\mu \tau_\nu - \Gamma^\rho_{\mu\nu} \tau_\rho - \Omega_{\mu a} e^a_\nu = 0, \tag{2.57a}$$
$$\partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho - \Omega_\mu{}^a{}_b e^b_\nu = 0. \tag{2.57b}$$

With (2.57a) and (2.57b) we can solve for the affine connection coefficients $\Gamma^{\rho}_{\mu\nu}$ using (2.47) and establish the following relation

$$\Gamma^\rho_{\mu\nu} = -v^\rho \partial_\mu \tau_\nu + v^\rho \Omega_{\mu a} e^a_\nu + e^\rho_a \partial_\mu e^a_\nu - \Omega_\mu{}^a{}_b e^\rho_a e^b_\nu. \tag{2.58}$$

It is then easy to check that the invariant tensors v^μ, $h_{\mu\nu}$ and e are covariantly conserved as they should be by construction i.e.

$$\nabla_\rho v^\mu = 0, \qquad \nabla_\rho h_{\mu\nu} = 0, \qquad \nabla_\rho e = 0, \tag{2.59}$$

where ∇ is the covariant derivative associated with $\Gamma^{\rho}_{\mu\nu}$[2]. As an example of the compatibility, we work out the details for the case of $\nabla_\mu v^\nu$

$$\begin{aligned} \nabla_\mu v^\nu &= \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho = \partial_\mu v^\nu - v^\nu v^\rho \partial_\mu \tau_\rho + e^\nu_a v^\rho \partial_\mu e^a_\nu \\ &= \partial_\mu v^\nu - (-v^\nu \tau_\rho + e^\nu_a e^a_\nu) \partial_\mu v^\rho = 0, \end{aligned} \tag{2.60}$$

where we used the identity (2.37).

The vielbein postulate can also be used to solve for the usual torsion $T^\rho{}_{\mu\nu}$ and Riemann curvature $R_{\mu\nu\sigma}{}^\rho$ tensor in terms of the frame bundle counterparts (2.32) and (2.33)

$$T^\rho{}_{\mu\nu} = -v^\rho T_{\mu\nu}(H) + e^\rho_a T_{\mu\nu}{}^a(P), \tag{2.61a}$$
$$R_{\mu\nu\sigma}{}^\rho = -v^\rho e_{\sigma a} R_{\mu\nu}{}^a(C) - e_{\sigma a} e^\rho_b R_{\mu\nu}{}^{ab}(J), \tag{2.61b}$$

where the torsion and curvature of an affine connection is defined as (3.31) and (3.32), respectively.

[2] Note that e is a density and the covariant derivative of a scalar density σ of weight w takes the form $\nabla_\mu \sigma = \partial_\mu \sigma + w\sigma \Gamma^\rho_{\mu\rho}$.

2.3 Carroll geometry on null hypersurfaces

This section is based on the construction presented in [18]. An alternate and perhaps better motivated way of obtaining Carroll geometry is as the induced geometry on a null hypersurface embedded in a Lorentzian structure in 1 dimension higher. The degenerate Carroll structure emerges due to the fact that the normal vector of a null hypersurface is also tangent, which renders the pullback metric degenerate.

To make this construction, we consider a $d+2$ dimensional Lorentzian spacetime equipped with a non-degenerate metric of signature $(-1,+1,\ldots,+1)$. Further, we assume that there exists adapted coordinates (u,x^μ) such that hypersurfaces of constant u are null, that is

$$g^{AB}\partial_A u\partial_B u = 0 \qquad \Rightarrow \qquad g^{uu}=0, \tag{2.62}$$

where we in this section let $A,B,\ldots = u,0,1,\ldots d$. The condition that all hypersurfaces with constant u are null is here chosen for simplicity and can be weakened such that a null hypersurface only occurs at a specific value e.g. $u=0$. One can then consider a family of hypersurfaces Σ_u and characterize the induced geometry for each u. Taking the limit $u\to 0$ then corresponds to an effective ultra-relativistic limit of the geometry on Σ_u. An example of this is to consider the limiting procedure of bringing a stretched black hole horizon into coincidence with the event horizon [28].

Returning to the strong assumption of a null foliation on the entire embedding spacetime, we can write down a parameterization of the most general metric with $g^{uu}=0$

$$ds^2 = g_{AB}dx^A dx^B = du(2\bar{\Phi}du - 2\hat{\tau}_\mu dx^\mu) + h_{\mu\nu}dx^\mu dx^\nu, \tag{2.63}$$

and its inverse

$$g^{uu}=0, \qquad g^{\mu u} = v^\mu, \qquad g^{\mu\nu} = \bar{h}^{\mu\nu}. \tag{2.64}$$

In (2.63) and (2.64) we have defined the following tensors

$$\hat{\tau}_\mu = \tau_\mu - h_{\mu\nu}M^\nu, \tag{2.65}$$
$$\bar{h}^{\mu\nu} = h^{\mu\nu} - M^\mu v^\nu - M^\nu v^\mu, \tag{2.66}$$
$$\bar{\Phi} = -M^\mu\tau_\mu + \frac{1}{2}h_{\mu\nu}M^\mu M^\nu, \tag{2.67}$$

where v^μ, τ_μ, $h^{\mu\nu}$ and $h_{\mu\nu}$ satisfy the completeness relations (2.47) and M^μ is some vector field. If we let v^μ, τ_μ, $h^{\mu\nu}$ and $h_{\mu\nu}$ transform under local Carroll boosts, then the vector field M^μ must transform as $\delta M^\mu = \lambda^\mu$ for the metric $g_{\mu\nu}$ to remain inert. A vector field transforming as M^μ occurs naturally when considering sub-leading Carroll structures cf. (4.10a) and (4.14a), but here it can just be seen as a further freedom of a metric satisfying (2.62).

Unlike the case of timelike and spacelike hypersurfaces, we are not given a natural projector onto a null hypersurface as the normal vector is tangent rather than transverse. Thus, we can pull back the metric g_{AB}, but we cannot project the null normal vector $U^A = g^{AB}\partial_B u = (0, v^\mu)$ without more structure. The additional information that needs to be supplied is an extra transverse null vector field V^A [41] satisfying

$$V^A V_A = 0, \qquad V^A U_A = -1. \tag{2.68}$$

Notice that we only have 2 constraints in (2.68), but $d+2$ degrees of freedom in V^A. Thus, we have d components unaccounted for, which can be interpreted as the boost gauge symmetry of

Carroll geometry. The full metric g_{AB} can then be decomposed in terms of the nullbeine U^A and V^A as

$$g_{AB} = -U_A V_B - U_B V_A + \Pi_{AB}, \tag{2.69}$$

where (2.69) also serves as the definition of Π_{AB}. It is easily seen from (2.69) that Π_{AB} satisfies

$$U^A \Pi_{AB} = 0, \qquad V^A \Pi_{AB} = 0. \tag{2.70}$$

With this extra structure, we can define a projector $P^A{}_B$ on the tangent space as

$$P^A{}_B \equiv \delta^A_B + V^A U_B, \tag{2.71}$$

which clearly has the properties

$$P^A{}_B P^B{}_C = P^A{}_C, \qquad P^A{}_B V^B = 0 \qquad P^A{}_B U^B = U^A, \qquad U_A P^A{}_B = 0. \tag{2.72}$$

The last property shows that the output of the projection is always tangent to the null hypersurface and hence we can restrict the indices as $P^\mu{}_A$. By introducing the transverse vector V^A, we can now map vectors to the hypersurface using the projector $P^A{}_B$ and co-vectors by the pull-back associated with the embedding map [42]. In the adapted coordinates, we can write the pull-back as

$$\Phi_\mu{}^A \equiv \delta^A_\mu. \tag{2.73}$$

Having the above construction and the previous sections in mind, a natural of choice of transverse vector is $V^A = (-1, M^\mu)$ implying

$$V^u = -1, \quad V^\mu = M^\mu, \quad U^u = 0, \quad U^\mu = v^\mu \quad \Pi^{uA} = 0, \quad \Pi^{\mu\nu} = h^{\mu\nu}. \tag{2.74}$$

Lowering the indices using (2.63) yields

$$V_u = \tau_\mu M^\mu, \qquad V_\mu = \tau_\mu, \qquad U_u = 1, \qquad U_\mu = 0, \tag{2.75a}$$

$$\Pi_{uu} = h_{\mu\nu} M^\mu M^\nu, \qquad \Pi_{u\mu} = h_{\mu\nu} M^\nu, \qquad \Pi_{\mu\nu} = h_{\mu\nu}. \tag{2.75b}$$

Finally, we can apply the projection operator $P^A{}_B$ on contra-variant tensors and the pull-back $\Phi_B{}^A$ on co-variant tensors in ambient space, which results in the following non-zero quantities on the hypersurface

$$P^\mu{}_A U^A = v^\mu, \qquad P^\mu{}_A P^\nu{}_B g^{AB} = h^{\mu\nu}, \qquad \Phi_\mu{}^A V_A = \tau_\mu, \qquad \Phi^A{}_\mu \Phi^B{}_\nu g_{AB} = h_{\mu\nu}. \tag{2.76}$$

Hence, we see that we get the defining objects of Carroll geometry v^μ and $h_{\mu\nu}$, but also τ_μ and $h^{\mu\nu}$ which transform under boosts corresponding to the non-uniqueness of V^A. Importantly, the induced Carroll geometry v^μ and $h_{\mu\nu}$ are independent of any choice of V^A.

2.4 Metric responses and Energy-Momentum tensors

Any action governing a field theory coupling to a general Carrollian geometry can be written in the form

$$S = \int d^{d+1}x \, e \mathcal{L}[\varphi^I, v^\mu, h^{\mu\nu}], \tag{2.77}$$

where φ^I represents possible matter fields, $d^{d+1}x\, e$ is the Carrollian invariant measure and $\mathcal{L}$ is a Lagrangian scalar. Additionally, it must be invariant under the action of the Carroll group which amounts to invariance under boosts (2.48a) and (2.48d). We can then consider the response to variation of the metric data

$$\delta S = \int d^{d+1}x\, e \left[-T^v_\mu \delta v^\mu - \frac{1}{2} T^h_{\mu\nu} \delta h^{\mu\nu} \right], \tag{2.78}$$

where we defined the metric responses or "Carrollian momenta"

$$T^v_\mu \equiv -e^{-1} \frac{\delta S}{\delta v^\mu}, \tag{2.79a}$$
$$T^h_{\mu\nu} \equiv -2e^{-1} \frac{\delta S}{\delta h^{\mu\nu}}. \tag{2.79b}$$

One could have defined other Carrollian momenta by considering the response to varying with respect to different parameterizations of the metric data. We could equivalently have used the parameterization $(\tau_\mu, h_{\mu\nu})$ or as in e.g. [43] used adapted coordinates $(x^0, \ldots, x^d)$ such that

$$\tau = \Omega(-dx^0 + b_a dx^a), \qquad h = h_{ab} dx^a \otimes dx^b, \tag{2.80}$$

with $a, b = 1, \ldots d$ and the metric data being the triple (Ω, b_i, h_{ij}). The responses to varying with respect to each parameterization give different descriptions of the same information.

T^v_μ and $T^h_{\mu\nu}$ cannot both be boost invariant because $h^{\mu\nu}$ transforms under boosts. Thus, we cannot directly interpret them as energy-momentum tensors. To find the transformation laws for (2.79a) and (2.79b), we consider an infinitesimal boost

$$v'^\mu = v^\mu, \tag{2.81a}$$
$$h'^{\mu\nu} = h^{\mu\nu} + 2v^{(\mu}\lambda^{\mu)}, \tag{2.81b}$$

where λ^μ again is spatial i.e. $\lambda^\mu \tau_\mu = 0$. Consequently, general variations δ of the primed and un-primed variables are related by

$$\delta v^\mu = \delta v'^\mu, \tag{2.82a}$$
$$\delta h^{\mu\nu} = \delta h'^{\mu\nu} - 2\delta v'^{(\mu}\lambda^{\mu)}. \tag{2.82b}$$

Inserting this into (2.78) we find

$$\begin{aligned}\delta S &= \int d^{d+1}x\, e \left[-T^v_\mu \delta v^\mu - \frac{1}{2} T^h_{\mu\nu} \delta h^{\mu\nu} \right] \\ &= \int d^{d+1}x\, e \left[-(T^v_\mu - \lambda^\nu T^h_{\nu\mu}) \delta v'^\mu - \frac{1}{2} T^h_{\mu\nu} \delta h'^{\mu\nu} \right],\end{aligned} \tag{2.83}$$

from which we can read off the infinitesimal Carroll boosts

$$\delta_C T^v_\mu = -\lambda^\nu T^h_{\nu\mu}, \tag{2.84a}$$
$$\delta_C T^h_{\mu\nu} = 0. \tag{2.84b}$$

These transformation laws imply that we can make the following boost-invariant combination

$$T^\mu{}_\nu \equiv -v^\mu T^v_\nu - h^{\mu\rho} T^h_{\rho\nu}. \tag{2.85}$$

We will interpret $T^\mu{}_\nu$ as an energy-momentum tensor (EMT), which we will further argue for in section 2.4.2. The boost invariance can be confirmed by the following computation

$$T'^\mu{}_\nu = -v^\mu(T^v_\nu - \lambda^\rho T^h_{\rho\nu}) - (h^{\mu\rho} + v^\mu\lambda^\rho + v^\rho\lambda^\mu)T^h_{\rho\nu} = T^\mu{}_\nu - \lambda^\mu v^\rho T^h_{\rho\nu}, \tag{2.86}$$

which shows boost-invariance given that $v^\rho T^h_{\rho\nu}$ vanishes. This is indeed the case by virtue of the boost Ward identity (2.89), which we will prove in the next subsection. Note that this also shows that we can recover T^v_μ and $T^h_{\mu\nu}$ from $T^\mu{}_\nu$, such that no information is lost in constructing the EMT (2.85). The boost Ward identity $v^\rho T^h_{\rho\nu} = 0$ further implies that the following projection vanishes

$$h^\mu_\nu v^\rho T^\nu{}_\rho = 0, \tag{2.87}$$

which we can interpret as the vanishing of the energy current for any Carrollian field theory.

2.4.1 Ward identities

The action (2.77) is in addition to being boost invariant also invariant under spatial rotations and diffeomorphisms. By inserting the boost transformations (2.48a) and (2.48d) into (2.78) and demanding invariance for any boost λ^μ, we immediately find the corresponding Ward identity

$$T^h_{\mu\nu} v^\mu h^{\nu\rho} = 0. \tag{2.88}$$

The statement of (2.88) can be made stronger by noticing that $\tau_\mu\tau_\nu\delta h^{\mu\nu} = 0$ and thus the spatial metric response $T^h_{\mu\nu}$ contains no component $\sim \tau_\mu\tau_\nu$. This results in the boost Ward identity

$$T^h_{\mu\nu} v^\mu = 0, \tag{2.89}$$

which can be interpreted as $T^h_{\mu\nu}$ being purely spatial.

The Ward identity related to the remaining internal gauge transformations i.e. spatial rotations can be understood as the symmetry in the indices of $T^h_{\mu\nu}$. This follows by considering the response to varying the vielbein e^μ_a and then constructing a rotationally invariant momentum from these.

Finally, we can consider the diffeomorphism Ward identity by demanding invariance of the Lagrangian density under diffeomorphism up to a total derivative

$$\begin{aligned} e&\left[-T^v_\mu \mathcal{L}_\xi v^\mu - \frac{1}{2}T^h_{\mu\nu}\mathcal{L}_\xi h^{\mu\nu}\right] \\ &\qquad = e\left[-T^v_\mu(\xi^\rho\partial_\rho v^\mu - \partial_\rho\xi^\mu v^\rho) - \frac{1}{2}T^h_{\mu\nu}(\xi^\rho\partial_\rho h^{\mu\nu} - 2\partial_\rho\xi^\mu h^{\rho\nu})\right] \\ &\qquad \approx -\xi^\rho\left[eT^v_\mu\partial_\rho v^\mu + \partial_\mu(ev^\mu T^v_\rho) + \frac{e}{2}T^h_{\mu\nu}\partial_\rho h^{\mu\nu} + \partial_\mu(eT^h_{\rho\nu}h^{\mu\nu})\right] \stackrel{!}{=} 0, \end{aligned} \tag{2.90}$$

where $\approx$ denotes equality up to a total derivative. For (2.90) to hold for all choices of ξ^μ we need to have

$$\partial_\mu(eT^\mu{}_\nu) - eT^v_\mu\partial_\nu v^\mu - \frac{e}{2}T^h_{\rho\sigma}\partial_\nu h^{\rho\sigma} = 0. \tag{2.91}$$

The relation (2.91) is not manifestly covariant and cannot be made so without the introduction of a connection. We will address the choice of connection in chapter 3.

2.4.2 Comparison to the canonical energy-momentum tensor

We can consider the so-called flat Carroll manifold [22] with $M = \mathbb{R}^{d+1}$ where

$$v = \partial_0, \qquad h_{\mu\nu} = \delta_{ab}\, dx^a \otimes dx^b, \tag{2.92}$$

which also implies that $e = 1$. The flat Carroll structure is analogous to Minkowski space and can indeed be seen as the $c \to 0$ limit of it. Due to the structure being flat, translations are global symmetries and hence have associated Noether currents. These can be combined into the canonical energy-momentum tensor following the usual procedure from field theory [44]. In particular, the canonical EMT for a single scalar field ϕ with Lagrangian $\mathcal{L}[\phi, \partial_\mu\phi]$ takes the form

$$\tau^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}. \tag{2.93}$$

This EMT is conserved in the sense $\partial_\mu\tau^\mu{}_\nu = 0$ as a direct consequence of the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{2.94}$$

We can then investigate whether $\tau^\mu{}_\nu$ corresponds to the EMT (2.85) when evaluated in the flat limit i.e. (2.92). In order to do this, we must consider the metric data as general to perform the derivatives (2.79a)-(2.79b) and then at the end of the calculation evaluate at the flat structure. For simplicity, we assume that ϕ only couples to the v^μ and $h^{\mu\nu}$ and not their derivatives. Note that couplings to derivatives of the metric data also exist and occur naturally in covariant derivatives, curvatures, etc. The assumption of no derivatives implies that

$$\frac{\delta S}{\delta v^\mu} = \frac{\partial(e\mathcal{L})}{\partial v^\mu}, \qquad \frac{\delta S}{\delta h^{\mu\nu}} = \frac{\partial(e\mathcal{L})}{\partial h^{\mu\nu}}, \tag{2.95}$$

where the factors of e are due to the general background. Having this in mind, we can compute the total spacetime derivative of the Lagrangian

$$\begin{aligned}\partial_\mu(e\mathcal{L}[\phi, \partial_\mu\phi, v^\mu, h^{\mu\nu}]) &= \frac{\partial(e\mathcal{L})}{\partial\phi}\partial_\mu\phi + \frac{\partial(e\mathcal{L})}{\partial(\partial_\nu\phi)}\partial_\mu\partial_\nu\phi + \frac{\partial(e\mathcal{L})}{\partial v^\rho}\partial_\mu v^\rho + \frac{\partial(e\mathcal{L})}{\partial h^{\rho\sigma}}\partial_\mu h^{\rho\sigma} \\ &= \partial_\nu\left(\frac{\partial(e\mathcal{L})}{\partial(\partial_\nu\phi)}\partial_\mu\phi\right) - eT^v_\rho\partial_\mu v^\rho - \frac{e}{2}T^h_{\rho\sigma}\partial_\mu h^{\rho\sigma},\end{aligned} \tag{2.96}$$

where we for the second line used (2.94) (taking into account the non-trivial measure) before invoking the product rule and for the last two terms the definitions (2.79a)-(2.79b). We can then rearrange the terms and apply the diffeomorphism Ward identity (2.91) to find

$$\partial_\nu\left(\frac{\partial(e\mathcal{L})}{\partial(\partial_\nu\phi)}\partial_\mu\phi - \delta^\nu_\mu e\mathcal{L} - eT^\nu{}_\mu\right) = 0. \tag{2.97}$$

Setting the Carroll data to be flat, we see that the canonical EMT $\tau^\mu{}_\nu$ differs from the EMT $T^\mu{}_\nu$ only by total derivative terms. Thus, we conclude that the definition (2.85) can indeed be interpreted as a Carrollian EMT. Had we included couplings to derivatives of the metric data, it would have resulted in correction to (2.97), but they would vanish in the flat limit (2.92).

2.4.3 Carrollian Killing vector and conserved currents

In a curved Lorentzian background $g_{\mu\nu}$, one can find isometry generating vector fields by the Killing equation

$$\mathcal{L}_\xi g_{\mu\nu} = 0. \tag{2.98}$$

This, together with the relativistic energy-momentum tensor $T^{\text{rel}}_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$, allows one to construct currents $J^\mu_{\text{rel}} = T^{\mu\nu}_{\text{rel}}\xi_\nu$ that are covariantly conserved $\nabla_\mu J^\mu_{\text{rel}} = 0$. This conservation is exactly due to the diffeomorphism Ward identity of the relativistic matter Lagrangian.

Some work has already been done in the direction of conserved currents related to Carrollian Killing vectors [43]. However, the authors of [43] work in the adapted coordinates (2.80) and prove the conservation by a limit of the relativistic formulae. We will take an intrinsic and fully covariant approach and mirror the relativistic derivation for a theory coupled to Carroll background. That is, we define a current

$$J^\mu = T^\mu{}_\nu \xi^\nu, \tag{2.99}$$

where $T^\mu{}_\nu$ is the energy-momentum tensor (2.85) and ξ^μ is some vector field to be determined. As we do not have a connection, we have to formulate the conservation criteria as[3] $\partial_\mu(eJ^\mu) = 0$. We can then straightforwardly calculate

$$\begin{aligned}
\partial_\mu(eJ^\mu) &= \partial_\mu(eT^\mu{}_\nu)\xi^\nu - e(v^\mu T^v_\nu + h^{\mu\rho}T^h_{\rho\nu})\partial_\mu\xi^\nu \\
&= eT^v_\nu(\xi^\mu\partial_\mu v^\nu - v^\mu\partial_\mu\xi^\nu) + \frac{e}{2}T^h_{\mu\nu}(\xi^\rho\partial_\rho h^{\mu\nu} - 2h^{\mu\rho}\partial_\rho\xi^\nu) \\
&= eT^v_\mu\mathcal{L}_\xi v^\mu + \frac{e}{2}T^h_{\mu\nu}\mathcal{L}_\xi h^{\mu\nu},
\end{aligned} \tag{2.100}$$

where we in the second equality used the diffeomorphism Ward identity (2.91). A sufficient condition for the conservation of J^μ is that the two terms of (2.100) vanish individually. For the second term of (2.100) to be zero it is sufficient that the spatial-spatial projection of $\mathcal{L}_\xi h^{\mu\nu}$ vanish due to $T^h_{\mu\nu}$ being purely spatial

$$h_{\mu\sigma}h_{\nu\rho}\mathcal{L}_\xi h^{\sigma\rho} = -h_{\mu\sigma}h^{\sigma\rho}\mathcal{L}_\xi h_{\nu\rho} = -\mathcal{L}_\xi h_{\mu\nu} + \tau_\mu v^\rho \mathcal{L}_\xi h_{\rho\nu} = -\mathcal{L}_\xi h_{\mu\nu} - \tau_\mu h_{\nu\rho}\mathcal{L}_\xi v^\rho \overset{!}{=} 0, \tag{2.101}$$

where we used the completeness relations (2.47). From (2.100) and (2.101) we see that a sufficient condition for conservation of J^μ is the "Carrollian Killing equations"

$$\mathcal{L}_\xi v^\mu = 0 \tag{2.102a}$$
$$\mathcal{L}_\xi h_{\mu\nu} = 0. \tag{2.102b}$$

This is a natural definition of the infinitesimal isometries as the Carroll geometry is defined by v^μ and $h_{\mu\nu}$. The "inverse vielbeine" τ_μ and $h^{\mu\nu}$ do not contribute more information, only gauge, and consequently do not have to be preserved under Carroll isometries.

One can take the idea of Carrollian Killing vectors further and relax the conditions (2.102a)-(2.102b) to include conformal rescalings. This idea is pursued in e.g. [24] where the authors show that under certain conditions the algebra of the conformal Carrollian Killing vectors is isomorphic to the BMS algebra. This supplies further evidence of the connection between Carroll geometry and lightlike infinity of asymptotically flat spacetimes [22, 23].

[3] This form comes from considering the tensor density dual to a d-form, as described in appendix A. In this light, $\partial_\mu(eJ^\mu)$ is an exterior derivative and the conservation criteria is simply demanding the current d-form to be closed. This is equivalent to being covariantly conserved in the relativistic theory due to $\sqrt{-g}\nabla_\mu J^\mu = \partial_\mu(\sqrt{-g}J^\mu)$.

2.4.4 Example of a Carrollian field theory

As an example of the above machinery, let us consider a Carrollian field theory governed by the action

$$S_\phi = \int d^{d+1}x\, e\mathcal{L}_\phi = \frac{1}{2}\int d^{d+1}x\, e(v^\mu\partial_\mu\phi)^2. \tag{2.103}$$

The action (2.103) is trivially boost invariant as it only couples to e and v^μ. We can compute the EOM using the Euler-Lagrange equation (2.94)

$$\partial_\mu(ev^\mu v^\nu\partial_\nu\phi) = 0, \tag{2.104}$$

which for a curved background is not manifestly covariant. As for the diffeomorphism Ward identity (2.91), it is also possible to write this covariantly upon the introduction of a connection using (3.28). The metric responses can be computed as

$$\delta\mathcal{L}_\phi = e\mathcal{L}_\phi\left(\tau_\mu\delta v^\mu - \frac{1}{2}h_{\mu\nu}\delta h^{\mu\nu}\right) + ev^\mu\partial_\mu\phi\partial_\nu\phi\delta v^\nu, \tag{2.105}$$

from which we can read off the momenta according to (2.79a) and (2.79b)

$$T^v_\mu = -\tau_\mu\mathcal{L}_\phi - v^\nu\partial_\nu\phi\partial_\mu\phi, \qquad T^h_{\mu\nu} = \mathcal{L}_\phi h_{\mu\nu}. \tag{2.106}$$

Using these, we can form the EMT (2.85)

$$T^\mu{}_\nu = -v^\mu(-\tau_\nu\mathcal{L}_\phi - v^\rho\partial_\rho\phi\partial_\nu\phi) - h^{\mu\nu}\mathcal{L}_\phi h_{\rho\nu} = v^\mu v^\rho\partial_\rho\phi\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}_\phi, \tag{2.107}$$

where we used the completeness relation (2.47). Further, if we go to the flat Carroll structure (2.92), then we see that the first term of (2.107) becomes the derivative appearing in the canonical EMT (2.93). Hence we have $T^\mu{}_\nu = \tau^\mu{}_\nu$ for the action (2.103) in the flat limit and the statement (2.97) holds.

Returning to a general Carroll structure, we can also check the Ward identities. The boost Ward identity (2.89) is readily seen to hold because $T^h_{\mu\nu} \sim \mathcal{L}_\phi h_{\mu\nu}$ and is thus annihilated by v^μ. For the diffeomorphism Ward identity (2.91), we need to perform the following computation

$$\begin{aligned}\partial_\mu(eT^\mu{}_\nu) &= \partial_\mu(ev^\mu v^\rho\partial_\rho\phi)\partial_\nu\phi + ev^\mu v^\rho\partial_\rho\phi\partial_\mu\partial_\nu\phi - \partial_\nu(e\mathcal{L}_\phi) \\ &= ev^\mu v^\rho\partial_\rho\phi\partial_\mu\partial_\nu v^\rho + eT^v_\rho\partial_\nu\phi + \frac{e}{2}T^h_{\rho\sigma}\partial_\nu h^{\rho\sigma} - \frac{\partial(e\mathcal{L}_\phi)}{\partial(\partial_\rho\phi)}\partial_\nu\partial_\rho\phi \\ &= eT^v_\rho\partial_\nu v^\rho + \frac{e}{2}T^h_{\rho\sigma}\partial_\nu h^{\rho\sigma}.\end{aligned} \tag{2.108}$$

In the second equality we used the EOM (2.104), and the fact that there are no couplings to the derivatives of the metric data, so we can write out the spacetime derivative as in (2.96) using (2.95). The calculation (2.108) shows that the Ward identity (2.91) holds for the field theory (2.103).

Chapter 3

Connections on Carroll Structures

In GR one studies Lorentzian structures where the geometry is encoded in a non-degenerate metric $g_{\mu\nu}$. Hence, it is natural to consider connections that are compatible with this structure i.e. connections where the metric is parallel

$$\nabla_\rho g_{\mu\nu} = 0. \tag{3.1}$$

The connection resulting from the condition (3.1) is equivalent to that of the gauging procedure of section 2.2.2 in the sense that the two approaches fix the same components of a general connection $\Gamma^\rho_{\mu\nu}$. However, (3.1) does not entirely fix the connection and this ambiguity is described by the torsion $T^\rho{}_{\mu\nu} = 2\Gamma^\rho_{[\mu\nu]}$, which one is free to specify. In particular, one can choose a unique metric-compatible and torsionless connection, the Levi-Civita connection, which is the content of the fundamental theorem of Riemannian geometry.

A Carrollian geometry is described by the pair of tensors $(v^\mu, h_{\mu\nu})$ and the metricity-condition is then accordingly

$$\nabla_\rho v^\mu = 0, \tag{3.2a}$$
$$\nabla_\rho h_{\mu\nu} = 0. \tag{3.2b}$$

One can then proceed along the lines of Riemannian geometry and try to characterize compatible connections by their torsion. However, it turns out that the compatibility with the degenerate Carroll structure implies some "intrinsic torsion" [45] which can not be chosen to vanish. This also implies that fixing the free or non-intrinsic part of the torsion is not sufficient to fix the remaining freedom. Finally, the intrinsic torsion can also be used to classify Carroll structures as it is inherent to the geometry i.e. $(v^\mu, h_{\mu\nu})$ and does not depend on the choice of connection.

In this chapter, we will in section 3.1 review the characterization of connections by intrinsic torsion and show how this leads to a classification of Carroll structures. Then we will in section 3.2 introduce a new procedure to fix the free torsion and the ambiguity in the connection which is left by this choice of torsion. We will thereby single out a Carroll connection analogous to the Levi-Civita connection. Finally, in section 3.3, we will derive important properties of this natural Carroll connection.

3.1 Intrinsic torsion of adapted connections

This section is based on [45], though we use a slightly different formalism in order to match the notation of chapter 2. In section 2.2, we saw how different kinematical or structure groups give rise to different notions of geometry through a gauging procedure. In particular, connections associated with a structure group G are adapted in the sense that any tensor left invariant at the level of the group induces a tensor field that is parallel in those connections. One way to think about intrinsic torsion is exactly as a consequence of these compatibility conditions. In the following, we are going to review a method of systematically investigating these structures.

In the frame bundle, we saw in section 2.2, that we can compute the torsion 1-form with Cartan's first structure equation

$$T = d\theta + [\omega \wedge \theta], \tag{3.3}$$

where θ is the co-frame (2.27a). In particular, one can observe that the change in torsion due to a shift in connection $\kappa = \omega' - \omega$ (κ is sometimes called the contorsion) is

$$T' - T = [\kappa \wedge \theta]. \tag{3.4}$$

In (3.4) the exterior derivative dropped out and the relation is purely algebraic. Hence, we can analyze the change in torsion as a map between vector spaces without worrying about the value of fields at more than one point at a time. More properly, we will deal with $\mathfrak{g}$-modules, which can be thought of as a generalization of a vector space with the scalar multiplication replaced by an action of the Lie algebra $\mathfrak{g}$. In doing this, the structures we identify are automatically compatible with the group structure and consequently invariant under gauge transformations.

3.1.1 The Spencer differential

To analyze the change in torsions, we define the mapping (3.4) as the so-called Spencer differential

$$\begin{aligned} \partial : \Omega(M, \mathfrak{g}) &\longrightarrow \Omega^2(M, \mathfrak{g}), \\ \kappa &\longmapsto [\kappa \wedge \theta], \end{aligned} \tag{3.5}$$

where we remind ourselves that $\Omega^p(M, \mathfrak{g}) = \Omega^p(M) \otimes \mathfrak{g}$ are p-forms valued in $\mathfrak{g}$. Having defined the Spencer differential, we can write down the following exact sequence

$$0 \longrightarrow \ker \partial \longrightarrow \Omega(M, \mathfrak{g}) \xrightarrow{\partial} \Omega^2(M, \mathfrak{g}) \longrightarrow \operatorname{coker} \partial \longrightarrow 0, \tag{3.6}$$

with the co-kernel defined as $\operatorname{coker} \partial = \Omega^2(M, \mathfrak{g}) / \operatorname{im} \partial$. These spaces can be interpreted as follows:

- $\ker \partial$: The kernel of the Spencer differential represents the components of the contorsion κ that do not affect the torsion. In particular, it can be seen as the obstruction to defining a unique connection from torsion constraints.
- $\Omega(M, \mathfrak{g})$: The space of contorsions i.e. the difference between two adapted connections.
- $\Omega^2(M, \mathfrak{g})$: The torsion or the change in torsion lives in this space.
- $\operatorname{coker} \partial$: The co-kernel measures the part of the domain (i.e. the torsion) that is not altered under a change of connection. Hence, this can be interpreted as an "intrinsic torsion" that follows from the G-structure regardless of the choice of connection. Further, one can consider what subspace of the co-kernel the torsion of a specific realization falls in and thereby classify the G-structure by its intrinsic torsion.

One may wonder why the above considerations play no role in GR and Riemannian geometry. The reason is that for the structure group of pseudo-Riemannian geometry $G = SO(d, 1)$ the Spencer differential turns out to be an isomorphism, and consequently $\ker \partial \cong \operatorname{coker} \partial \cong 0$.

Using the language from above, there is no intrinsic torsion (coker $\partial \cong 0$) nor any obstruction to defining the connection by fixing the torsion ($\ker \partial \cong 0$). Concretely, this can be interpreted as the explanation of why a Riemann-Cartan connection, i.e any connection satisfying (3.1), can be written as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) + \frac{1}{2} g^{\rho\lambda}(g_{\mu\sigma} T^{\sigma}{}_{\lambda\nu} + g_{\nu\sigma} T^{\sigma}{}_{\lambda\mu} + g_{\lambda\sigma} T^{\sigma}{}_{\mu\nu}), \tag{3.7}$$

with the torsion given by $T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{[\mu\nu]}$. It is clear that the connection (3.7) is fixed by the torsion and vice versa, which is the practical consequence of the Spencer differential being an isomorphism. As any connection can be modified by an appropriate choice of contorsion κ to have vanishing torsion, we can define a unique torsionless connection, the Levi-Civita connection. Hence, the above analysis is equivalent to the fundamental theorem of Riemannian geometry.

3.1.2 Intrinsic torsion of Carroll structures

In order to carry out the procedure described in section 3.1.1, we first need to compute the action of the Spencer differential ∂ in the case of the Carroll structure. This is simply done using its definition (3.5) and the Carroll algebra's (2.5a)-(2.5d) action on a local basis of $\Omega(M, \mathfrak{g})$

$$\partial(\tau \otimes C_a) = [\tau \otimes C_a \wedge \theta] = \tau \wedge e^b [C_a, P_b] = \delta_{ab} e^b \wedge \tau \otimes H, \tag{3.8a}$$
$$\partial(e^a \otimes C_b) = \delta_{bc} e^c \wedge e^a \otimes H, \tag{3.8b}$$
$$\partial(\tau \otimes J_{ab}) = 2\delta_{c[b} e^c \wedge \tau \otimes P_{a]}, \tag{3.8c}$$
$$\partial(e^a \otimes J_{bc}) = 2\delta_{d[c} e^d \wedge e^a \otimes P_{b]}. \tag{3.8d}$$

By inspection of (3.8a)-(3.8d), one can conclude that the kernel is given by

$$\ker \partial = \langle \delta_{c(a} e^c \otimes C_{b)} \rangle , \tag{3.9}$$

where $\langle \dots \rangle$ denotes the span. Further, we can also check that (3.9) is a $\mathfrak{g}$-submodule that is stable under gauge transformations. The action of a gauge transformation Σ, as defined in (2.28), on e^a is given by (2.31b). For the transformation of C_a, we need to be careful because under the gauge transform we here want to interpret e^a and C_a as a vector and a co-vector, respectively. However, the action on e^a was defined in (2.29) as the adjoint to the action on P_a rather than the inverse. Thus, to get the correct transformation we need to use $\delta C_a = [C_a, -\Sigma]$, which then yields the following change in (3.9)

$$\delta(e_{(a} \otimes C_{b)}) = -\lambda_{c(a} e^c \otimes C_{b)} - \lambda^c{}_{(a} e_{b)} \otimes C_c = -(\lambda^{(c}{}_a \delta^{d)}_b + \lambda^{(c}{}_b \delta^{d)}_a) e_c \otimes C_d \in \ker \partial. \tag{3.10}$$

From the computation (3.10), we see that a gauge transformation stays in the kernel. Likewise, we can consider the RHS of (3.8a)-(3.8d) the image of the Spencer differential and deduce that the co-kernel is spanned by

$$\operatorname{coker} \partial = \langle [(\delta_{c(a} \tau \wedge e^c \otimes P_{b)}] \rangle , \tag{3.11}$$

where $[\dots]$ is the equivalence class modulo the image $\operatorname{im} \partial$. Strictly speaking, the symmetrization in (3.11) is not necessary to write explicitly, as the anti-symmetric part is what we quotient by. It may seem that the co-kernel is missing the symmetric complement of (3.8d), but that would be a rank 3 tensor symmetric in the two first indices and anti-symmetric in the first and third indices, and such a tensor must vanish. The co-kernel can also be shown to be invariant under gauge transformations. The above analysis shows that Carroll structures unlike Lorentzian structures can contain intrinsic torsion.

To further characterize the space $\operatorname{coker}\partial$, we can consider the map Φ defined by

$$\begin{aligned}\Phi : \operatorname{coker}\partial &\longrightarrow \langle e^{(a} \otimes e^{b)}\rangle\,, \\ [\tau \wedge e^a \otimes P_b] &\longmapsto e^{(a} \otimes e^{c)}\delta_{cb},\end{aligned} \tag{3.12}$$

and extend it by linearity. The map Φ can be shown to be a G-equivariant isomorphism i.e. $\operatorname{coker}\partial \cong \langle e^{(a} \otimes e^{b)}\rangle$. In particular, $\langle e^{(a} \otimes e^{b)}\rangle$ is the space of spatial, symmetric (0,2) tensors which transform only under the subgroup SO(d). Thus, it breaks into the usual traceless and trace irreducible submodules

$$\operatorname{coker}\partial \cong \mathcal{C}_1 \oplus \mathcal{C}_2, \qquad \text{with} \qquad \begin{aligned}\mathcal{C}_1 &= \langle e^{(a} \otimes e^{b)} - \tfrac{1}{d}\delta_{cd}e^c \otimes e^d\rangle\,, \\ \mathcal{C}_2 &= \langle \delta_{ab} e^a \otimes e^b\rangle\,.\end{aligned} \tag{3.13}$$

As we have a split into two irreducible submodules, we get a total of $2^2 = 4$ classes of intrinsic torsion.

To figure out which class a specific Carrollian geometry falls in, we simply apply the map Φ to the equivalence class of the torsion (2.32) $[T]$

$$\begin{aligned}\Phi([T]) &= \Phi([(de^a - e_b \wedge \Omega^{ba}) \otimes P_a]) = \Phi([de^a \otimes P_a]) \\ &= \Phi([\tfrac{1}{2}de^a(\vartheta_b, \vartheta_c)e^b \wedge e^c \otimes P_a - de^a(v, \vartheta_b)\tau \wedge e^b \otimes P_a]) \\ &= -de_{(a}(v, \vartheta_{b)})e^a \otimes e^b.\end{aligned} \tag{3.14}$$

For the first equality, we used that the $(\dots) \otimes H$ is modded out, because all 2-forms tensored with H are in the image (3.8a)-(3.8b). In the second equality, we utilized that the anti-symmetry in the flat indices of Ω^{ab} puts in the kernel of Φ, and finally we expand the 2-form in the basis (τ_μ, e^a_μ). The object (3.14) does not seem to have a direct relation to any of the invariant Carroll tensors presented in section 2.2.4. However, recalling that the extrinsic curvature $K_{\mu\nu}$ is spatial, we can compute its components in the frame bundle as

$$\vartheta^\mu_a \vartheta^\nu_b K_{\mu\nu} = -\vartheta^\mu_{(a}\mathcal{L}_v e_{b)\mu} = -\vartheta^\mu_{(a} v^\nu (de_{b)})_{\nu\mu} = -de_{(a}(v, \vartheta_{b)}), \tag{3.15}$$

where we used Cartan's magic formula. Hence, we have shown that the extrinsic curvature exactly captures the intrinsic torsion of the Carroll structure. In this light, we can state the four classes of intrinsic torsion in terms of $K_{\mu\nu}$:

($\mathcal{C}_0$) $K_{\mu\nu} = 0$ (the trivial submodule),

($\mathcal{C}_1$) $K = h^{\mu\nu}K_{\mu\nu} = 0$ (traceless submodule),

($\mathcal{C}_2$) $K_{\mu\nu} = f h_{\mu\nu}$ ($f \neq 0$, trace submodule),

($\mathcal{C}_3$) None of the above, (full co-kernel).

In the original paper [45], a similar analysis is performed of Newton-Cartan or Galilean structures, which are defined in terms of the gauge invariant tensors $(\tau_\mu, h^{\mu\nu})$ (note the duality to Carroll $(v^\mu, h_{\mu\nu})$). This analysis shows that the intrinsic torsion is captured by $d\tau$ and a classification by the intrinsic torsion works out as the well-known classes:

($\mathcal{G}_0$) $d\tau = 0$, torsionless Newton–Cartan geometry (NC),

($\mathcal{G}_1$) $d\tau \neq 0$ and $d\tau \wedge \tau = 0$, twistless torsional Newton–Cartan geometry (TTNC),

($\mathcal{G}_2$) None of the above, torsional Newton–Cartan geometry (TNC).

3.2 Resolving the ambiguity

The goal of this chapter is to find a suitable Carrollian connection whose properties resemble those of the Levi-Civita connection. The analysis of 3.1 shows that we cannot uniquely single out a Carroll-compatible connection by considering only the torsion. Also, we can in general not choose a torsionless connection, as part of the torsion is intrinsic. Thus, we can only hope to set the "non-intrinsic" torsion to zero. However, there is no way of doing this in a Carroll-invariant manner, as one cannot define a $\mathfrak{g}$-stable complement to the image $\operatorname{im}\partial$ in $\Omega^2(M,\mathfrak{g})$. The space of torsions does split as a vector space for any choice of basis (v^μ, e_a^μ), but the complement will not be invariant under boosts. On the other hand, it turns out one cannot construct a boost invariant connection using only the metric data $(v^\mu, \tau_\mu, h^{\mu\nu}, h_{\mu\nu})$ [18], and thus we are anyway forced to consider connections that transform under boosts.

The first condition for the Carroll analog of the Levi-Civita connection is thus to set the non-intrinsic torsion to zero. This is implemented by setting the projection onto the vector space complement of $\operatorname{im}\partial$ to zero. This connection can hence be regarded as a "minimal torsion connection". The remaining degrees of freedom in the connection i.e. (3.9) is still undetermined after the choice of torsion. This can be fixed by mimicking the property that the Levi-Civita connection is also compatible with the inverse Lorentzian metric, i.e. $\nabla_\rho g^{\mu\nu} = 0$. In the Carroll case, the remaining freedom is not enough to ensure full compatibility, but we can improve the compatibility with the inverse vielbein τ_μ. That is, we can decompose $\nabla_\mu \tau_\nu$ into the subspaces defined by the spatial and temporal projectors (2.49) and index symmetries and then exhaust the remaining freedom by setting projections to zero. It may seem like an arbitrary choice that we choose to improve the derivative of τ_μ rather than the derivative of $h^{\mu\nu}$, but either choice is equivalent. To see this, we note the identities

$$\nabla_\rho h^{\mu\nu} = 2v^{(\mu}h^{\nu)\sigma}\nabla_\rho \tau_\sigma, \qquad \nabla_\rho \tau_\mu = -h_{\mu\sigma}\tau_\lambda \nabla_\rho h^{\sigma\lambda}, \tag{3.16}$$

which hold for any Carroll-compatible connection. This shows that knowing the derivative of τ_μ is the same as knowing the derivative of $h^{\mu\nu}$. Consequently, improving the one derivative improves the other equally.

3.2.1 Solving for the connection

To better understand where each part of the torsion falls, we can write out all projections of the torsion (2.32)

$$T(H)(v,\vartheta_a) = d\tau(v,\vartheta_a) - \Omega_a(v), \tag{3.17a}$$

$$T(H)(\vartheta_a,\vartheta_b) = d\tau(\vartheta_a,\vartheta_b) + 2\Omega_{[a}(\vartheta_{b]}), \tag{3.17b}$$

$$T_a(P)(v,\vartheta_b) = de_a(v,\vartheta_b) - \Omega_{ab}(v), \tag{3.17c}$$

$$T_a(P)(\vartheta_b,\vartheta_c) = de_a(\vartheta_b,\vartheta_c) + 2\Omega_{a[b}(\vartheta_{c]}), \tag{3.17d}$$

where $T(H)$ and $T_a(P)$ are defined in (2.34a) and $\vartheta_a = e_a^\mu \partial_\mu$. Note that the symmetric part of (3.17c) is exactly the intrinsic torsion as computed in (3.14). Setting the non-intrinsic torsion to zero then amounts to demanding that (3.17a), (3.17b), (3.17d) along with the anti-symmetric part of (3.17c) vanish

$$T(H)(v,\vartheta_a) = 0, \qquad T(H)(\vartheta_a,\vartheta_b) = 0, \qquad T_{[a}(P)(v,\vartheta_{b]}) = 0, \qquad T_a(P)(\vartheta_b,\vartheta_c) = 0. \tag{3.18}$$

These equations allow us to solve for the connection in terms of derivatives of the co-frame and $\Omega_{(a}(\vartheta_{b)})$

$$\Omega_{ab}(\vartheta_c) = -\frac{1}{2}(de_a(\vartheta_b, \vartheta_c) + de_b(\vartheta_c, \vartheta_a) - de_c(\vartheta_a, \vartheta_b)), \tag{3.19a}$$

$$\Omega_{ab}(v) = -de_{[a}(\vartheta_{b]}, v), \tag{3.19b}$$

$$\Omega_a(\vartheta_b) = -\frac{1}{2}d\tau(\vartheta_a, \vartheta_b) + \Omega_{(a}(\vartheta_{b)}), \tag{3.19c}$$

$$\Omega_a(v) = d\tau(v, \vartheta_a). \tag{3.19d}$$

We note here that the remaining ambiguity $\Omega_{(a}(\vartheta_{b)})$ is precisely the part of the connection that lies in the kernel of the Spencer differential (3.9).

As discussed, we want to fix this residual freedom by considering the derivative of τ_μ which can be computed using the connection coefficients (2.58)

$$\nabla_\mu \tau_\nu = \partial_\mu \tau_\nu - \Gamma^\rho_{\mu\nu}\tau_\rho = \Omega_{\mu a} e^a_\nu. \tag{3.20}$$

For completeness, we also compute the derivative in the frame bundle where τ_μ is simply represented by $-\eta$ cf. (2.40). The covariant derivative can then be computed in the frame bundle using (2.41)

$$D(-\eta) = \Omega_a \otimes \gamma^a, \tag{3.21}$$

which agrees with (3.20). From both (3.20) and (3.21) it is clear that setting the symmetric part of (3.19c) to zero improves compatibility as it only leaves the anti-symmetric part of (3.21). With this choice, we can solve for the remaining part of the connection

$$\Omega_{(a}(\vartheta_{b)}) = 0 \qquad \Rightarrow \qquad \Omega_a(\vartheta_b) = -\frac{1}{2}d\tau(\vartheta_a, \vartheta_b). \tag{3.22}$$

The projection (3.22) together with (3.19a)-(3.19d) fixes all the components of the connection. As we will primarily work in the second-order formalism, we further derive the corresponding affine connection using (2.58)

$$\tilde{\Gamma}^\rho_{\mu\nu} \equiv -v^\rho \partial_{(\mu}\tau_{\nu)} + h^{\rho\lambda}(\partial_{(\mu}h_{\nu)\lambda} - \frac{1}{2}\partial_\lambda h_{\mu\nu}) - K^\rho{}_\mu \tau_\nu + v^\rho \tau_{(\mu}\tau_{\nu)\lambda} v^\lambda, \tag{3.23}$$

where we defined $\tau_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$. We denote this special minimal torsion connection by $\tilde{\Gamma}$ and the corresponding covariant derivative as $\tilde{\nabla}$.

Alternatively, the derivation of (3.23) can be done entirely in the second-order formalism, which consists of the above steps but without reference to the natural Carrollian structure. In particular, one can write down the most general Carroll compatible connection i.e. satisfying (3.2a) and (3.2b), see [18]. Then one can set as many projections of the torsion to zero as possible which will leave only the intrinsic torsion. Finally, one can compute the derivative $\nabla_\mu \tau_\nu$ and use the remaining freedom to improve the compatibility as above. This procedure will yield the same connection as derived through the first-order formalism (3.23). The $\tilde{\Gamma}$-connection is also considered as a natural analog of the Levi-Civita connection in Carrollian geometry in [40] (see footnote 76).

A final note on this procedure of minimizing the torsion and subsequently improving compatibility with the inverse metric data: If one applies it to a Galilean structure then the resulting connection coincides with the one found to be convenient in [29]. This connection is also described in [46] as a natural Levi-Civita analog for Galilean structures.

3.3 Properties of the $\tilde{\Gamma}$-connection

The connection (3.23) by definition satisfies the Carrollian compatibility conditions (3.2a) and (3.2b) and to ease future computations we will in this section derive a number of relations for the $\tilde{\Gamma}$-connection. As mentioned, one cannot write down a boost-invariant connection using only the vielbeine and hence the choice of connection (3.23) must transform. Specifically, the $\tilde{\Gamma}$-connection changes under a boost transformation λ^μ according to

$$\delta\tilde{\Gamma}^\rho_{\mu\nu} = v^\rho\tau_\mu\lambda^\sigma K_{\sigma\nu} + \lambda^\rho K_{\mu\nu} - K^\rho{}_\mu\lambda_\nu - v^\rho(\tilde{\nabla}_{(\mu}\lambda_{\nu)} - \lambda_{(\mu}\tau_{\nu)\sigma}v^\sigma + v^\sigma\tau_{(\mu}\tilde{\nabla}_{|\sigma|}\lambda_{\nu)}). \tag{3.24}$$

3.3.1 Covariant derivatives of extended metric data

A commonly occurring object is the inverse vielbein τ_μ, for which we can compute the covariant derivative by direct computation

$$\begin{aligned}\tilde{\nabla}_\mu\tau_\nu &= \partial_\mu\tau_\nu - \tilde{\Gamma}^\lambda_{\mu\nu}\tau_\rho = \partial_\mu\tau_\nu - \partial_{(\mu}\tau_{\nu)} + \tau_{(\mu}\tau_{\nu)\lambda}v^\lambda \\ &= \frac{1}{2}\tau_{\mu\nu} - v^\rho\tau_{\rho(\mu}\tau_{\nu)}.\end{aligned} \tag{3.25}$$

Alternatively, we could have found the expression from its first-order form (3.21). To see that these approaches are equivalent we can compute the projections of (3.25)

$$v^\mu e^\nu_a\tilde{\nabla}_\mu\tau_\nu = v^\mu e^\nu_a\tau_{\mu\nu}, \tag{3.26a}$$
$$e^\mu_a e^\nu_b\tilde{\nabla}_\mu\tau_\nu = \frac{1}{2}e^\mu_a e^\nu_b\tau_{\mu\nu}, \tag{3.26b}$$

which can be seen to be in agreement with (3.19d) and (3.22). Using either approach we can further compute the covariant derivative of $h^{\mu\nu}$ and its contraction

$$\tilde{\nabla}_\rho h^{\mu\nu} = v^{(\mu}h^{\nu)\sigma}(\delta^\gamma_\rho - \tau_\rho v^\gamma)\tau_{\gamma\sigma}, \tag{3.27a}$$
$$\tilde{\nabla}_\mu h^{\mu\nu} = h^{\nu\sigma}v^\rho\tau_{\rho\sigma}. \tag{3.27b}$$

Finally, in deriving equations of motion (see sections 4.4-4.5), we need to integrate by parts which relies on Stokes' theorem, and so we need to relate the $\tilde{\Gamma}$-derivative to the exterior derivative in the sense of appendix A, in particular (A.7). For a vector field X^μ the relevant identity is

$$\partial_\mu(eX^\mu) = e(\tilde{\nabla}_\mu X^\mu + \tau_\mu X^\mu K), \tag{3.28}$$

where, in the language of appendix A, eX^μ is dual to a d-form. We can also compute the identity for a tensor $Q^{[\mu\nu]}$ related to a $(d-1)$-form

$$\partial_\nu(eQ^{[\mu\nu]}) = e(\tilde{\nabla}_\nu Q^{[\mu\nu]} + \tau_\sigma K^\mu{}_\rho Q^{[\sigma\rho]} + K\tau_\nu Q^{[\mu\nu]}). \tag{3.29}$$

3.3.2 Torsion and curvature

The torsion $T^\rho{}_{\mu\nu}$ and Riemann curvature tensor $R_{\mu\nu\sigma}{}^\rho$ is as usual defined through the relation

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]X^\rho = -\tilde{R}_{\mu\nu\sigma}{}^\rho X^\sigma - \tilde{T}^\sigma{}_{\mu\nu}\tilde{\nabla}_\sigma X^\rho \tag{3.30}$$

for any vector field X^μ. These can be solved to obtain the explicit formulas

$$\tilde{T}^\rho{}_{\mu\nu} = 2\tilde{\Gamma}^\rho_{[\mu\nu]}, \tag{3.31}$$
$$\tilde{R}_{\mu\nu\sigma}{}^\rho = -\partial_\mu\tilde{\Gamma}^\rho_{\nu\sigma} + \partial_\nu\tilde{\Gamma}^\rho_{\mu\sigma} - \tilde{\Gamma}^\rho_{\mu\lambda}\tilde{\Gamma}^\lambda_{\nu\sigma} + \tilde{\Gamma}^\rho_{\nu\lambda}\tilde{\Gamma}^\lambda_{\mu\sigma}. \tag{3.32}$$

The definitions (3.31) and (3.32) are equivalent to calculating the torsion and curvature in the frame bundle and then using the relations (2.61a) and (2.61b) to translate the algebra valued 2-forms to tensors in the tangent bundle. From the connection coefficients (3.23) we can compute the torsion

$$\tilde{T}^{\rho}{}_{\mu\nu} = 2\tilde{\Gamma}^{\rho}_{[\mu\nu]} = 2\tau_{[\mu}K_{\nu]}{}^{\rho}. \tag{3.33}$$

The second-order form of the torsion (3.33) highlights the minimal torsion feature of the connection, as the torsion is exactly given by the intrinsic torsion as encoded by $K_{\mu\nu}$.

One cannot write out the full curvature tensor in a manifestly covariant manner in terms of vielbeine, but it is worth noting a few consequences of the compatibility conditions (3.2a) and (3.2b). In particular, we find the following relations using the compatibility conditions and the definition (3.30)

$$\tilde{R}_{\mu\nu\sigma}{}^{\rho}v^{\sigma} = 0, \tag{3.34a}$$
$$\tilde{R}_{\mu\nu\sigma}{}^{\lambda}h_{\lambda\rho} + \tilde{R}_{\mu\nu\rho}{}^{\lambda}h_{\lambda\sigma} = 0. \tag{3.34b}$$

Alternatively, these properties can be read off directly from the frame bundle formulation (2.61b). If we further define the Ricci tensor as the contraction

$$\tilde{R}_{\mu\nu} \equiv \tilde{R}_{\mu\sigma\nu}{}^{\sigma}, \tag{3.35}$$

we immediately see that

$$\tilde{R}_{\mu\nu}v^{\nu} = 0. \tag{3.36}$$

It is important to note that the Ricci tensor is not a symmetric tensor as it is in the case for the Levi-Civita connection. Also for the Ricci tensor we cannot covariantly write down the full tensor in terms of vielbeine, but we can explicitly compute the anti-symmetric part using the first Bianchi identity

$$\tilde{R}_{[\mu\nu\sigma]}{}^{\rho} = \tilde{T}^{\lambda}{}_{[\mu\nu}\tilde{T}^{\rho}{}_{\sigma]\lambda} - \tilde{\nabla}_{[\mu}\tilde{T}^{\rho}{}_{\nu\sigma]}. \tag{3.37}$$

Contracting ν and ρ of (3.37) yields

$$\tilde{R}_{[\mu\nu]} = \frac{1}{2}(\tilde{R}_{\mu\nu\sigma}{}^{\sigma} + \tilde{T}^{\lambda}{}_{\mu\nu}\tilde{T}^{\sigma}{}_{\lambda\sigma} + \nabla_{\mu}\tilde{T}^{\lambda}{}_{\nu\lambda} - \nabla_{\nu}\tilde{T}^{\lambda}{}_{\mu\lambda} + \nabla_{\lambda}\tilde{T}^{\lambda}{}_{\mu\nu}). \tag{3.38}$$

Combining (3.34a) and (3.34b) shows that the contraction $\tilde{R}_{\mu\nu\sigma}{}^{\sigma}$ vanishes, and we obtain the following expression for the anti-symmetric part of the Ricci tensor

$$\tilde{R}_{[\mu\nu]} = \tilde{\nabla}_{[\mu}(\tau_{\nu]}K) + \tilde{\nabla}_{\rho}(\tau_{[\mu}K^{\rho}{}_{\nu]}), \tag{3.39}$$

where we used that $\tilde{T}^{\rho}{}_{\mu\rho} = K\tau_{\mu}$.

Chapter 4

Ultra-Relativistic Expansion of the Einstein-Hilbert Action

One of the greatest accomplishments of modern physics is understanding how matter sources curvature in the geometry of spacetime through Einstein's theory of General Relativity. The content of Einstein's theory is elegantly captured by the Einstein-Hilbert (EH) action

$$S_{\mathrm{EH}} = \frac{c^3}{16\pi G_N} \int d^{d+1}x \sqrt{-g} R. \tag{4.1}$$

A natural question to ask is how Newtonian gravity emerges as a non-relativistic limit of GR and in particular if it can be extracted directly from (4.1). Much work has been done on geometrizing Newtonian gravity since Cartan originally formulated the framework of Newton-Cartan geometry in 1923 [1, 2]. Recently, in the work [29], the authors derived an action principle for Newtonian gravity directly from a careful large c expansion of the EH action (4.1).

In this chapter, we will investigate the converse ultra-relativistic (UR) small c expansion to obtain an action for a Carrollian theory of gravity order by order in c^2. The direction of ultra-relativistic gravity as a strict limit of GR has already been pursued at both the level of the EOM [32] and using a Hamiltonian approach [33]. The action derived from the latter will turn out to agree with the leading-order action derived in this thesis. A Carrollian limit of the Palatini action has also been considered in [31], which is distinct from the limit that we will consider. This is due to the authors of [31] choosing a scaling in c of the spin connection that appears natural in the first-order formalism. We will instead work in the second-order formalism and derive the scaling in c of the connection from the vielbeine and metric.

The chapter is structured as follows: In section 4.1 we set up the framework for the expansion and demonstrate how Carroll metric data can be obtained as a limit of the relativistic metric data. We shall in section 4.2 rewrite the action (4.1) in terms of more convenient variables, before computing the action and EOM at leading order (LO) and next-to-leading order (NLO) in section 4.4 and 4.5, respectively.

4.1 Ultra-relativistic expansion of the vielbeine

We want to mimic the methods of [29], where the authors expand the EH action in a Galilean limit, i.e. $c \to \infty$. In particular, they do this by reinstating factors of c and carefully choosing the scaling in c of every object occurring in the EH action (4.1). Having primed the EH action in this way, the authors can compute the Galilean action order by order in c^{-2}, which is a self-consistent sub-sector of theory [29].

We will, instead of taking $c^{-1} \to 0$, which corresponds to a non-relativistic limit, consider $c \to 0$. Specifically, we will be expanding in powers of c^2 rather than powers of c. This can be done consistently as it turns out only even powers appear in the EH action when fully written out. To clarify what is meant by expanding in a dimensionful constant c, we can redefine $c \equiv \sqrt{\sigma}\hat{c}$, where $\hat{c} = 299\,792\,458\,\mathrm{m\,s^{-1}}$ is the physical speed of light and σ is a dimensionless expansion parameter. We can then for simplicity choose to work in units with $\hat{c} = 1$, such that we effectively expand in $\sigma = c^2$ but in a mathematically meaningful way.

For this ultra-relativistic expansion to be possible, we need to assume that up to an overall power of c, which we will factor out, the relativistic fields are analytic in $\sigma = c^2$

$$\phi^I = \phi^I_{(0)} + \sigma\phi^I_{(2)} + \sigma^2\phi^I_{(4)} + \mathcal{O}(\sigma^3), \tag{4.2}$$

where ϕ^I represent all the dynamical fields in our theory and $\phi^I_{(n)}$ are their expansion coefficients. This analyticity assumption also implies that equivalent relativistic spacetimes can become inequivalent in the $c \to 0$ limit, if they are related by coordinate-transformations not analytic in c^2. For a detailed discussion of the analogous phenomena in the setting of the $c \to \infty$ limit, see [29].

4.1.1 Vielbeine and tangent space transformation

We consider a Lorentzian spacetime modeled by a $(d+1)$-manifold equipped with a pseudo-Riemannian metric, which can be described in terms of the relativistic vielbeine E^A_μ. As in chapter 2, the curved indices take values $\mu = 0, \ldots, d$ and the upper case flat indices $A = 0, \ldots, d$. If we restrict to only the spatial indices, then we write lower case $a = 1, \ldots d$.

By considering the light-cone structure of the tangent space, one can conclude that the timelike vielbein must scale differently in c as compared to the spacelike. Thus, we may also assume that the vielbeine and their inverses can be written as

$$E^A_\mu = cT_\mu\delta^A_0 + \mathcal{E}^a_\mu\delta^A_a, \tag{4.3a}$$

$$E^\mu_A = -c^{-1}T^\mu\delta^0_A + \mathcal{E}^\mu_a\delta^a_A, \tag{4.3b}$$

to reflect this inhomogeneous scaling in c. We will dub the variables T_μ, $\mathcal{E}^a_\mu$, T^μ and $\mathcal{E}^\mu_a$ "Pre-Ultra-Relativistic" (PUR) variables in direct analogy to the Pre-Non-Relativistic vielbeine of [29]. The PUR variables are a convenient starting point for our $c \to 0$ expansion, as we have already singled out the time and space components in anticipation that they will decouple in the limit. As a consequence of the relativistic completeness relations $E^A_\mu E^\mu_B = \delta^A_B$ and $E^A_\mu E^\nu_A = \delta^\nu_\mu$, the PUR vielbeine satisfy completeness relations analogous to (2.36)

$$T^\mu T_\mu = -1, \qquad T^\mu\mathcal{E}^a_\mu = 0, \qquad T_\mu\mathcal{E}^\mu_a = 0, \qquad \mathcal{E}^a_\mu\mathcal{E}^\mu_b = \delta^a_b. \tag{4.4}$$

Under local Lorentz transformations, the relativistic vielbeine transform as $\delta E^A_\mu = \Lambda^A{}_B E^B_\mu$. We can decompose $\Lambda^A{}_B$ into temporal and spatial projections and choose the scaling of each component to be

$$\Lambda^A{}_B = c\Lambda_b\delta^A_0\delta^b_B + c\Lambda^a\delta^A_a\delta^0_B + \Lambda^a{}_b\delta^A_a\delta^b_B, \tag{4.5}$$

with $\Lambda_{ab} = -\Lambda_{ba}$. In the parameterization (4.5), both the boost- (Λ_a) and rotation-parameters ($\Lambda^a{}_b$) are taken to be order c^0. Considering (2.1), the factors of c in the transformation (4.5) may seem strange. However, they are chosen such that new factors, appearing due to transformation, in (4.3a)-(4.3b) are sub-leading in a $c \to 0$ expansion. Alternately, this can be seen as a kind of renormalization, because the scaling (2.1) diverges in the $c \to 0$ limit. Written out for each of the decomposed vielbeine, we find the transformations

$$\delta T_\mu = \Lambda_b \mathcal{E}^b_\mu, \tag{4.6a}$$
$$\delta T^\mu = c^2 \Lambda^a \mathcal{E}^\mu_a, \tag{4.6b}$$
$$\delta \mathcal{E}^a_\mu = \Lambda^a{}_b \mathcal{E}^b_\mu + c^2 \Lambda^a T_\mu, , \tag{4.6c}$$
$$\delta \mathcal{E}^\mu_a = -\Lambda^b{}_a \mathcal{E}^\mu_b + \Lambda_a T^\mu. \tag{4.6d}$$

From the PUR variables, we can construct a degenerate spatial metric $\Pi_{\mu\nu}$ and its projective inverse as

$$\Pi_{\mu\nu} = \delta_{ab} \mathcal{E}^a_\mu \mathcal{E}^b_\nu, \tag{4.7a}$$
$$\Pi^{\mu\nu} = \delta^{ab} \mathcal{E}^\mu_a \mathcal{E}^\nu_b, \tag{4.7b}$$

which in the $c \to 0$ limit will become $h_{\mu\nu}$ and $h^{\mu\nu}$, respectively. From (4.4) we can deduce completeness relations for the spatial metrics that mirror the relations (2.47)

$$T^\mu T_\nu = -1, \qquad T^\mu \Pi_{\mu\nu} = 0, \qquad T_\mu \Pi^{\mu\nu} = 0, \qquad \Pi_{\mu\rho}\Pi^{\rho\nu} - T_\mu T^\nu = \delta^\nu_\mu. \tag{4.8}$$

Further, the transformation laws (4.6c) and (4.6d) imply the following transformations for the spatial metric

$$\delta \Pi_{\mu\nu} = 2c^2 \Lambda_a T_{(\mu} \mathcal{E}^a_{\nu)}, \tag{4.9a}$$
$$\delta \Pi^{\mu\nu} = 2\Lambda^a T^{(\mu} \mathcal{E}^{\nu)}_a. \tag{4.9b}$$

4.1.2 Expanding the PUR variables

Having established the relativistic PUR variables, we can use the analyticity assumption (4.2) to write the expansion of each vielbein. We choose the contra-variant vielbeine $(T^\mu, \mathcal{E}^\mu_a)$ to be defining in the sense that the sub-leading orders of the co-variant vielbeine $(T_\mu, \mathcal{E}^a_\mu)$ can be expressed in terms of the sub-leading contra-variant vielbeine. With this setup, the expansions of the vielbeine become

$$T^\mu = v^\mu + c^2 M^\mu + \mathcal{O}(c^4), \tag{4.10a}$$
$$T_\mu = \tau_\mu + c^2(\tau_\mu \tau_\nu M^\nu - e^a_\mu \tau_\nu \pi^\nu_a) + \mathcal{O}(c^4), \tag{4.10b}$$
$$\mathcal{E}^\mu_a = e^\mu_a + c^2 \pi^\mu_a + \mathcal{O}(c^4), \tag{4.10c}$$
$$\mathcal{E}^a_\mu = e^a_\mu + c^2(\tau_\mu M^\nu e^a_\nu - e^b_\mu e^a_\nu \pi^\nu_b) + \mathcal{O}(c^4), \tag{4.10d}$$

where we used the completeness relation (4.4) to solve for the sub-leading part of the covariant vielbeine. We can likewise expand the parameters of the Lorentz transform (4.5) in orders of c^2

$$\Lambda^a = \lambda^a + c^2 \eta^a + \mathcal{O}(c^4), \tag{4.11a}$$
$$\Lambda^a{}_b = \lambda^a{}_b + c^2 \sigma^a{}_b + \mathcal{O}(c^4). \tag{4.11b}$$

Using this expansion, we can derive the transformation rules for each of the expanded vielbeine. As an example, we consider the transformation of T^μ

$$\delta T^\mu = c^2 \Lambda^a \mathcal{E}^\mu_a = c^2 \lambda^a e^\mu_a + \mathcal{O}(c^4), \tag{4.12}$$

which can be interpreted as $\delta v^\mu = 0$ and $\delta M^\mu = \lambda^a e^\mu_a$. Repeating this for the other leading order (LO) vielbeine yields

$$\delta v^\mu = 0, \tag{4.13a}$$
$$\delta \tau_\mu = \lambda_a e^a_\mu \tag{4.13b}$$
$$\delta e^\mu_a = -\lambda^b{}_a e^\mu_b + \lambda_a v^\mu, \tag{4.13c}$$
$$\delta e^a_\mu = \lambda^a{}_b e^b_\mu, \tag{4.13d}$$

which coincides with the transformation rules derived in the gauging procedure (2.31a)-(2.31b) and (2.38a)-(2.38b). Similarly, we also calculate the transformations for the NLO fields

$$\delta M^\mu = \lambda^a e^\mu_a, \tag{4.14a}$$
$$\delta \pi^\mu_a = -\lambda^b{}_a \pi^\mu_b - \sigma^b{}_a e^\mu_b + \lambda_a M^\mu + \eta_a v^\nu, \tag{4.14b}$$

which do not have counter-parts in chapter 2, because there we only considered a strict $c \to 0$ limit.

We can further compute the corresponding c^2-expansion for the composite objects $\Pi^{\mu\nu}$ and $\Pi_{\mu\nu}$. Specifically, we find

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^2 \Phi^{\mu\nu} + \mathcal{O}(c^4), \tag{4.15a}$$
$$\Pi_{\mu\nu} = h_{\mu\nu} + 2c^2 \delta_{ab} e^a_{(\mu}(\tau_{\nu)} M^\rho e^a_\rho - e^c_{\nu)} e^b_\rho \pi^\rho_c) + \mathcal{O}(c^4), \tag{4.15b}$$

where we defined the sub-leading part of the inverse spatial metric

$$\Phi^{\mu\nu} \equiv \delta^{ab}(e^\mu_a \pi^\nu_a + e^\nu_a \pi^\mu_b). \tag{4.16}$$

The spatial metric also transforms under Lorentz transformation (4.11a)-(4.11b) and the rules are given by

$$\delta h^{\mu\nu} = 2 v^{(\mu} e^{\nu)}_a \lambda^a, \tag{4.17a}$$
$$\delta h_{\mu\nu} = 0. \tag{4.17b}$$

Similarly, one can also compute the sub-leading part of the spatial metrics (4.15a)-(4.15b).

The above procedure gives an alternate way, compared to section 2.2, of deriving geometric objects associated with the Carroll limit. Further, it enables one to consider the sub-leading structure of the limit. As a final note, one could also have included diffeomorphisms in the discussion i.e. the PUR vielbeine would transform as $\delta E^A_\mu = \mathcal{L}_\Xi E^B_\mu + \Lambda^A{}_B E^B_\mu$ with Ξ^μ being a diffeomorphism generating vector field. The vector field Ξ^μ is also expanded as $\Xi^\mu = \xi^\mu + c^2\zeta^\mu + \mathcal{O}(c^4)$, and $\mathcal{L}_\xi$-derivative should then be understood as usual Lie derivatives, while the $\mathcal{L}_\zeta$-derivatives appearing at sub-leading order can be interpreted as a gauge transformation [29].

4.2 Pre-Ultra-Relativistic parameterization

In this section, we will recast the usual geometric objects of GR into a form that reflects the decoupling of space and time using the PUR parameterization. Specifically, we aim to reexpress the EH action in PUR variables, and thus we need to work through all the geometric objects that go into (4.1). In doing this, we retain the full content of the relativistic theory, but by re-expressing the theory using explicit factors of c, we set up the action for the subsequent $c \to 0$ expansion. This follows the same approach as the Pre-Non-Relativistic parameterization of [29], and in many cases the below results are equivalent up to the fact that the counting is reversed in going between $c^2 \leftrightarrow c^{-2}$.

4.2.1 Metric

As we will be working in the second-order formalism, the onset of this PUR parameterization is the metric and inverse metric, which are constructed from (4.3a)-(4.3b)

$$g_{\mu\nu} \equiv \eta_{AB} E^A_\mu E^B_\nu = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \tag{4.18a}$$
$$g^{\mu\nu} \equiv \eta^{AB} E^\mu_A E^\nu_B = -c^{-2} T^\mu T^\nu + \Pi^{\mu\nu}. \tag{4.18b}$$

The metric satisfies the usual completeness relations $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$ due to (4.4). Another object, one can construct directly from the metric is its determinant g, which we can write out in PUR variables as

$$g = \det(g_{\mu\nu}) = \det(-c^2 T_\mu T_\nu + \Pi_{\mu\nu}) = -c^2 \det(T_\mu T_\nu + \Pi_{\mu\nu}), \tag{4.19}$$

where we were able to pull out the $-c^2$ using the matrix determinant lemma, relying on the fact that $\det(\Pi_{\mu\nu}) = 0$. This leads us to define the density

$$E \equiv \sqrt{\det(T_\mu T_\nu + \Pi_{\mu\nu})}, \tag{4.20}$$

whose $c \to 0$ limit coincides with e, as defined in (2.50). The definition (4.20) allows us to write the usual square root metric determinant as

$$\sqrt{-g} = cE, \tag{4.21}$$

thereby transitioning to PUR variables and explicitly getting the scaling in c.

4.2.2 Levi-Civita connection

The conventional connection of GR is the Levi-Civita connection. However, its scaling in c is not clear, and we will indeed see that it contains many different orders of c. Furthermore, once we start expanding, the natural structure will be Carrollian rather than the original Lorentzian structure. Hence, it is advantageous to choose a Carroll-compatible connection (realized on the PUR variables) that satisfies (3.2a) and (3.2b). This is accomplished by splitting the original connection coefficient into a new Carrollian connection coefficient and a tensorial shift. In the following, we will write out the Levi-Civita connection in explicit orders of c and shift it such that we can write covariant derivatives and curvatures in terms of a connection of the type (3.23).

The Levi-Civita connection associated with the metric $g_{\mu\nu}$ is given by the usual connection coefficients

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{4.22}$$

Writing out (4.22) in terms of the PUR variables and keeping track of the factors of c, one finds an expansion of the kind

$$\Gamma^\rho_{\mu\nu} = c^{-2} \overset{(-2)}{C}{}^\rho_{\mu\nu} + \overset{(0)}{C}{}^\rho_{\mu\nu} + c^2 \overset{(2)}{C}{}^\rho_{\mu\nu} \tag{4.23}$$

with the various orders given by

$$\overset{(-2)}{C}{}^\rho_{\mu\nu} = \frac{1}{2} T^\rho \mathcal{L}_T \Pi_{\mu\nu}, \tag{4.24a}$$
$$\overset{(0)}{C}{}^\rho_{\mu\nu} = -T^\rho \partial_{(\mu} T_{\nu)} - T^\rho T_{(\mu} \mathcal{L}_T T_{\nu)} + \Pi^{\rho\lambda} (\partial_{(\mu} \Pi_{\nu)\lambda} - \tfrac{1}{2} \partial_\lambda \Pi_{\mu\nu}), \tag{4.24b}$$
$$\overset{(2)}{C}{}^\rho_{\mu\nu} = \Pi^{\rho\lambda} (T_\mu \partial_{[\lambda} T_{\nu]} + T_\nu \partial_{[\lambda} T_{\mu]}). \tag{4.24c}$$

Note that only the order c^0 part $\overset{(0)}{C}{}^{\rho}_{\mu\nu}$ transforms as connection coefficients under a diffeomorphism ξ^μ i.e.

$$\delta_\xi \overset{(0)}{C}{}^{\rho}_{\mu\nu} = \partial_\mu\partial_\nu\xi^\rho + \xi^\sigma\partial_\sigma\overset{(0)}{C}{}^{\rho}_{\mu\nu} + \overset{(0)}{C}{}^{\rho}_{\sigma\nu}\partial_\mu\xi^\sigma + \overset{(0)}{C}{}^{\rho}_{\mu\sigma}\partial_\nu\xi^\sigma - \overset{(0)}{C}{}^{\sigma}_{\mu\nu}\partial_\sigma\xi^\rho, \tag{4.25}$$

while the remaining terms transform tensorially. To make the connection to the minimal torsion connection $\tilde{\Gamma}$ of chapter 3, we split the order c^0 part of the Levi-Civita connection as

$$\overset{(0)}{C}{}^{\rho}_{\mu\nu} = \tilde{C}^{\rho}_{\mu\nu} + \overset{(0)}{S}{}^{\rho}_{\mu\nu}, \tag{4.26}$$

where the tensorial shift $\overset{(0)}{S}{}^{\rho}_{\mu\nu}$ is given by

$$\overset{(0)}{S}{}^{\rho}_{\mu\nu} \equiv -\frac{1}{2}\Pi^{\rho\lambda}T_\nu\mathcal{L}_T\Pi_{\mu\lambda}. \tag{4.27}$$

The shift $\overset{(0)}{S}{}^{\rho}_{\mu\nu}$ is chosen such that the connection coefficients $\tilde{C}^{\rho}_{\mu\nu}$ are the $\tilde{\Gamma}$-connection (3.23) realized on the PUR variables $(T^\mu, T_\mu, \Pi_{\mu\nu}, \Pi^{\mu\nu})$ rather than the LO Carrolian variables $(v^\mu, \tau_\mu, h_{\mu\nu}, h^{\mu\nu})$, as in chapter 3. Concretely, the connection takes the form

$$\tilde{C}^{\rho}_{\mu\nu} = -T^\rho\partial_{(\mu}T_{\nu)} + \Pi^{\rho\lambda}(\partial_{(\mu}\Pi_{\nu)\lambda} - \frac{1}{2}\partial_\lambda\Pi_{\mu\nu}) + \frac{1}{2}\Pi^{\rho\lambda}T_\nu\mathcal{L}_T\Pi_{\mu\lambda} - T^\rho T_{(\mu}\mathcal{L}_T T_{\nu)}, \tag{4.28}$$

and importantly the limit works out to be the Carrollian connection i.e.

$$\tilde{C}^{\rho}_{\mu\nu}\Big|_{\sigma=0} = \tilde{\Gamma}^{\rho}_{\mu\nu}. \tag{4.29}$$

Also, we note that all the properties presented in section 3.3 for the $\tilde{\Gamma}$-connection hold for the $\tilde{C}$-connection when the LO variables are swapped for the PUR variables.

4.2.3 Ricci tensor

The Ricci tensor is given by the contraction of the Riemann tensor $R_{\mu\nu} \equiv R_{\mu\sigma\nu}{}^{\sigma}$, which in terms of the connection coefficients take the form

$$R_{\mu\nu} = -\partial_\mu\Gamma^{\rho}_{\rho\nu} + \partial_\rho\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\rho\nu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu}. \tag{4.30}$$

We then follow the same procedure as for the Levi-Civita connection and write out the explicit orders of c as

$$R_{\mu\nu} = c^{-4}\overset{(-4)}{R}_{\mu\nu} + c^{-2}\overset{(-2)}{R}_{\mu\nu} + \overset{(0)}{R}_{\mu\nu} + c^2\overset{(2)}{R}_{\mu\nu} + c^4\overset{(4)}{R}_{\mu\nu}. \tag{4.31}$$

The expansion naively starts at c^{-4}, but the leading term turns out to vanish, and hence the expansion starts at c^{-2}. Expanding one finds

$$\overset{(-4)}{R}_{\mu\nu} = 0, \tag{4.32a}$$

$$\overset{(-2)}{R}_{\mu\nu} = \overset{(\tilde{C})}{\nabla}_\rho\overset{(-2)}{C}{}^{\rho}_{\mu\nu} + \frac{1}{4}\Pi^{\rho\sigma}\mathcal{L}_T\Pi_{\rho\sigma}\mathcal{L}_T\Pi_{\mu\nu}, \tag{4.32b}$$

$$\overset{(0)}{R}_{\mu\nu} = \overset{(\tilde{C})}{R}_{\mu\nu} - \overset{(\tilde{C})}{\nabla}_\mu\overset{(0)}{S}{}^{\rho}_{\rho\nu} + \overset{(\tilde{C})}{\nabla}_\rho\overset{(0)}{S}{}^{\rho}_{\mu\nu} + 2\tilde{C}^{\lambda}_{[\rho\mu]}\overset{(0)}{S}{}^{\rho}_{\lambda\nu} - \overset{(2)}{C}{}^{\rho}_{\mu\lambda}\overset{(-2)}{C}{}^{\lambda}_{\rho\nu} - \overset{(-2)}{C}{}^{\rho}_{\mu\lambda}\overset{(2)}{C}{}^{\lambda}_{\rho\nu}, \tag{4.32c}$$

$$\overset{(2)}{R}_{\mu\nu} = \overset{(\tilde{C})}{\nabla}_\rho\overset{(2)}{C}{}^{\rho}_{\mu\nu}, \tag{4.32d}$$

$$\overset{(4)}{R}_{\mu\nu} = -T_\mu T_\nu\Pi^{\rho\sigma}\Pi^{\lambda\gamma}\partial_{[\lambda}T_{\sigma]}\partial_{[\rho}T_{\gamma]}, \tag{4.32e}$$

where $\overset{(\tilde{C})}{\nabla}_\mu$ and $\overset{(\tilde{C})}{R}_{\mu\nu}$ are the covariant derivative and the Ricci tensor associated with the $\tilde{C}$-connection, respectively. The relevant projections of the Ricci tensor for the action expansion are

$$\begin{aligned}
T^\mu T^\nu \overset{(-2)}{R}_{\mu\nu} &= 0, && (4.33a)\\
\Pi^{\mu\nu} \overset{(-2)}{R}_{\mu\nu} &= -T^\rho \partial_\rho \overset{(0)}{K} + \overset{(0)}{K}{}^2, && (4.33b)\\
T^\mu T^\nu \overset{(0)}{R}_{\mu\nu} &= T^\mu T^\nu \overset{(\tilde{C})}{R}_{\mu\nu} + T^\mu \partial_\mu \overset{(0)}{K} - \overset{(0)}{K}{}^{\mu\nu} \overset{(0)}{K}_{\mu\nu}, && (4.33c)\\
\Pi^{\mu\nu} \overset{(0)}{R}_{\mu\nu} &= \Pi^{\mu\nu} \overset{(\tilde{C})}{R}_{\mu\nu}, && (4.33d)\\
T^\mu T^\nu \overset{(2)}{R}_{\mu\nu} &= 2\tilde{\nabla}_\rho (T^\mu \Pi^{\rho\sigma} \partial_{[\mu} T_{\sigma]}), && (4.33e)\\
\Pi^{\mu\nu} \overset{(2)}{R}_{\mu\nu} &= 2\Pi^{\mu\nu}\Pi^{\rho\sigma} \partial_{[\mu} T_{\rho]} \partial_{[\nu} T_{\sigma]}, && (4.33f)\\
T^\mu T^\nu \overset{(4)}{R}_{\mu\nu} &= \Pi^{\mu\nu}\Pi^{\rho\sigma} \partial_{[\mu} T_{\rho]} \partial_{[\nu} T_{\sigma]}, && (4.33g)\\
\Pi^{\mu\nu} \overset{(4)}{R}_{\mu\nu} &= 0, && (4.33h)
\end{aligned}$$

where we defined $\overset{(0)}{K}_{\mu\nu} = -\frac{1}{2}\mathcal{L}_T \Pi_{\mu\nu}$ such that the LO term in the expansion is the Carroll extrinsic curvature (2.52) i.e. $\overset{(0)}{K}_{\mu\nu}|_{\sigma=0} = K_{\mu\nu}$. Furthermore, we note that the term $T^\mu T^\nu \overset{(\tilde{C})}{R}_{\mu\nu}$ in (4.33c) vanishes due to the Carroll compatibility of the $\tilde{C}$-connection, as shown in (3.36).

4.2.4 Ricci scalar

The Ricci scalar is given by the trace of the Ricci tensor with the full inverse metric $g^{\mu\nu}$

$$R \equiv g^{\mu\nu} R_{\mu\nu} = \left(-\frac{1}{c^2} T^\mu T^\nu + \Pi^{\mu\nu}\right) R_{\mu\nu}. \tag{4.34}$$

Using the projections of the Ricci tensor (4.33a)-(4.33h), we find that the Ricci scalar can be written as

$$\begin{aligned}
R =& \frac{1}{c^2}\left[\overset{(0)}{K}{}^{\mu\nu}\overset{(0)}{K}_{\mu\nu} - \overset{(0)}{K}{}^2 - 2E^{-1}\partial_\rho(E T^\rho \overset{(0)}{K})\right] \\
&+ \left[\Pi^{\mu\nu}\overset{(\tilde{C})}{R}_{\mu\nu} - \overset{(\tilde{C})}{\nabla}_\rho (T^\mu \Pi^{\rho\sigma} T_{\mu\sigma})\right] + \frac{c^2}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma} T_{\mu\rho} T_{\nu\sigma},
\end{aligned} \tag{4.35}$$

where we defined $T_{\mu\nu} = 2\partial_{[\mu}T_{\nu]}$. As we are not concerned in this thesis with boundary terms of the action (4.1), we can simplify (4.35) by discarding total derivatives using (3.28). In particular, if we denote equality up to a total derivative by $\approx$, then we find

$$ER \approx \frac{E}{c^2}[\overset{(0)}{K}{}^{\mu\nu}\overset{(0)}{K}_{\mu\nu} - \overset{(0)}{K}{}^2] + E\Pi^{\mu\nu}\overset{(\tilde{C})}{R}_{\mu\nu} + \frac{Ec^2}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma}. \tag{4.36}$$

Note that the form (4.36) highlights that the LO EH action in both the Carrollian and Galilean limits, i.e. the leading term in c^2 and c^{-2}, can be written manifestly covariantly without the added structure of a connection.

4.2.5 Einstein-Hilbert action

We are now in a position to write the full EH action (4.1) using the PUR variables with all factors of c explicitly accounted for. Specifically, using the decomposition (4.36) of the Ricci scalar and remembering that the covariant measure (4.21) carries an extra factor of c, we find that the Lagrangian density of the EH action can be written as

$$\begin{aligned}\mathcal{L}_{\mathrm{EH}} &= \frac{c^4}{16\pi G_N} E R \\ &= \frac{c^2}{16\pi G_N} E[\overset{(0)}{K}{}^{\mu\nu} \overset{(0)}{K}_{\mu\nu} - \overset{(0)}{K}{}^2] + \frac{c^4}{16\pi G_N} E\Pi^{\mu\nu} \overset{(\tilde{C})}{R}_{\mu\nu} + \frac{c^6}{64\pi G_N} E\Pi^{\mu\nu}\Pi^{\rho\sigma} T_{\mu\rho} T_{\nu\sigma}.\end{aligned} \tag{4.37}$$

From (4.37) it is apparent that the LO Carroll action starts at order c^2. This result is also consistent with the LO Galilean expansion found in [29] starting at order c^6.

4.3 Expanding the Einstein-Hilbert and matter action

Having determined the Lagrangian in terms of PUR variables, we can now turn to the actual expansion of the theory, as described in appendix B. If we factor out the pre-leading factors of c, we obtain the following expansion

$$\mathcal{L}_{\mathrm{EH}} = c^2(\overset{(2)}{\mathcal{L}}_{\mathrm{LO}} + \sigma \overset{(4)}{\mathcal{L}}_{\mathrm{NLO}} + \mathcal{O}(\sigma^2)). \tag{4.38}$$

The next step is then to compute the corresponding EOM at each level of the expansion. We can define the responses to the variations of v^μ and $h^{\mu\nu}$ as

$$\overset{(2n+2)}{G^{v}_{\mu}} \equiv 8\pi G_N e^{-1} \frac{\delta \overset{(2n+2)}{\mathcal{L}}_{\mathrm{N}^n\mathrm{LO}}}{\delta v^\mu}, \tag{4.39a}$$

$$\overset{(2n+2)}{G^{h}_{\mu\nu}} \equiv 16\pi G_N e^{-1} \frac{\delta \overset{(2n+2)}{\mathcal{L}}_{\mathrm{N}^n\mathrm{LO}}}{\delta h^{\mu\nu}}, \tag{4.39b}$$

where $\frac{\delta}{\delta v^\mu}$ is the Euler-Lagrange derivative with respect to v^μ and similarly for $h^{\mu\nu}$. We could of course analogously define the EOM for the sub-leading fields M^μ and $\Phi^{\mu\nu}$, but this thesis will only deal with the EH action to NLO and hence, as explained in appendix B, they will only reproduce already known LO equations.

We can also extend the theory by including matter, which we implement by adding a corresponding term to the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{mat}}. \tag{4.40}$$

We assume that the matter Lagrangian has M pre-leading orders of c

$$\mathcal{L}_{\mathrm{mat}} = c^M(\overset{(M)}{\mathcal{L}}_{\mathrm{mat,LO}} + \sigma \overset{(M+2)}{\mathcal{L}}_{\mathrm{mat,NLO}} + \mathcal{O}(\sigma^2)). \tag{4.41}$$

With this, we can define the metric responses order by order in analogy to the Carrollian momenta (2.79a) and (2.79b)

$$\overset{(2n+M)}{T^{v}_{\mu}} \equiv -e^{-1} \frac{\delta \overset{(2n+M)}{\mathcal{L}}_{\mathrm{mat,N}^n\mathrm{LO}}}{\delta v^\mu}, \tag{4.42a}$$

$$\overset{(2n+M)}{T^{h}_{\mu\nu}} \equiv -2e^{-1} \frac{\delta \overset{(2n+M)}{\mathcal{L}}_{\mathrm{mat,N}^n\mathrm{LO}}}{\delta h^{\mu\nu}}. \tag{4.42b}$$

The UR analog of the Einstein equation at each order of the expansion then becomes

$$\overset{(2n)}{G^{v}_{\mu}} = 8\pi G_N \overset{(2n)}{T^{v}_{\mu}}, \tag{4.43a}$$

$$\overset{(2n)}{G^{h}_{\mu\nu}} = 8\pi G_N \overset{(2n)}{T^{h}_{\mu\nu}}. \tag{4.43b}$$

Again, there are similar matter momenta and Einstein equations for the sub-leading fields M^μ and $\Phi^{\mu\nu}$, but as they play no role in this thesis they are omitted.

4.4 LO theory

The LO Lagrangian is simply obtained by setting $\sigma = 0$ in (4.37) and keeping in mind the pre-leading factor of c^2

$$\overset{(2)}{\mathcal{L}}_{\mathrm{LO}} = \frac{E}{16\pi G_N}[\overset{(0)}{K}{}^{\mu\nu}\overset{(0)}{K}_{\mu\nu} - \overset{(0)}{K}{}^2]\Big|_{\sigma=0} = \frac{e}{16\pi G_N}[K^{\mu\nu}K_{\mu\nu} - K^2]. \tag{4.44}$$

Note that the Lagrangian (4.44) is invariant under boosts, as it consists only of Carroll invariants. This was of course to be expected, as the EH action (4.1) is invariant under local Lorentz transformations, and the expansion then inherits this invariance order by order. Another note is that the action (4.44) has appeared in [33] as the strict $c \to 0$ limit of the EH action, and it is contained in the effective action derived in [18] by an effective field theory approach.

Using the definitions (4.39a) and (4.39b), we can write the variation of the LO Lagrangian in terms of the metric responses as

$$\delta\overset{(2)}{\mathcal{L}}_{\mathrm{LO}} \approx \frac{e}{8\pi G_N}\left[\overset{(2)}{G}{}^{v}_{\mu}\delta v^{\mu} + \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu}\delta h^{\mu\nu}\right], \tag{4.45}$$

which again only holds up to total derivatives.

4.4.1 Variational calculus

To derive the LO EOM $\overset{(2)}{G}{}^{v}_{\mu}$ and $\overset{(2)}{G}{}^{h}_{\mu\nu}$ we need a number of variational identities. As an example, we will explicitly compute δK

$$\delta K = \delta h^{\mu\nu}K_{\mu\nu} - \frac{1}{2}h^{\mu\nu}(\mathcal{L}_{\delta v}h_{\mu\nu} + \mathcal{L}_{v}\delta h_{\mu\nu}). \tag{4.46}$$

The δv^{μ} in the first Lie derivative of (4.46) cannot be factored out in a covariant manner without a connection and thus shows the necessity of introducing the $\tilde{\Gamma}$-connection even at LO. Hence, we want to swap the Lie derivative for the $\tilde{\Gamma}$-covariant derivative using the following general identity for any connection

$$\begin{aligned}\mathcal{L}_{\xi}X^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell} = {} & \xi^{\lambda}\nabla_{\lambda}X^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell} \\ & - \nabla_{\lambda}\xi^{\mu_1}X^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell} - \ldots - \xi^{\sigma}T^{\mu_1}{}_{\sigma\lambda}X^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell} - \ldots \\ & + \nabla_{\nu_1}\xi^{\lambda}X^{\mu_1\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_\ell} + \ldots + \xi^{\sigma}T^{\lambda}{}_{\sigma\nu_1}X^{\mu_1\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_\ell} + \ldots .\end{aligned} \tag{4.47}$$

Applying this identity to (4.46), we obtain

$$\begin{aligned}\delta K = {} & \delta h^{\mu\nu}K_{\mu\nu} - h^{\mu\nu}(h_{\rho(\mu}\tilde{\nabla}_{\nu)}\delta v^{\rho} - K_{\rho(\mu}\tau_{\nu)}\delta v^{\rho} + K_{\mu\nu}\tau_{\rho}\delta v^{\rho}) \\ & - \frac{1}{2}(v^{\rho}\tilde{\nabla}_{\rho}\delta h_{\mu\nu} - 2K_{(\mu}{}^{\rho}\delta h_{\nu)\rho}).\end{aligned} \tag{4.48}$$

We want the response to δv^{μ} and $\delta h^{\mu\nu}$, and thus to get rid of the $\delta h_{\mu\nu}$ we invoke the relation

$$\delta h_{\mu\nu} = -h_{\mu\sigma}h_{\nu\rho}\delta h^{\sigma\rho} + 2h_{\rho(\mu}\tau_{\nu)}\delta v^{\rho}, \tag{4.49}$$

which follows from the completeness relations (2.47). Inserting the variation (4.49) into (4.48) and using that all temporal components are projected out, we find

$$\begin{aligned}\delta K &= \delta h^{\mu\nu}K_{\mu\nu} - h^{\nu}_{\rho}\tilde{\nabla}_{\nu}\delta v^{\rho} - K\tau_{\rho}\delta v^{\rho} - K_{\mu\nu}\delta h^{\mu\nu} - v^{\rho}\tau_{\rho\sigma}\delta v^{\rho} + \frac{1}{2}v^{\rho}h_{\mu\nu}\tilde{\nabla}_{\rho}\delta h^{\mu\nu} \\ &= -(v^{\nu}\tau_{\nu\mu} + K\tau_{\mu})\delta v^{\mu} - h^{\nu}_{\mu}\tilde{\nabla}_{\nu}\delta v^{\mu} + \frac{1}{2}v^{\rho}h_{\mu\nu}\tilde{\nabla}_{\rho}h^{\mu\nu}.\end{aligned} \tag{4.50}$$

To determine the EOM (4.45), we further need the variations

$$\delta e = e(\tau_\mu \delta v^\mu - \tfrac{1}{2} h_{\mu\nu} \delta h^{\mu\nu}), \tag{4.51a}$$

$$\delta(K^{\mu\nu} K_{\mu\nu}) = -2(v^\nu \tau_{\nu\rho} K^\rho{}_\mu + K^{\rho\sigma} K_{\rho\sigma} \tau_\mu)\delta v^\mu - 2K^\nu{}_\mu \tilde{\nabla}_\nu \delta v^\mu + K_{\mu\nu} v^\rho \tilde{\nabla} \delta h^{\mu\nu}, \tag{4.51b}$$

where (4.51a) follows from the variation of a determinant $\delta \det A = \det A \operatorname{tr} A^{-1} \delta A$, and (4.51b) is derived analogously to (4.50).

4.4.2 LO equations of motion

Putting together the variations (4.50), (4.51a), (4.51b) and using the integration-by-parts identity (3.28), we arrive at the EOM of the LO theory

$$\overset{(2)}{G}{}^v_\mu = -\frac{1}{2}\tau_\mu (K^{\rho\sigma} K_{\rho\sigma} - K^2) + h^{\gamma\lambda} \tilde{\nabla}_\lambda (K_{\mu\gamma} - K h_{\mu\gamma}), \tag{4.52a}$$

$$\overset{(2)}{G}{}^h_{\mu\nu} = -\frac{1}{2} h_{\mu\nu} (K^{\rho\sigma} K_{\rho\sigma} - K^2) + K(K_{\mu\nu} - K h_{\mu\nu}) - v^\rho \tilde{\nabla}_\rho (K_{\mu\nu} - K h_{\mu\nu}). \tag{4.52b}$$

These EOM are of a different nature as compared to the ones obtained at LO in the Galilean expansion, which turns out to be a geometrical constraint ($\mathcal{G}_1$ (TTNC) in the classification of section 3.1.2) guaranteeing the existence of a foliation of spacetime into time slices [29]. One clear indication that the EOM (4.52a)-(4.52b) are not only constraints, is the "time"-derivative $v^\rho \tilde{\nabla}_\rho$ in (4.52b). This will in turn lead to the LO theory being dynamical, which we will explore further in chapter 6. However, we still expect the equations to be very different from those of GR as the light-cone is completely collapsed at LO. Hence, spatially separated points should not be in causal contact and evolve independently. We will confirm this interpretation in section 6.2.

4.4.3 LO Cosmological constant

To include a cosmological constant Λ in Einstein gravity, we simply add the following term to the EH action (4.1)

$$S_\Lambda = \frac{c^4}{16\pi G_N} \int d^{d+1}x \, E(-2\Lambda). \tag{4.53}$$

As the analysis of section 4.2.5 shows that the EH action starts at order c^2 (4.37), we see that a cosmological constant with no scaling in c will not contribute at LO. Thus, we need to choose the scaling $\Lambda = c^{-2}\tilde{\Lambda}$ for the action (4.53) to enter at LO. The change in LO Lagrangian is then

$$\tilde{\mathcal{L}}_\Lambda(\sigma = 0) = -\frac{e\tilde{\Lambda}}{8\pi G_N}, \tag{4.54}$$

where we in $\tilde{\mathcal{L}}_\Lambda$ have stripped off the pre-leading factor of c^2. The correction to the EOM (4.52a) and (4.52b) are easily determined using (4.51a)

$$\overset{(2)}{G}{}^{v\Lambda}_\mu = -\frac{1}{2}\tau_\mu (K^{\rho\sigma} K_{\rho\sigma} - K^2 + 2\tilde{\Lambda}) + h^{\gamma\lambda} \tilde{\nabla}_\lambda (K_{\mu\gamma} - K h_{\mu\gamma}), \tag{4.55a}$$

$$\overset{(2)}{G}{}^{h\Lambda}_{\mu\nu} = -\frac{1}{2} h_{\mu\nu} (K^{\rho\sigma} K_{\rho\sigma} - K^2 - 2\tilde{\Lambda}) + K(K_{\mu\nu} - K h_{\mu\nu}) - v^\rho \tilde{\nabla}_\rho (K_{\mu\nu} - K h_{\mu\nu}). \tag{4.55b}$$

4.5 NLO theory

We can then go on to consider the EH action at NLO. According to (B.6b) of appendix B, we only need the following partial derivative to compute the NLO action

$$\left.\frac{\partial\tilde{\mathcal{L}}}{\partial\sigma}\right|_{\sigma=0} = \frac{e}{16\pi G_N} h^{\mu\nu}\tilde{R}_{\mu\nu}. \tag{4.56}$$

We can then write down the NLO action (B.6b), whose structure is defined by the nature of the expansion

$$\overset{(4)}{\mathcal{L}}_{\text{NLO}} = \frac{e}{8\pi G_N}\left[\frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu} + \overset{(2)}{G}{}^{v}_{\mu} M^{\mu} + \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu}\Phi^{\mu\nu}\right]. \tag{4.57}$$

The NLO action (4.57) is a novel result of this thesis. At NLO we introduce the sub-leading fields M^{μ} and $\Phi^{\mu\nu}$. From the form (4.57), one readily sees that the EOM of the NLO fields work out to be the LO EOM. As described in appendix B, this is a general feature of this kind of expansion. In particular, all previous EOM are repeated at each level of the expansion.

From here it is in principle straightforward to compute the NLO EOM by varying the action (4.57). However, as this is quite tedious, we here only present the EOM for an expansion that truncates at LO i.e. $M^{\mu} = 0$, $\Phi^{\mu\nu} = 0$, such that the two last terms of (4.57) drop out.

4.5.1 Variational calculus

Varying the NLO action (4.57) poses some technical difficulties, as we now need the variations of objects that depend on the $\tilde{\Gamma}$-connection. In particular, to obtain the full NLO EOM (including sub-leading fields) we would need to vary both $h^{\mu\nu}\tilde{R}_{\mu\nu}$ and the covariant derivative in the LO EOM $\overset{(2)}{G}{}^{v}_{\mu}$ and $\overset{(2)}{G}{}^{h}_{\mu\nu}$.

We will now go through the calculation of the variation of the Ricci-like scalar $h^{\mu\nu}\tilde{R}_{\mu\nu}$. In general, for any choice of connection, one can show directly from the definition (3.32) that the variation of the Ricci tensor can be written as

$$\delta\tilde{R}_{\mu\nu} = \tilde{\nabla}_{\rho}\delta\tilde{\Gamma}^{\rho}_{\mu\nu} - \tilde{\nabla}_{\mu}\delta\tilde{\Gamma}^{\rho}_{\rho\nu} - 2\tilde{\Gamma}^{\lambda}_{[\mu\rho]}\delta\Gamma^{\rho}_{\lambda\nu}. \tag{4.58}$$

We then contract with $eh^{\mu\nu}$ and integrate by parts

$$\begin{aligned} eh^{\mu\nu}\delta\tilde{R}_{\mu\nu} &= e(h^{\mu\nu}\tilde{\nabla}_{\rho}\delta\tilde{\Gamma}^{\rho}_{\mu\nu} - h^{\mu\nu}\tilde{\nabla}_{\mu}\delta\tilde{\Gamma}^{\rho}_{\rho\nu} - 2h^{\mu\nu}\tilde{\Gamma}^{\lambda}_{[\mu\rho]}\delta\tilde{\Gamma}^{\rho}_{\lambda\nu}) \\ &\approx e(-\delta\tilde{\Gamma}^{\rho}_{\mu\nu}\tilde{\nabla}_{\rho}h^{\mu\nu} + \delta\tilde{\Gamma}^{\rho}_{\rho\nu}\tilde{\nabla}_{\mu}h^{\mu\nu} - 2h^{\mu\nu}\tilde{\Gamma}^{\lambda}_{[\mu\rho]}\delta\tilde{\Gamma}^{\rho}_{\lambda\nu} - \tau_{\rho}Kh^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\mu\nu}) \\ &= e(-\delta\tilde{\Gamma}^{\rho}_{(\mu\nu)}v^{\mu}h^{\nu\sigma}(\delta^{\gamma}_{\rho} - \tau_{\rho}v^{\gamma})\tau_{\gamma\sigma} + \delta\tilde{\Gamma}^{\rho}_{\rho\nu}h^{\nu\sigma}v^{\gamma}\tau_{\gamma\sigma} + K^{\nu\lambda}\tau_{\rho}\delta\tilde{\Gamma}^{\rho}_{\lambda\nu} - \tau_{\rho}Kh^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\mu\nu}), \end{aligned} \tag{4.59}$$

where we again used the identity (3.28) together with Carroll compatibility of $\tilde{\nabla}$. By rewriting the Ricci-like scalar as (4.59), we have reduced the problem of finding its variation to determining projections of $\delta\tilde{\Gamma}^{\rho}_{\mu\nu}$. The needed projections can be obtained either by direct computation or in a more covariant manner by considering covariant derivatives of the metric data. As an example of this, let's consider how $h^{\mu\nu}\tau_{\rho}\delta\tilde{\Gamma}^{\rho}_{\mu\nu}$ can be determined from the relations (3.25). Upon tracing (3.25) with $h^{\mu\nu}$, the derivative $\tilde{\nabla}_{\mu}\tau_{\nu}$ becomes

$$h^{\mu\nu}\tilde{\nabla}_{\mu}\tau_{\nu} = 0. \tag{4.60}$$

We can then vary both sides of (4.60) to obtain

$$\delta h^{\mu\nu}\tilde{\nabla}_{\mu}\tau_{\nu} + h^{\mu\nu}(\tilde{\nabla}_{\mu}\delta\tau_{\nu} - \delta\tilde{\Gamma}^{\rho}_{\mu\nu}\tau_{\rho}) = 0, \tag{4.61}$$

which we can solve for the projection $h^{\mu\nu}\tau_\rho\delta\tilde{\Gamma}^\rho_{\mu\nu}$ yielding

$$h^{\mu\nu}\tau_\rho\delta\tilde{\Gamma}^\rho_{\mu\nu} = -\frac{1}{2}\delta h^{\mu\nu}\tau_{\rho\mu}(v^\rho\tau_\nu + h^\rho_\nu) - h^\mu_\sigma\tau_\lambda\tilde{\nabla}_\mu\delta h^{\sigma\lambda}. \tag{4.62}$$

By similar considerations of covariant derivatives, one can compute the remaining projections appearing in (4.59) and combine them to find the needed variation

$$\begin{aligned} eh^{\mu\nu}\delta\tilde{R}_{\mu\nu} \approx \delta v^\mu\frac{1}{2}h^{\nu\sigma}h^\lambda_\mu\tau_{\nu\lambda}v^\kappa\tau_{\kappa\sigma} + \delta h^{\mu\nu}[2h^\sigma_\mu\tau_\nu K v^\kappa\tau_{\kappa\sigma} + K_\mu{}^\sigma\tau_{\sigma\rho}(v^\rho\tau_\nu - \delta^\rho_\nu)] \\ + \tilde{\nabla}_\nu\delta v^\lambda h^{\nu\sigma}\tau_{\lambda\sigma} + \tilde{\nabla}_\mu\delta h^{\sigma\lambda}[Kh^\mu_\sigma\tau_\lambda - K^\mu{}_\sigma\tau_\lambda - \frac{1}{2}h^{\mu\gamma}h_{\sigma\lambda}v^\kappa\tau_{\kappa\gamma}]. \end{aligned} \tag{4.63}$$

The variation (4.63) together with those of section 4.4.1 is sufficient to compute the EOM for $M^\mu = 0$ and $\Phi^{\mu\nu} = 0$.

4.5.2 NLO equations of motion

With the results of the previous section, we can compute the NLO EOM under the aforementioned restrictions by combining (4.63) and (4.51a) and integrating by parts to find

$$\overset{(4)}{G}{}^v_\mu\Big|_{M^\mu=\Phi^{\mu\nu}=0} = \tau_\mu h^{\rho\sigma}\tilde{R}_{\rho\sigma} + \frac{1}{2}h^{\nu\sigma}h^\lambda_\mu\tau_{\nu\lambda}v^\kappa\tau_{\kappa\sigma} - \tilde{\nabla}_\nu(h^{\nu\sigma}\tau_{\mu\sigma}), \tag{4.64a}$$

$$\begin{aligned} \overset{(4)}{G}{}^h_{\mu\nu}\Big|_{M^\mu=\Phi^{\mu\nu}=0} = \tilde{R}_{(\mu\nu)} - \frac{1}{2}h_{\mu\nu}h^{\sigma\rho}\tilde{R}_{\sigma\rho} + 2Kv^\kappa\tau_{\kappa(\mu}\tau_{\nu)} + \tau_{\sigma\rho}K_{(\mu}{}^\sigma(v^\rho\tau_{\nu)} - \delta^\rho_{\nu)}) \\ - \tilde{\nabla}_\lambda(Kh^\lambda_{(\mu}\tau_{\nu)} - K^\lambda{}_{(\mu}\tau_{\nu)} - \frac{1}{2}h_{\mu\nu}h^{\lambda\gamma}v^\kappa\tau_{\kappa\gamma}). \end{aligned} \tag{4.64b}$$

The remaining part of the NLO EOM, stemming from the variation of the LO EOM, i.e. the last two terms of (4.57), can also be computed. However, as this is technically challenging and will not be used in this thesis, they are omitted.

Chapter 5

3+1 Decomposition of GR

In this chapter, we take a detour into the fully relativistic realm and review some of the basic concepts of the 3+1 or ADM decomposition[1] of GR. Conventionally, there are two main reasons for breaking the general covariance and singling out space and time to obtain a more conventional initial-value problem: Firstly, to carry out a Hamiltonian analysis [41], where the framework depends explicitly on a choice of time, which in relativity is inherently non-unique. Secondly, if one wants to solve the equations of GR numerically [48], the natural approach is to prepare some initial data and evolve that forward in time. Another advantage of the 3+1 decomposition is that the gauge freedom stemming from diffeomorphism invariance becomes more transparent.

We have already seen that in Carroll geometry this split of space and time is naturally realized in terms of the invariant objects $h_{\mu\nu}$ and v^μ. As we will see in chapter 6, the EOM of the LO theory (4.52a)-(4.52b) have many similar features to the equations of the 3+1 decomposition. This is of course in some sense no surprise, as the Carroll equations are an expansion of GR. Hence, we can apply adapted methods from the 3+1 decomposition to solve problems in the LO theory. In particular, the alternative perspective of first constructing initial data and then worrying about the time evolution will turn out to be particularly well-suited for the LO Carroll theory.

In section 5.1, we will first review and then implement the necessary differential geometry on hypersurfaces to realize the 3+1 split of Einstein's equation. The resulting equations are in part constraints, and hence section 5.2 deals with methods for constructing consistent initial data. Finally, in section 5.3, we show that the EOM derived from the action expansion of chapter 4 can be obtained directly from the 3+1 decomposition.

5.1 Foliations and hypersurface geometry

The review of the 3+1 decomposition in this and the next section is based on the references [48, 49]. The ability to write GR as an initial-value problem of course depends on the existence of some time function to keep track of the evolution of our dynamical fields i.e. the metric $g_{\mu\nu}$. More formally, this means that we assume the existence of a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ such that the union of all slices or leaves Σ_t is the entire spacetime manifold M

$$M = \bigcup_{t\in\mathbb{R}} \Sigma_t. \tag{5.1}$$

[1] The name "3+1" is of course due to the most relevant case of spatial dimension $d = 3$. However, the methods described in section 5.1 generalize readily to any spatial dimension d. The initials ADM are due to Arnowitt, Deser and Misner who pioneered the canonical formalism of GR [47].

Along with each slice, we have a hypersurface tangent space defined by ker dt. Each hypersurface can be understood as a time slice, and the initial-value problem consists of propagating fields living on Σ_t to $\Sigma_{t+\delta t}$.

5.1.1 Lapse function and the spatial metric

For each of the time slices Σ_t, we can compute a future-directed timelike normal vector n^μ as

$$n^\mu \equiv -\alpha g^{\mu\nu}\partial_\nu t, \qquad n_\mu \equiv g_{\mu\nu}n^\nu, \tag{5.2}$$

where we defined the lapse function α according to

$$\alpha \equiv (-g^{\mu\nu}\partial_\mu t\partial_\nu t)^{-1/2}. \tag{5.3}$$

The lapse function has the interpretation of the rate of proper time elapsed per coordinate time t. The sign of (5.2) is chosen such that $n^\mu\partial_\mu t = \alpha^{-1} > 0$ i.e. n^μ is indeed future-directed. Also note that $n^\mu n_\mu = -1$, which corresponds to the fact that n^μ is a timelike normal. The (co-)normal allow us to define a projector onto the tangent space of the hypersurface Σ_t given by

$$\gamma^\mu_\nu \equiv n^\mu n_\nu + \delta^\mu_\nu. \tag{5.4}$$

The tensor (5.4) satisfies the needed conditions to be a projection

$$\gamma^\mu_\rho\gamma^\rho_\nu = \gamma^\mu_\nu, \qquad \gamma^\mu_\nu n^\nu = 0, \qquad \gamma^\nu_\mu n_\nu = 0. \tag{5.5}$$

We call indices annihilated by γ^μ_ν temporal and indices annihilated by $-n^\mu n_\nu$ spatial. The next natural task is to compute the projection of the metric

$$\gamma_{\mu\nu} \equiv \gamma^\rho_\mu\gamma^\sigma_\nu g_{\rho\sigma} = n_\mu n_\nu + g_{\mu\nu}, \tag{5.6a}$$
$$\gamma^{\mu\nu} \equiv \gamma^\mu_\rho\gamma^\nu_\sigma g^{\rho\sigma} = n^\mu n^\nu + g^{\mu\nu}, \tag{5.6b}$$

which is called the spatial metric or sometimes the induced metric, as it coincides with what would have been obtained by pulling back $g_{\mu\nu}$ to Σ_t. The spatial metric γ can be used to raise and lower indices of spatial tensors e.g

$$X^\mu \equiv \gamma^{\mu\nu}X_\nu, \tag{5.7}$$

for X_μ spatial.

Associated with the metric $g_{\mu\nu}$, we have a unique torsionless Levi-Civita connection ∇. We also have a metric $\gamma_{\mu\nu}$ living on the hypersurface Σ_t, and hence we can wonder if we can associate a torsion-free, γ-compatible connection to that as well. It is indeed possible to define a Levi-Civita connection $\hat{\nabla}$ associated with γ that acts on spatial tensors. In particular, for a general spatial tensor $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_\ell}$ we define

$$\hat{\nabla}_\rho T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_\ell} \equiv \gamma^\gamma_\rho\gamma^{\mu_1}_{\alpha_1}\ldots\gamma^{\mu_k}_{\alpha_k}\gamma^{\beta_1}_{\nu_1}\ldots\gamma^{\beta_\ell}_{\nu_\ell}\nabla_\gamma T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}. \tag{5.8}$$

A short computation shows that this connection is both torsion-free and γ-compatible i.e.

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]f = 0, \qquad \hat{\nabla}_\rho\gamma_{\mu\nu} = 0, \tag{5.9}$$

and $\hat{\nabla}$ can thus rightfully be called the Levi-Civita connection associated with $\gamma_{\mu\nu}$. We can further define the 3-Riemann tensor associated with $\hat{\nabla}$ using the usual implicit definition

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]X^\sigma = -\hat{R}_{\mu\nu\rho}{}^\sigma X^\rho, \qquad \hat{R}_{\mu\nu\rho}{}^\sigma n_\sigma = 0, \tag{5.10}$$

where X^μ is any spatial vector. The relation between the 3-curvature and the full Riemann tensor is the subject of section 5.1.3.

5.1.2 Extrinsic curvature and acceleration

Another central object in the 3+1 decomposition is the extrinsic curvature, which we define as

$$K_{\mu\nu} \equiv -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}. \tag{5.11}$$

Note that we use the same symbol K for the extrinsic curvature in the 3+1 decomposition as in Carroll geometry (2.52), hence unless stated otherwise $K_{\mu\nu}$ in this chapter means (5.11). There are many equivalent forms of the $K_{\mu\nu}$ of which a notable one is

$$K_{\mu\nu} = -\gamma^\rho_\mu \gamma^\sigma_\nu \nabla_\rho n_\sigma. \tag{5.12}$$

The expression (5.12) measures the change in the co-normal along the hypersurface. Thus, it explains how $K_{\mu\nu}$ is connected to curvature stemming from the embedding. On the other hand, the form (5.11) highlights that $K_{\mu\nu}$ can be thought of as a velocity of $\gamma_{\mu\nu}$. It will indeed turn out that we need to specify exactly $(\gamma_{\mu\nu}, K_{\mu\nu})$ as initial data to propagate from one time slice to the next. This is analogous to how, for a second order differential equation, one would specify $(x, \dot{x})$ at an initial time. The expression for the extrinsic curvature (5.11) also readily shows that $K_{\mu\nu}$ is a spatial, symmetric tensor.

If we consider the vector field n^μ as the 4-velocity of some test particle, then we can compute the acceleration

$$a_\mu \equiv n^\nu \nabla_\nu n_\mu, \tag{5.13}$$

that measures how far n^μ is from pointing along a geodesic. Equivalently, we can write the acceleration as a Lie derivative

$$a_\mu = n^\nu \nabla_\nu n_\mu + \underbrace{n_\nu \nabla_\mu n^\nu}_{=0} = \mathcal{L}_n n_\mu = n^\nu (dn)_{\nu\mu}, \tag{5.14}$$

which again makes it apparent that a_μ is a spatial co-vector. The acceleration is closely related to the lapse function and in particular one can show from (5.2) and (5.14) that

$$a_\mu = \frac{1}{\alpha}\hat{\nabla}_\mu \alpha. \tag{5.15}$$

It will also be convenient to consider the second covariant derivative of the lapse α, which works out to be

$$\frac{1}{\alpha}\hat{\nabla}_\mu \hat{\nabla}_\nu \alpha = \hat{\nabla}_\mu a_\nu + a_\mu a_\nu. \tag{5.16}$$

5.1.3 Gauss, Codazzi and Ricci equations

To make the connection between $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ living on hypersurface Σ_t and the Einstein equation, which relate the 4-Einstein tensor and some matter source, we need to find a relation between the 3-curvature and the 4-curvature. This question is exactly answered by the equations of Gauss, Codazzi and Ricci, which relates the 3-curvature and extrinsic curvature to different projections of the 4-Riemann tensor.

To derive the Gauss equation, one considers the commutator of $\hat{\nabla}$ with itself acting on a spatial vector by on the one hand using the definition (5.10) and on the other hand swapping the $\hat{\nabla}$-derivative for the ∇-derivative. This allows one to relate the Riemann tensors associated with each of the derivatives in the so-called Gauss relation

$$\hat{R}_{\mu\nu\rho}{}^\sigma = \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\rho \gamma^\sigma_\delta R_{\alpha\beta\gamma}{}^\delta - K_{\mu\rho} K_\nu{}^\sigma + K_{\nu\rho} K_\mu{}^\sigma, \tag{5.17}$$

The equation (5.17) shows that the induced intrinsic curvature on the hypersurface is partly due to the embedding and partly the intrinsic curvature of the embedding space. From (5.17) we contract twice using $\gamma^{\mu\nu}$, yielding a relation for the Ricci scalar

$$\hat{R} = 2n^\mu n^\nu G_{\mu\nu} - K^2 + K_{\mu\nu}K^{\mu\nu}, \tag{5.18}$$

where we used $\gamma^{\mu\rho}\gamma^\nu_\sigma R_{\mu\nu\rho}{}^\sigma = 2n^\mu n^\nu G_{\mu\nu}$ with $G_{\mu\nu}$ being the Einstein tensor defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{5.19}$$

The Codazzi equation is found by computing the $\hat{\nabla}$-derivative of $K_{\mu\nu}$ using (5.12) and subsequently anti-symmetrizing it to relate it to the 4-Riemann tensor. This calculation yields the Codazzi equation

$$\hat{\nabla}_\nu K_{\mu\rho} - \hat{\nabla}_\mu K_{\nu\rho} = \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\rho n^\sigma R_{\alpha\beta\gamma\sigma}. \tag{5.20}$$

However, we are again interested in the trace, which we compute by contracting with $\gamma^{\nu\rho}$

$$\hat{\nabla}_\nu K^\nu{}_\mu - \hat{\nabla}_\mu K = -\gamma^\nu_\mu G_{\nu\rho} n^\rho, \tag{5.21}$$

noting the identity $\gamma^\alpha_\mu \gamma^{\nu\rho} n^\sigma R_{\alpha\nu\rho\sigma} = -\gamma^\rho_\mu n^\nu G_{\rho\nu}$.

The final theorem we need is the so-called Ricci equation, which relates the Lie derivative along n^μ of extrinsic curvature to the 4-curvature. The derivation again proceeds by direct computation, where one uses that Lie derivatives can be written in terms of covariant derivatives (4.47), which through (5.12) gives a double covariant derivative that can be related to the 4-Riemann tensor

$$\mathcal{L}_n K_{\mu\nu} = n^\alpha n^\gamma \gamma^\beta_\mu \gamma^\delta_\nu R_{\alpha\beta\gamma\delta} - a_\mu a_\nu - \hat{\nabla}_\mu a_\nu - K_\mu{}^\rho K_{\rho\nu}. \tag{5.22}$$

The Ricci equation can also be directly related to the Einstein tensor by the identity

$$n^\alpha n^\gamma \gamma^\beta_\mu \gamma^\delta_\nu R_{\alpha\beta\gamma\delta} = \gamma^{\alpha\gamma}\gamma^\beta_\mu \gamma^\delta_\nu R_{\alpha\beta\gamma\delta} - \gamma^\alpha_\mu \gamma^\beta_\nu R_{\alpha\beta}, \tag{5.23}$$

where the first term on the RHS can be substituted with the Gauss equation and the second by $R_{\mu\nu} = G_{\mu\nu} + \frac{1}{d-1}Gg_{\mu\nu}$. Note that one application of (5.16) is to simplify the second term of the RHS of the Ricci equation.

5.1.4 Decomposing the Einstein equation

In the previous sections, we have not yet said anything about physics, we have merely related geometrical quantities. The physical input comes from the Einstein equation[2]

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{5.24}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and $G_{\mu\nu}$ the Einstein tensor. As the Einstein equation (5.24) is symmetric and we have two projectors, we can make three distinct projections of (5.24). To write out each of the three projections of the Einstein equation, we need to define the projections of the energy-momentum tensor, which we will do in the following way

$$\rho = n^\mu n^\nu T_{\mu\nu}, \qquad S_\mu = -\gamma^\rho_\mu n^\nu T_{\rho\nu}, \qquad S_{\mu\nu} = \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta}. \tag{5.25}$$

[2] In this section, we use units with $c = 1$.

With these, it is a simple matter to combine the geometric equations of Gauss (5.18), Codazzi (5.21) and Ricci (5.22) with the physical Einstein equation (5.24). After a some algebra, we finally obtain the Einstein equation in its 3+1 decomposed form

$$\hat{R} + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi G_N \rho \tag{5.26a}$$

$$\hat{\nabla}_\nu K^\nu{}_\mu - \hat{\nabla}_\mu K = 8\pi G_N S_\mu \tag{5.26b}$$

$$\mathcal{L}_n K_{\mu\nu} = \hat{R}_{\mu\nu} - 2K_\mu{}^\rho K_{\rho\nu} + KK_{\mu\nu} - \hat{\nabla}_\mu a_\nu - a_\mu a_\nu - 8\pi G_N(S_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}(S-\rho)), \tag{5.26c}$$

with $S = \gamma^{\mu\nu}S_{\mu\nu}$. The equations are conventionally named the Hamiltonian constraint (5.26a), the momentum constraint (5.26b) and the evolution equation (5.26c). As the names suggest, equations (5.26a) and (5.26b) are different in nature as compared to (5.26c), in that they are constraints. This is to be understood in the sense that the constraint equations consist of spatial tensors, and the constraints can be checked on a single time slice. The evolution equation (5.26c) can then be used to propagate the data to the next time slice. Further, one can show that the constraint equations (5.26a) and (5.26b) are preserved under time evolution by (5.26c) due to the diffeomorphism Ward identity of GR, the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$. Thus, to solve the 3+1 EOM, we are first to find some initial data $(\gamma_{\mu\nu}, K_{\mu\nu})$ satisfying the constraint equations, and then we evolve the extrinsic curvature using (5.26c) and the spatial metric using the definition (5.11).

Gauge choices and the shift vector

For completeness, we also consider gauge choices in the 3+1 decomposition which do not directly relate to Carroll geometry. Integrating forwards in time in principle works as described above, but the Lie derivative with respect to n^μ is not natural for our choice of coordinate t due to the fact $n^\mu\partial_\mu t = \alpha^{-1} \neq 1$. Also, we have not explicitly addressed the question of diffeomorphism invariance. It turns out we can resolve both at once by considering the vector field

$$t^\mu = \alpha n^\mu + \beta^\mu, \tag{5.27}$$

where α is the lapse function, and β^μ is the so-called shift vector, which is spatial $n_\mu\beta^\mu = 0$. Derivatives along t^μ are natural in the sense that $t^\mu\partial_\mu t = 1$. This also tackles the problem of diffeomorphism invariance as we can take t^μ to connect the same spatial coordinate from one time slice to the next. That is, α represents our ability to reparameterize the distribution of time slices, while β^μ specifies how the spatial coordinates "drift" along spatial directions. These 4 or $d+1$ degrees of freedom are exactly the gauge freedom of diffeomorphism invariance and are thus completely arbitrary. However, for practical purposes in numerical relativity, there is a range of standard choices addressing problems arising in numerical computations such as stability, dealing with singularities, etc. [48]. To implement this new time-derivative along t^μ, one uses the relations

$$\mathcal{L}_t\gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta\gamma_{\mu\nu}, \tag{5.28a}$$

$$\mathcal{L}_t K_{\mu\nu} = \alpha\mathcal{L}_n K_{\mu\nu} + \mathcal{L}_\beta K_{\mu\nu}, \tag{5.28b}$$

which can be derived using $n^\mu\gamma_{\mu\nu} = n^\mu K_{\mu\nu} = 0$.

5.2 Initial data for 3+1 GR

The 3+1 decomposition takes a different approach to solving the Einstein equation (5.24), in that we first come up with some initial data and then worry about the time evolution afterwards.

This is opposed to how one usually derives e.g. the Schwarzschild metric, where one solves for both the space and the time direction simultaneously. However, as we have seen, not all initial data are valid, in particular it needs to satisfy the Hamiltonian constraint (5.26a) and the momentum constraint (5.26b). A further complication is that the constraint equations only allow for the determination of 4 out of the 12 components of $\gamma_{\mu\nu}$ and $K_{\mu\nu}$, and a priori there is no natural split between free and constrained components.

There has been extensive work done, see [49], on methods to construct initial data and how to overcome these problems. In this section, we will review some aspects of the so-called conformal decomposition. The results presented in this section only hold for $d = 3$, while the results of the previous sections hold for any d. Further, we go to adapted coordinates $(x^0, x^1, \dots, x^d)$ such that $t \equiv x^0$ parameterizes the time-slices and x^a are spatial coordinates on the hypersurface with $a, b, \dots = 1, \dots, d$. In particular, this means that spatial projections amount to restricting indices to run only over $1, \dots, d$.

5.2.1 Conformal and traceless-transverse decomposition

The idea of the conformal decomposition is to write the 3-metric γ_{ab} as a product of a scale factor ψ and a background metric $\bar{\gamma}_{ab}$ as

$$\gamma_{ab} = \psi^4 \bar{\gamma}_{ab}. \tag{5.29}$$

The exponent of ψ is in principle arbitrary, but for $d = 3$ the exponent ψ^4 turns out to be a convenient choice. It follows that the inverse metric is also conformally related according to

$$\gamma^{ab} = \psi^{-4} \bar{\gamma}^{ab}. \tag{5.30}$$

As described, there exists a Levi-Civita connection $\hat{\nabla}$ associated with γ_{ab}, and equivalently there exists a Levi-Civita connection $\bar{\nabla}$ that is compatible with $\bar{\gamma}_{ab}$. One can show that the connection coefficients[3] of the two connections are related by

$$\hat{\Gamma}^c_{ab} = \bar{\Gamma}^c_{ab} + 2(2\delta^c_{(a} \bar{\nabla}_{b)} \log \psi - \bar{\gamma}_{ab} \bar{\gamma}^{cd} \bar{\nabla}_d \log \psi). \tag{5.31}$$

We can further relate the Ricci tensors of the two connections by

$$\begin{aligned} \hat{R}_{ab} = \bar{R}_{ab} &- 2(\bar{\nabla}_a \bar{\nabla}_b \log \psi + \bar{\gamma}_{ab} \bar{\gamma}^{cd} \bar{\nabla}_c \bar{\nabla}_d \log \psi) \\ &+ 4(\bar{\nabla}_a \log \psi \bar{\nabla}_b \log \psi - \bar{\gamma}_{ab} \bar{\gamma}^{cd} \bar{\nabla}_c \log \psi \bar{\nabla}_d \log \psi), \end{aligned} \tag{5.32}$$

which implies that for the Ricci scalar we have

$$R = \psi^{-4} \bar{R} - 8\psi^{-5} \bar{\nabla}^2 \psi, \tag{5.33}$$

with $\bar{\nabla}^2 \psi = \bar{\gamma}^{ab} \bar{\nabla}_a \bar{\nabla}_b \psi$. We can then turn to the extrinsic curvature, which we can decompose into its traceless part A_{ab} and trace K according to

$$K_{ab} = A_{ab} + \frac{1}{3} \gamma_{ab} K. \tag{5.34}$$

As the extrinsic curvature can be seen as a free variable, we can give each part of the decomposition its own scaling in ψ

$$A_{ab} = \psi^\alpha \bar{A}_{ab}, \qquad K = \psi^\beta \bar{K}. \tag{5.35}$$

[3]Spatial connection coefficients work as usual in the adapted coordinates e.g. $\hat{\nabla}_a X^b = \partial_a X^b + \hat{\Gamma}^b_{ac} X^c$. The connection coefficients can also be used in general coordinates if we subsequently project e.g. $\hat{\nabla}_\mu X^\nu = \gamma^\rho_\mu \gamma^\nu_\sigma (\partial_\rho X^\sigma + \hat{\Gamma}^\sigma_{\rho\lambda} X^\lambda)$, for any spatial X^μ.

Different choices of α and β give different simplifications of the equations, and for our purposes it will turn out to be advantageous to choose

$$\alpha = -2, \qquad \beta = 0. \tag{5.36}$$

In particular, the choice (5.36) simplifies the momentum constraint. A final rewriting we are going to make is breaking up the traceless part of the extrinsic curvature $\bar{A}_{ab}$ by doing a so-called traceless-transverse decomposition. Specifically, we write $\bar{A}_{ab}$ as

$$\bar{A}_{ab} = (\bar{L}X)_{ab} + \bar{A}^{\mathrm{TT}}_{ab} \tag{5.37}$$

where A^{TT}_{ab} is a tensor field satisfying

$$\bar{\gamma}^{ab}\bar{A}^{\mathrm{TT}}_{ab} = 0, \qquad \bar{\nabla}^a \bar{A}^{\mathrm{TT}}_{ab} = 0, \tag{5.38}$$

hence justifying the name, as having vanishing divergence translates to the field being transverse to the wave vector in Fourier space. The other component of (5.37) is a vector field X^μ acted upon by the conformal Killing operator $\bar{L}$

$$(\bar{L}X)^{ab} = 2\bar{\nabla}^{(a}X^{b)} - \frac{2}{3}\bar{\nabla}_c X^c \bar{\gamma}^{ab}. \tag{5.39}$$

The kernel of $\bar{L}$ is precisely the conformal Killing vectors of the 3-metric $\bar{\gamma}_{ab}$. The decomposition (5.37) is only guaranteed to be unique when dealing with asymptotically flat or compact, boundaryless manifolds. However, when the decomposition is well-defined, we can write the Hamiltonian (5.26a) and momentum (5.26b) constraints as

$$\bar{\nabla}^2\psi - \frac{1}{8}\bar{R}\psi + \frac{1}{8}(\bar{L}X_{ab} + \bar{A}^{\mathrm{TT}}_{ab})(\bar{L}X^{ab} + \bar{A}^{ab}_{\mathrm{TT}})\psi^{-7} - \frac{1}{12}K^2\psi^5 = -2\pi\rho\psi^5, \tag{5.40a}$$

$$\bar{\Delta}_L X^a - \frac{2}{3}\psi^6\bar{\nabla}^a K = 8\pi\psi^{10}S^a. \tag{5.40b}$$

The operator $\bar{\Delta}_L$ is the conformal vector Laplacian given by

$$\bar{\Delta}_L X^a \equiv \bar{\nabla}_b(\bar{L}X)^{ba} = \bar{\nabla}_b\bar{\nabla}^b X^a + \frac{1}{3}\bar{\nabla}^a\bar{\nabla}_b X^b + \bar{R}^a{}_b X^b. \tag{5.41}$$

The decompositions (5.40a) and (5.40b) are under the stated assumptions completely equivalent to the original Hamiltonian (5.26a) and momentum (5.26b) constraints. This formulation however clarifies what is "free" and "constrained" data. In particular, we can choose

$$\bar{\gamma}_{ab},\, \bar{A}^{\mathrm{TT}}_{ab},\, K,\, \rho\,, S_a \qquad \overset{\text{solve for}}{\Longrightarrow} \qquad \psi,\, X^a. \tag{5.42}$$

Furthermore, from a technical point of view both equations (5.40a) and (5.40b) can be shown to be elliptic PDEs [49]. Thus, the problem of determining initial data through the conformal traceless-transverse method is a well-defined problem.

5.2.2 Bowen-York solutions

With the equations (5.40a) and (5.40b), one can start looking for simple solutions. One such example is the family of solutions known as Bowen-York, for which one chooses the free data

$$\bar{\gamma}_{ab} = \delta_{ab}, \qquad \bar{A}^{\mathrm{TT}}_{ab} = 0, \qquad K = 0, \qquad \rho = 0, \qquad S_a = 0, \tag{5.43}$$

where δ_{ab} is the flat metric in Cartesian coordinates. That is, we consider a conformally flat metric with no matter sources. Further, we take the initial time slice to be

$$\Sigma_0 = \mathbb{R}^3 \setminus \{0\}. \tag{5.44}$$

The choice $K = 0$, which is known as the maximal slicing condition, decouples the equations (5.40b) from (5.40a). This implies that (5.40b) takes the form

$$\partial^b \partial_b X^a + \frac{1}{3} \partial^a \partial_b X^b = 0. \tag{5.45}$$

On the domain (5.44), one can derive the following 6-parameter family of solutions to (5.45)

$$X^a = -\frac{1}{4r} \left[7\delta^{ab} P_b + \frac{P_b x^b x^a}{r^2} \right] - \frac{1}{r^3} \epsilon^{ab}{}_c J_b x^c, \tag{5.46}$$

where $r = \sqrt{\delta_{ab} x^a x^b}$ and ϵ_{abc} is the totally anti-symmetric symbol defined by $\epsilon_{123} = 1$. To get the full initial data, one further has to compute $\bar{L}X^{ab}$ and solve (5.40a) for ψ, where the latter cannot be done analytically. Even though we are not able to find the full initial data analytically, we can still characterize the parameters P_a and J_a using the ADM linear and angular momentum [41]. Specifically, by imposing asymptotic flatness in the form $\psi \to 1$ for $r \to \infty$, all dependence on ψ drops out of the ADM integrals and one can show that [49]

$$P_a^{\mathrm{ADM}} = P_a, \qquad J_a^{\mathrm{ADM}} = J_a. \tag{5.47}$$

That is, the Bowen-York initial data (5.43) and (5.46) describes some boosted and rotating black hole. Consequently, one might expect it to be a slice of the Kerr spacetime. However, it turns out that the Bowen-York data also contains gravitational radiation, and hence corresponds to a solution of the Einstein equation distinct from the Kerr metric [50].

5.3 Carroll gravity from the 3+1 decomposition

As stated at the beginning of the chapter, Carroll geometry bears much resemblance to the 3+1 decomposition, as it also has a privileged direction of time. The Carrollian vector field v^μ is of course fundamentally different as it is physical as opposed to the n^μ of the 3+1 formalism, which depends on the choice of foliation. This resemblance is readily seen by comparing the decomposition of the 4-metric (5.4) into a spatial metric and co-normal and the decomposition of the relativistic metric (4.18a) into PUR variables. This similarity suggests that the 3+1 decomposition serves as a good starting point for a small c expansion. Thus, we will in this section develop an alternative method of obtaining the EOM of chapter 4 through the 3+1 formalism.

The 3+1 and PUR decompositions can be made equivalent under the identification

$$\gamma_{\mu\nu} = \Pi_{\mu\nu}, \qquad \gamma^{\mu\nu} = \Pi^{\mu\nu}, \qquad n_\mu = cT_\mu, \qquad n^\mu = c^{-1} T^\mu. \tag{5.48}$$

However, in addition to the identification (5.48), there is a more subtle assumption we need to consider if we want to perform a small c expansion of GR using the equations of the 3+1 decomposition. Specifically, the construction of the 3+1 formalism starts by assuming a foliation. This condition can be expressed locally by the Frobenius condition for hypersurface integrability [39]

$$n \wedge dn = 0. \tag{5.49}$$

Thus, under the identification (5.48) this becomes

$$0 = T \wedge dT = \tau \wedge d\tau + c^2(\dots) + \mathcal{O}(c^4). \tag{5.50}$$

That is, we get an extra condition for each level of the expansion, as compared to the analysis of chapter 4, when using the 3+1 decomposition as a starting point. Consequently, the Carroll theory we obtain from the 3+1 approach is only a sub-sector of the theory we get by expanding the action directly, as in chapter 4.

5.3.1 Changing parameterization

The EOM of chapter 4 are defined as the response to varying the timelike vielbein and the spatial metric. Hence, we need to relate the results of section 5.1 to these responses. A convenient way of doing this is by considering the variation of the full EH Lagrangian (4.37)

$$\delta\mathcal{L}_{\text{EH}} = \frac{c^4 E}{16\pi G_N} G_{\mu\nu}\delta g^{\mu\nu}, \tag{5.51}$$

where we used the well-known variation of the relativistic $\sqrt{-g}R$ and reinstated factors of c. We can then relate the variation of the inverse metric to the variations of the PUR variables

$$\delta g^{\mu\nu} = -2c^{-2}T^{(\mu}\delta T^{\nu)} + \delta\Pi^{\mu\nu}. \tag{5.52}$$

Inserting this into (5.51) and applying (4.8) to write out projections, we find

$$\begin{aligned}\delta\mathcal{L}_{\text{EH}} = &-\frac{c^2 E}{8\pi G_N}(G_{\mu\nu}T^\mu\Pi^\nu_\rho - G_{\mu\nu}T^\mu T^\nu T_\rho)\delta T^\rho \\ &+ \frac{c^4 E}{16\pi G_N}(G_{\mu\nu}\Pi^\mu_\rho\Pi^\nu_\sigma - 2G_{\mu\nu}T^\mu T_{(\rho}\Pi^\nu_{\sigma)})\delta\Pi^{\rho\sigma}.\end{aligned} \tag{5.53}$$

Hence, we have reduced the problem to determining projections of $G_{\mu\nu}$, which is exactly accomplished by the Gauss, Codazzi and Ricci equations. To obtain the expanded EOM, we simply need to expand (5.53) order by order in c^2. To find the correct factor of c in the geometrical equations of section 5.1, we further need to substitute the 3+1 variable for the PUR variables according to (5.48) e.g. for the extrinsic curvature

$$-\frac{1}{2}\mathcal{L}_n\gamma_{\mu\nu} = -\frac{c^{-1}}{2}\mathcal{L}_T\Pi_{\mu\nu} = c^{-1}\overset{(0)}{K}_{\mu\nu}. \tag{5.54}$$

In (5.54) we did not use $K_{\mu\nu}$ for the LHS because we in this section reserve the symbol for the Carrollian extrinsic curvature $K_{\mu\nu} \equiv -\frac{1}{2}\mathcal{L}_v h_{\mu\nu}$. Also, we recall the definition $\overset{(0)}{K}_{\mu\nu} \equiv -\frac{1}{2}\mathcal{L}_T\Pi_{\mu\nu}$, which implies $\overset{(0)}{K}_{\mu\nu} = K_{\mu\nu} + \mathcal{O}(c^2)$. We can repeat this substitution by PUR variables for each term in Gauss, Codazzi and Ricci equations to establish the following relations

$$T^\mu T^\nu G_{\mu\nu} = \frac{1}{2}(\overset{(0)}{K}{}^2 - \overset{(0)}{K}_{\mu\nu}\overset{(0)}{K}{}^{\mu\nu} + c^2\hat{R}), \tag{5.55a}$$

$$\Pi^\mu_\rho T^\nu G_{\mu\nu} = \hat{\nabla}_\rho\overset{(0)}{K} - \hat{\nabla}_\sigma\overset{(0)}{K}{}^\sigma{}_\rho, \tag{5.55b}$$

$$\begin{aligned}\Pi^\rho_\mu\Pi^\sigma_\nu G_{\rho\sigma} = &\hat{R}_{\mu\nu} - \hat{\nabla}_\mu a_\nu - a_\mu a_\nu + c^{-2}(\overset{(0)}{K}\overset{(0)}{K}_{\mu\nu} - 2\overset{(0)}{K}_{\mu\rho}\overset{(0)}{K}{}^\rho{}_\nu - \mathcal{L}_T\overset{(0)}{K}_{\mu\nu}) \\ &- \frac{1}{2}\Pi_{\mu\nu}(\hat{R} - 2\hat{\nabla}_\mu a^\mu - 2a_\mu a^\mu + c^{-2}[\overset{(0)}{K}{}^2 + \overset{(0)}{K}_{\mu\nu}\overset{(0)}{K}{}^{\mu\nu} - 2\mathcal{L}_T\overset{(0)}{K}]),\end{aligned} \tag{5.55c}$$

where the Ricci equation was combined with the Gauss equation to obtain (5.55c). Considering both (5.53) and (5.55a)-(5.55c), we again see that the expansion starts at order c^2. We then compute the LO EOM by taking $\sigma = 0$ in the variation of $\tilde{\mathcal{L}}_{\text{EH}} \equiv c^{-2}\mathcal{L}_{\text{EH}}$

$$\begin{aligned}
\delta\tilde{\mathcal{L}}_{\text{EH}}|_{\sigma=0} &= \frac{e}{8\pi G_n}\delta v^\mu (G_{\mu\nu}T^\mu T^\nu T_\rho - G_{\mu\nu}T^\mu \Pi^\nu_\rho)|_{\sigma=0} \\
&\quad + \frac{e}{16\pi G_N}\delta h^{\mu\nu}(c^2 G_{\mu\nu}\Pi^\mu_\rho \Pi^\nu_\sigma - 2c^2 G_{\mu\nu}T^\mu T_{(\rho}\Pi^\nu_{\sigma)})|_{\sigma=0} \\
&= \frac{e}{8\pi G_n}\delta v^\mu (\tfrac{1}{2}(K^2 - K_{\mu\nu}K^{\mu\nu})\tau_\mu - (\hat{\nabla}_\rho \overset{(0)}{K} - \hat{\nabla}_\sigma \overset{(0)}{K}{}^\sigma{}_\rho)|_{\sigma=0} \\
&\quad + \frac{e}{16\pi G_N}\delta h^{\mu\nu}(KK_{\mu\nu} - 2K_{\mu\rho}K^\rho{}_\nu - \mathcal{L}_v K_{\mu\nu} - \tfrac{1}{2}h_{\mu\nu}(K^2 + K_{\rho\sigma}K^{\rho\sigma} - 2\mathcal{L}_v K)).
\end{aligned} \tag{5.56}$$

Comparing this to the definition of the EOM $\overset{(2)}{G}{}^v_\mu$ (4.39a) and $\overset{(2)}{G}{}^h_{\mu\nu}$ (4.39b), we see that the contents of the brackets on the first and second line of (5.56) should exactly be the LO EOM (4.52a) and (4.52b), respectively. Though it looks very promising, it is not immediately clear that these expressions agree. However, if we consider the full Levi-Civita connection written out in terms of the PUR variables (4.24a)-(4.24c) and the shift $\overset{(0)}{S}{}^\rho_{\mu\nu}$, it is easy to see that for a spatial vector X^μ

$$\hat{\nabla}_\mu X^\nu \equiv \Pi^\rho_\mu \Pi^\nu_\sigma \nabla_\rho X^\sigma = \Pi^\rho_\mu \Pi^\nu_\sigma \overset{(\tilde{C})}{\nabla}_\rho X^\sigma, \tag{5.57}$$

because the differences between $\Gamma^\rho_{\mu\nu}$ and $\tilde{C}^\rho_{\mu\nu}$ are projected out. From this, we conclude that

$$-(\hat{\nabla}_\rho \overset{(0)}{K} - \hat{\nabla}_\sigma \overset{(0)}{K}{}^\sigma{}_\rho)|_{\sigma=0} = h^{\gamma\lambda}\tilde{\nabla}_\lambda (K_{\mu\gamma} - K h_{\mu\gamma}), \tag{5.58}$$

and hence the expressions for the EOM $\overset{(2)}{G}{}^v_\mu$ do indeed agree. To match up $\overset{(2)}{G}{}^h_{\mu\nu}$ we need the Lie derivative identity

$$\mathcal{L}_v K_{\mu\nu} = v^\rho \tilde{\nabla}_\rho K_{\mu\nu} - 2K_\mu{}^\rho K_{\rho\nu}, \tag{5.59}$$

which can be derived using (4.47). Applying (5.59) to the second line of (5.56), we find agreement with (4.52b). A priori, it is not given that the EOM one can read off from (5.56), should immediately agree with the EOM of chapter 5, because we have not considered the integrability condition (5.50). However, as the LO EOM from the action expansion do not explicitly depend on the LO Frobenius condition $\tau \wedge d\tau = 0$, we find agreement without further considerations. The equations obtained from the 3+1 approach should in general be supplemented by the Frobenius condition expanded to the appropriate order. To obtain the NLO EOM one expands (5.53) to order c^0 and reads off the responses to the variation. This procedure can of course be repeated ad infinitum to obtain further orders.

Thus, we see that the LO and sub-leading Carroll EOM can be derived using the 3+1 formalism. Furthermore, this approach automatically casts the equations in terms of variables such as $\Pi_{\mu\nu}$, $\overset{(0)}{K}_{\mu\nu}$ etc. that in the $c \to 0$ limit correspond to natural Carroll tensors. This novel approach of expanding the EOM of GR is equivalent to the action expansion up to the added constraint of the integrability condition (5.50).

This procedure can of course also be applied to the Galilean limit where the caveat of hypersurface integrability at the level of the PUR (or in that case pre-non-relativistic) variables (5.50) plays a bigger role. Note that the Galilean expansion of $T \wedge dT$ would be in orders of c^{-2} rather than c^2, as it is done in (5.50). As mentioned in chapter 4, the LO (i.e. c^6) EOM from a Galilean action expansion is exactly the Frobenius condition on the clock-form τ_μ [29], i.e.

$\tau \wedge d\tau = 0$. However, the 3+1 formalism assumes the integrability condition, and the LO EOM from the 3+1 expansion is identically zero. That is, the LO EOM from the two approaches only agree when we impose the LO Frobenius condition. Likewise, the NLO EOM from the action expansion will also disagree with the 3+1 NLO EOM until we apply the NLO Frobenius condition. This underlines that it is coincidental that the Carroll LO EOM come out the same from both the action expansion and the 3+1 decomposition without imposing the Frobenius condition (5.50), and one should not expect this to hold true at sub-leading orders.

Chapter 6

Leading Order Theory Solutions and Charges

The LO Carroll theory, as derived in chapter 4, is in itself a dynamical field theory governing the Carroll metric data and is thus interesting in its own right. The gravity theory is of course expected to be somewhat strange due to the Carrollian causal structure, which forces isolated dynamics on each integral line of v^μ. This property would limit the possible dynamics but on the other hand allow for more problems to be exactly solvable. This chapter is dedicated to exploring novel approaches to finding solutions to the LO theory and characterizing them through e.g. boundary charges. We will show that the LO EOM can be cast in a form very similar to that of the 3+1 decomposed Einstein equation, as presented in chapter 5. Hence, we will also adapt some of the methods of the 3+1 decomposition to solve the LO Carroll theory. The results on LO Carroll gravity and its solutions presented in this chapter are original.

Specifically, we will in section 6.1 derive Ward identities for the LO theory and in section 6.2 rewrite the LO EOM to make them comparable to the equations of chapter 5. Then, in section 6.3 and 6.4, we will address the problems of time evolution and preparing initial data, respectively. Next, in section 6.5, we will compare our results to earlier work of Dautcourt [32] before we in section 6.6 derive boundary charges for the theory. Finally, we will in section 6.7 work through a number of examples of solutions to LO Carroll gravity.

6.1 Ward identities

As the diffeomorphism Ward identity appears in both the analysis of the EOM in section 6.2 and the derivation of the LO boundary charges in section 6.6, we will in this section derive the LO Ward identities. To derive the Ward identities, we follow the same procedure as in section 2.4.1. That is, to consider a general variation (4.45) and deduce the consequences of invariance when the variation is chosen to be a local symmetry. In particular, the Ward identities will work out to have the same form as in section 2.4.1 up to constants due to their definitions, compare (2.78) and (4.45). Hence, it is easily seen that the boost Ward identity (2.89) translates to

$$v^\mu \overset{(2)}{G}{}^{h\Lambda}_{\mu\nu} = 0, \tag{6.1}$$

where $\overset{(2)}{G}{}^{h\Lambda}_{\mu\nu}$ is the response of the LO action to varying with respect to $h_{\mu\nu}$, as defined in (4.39b). This is consistent with the explicit expression (4.52b) derived in chapter 4.

The diffeomorphism Ward identity also follows in the same way as it did for a Carroll field theory. Carrying out the same steps now utilizing the $\tilde{\Gamma}$-connection, we find the following diffeomorphism Ward identity

$$\frac{1}{2}v^\rho \overset{(2)}{G}{}^{h\Lambda}_{\rho\sigma} h^{\sigma\lambda}\tau_{\lambda\gamma}(v^\gamma\tau_\mu - \delta^\gamma_\mu) - 2\overset{(2)}{G}{}^{v\Lambda}_{(\mu}K_{\nu)}{}^\nu - K^{\rho\sigma}\overset{(2)}{G}{}^{h\Lambda}_{\rho\sigma}\tau_\mu + v^\nu\tilde{\nabla}_\nu \overset{(2)}{G}{}^{v\Lambda}_\mu + \tilde{\nabla}_\nu(h^{\nu\rho}\overset{(2)}{G}{}^{h\Lambda}_{\rho\mu}) = 0, \tag{6.2}$$

where the LO EOM $\overset{(2)}{G}{}^{v\Lambda}_\mu$ is defined in (4.39a). This can be further simplified by noticing that the first term drops out by imposing the boost Ward identity (6.1) yielding the combined relation

$$-2\overset{(2)}{G}{}^{v\Lambda}_{(\mu}K_{\nu)}{}^\nu - K^{\rho\sigma}\overset{(2)}{G}{}^{h\Lambda}_{\rho\sigma}\tau_\mu + v^\nu\tilde{\nabla}_\nu \overset{(2)}{G}{}^{v\Lambda}_\mu + \tilde{\nabla}_\nu(h^{\nu\rho}\overset{(2)}{G}{}^{h\Lambda}_{\rho\mu}) = 0. \tag{6.3}$$

We note, that the diffeomorphism Ward identity of GR is the contracted Bianchi identity $\nabla_\mu G^{\mu\nu}$, which in this sense can be considered analogous to (6.2).

6.2 Vacuum equations of motions

We have already derived the vacuum EOM both with and without a cosmological constant (4.52a)-(4.52b) and (4.55a)-(4.55b), respectively. In particular, we found that the LO EOM with a cosmological constant are

$$\overset{(2)}{G}{}^{v\Lambda}_\mu = -\frac{1}{2}\tau_\mu(K^{\rho\sigma}K_{\rho\sigma} - K^2 + 2\tilde{\Lambda}) + h^{\gamma\lambda}\tilde{\nabla}_\lambda(K_{\mu\gamma} - Kh_{\mu\gamma}), \tag{6.4a}$$

$$\overset{(2)}{G}{}^{h\Lambda}_{\mu\nu} = -\frac{1}{2}h_{\mu\nu}(K^{\rho\sigma}K_{\rho\sigma} - K^2 - 2\tilde{\Lambda}) + K(K_{\mu\nu} - Kh_{\mu\nu}) - v^\rho\tilde{\nabla}_\rho(K_{\mu\nu} - Kh_{\mu\nu}). \tag{6.4b}$$

To better understand the nature of these equations, it is instructive to rewrite them in a form that resembles the 3+1 EOM (5.26a)-(5.26c). This will allow us to interpret the LO Carroll EOM analogously to the 3+1 EOM. For generality, we consider the LO EOM with a cosmological constant, which simply can be written as

$$\overset{(2)}{G}{}^{v\Lambda}_\mu = 0, \qquad \overset{(2)}{G}{}^{h\Lambda}_{\mu\nu} = 0. \tag{6.5}$$

Taking the temporal projection of $\overset{(2)}{G}{}^{v\Lambda}_\mu = 0$, i.e. contracting (6.4a) with v^μ, we find the first EOM

$$K^{\mu\nu}K_{\mu\nu} - K^2 = -2\tilde{\Lambda}. \tag{6.6}$$

The second term of (6.4a) does not contribute to (6.6) because it is annihilated by v^μ. On the other hand, the spatial projection of $\overset{(2)}{G}{}^{v\Lambda}_\mu = 0$ yields the second equation

$$h^{\rho\sigma}\tilde{\nabla}_\rho(K_{\sigma\mu} - Kh_{\sigma\mu}) = 0. \tag{6.7}$$

Finally, we turn to (6.4b) where the first term simplifies upon imposing (6.6), leaving only

$$2\tilde{\Lambda}h_{\mu\nu} + K(K_{\mu\nu} - Kh_{\mu\nu}) - v^\rho\tilde{\nabla}_\rho(K_{\mu\nu} - Kh_{\mu\nu}) = 0. \tag{6.8}$$

Note that $\overset{(2)}{G}{}^{h\Lambda}_{\mu\nu}$ only has a spatial part as consequence of the boost Ward identity (6.1). We then compute the trace of (6.8) using $h^{\mu\nu}$

$$2d\tilde{\Lambda} + (1-d)K^2 - (1-d)v^\rho\tilde{\nabla}_\rho K = 0, \tag{6.9}$$

where we used $h^{\mu\nu}v^\rho\tilde{\nabla}_\rho X_{\mu\nu} = v^\rho\tilde{\nabla}_\rho(h^{\mu\nu}X_{\mu\nu})$ for $X_{\mu\nu}$ spatial, which follows directly from (3.27a). If we consider $d \neq 1$ then (6.9) implies

$$v^\rho\tilde{\nabla}_\rho K - K^2 = -\frac{2d}{d-1}\tilde{\Lambda}, \tag{6.10}$$

which we can insert into (6.7) together with (6.6) to find the third EOM

$$v^\rho\tilde{\nabla}_\rho K_{\mu\nu} - KK_{\mu\nu} = -\frac{2}{d-1}\tilde{\Lambda}h_{\mu\nu}. \tag{6.11}$$

To summarize, we have rewritten the full content of the LO EOM in vacuum (6.4a)-(6.4b) as

$$K^{\mu\nu}K_{\mu\nu} - K^2 = -2\tilde{\Lambda}, \tag{6.12a}$$

$$h^{\rho\sigma}\tilde{\nabla}_\rho(K_{\sigma\mu} - Kh_{\sigma\mu}) = 0, \tag{6.12b}$$

$$\mathcal{L}_v K_{\mu\nu} = -2K_\mu{}^\rho K_{\rho\nu} + KK_{\mu\nu} - \frac{2}{d-1}\tilde{\Lambda}h_{\mu\nu}, \tag{6.12c}$$

where, for the last EOM (6.11), we used the Lie derivative identity (5.59) to get the form (6.12c).

Before we move on, let us check that the EOM are boost invariant. The EOM (6.12a) and (6.12c) are easily seen to be invariant under boosts, because they only consist of Carroll invariants. The second constraint (6.12b) is not so straightforward and requires a little computation

$$\begin{aligned}
&\delta_\lambda[h^{\rho\sigma}\tilde{\nabla}_\rho(K_{\sigma\mu} - Kh_{\sigma\mu})] \\
&\quad= v^\rho\lambda^\sigma\tilde{\nabla}_\rho(K_{\sigma\mu} - Kh_{\sigma\mu}) - h^{\rho\sigma}\delta\tilde{\Gamma}^\lambda_{\rho\sigma}(K_{\lambda\mu} - Kh_{\lambda\mu}) - h^{\rho\sigma}\delta\tilde{\Gamma}^\lambda_{\rho\mu}(K_{\sigma\lambda} - Kh_{\sigma\lambda}) \\
&\quad= 2\tilde{\Lambda}\lambda_\mu + K\lambda^\nu(K_{\mu\nu} - Kh_{\mu\nu}) - (\lambda^\lambda K - \lambda^\sigma K_\sigma{}^\lambda)(K_{\lambda\mu} - Kh_{\lambda\mu}) \\
&\qquad\qquad - (\lambda^\sigma K^\lambda{}_\mu - K^{\sigma\lambda}\lambda_\mu)(K_{\sigma\lambda} - Kh_{\sigma\lambda}) \\
&\quad= 0.
\end{aligned} \tag{6.13}$$

In the above, we for the second equality used the EOM (6.8) along with the transformation (3.24), and for the third equality we used (6.12a). Hence, we see that the vacuum EOM (6.12a), (6.12b) and (6.12c) are all invariant under boosts, as they must, because they derive from the Carrollian invariant Lagrangian (4.44).

The LO vacuum EOM (6.12a), (6.12b) and (6.12c) somewhat resemble (5.26a), (5.26b) and (5.26c) of the 3+1 decomposition. The LO EOM are in particular susceptible to a similar initial-value problem interpretation with dynamical variables $(h_{\mu\nu}, K_{\mu\nu})$ (position and velocity) and v^μ dictating how spatial points are embedded in the Carrollian spacetime. The equations (6.12a) and (6.12b) resemble the Hamiltonian and momentum constraints, respectively, and can be understood as constraints that have to be satisfied for the initial data to be valid. Because the constraint equations are similar in structure to those of the 3+1 decomposition, we can employ some of the same methods to construct initial data. We will pursue this idea in section 6.4

The evolution equations of the initial-value problem are (6.12c) along with the definition of the extrinsic curvature i.e.

$$\mathcal{L}_v h_{\mu\nu} = -2K_{\mu\nu}, \tag{6.14}$$

which gives the $\mathcal{L}_v$-derivative or "time" derivative of the dynamical variables. Importantly, neither evolution equation contains spatial derivatives, and thus we conclude that points on different integral lines of v^μ cannot affect each other and evolve separately. This confirms the interpretation that at LO the light-cone has collapsed, and spatially separated points are causally disconnected.

Mathematically, this means that the evolution equations are ordinary differential equations (ODEs) rather than partial differential equations (PDEs), which offers an enormous simplification as compared to GR. In particular, this means that we can analytically solve the evolution equation for a large class of Carrollian spacetimes, which we will investigate in section 6.3.

6.2.1 Consistency of the constraint equations

For (6.12a)-(6.12c) to be a complete self-consistent set of equations, we need to check that the constraints are conserved under the time evolution of (6.12c). That is, we assume that the evolution equation (6.12c) holds everywhere and then compute the Lie derivative of the constraints along v^μ. For the constraint (6.12a) the computation is straightforward

$$\begin{aligned}\mathcal{L}_v(K^{\mu\nu}K_{\mu\nu} - K^2 + 2\tilde{\Lambda}) &= v^\rho\tilde{\nabla}_\rho(K^{\mu\nu}K_{\mu\nu} - K^2) = 2v^\rho(K^{\mu\nu}\tilde{\nabla}_\rho K_{\mu\nu} - K\tilde{\nabla}_\rho K)\\ &= 2K^{\mu\nu}\left(KK_{\mu\nu} - \frac{2}{d-1}\tilde{\Lambda}h_{\mu\nu}\right) - 2K\left(K^2 - \frac{2d}{d-1}\tilde{\Lambda}\right)\\ &= 2K(K^{\mu\nu}K_{\mu\nu} - K^2 + 2\tilde{\Lambda}),\end{aligned}\tag{6.15}$$

where we used $\tilde{\nabla}_\rho(K^{\mu\nu}K_{\mu\nu}) = 2K^{\mu\nu}\tilde{\nabla}_\rho K_{\mu\nu}$ for the first equality and the trace form of the evolution equation (6.10) for the third. Hence, we see that the constraint (6.12a) is preserved under time evolution i.e. if (6.12a) is satisfied at time t then it is also satisfied at time $t + \delta t$.

In principle, the calculation for the second constraint (6.12b) follows in the same manner, but it is more technically challenging due to the covariant derivative. Thus, we want to connect to the Ward identity (6.3), which we can do by recalling that the constraint (6.12b) can be written as $h^\nu_\mu \overset{(2)}{G}{}^{v\Lambda}_\nu$, where $h^\mu_\nu \equiv h_{\nu\rho}h^{\rho\mu}$ is the spatial projector, and noting the Lie derivative identity

$$\mathcal{L}_v(h^\nu_\mu \overset{(2)}{G}{}^{v\Lambda}_\nu) = h^\nu_\mu v^\rho \tilde{\nabla}_\rho \overset{(2)}{G}{}^{v\Lambda}_\nu - K_\mu{}^\nu \overset{(2)}{G}{}^{v\Lambda}_\nu + v^\rho \overset{(2)}{G}{}^{v\Lambda}_\rho v^\nu \tau_{\nu\mu}.\tag{6.16}$$

Taking the spatial projection of the Ward identity (6.3) and substituting in (6.16), we obtain the following expression for the $\mathcal{L}_v$-derivative of the constraint (6.12b)

$$\mathcal{L}_v(h^\nu_\mu \overset{(2)}{G}{}^{v\Lambda}_\nu) = h^\nu_\mu \overset{(2)}{G}{}^{v\Lambda}_\nu K - \tilde{\nabla}_\nu(h^{\nu\rho}\overset{(2)}{G}{}^{h\Lambda}_{\rho\mu}) + v^\rho \overset{(2)}{G}{}^{v\Lambda}_\rho v^\nu \tau_{\nu\mu}.\tag{6.17}$$

As we assume the constraints hold at a time t, the first and last term of the RHS of (6.17) clearly drop out. However, the middle term also vanishes because $\overset{(2)}{G}{}^{h\Lambda}_{\mu\nu}$ can be seen as a combination of the evolution equation (6.12c), which vanishes by assumption, and the constraint (6.12a) that we showed also vanishes everywhere (6.15).

6.2.2 LO Carroll gravity and classification by intrinsic torsion

In the light of the equation (6.12a), which is an algebraic constraint on $K_{\mu\nu}$, it is interesting to revisit the classification of Carroll spacetimes by intrinsic torsion, as presented in section 3.1.2. In particular, let us consider the LO EOM with $\tilde{\Lambda} = 0$ and deduce the consequences of (6.12a).

($\mathcal{C}_0$) $K_{\mu\nu} = 0$: This class trivially satisfies the LO EOM and is thus consistent with the leading order theory.

($\mathcal{C}_1$) $K = 0$: By the EOM (6.12a), we see that $K = 0$ implies $K_{\mu\nu} = 0$ because $K_{\mu\nu}$ is purely spatial, and $h^{\mu\nu}$ is positive definite on the spatial subspace. Thus, on-shell $\mathcal{C}_1$ implies $\mathcal{C}_0$.

($\mathcal{C}_2$) $K_{\mu\nu} = fh_{\mu\nu}$: That is, the extrinsic curvature is only trace $K_{\mu\nu} = \frac{K}{d}h_{\mu\nu}$. However, this contradicts (6.12a) for $d > 1$, as can be seen by contracting with $K^{\mu\nu}$. Consequently, we cannot have $\mathcal{C}_2$ geometries in the LO vacuum theory.

($\mathcal{C}_3$) Puts no restrictions on $K_{\mu\nu}$ and is thus trivially consistent with the LO EOM.

Hence, we see that only the classes $\mathcal{C}_0$ and $\mathcal{C}_3$ are possible when considering the LO vacuum theory. However, once one introduces matter (4.43a)-(4.43b), these considerations can become invalid as matter, depending on the scaling in c, will appear as sources on the RHS of (6.12a)-(6.12c). If we assume that the power counting for matter of the Galilean expansion [29] holds in the Carroll limit, then many kinds of matter start at order c^2 and spoil the above consideration e.g. point particles and electromagnetism.

This is also unlike what one might expect from the results of the Galilean limit [29] where the leading-order EOM directly corresponds to the $\mathcal{G}_1$ class of intrinsic torsion of Galilean structures (see the classification in section 3.1.2). Further, as can be seen from (4.37) the Galilean LO EOM is of order c^6, and no reasonable Galilean expansion of matter starts at that order. Thus, the geometric constraint of the LO EOM in the $c \to \infty$ limit is a general property of all solutions in non-relativistic gravity regardless of matter.

6.3 Solving the evolution equation

To explicitly solve the evolution equation (6.12c), we need to choose a partially adapted coordinate system. In particular, we want to construct a coordinate system $(t \equiv x^0, \ldots, x^{d+1})$ such that the Carrollian vector field is parallel to the t-direction i.e. $v \sim \partial_t$. One can always choose such coordinates, at least locally [51]. Hence, this construction can be made without loss of generality. More precisely this means that the vector field v has the form

$$v = e^{-\frac{H}{2}}\partial_t, \qquad v^\mu = (e^{-\frac{H}{2}}, 0, \ldots, 0), \tag{6.18}$$

where H can be interpreted as a sort of Carrollian lapse function. The precise form of (6.18) is chosen to match the coordinates used by Dautcourt in [32], which also facilitates comparison between the results of this thesis and Dautcourt, see section 6.5. Furthermore, the vector field v defines a spatial subspace of T^*M, and consequently spatial co-variant tensors e.g $h_{\mu\nu}$ and $K_{\mu\nu}$ cannot have non-zero dt-components. Thus, we will restrict the indices of these tensors to run only over $a, b, \ldots = 1, \ldots, d$. In the following, we will also use the inverse spatial metric as h^{ab}, even though this in principle presupposes more structure i.e. a choice of τ_μ. However, if all indices of h^{ab} are always contracted with a spatial co-variant tensor, then any ambiguity is projected out.

First, we determine the extrinsic curvature in these coordinates where we can immediately restrict the indices to be spatial

$$K_{ab} = -\frac{1}{2}\mathcal{L}_v h_{ab} = -\frac{e^{-\frac{H}{2}}}{2}\mathcal{L}_{\partial_0} h_{ab} = -\frac{e^{-\frac{H}{2}}}{2}\dot{h}_{ab}. \tag{6.19}$$

Here, the over-dot $\dot{f}$ denotes differentiation with respect to t. The Lie derivative of the extrinsic curvature works out in much the same way,

$$\mathcal{L}_v K_{ab} = -\frac{e^{-H}}{2}\left(\ddot{h}_{ab} - \frac{\dot{H}}{2}\dot{h}_{ab}\right). \tag{6.20}$$

Using the rewritings (6.19) and (6.20), we obtain the following form of the evolution equation (6.12c)

$$\ddot{h}_{ab} + \frac{1}{2}\dot{h}_{ab}(h^{cd}\dot{h}_{cd} - \dot{H}) - \dot{h}_{ac}h^{cd}\dot{h}_{db} = \frac{4}{d-1}\tilde{\Lambda}e^{H}h_{ab}. \tag{6.21}$$

From the form (6.21) we clearly see that the evolution equation indeed is an ODE. Further, we note that the function H represents gauge freedom i.e. in the adapted coordinates we are free to reparameterize the time variable t, which corresponds to a new choice of H.

6.3.1 Zero cosmological constant

If we consider the case of $\tilde{\Lambda} = 0$, we already get a simpler ODE. To solve the problem, we first need to choose a gauge. Considering (6.21), a natural choice seems to be

$$\dot{H} = h^{ab}\dot{h}_{ab}, \tag{6.22}$$

which indeed offers a great deal of simplification as it removes all reference to the lapse function H. With these choices, the problem (6.21) becomes

$$\ddot{h}_{ab} - \dot{h}_{ac}h^{cd}\dot{h}_{db} = 0. \tag{6.23}$$

We then contract with h^{ca} to obtain

$$0 = h^{ca}\ddot{h}_{ab} - h^{ca}\dot{h}_{ae}h^{ed}\dot{h}_{db} = h^{ca}\ddot{h}_{ab} + \dot{h}^{ca}\dot{h}_{ab} = \frac{d}{dt}(h^{ca}\dot{h}_{ab}), \tag{6.24}$$

where we used $\dot{h}^{ab} = -h^{ac}\dot{h}_{cd}h^{db}$. Integrating (6.24), we get a first order ODE for h_{ab}

$$\dot{h}_{ab} = h_{ac}C^{c}{}_{b}, \tag{6.25}$$

with $C^{a}{}_{b}$ being an integration constant. The equation (6.25) is a linear system of ODEs and is thus solved by a matrix exponential

$$h_{ab} = D_{ac}\exp[tC^{c}{}_{b}], \tag{6.26}$$

where $\exp[A^{a}{}_{b}]$ denotes the matrix exponential of the map $A^{a}{}_{b}$, and D_{ac} is an additional matrix of integration constant. Having solved for the general solution (6.26), we can impose some initial conditions

$$h_{ab}(t=0) = h_{0,ab}, \qquad \dot{h}_{ab}(t=0) = \dot{h}_{0,ab}, \tag{6.27}$$

corresponding to an initial choice of spatial metric and extrinsic curvature. Finally, we solve for the integration constants of (6.26), using (6.27) to obtain the solution to the initial value problem

$$h_{ab}(t) = h_{0,ac}\exp[t\,h_0^{cd}\dot{h}_{0,db}] = h_{0,ac}\exp[-2t\,h_0^{cd}K_{0,db}], \tag{6.28}$$

where we expressed the solution in terms of the extrinsic curvature (6.19) at $t = 0$. Despite appearances, the solution (6.28) is a symmetric tensor due to the following matrix calculation

$$\left[Ae^{A^{-1}B}\right]^{T} = e^{BA^{-1}}A = e^{AA^{-1}BA^{-1}}A = Ae^{A^{-1}B}, \tag{6.29}$$

where A and B are symmetric matrices, and we used the identity $Ae^{BA} = e^{AB}A$. Under boost transformations, the solution (6.28) is invariant because the only object that transforms is

h_0^{cd}, but it is fully projected with spatial co-variant tensors. Also, note that there is an implicit dependence on the spatial coordinates in (6.28) i.e. every point in space evolves along an integral line of v^μ according to its own matrix exponential.

With the solution at hand, we can also deduce the consequences of our gauge choice (6.22)

$$\dot{H} = h^{ab}(t)\dot{h}_{ab}(t) = h_0^{ab}\dot{h}_{0,ab}. \tag{6.30}$$

Integrating this again yields an integration constant, which can be set to zero using the remaining gauge freedom i.e. $H(t=0)=0$. This yields

$$H(t) = -2t\, h_0^{ab}K_{0,ab}. \tag{6.31}$$

Thus, we see that for any vacuum initial data of LO Carroll gravity, the evolution equation (6.12c) can be solved in terms of a matrix exponential.

6.3.2 Non-zero cosmological constant

We will now extend the methods of the previous section to include the case of $\tilde{\Lambda} \neq 0$. We again consider the gauge choice (6.22) and the h^{ab} contraction of (6.21)

$$\frac{d}{dt}(h^{ac}\dot{h}_{cb}) = \kappa e^H \delta_b^a, \tag{6.32}$$

where we defined the constant $\kappa \equiv \frac{4\tilde{\Lambda}}{d-1}$. Initially, it seems that we are stuck, as we have not been able to eliminate the unknown function H. However, if we trace (6.32), we obtain an equation for H

$$\ddot{H} = d\kappa e^H. \tag{6.33}$$

It is always possible to solve (6.33), but it does not depend in a simple way on the initial conditions. Thus, the explicit solution is omitted here. With this, we can solve for H using $H(t=0)=0$ (arbitrary gauge choice) and $\dot{H}(t=0) = h_0^{ab}h_{0,ab}$ (as follows from (6.22)) and reinsert into (6.32). Because the expression on the RHS of (6.32) is now a known function of t, we can solve it using basically the same steps as for (6.24). Specifically, to obtain the time evolution of the spatial metric, we need to determine a function $F(t)$ satisfying

$$\ddot{F} = \kappa e^H, \qquad F(t=0)=0, \qquad \dot{F}(t=0)=0, \tag{6.34}$$

which is computed by direct integration, because e^H is now a known function of t. With these components, it easy to verify that the full solution to (6.21) for initial data (6.27) is

$$h_{ab}(t) = h_{0,ac}\exp[-2t\, h_0^{cd}K_{0,db}]e^{F(t)}. \tag{6.35}$$

We conclude that also for a non-zero cosmological constant we are able to solve the LO evolution analytically.

6.3.3 On the existence of oscillatory solutions to the LO theory

In the previous two subsections, we have seen that the possible time evolution in the LO vacuum theory is very limited. That is, we can always choose coordinates such that solutions can be brought on the form (6.28) or (6.35) depending on the presence of a cosmological constant $\tilde{\Lambda}$. An important class of physical solutions of GR is gravitational waves, and a natural question is then if the vacuum LO theory allows for solutions that oscillate in the time coordinate t? A priori, the solutions (6.28) and (6.35) do have the potential to produce such oscillatory solutions if the matrix in the exponent has any imaginary eigenvalues.

To determine whether $h_0^{ac}\dot{h}_{0,cb}$ has any imaginary eigenvalues, we temporally switch to matrix notation according to

$$h_{0,ab} \to h_0, \qquad h_0^{ab} \to h_0^{-1}, \qquad \dot{h}_{0,ab} \to \dot{h}_0. \tag{6.36}$$

The matrices $h_0, h_0^{-1}, \dot{h}_0$ are all symmetric and h_0, h_0^{-1} are positive definite. Due to $h_0 > 0$, we have a unique positive square root $h_0^{1/2}$ satisfying $h_0 = h_0^{1/2}h_0^{1/2}$ and likewise for h_0^{-1}. With these properties in mind, we can rewrite the matrix of interest as

$$h_0^{-1}\dot{h}_0 = h_0^{-1/2}(h_0^{-1/2}\dot{h}_0 h_0^{-1/2})h_0^{1/2}. \tag{6.37}$$

The rewriting (6.37) shows that $h_0^{-1}\dot{h}_0$ is similar to the matrix $h_0^{-1/2}\dot{h}_0 h_0^{-1/2}$, and thus they have the same eigenvalues. Furthermore, it is easily seen that $h_0^{-1/2}\dot{h}_0 h_0^{-1/2}$ is symmetric and hence by the spectral theorem has real eigenvalues.

The above linear algebra argument demonstrates that it is not possible to get an oscillatory behavior from the matrix exponentials of (6.28) and (6.35). The solution for non-zero cosmological constant (6.28) also contains the function $F(t)$. However, $F(t)$ is readily seen to be a monotone function from its definition (6.34). Hence, we must conclude that the solutions (6.28) and (6.35) of the vacuum theory do not contain any oscillatory solutions.

6.4 Initial data for Carroll gravity

The Carroll constraint equations (6.12a) and (6.12b) are very similar to those of the 3+1 decomposition. In particular, it turns out that (6.12b) takes the exact same form as the momentum constraint (5.26b), while (6.12a) is a simplified version of the Hamiltonian constraint (5.26a), as the 3-Ricci scalar is suppressed in the $c \to 0$ limit cf. (5.55a). Thus, it is interesting to ask if one can adapt the methods for creating initial data in 3+1 decomposition outlined in section 5.2. This is indeed the case, and we will see that the Bowen-York type solutions can be analytically solved for the 3-metric in LO theory as opposed to the relativistic case (section 5.2.2) where one still has to solve a non-linear PDE for ψ.

In this section, we will again be working in adapted coordinates (6.18). However, for the 3+1 formulation methods to make sense, we further need to assume a foliation such that we can prepare the initial data on a specific time slice. That is, in these coordinates we can spatially project both co-variant and contra-variant tensors by restricting the index to $a, b, \ldots = 1, \ldots d$. In this section, the particular forms of the timelike vielbeine v^μ, τ_μ are of no direct interest, as they govern time evolution and define the spatial vector subspace, respectively. Also, the methods described in this section build on section 5.2. Thus, the results stated hold only for $d = 3$.

6.4.1 Conformal traceless-transverse decomposition

The conformal decomposition of the spatial Carroll objects proceeds completely analogously to section 5.2.1. Specifically, we again define conformally related 3-metric and extrinsic curvature

$$h_{ab} = \psi^4 \bar{h}_{ab}, \qquad K_{ab} = \psi^{-2}\bar{A}_{ab} + \frac{1}{3}h_{ab}K. \tag{6.38}$$

Along with the 3-metric $\bar{h}_{ab}$, we again get an associated Levi-Civita connection $\bar{\nabla}$, which inherits all the nice properties described in section 5.2.1. We want to identify the covariant derivative in equation (6.12b) with a spatial covariant derivative to match it with the momentum constraint (5.26b). This identification holds because all differences between the $\tilde{\Gamma}$-connection and the Levi-Civita connection vanish when fully spatially projected (cf. argument above (5.57)). Also, we can observe that (6.12b) is spatial i.e. annihilated by v^μ. Hence, the $\tilde{\nabla}$-derivative in (6.12b) is indeed fully spatially projected. With these facts, it is clear that we can equivalently use the spatially projected $\tilde{\nabla}$-derivative or the $\hat{\nabla}$-derivative in (6.12b).

Using the definitions and results of the 3+1 decomposition presented in section 5.2.1 (swapping $\gamma_{ab} \to h_{ab}$), we are in a position where we can write down the LO Carroll vacuum constraint equations (6.12a) and (6.12b) in a conformally decomposed form

$$(\bar{L}X_{ab} + \bar{A}^{\mathrm{TT}}_{ab})(\bar{L}X^{ab} + \bar{A}^{ab}_{\mathrm{TT}}) - \frac{2}{3}K^2\psi^{12} = 0, \tag{6.39a}$$

$$\bar{\Delta}_L X^a - \frac{2}{3}\psi^6\bar{\nabla}^a K = 0, \tag{6.39b}$$

where we further used the traceless-transverse decomposition $\bar{A}_{ab} = \bar{L}X_{ab} + \bar{A}^{\mathrm{TT}}_{ab}$. Comparing (6.39a) and (6.39b) to the conformally decomposed 3+1 constraints (5.40a) and (5.40b) in vacuum, it is again observed that the Hamiltonian constraint is simplified to (6.39a) by the vanishing of the curvature term, while the momentum constraint agrees with (6.39b). As discussed in section 6.2, we can only have solutions with non-zero extrinsic curvature to the LO theory if the mean curvature K is non-zero due to the constraint (6.12a). Thus, we assume $K \neq 0$, which allows us to solve (6.39a) for ψ in terms of the traceless part $\bar{A}_{ab}$

$$\psi = \left[\frac{3}{2K^2}(\bar{L}X_{ab} + \bar{A}^{\mathrm{TT}}_{ab})(\bar{L}X^{ab} + \bar{A}^{ab}_{\mathrm{TT}})\right]^{1/12}, \tag{6.40}$$

where indices are raised and lowered using $\bar{h}_{ab}$. Note that it is not possible to explicitly solve the relativistic counterpart of (6.40) as the equation only becomes algebraic in the UR limit. The solution for ψ can then be reinserted into (6.39b) to get a single equation for X^a. This is of course a simplification as compared to the relativistic equations, but the remaining equation for X^a is still a highly non-linear elliptic PDE, which does not admit an analytical solution.

6.4.2 Bowen-York type solutions

The equation (6.39b) does, however, become tractable if we consider a solution with h_{ab} conformally related to the flat metric and $K = \text{const.}$ such that (6.39b) completely decouples from ψ. This is of course very similar to the setup for the Bowen-York initial data (5.43), but there is a conceptual difference in the interpretation of choosing the value of K. In the 3+1 formalism, $K = 0$ is a gauge choice, that is we can always get $K = 0$ by choosing a maximally slicing foliation [48]. On the other hand, K in the Carroll theory is gauge invariant and consequently observable. Thus, fixing its value specializes our survey to a subset of solutions.

Having this in mind, we can mimic the free data of the Bowen-York relativistic solutions, but with a non-zero constant K

$$\bar{h}_{ab} = \delta_{ab}, \qquad \bar{A}^{\mathrm{TT}}_{ab} = 0, \qquad K = K_0, \tag{6.41}$$

With these choices, the equation (6.39b) becomes the same equation as in the relativistic case (5.45) i.e.

$$\partial^b\partial_b X^a + \frac{1}{3}\partial^a\partial_b X^b = 0. \tag{6.42}$$

Because the equations are the same as in the 3+1 decomposition, we can reuse the Bowen-York family of solutions (5.46). Furthermore, we can compute the traceless part of the conformally related extrinsic curvature acting with the conformal Killing operator (5.39) on the Bowen-York solution X^a (5.46)

$$\bar{L}X^{ab} = \frac{3}{2r^3}\left[x^a P^b + x^b P^a - \left(\delta^{ab} - \frac{x^a x^b}{r^2}\right) P_c x^c\right] + \frac{3}{r^5}\left[\epsilon^{ac}{}_d J_c x^d x^b + \epsilon^{bc}{}_d J_c x^d x^a\right], \tag{6.43}$$

with $r = \sqrt{\delta_{ab} x^a x^b}$. Unlike in the relativistic problem, this immediately gives the solution for ψ through (6.40), which completes the initial data:

$$\psi = \left[\frac{3}{2K^2}\bar{L}X_{ab}\bar{L}X^{ab}\right]^{1/12}, \qquad h_{0,ab} = \psi^4\delta_{ab}, \qquad K_{0,ab} = \psi^{-2}\bar{L}X_{ab} + \frac{1}{3}\psi^4\delta_{ab}. \tag{6.44}$$

As we have shown that the LO evolution equations (6.12c) can be solved in terms of matrix exponentials (6.28), we can use this result to write down a family of full spacetime solutions corresponding to the Bowen-York type initial data. In particular, the function H in (6.31) takes the form

$$H(t) = -2t\, h_0^{ab} K_{0,ab} = -2K_0 t, \tag{6.45}$$

where $\bar{A}_{ab}$ does not contribute because it is traceless. We can then write out the full time-dependent solution using (6.28) and (6.18)

$$v^\mu = e^{-\frac{H}{2}}\partial_t = e^{K_0 t}\partial_t, \tag{6.46a}$$

$$h_{ab}(t) = h_{0,ac}\exp[-2t\, h_0^{cd} K_{0,db}] = \psi^4\delta_{ac}\exp\left[-2t\,\psi^{-6}\bar{L}X^c{}_b\right] e^{-\frac{2t}{3}K_0}, \tag{6.46b}$$

where the indices of $\bar{L}X^{ab}$ are lowered using δ^{ab}. Thus, for a given set of parameters (P_a, J_a), one can construct a Bowen-York type solution to the LO vacuum theory using (6.46a) and (6.46b). The above construction of Bowen-York type solutions exemplifies the simplification that the LO equations present: In the relativistic problem, neither the Hamiltonian constraint nor the evolution equation can be solved, while for the LO analog we can solve the entire set of equations.

Finally, we can consider (6.44) to conclude that $\psi^{-6} \sim (\bar{L}A^{ab})^{-1}$, which implies that the exponent of (6.46b) is of order r^0 for large r. Hence, the entire scaling for large r comes from the ψ^4 factor in (6.46b), which falls off as either $r^{-4/3}$ or r^{-2} depending on the parameters, as can be deduced from (6.43) and (6.44). In particular, this implies that the spatial metric h_{ab} vanishes at spatial infinity, as opposed to being some sort of asymptotically flat Carroll geometry.

6.5 Comparison to Dautcourt

In the paper [32], Dautcourt performs an ultra-relativistic limit of GR from the EOM i.e. the Einstein equation. In doing this, Dautcourt chooses adapted coordinates corresponding to (6.18) along with a choice of spatial subspace, as in section 6.4. We have already derived the evolution equation in these coordinates (6.21). To make the connection to Dautcourt's notation, we also rewrite the constraint equations (6.12a)-(6.12b) in adapted coordinates using the form (6.19) of the extrinsic curvature and set $\tilde{\Lambda} = 0$. In particular, we find the following set of equations corresponding to (6.12a), (6.12b) and (6.12c), respectively

$$\dot{h}_{ab}\dot{h}^{ab} + (h^{ab}\dot{h}_{ab})^2 = 0, \tag{6.47a}$$

$$h^{bc}\dot{h}_{bc|a} - h^{bc}\dot{h}_{ba|c} - \frac{1}{2}H_{,a}h^{bc}\dot{h}_{bc} + \underbrace{\frac{1}{2}H_{,b}h^{bc}\dot{h}_{ca}} = 0, \tag{6.47b}$$

$$\ddot{h}_{ab} + \frac{1}{2}\dot{h}_{ab}(h^{cd}\dot{h}_{cd} - \dot{H}) - \dot{h}_{ac}h^{cd}\dot{h}_{db} = 0. \tag{6.47c}$$

Note that here comma denotes partial derivative and stroke denotes the Levi-Civita covariant derivative associated with h_{ab}. The equations (6.47a)-(6.47c) are to be compared with (25)-(27) of [32]. The EOM (6.47a) and (6.47c) are in agreement with the findings of Dautcourt. However, the equation (6.47b) differs by the term marked with a bracket as compared to the one presented in [32]. This missing term leads Dautcourt to conclude that to obtain a consistent set of equations one must impose an additional constraint. This is in contrast to the EOM (6.47a)-(6.47c), as we have shown their consistency in section 6.2.1. In the paper [32], Dautcourt further considers dust matter for which the analysis also needs to be altered to take into account the extra term in (6.47b).

6.6 Boundary charges

A convenient way of characterizing solutions to gravitational theories are boundary charges that e.g. in GR, allows one to ascribe a mass and angular momentum to a black hole as seen by a distant observer. One can also use the existence of non-zero boundary charges to conclude that solutions are not pure gauge. Thus, in this section we shall derive the boundary charges for the LO Carroll theory.

As described in appendix C, the covariant phase space formalism supplies an elegant and fully covariant way of deriving boundary charges for gravitational theories. From an operational point of view, we need to compute a number of quantities to derive the diffeomorphism charge integrand $k^{[\mu\nu]}$ corresponding to the LO action (4.44). Specifically, we have to calculate the presymplectic potential Θ^μ (C.12), the current related to Noether's second theorem S^μ (C.15), the current M^μ (C.11) and the Noether-Wald charge $Q^{[\mu\nu]}$ (C.14).

6.6.1 The Noether-Wald charge

To determine the pre-symplectic potential, we use the defining relations (C.12). That is, we need to recompute (4.45) and keep track of total derivatives

$$\begin{aligned}\delta\overset{(2)}{\mathcal{L}}_{\mathrm{LO}} &\equiv \frac{e}{8\pi G_N}\left[\overset{(2)}{G}{}^{v}_{\mu}\delta v^\mu + \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu}\delta h^{\mu\nu}\right] + \partial_\mu\Theta^\mu \\ &= (\ldots)_\mu\delta v^\mu + (\ldots)_{\mu\nu}\delta h^{\mu\nu} + \frac{e}{8\pi G_N}\left[(Kh^\nu_\mu - K^\nu{}_\mu)\tilde{\nabla}_\nu\delta v^\mu - \frac{1}{2}(Kh_{\mu\nu} - K_{\mu\nu})v^\rho\tilde{\nabla}_\rho\delta h^{\mu\nu}\right],\end{aligned} \tag{6.48}$$

where the ellipses $(\ldots)$ signify terms with no derivatives of δv^μ or $\delta h^{\mu\nu}$. We then use the integration-by-parts identity (3.28) to write the last term as a total derivative and identify

$$\Theta^\mu = \frac{e}{8\pi G_N}\left[(Kh^\mu_\nu - K^\mu{}_\nu)\delta v^\nu - \frac{1}{2}(Kh_{\sigma\rho} - K_{\sigma\rho})v^\mu\delta h^{\sigma\rho}\right]. \tag{6.49}$$

We also compute the pre-potential evaluated on a diffeomorphism ξ^μ i.e. $\delta v^\mu = \mathcal{L}_\xi v^\mu$ and $\delta h^{\mu\nu} = \mathcal{L}_\xi h^{\mu\nu}$

$$\begin{aligned}\Theta^\mu_\xi =& \frac{e}{8\pi G_N}\left[(K^\mu{}_\rho K_{\rho\sigma} - KK^\mu{}_\sigma)\xi^\sigma + (K^2 - K^{\rho\sigma}K_{\rho\sigma})v^\mu\tau_\nu\xi^\nu\right] \\ &+ \frac{e}{8\pi G_N}\left[2Kv^{[\mu}\tilde{\nabla}_\nu\xi^{\nu]} + 2v^{[\nu}K^{\mu]}{}_\rho\tilde{\nabla}_\nu\xi^\rho\right].\end{aligned} \tag{6.50}$$

To derive the S^μ_ξ current, we need to determine the total derivative (C.15) arising when computing the diffeomorphism Ward/Noether identity. Hence, we repeat the steps that lead to (6.2)

$$\begin{aligned}\frac{1}{8\pi G_N}\left[\overset{(2)}{G}{}^v_\mu\mathcal{L}_\xi v^\mu + \frac{1}{2}\overset{(2)}{G}{}^h_{\mu\nu}\mathcal{L}_\xi h^{\mu\nu}\right] &\equiv (\ldots)_\mu\xi^\mu + +\partial_\mu S^\mu_\xi \\ = (\ldots)_\mu\xi^\mu + \frac{e}{8\pi G_N}\tilde{\nabla}_\mu&\left[\frac{1}{2}(K^2 + K^{\rho\sigma}K_{\rho\sigma})\xi^\mu + (K^{\rho\sigma}K_{\rho\sigma}\tau_\nu v^\mu - KK^\mu{}_\nu)\xi^\nu\right] \\ + \frac{e}{8\pi G_N}\tilde{\nabla}_\mu&\left[2v^{[\nu}h^{\mu]\sigma}\xi^\rho\tilde{\nabla}_\nu K_{\sigma\rho} + 2v^{[\mu}\xi^{\nu]}\tilde{\nabla}_\nu K\right],\end{aligned} \tag{6.51}$$

where the ellipses $(\ldots)_\mu$ denote the terms with no derivatives of ξ^μ i.e. the Ward identity (6.2). Again using the identity (3.28), we can read off the current

$$\begin{aligned}S^\mu_\xi =& \frac{e}{8\pi G_N}\left[\frac{1}{2}(K^2 + K^{\rho\sigma}K_{\rho\sigma})\xi^\mu + (K^{\rho\sigma}K_{\rho\sigma}\tau_\nu v^\mu - KK^\mu{}_\nu)\xi^\nu\right] \\ &+ \frac{e}{8\pi G_N}\left[2v^{[\nu}h^{\mu]\sigma}\xi^\rho\tilde{\nabla}_\nu K_{\sigma\rho} + 2v^{[\mu}\xi^{\nu]}\tilde{\nabla}_\nu K\right].\end{aligned} \tag{6.52}$$

Finally, we calculate the remaining current $M_\xi = \xi \lrcorner \mathcal{L}$

$$M^\mu_\xi = \xi^\mu\overset{(2)}{\mathcal{L}}_{\text{LO}} = \frac{e}{16\pi G_N}[K^{\mu\nu}K_{\mu\nu} - K^2]\xi^\mu, \tag{6.53}$$

where we used the rule (A.8). This puts us in a position to calculate the Noether-Wald charge (C.14) as

$$\begin{aligned}\partial_\nu Q^{[\mu\nu]}_\xi &\equiv M^\mu_\xi - \Theta^\mu_\xi - S^\mu_\xi \\ &= \frac{e}{8\pi G_N}\left[2KK^\mu{}_\nu\xi^\nu - K^\mu{}_\rho K^\rho{}_\nu - K^2h^\mu_\nu\xi^\nu\right] \\ &\quad - \frac{e}{4\pi G_N}\left[v^{[\nu}h^{\mu]\sigma}\xi^\rho\tilde{\nabla}_\nu K_{\sigma\rho} + v^{[\mu}\xi^{\nu]}\tilde{\nabla}_\nu K + Kv^{[\mu}\tilde{\nabla}_\nu\xi^{\nu]} + v^{[\nu}K^{\mu]}{}_\rho\tilde{\nabla}_\nu\xi^\rho\right].\end{aligned} \tag{6.54}$$

Equation (6.54) does not look like an exterior derivative, but using the derivatives of ξ^μ as a guide and the formula (3.29) we can identify

$$Q^{[\mu\nu]} = \frac{e}{4\pi G_N}(v^{[\mu}K^{\nu]}{}_\sigma\xi^\sigma - v^{[\mu}\xi^{\nu]}K), \tag{6.55}$$

which completes the derivation of the Noether-Wald charge.

6.6.2 Charge integrand for the LO theory

We have now derived all the necessary components to write down the charge integrand, which is given by (C.27)

$$k^{[\mu\nu]}_\xi = -\delta_h Q^{\mu\nu} + 2\xi^{[\mu}\Theta^{\nu]}_h, \tag{6.56}$$

where $\delta_h Q$ is an on-shell variation of (6.55) and Θ^μ_h is (6.49) evaluated with the same on-shell variation. This allows us to define a charge

$$\oint H_\xi = \int_S k^{\mu\nu} (d^{d-1}x)_{\mu\nu}, \tag{6.57}$$

where S is some co-dimension 2 surface and the differential $(d^{d-1}x)_{\mu\nu}$ is defined in (A.5). We will take S to be a sphere at infinity as that is typically the only natural surface in the problem. The notation $\oint$ serves to remind us that the charge is not necessarily integrable. The charge H_ξ is conserved by the fundamental theorem of covariant phase space formalism (C.22) if

$$\delta v^\mu = \mathcal{L}_\xi v^\mu = 0, \qquad \delta h^{\mu\nu} = \mathcal{L}_\xi h^{\mu\nu} = 0, \tag{6.58}$$

such that the pre-symplectic form vanishes. Considering the results on Carrollian Killing vectors of section 2.4.3, one would expect $\mathcal{L}_\xi h_{\mu\nu} = 0$ to be a sufficient condition for conservation. This is indeed the case, which can be seen by the following computation

$$\begin{aligned}\mathcal{L}_\xi h^{\mu\nu} &= h^{\mu\sigma} h_{\sigma\rho} \mathcal{L}_\xi h^{\rho\nu} - v^\mu \tau_\rho \mathcal{L}_\xi h^{\rho\nu} = -h^{\mu\sigma} h^{\nu\rho} \mathcal{L}_\xi h_{\sigma\rho} + 2v^{(\mu} h^{\nu)\rho} \mathcal{L}_\xi \tau_\rho \\ &= -h^{\mu\sigma} h^{\nu\rho} \mathcal{L}_\xi h_{\sigma\rho} + 2v^{(\mu} \lambda^{\nu)},\end{aligned} \tag{6.59}$$

where we defined $\lambda^\mu = h^{\mu\nu} \mathcal{L}_\xi \tau_\nu$. Equation (6.59) shows that if $\mathcal{L}_\xi h_{\mu\nu} = 0$ then $\mathcal{L}_\xi h^{\mu\nu}$ is only a gauge transformation (2.48d), which should be in the kernel of the pre-symplectic form. One can show by direct computation that this is the case, and consequently H_ξ is a conserved charge when

$$\mathcal{L}_\xi v^\mu = 0, \qquad \mathcal{L}_\xi h_{\mu\nu} = 0, \tag{6.60}$$

which coincides with the definition of Carrollian Killing vectors (2.102a)-(2.102b). One can also partially relax the criteria (6.60) to include so-called asymptotic symmetries such that the Carrollian Killing equations only have to vanish in an asymptotic sense at spatial infinity. For asymptotic symmetries it is important that the surface S, which we integrate over in (6.57), is chosen to be at infinity.

As the Carroll geometry automatically supplies us with the vector field v^μ, one may wonder if we can have an associated charge H_v. For the charge to be conserved, we need to have (6.60) satisfied: $\mathcal{L}_v v^\mu$ is identically zero, while the second condition translates to $K_{\mu\nu} = 0$. However, all terms in the charge integrand $k^{\mu\nu}_\xi$ are proportional to $K_{\mu\nu}$, and we thus conclude that we can not have a non-zero conserved charge associated with v^μ. This can be interpreted as the absence of a charge associated with time translation. In a relativistic theory, the time translation charge would be contributed to mass or energy. Thus, we conclude that the LO theory does not contain a notion of Carrollian mass or energy. As we will see in section 6.7.2, we can on the other hand have Carrollian versions of both linear and angular momentum.

6.7 Examples of Carroll spacetimes

We will in this section consider several solutions to the LO vacuum theory using the methods developed in the previous sections.

6.7.1 Ultra-relativistic Schwarzschild black hole

In this chapter, we have presented methods to directly solve the LO EOM. However, maybe the simplest way to obtain solutions to the Carroll theory to any order is by expanding a solution of the full Einstein equations. If we can write a GR vacuum solution in the PUR decomposed form (4.18a)-(4.18b), such that the PUR variables start at order c^0 and are analytic in c^2, then the PUR variables can be expanded in c^2 to obtain the Carrollian data according to the expansions (4.10a) and (4.15b).

To obtain a simple example of a solution to LO theory, we can consider the relativistic Schwarzschild metric with factors of c reinstated

$$ds^2 = -c^2\left(1 - \frac{2G_N m}{c^2 r}\right)dt^2 + \left(1 - \frac{2G_N m}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{6.61}$$

where $d\Omega^2$ is the metric on the round 2-sphere. The metric (6.61) does not immediately admit an expansion, because what is to be identified with T_μ and $\Pi_{\mu\nu}$ is not analytic in c^2. The expansion, in the sense of (4.18a), does however exist if we take the $c \to 0$ limit with $M = mc^{-2}$ kept constant. This can be understood as keeping the position of the "event horizon" fixed rather than sending it to infinity. Substituting the mass, we see that the expansion terminates after LO, leaving only

$$v^\mu \partial_\mu = \left(1 - \frac{2G_N M}{r}\right)^{-1/2} \partial_t, \tag{6.62a}$$

$$h_{\mu\nu} dx^\mu \otimes dx^\nu = \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{6.62b}$$

From this, it is easily seen that $K_{\mu\nu} = 0$, and consequently all the LO EOM are trivially satisfied. Furthermore, as the expansion terminates after LO, the partial NLO EOM (4.64a)-(4.64b) should be satisfied. This is indeed the case and is easily observed once one calculates that

$$\tilde{R}_{\mu\nu} = 0, \qquad \tilde{\nabla}_\rho(h^{\rho\nu}\tau_{\mu\nu}) = 0, \qquad h^{\mu\rho}h^{\nu\sigma}\tau_{\rho\sigma} = 0, \tag{6.63}$$

which follows by direct calculation from (6.62a) and (6.62b).

It is interesting to compare this with the non-relativistic limit $c^{-1} \to 0$ as considered in [29, 30]. In the NR limit, the original scaling, m kept constant in c, realizes a weak-field limit because the singular structure is suppressed by factors of c^{-2} i.e. the event horizon is pulled in to $r = 0$ rather than pushed to $r = \infty$. Using the same scaling as above, that is $M = mc^{-2}$ constant, has the opposite effect in the NR limit and can be understood as a strong field limit.

One might then also call the Carrollian limit above a strong field limit, as it fixes the position of the singular structure. Additionally, we can also get a Carrollian weak field limit by further suppressing the mass by keeping $M' = mc^{-4}$ constant which at LO would be the flat Carrollian structure (2.92).

6.7.2 Bowen-York type solutions

As an example of the Bowen-York type solutions described in section 6.4.2, we here work out a simple example. Specifically, we can consider the parameters $P_a = (0, 0, 0)$ and $J_a = (0, 0, s)$ for which the matrix exponential of (6.46b) can be explicitly computed. As this choice of parameters singles out the z-axis, it is convenient to go to standard spherical coordinate (t, r, θ, ϕ) in which the solution (6.46b) takes the form

$$v^\mu = e^{K_0 t}\delta^\mu_t, \tag{6.64a}$$

$$h_{\mu\nu} = 3e^{-\frac{2K_0 t}{3}}\left(\frac{s\sin\theta}{K_0}\right)^{2/3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\cosh\frac{2K_0 t}{\sqrt{3}}}{r^2} & 0 & -\frac{\sin\theta\sinh\frac{2K_0 t}{\sqrt{3}}}{r} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{\sin\theta\sinh\frac{2K_0 t}{\sqrt{3}}}{r} & 0 & \sin^2\theta\cosh\frac{2K_0 t}{\sqrt{3}} \end{bmatrix}. \tag{6.64b}$$

The metric data (6.64a)-(6.64b) can be checked to satisfy the LO EOM (6.12a), (6.12b) and (6.11).

Based on the interpretation of the relativistic Bowen-York solutions, we expect (6.64a)-(6.64b) to carry some kind of angular momentum about the z-axis. The solution does indeed possess a Carrollian Killing vector generating the rotation around the z-axis, that is

$$\mathcal{L}_{-\partial_\phi} v^\mu = 0, \qquad \mathcal{L}_{-\partial_\phi} h_{\mu\nu} = 0, \tag{6.65}$$

where the sign is a conventional choice. As this is the conservation criteria for (6.60), we can go ahead and compute the charge integrand (6.56) associated with the Carrollian Killing vector $-\partial_\phi$

$$k_{-\partial_\phi} = \delta s \frac{3\sin^3\theta}{8\pi G_N} d\theta \wedge d\phi. \tag{6.66}$$

Further, we can integrate over a 2-sphere at infinity to obtain the variation of the boundary charge

$$\not{\delta} H_{-\partial_\phi} = \int_{S^2} k_{\partial_\phi} = \frac{\delta s}{G_N}. \tag{6.67}$$

The charge (6.67) is clearly integrable yielding

$$H_{-\partial_\phi} = \frac{s}{G_N}, \tag{6.68}$$

which is conserved in time. This computation shows that the parameter s can be interpreted analogously to the relativistic case in that it corresponds to some notion of Carrollian angular momentum.

One can equivalently consider a solution with $P_a \neq 0$ in which case one would expect to find a linear momentum charge. To compute such a charge, one needs to consider the Cartesian basis vectors $(\partial_x, \partial_y, \partial_z)$ as the generators of translations. These are only Carrollian Killing vectors in the asymptotic sense and their Lie derivatives (6.60) do not vanish in the bulk, but only at the boundary. However, if one integrates the charge integrand over a sphere at infinity, one does indeed recover the parameters $\frac{P_a}{G_N}$ as the asymptotic symmetry charges associated with $-\partial_a$.

6.7.3 Positive cosmological constant

As a further example of the solvability of LO theory, we will now consider time evolution with a positive cosmological constant of spatially isotropic and homogeneous initial data. Specifically, we choose Cartesian coordinates and the initial metric and extrinsic curvature to be

$$h_{0,ab} = \delta_{ab}, \qquad K_{0,ab} = -h\delta_{ab}, \tag{6.69}$$

where δ_{ab} is the flat metric. We can check that the initial data is consistent by calculating

$$K_0^{ab}K_{0,ab} - K_0^2 = -2\frac{d(d-1)}{2}h^2, \tag{6.70}$$

which shows by comparison to (6.12a) that (6.69) corresponds to $\tilde{\Lambda} = \frac{d(d-1)}{2}h^2 > 0$. The second constraint (6.12b) is trivially satisfied, as the spatial covariant derivative coincides with the partial derivative for (6.69) and hence annihilates the constant components. Note that we could not have prepared initial data for a negative cosmological constant analogous to (6.69), because the RHS of (6.12a) is always negative when $h_{0,ab}$ and $K_{0,ab}$ are proportional to δ_{ab}.

As the initial data (6.69) is valid, we can go on and apply the methods of section 6.3.2 to obtain the time evolution. First, we seek the solution to the ODE for H (6.33), which works out to be

$$H(t) = -2\log\left[1 - dht\right]. \tag{6.71}$$

With the lapse function $H(t)$, we can go on to compute the auxiliary function $F(t)$ using (6.34)

$$F(t) = -2\frac{dht + \log(1-dht)}{d}. \tag{6.72}$$

Having determined both $H(t)$ and $F(t)$, we can write down the time evolved solutions using (6.18) and (6.35)

$$v = (1-dht)\partial_t, \qquad h_{ab}(t) = \frac{1}{(1-dht)^{2/d}}\delta_{ab}. \tag{6.73}$$

The solution (6.73) corresponds to a spatially isotropic and homogeneous space with a positive cosmological constant. Thus, one may wonder if this is connected to a limit of de Sitter space. The answer is affirmative, and this can be seen if we reparameterize time such that $v = \partial_{t'}$. In particular, we need to find a new time coordinate t' that satisfies

$$\frac{dt'}{dt} = e^{H(t)/2}, \qquad t'(t=0) = 0, \tag{6.74}$$

where the initial condition is an arbitrary choice. The ODE (6.74) can easily be solved to find

$$t = \frac{1-e^{-dht'}}{dh}. \tag{6.75}$$

Finally, we can transform the solution (6.73) into these coordinates where it takes the simple form

$$v = \partial_t, \qquad h_{ab} = e^{2ht}\delta_{ab}, \tag{6.76}$$

and we dropped the primes on t'. This is exactly the Carrollian structure we would have found if we considered the de Sitter metric in planar coordinates

$$ds^2 = -c^2dt^2 + e^{2ht}dx^idx^i, \tag{6.77}$$

where we have chosen the cosmological constant to scale as $\Lambda = \frac{d(d-1)}{2}\frac{h^2}{c^2}$. Having this connection to a maximally symmetric pseudo-Riemannian space in mind, it is interesting to investigate the

Carrollian Killing vectors of (6.76). That is, we seek solutions to (2.102a)-(2.102b), which result in the following Carrollian Killing vectors

$$
\begin{aligned}
P_a &= \partial_a, && (6.78a)\\
J_{ab} &= x_b\partial_a - x_a\partial_b, && (6.78b)\\
H &= \partial_0 - hx^a\partial_a, && (6.78c)\\
C_a &= x_a\partial_0 - hx_a x^b\partial_b + \tfrac{h}{2}x^b x_b\partial_a, && (6.78d)
\end{aligned}
$$

where indices are raised and lowered using δ_{ab}. By direct computation, one can show that the Killing vectors (6.78a)-(6.78d) satisfy the Carroll algebra (2.5a)-(2.5c) in addition to the following non-zero Lie brackets

$$
\begin{aligned}
[P_a, H] &= -hP_a, && (6.79a)\\
[C_a, H] &= hC_a, && (6.79b)\\
[P_a, C_b] &= \delta_{ab}H + hJ_{ab}, && (6.79c)
\end{aligned}
$$

where (6.79c) replaces (2.5d). These non-zero commutators parameterized by the cosmological constant h (up to a multiplicative constant) look familiar to the de Sitter algebra. One can indeed make a Carrollian contraction of the relativistic de Sitter algebra [52] analogous to the procedure in section 2.1. However, the resulting Carrollian de Sitter algebra is distinct from the algebra (6.79a)-(6.79c). This may be related to the fact that equivalent Lorentzian spacetimes can become inequivalent in the $c \to 0$ limit if the coordinate transform is not analytic in c^2. In particular, the planar embedding coordinates for relativistic de Sitter space are not analytic in the cosmological constant Λ^{-1}, which implies non-analyticity in c^2 due to $\Lambda^{-1} \sim c^2$.

The algebra (6.79a)-(6.79c) does appear in the classification [53] of spatially isotropic homogeneous spacetimes, cf. LP# 17 in table 5. Hence we can rightfully call the data (6.76) a spatially isotropic homogeneous Carrollian manifold.

Chapter 7

Conclusion and Outlook

In this thesis, we have studied Carrollian geometry and field theories from a number of perspectives. First, we reviewed the intrinsic construction of Carroll geometry from a gauging procedure along with its emergence on null hypersurfaces in Lorentzian theories. We then analyzed general aspects of classical Carrollian field theories through a fully covariant treatment of Carrollian energy-momentum tensors and their relation to the canonical energy-momentum tensor and Carrollian Killing vectors. We further reviewed the notion of intrinsic torsion, which allowed us to advocate for a, in some appropriate sense, natural minimal torsion connection on Carroll structures. We then carried out a novel small c expansion of the Einstein-Hilbert action in powers of c^2 to obtain a LO and NLO action principle for Carrollian gravity. Upon reviewing the relativistic 3+1 decomposition, we established a new link between ultra- and non-relativistic expansions of GR and the 3+1 formalism. Furthermore, we performed an original analysis of the LO theory and demonstrated that the LO EOM realizes an ultra-local causal structure by rendering the evolution equation of the theory an ODE. The simplifying causal structure additionally allowed us to explore methods for analytically solving the LO theory. In particular, we showed that the time evolution of the vacuum LO EOM is completely solvable in terms of matrix exponentials. We then derived the boundary charges associated with diffeomorphism symmetries for the LO action, which showed the absence of a notion of mass in terms of a time translation charge. Finally, we presented a few solutions showcasing the solvability of the LO theory and demonstrating the existence of linear and angular momentum in Carrollian gravity.

7.1 Outlook

One apparent direction to pursue is completing the NLO EOM (4.64b)-(4.64a) to include the sub-leading fields M^μ and $\Phi^{\mu\nu}$ in some tractable form. This can be done either in general by varying the NLO action (4.57) or through the 3+1 decomposition, as described in section 5.3. However, one needs to be aware that in the 3+1 approach the NLO Frobenius condition (5.50) has to be taken into account.

The Galilean expansion performed in [29] uses the same action expansion methods employed in chapter 4, hence it would be interesting to apply 3+1 methods in the large c limit. In particular, the authors of [29] already impose the LO Frobenius condition $\tau \wedge d\tau = 0$ as it is the LO EOM in the Galilean expansion, but it may be worthwhile to examine what simplification the sub-leading Frobenius condition gives rise to. Finally, the methods for constructing solutions in the LO Carroll theory may also have utility in the next-to-next-to-leading-order (NNLO) action of the Galilean expansion as they are both order c^2. This means that the LO Carroll action studied in this thesis appears as part of the NNLO Galilean action.

Another interesting question is the role of mass and energy in the Carrollian expansion of the EH action. The analysis of section 6.6 shows that to leading order in c^2, it is not possible to attribute a non-zero charge to time translation. Thus, a concept of mass presumably enters at sub-leading order in the action expansion. In particular, it would be interesting to carry out the analysis of boundary charges for the NLO action (4.57) to determine whether concepts of mass and energy arise at NLO.

It would also be helpful to fully understand the connection to the first-order formalism approach in the previous work of Bergshoeff et al. [31]. In the Carrollian theory of gravity presented in the paper by Bergshoeff et al., the EOM imply that $K_{\mu\nu} = 0$, and hence it is clearly a distinct theory from the one presented in this thesis. This is presumably due to another choice of scaling of the spin connection, which appears natural in the first-order formulation. Nevertheless, it would be interesting to understand under which conditions the two expansions can be identified. For a Galilean first-order expansion approach equivalent to the methods of [29], which the methods in this thesis is built on, see [54].

The LO vacuum theory allows for the time evolution to be solved analytically and non-trivial initial data to be constructed in closed form in terms of the Bowen-York type solutions. This suggests that the LO Carrollian theory might be an interesting point to perturb around in GR. The utility of such a "post-ultra-relativistic" expansion crucially depends on the complexity of the lowest-order perturbation theory, i.e. the NLO theory. This would of course require that we have the full NLO EOM, and hence we can only speculate on their complexity. Under the assumption that a tractable perturbation theory can be developed, Bowen-York type solutions would be interesting to perturb around because their relativistic counterpart in numerical relativity is a common method for constructing binary black hole initial data. Specifically, the linearity of the constraint equation (6.42) allows for superposition. Thus, one can readily construct LO Carroll solutions with multiple "Carrollian black holes" (recall that there is no concept of mass at LO). One major conceptual issue is that the Carrollian causality of the LO theory forces everything to be stationary, which does not seem easily reconcilable with the dynamics expected from e.g. the scattering of black holes. Nevertheless, a post-ultra-relativistic expansion would be an interesting direction to further explore the ultra-relativistic limit of GR.

Appendix A

Differential Forms and Duality

From the variation of actions on a n-manifold, we often encounter expressions like

$$\omega = \partial_\mu X^\mu \, d^n x = \frac{1}{n!}\partial_\mu X^\mu \epsilon_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}, \tag{A.1}$$

where ϵ is the Levi-Civita tensor density taking values -1, 0, 1. The object ω is a n-form, but it is more efficiently represented by $\partial_\mu X^\mu$. Further, we can conclude that $\partial_\mu X^\mu$ is scalar density of weight -1 in order for ω to be a differential form. This suggests a construction much like Hodge duality, but rather than relating p-forms to $(n-p)$-forms, we want to dualize p-forms to totally anti-symmetric $(n-p)$-fold contravariant tensor densities. This construction does not rely on a metric, only a top form. Hence this duality also applies to non-Lorentzian structures unlike the Hodge duality.

To set up the formulae, we first remind ourselves of the identity

$$\epsilon_{\mu_1 \ldots \mu_p \nu_{p+1} \ldots \nu_n} \epsilon^{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_n} = p!(n-p)! \delta^{[\mu_{p+1}}_{\nu_{p+1}} \ldots \delta^{\mu_n]}_{\nu_n}. \tag{A.2}$$

Using (A.2), we can easily see that any $(n-p)$-form can be written as

$$\omega_{\mu_{p+1} \ldots \mu_n} = \frac{1}{p!} X^{[\mu_1 \ldots \mu_p]} \epsilon_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_n}, \tag{A.3a}$$

$$X^{[\mu_1 \ldots \mu_p]} = \frac{1}{(n-p)!} \omega_{\mu_{p+1} \ldots \mu_n} \epsilon^{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_n}. \tag{A.3b}$$

We can also write out in an abstract form

$$\omega = X^{\mu_1 \ldots \mu_p} (d^{n-p} x)_{\mu_1 \ldots \mu_p}, \tag{A.4}$$

where we defined

$$(d^{n-p} x)_{\mu_1 \ldots \mu_p} = \frac{1}{p!(n-p)!} \epsilon_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_n} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}. \tag{A.5}$$

Note that if we combine (A.3a) and (A.4), then we get something that is consistent with usual convention

$$\omega = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \qquad \omega \in \Omega^p(M). \tag{A.6}$$

Another common operation we need is the exterior derivative, and how it acts in the dual formulation. By direct application of the definitions and (A.2) on the dual of $X^{\mu_1 \cdots \mu_p}$, we find

$$\begin{aligned}(dX)^{\mu_1 \ldots \mu_{p-1}} &= \frac{n-p+1}{p!(n-p+1)!} \epsilon^{\mu_1 \ldots \mu_{p-1} \mu_p \ldots \mu_n} \partial_{\mu_p} X^{\nu_1 \ldots \nu_p} \epsilon_{\nu_1 \ldots \nu_p \mu_{p+1} \ldots \mu_n} = \partial_{\mu_p} X^{\nu_1 \ldots \nu_p} \delta^{[\mu_1}_{\nu_1} \ldots \delta^{\mu_p]}_{\nu_p} \\ &= \partial_{\mu_p} X^{\mu_1 \ldots \mu_{p-1} \mu_p}.\end{aligned} \tag{A.7}$$

Likewise, we can also do interior products with a vector ξ and $X^{\mu_1\cdots\mu_p}$ representing a $(n-p)$-form

$$\begin{aligned}(\xi \lrcorner X)^{\mu_1\cdots\mu_{p+1}} &= \frac{1}{p!(n-p-1)!}\epsilon^{\mu_1\cdots\mu_{p+1}\mu_{p+2}\cdots\mu_n}\xi^{\nu_{p+1}}X^{\nu_1\cdots\nu_p}\epsilon_{\nu_1\ldots\nu_p\nu_{p+1}\nu_{p+2}\ldots\nu_n} \\ &= \frac{(p+1)!(n-p-1)!}{p!(n-p-1)!}\xi^{\nu_{p+1}}X^{\nu_1\cdots\nu_p}\delta^{[\mu_1}_{\nu_1}\ldots\delta^{\mu_{p+1}]}_{\nu_{p+1}} \\ &= (p+1)X^{[\mu_1\cdots\mu_p}\xi^{\mu_{p+1}]}. \end{aligned} \tag{A.8}$$

Curiously, but maybe not surprisingly, the operational rules of the exterior derivative and interior product switch in the dual formulation.

Appendix B

Action Expansion Generalities

This appendix describes a general framework for an ultra-relativistic expansion of a relativistic action following the approach for the non-relativistic expansion of [29]. We consider a theory with dynamical fields ϕ^I governed by the Lagrangian $\mathcal{L}(\phi^I, \partial_\mu \phi^I; \sigma)$, and we want to expand in orders of $\sigma = c^2$. We assume that the fields ϕ^I are analytical in σ and their expansions start at order σ^0 i.e.

$$\phi^I = \phi^I_{(0)} + \sigma \phi^I_{(2)} + \sigma^2 \phi^I_{(4)} + \mathcal{O}(\sigma^3). \tag{B.1}$$

The Lagrangian may have pre-leading orders of c, which we factor out

$$\mathcal{L} = c^N \tilde{\mathcal{L}}, \tag{B.2}$$

such that the Lagrangian $\tilde{\mathcal{L}}$ starts at order σ^0. We can then write down the Taylor series for $\tilde{\mathcal{L}}$ as

$$\tilde{\mathcal{L}}(\sigma) = \tilde{\mathcal{L}}(0) + \sigma \tilde{\mathcal{L}}'(0) + \frac{\sigma^2}{2} \tilde{\mathcal{L}}''(0) + \mathcal{O}(\sigma^3), \tag{B.3}$$

where the prime denotes the total derivative

$$\frac{d}{d\sigma} = \frac{\partial}{\partial \sigma} + \frac{\partial \phi^I}{\partial \sigma} \frac{\partial}{\partial \phi^I} + \frac{\partial(\partial_\mu \phi^I)}{\partial \sigma} \frac{\partial}{\partial(\partial_\mu \phi^I)}. \tag{B.4}$$

This leads to the following expansion of $\mathcal{L}$

$$\mathcal{L} = c^N \overset{(N)}{\mathcal{L}}_{\text{LO}} + c^{N+2} \overset{(N+2)}{\mathcal{L}}_{\text{NLO}} + \mathcal{O}(c^{N+4}). \tag{B.5}$$

In particular, we find that

$$\overset{(N)}{\mathcal{L}}_{\text{LO}} = \tilde{\mathcal{L}}(0) = \tilde{\mathcal{L}}(\phi^I_{(0)}, \partial_\mu \phi^I_{(0)}; \sigma = 0), \tag{B.6a}$$

$$\begin{aligned} \overset{(N+2)}{\mathcal{L}}_{\text{NLO}} = \tilde{\mathcal{L}}'(0) &= \left.\frac{\partial \tilde{\mathcal{L}}}{\partial \sigma}\right|_{\sigma=0} + \phi^I_{(2)} \frac{\partial \overset{(N)}{\mathcal{L}}_{\text{LO}}}{\partial \phi^I_{(0)}} + \partial_\mu \phi^I_{(2)} \frac{\partial \overset{(N)}{\mathcal{L}}_{\text{LO}}}{\partial(\partial_\mu \phi^I_{(0)})} \\ &\approx \left.\frac{\partial \tilde{\mathcal{L}}}{\partial \sigma}\right|_{\sigma=0} + \phi^I_{(2)} \frac{\delta \overset{(N)}{\mathcal{L}}_{\text{LO}}}{\delta \phi^I_{(0)}}, \end{aligned} \tag{B.6b}$$

where we integrated by parts to get to the last line, and $\frac{\delta}{\delta \phi^I_{(0)}}$ is the Euler-Lagrange derivative. Notice that the factor multiplying $\phi^I_{(2)}$ in (B.6b) is exactly the LO EOM. Thus, the NLO EOM of $\phi^I_{(2)}$ is the LO EOM of $\phi^I_{(0)}$. This phenomenon occurs at each level of the expansion where we introduce new fields, but at the same time we reproduce all EOM of the previous levels. Written out for the NLO level this means

$$\frac{\delta \overset{(N+2)}{\mathcal{L}}_{\text{NLO}}}{\delta \phi^I_{(2)}} = \frac{\delta \overset{(N)}{\mathcal{L}}_{\text{LO}}}{\delta \phi^I_{(0)}}. \tag{B.6c}$$

For the purposes of the this thesis, the unexpanded variables are $\phi^I = (T^\mu, \Pi^{\mu\nu})$, while the LO and NLO variables are $\phi^I_{(0)} = (\tau^\mu, h^{\mu\nu})$ and $\phi^I_{(2)} = (M^\mu, \Phi^{\mu\nu})$, respectively.

Appendix C

Covariant Phase Space Formalism

Conventionally, the Hamilitonian formalism nicely exposes the close relationship between symmetries and conserved charges [41]. However, the formalism critically depends on a privileged time, thus for relativistic theories one has to choose a non-canonical time function and break covariance. One solution to this problem is the covariant phase space formalism, which establishes the symplectic structure on the phase space manifold without giving up the manifest covariance. Importantly, it can be used to compute charges associated with symmetry transformations through the so-called fundamental theorem of covariant phase space formalism. The following review of the variational bi-complex is based on [55] and the subsequent derivation of the fundamental theorem of covariant phase space formalism is based on [56].

C.1 Phase space and jet bundles

In order to develop a consistent framework to compute symmetry charges, it is convenient to formalize the notion of varying with respect to a field. This can be done by considering a fiber bundle $E \overset{\pi_E}{\to} M$ with some spacetime n-manifold M as base and typical fiber corresponding to the target space of the dynamical fields. The idea is then to extend this to a so-called jet bundle, which one then equips with two different exterior derivatives: d which acts as usual and δ which formalizes what is meant by a variation. The following briefly explains the most important objects and calculation rules:

- *Jet bundle:* The idea of a jet bundle is to extend the fiber bundle $E \overset{\pi_E}{\to} M$ to a fiber bundle $J^\infty(E) \overset{\pi^\infty}{\to} M$ with typical fibers being not only the target space but also all its derivatives. Thus, for a scalar field ϕ, a local coordinate chart looks like $(x^\mu, \phi, \phi_\mu, \phi_{\mu\nu}, \ldots)$, where the subscripts indicate partial derivatives, which of course are totally symmetric.

- *Contact structure:* One can define the map $j^\infty : \Gamma(E) \to \Gamma(J^\infty(E))$ that maps sections of E to their corresponding jet i.e. all its derivatives. Forms that satisfy

$$[j^\infty(s)]^*\omega = 0, \tag{C.1}$$

  for all sections s are known as contact forms. These can be thought of as forms measuring the change in the field that does not stem from moving along the spacetime manifold. In local coordinates for the scalar ϕ, the space of contact forms are spanned by

$$\theta_{\mu_1\ldots\mu_k} = d\phi_{\mu_1\ldots\mu_k} - \phi_{\mu_1\ldots\mu_k\nu}dx^\nu \tag{C.2}$$

  for any integer k. Thus, these measure what we usually think of as variations of a field.

- *(r, s)-forms:* The space of vector fields on $J^\infty(E)$ naturally splits into a vertical part (annihilated by the projection push-forward $(\pi^\infty)_*$), which corresponds to what we usually think of as variations, and a horizontal part (annihilated by all contact forms), which represents usual spacetime vector fields extended to the jet bundle. A $(r+s)$-form that is zero for more than r horizontal vectors and more than s vertical vectors is said to be a (r, s)-form. This splits the vector space of p-forms into a direct sum of spaces of (r, s)-forms such that $r + s = p$.

- *Wedge products:* As we split up the usual vector space of differential forms into a direct sum, wedge products have the usual graded commutation i.e. for $\omega \in \Omega^{r_1,s_1}, \eta \in \Omega^{r_2,s_2}$

$$\omega \wedge \eta = (-1)^{(r_1+s_1)(r_2+s_2)} \eta \wedge \omega. \tag{C.3}$$

- *Exterior derivatives:* The exterior derivative $\boldsymbol{d} = d + \delta$ on the jet bundle splits into a "spacetime" d and a vertical δ satisfying

$$d^2 = 0, \qquad d\delta = -\delta d, \qquad \delta^2 = 0, \tag{C.4}$$

  which follows from $\boldsymbol{d}^2 = 0$

- *Interior products:* Vector fields on the jet bundle can be decomposed into horizontal and veritcal parts. Interior products with vertical vector fields anti-commute with the horizontal derivative and vice versa i.e. for ξ horizontal and X vertical we have

$$\xi \lrcorner \delta = -\delta \xi \lrcorner, \qquad X \lrcorner d = -dX \lrcorner\,. \tag{C.5}$$

  This also allows us to write down, what we usually think of as the variation of some functional $F[\phi]$

$$\text{"}\delta F\text{"} = X_\phi \lrcorner \delta F, \tag{C.6}$$

  where X_ϕ is a vector field that specifies the actual variation.

- *Lie derivatives:* An equivalent statement of (C.5) is the Cartan's magic formula for a horizontal vector field ξ and a vertical vector field X, respectively

$$\mathcal{L}_\xi \omega = d(\xi \lrcorner \omega) + \xi \lrcorner d\omega, \qquad \mathcal{L}_X \omega = \delta(X \lrcorner \omega) + X \lrcorner \delta\omega. \tag{C.7}$$

  This further implies that the Lie derivative in general commutes with both d and δ i.e. for any vector field X

$$[\mathcal{L}_X, d] = 0, \qquad [\mathcal{L}_X, \delta] = 0. \tag{C.8}$$

- *The Interior Euler operator:* When deriving the Euler-Lagrange equation, one has to integrate-by-parts to find the equations of motion. This operation of integrating-by-parts is formalized in the jet bundle by the interior Euler operator $I(\cdot)$ (one can write down an explicit expression for it, but we do not need it). Importantly, it for $\omega \in \Omega^{n,r}$ satisfies

$$\omega = I(\omega) + d\eta, \tag{C.9}$$

  for some $\eta \in \Omega^{n-1,r}$. Further, it is a projection $I^2 = I$ and it annihilates horizontal, exact forms $I(d\omega) = 0$. Finally, it also commutes with the Lie derivative along a vertical vector field X

$$\mathcal{L}_X I(\omega) = I(\mathcal{L}_X \omega). \tag{C.10}$$

C.2 Noether's theorems

The fundamental theorem of covariant phase space formalism builds on Noether's first and second theorem. We consider a vertical vector field X in the jet bundle, which is said to be a symmetry if it perseveres the Lagrangian form $L \in \Omega^{n,0}$ up to a total derivative

$$\mathcal{L}_X L = \underbrace{\delta(X \lrcorner L)}_{=0} + X \lrcorner \delta L = dM_X, \tag{C.11}$$

where the first term is zero due to L being a $(n, 0)$-form, i.e. horizontal. We can also consider the vertical derivative of the Lagrangian form

$$\delta L = I(\delta L) - d\Theta = E(L) - d\Theta, \tag{C.12}$$

where we used the interior Euler operator $I(\cdot)$ to define the Euler-Lagrange form (or EOM) $E(L)$ and the pre-symplectic potential Θ. Note the unusual sign of $d\Theta$, which is due to the fact that the interior product with a vertical vector field anti-commutes with d. If we contract the general δ-derivative with the symmetry X, we can equate (C.11) and (C.12) to find

$$X \lrcorner E(L) - X \lrcorner d\Theta = dM_X \qquad \Rightarrow \qquad d(M_X - X \lrcorner \Theta) = X \lrcorner E(L), \tag{C.13}$$

where we pick up a sign when anti-commuting d and $X\lrcorner$. This is the statement of Noether's first theorem, i.e. we have a current $J_X = M_X - X \lrcorner \Theta$ associated with the symmetry X that is conserved on-shell, $E(L) = 0$.

Where the first theorem applies to both global and local symmetries, the second theorem regards only local symmetries. It establishes so-called Noether identities (or Ward identities) which hold off-shell and hence render the EOM dependent. Thus, Noether's second theorem can be interpreted to say: local symmetry implies gauge. The part of the theorem we will use is that the Noether current J_X splits up in an exact part dQ_X and a part S_X that vanishes on-shell

$$J_X = S_X + dQ_X \qquad \Rightarrow \qquad dQ_X = M_X - X \lrcorner \Theta - S_X. \tag{C.14}$$

The $(n-2)$-form Q is called the Noether-Wald charge. To determine S_λ one contracts the Euler-Lagrange form $E(L)$ with the symmetry generating vector field X_λ parameterized by some spacetime dependent function λ and performs integration-by-parts to obtain

$$X_\lambda \lrcorner E(L) = \lambda \Delta(L) + dS_\lambda. \tag{C.15}$$

The assertion $\Delta(L) = 0$ are the aforementioned Noether identities.

C.3 Fundamental Theorem of the Covariant Phase Space Formalism

We now specialize the gauge transformation to be diffeomorphisms. The diffeomorphisms, we are considering, are only defined on the base manifold, and thus we need to extend it onto jet bundle. This can be done in two ways: First, by the so-called total vector field

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} + \xi^\mu \phi_\mu \frac{\partial}{\partial \phi} + \xi^\mu \phi_{\mu\nu} \frac{\partial}{\partial \phi_\nu} + \dots, \tag{C.16}$$

where ξ^μ are the components with respect to the base manifold coordinates. The total vector field is horizontal and $\mathcal{L}_\xi$ can be thought of as a usual Lie derivative. An alternate way of thinking about the change in a field under diffeomorphism is through a vector field X_ξ that corresponds to the vertical change due to a Lie derivative. If we for example consider a metric field $g_{\mu\nu}$ as our degree of freedom, the form of X_ξ is

$$X_\xi = \mathcal{L}_\xi g_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}} + \mathcal{L}_\xi \partial_\rho g_{\mu\nu} \frac{\partial}{\partial(\partial_\rho g_{\mu\nu})} + \ldots, \tag{C.17}$$

where the Lie derivatives are computed with the usual spacetime formula evaluated on the jet bundle coordinates $g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \ldots$. In particular, we have

$$\mathcal{L}_\xi \omega = \mathcal{L}_{X_\xi} \omega, \tag{C.18}$$

that is, we can equivalently view the change under a diffeomorphism as a horizontal dragging ξ or as the corresponding vertical change X_ξ of the fields.

As the Lagrangian is a $(n, 0)$ form, we have

$$\mathcal{L}_{X_\xi} L = \mathcal{L}_\xi L = d(\xi \lrcorner L) + \xi \lrcorner dL = d(\xi \lrcorner L). \tag{C.19}$$

Thus, we identify $M_\xi = \xi \lrcorner L$. A final definition we need is the pre-symplectic form

$$\omega = \delta\Theta. \tag{C.20}$$

Now it is straightforward to show the fundamental theorem of covariant phase space formalism

$$\begin{aligned} X_\xi \lrcorner \omega = X_\xi \lrcorner \delta\Theta &= -\delta(X_\xi \lrcorner \Theta) + \mathcal{L}_{X_\xi}\Theta \\ &= \delta(S_\xi + dQ_\xi - M_\xi) + \mathcal{L}_\xi \Theta \\ &= \delta S_\xi - d\delta Q_\xi - \delta(\xi \lrcorner L) + \mathcal{L}_\xi \Theta \\ &= \delta S_\xi - d\delta Q_\xi + \xi \lrcorner \delta L + \mathcal{L}_\xi \Theta \\ &= \delta S_\xi - d\delta Q_\xi + \xi \lrcorner (E(L) - d\Theta) + \mathcal{L}_\xi \Theta \\ &= \delta S_\xi + \xi \lrcorner E(L) + d(-\delta Q_\xi + \xi \lrcorner \Theta). \end{aligned} \tag{C.21}$$

In the first and last line we used the two different Cartan magic formulas (C.7). We can then contract with a vector field X_ϕ that stays on the "mass shell" i.e. $\mathcal{L}_{X_\phi} E(L) = 0$ in addition to evaluating the equality on-shell such that the first two terms on the RHS vanish

$$X_\phi \lrcorner X_\xi \lrcorner \omega = d(X_\phi \lrcorner \delta Q_\xi + \xi \lrcorner X_\phi \lrcorner \Theta) = -dk_\xi. \tag{C.22}$$

What we learn from the theorem is then that we have a current

$$k_\xi = -X_\phi \lrcorner \delta Q_\xi - \xi \lrcorner X_\phi \lrcorner \Theta, \tag{C.23}$$

which is closed iff $X_\phi \lrcorner X_\xi \lrcorner \omega$ vanishes on-shell. We can further define a charge

$$\not{\delta} H_\xi = \int_S k_\xi, \tag{C.24}$$

with S is some surface of co-dimension 2 e.g. a 2-sphere at spatial infinity in 3+1 dimensions. The $\not{\delta}$ is used to emphasize that the charge H_ξ is not necessarily integrable. By Stokes' theorem the charge H_ξ is then conserved in time, provided $dk_\xi = 0$. There are further considerations to be had about ambiguities having to do with adding closed forms to Θ for which we refer to [56].

From a more operational point of view, we can work exclusively with horizontal forms by always considering the contractions with some undetermined vector field X_ϕ. As an example, if we vary a Lagrangian $L[\phi]$ in the usual (commuting δ's and d's) sense

$$\delta L = (\ldots)\delta\phi + \partial_\mu \Theta^\mu_\phi, \tag{C.25}$$

then the components Θ^μ correspond to

$$X_\phi \lrcorner \Theta = \Theta^\mu_\phi (d^{n-1}x)_\mu, \tag{C.26}$$

with $(d^{n-1}x)_\mu$ defined in (A.5). Hence, the charge integrand $k^{[\mu\nu]}$ takes the form

$$k^{[\mu\nu]} = -\delta_h Q^{\mu\nu}_\xi + 2\xi^{[\mu}\Theta^{\nu]}_h, \tag{C.27}$$

where we used the interior product (A.8)

C.4 Charge integrand for GR

As an example, we will now compute $k^{[\mu\nu]}$ for Einstein gravity. The Lagrangian of GR is given by the EH action ($c = 1$)

$$\mathcal{L} = \frac{1}{16\pi G}\sqrt{-g}R. \tag{C.28}$$

Using standard variational identities (see e.g. [39]) we find that

$$\delta\mathcal{L} = \frac{\sqrt{-g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \frac{\sqrt{-g}}{16\pi G}\nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\nu_{\mu\nu}). \tag{C.29}$$

If we further employ the equalities

$$\sqrt{-g}\nabla_\mu X^\mu = \partial_\mu(\sqrt{-g}X^\mu), \qquad \delta\Gamma^\rho_{\mu\nu} = g^{\rho\lambda}\left(\nabla_{(\mu}\delta g_{\nu)\lambda} - \frac{1}{2}\nabla_\lambda\delta g_{\mu\nu}\right), \tag{C.30}$$

we can conclude that the pre-symplectic potential is given by

$$\Theta^\rho = \frac{\sqrt{-g}}{16\pi G}\left[\nabla_\mu\delta g^{\mu\rho} - \nabla^\rho(g^{\mu\nu}\delta g_{\mu\nu})\right]. \tag{C.31}$$

We then specialize to diffeomorphism i.e. $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$

$$\Theta^\mu = \frac{\sqrt{-g}}{8\pi G}\left[\nabla_\nu\nabla^{(\nu}\xi^{\mu)} - \nabla^\mu\nabla_\nu\xi^\nu\right]. \tag{C.32}$$

To obtain S^μ we need to integrate-by-parts, as prescribed in (C.15) starting with the EOM and $\delta g^{\mu\nu} = -2\nabla^{(\mu}\xi^{\nu)}$

$$\begin{aligned} E_{\mu\nu}(g)\delta g^{\mu\nu} &= \frac{\sqrt{-g}}{16\pi G}G_{\mu\nu}(-2\nabla^{(\mu}\xi^{\nu)}) = \frac{\sqrt{-g}}{8\pi G}[\underbrace{\nabla^\mu G_{\mu\nu}}_{\Delta(g)=0}\xi^\nu - \nabla^\mu(G_{\mu\nu}\xi^\nu)] \\ &= -\frac{1}{8\pi G}\partial_\mu(\sqrt{-g}G^\mu{}_\nu\xi^\nu). \end{aligned} \tag{C.33}$$

Thus, we find that the Noether identity is the contracted Bianchi identity, and

$$S^\mu = -\frac{\sqrt{-g}}{8\pi G}G^\mu{}_\nu\xi^\nu. \tag{C.34}$$

The M_ξ-form is easily obtained using $M_\xi = \xi \lrcorner L$ or in components

$$M^\mu_\xi = \xi^\mu\mathcal{L} = \xi^\mu\frac{\sqrt{-g}R}{16\pi G}. \tag{C.35}$$

Before we can derive the Noether-Wald charge, we need to note the following identity for torsionless connections

$$[\nabla_\mu, \nabla_\nu]\xi^\rho = -R_{\mu\nu\sigma}{}^\rho \xi^\sigma \qquad \Rightarrow \qquad \nabla_\mu \nabla_\nu \xi^\mu = R_{\sigma\nu}\xi^\sigma + \nabla_\nu \nabla_\mu \xi^\mu, \tag{C.36}$$

With this in mind, we can compute dQ using (C.14)

$$\begin{aligned}\partial_\nu Q^{[\mu\nu]} &= M^\mu - \Theta^\mu - S^\mu = \frac{\sqrt{-g}}{8\pi G}\left[\frac{1}{2}R\xi^\mu - \nabla_\nu \nabla^{(\nu}\xi^{\mu)} + \nabla^\mu \nabla_\nu \xi^\nu + G^\mu{}_\nu \xi^\nu\right] \\ &= \frac{\sqrt{-g}}{8\pi G}\left[\frac{1}{2}R\xi^\mu - \nabla_\nu \nabla^{(\nu}\xi^{\mu)} + \nabla_\nu \nabla^\mu \xi^\nu - R^\mu{}_\nu \xi^\nu + G^\mu{}_\nu \xi^\nu\right] \\ &= \partial_\nu\left(\frac{\sqrt{-g}}{8\pi G}\nabla^{[\mu}\xi^{\nu]}\right) - \frac{\sqrt{-g}}{8\pi G}\xi^\nu \underbrace{\left[R^\mu{}_\nu - \frac{1}{2}R\delta^\mu_\nu - G^\mu{}_\nu\right]}_{=0},\end{aligned} \tag{C.37}$$

where we used that $\partial_{\mu_1}(\sqrt{-g}X^{[\mu_1\cdots\mu_p]}) = \sqrt{-g}\nabla_{\mu_1}X^{[\mu_1\cdots\mu_p]}$. Thus, we see that Noether's second theorem holds with

$$Q^{\mu\nu} = \frac{\sqrt{-g}}{8\pi G}\nabla^{[\mu}\xi^{\nu]}. \tag{C.38}$$

To compute the charge integrand $k^{\mu\nu}$ (C.27), we need the variation $\delta_h Q^{\mu\nu}$ with $h_{\mu\nu}$ being a variation of $g_{\mu\nu}$

$$\delta_h Q^{\mu\nu} = \frac{\sqrt{-g}}{8\pi G}\left[\frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} - h^{\rho[\mu}\nabla_\rho \xi^{\nu]} + \xi_\lambda \nabla^{[\mu}h^{\nu]\lambda}\right], \tag{C.39}$$

where $h \equiv g^{\mu\nu}h_{\mu\nu}$ and $h^{\mu\nu} \equiv g^{\mu\sigma}g^{\nu\rho}h_{\sigma\rho}$. Finally, we can put together the pieces to obtain the charge integrand

$$k_\xi^{[\mu\nu]} = \frac{\sqrt{-g}}{8\pi G}\left[-\frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} + h^{\rho[\mu}\nabla_\rho \xi^{\nu]} - \xi_\lambda \nabla^{[\mu}h^{\nu]\lambda} - \nabla_\rho h^{\rho[\mu}\xi^{\nu]} - \xi^{[\mu}\nabla^{\nu]}h\right]. \tag{C.40}$$

To obtain charges, we of course first need to integrate $k^{\mu\nu}$ over e.g. a sphere at infinity

$$\int_{S^2_\infty} k^{\mu\nu}(d^2x)_{\mu\nu}. \tag{C.41}$$

These charges are conserved if $\omega(\delta_h g_{\mu\nu}, \delta_\xi g_{\mu\nu})$ vanishes, which happens if ξ is a Killing vector, because then $\delta_\xi g_{\mu\nu} = 0$. It turns out that in the case of GR, there are no ambiguities for charges associated with Killing vectors [56].

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