Edits to thesis

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Edits

use can → use section 1.1

$\omega \to \omega_R$ eq 3.5

$\omega \to \omega_R$ eq 3.11

reduces as *to* reduced: under eq 3.65

$i\omega_q \to i\omega_m$: under eq 3.134

$\omega_q \to \omega_m$ under 3.140

$\omega_k \to \omega_n$ and $\omega_q \to \omega_m$ in eq. 3.117

Abstract

Superconducting resonators used to build quantum devices can emulate harmonic oscillators. Their energies match the energies in semi-conducting Double Quantum Dots (DQDs), that can emulate atoms. Motivated by energy transport through systems of atoms and oscillators we studied theoretical systems of double quantum dots and superconducting resonators. A setup with a chain of DQDs and resonators was derived and transformed into a Transverse Field Ising model and further into a 1D spin-less p-wave superconductor using a Jordan-Wigner transformation. With a Bogoliubov transformation a classical solution to a single resonator field is found that minimizes the classical potential. Going to the imaginary time path integral formalism it was possible to study many resonators and their collective excitations. A classical solution to the resonator field was found for a single resonator and many DQD's. In the imaginary time path integral formalism, a saddle point equation is found for many resonators and DQD's and an approximate solution found in the fine tuned regime with weak coupling. It was shown that a uniform solution solution could always minimize the action. Fluctuations around the uniform resonator field are then studied and the spectrum and spectral function was found. From the spectrum the group velocity could be estimated. In conclusion this study leads to an understanding of how coupling a chain of DQDs to resonators will affect the resonator states. The finding of a dispersion relation gives an understanding of the dynamics of the collective oscillator modes.

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Chapter 1

Introduction

1.1 Energy transport

The topic of energy transport is one especially interesting in the physics of biology. Davydov proposed a quantum model to study the contraction of muscle fibres from the release of energy from ATP [1]. From the model he found that a bosonic excitation accompanied by a local distortion could propagate down one protein, the myosin filament, leading to a sliding movement along another protein, the actin filament, which would lead to a contraction of the muscle fibre. The local excitations could propagate as soliton waves without dispersion. The transport of energy is also relevant in a variety of other biological processes including muscle contractions, DNA replication, neuro-electric pulse transfer and more [2]. Biology is thus one subject where quantum transport models are of interest.

Where biology is made up of atoms and vibrations in the eV energy range, quantum devices use quantum dots and superconducting resonators that can work in the μeV energy range [3]. The quantum dots behave as artificial atoms and superconducting resonators as harmonic oscillators. The building blocks of biology and quantum devices are thus similar but work at different energy scales, and we could hope to find interesting dynamics in these quantum devices too. Theoretical work on energy transport in quantum systems is an interesting subject that is important to study as it has applications outside quantum devices too.

This thesis will study systems of Double Quantum Dots (DQDs) and superconducting resonators coupled together. The DQDs will be electronic two-level systems and the position coordinate of the superconducting resonators will couple to the DQDs. First we will study the DQD chain and the effect on the resonator. After that we will study many resonators and how energy might disperse through the system. We will apply tools from many-body quantum physics and field theory to understand the collective excitations of the resonators and the states of the DQD system.

1.2 Davydov's soliton in a Transverse field Ising model

We now want to motivate the study of transverse field Ising chains coupled to coherent state resonators. We start from the transverse field Ising chain:

$$H = -h\sum_i \sigma_i^z - J\sum_i \sigma_i^x\sigma_{i+1}^x, \tag{1.1}$$

with J and h the longitudinal and transverse fields. In the case of $h > J$

1.2.1 J small

For $J \ll h$ we divide the system into a quadratic Hamiltonian and a perturbation:

$$H_0 = -h\sum_i \sigma_i^z, \tag{1.2}$$

with groundstate energy $-hN = E_0$ and groundstate $|gs\rangle = \prod_i |\uparrow\rangle_i = \dots \otimes |\uparrow\rangle_i \otimes |\uparrow\rangle_{i+1} \otimes \dots$. The system is in the paramagnetic phase. We treat the longitudinal field as a pertubation:

$$V = -J\sum_i \sigma_i^x\sigma_{i+1}^x. \tag{1.3}$$

An excitation to the non-perturbed ground state is a flipped spin where each flipped spin costs $2h$ energy. The single flipped spin state will be denoted $|i\rangle = \dots \otimes |\uparrow\rangle_{i-1} |\downarrow\rangle_i |\uparrow\rangle_{i+1} \dots = |\dots \uparrow\uparrow\downarrow_i\uparrow\uparrow \dots\rangle$. We want to create a low-energy approximation where we concentrate on the subspace of a single spin flip. We follow Löwdin theory [4] to generate an approximate Hamiltonian where the subspace of a single spin flip is separated from the rest of the Hilbert space to first order in the perturbation. This will be equivalent to the perturbation being allowed to move a single spin flip one position. Since $J/h \ll 1$ we assume that the set of single spin flips, $\mathcal{M} = \{|i\rangle, i \in \mathbb{Z}\}$ only interacts weakly with the rest of the Hilbert space of zero, two or more spin flips as these state are separated by and energy gap of order h. In the following $m, m' \in \mathcal{M}$ and we find:

$$H^0_{m,m'} = (E_g + 2h)\delta_{m,m'}. \tag{1.4}$$

We now look for the first order correction in J. The effect of the longitudinal term is to flip two spins such that:

$$\begin{aligned} -J\sum_j \sigma_j^x\sigma_{j+1}^x |i\rangle &= -J\left(\dots |\dots \downarrow\downarrow\downarrow\uparrow\uparrow \dots\rangle + |\dots \uparrow\downarrow\uparrow\uparrow\uparrow \dots\rangle + |\dots \uparrow\uparrow\uparrow\downarrow\uparrow \dots\rangle + |\dots \uparrow\uparrow\downarrow\downarrow\downarrow \dots\rangle + \dots\right) \\ &= -J\left(\dots |\dots \downarrow\downarrow\downarrow\uparrow\uparrow \dots\rangle + |i-1\rangle + |i+1\rangle + |\dots \uparrow\uparrow\downarrow\downarrow\downarrow \dots\rangle + \dots\right), \end{aligned} \tag{1.5}$$

giving the first order correction:

$$\begin{aligned} H^1_{m,m'} &= \langle m| V |m'\rangle \\ &= -J\bigg(\delta_{m,m'-1} + \delta_{m,m'+1}\bigg). \end{aligned} \tag{1.6}$$

To first order in J we thus have the following Hamiltonian for the system of a single spin:

$$H = \sum_i |i\rangle \left(E_g + 2h\right) \langle i| - J \sum_i \bigg(|i-1\rangle \langle i| + |i+1\rangle \langle i| \bigg) + \mathcal{O}\big((J/h)^2\big). \tag{1.7}$$

We see that the low-energy model is a simple tight binding model for the single spin flip, that can jump to the nearest neighboring sites. We also notice that the single spin flip states do not interact with the zero or two flipped spin states as the longitudinal field only flips pairs of spins.

1.2.2 h small

We also look at the ferromagnetic regime where $h \ll J$, where we have the unperturbed Hamiltonian:

$$H_0 = -J \sum_i \sigma_i^x \sigma_{i+1}^x. \tag{1.8}$$

The longitudinal term will favor spins aligned along x and ground state is two times degenerate. We will assume the system spontaneously chooses one groundstate denoted by $|gs\rangle$ with energy $-(N-1)J = E_g$. The lowest energy excitation is a domain wall denoted by the position as $|i\rangle = |\cdots + + + - - - \ldots\rangle$ where the $-$ starts at position i. The energy of a domain wall is $2J$. $|\pm\rangle_i$ is an eigenstate of σ_i^x. The transverse field can create or move domain walls similar to how the longitudinal field did to the spin flips before. In the subspace of single domain walls we get:

$$H^0_{m,m'} = (E_g + 2J)\delta_{m,m'}. \tag{1.9}$$

Similar to before the effect of the perturbation is:

$$\begin{aligned} V\,|+++---\rangle &= -h\big(\ldots|+-+---\rangle + |++----\rangle + |++++--\rangle + |+++-+-\rangle \ldots\big) \\ &= -h\big(\ldots|+-+---\rangle + |i-1\rangle + |i+1\rangle + |+++-+-\rangle \ldots\big). \end{aligned} \tag{1.10}$$

The resulting Hamiltonian to first order in the interaction with $E_0 = E_g + 2J$ is then:

$$H = \sum_i |i\rangle E_0 \langle i| - h \sum_i \big(|i-1\rangle \langle i| + |i+1\rangle \langle i|\big). \tag{1.11}$$

Again, we get a tight binding model for the domain wall case to lowest order in h/J.

1.2.3 Coupling harmonic oscillators

We will now couple a harmonic oscillator position coordinate to the quasi-particle density with the following interaction term:

$$H_{\text{int}} = g \sum_i \big(a_i + a_i^\dagger\big) |i\rangle \langle i|. \tag{1.12}$$

The Hilbert space is then made up of $|i\rangle \otimes |\ldots n_i, n_{i+1}, \ldots\rangle$ where n_i is the oscillator quantum number at site i. This gives a Hamiltonian on the form:

$$H = \sum_i \bigg(E_0 + g\left(a_i + a_i^\dagger\right)\bigg) |i\rangle \langle i| - t \sum_i \bigg(|i-1\rangle \langle i| + |i+1\rangle \langle i|\bigg) + \omega_R \sum_i a_i^\dagger a_i, \tag{1.13}$$

where the hopping term, t, is the transverse field, h, if we work with domain walls, and the longitudinal field, J, if we work with spin flips.

1.2.4 Davydov ansatz and equations of motion

Following the approach of Davydov [5][6] we work with the ansatz state:

$$|\Psi(t)\rangle = \sum_n c_n(t) |n\rangle \otimes |\alpha\rangle = |\phi\rangle |\alpha\rangle, \tag{1.14}$$

where the state $|n\rangle$ is a spin flip or domain wall at site n and $|\alpha\rangle = e^{\sum_n \left(\alpha_n \hat{a}_n^\dagger - \alpha_n^* \hat{a}_n\right)} |0\rangle$ is a bosonic coherent state with $|0\rangle$ the vacuum state. The coherent state is an eigenstate of the annihilation operator, $\hat{a}_n$, with eigenvalue α_n. Coherent states are described in appendix B. All time dependence of the ansatz state is put in the complex factors such that the basis kets $|n\rangle$, that form a complete set, are not time dependent but they reference the system from some time t_0. Using the Ehrenfest theorem we find the equations of motion for the coherent state:

$$\begin{aligned} i\hbar\partial_t \langle a_n\rangle &= \langle [a_n, H]\rangle \\ &= \langle (g |n\rangle \langle n| + \omega_R a_n)\rangle \\ &= g|c_n|^2 + \omega_R \alpha_n = i\hbar\dot{\alpha}_n, \end{aligned} \tag{1.15}$$

where it was used that $\langle \hat{a}_n\rangle = \alpha_n$ due to the the coherent state. From this we find the equations of motion for the resonators coordinates by adding or subtracting the complex conjugate. We define $x_n = \alpha_n + \alpha_n^*$ and $p_n = \alpha_n - \alpha_n^*$, giving that:

$$i\hbar\dot{x}_n = \omega_R p_n, \tag{1.16}$$

$$i\hbar\dot{p}_n = 2g|c_n|^2 + \omega_R x_n. \tag{1.17}$$

As these are now just complex numbers we differentiate once more to get:

$$-\frac{\hbar^2}{\omega_R}\ddot{x}_n = 2g|c_n|^2 + \omega_R x_n, \tag{1.18}$$

giving us equations of motion for the coherent oscillators. We then differentiate the ansatz state:

$$\begin{aligned} i\hbar\partial_t |\Psi(t)\rangle &= i\hbar \sum_n \dot{c_n}(t) |n\rangle |\alpha\rangle + i\hbar \sum_n c_n |n\rangle \partial_t \left(e^{\sum_n \left(\alpha_n \hat{a}_n^\dagger - \alpha_n^* \hat{a}_n\right)}\right) |0\rangle \\ &= H |\Psi(t)\rangle. \end{aligned} \tag{1.19}$$

Evaluating $\partial_t \left(e^{\sum_n \left(\alpha_n \hat{a}_n^\dagger - \alpha_n^* \hat{a}_n\right)}\right)$ is done as follows [7]: We define $A(t) = \sum_n \left(\alpha_n(t)\hat{a}_n^\dagger - \alpha_n^*(t)\hat{a}_n\right)$, an operator that does not necessarily commute at different times. We also define $B(t) = e^{A(t)}$, and then evaluate:

$$B(t+\delta t) - B(t) = e^{A(t+\delta t) - A(t) + A(t)} - e^{A(t)}. \tag{1.20}$$

Using the Baker–Campbell–Hausdorff formula we split the first term:

$$e^{A(t+\delta t) - A(t) + A(t)} = e^{A(t+\delta t) - A(t)} e^{A(t)} e^{-\frac{1}{2}[A(t+\delta t) - A(t), A(t)]}, \tag{1.21}$$

as the operators will commute to a number. We find:

$$
\begin{aligned}
[A(t+\delta t)-A(t), A(t)] &= [A(t+\delta t, A(t)] \\
&= \sum_{n,m}\left[\alpha_n(t+\delta t)a_n^\dagger - \alpha_n^*(t+\delta t)a_n, \alpha_m(t)\hat{a}_m^\dagger - \alpha_m^*(t)\hat{a}_m\right] \\
&= \sum_n \Big(\alpha_n(t+\delta t)\alpha_n^*(t) - \alpha_n^*(t+\delta t)\alpha_n(t)\Big) = C \in \mathbb{C}.
\end{aligned}
\tag{1.22}
$$

We confirm that the commutator vanishes in the limit of $\delta t \to 0$. We are then left with:

$$
B(t+\delta t) - B(t) = \left(e^{A(t+\delta t)-A(t)}e^{-\frac{1}{2}C} - 1\right)e^{A(t)}. \tag{1.23}
$$

We expand to lowest order in δt:

$$
e^{A(t+\delta t)-A(t)} = 1 + A(t+\delta t) - A(t) + \frac{1}{2}\left(A(t+\delta t)-A(t)\right)^2 + \ldots, \tag{1.24}
$$

and use again to lowest order that $A(t+\delta t) - A(t) = \dot{A}(t)\delta t$ giving:

$$
= 1 + \dot{A}(t)\delta t + \mathcal{O}\left(\delta t^2\right). \tag{1.25}
$$

For the next term we expand:

$$
e^{-\frac{C}{2}} = 1 - \frac{C}{2} + \ldots, \tag{1.26}
$$

where we use that after a Taylor expansion:

$$
\begin{aligned}
C &= \sum_n \Big(\alpha_n(t+\delta t)\alpha_n^*(t) - \alpha_n^*(t+\delta t)\alpha_n(t)\Big) \\
&= \sum_n \Big(\big(\alpha_n(t) + \dot{\alpha}_n(t)\delta t\big)\alpha_n^*(t) - \big(\alpha_n^*(t) + \dot{\alpha}_n^*(t)\delta t\big)\alpha_n(t)\Big) + \mathcal{O}\left(\delta t^2\right) \\
&= \sum_n \Big(\dot{\alpha}_n(t)\alpha_n^*(t) - \dot{\alpha}_n^*(t)\alpha_n(t)\Big)\delta t + \mathcal{O}\left(\delta t^2\right).
\end{aligned}
\tag{1.27}
$$

Collecting everything to the lowest order in δt we get:

$$
B(t+\delta t) - B(t) = \left(\dot{A}(t) - \frac{1}{2}\sum_n \Big(\dot{\alpha}_n(t)\alpha_n^*(t) - \dot{\alpha}_n^*(t)\alpha_n(t)\Big)\right)\delta t e^{A(t)} + \mathcal{O}(\delta t^2). \tag{1.28}
$$

We then divide by δt and take the limit of $\delta t \to 0$, giving:

$$
\partial_t e^{A(t)} = \sum_n \left(\dot{\alpha}_n(t)\hat{a}_n^\dagger(t) - \dot{\alpha}_n^*(t)\hat{a}_n - \frac{1}{2}\Big[\dot{\alpha}_n(t)\alpha_n^*(t) - \dot{\alpha}_n^*(t)\alpha_n(t)\Big]\right)e^{A(t)}. \tag{1.29}
$$

This gives the Schrödinger equation on the ansatz state as:

$$
i\hbar\sum_n \dot{c}_n(t)\left|n\right\rangle\left|\alpha\right\rangle + i\hbar\left|\phi\right\rangle\sum_n\left(\dot{\alpha}_n(t)\hat{a}_n^\dagger(t) - \dot{\alpha}_n^*(t)\hat{a}_n - \frac{1}{2}\Big[\dot{\alpha}_n(t)\alpha_n^*(t) - \dot{\alpha}_n^*(t)\alpha_n(t)\Big]\right)\left|\alpha\right\rangle = H\left|\Psi(t)\right\rangle. \tag{1.30}
$$

We would like to apply the bra $\langle\alpha|\langle n|$ to get an equation for the $c_n(t)$. On the left hand side we get:

$$
\begin{aligned}
i\hbar\langle\alpha|\langle n|\partial_t|\Psi\rangle &= i\hbar\dot{c}_n + i\hbar c_n\frac{1}{2}\sum_m\Big(\dot{\alpha}_m(t)\alpha_m^*(t) - \dot{\alpha}_m^*(t)\alpha_m(t)\Big) \\
&= i\hbar\dot{c}_n + c_n\gamma(t),
\end{aligned} \tag{1.31}
$$

where we defined $\gamma(t) = i\hbar\frac{1}{2}\sum_m\Big(\dot{\alpha}_m(t)\alpha_m^*(t) - \dot{\alpha}_m^*(t)\alpha_m(t)\Big)$. Using that $\langle m|\phi\rangle = c_m$, the right hand side gives:

$$
\begin{aligned}
\langle\alpha|\langle n|H|\Psi\rangle &= \langle n|\sum_m\bigg(\big(E_0 + g\left(\alpha_m + \alpha_m^*\right)\big)|m\rangle c_m - t\big(|m\rangle c_{m-1} + |m\rangle c_{m+1}\big) + \omega_R|\alpha_m|^2\sum_j c_j|j\rangle\bigg) \\
&= \big(E_0 + g\left(\alpha_n + \alpha_n^*\right)\big)c_n - t\big(c_{n-1} + c_{n+1}\big) + \omega_R c_n\sum_n|\alpha_n|^2 \\
&= \big(E_0 + W(t) + g\left(\alpha_n + \alpha_n^*\right)\big)c_n - t\big(c_{n-1} + c_{n+1}\big),
\end{aligned} \tag{1.32}
$$

where we defined $W(t) = \omega_R\sum_n|\alpha_n|^2$. Collecting the left and right hand sides we get:

$$
i\hbar\dot{c}_n + \gamma(t)c_n = \big(E_0 + W(t) + g\left(\alpha_n + \alpha_n^*\right)\big)c_n - t\big(c_{n-1} + c_{n+1}\big). \tag{1.33}
$$

We define:

$$
c_n(t) = e^{\theta(t)}\phi_n(t), \tag{1.34}
$$

with the global phase $\theta(t) = \frac{1}{i\hbar}\int_{-\infty}^{t}\left(E_0 + W(t) - \gamma(t) - 2h\right)dt$, which gives:

$$
i\hbar\dot{\phi}_n = g\left(\alpha_n + \alpha_n^*\right)\phi_n - t\big(\phi_{n-1} - 2\phi_n + \phi_{n+1}\big). \tag{1.35}
$$

Combined with (1.18) we get the equations of motion:

$$
i\hbar\dot{\phi}_n = gx_n\phi_n - t\big(\phi_{n-1} - 2\phi_n + \phi_{n+1}\big), \tag{1.36}
$$

$$
-\frac{\hbar^2}{\omega_R}\ddot{x}_n = 2g|c_n|^2 + \omega_R x_n. \tag{1.37}
$$

We now assume that the LHS in (1.18) is zero. This could be due to $\ddot{x} = \dot{p}$ representing the magnetic flux change in a superconducting resonator being small compared to the electric potential. This gives the solution:

$$
\begin{aligned}
x_n &= -\frac{2g}{\omega_R}|\phi_n|^2 \\
\Rightarrow i\hbar\dot{\phi}_n &= -\frac{2g^2}{\omega_R}|\phi_n|^2\phi_n - t\partial_n^2\phi_n,
\end{aligned} \tag{1.38}
$$

where we used the discrete ($\delta n = 1$), $\partial_n^2\phi_n = h\big(\phi_{n-1} - 2\phi_n + \phi_{n+1}\big)$, only really valid in the continuum limit. We have thus arrived at a non-linear Schrödinger equation for the spin flips or domain walls. We now define $\frac{\omega_R t}{2g^2} = \sigma_0$ and get:

$$
i\frac{\hbar}{t}\dot{\phi}_n + \sigma_0^{-1}|\phi_n|^2\phi_n + \left(\phi_{n-1} - 2\phi_n + \phi_{n+1}\right) = 0. \tag{1.39}
$$

According to [5] we find an approximate stationary solution when $\sigma_0 \gg 1$ and $N \gg 1$ as:

$$|\phi_n|^2 = \frac{1}{8\sigma_0}\operatorname{sech}^2\left(\frac{n-n_0}{4\sigma_0}\right). \tag{1.40}$$

The limit of $\sigma_0 \gg 1 \Rightarrow \frac{t}{\omega_R} \gg 2\frac{g^2}{\omega_R^2}$. To get to the low energy model we already assumed that t was small so we must also assume that the interaction term between the harmonic oscillator and spin flip or domain wall is even smaller in terms of ω_R.

1.2.5 Time evolution of the ansatz state

We define the time unit $t = \hbar/t_{hopping}$. Using $\frac{df_t}{dt} \approx \frac{f_{t+dt}-f_t}{dt}$ gives:

$$\phi_{n,t+dt} \approx \phi_{n,t} + i\Big(\sigma_0^{-1}|\phi_n|^2\phi_n + (\phi_{n-1} - 2\phi_n + \phi_{n+1})\Big)\frac{dt}{t}, \tag{1.41}$$

and defining $\hbar = 1$, we require $t_{hopping}\delta t \ll 1$. Setting $t_{hopping} = 1$ we require $\delta t \ll 1$. The only parameter left is $\sigma_0 = \frac{\omega_R}{2g^2}$ which is large for weak couplings and small for strong couplings. We simulate the system, where an initial excitation is placed at the first site, meaning $\phi_n(t=0) = \delta_n$. The boundary conditions are taken such that:

$$\phi_{0,t+\delta t} = \phi_{0,t} + i\Big(\sigma_0^{-1}|\phi_0|^2\phi_0 + (\phi_1 - 2\phi_0)\Big)\frac{\delta t}{t_{hopping}}. \tag{1.42}$$

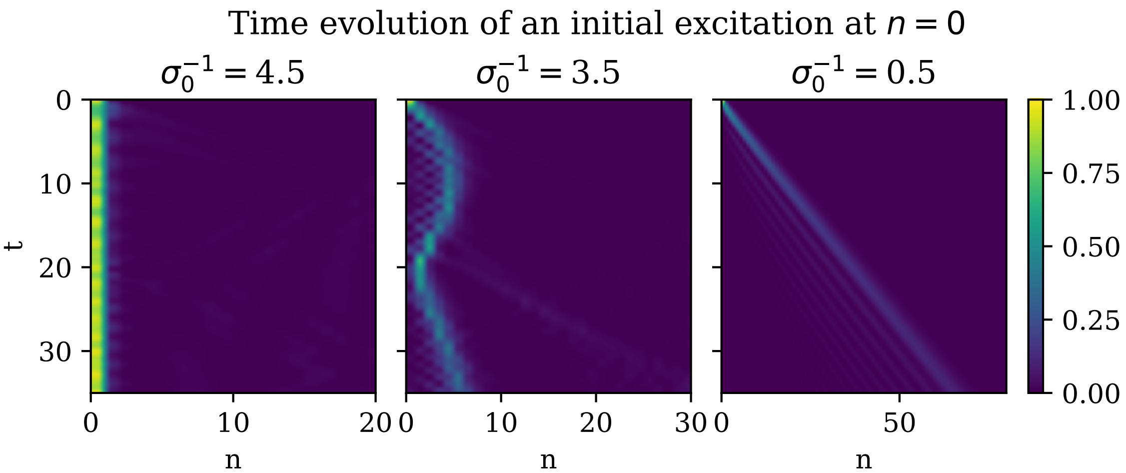

Figure 1.1: Simulation of finite system of 30 spins in weak and strong coupling regime. Color indicates the probability of the excitation to be at site n as $|\phi_n|^2$.

For strong coupling, $\sigma_0^{-1} > 1$, the excitation is stationary and localized around the initial $n_0 = 1$. For weak coupling, $\sigma_0^{-1} < 1$, the initial excitation travels along the chain with minimal dispersion. For couplings in between, the excitation is less strongly located at $n_0 = 1$.

Even this very simple system consisting only of a low-energy tight binding chain coupled to harmonic oscillators was shown with simple methods to exhibit interesting collective phenomena. We will go on to study resonators coupled to double quantum dot chains that can be transformed into spin chains.

Chapter 2

Quantum Ising model from quantum dots

2.1 Double quantum dot chain

We imagine having a chain of DQDs with a single electron on each DQD that can tunnel between the left and right site on the DQD, but not between DQDs. The DQDs interact capacitively with each other.

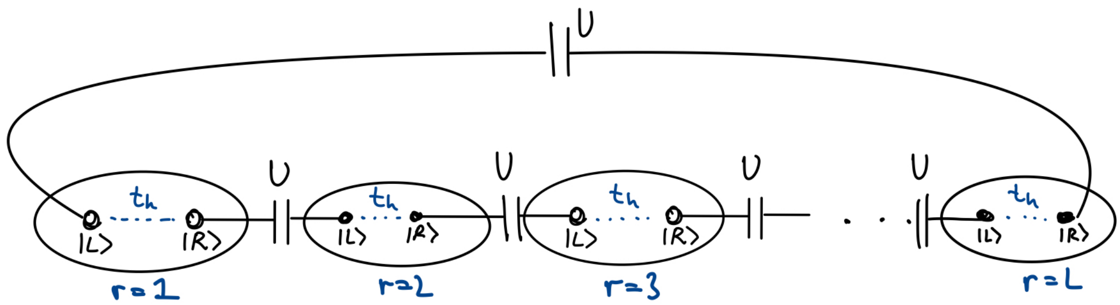

Figure 2.1: Chain of DQDs

Each DQD is represented as a lattice site, with the total number of sites in the lattice being N. The lattice spacing is a, which is the distance between the DQDs. The length of the DQD chain is then $L = Na$. We take $a = 1$ throughout this thesis. The DQD sites are represented by L and R for left and right. The detuning between the left and right site in each DQD can be adjusted with an electrostatic potential from a local gate. The Hamiltonian for this chain with periodic boundary conditions (PBC) is:

$$H_0 = \sum_{r=1}^{L} \left(\frac{\Delta_r}{2} \left[d^\dagger_{r,L} d_{r,L} - d^\dagger_{r,R} d_{r,R} \right] - t_r \left[d^\dagger_{r,L} d_{r,R} + d^\dagger_{r,R} d_{r,L} \right] + 4U_{r,r+1} d^\dagger_{r,R} d_{r,R} d^\dagger_{r+1,L} d_{r+1,L} \right). \quad (2.1)$$

The first term in the Hamiltonian represents a detuning in the DQD at site r. The second term represents the hopping matrix element in a DQD at site r and the last term represents the Coulomb interaction between DQD at site r and $r+1$. Due to the PBC we require $d_{1+L,\alpha} = d_{1,\alpha}$. We want to connect the DQDs to a resonator mode that is modeled as a harmonic oscillator. The resonator interacts capacitatively with the DQDs. We imagine either coupling the resonator to one of the sites on the DQD or to the hopping matrix element as shown in figure (2.2). Resonators and their coupling to quantum dots are described in appendix C.

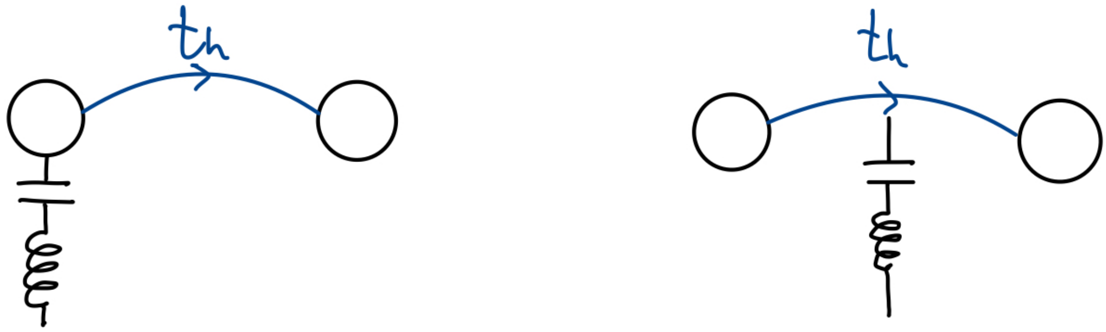

Figure 2.2: Resonator-DQD couplings. On the left is a resonator coupled to the left site of a DQD represented with a term in the Hamiltonian $H^{density}_{int} = g\left(a^\dagger + a\right) d^\dagger_L d_L$. The right is a coupling to the tunneling matrix element. This is represented by the term $H^{hopping}_{int} = g\left(a^\dagger + a\right)\left(d^\dagger_L d_R + d^\dagger_R d_L\right)$.

We then imagine coupling a resonator to each tunneling matrix element in the DQD chain. This will add a resonator Hamiltonian:

$$H_r = \sum_{r=1}^{L} \omega_r a^\dagger_r a_r, \quad (2.2)$$

and we will have an interaction term between the resonator modes and the DQD tunneling matrix elements:

$$H^{hopping}_{int} = \sum_{r=1}^{L} g_r \left(a_r + a^\dagger_r\right)\left(d^\dagger_{r,L} d_{r,R} + d^\dagger_{r,R} d_{r,L}\right). \quad (2.3)$$

This coupling was chosen as it will later allow us to get to a quadratic model. The full Hamiltonian is then:

$$H = H_0 + H_r + H_{int}. \quad (2.4)$$

The chemical potential is adjusted such that there is only one electron in each DQD. The Hilbert space of each DQD is now two-dimensional and we can write it in a basis of one electron on the left or right site:

$\{|L\rangle_r, |R\rangle_r\}$, with $d^\dagger_{r,L/R}|0\rangle = |L/R\rangle_r$. Since the Hilbert space is now two-dimensional we can represent operators on the space with Pauli matrices. We define the vectors of the two-dimensional Hilbert space as eigenstates of the σ^z-operator such that $\sigma^z_r|L\rangle_r = |L\rangle_r$ and $\sigma^z_r|R\rangle_r = -|R\rangle_r$. This gives the Pauli matrices at each site of the DQD chain as:

$$\begin{aligned}\sigma^z_r &= d^\dagger_{r,L}d_{r,L} - d^\dagger_{r,R}d_{r,R},\\ \sigma^x_r &= d^\dagger_{r,L}d_{r,R} + d^\dagger_{r,R}d_{r,L},\\ \sigma^y_r &= -id^\dagger_{r,L}d_{r,R} + id^\dagger_{r,R}d_{r,L}.\end{aligned}$$

Since the DQDs are distinguishable the operators commute at different sites:

$$\left[\sigma^\alpha_r, \sigma^{\alpha'}_{r'}\right] = 0 \quad r \neq r'. \tag{2.5}$$

On the same site we can make use of the Pauli matrix relations:

$$\left[\sigma^\alpha_r, \sigma^\beta_r\right] = 2i\epsilon_{\alpha\beta\gamma}\sigma^\gamma_r, \tag{2.6}$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita epsilon. To mimic particle creation and annihilation operators we define raising and lowering operators and require $\sigma^+_r|R\rangle_r = |L\rangle_r$ and $\sigma^-_r|L\rangle_r = |R\rangle_r$:

$$\sigma^+_r = d^\dagger_{r,L}d_{r,R}, \qquad \sigma^-_r = d^\dagger_{r,R}d_{r,L}, \tag{2.7}$$

and it is seen that:

$$\sigma^\pm_r = \frac{\sigma^x_r \pm i\sigma^y_r}{2}. \tag{2.8}$$

These operators anticommute on the same site, but they commute on different sites:

$$\{\sigma^+_r, \sigma^-_{r'}\} = 1 \quad , \quad r = r' \tag{2.9}$$

$$[\sigma^+_r, \sigma^-_{r'}] = 0 \quad , \quad r \neq r' \tag{2.10}$$

That they commute on different sites is seen, as the single fermion operators anticommute on different sites. To derive the anticommutation on the same site we use that the Hilbert space is two-dimensional and the following two identities:

$$\begin{aligned}[AB, C] &= ABC - CAB + (ACB - ACB)\\ &= A\{B, C\} - \{A, C\}B,\end{aligned} \tag{2.11}$$

$$\begin{aligned}\{A, BC\} = \{BC, A\} &= ABC + BCA + (BAC - BAC)\\ &= [A, B]C + B\{A, C\}.\end{aligned} \tag{2.12}$$

On the same site where $r = r'$ the raising and lowering operator will anticommute to one:

$$\begin{aligned}
\{\sigma^+, \sigma^-\} &= \{d_L^\dagger d_R, d_R^\dagger d_L\} \\
&= \left[d_L^\dagger d_R, d_R^\dagger\right] d_L + d_R^\dagger \left\{d_L^\dagger d_R, d_L\right\} \\
&= d_L^\dagger \left\{d_R, d_R^\dagger\right\} d_L - \left\{d_L^\dagger, d_R^\dagger\right\} d_R d_L + d_R^\dagger \left[d_L, d_L^\dagger\right] d_R + d_R^\dagger d_L^\dagger \left\{d_L, d_R\right\} \\
&= d_L^\dagger d_L + (1 - 2 d_L^\dagger d_L) d_R^\dagger d_R \\
&= 1.
\end{aligned} \tag{2.13}$$

To get to the last line we used that the Hilbert space of each DQD with one electron is spanned by $\{|L\rangle, |R\rangle\}$, such that the state of each DQD can be written generally as $|\Psi\rangle = \alpha |L\rangle + \beta |R\rangle$. It is then found that:

$$\left(d_L^\dagger d_L + d_R^\dagger d_R\right) |\Psi\rangle = |\Psi\rangle \Rightarrow d_L^\dagger d_L + d_R^\dagger d_R = \mathbb{1}, \tag{2.14}$$

and

$$d_L^\dagger d_L d_R^\dagger d_R |\Psi\rangle = 0. \tag{2.15}$$

This shows that the spin raising and lowering operators do not behave as either fermions or bosons. Before we can rewrite the Hamiltonian fully in terms of spins we use that:

$$\begin{aligned}
d_{r,L}^\dagger d_{r,L} &= \frac{d_{r,L}^\dagger d_{r,L} + d_{r,R}^\dagger d_{r,R}}{2} + \frac{d_{r,L}^\dagger d_{r,L} - d_{r,R}^\dagger d_{r,R}}{2} \\
&= \frac{1 + \sigma_r^z}{2},
\end{aligned} \tag{2.16}$$

where again it was use that $d_L^\dagger d_L + d_R^\dagger d_R = 1$. Similarly we have $\frac{1-\sigma_r^z}{2} = d_R^\dagger d_R$. The detuning and hopping term is straight forward and for the interaction term we get from (2.16):

$$\begin{aligned}
d_{r,R}^\dagger d_{r,R} d_{r+1,L}^\dagger d_{r+1,L} &= \left(\frac{1 + \sigma_r^z}{2}\right)\left(\frac{1 - \sigma_{r+1}^z}{2}\right) \\
&= \frac{1}{4}\left(1 + \left[\sigma_{r+1}^z - \sigma_r^z\right] - \sigma_r^z \sigma_{r+1}^z\right).
\end{aligned} \tag{2.17}$$

We can now write our DQD chain Hamiltonian (2.1) in terms of spin operators operating on the two-dimensional Hilbert space of each DQD:

$$\begin{aligned}
H_0 &= \sum_{r=1}^{L} \left\{ \frac{\Delta_r}{2} \sigma_r^z - t_r \sigma_r^x + U_{r+1,r} \left(\left[\sigma_{r+1}^z - \sigma_r^z\right] - \sigma_r^z \sigma_{r+1}^z \right) \right\} \\
&= \sum_{r=1}^{L} \left\{ \left(\frac{\Delta_r}{2} + U_{r,r-1} - U_{r+1,r} \right) \sigma_r^z - t_r \sigma_r^x + U_{r+1,r} \sigma_r^z \sigma_{r+1}^z \right\},
\end{aligned} \tag{2.18}$$

where the constant energy term $\sum_{r=1}^{L} U_{r+1,r}$ is neglected since it only contributes with and overall phase. The interaction term (2.3) is now:

$$H_{int}^{hopping} = \sum_{r=1}^{L} g_r^h \left(a_r + a_r^\dagger\right) \sigma_r^x. \tag{2.19}$$

The resonator Hamiltonian is unchanged. If we had chosen to couple to the DQD densities we would get an interaction term on the form:

$$H_{int}^{density} = \sum_{r=1}^{L} g_r^d \left(a_r + a_r^\dagger\right) \sigma_r^z. \tag{2.20}$$

To get a transverse field Ising (TFI) model where the transverse field is in σ^z, we rotate our Hamiltonian by applying a unitary rotation in the Pauli matrices. A general rotation of the Pauli matrices is written as $R_r^\alpha(\theta) = e^{i\frac{\theta}{2}\sigma_r^\alpha}$, which rotates the spin at site r by and angle θ around the α axis. We use that $R_r^\alpha = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}\sigma_r^\alpha$ when calculating the effect of the rotation.

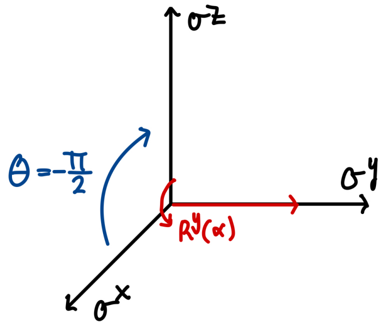

Figure 2.3: Rotating $-\frac{\pi}{2}$ around σ^y

$R_r^y(\frac{-\pi}{2}) = e^{-i\frac{\pi}{2}\sigma_r^y}$ is a rotation around the Pauli y-axis resulting in:

$$\sigma_r^x \rightarrow \sigma_r^z \quad , \quad \sigma_r^z \rightarrow -\sigma_r^x. \tag{2.21}$$

This can be shown by applying the unitary operator $U = \prod_{j=1}^{L} R_r^y(-\frac{\pi}{2}) = e^{-i\frac{\pi}{4}\sum_j \sigma_j^y} = \prod_{j=1}^{L} \frac{1-i\sigma_j^y}{\sqrt{2}}$ to the Hamiltonian $H' = U^\dagger H U$. The unitary operator does not depend on time so the Scrödinger equation is left unchanged as the partial derivative with respect to time will be zero. Only the state will be transformed as $\psi \rightarrow \psi' = U^\dagger \psi$. We end up with a rotated Hamiltonian on the form:

$$H_0' = \sum_{r=1}^{L} \left\{ -\left(\frac{\Delta_r}{2} + U_{r,r-1} - U_{r+1,r}\right)\sigma_r^x - t_r\sigma_r^z + U_{r+1,r}\sigma_r^x\sigma_{r+1}^x \right\}, \tag{2.22}$$

$$H'^{hopping}_{int} = \sum_{r=1}^{L} g_r^h (a_r + a_r^\dagger)\sigma_r^z, \tag{2.23}$$

$$H'^{density}_{int} = -\sum_{r=1}^{L} g_r^d (a_r + a_r^\dagger)\sigma_r^x. \tag{2.24}$$

Longitudinal terms with a single σ^x will break the $\mathbb{Z}_2$ symmetry of the system. The $\mathbb{Z}_2$ symmetry is the symmetry under flipping spins. The parity operator is defined as:

$$\mathcal{P} = \prod_{j=1}^{L} (-i) R_r^z(\pi) = \prod_{j=1}^{L} \sigma_r^z. \tag{2.25}$$

The parity operator flips the x and y-spins since $\sigma^z \sigma^{x,y} \sigma^z = -\sigma^{x,y}$ and leaves the z-spins unchanged. The parity operator is its own inverse $\mathcal{P}^{-1} = \mathcal{P}$ since $(\sigma_r^z)^2 = 1$. Without the longitudinal field we would have $[H_0', \mathcal{P}] = \left[H_{int}^{hopping}, \mathcal{P}\right] = 0$. From quantum mechanics we know that H' and $\mathcal{P}$ then possess a common eigenbasis. The parity operator measures whether the number of spins flipped is even or uneven. A state with even(uneven) parity will have eigenvalue 1(-1). This means that the system will be block diagonal in parity and have eigenstates that live in either the even or odd parity subspace. We see that by considering an eigenvector of $\mathcal{P} |\lambda\rangle = \lambda |\lambda\rangle$. We have from $[H, \mathcal{P}] |\lambda\rangle = 0 \Rightarrow \lambda H |\lambda\rangle = \mathcal{P} H |\lambda\rangle$, that $H |\lambda\rangle$ is also an eigenvalue of $\mathcal{P}$ with eigenvalue λ. If a vector $|\lambda\rangle$ and $H |\lambda\rangle$ both have eigenvalue λ we then conclude that H must be block diagonal in parity. To get to the usual 1D spin model notation we define the fields:

$$h_r^x = \left(\frac{\Delta_r}{2} + U_{r,r-1} - U_{r+1,r}\right), \qquad h_r^z = t_r, \qquad J_r^x = U_{r+1,r}, \tag{2.26}$$

such that our DQD chain Hamiltonian becomes:

$$H_0' = -\sum_{r=1}^{L} \left(J_r^x \sigma_r^x \sigma_{r+1}^x + h_r^x \sigma_r^x + h_r^z \sigma_r^z\right). \tag{2.27}$$

The DQD Hamiltonian is now a transverse and longitudinal field Ising model. This model is in general not integrable except for the specific case when the system has no disorder, $h^z = J^z$ and $h^x \to 0$ [8]. The resonator Hamiltonian is unchanged since the transformation does not effect the resonator operators. In the case of no longitudinal field, $h_r^x = 0$, we get the TFI model, which is an integrable model (not considering the resonator interaction so far). For $h_r^z = 0$ we get a classical 1D Ising model. We solve the TFI model by applying a Jordan-Wigner transformation, mapping the spins to non-local spinless fermions[9].

2.2 From transverse field Ising to spinless fermions

2.2.1 Jordan-Wigner transformation

The Jordan-Wigner transformation maps the local spins to non-local fermions. We define the fermion number operator:

$$c_r^\dagger c_r = \frac{1 - \sigma_r^z}{2}, \tag{2.28}$$

that is either zero or one. We then define the string operator:

$$\begin{aligned} \mathcal{L}_r &= \prod_{j=1}^{r-1} \sigma_j^z \\ &= \prod_{j=1}^{r-1} \left(1 - 2c_j^\dagger c_j\right) \\ &= (-1)^{\sum_{i=1}^{r-1} c_j^\dagger c_j}, \end{aligned} \tag{2.29}$$

where it was used that $1 - 2c^\dagger c = (-1)^{c^\dagger c}$, with -1 if the site is occupied and 1 if not. It is worth mentioning that $\mathcal{L}_r^2 = 1$, $\mathcal{L}_r^\dagger = \mathcal{L}_r$ and also that $[\mathcal{L}_r, \sigma_r^\pm] = 0$, since the spin-operators commute on different sites. With this we define the creation operator:

$$c_r^\dagger = \sigma_r^- \mathcal{L}_r, \tag{2.30}$$

and the annihilation operator follows from complex conjugation. The spin operators are given as $\sigma_r^- = \mathcal{L}_r c_r^\dagger$. This gives the correct fermion statistics as the fermion operators anticommute on different sites:

$$\left\{c_r^\dagger, c_s\right\} = 0. \tag{2.31}$$

The number operator is $c_r^\dagger c_r = \frac{1-\sigma_r^z}{2}$. Using (2.8) we calculate σ_r^x:

$$\begin{aligned} \sigma_r^x &= \sigma_r^+ + \sigma_r^- \\ &= \mathcal{L}_r(c_r^\dagger + c_r). \end{aligned} \tag{2.32}$$

We now want to find the interaction term, $\sigma_r^x \sigma_{r+1}^x$, in terms of the fermion operators. We get from (2.32):

$$\sigma_r^x \sigma_{r+1}^x = \mathcal{L}_r(c_r^\dagger + c_r)\mathcal{L}_{r+1}(c_{r+1}^\dagger + c_{r+1}). \tag{2.33}$$

We have $\mathcal{L}_r^2 = 1$ and $[\mathcal{L}_r, c_r] = [\mathcal{L}_r, c_r^\dagger] = 0$ such that:

$$\sigma_r^x \sigma_{r+1}^x = (c_r^\dagger + c_r)\mathcal{L}_r\mathcal{L}_{r+1}(c_{r+1}^\dagger + c_{r+1}). \tag{2.34}$$

Using that $[\sigma_r^z, \sigma_s^z] = 0$ for $r \neq s$ and $\left(\sigma_j^z\right)^2 = 1$ such that $\mathcal{L}_r\mathcal{L}_{r+1} = \sigma_r^z = 1 - 2c_r^\dagger c_r$ we get:

$$\begin{aligned} \sigma_r^x \sigma_{r+1}^x &= \left(c_r^\dagger + c_r\right)\left(1 - 2c_r^\dagger c_r\right)\left(c_{r+1}^\dagger + c_{r+1}\right) \\ &= \left(c_r^\dagger - c_r\right)\left(c_{r+1} + c_{r+1}^\dagger\right). \end{aligned} \tag{2.35}$$

In the fermion operators we get anomalous terms that do not conserve particle number but conserve parity. The Jordan-Wigner transformation can be summarized:

For the Jordan-Wigner transformation we defined the string operator:

$$\mathcal{L}_r = \prod_{j=1}^{r-1} \sigma_r^z. \tag{2.36}$$

With the string operator we could define the fermion raising and lowering operators:

$$c_r^\dagger = \sigma_r^- \mathcal{L}_r \qquad\qquad c_r = \sigma_r^+ \mathcal{L}_r. \tag{2.37}$$

Which enabled us to transform the spin operators:

$$\begin{aligned} \sigma_r^x &= \mathcal{L}_r \left(c_r^\dagger + c_r\right), \\ \sigma_r^z &= 1 - 2c_r^\dagger c_r. \end{aligned} \tag{2.38}$$

With these we could get the spin-spin interaction term in the Hamiltonian on a quadratic form:

$$\sigma_r^x \sigma_{r+1}^x = \left(c_r^\dagger - c_r\right)\left(c_{r+1} + c_{r+1}^\dagger\right). \tag{2.39}$$

2.2.2 Mapping spins to fermions

To get the TFI model from (2.27), we require that $h_r^x = 0$. We also need to have the resonator coupled to the tunneling matrix element as described by (2.23). One way to get $h_r^x = 0$ is to require the Coulomb interaction to be uniform, such that $U_{r+1,r} = U \Rightarrow h_r^x = \frac{\Delta_r}{2}$. Then we would also have to require the detuning to be zero. In the more realistic case the system is not uniform. We could then adjust the detuning locally at each site, such that it is equal to the capacitance between the neighboring DQDs. That would give $\frac{\Delta_r}{2} = U_{r+1,r} - U_{r,r-1}$ resulting in $h_r^x = 0$. There could therefore be a physical way of achieving a TFI model in a lab with a chain of DQDs.

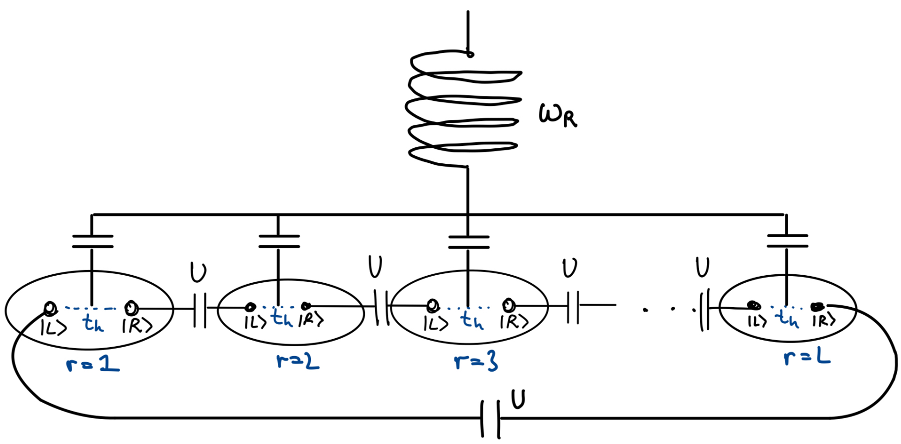

Figure 2.4: Chain of DQDs with a single resonator coupling uniformly to all DQD hopping elements

We will work with a uniform DQD chain with a single resonator coupling uniformly to all the DQD hopping terms. The system is shown in figure 2.4. We take $J_r^x \to J$, $h_r^z \to h$ and $g_r^h \to g$ and the detuning is adjusted such that $h_r^x = 0$ to get a uniform TFI model. The physical chain will have PBC such that $\sigma_{L+1}^\alpha = \sigma_1^\alpha$. This results in a Hamiltonian in terms of spin operators:

$$\begin{aligned} H_0^{TFI} &= -J \sum_{r=1}^{L} \sigma_r^x \sigma_{r+1}^x - h \sum_{r=1}^{L} \sigma_r^z, \\ H_{int}^{hop} &= g\left(a + a^\dagger\right) \sum_{r=1}^{L} \sigma_r^z \\ H_r &= \omega_R a^\dagger a. \end{aligned} \tag{2.40}$$

Remembering that the spin system is rotated, a spin in σ^z corresponds to $-\sigma^x$ in the physical system according to (2.21). Inserting (2.38) and (2.39) into this Hamiltonian gives a Hamiltonian that is quadratic in terms of the fermion operators:

$$\tilde{H} \;=\; \omega_R a_r^\dagger a_r \;-\; \sum_{r=1}^{L} \left(h - g[\,a^\dagger \;+\; a\,]\right)\left(1 - 2c_r^\dagger c_r\right) \;-\; J \sum_{r=1}^{L-1} \left(c_r^\dagger - c_r\right)\left(c_{r+1} + c_{r+1}^\dagger\right) \;-\; J\sigma_L^x \sigma_1^x. \tag{2.41}$$

The last term comes from the PBC for the spins. The other terms give the Hamiltonian for open boundary conditions:

$$\tilde{H} = H_{OBC} - J\sigma_L^x \sigma_1^x. \tag{2.42}$$

To work out the last term in terms of Jordan-Wigner fermions, we consider the boundery conditions. We have $\sigma_L^x \sigma_{L+1}^x = \sigma_L^x \sigma_1^x$, due to the PBC. From the Jordan-Wigner transformation we find:

$$\begin{aligned} \mathcal{L}_L &= \prod_{j=1}^{L-1} (1 - 2c_j^\dagger c_j) \\ &= \left(\prod_{j=1}^{L} (1 - 2c_j^\dagger c_j)\right)(1 - 2c_L^\dagger c_L) \\ &= \mathcal{L}_{L+1}(1 - 2c_L^\dagger c_L). \end{aligned} \tag{2.43}$$

The string operator $\mathcal{L}_{L+1}$ measures the parity of the number of fermions in the system, as $\mathcal{L}_{L+1} = (-1)^{\sum_{j=1}^{L} c_j^\dagger c_j} = (-1)^{\hat{N}}$, which is equal to $\mathcal{P}$. This is the same operator as the spin parity and we have shown that the eigenstates of the Hamiltonian will also be eigenstates of the parity operator. $\mathcal{P}$ will have eigenvalues ± 1 depending on the subspace we are in. The boundary terms give:

$$\begin{aligned} \sigma_L^x \sigma_1^x &= \mathcal{L}_L (c_L^\dagger + c_L) \mathcal{L}_1 (c_1^\dagger + c_1) \\ &= \mathcal{P}(1 - 2c_L^\dagger c_L)(c_L^\dagger + c_L)(c_1^\dagger + c_1) \\ &= -\mathcal{P}\left(c_L^\dagger - c_L\right)\left(c_1 + c_1^\dagger\right). \end{aligned} \tag{2.44}$$

The Hamiltonian then becomes:

$$\tilde{H} = H_{OBC} + \mathcal{P}\, J \left(c_L^\dagger - c_L\right)\left(c_1 + c_1^\dagger\right). \tag{2.45}$$

The P can be removed by considering the boundary conditions of the fermions:

$$\begin{aligned}\sigma^+_{L+1} &= \mathcal{L}_{L+1}c_{L+1} = \mathcal{P}c_{L+1} \\ &= \sigma^+_1 = c_1 \\ &\Rightarrow c_{L+1} = \mathcal{P}c_1,\end{aligned} \tag{2.46}$$

which when inserted into the Hamiltonian gives:

$$\tilde{H} = H_{OBC} + J\mathcal{P}\left(c^\dagger_L - c_L\right)\mathcal{P}\left(c_{L+1} + c^\dagger_{L+1}\right). \tag{2.47}$$

Evaluating $\mathcal{P}\left(c^\dagger_L - c_L\right)\mathcal{P}$ is quick if we just consider that $c^\dagger_L - c_L = -i\mathcal{L}_L\left(i\sigma^-_L - i\sigma^+_L\right) = -i\mathcal{L}_L\sigma^y_L$. $\mathcal{P}$ commutes with $\mathcal{L}_L$ which gives:

$$\begin{aligned}\mathcal{P}\left(c^\dagger_L - c_L\right)\mathcal{P} &= -i\mathcal{L}_L\prod_{j=1}^{L}\sigma^z_j\sigma^y_L\prod_{k=1}^{L}\sigma^z_k \\ &= +i\mathcal{L}_L\sigma^y_L \\ &= -\left(c^\dagger_L - c_L\right),\end{aligned} \tag{2.48}$$

where it was used that the spin-operators commute on different sites and they anticommute on the same site. We then have the Hamiltonian:

$$\begin{aligned}\tilde{H} &= H_{OBC} - J\left(c^\dagger_L - c_L\right)\left(c^\dagger_{L+1} + c_{L+1}\right) \\ &= \omega_R a^\dagger a - \sum_{r=1}^{L}\left(h - g[\,a^\dagger + a\,]\right)\left(1 - 2c^\dagger_r c_r\right) - J\sum_{r=1}^{L}\left(c^\dagger_r - c_r\right)\left(c_{r+1} + c^\dagger_{r+1}\right),\end{aligned} \tag{2.49}$$

where the fermion boundary conditions depend on the parity operator $\mathcal{P}$ through (2.46). We have shown that the Hamiltonian without a longitudinal field is symmetric under the total fermion parity $\mathcal{P}$ and we can project it onto an even or odd subspace using the projectors:

$$P_{even/odd} = \frac{1 \pm \mathcal{P}}{2}. \tag{2.50}$$

Since $\mathcal{P}^2 = 1$ we have $P^2_{even/odd} = P_{even/odd}$ which shows that $P_{even/odd}$ is indeed a projector. We can now project the Hamiltonian living in a 2^L dimensional Hilbert space onto an even or odd parity subspace of dimension 2^{L-1}:

$$\begin{aligned}H_{even/odd} &= P_{even/odd}\tilde{H}P_{even/odd} \\ &= \frac{1 \pm \mathcal{P}}{2}\tilde{H}\frac{1 \pm \mathcal{P}}{2} \\ &= \frac{\tilde{H} + \mathcal{P}\tilde{H}\mathcal{P} - \mathcal{P}\tilde{H} - \tilde{H}\mathcal{P}}{4} \\ &= \tilde{H}P_{even/odd},\end{aligned} \tag{2.51}$$

where in the last equation it was used that $\mathcal{P}\tilde{H}\mathcal{P} = \tilde{H}$ since they commute. The can thus be written in block diagonal form:

$$\tilde{H} = \begin{pmatrix} H_{even} & 0 \\ 0 & H_{odd} \end{pmatrix}. \tag{2.52}$$

The even and odd subspace Hamiltonian is just (2.49) but where the Hilbert space it operates on is made up of only even or odd parity states in the Fock space. It is now possible to work out the fermion boundary conditions. Understanding that the Hamiltonian will operate on either an odd or even parity subspace we define a test vector, $|f\rangle$, that belongs to one of those subspaces such that $\mathcal{P}|f\rangle = p|f\rangle$, with $p = \pm 1$. We commute $\mathcal{P}$ through c:

$$\begin{aligned}
\mathcal{P}c_r &= (1-2c_1^\dagger c_1)\cdots(1-2c_L^\dagger c_L)c_r \\
&= (1-2c_1^\dagger c_1)\cdots(1-2c_r^\dagger c_r)c_r\cdots(1-2c_L^\dagger c_L) \\
&= (1-2c_1^\dagger c_1)\cdots c_r\cdots(1-2c_L^\dagger c_L) \\
&= c_r\prod_{j\neq r}^{L}(1-2c_j^\dagger c_j).
\end{aligned} \tag{2.53}$$

The boundary condition from (2.46) is generalized to:

$$\begin{aligned}
\sigma_{r+L}^{+} &= \sigma_r \\
\Rightarrow \mathcal{L}_{r+L}c_{r+L} &= \mathcal{L}_r c_r \\
\Rightarrow c_{r+L} &= \mathcal{P}c_r,
\end{aligned} \tag{2.54}$$

where it was used that $\mathcal{L}_{r+L} = \prod_{j=1}^{r+L-1}\sigma_j^z = \prod_{j=1}^{L}\sigma_j^z\prod_{k=L+1}^{r-1+L}\sigma_k^z$, then since $\sigma_{x+L}^z = \sigma_x$ we get $\mathcal{L}_{r+L} = \mathcal{P}\mathcal{L}_r$. Applying $|f\rangle$ on the boundary condition equation gives:

$$\mathcal{P}c_r|f\rangle = c_r\prod_{j\neq r}^{L}(1-2c_j^\dagger c_j)|f\rangle. \tag{2.55}$$

The site r is either occupied or not. If $c_r^\dagger c_r|f\rangle = 0$ we have:

$$\mathcal{P}c_r|f\rangle = 0, \tag{2.56}$$

and the boundary condition is not important. If the site is occupied, then $\prod_{j\neq r}^{L}(1-2c_j^\dagger c_j)$ will measure the opposite parity of $\mathcal{P}$ since one less fermion is present, giving:

$$\begin{aligned}
\mathcal{P}c_r|f\rangle &= c_r(-1)^{\hat{N}-1}|f\rangle \\
&= -c_r p|f\rangle \\
\Rightarrow \mathcal{P}c_r &= -pc_r,
\end{aligned} \tag{2.57}$$

where it was used that $|f\rangle$ is an eigenvector of $\mathcal{P}$ with eigenvalue $p = \pm 1$ since it lived in the even or odd subspace. Therefore we get the boundary conditions:

$$c_{r+L} = -pc_r. \tag{2.58}$$

If the fermion parity is even with $p = 1$, then the fermions in that subspace will be required to have antiperiodic boundary conditions (APBC). If on the other hand we are in the odd parity subspace the fermions are required to have PBC. This will affect the Fourier transform.

2.2.3 Fourier transform

With the system being translationally invariant we look to perform a Fourier transformation. We will assume that we have an even number of DQDs. We define the Fourier transformation:

$$\begin{aligned} c_k &= \frac{1}{\sqrt{N}} \sum_{r=1}^{L} e^{-ikr} c_r \\ \Rightarrow c_r &= \frac{1}{\sqrt{N}} \sum_{k} e^{ikr} c_k. \end{aligned} \tag{2.59}$$

The wavenumbers, k, of the Fourier transformation depend on the boundary conditions through the fermion parity of the system as given by (2.44). If the fermion parity is odd with $p = -1$, we have PBC $c_{L+r} = c_r$. This gives from (2.59) that $e^{ikL} = 1 \Rightarrow kL = 2\pi n$ with n an integer. When the number of sites, N is even, it is convenient to choose k for PBC such that:

$$\text{p = -1} \Rightarrow \mathcal{K}_{odd} = \left\{ k = \frac{2\pi n}{L} \ , \ n = 1 - \frac{N}{2}, \ldots, 0 \ldots, \frac{N}{2} \right\}. \tag{2.60}$$

Except for the $k = 0, \pi$, all wavenumbers have a negative partner $-k$. If the fermion parity is even with $p = 1$, we have APBC $c_{L+1} = -c_1$. Following the same approach gives $e^{ikL} = -1 \Rightarrow kL = (2n-1)\pi$ with n an integer. It is convenient to choose k for APBC such that it is symmetric around 0:

$$\text{p=1} \Rightarrow \mathcal{K}_{even} = \left\{ k = \pm\frac{(2n-1)\pi}{L} \ , \ n = 1, \ldots, \frac{N}{2} \right\}. \tag{2.61}$$

This is equivalent to choosing $n = 1 - \frac{N}{2}, \ldots, 0 \ldots, \frac{N}{2}$ but makes summations easier when working with terms of both k and $-k$. For even parity all wavenumbers have negative partners. It was assumed that N is even.

If an uneven number of sites was chosen, then $\mathcal{K}_{odd} = \left\{ k = \frac{2\pi}{L} n \ , \ n = -\frac{N-1}{2} \ldots, 0, \frac{N-1}{2} \right\}$, so only $k = 0$ is unpaired. For the even case a k has an unpaired partner too. If we choose $n = -\frac{N-1}{2} \ldots, 0, \ldots \frac{N-1}{2}$ we get $\mathcal{K}_{even} = \left\{ k = \frac{2n-1}{L}\pi \ , \ n = -\frac{N-1}{2} \ldots, 0, \frac{N-1}{2} \right\}$ which leaves the $k = -\pi$ unpaired with a negative partner.

For both APBC and PBC we have the relation $\frac{1}{N} \sum_{r=1}^{L} e^{i(k-k')r} = \delta_{k,k'}$. By Fourier transforming the Hamiltonian (2.49) we arrive at:

$$\begin{aligned} \tilde{H} = \omega_R a^\dagger a + Ng\left(a^\dagger + a\right) + \sum_k \Big\{ &\left(2h - 2g\left[a^\dagger + a\right]\right) c_k^\dagger c_k \\ &- 2J\cos(k) c_k^\dagger c_k - iJ\sin(k)\left(c_k^\dagger c_{-k}^\dagger + c_k c_{-k}\right) \Big\}. \end{aligned} \tag{2.62}$$

To get the $i\sin(k)$ term we insert (2.59) into (2.49) to get:

$$\begin{aligned} \sum_{r=1}^{L} \left(c_r^\dagger c_{r+1}^\dagger + c_{r+1} c_r \right) &= \frac{1}{N} \sum_{k,k'} \sum_{r=1}^{L} \left(e^{-ikr} e^{-ik'(r+1)} c_k^\dagger c_{k'}^\dagger + e^{ik(r+1)} e^{ik'r} c_k c_{k'} \right) \\ &= \sum_{k,k'} \left(\delta_{k,-k'} e^{-ik'} c_k^\dagger c_{k'}^\dagger + \delta_{k,-k'} e^{ik} c_k c_{k'} \right) \\ &= \sum_{k} \left(e^{ik} c_k^\dagger c_{-k}^\dagger + e^{ik} c_k c_{-k} \right). \end{aligned}$$

The delta function will pick $k = 0$ and $k = \pi$ as their own negative partners. This is seen as $\sum_r e^{-i(0+k')r}$ is zero unless k' is also 0 and similar for $\sum_r e^{-i(\pi+k')r}$ with $k' = \pi$. The wavenumber $-\pi$ is not an element in the k-space. Since two fermions are annihilated or created with the same momentum, the $k = 0, \pi$ terms will be zero in the sum over anomalous terms. Making use of the operators anticommuting, we can subtract $0 = \frac{1}{2}\{c_k, c_{k'}\} = \frac{1}{2}\left\{c_k^\dagger, c_{k'}^\dagger\right\}$, thus getting:

$$= \sum_k \left\{ \frac{1}{2}\left(e^{ik} c_k^\dagger c_{-k}^\dagger - e^{ik} c_{-k}^\dagger c_k^\dagger\right) + \frac{1}{2}\left(e^{ik} c_k c_{-k} - e^{ik} c_{-k} c_k\right) \right\},$$

and upon changing dummy index on half the terms so $k \to -k$ we get:

$$\begin{aligned} &= \sum_k \left\{ \frac{1}{2}\left(e^{-ik} - e^{ik}\right) c_{-k}^\dagger c_k^\dagger + \frac{1}{2}\left(e^{-ik} - e^{ik}\right) c_{-k} c_k \right\} \\ &= -i \sum_k \sin(k) \left(c_{-k}^\dagger c_k^\dagger + c_{-k} c_k\right). \end{aligned} \tag{2.63}$$

Splitting the sum into one for positive and one for negative ks give:

$$iJ \sum_k \sin(k) c_{-k} c_k = iJ \left(\sum_{k>0} \sin(k) c_{-k} c_k + \sum_{k<0} \sin(k) c_{-k} c_k \right). \tag{2.64}$$

Changing the dummy index from $k \to -k$ and using the anticommutation relations gives:

$$\begin{aligned} &= iJ \left(\sum_{k>0} \sin(k) c_{-k} c_k + \sum_{k>0} \sin(-k) c_k c_{-k} \right) \\ &= 2iJ \sum_{k>0} \sin(k) c_{-k} c_k. \end{aligned} \tag{2.65}$$

We can then write a Hamiltonian for $k > 0$:

$$H_k = \left[2h - 2g\left[a^\dagger + a\right] - 2J\cos(k)\right]\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}\right) - 2iJ\sin(k)\left(c_k c_{-k} + c_k^\dagger c_{-k}^\dagger\right). \tag{2.66}$$

Using the sets defined in (2.60) and (2.61) we define the positive k-values as:

$$\begin{aligned} \mathcal{K}_{even}^+ &= \left\{ k = \frac{2n-1}{L}\pi, n = 1, \ldots, \frac{N}{2} \right\} \\ \mathcal{K}_{odd}^+ &= \left\{ k = \frac{2n}{L}\pi, n = 1, \ldots, \frac{N}{2} - 1 \right\}, \end{aligned} \tag{2.67}$$

such that the Hamiltonian (2.62) can be rewritten for even parity as:

$$\tilde{H}_{even} = \omega_R a^\dagger a + Ng[a + a^\dagger] + \sum_k^{\mathcal{K}_{even}^+} H_k, \tag{2.68}$$

and for odd parity we include the $k = 0, \pi$ terms:

$$\begin{aligned} \tilde{H}_{odd} = \omega_R a^\dagger a + Ng[a + a^\dagger] + \sum_k^{\mathcal{K}_{odd}^+} H_k + \left(2h - 2g[a^\dagger + a] - 2J\right) c_0^\dagger c_0 \\ + \left(2h - 2g[a^\dagger + a] + 2J\right) c_\pi^\dagger c_\pi, \end{aligned} \tag{2.69}$$

where the difference in the sign of $2J$ due to k being 0 or π in $\cos k$ and $\sin k = 0$. To ease notation we define:

$$H_{0,\pi} = \left(2h - 2g[a^\dagger + a] - 2J\right) c_0^\dagger c_0 + \left(2h - 2g[a^\dagger + a] + 2J\right) c_\pi^\dagger c_\pi. \tag{2.70}$$

Preparing for a Bogoliubov transformation we rearrange the fermion operators such that H_k can be written as two by two matrix. We then also define the vector $\vec{c}_k^\dagger = (c_k^\dagger, c_{-k})$. Rearranging gives:

$$\begin{aligned}\sum_k^{\mathcal{K}_{e/o}^+} H_k &= \sum_k^{\mathcal{K}_{e/o}^+} \left\{ \left[2h - 2g\left[a^\dagger + a\right] - 2J\cos(k)\right] \left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}\right) - 2iJ\sin(k)\left(c_k^\dagger c_{-k}^\dagger - c_{-k}c_k\right)\right\} \\ &= \sum_k^{\mathcal{K}_{e/o}^+} \left\{ \left[2h - 2g\left[a^\dagger + a\right] - 2J\cos(k)\right] \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger + 1\right) - 2iJ\sin(k)\left(c_k^\dagger c_{-k}^\dagger - c_{-k}c_k\right)\right\}.\end{aligned} \tag{2.71}$$

Evaluating the term with no fermion operators gives:

$$\begin{aligned}\sum_k^{\mathcal{K}_{e/o}^+} \left\{2h - 2g\left[a^\dagger + a\right] - 2J\cos(k)\right\} &= 2\sum_k^{\mathcal{K}_{e/o}^+} h - 2\sum_k^{\mathcal{K}_{e/o}^+} g\left(a + a^\dagger\right) - 2J\sum_k^{\mathcal{K}_{e/o}^+}\cos(k) \\ &= 2\sum_k^{\mathcal{K}_{e/o}^+} h - 2\sum_k^{\mathcal{K}_{e/o}^+} g\left(a + a^\dagger\right),\end{aligned} \tag{2.72}$$

where it was used that $\sum_k^{\mathcal{K}_{e/o}^+} \cos(k) \to \frac{L}{2\pi}\int_0^\pi \cos(k)dk = 0$ or it can be seen since cosine is odd around $\frac{\pi}{2}$. The sum $2\sum_k^{\mathcal{K}_{e/o}^+} h$ is equal to Nh for the even parity Hamiltonian and $h\,(N-2)$ for the odd. We can subtract Nh from the full Hamiltonian, since that only amounts to a global phase, and then have an extra $-2h$ on $\tilde{H}_{odd}$. The rearrange H_k is now:

$$H_k' = \left\{\left[2h - 2g\left[a^\dagger + a\right] - 2J\text{cos}(k)\right]\left(c_k^\dagger c_k - c_{-k}c_{-k}^\dagger\right) - 2iJ\text{sin}(k)\left(c_k^\dagger c_{-k}^\dagger - c_{-k}c_k\right)\right\}. \tag{2.73}$$

The even Hamiltonian (2.68) is then:

$$\begin{aligned}\tilde{H}_{even} &= \omega_R a^\dagger a + Ng[a + a^\dagger] + \sum_k^{\mathcal{K}_{even}^+}\left(H_k' - 2g\left[a + a^\dagger\right]\right) \\ &= \omega_R a^\dagger a + \sum_k^{\mathcal{K}_{even}^+} H_k'.\end{aligned} \tag{2.74}$$

For the odd Hamiltonian we get:

$$\begin{aligned}\tilde{H}_{odd} &= \omega_R a^\dagger a + Ng[a + a^\dagger] + \sum_k^{\mathcal{K}_{odd}^+}\left(H_k' - 2g\left[a + a^\dagger\right]\right) + H_{0,\pi} - 2h \\ &= \omega_R a^\dagger a + Ng[a + a^\dagger] - 2\left(\frac{N}{2} - 1\right) g\left[a + a^\dagger\right] + \sum_k^{\mathcal{K}_{odd}^+} H_k' + H_{0,\pi} - 2h \\ &= \omega_R a^\dagger a + 2g[a + a^\dagger] - 2h + \sum_k^{\mathcal{K}_{odd}^+} H_k' + H_{0,\pi}\end{aligned} \tag{2.75}$$

We can absorb the loose $-2h + 2g[a + a^\dagger]$ into $H_{0,\pi}$ such that:

$$H'_{0,\pi} = -2J\left(c_0^\dagger c_0 - c_\pi^\dagger c_\pi\right) + \left(2h - 2g\left[a^\dagger + a\right]\right)\left(c_0^\dagger c_0 + c_\pi^\dagger c_\pi - 1\right), \tag{2.76}$$

giving:

$$\tilde{H}_{odd} = \omega_R a^\dagger a + \sum_k^{\mathcal{K}^+_{odd}} H'_k + H'_{0,\pi}. \tag{2.77}$$

H'_k can then be written in matrix form:

$$H'_k = \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} 2h - 2g\left[a^\dagger + a\right] - 2J\text{cos}k & -2iJ\text{sin}(k) \\ 2iJ\text{sin}(k) & -\left(2h - 2g\left[a^\dagger + a\right] - 2J\text{cos}k\right) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}. \tag{2.78}$$

We define

$$\begin{aligned} \alpha_k &= 2h - 2g\left[a^\dagger + a\right] - 2J\text{cos}(k), \\ \beta_k &= 2J\text{sin}(k), \end{aligned} \tag{2.79}$$

for ease of notation. We define τ_i as Pauli matrices on the the two-dimensional space of the H_k matrix. We can not get further diagonalizing this matrix as there is no inverse to the boson creation and annihilation operators. If we integrate out the bosons, the matrix can be diagonalized by rotation in the $\{\tau_x, \tau_y, \tau_z\}$ space with a unitary transformation. Using the fermion vectors defined by:

$$\vec{c}_k^{\,\dagger} = (c_k^\dagger, c_{-k}) \quad , \quad \vec{c}_k = \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \tag{2.80}$$

we have H'_k as:

$$H'_k = \vec{c}_k^{\,\dagger} \left(\alpha_k \tau_z + \beta_k \tau_y\right) \vec{c}_k. \tag{2.81}$$

2.3 Coupling a coherent resonator to the uniform TFI

Since the term $a^\dagger + a$ in β_k is not invertible, we can not simply diagonalize the Hamiltonian even if it is quadratic in the fermion operators.

2.3.1 Integrating out the bosons

To diagonalize the Hamiltonian we assume that the bosonic system, the resonator, is prepared in a pure coherent state such that $\rho_R = |\Phi\rangle\langle\Phi|$. The coherent state has average photon number $N = \langle a^\dagger a\rangle$ and is an eigenstate to the annihilation operator $a|\Phi\rangle = \Phi|\Phi\rangle$. This might not be an unreasonable approximation since a resonator coupled to a classical feedline will be in a coherent state as shown in Appendix B. Since Φ is a complex number we have $\langle\Phi|\left(a + a^\dagger\right)|\Phi\rangle = 2\text{Re}\left[\Phi\right]$. We integrate out the bosons:

$$H_{\text{eff}} = \frac{1}{Z}\text{Tr}_r\left[\rho \tilde{H}_{odd/even}\right] = \langle\Phi|\tilde{H}_{odd/even}|\Phi\rangle. \tag{2.82}$$

For $\tilde{H}_{even}$ we then get:

$$H_{even,eff} = \omega_R |\Phi|^2 + \sum_{k}^{\mathcal{K}^+_{even}} \langle\Phi| H'_k |\Phi\rangle\,, \tag{2.83}$$

and for $\tilde{H}_{odd}$ we get:

$$H_{odd,eff} = \omega_R |\Phi|^2 + \sum_{k}^{\mathcal{K}^+_{odd}} \langle\Phi| H'_k |\Phi\rangle + \langle\Phi| H'_{0,\pi} |\Phi\rangle\,. \tag{2.84}$$

The average $\langle\Phi| H_{0,\pi} |\Phi\rangle$ gives:

$$\langle\Phi| H_{0,\pi} |\Phi\rangle = -2J\left(c_0^\dagger c_0 - c_\pi^\dagger c_\pi\right) + (2h - 4g\mathrm{Re}\,[\Phi])\left(c_0^\dagger c_0 + c_\pi^\dagger c_\pi - 1\right). \tag{2.85}$$

After integrating out the bosons, the β_k in H'_k that before had bosons operators in it, now only depends on the complex number Φ as $\langle\Phi| \beta_k |\Phi\rangle = 2h - 4g\mathrm{Re}\,[\Phi] - 2J\cos k$. The matrix $\langle\Phi| H'_k |\Phi\rangle$ is now diagonalizable. To do so we introduce the rotation in the Pauli matrices:

$$R_x(\theta) = e^{i\theta\frac{\tau_x}{2}} = \cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}\tau_x, \tag{2.86}$$

that rotates a Pauli matrix around the τ_x axis by θ. The matrix in (2.81) makes an angle $\tan\theta_k = \frac{\beta_k}{\alpha_k}$ with the τ_z axis so we rotate by θ_k. The rotation is unitary such that $R_x(\theta)^\dagger = R_x^{-1}(\theta)$. Inserting unities gives:

$$\langle\Phi| H'_k |\Phi\rangle = \vec{c}_k^{\,\dagger} R_x^\dagger(\theta_k) R_x(\theta_k)(\alpha_k\tau_z - \beta_k\tau_y) R_x^\dagger(\theta_k) R_x(\theta_k)\vec{c}_k, \tag{2.87}$$

with $\vec{c}_k^{\,\dagger} = \begin{pmatrix} c_k^\dagger & c_{-k}\end{pmatrix}$. The rotated matrix is found as:

$$R_x(\theta_k)(\alpha_k\tau_z - \beta_k\tau_y)R_x^\dagger(\theta_k) = \begin{pmatrix} \cos\frac{\theta_k}{2} & -i\sin\frac{\theta_k}{2} \\ -i\sin\frac{\theta_k}{2} & \cos\frac{\theta_k}{2}\end{pmatrix}\begin{pmatrix} \alpha_k & -i\beta_k \\ i\beta_k & -\alpha_k\end{pmatrix}\begin{pmatrix} \cos\frac{\theta_k}{2} & i\sin\frac{\theta_k}{2} \\ i\sin\frac{\theta_k}{2} & \cos\frac{\theta_k}{2}\end{pmatrix} = M. \tag{2.88}$$

Calculating the off-diagonal elements give:

$$\begin{aligned} M_{1,2} &= -2\alpha_k\cos\frac{\theta_k}{2}\sin\frac{\theta}{2} + i\beta_k\left(\cos^2\frac{\theta_k}{2} - \sin^2\frac{\theta}{2}\right) \\ &= -\alpha_k\sin\theta + i\beta_k\cos\theta, \end{aligned}$$

where it was used that $\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2}$ and $\cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 1 - 2\sin^2\frac{x}{2} = 1 - 2\left(\sqrt{\frac{1-\cos x}{2}}\right)^2 = \cos x$. The sine and cosine can be carried out using $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$ and $\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$. The two identities can be proven by drawing a right triangle with angle θ, adjacent side of length 1 and opposite side with length x. The off-diagonal elements are then found to give:

$$M_{1,2} = -i\alpha_k\frac{\frac{\beta_k}{\alpha_k}}{\sqrt{1 + (\frac{\beta_k}{\alpha_k})^2}} + i\beta_k\frac{1}{\sqrt{1 + (\frac{\beta_k}{\alpha_k})^2}} = 0.$$

By repeating the calculation it is found that also $M_{2,1} = -M_{1,2} = 0$. The diagonal elements are found to be:

$$\begin{aligned} M_{1,1} &= \alpha_k\cos\theta_k + \beta_k\sin\theta_k \\ &= \sqrt{\alpha_k^2 + \beta_k^2} \\ &= \sqrt{(2h - 4g\mathrm{Re}[\Phi] - 2J\cos k)^2 + (2J\sin k)^2}. \end{aligned}$$

This is defined as:

$$E_k = 2\sqrt{(h - 2g\text{Re}[\Phi] - J\cos k)^2 + (J\sin k)^2}. \tag{2.89}$$

The other diagonal is found to be $M_{2,2} = -E_k$. The rotated matrix is thus diagonal:

$$R_x(\theta_k)(\alpha_k\tau_z - \beta_k\tau_y)R_x^\dagger(\theta_k) = \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} = M. \tag{2.90}$$

The energy is seen to be symmetric in k, $E_k = E_{-k}$. We rotate the fermion vectors:

$$\begin{aligned} R_x(\theta_k)\vec{c}_k &= \begin{pmatrix} \cos\frac{\theta_k}{2} & -i\sin\frac{\theta_k}{2} \\ -i\sin\frac{\theta_k}{2} & \cos\frac{\theta_k}{2} \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \\ &= \vec{\gamma}_k = \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix}. \end{aligned} \tag{2.91}$$

The new operators still obey Fermi statistics. To show that they are fermions we first need to note that $\theta_{-k} = -\theta_k$. Using that $\gamma_k = \cos\frac{\theta_k}{2}c_k - i\sin\frac{\theta_k}{2}c_{-k}^\dagger$ we work out the anticommutation:

$$\begin{aligned} \left\{\gamma_k, \gamma_{k'}^\dagger\right\} &= \left\{\cos\frac{\theta_k}{2}c_k - i\sin\frac{\theta_k}{2}c_{-k}^\dagger, \cos\frac{\theta_{k'}}{2}c_{k'}^\dagger + i\sin\frac{\theta_k'}{2}c_{-k'}\right\} \\ &= \cos\frac{\theta_k}{2}\cos\frac{\theta_{k'}}{2}\delta_{k,k'} + \sin\frac{\theta_k}{2}\sin\frac{\theta_k'}{2}\delta_{k,k'} = \delta_{k,k'}. \end{aligned}$$

In the rotated basis the fermion contribution is thus:

$$\begin{aligned} \langle\Phi| H_k' |\Phi\rangle &= \vec{\gamma}_k^\dagger \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} \vec{\gamma}_k \\ &= E_k\gamma_k^\dagger\gamma_k - E_k\gamma_{-k}\gamma_{-k}^\dagger \\ &= E_k\gamma_k^\dagger\gamma_k + E_{-k}\gamma_{-k}^\dagger\gamma_{-k} - \frac{E_k}{2} - \frac{E_{-k}}{2} \\ &= E_k\left(\gamma_k^\dagger\gamma_k - \frac{1}{2}\right) + E_{-k}\left(\gamma_{-k}^\dagger\gamma_{-k} - \frac{1}{2}\right), \end{aligned}$$

where the anticommutation was used and $E_k = E_{-k} \Rightarrow E_{-k} = \frac{E_k}{2} + \frac{E_{-k}}{2}$.

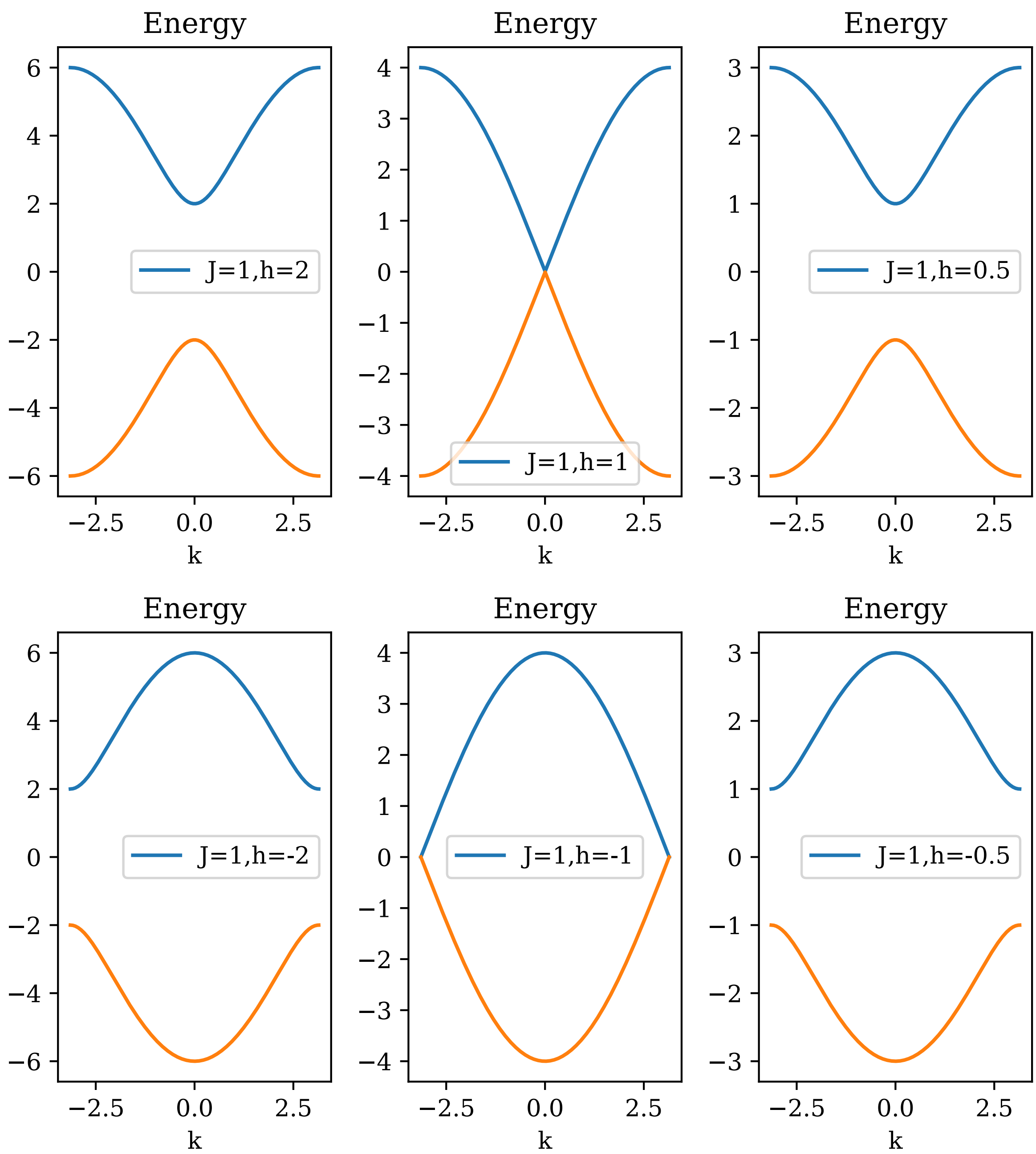

Figure 2.5: The energy spectrum E_k and $-E_k$ as function of k in the case of $g = 0$. If $g \neq 0$ the h would be shifted.

The term $H_{0,\pi}$ (2.70) from the odd parity Hamiltonian should also be written in the new basis. We had $\theta_k = \arctan \frac{\beta_k}{\alpha_k}$. For both $k = 0$ and $k = \pi$ we have $\beta_{k=0,\pi} = J \sin(0, \pi) = 0$. For α_k we have $\alpha_{k=0,\pi} = h - 2g\text{Re}[\Phi] \pm J$. Except for the case of $h - 2g\text{Re}[\Phi] \pm J = 0$ we have $\theta_{0,\pi} = 0$ such that:

$$\gamma_{k=0} = c_{k=0} \qquad \gamma_{k=\pi} = c_{k=\pi}. \tag{2.92}$$

If however we are in the fine tuned case where $h - 2g\text{Re}[\Phi] \pm J = 0$ we need to be more careful. In the case where $h - 2g\text{Re}[\Phi] = J$ we have for the term $\frac{\beta_k}{\alpha_k}$:

$$\lim_{k\to 0} \frac{J\text{sin}k}{h - 2g\text{Re}[\Phi] - J\text{cos}k} = \lim_{k\to 0} \frac{\text{sin}k}{1 - \text{cos}k} \\ = \lim_{k\to 0} \frac{\text{cos}k}{\text{sin}k} \to \infty, \tag{2.93}$$

by using L'Hospitals rule. Similarly if $h - 2g\text{Re}[\Phi] = -J$ we get the same for $k \to \pi$. We have that $\arctan(0) = 0$, and $\arctan(\infty) = \frac{\pi}{2}$. Following this recipe we find that $h - 2g\text{Re}\Phi = J$ results in:

$$\gamma_0 = \frac{c_0 - ic_0^\dagger}{\sqrt{2}}, \qquad \gamma_\pi = c_\pi, \tag{2.94}$$

and we get that $\gamma_0^\dagger = i\gamma_0$. In the case of $h - 2g\text{Re}\Phi = -J$:

$$\gamma_0 = c_0, \qquad \gamma_\pi = \frac{c_\pi + ic_\pi^\dagger}{\sqrt{2}}, \tag{2.95}$$

where we have $\gamma_\pi^\dagger = -i\gamma_\pi$.

2.3.2 Ground state

We can now find the even parity eigenstates from the APBC Hamiltonian (2.74) using the diagonal matrix (2.92). The ground state will be the rotated fermion vacuum. In the rotated fermion basis we have:

$$\begin{aligned} H_{even,eff} &= \omega_R|\Phi|^2 + \sum_k^{\mathcal{K}_{even}^+} \left\{ E_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) + E_{-k} \left(\gamma_{-k}^\dagger \gamma_{-k} - \frac{1}{2} \right) \right\} \\ &= \omega_R|\Phi|^2 + \sum_k^{\mathcal{K}_{even}^+} E_k \left(\gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k} - 1 \right) \\ &= \omega_R|\Phi|^2 + \sum_k^{\mathcal{K}_{even}} E_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right), \end{aligned} \tag{2.96}$$

where we now sum over all of the N k-values in $\mathcal{K}_{even}$. We see that the ground state is the state with no rotated fermions, the Bogoliubov vacuum, as the particle energy is strictly positive. The ground state by annihilating all rotated fermions from initial fermion vacuum. We get a BCS-like ground state on the form:

$$|\emptyset_\gamma\rangle = \prod_{k>0}^{\mathcal{K}_{even}^+} \frac{1}{A_k} \gamma_{-k}\gamma_k |0\rangle, \tag{2.97}$$

where A_k normalizes the state. We find the ground state in terms of the Jordan-Wigner fermions:

$$\begin{aligned} |\emptyset_\gamma\rangle &= \prod_{k>0} \frac{1}{A_k} \left(\cos\frac{\theta_k}{2} c_{-k} + i \sin\frac{\theta_k}{2} c_k^\dagger \right) \left(\cos\frac{\theta_k}{2} c_k - i \sin\frac{\theta_k}{2} c_{-k}^\dagger \right) |0\rangle \\ &= \prod_{k>0} \frac{1}{A_k} \left(-i \cos\frac{\theta_k}{2} \sin\frac{\theta_k}{2} + \sin^2\frac{\theta_k}{2} c_k^\dagger c_{-k}^\dagger \right) |0\rangle \\ &= \prod_{k>0} \frac{1}{A_k} \left(-\frac{i}{2} \sin\theta_k + \frac{1 - \cos\theta_k}{2} c_k^\dagger c_{-k}^\dagger \right) |0\rangle, \end{aligned} \tag{2.98}$$

where trigonometric identities where used for the $\sin\theta_k\cos\theta_k$ and $\sin^2\theta_k$ terms. We also used that $c_l\left|0\right\rangle = 0$ and $\theta_k = -\theta_{-k}$. To normalize the state we calculate:

$$
\begin{aligned}
&\left\langle 0\right| \left(i\cos\frac{\theta_k}{2}\sin\frac{\theta_k}{2} + \sin^2\frac{\theta_k}{2} c_{-k}c_k \right) \frac{1}{|A_k|^2} \left(-i\cos\frac{\theta_k}{2}\sin\frac{\theta_k}{2} + \sin^2\frac{\theta_k}{2} c_k^\dagger c_{-k}^\dagger \right) \left|0\right\rangle \\
=&\frac{1}{|A_k|^2}\left(\cos\frac{\theta_k}{2}\sin\frac{\theta_k}{2}\right)^2 + \frac{1}{|A_k|^2}\sin^4\frac{\theta_k}{2} \\
=&\frac{1}{|A_k|^2}\left(\cos^2\frac{\theta_k}{2} + \sin^2\frac{\theta_k}{2}\right)\sin^2\frac{\theta_k}{2} \\
=&\frac{|i\sin\frac{\theta_k}{2}|^2}{|A_k|^2},
\end{aligned} \tag{2.99}
$$

giving $A = -i\sin\frac{\theta_k}{2}$. The resulting normalized ground state is found to be:

$$
\left|\emptyset_\gamma^{even}\right\rangle = \prod_{k>0}^{\mathcal{K}_{even}^+} \left(\cos\frac{\theta_k}{2} + i\sin\frac{\theta_k}{2} c_k^\dagger c_{-k}^\dagger \right) \left|0\right\rangle . \tag{2.100}
$$

Since we sum over $k > 0$ due to the ABC there are no divergence in the $\cot\frac{\theta_k}{2}$ since θ_k will never be zero. It can be seen that the state (2.100) is even in fermion parity since it consists of pairs of fermion operators with opposite momentum and the fermion parity is thus conserved. The corresponding energy of the state is read directly from (2.96) as:

$$
E_0^{Even} = \omega_R|\Phi|^2 - \frac{1}{2}\sum_k^{\mathcal{K}_{even}} E_k, \tag{2.101}
$$

remembering that the sum is over all N values of k in $\mathcal{K}_{even}$. For odd parity an extra fermion is needed. We can construct a BCS ground state for $\sum_k^{\mathcal{K}_{odd}^+} H_k$ similar to the even parity case but that would also contain an even number of fermions. Luckily $H_{0,\pi}$ can contribute the needed fermion. We look for a single fermion from $H_{0,\pi}$.

For $J, (h - 2g\mathrm{Re}[\Phi]) > 0$ the lowest energy excitation will be $E_{k=0}$, and we add $c_{k=0}^\dagger$ to the ground state. If $J < 0$ and $h - 2g\mathrm{Re}[\Phi] > 0$, then the lowest energy excitation comes $E_{k=\pi}$. From now on we will work with $J, h > 0$ such that the single fermion in the odd parity state comes from $k = 0$. From the odd parity Hamiltonian:

$$
H_{odd,eff} = \omega_R|\Phi|^2 + \sum_k^{\mathcal{K}_{odd}^+} E_k\left(\gamma_k^\dagger\gamma_k + \gamma_{-k}^\dagger\gamma_{-k} - 1\right) + \left\langle\Phi\right| H_{0,\pi}\left|\Phi\right\rangle, \tag{2.102}
$$

the odd parity ground state is constructed:

$$
c_0^\dagger\left|\emptyset_\gamma^{odd}\right\rangle = c_0^\dagger\prod_{\pi>k>0}^{\mathcal{K}_{odd}^+}\left(\cos\frac{\theta_k}{2} + i\sin\frac{\theta_k}{2}c_k^\dagger c_{-k}^\dagger\right)\left|0\right\rangle . \tag{2.103}
$$

The energy is found from (2.84) and (2.85):

$$
\begin{aligned}
E_0^{odd} &= \omega_R|\Phi|^2 - \sum_k^{\mathcal{K}_{odd}\setminus\{0,\pi\}} E_k - 2J \\
&= \omega_R|\Phi|^2 - 2\sum_k^{\mathcal{K}_{odd}^+} E_k - 2J,
\end{aligned} \tag{2.104}
$$

where it was used that $\sum_k^{\mathcal{K}_{odd}\setminus\{0,\pi\}} E_k = 2\sum_k^{\mathcal{K}_{odd}^+} E_k$ since $E_k = E_{-k}$. The set $\mathcal{K}_{odd}$ without the $k = 0, \pi$ contains $N-2$ elements, two fewer than $\mathcal{K}_{even}$. The energy difference between the even and odd sector ground states is found to be::

$$\delta E_0 = E_0^{odd} - E_0^{even} = -2\sum_k^{\mathcal{K}_{odd}^+} E_k - 2J + 2\sum_k^{\mathcal{K}_{even}^+} E_k. \tag{2.105}$$

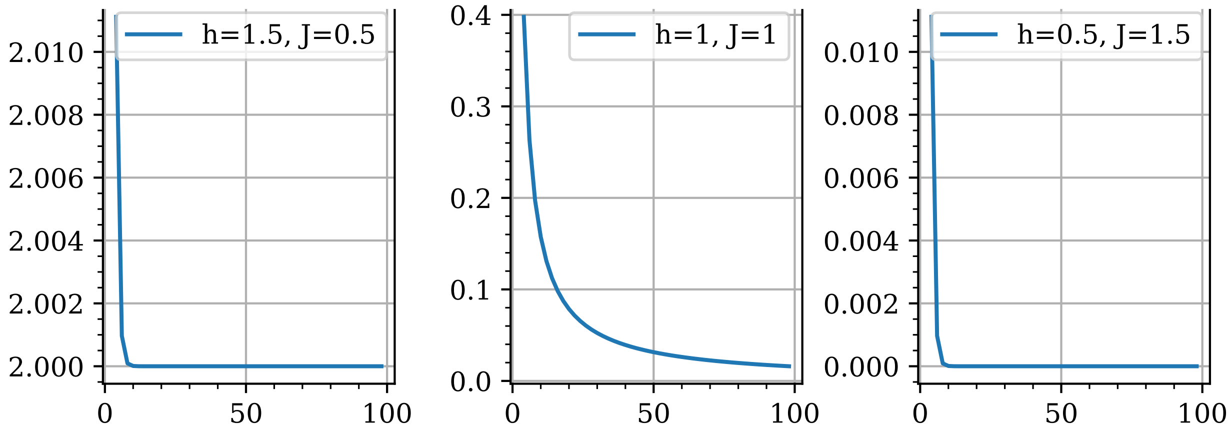

Figure 2.6: δE_0 as a function of L plotted for different values of J and h with $g = 0$. It is numerically seen that the difference in ground state energy between the even/odd sector is exponentially decreasing for $J \neq h$ but decreases as a power law for $J = h$.

We test out two limiting cases of the energy difference. In the paramagnetic case of only a transverse field h, we expect the energy difference to be $2h$, since we lose $-h$ from the aligned spin and have to pay h to flip it the other way. With $J = 0$ we have $E_k = 2h$ giving:

$$\begin{aligned}\delta E_0 &= -(N-2)2h + 2Nh \\ &= 2h.\end{aligned} \tag{2.106}$$

In the ferromagnetic case with $h = 0$, we have $E_k = 2J$ and we get $\delta E_0 = 0$, recovering the degenerate ground state.

2.3.3 Treating the resonator as a classical oscillator

Understanding the fermion ground state and energy, we can study the effect of the Jordan-Wigner fermions on the resonator state. We treat the resonator classically. For a harmonic oscillator we have $\langle a^\dagger + a\rangle = \sqrt{2}\frac{x}{\ell} = \sqrt{2}X$ with $\ell = \sqrt{\frac{\hbar}{m\omega}}$ the natural length of the oscillator and x the classical oscillator length. We also have $\mathrm{Im}\Phi = \frac{\langle a - a^\dagger\rangle}{2i} = \frac{P}{2i}\sqrt{\frac{2}{\hbar m\omega_R}} = \frac{P}{\sqrt{2\hbar m\omega_R}}$. Using the Hamiltonian for the even subspace (2.96) we have in terms of the classical position and momentum coordinates:

$$H_{eff,even} = \hbar\omega_R\frac{X^2}{2} + \frac{P^2}{2m} + 2\sum_k^{\mathcal{K}_{even}} E_k(X)\left(\gamma_k^\dagger\gamma_k - \frac{1}{2}\right). \tag{2.107}$$

The fermion energy supplies a non-quadratic term to the classical resonator potential. The energy written in the unitless $h/J = Y$ and X is:

$$E_k(X) = 2J\sqrt{\sin^2 k + \left(Y - \frac{\sqrt{2}gX}{J} - \cos k\right)^2}. \tag{2.108}$$

The classical potential is:

$$V(X) = \hbar\omega_R \frac{X^2}{2} + 2J \sum_{k}^{\mathcal{K}_{even}} \sqrt{\sin^2 k + \left(Y - \frac{\sqrt{2}gX}{J} - \cos k\right)^2} \left(\gamma_k^\dagger \gamma_k - \frac{1}{2}\right). \tag{2.109}$$

We see that each fermion will increase the non-quadratic term in the potential. For the odd subspace we would sum over $\mathcal{K}_{odd}$ and add the $-2J$ from the single extra fermion. To plot the potential we estimate some of the parameters. In appendix A the experimental parameter range is explored and it is not unreasonable to assume that we could have:

$$\omega_R \approx J \approx h \approx 10g. \tag{2.110}$$

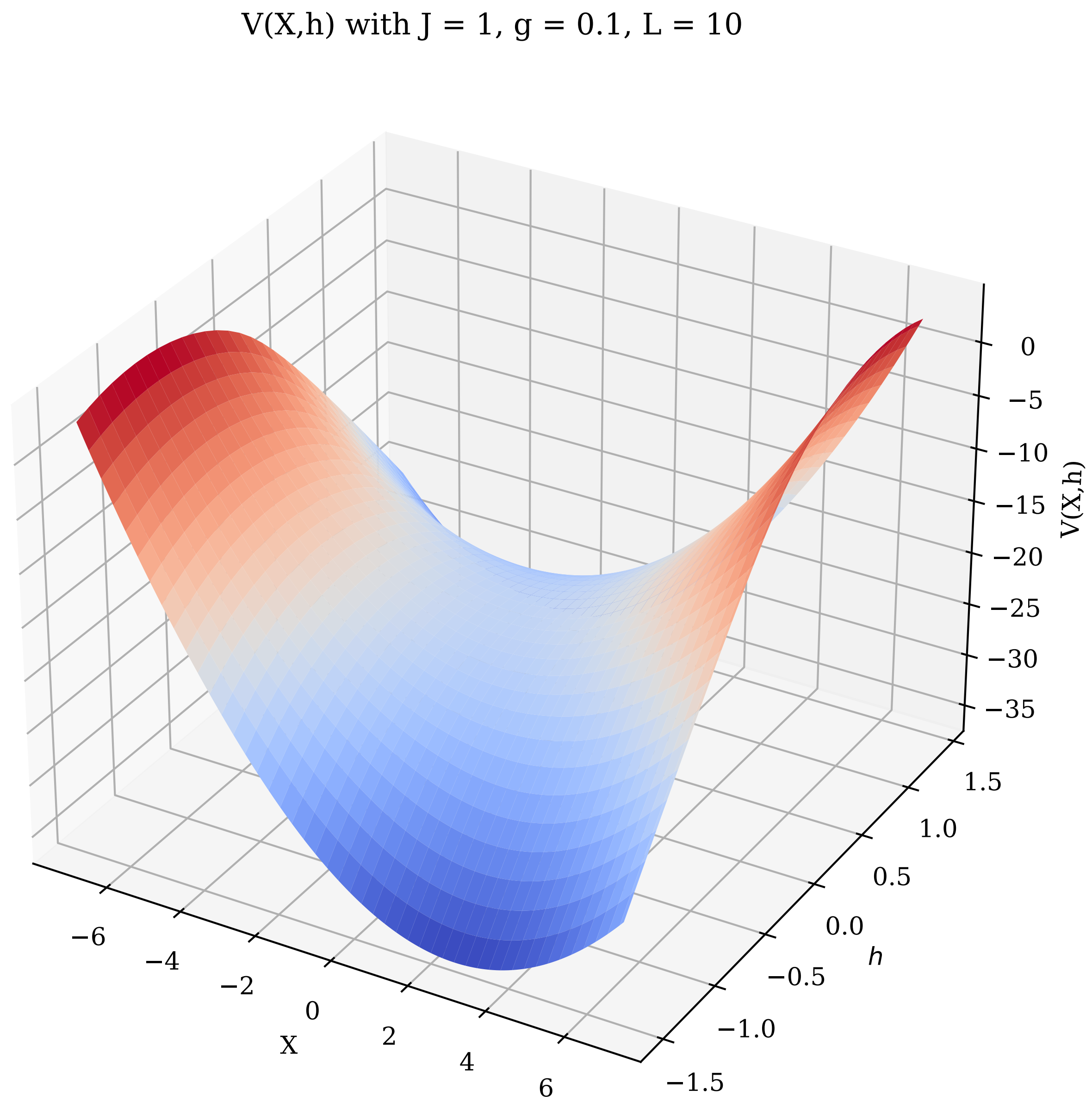

Figure 2.7: Surface plot of the classical spring potential where the units are in terms of ω_R with the fermions in the ground state

The energy integral is differentiable in all points. By differentiating the potential energy with respect to X we find the displacement that minimizes the potential energy from the fermion ground state. The potential is:

$$V(X) = \hbar\omega_R \frac{X^2}{2} - 2\sum_{k}^{\mathcal{K}_{even}} \sqrt{(J\sin k)^2 + \left(h - \sqrt{2}gX - J\cos k\right)^2}, \tag{2.111}$$

and differentiating with respect to the resonator coordinate, X we find:

$$\frac{\partial V}{\partial X} = \hbar\omega_R X + 2\sqrt{2}g \sum_{k}^{\mathcal{K}_{even}} \frac{h - \sqrt{2}gX - J\cos k}{\sqrt{\left(h - \sqrt{2}gX - J\cos k\right)^2 + (J\sin k)^2}}. \tag{2.112}$$

Solving for the extremum gives:

$$
\begin{aligned}
&\left.\frac{\partial V}{\partial X}\right|_{X=X_0} = 0 \\
&\Rightarrow X_0 = -2\sqrt{2}\frac{g}{\omega_R}\sum_k^{\mathcal{K}_{even}} \frac{h-\sqrt{2}gX_0 - J\cos k}{\sqrt{\left(h-\sqrt{2}gX_0-J\cos k\right)^2+\left(J\sin k\right)^2}}.
\end{aligned}
\tag{2.113}
$$

The result is an equation for the resonator position that minimizes the potential coming from coupling the resonator to the DQD-chain. This equation will require solving an elliptical integral so it must be done either numerically, with some approximations or in some limit. We found that the resonator will react to the state of the DQD-chain and we can understand the coupling between the resonator and chain from the resonators perspective.

We have solved the case of a TFI-model coupled to a single resonator in a coherent state. We found an equation for the position coordinate of the resonator that minimized the classical potential from the DQD chain. Getting to this result required the assumption that the resonator was in a coherent state. This motivates us to further understand the quantum resonator's response from coupling to a DQD system. In the next chapter we will study the case of many resonators coupled to many DQDs.

Chapter 3

Collective excitations of resonators coupled to a double quantum dot chain

3.1 The saddle point equation

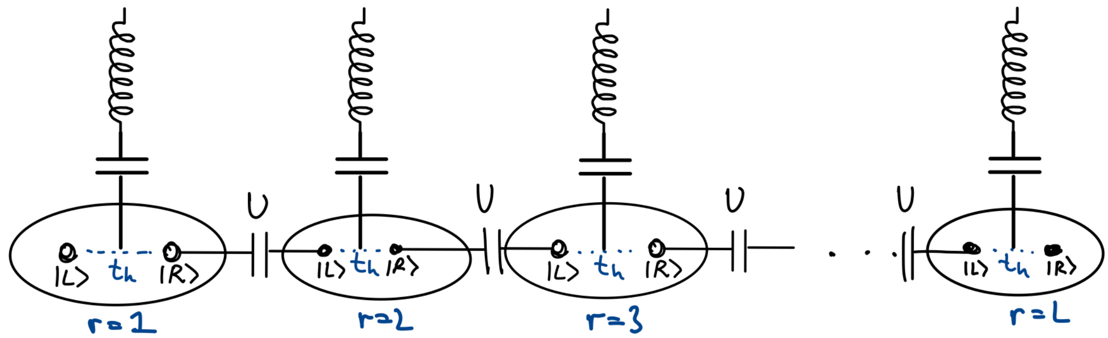

Figure 3.1: Physical system

We will study a periodic DQD chain with a resonator coupled locally to each DQD. We start from the transverse field Ising model in (2.49) that we got from a Jordan-Wigner transformation. The system is described by a Hamiltonian for the resonators:

$$H_R = \omega_R \sum_{r=1}^{L} a_r^\dagger a_r, \tag{3.1}$$

and for the DQD chain we have:

$$H_0 = \sum_{r=1}^{L} \left[-h \left(1 - 2c_r^\dagger c_r \right) - J \left(c_r^\dagger c_{r+1} + c_{r+1}^\dagger c_r \right) - J \left(c_r^\dagger c_{r+1}^\dagger + c_{r+1} c_r \right) \right]. \tag{3.2}$$

We will neglect the constant term coming from $-h$. The interaction between the two is described by:

$$H_{int} = \frac{g}{2} \sum_{r=1}^{L} \left(a_r^\dagger + a_r \right) \left(1 - 2c_r^\dagger c_r \right), \tag{3.3}$$

We study collective excitations from the imaginary time path integral with the action given by $S = \int_0^\beta d\tau \left[\bar{\phi} \partial_\tau \phi + H(\bar{\phi}, \phi) - \mu N(\bar{\phi}, \phi) \right]$ [10]:

$$\mathcal{Z} = \int \mathcal{D} \left[\bar{\psi}, \psi \right] \int \mathcal{D} \left[\bar{\phi}, \phi \right] e^{-S_R[\bar{\psi},\psi] - S_0[\bar{\phi},\phi] - S_{int}[\bar{\psi},\psi,\bar{\phi},\phi]}. \tag{3.4}$$

We have the action for the resonator:

$$S_R = \sum_{r=1}^{L} \int_0^\beta d\tau \bar{\phi}_r(\tau) \left(\partial_\tau + \hbar\omega_R \right) \phi_r(\tau) + g \sum_{r=1}^{L} \int_0^\beta d\tau \frac{\bar{\phi}_r(\tau) + \phi_r(\tau)}{2}, \tag{3.5}$$

the DQD chain:

$$S_0 = \sum_{r=1}^{L} \int_0^\beta d\tau \left\{ \bar{\psi}_r(\tau) \left(\partial_\tau + 2h \right) \psi_r(\tau) - J \left[\bar{\psi}_r(\tau) \psi_{r+1}(\tau) + \bar{\psi}_{r+1}(\tau) \psi_r(\tau) \right] \right.$$
$$\left. - J \left[\bar{\psi}_r(\tau) \bar{\psi}_{r+1}(\tau) + \psi_{r+1}(\tau) \psi_r(\tau) \right] \right\}, \tag{3.6}$$

and the interaction term:

$$S_{int} = -g \sum_{r=1}^{L} \int_0^\beta d\tau \left[\bar{\phi}_r(\tau) + \phi_r(\tau) \right] \bar{\psi}_r(\tau) \psi_r(\tau). \tag{3.7}$$

The ψ's are Grassmann fields and ϕ's are complex fields. The chemical potentials have been absorbed into ω_R and $2h$. The functional integral measure is defined as $\lim_{N\to\infty} \prod_n^N d(\bar{\psi}^n, \psi^n)$, where $d(\bar{\psi}^n, \psi^n) = \prod_\nu d\bar{\psi}_\nu d\phi_\nu$ for Grassmann numbers (for complex numbers divide by π in the product) and the ν is the quantum numbers which is r in our case, and N comes from the parameterization of the temperature. We have absorbed the term linear in the resonator fields into the resonator action. The transverse field, h acts as a chemical potential for the fermions.

Motivated by the effect of the DQD chain on a single classical resonator, our goal will be to integrate out the Grassmann fields to find an effective theory for the resonators. We begin by transforming from imaginary time and real space to Matsubara and Fourier space. We use the Matsubara frequencies $\omega_m = \frac{2\pi m}{\beta}$ for bosons and $\omega_n = \frac{(2n+1)\pi}{\beta}$ for fermions, with $n \in \mathbb{Z}$:

$$\chi(\tau) = \frac{1}{\sqrt{\beta}} \sum_n \chi_n e^{-i\omega_n \tau}, \qquad \bar{\chi}(\tau) = \frac{1}{\sqrt{\beta}} \sum_n \bar{\chi}_n e^{i\omega_n \tau}, \tag{3.8}$$

where χ could be a fermion or boson field and one would have to adjust the frequency used accordingly. Throughout the thesis, $i\omega_n$ with an n will refer to a fermionic Matsubara frequency and $i\omega_m$ with an m will refer to a bosonic one. For the Fourier transformation we use:

$$\chi_r = \frac{1}{\sqrt{N}} \sum_k \chi_k e^{ikr}, \tag{3.9}$$

where the sum is taken from some set of wave numbers. With this convention both the χ_k and χ_r fields will be unitless. The fields χ_n will have units of $\sqrt{\beta}$. The momentum is $p = \hbar k$ from the momentum operator $\hat{p} = -i\hbar\partial_r$.

We have to consider the boundary conditions. We chose a chain with an even number of sites. We know that the Hilbert space is block diagonal in parity, so the spinless fermion states will be in either of the sub-spaces and they do not interact if nothing breaks the symmetry. We choose anti-periodic boundary conditions for the fermions in real space and only work with odd parity Jordan-Wigner fermions. This gives a k-space for the fermions defined as:

$$\mathcal{K}_{\text{even}} = \left\{ k = \pm \frac{2n-1}{L}\pi \quad , \quad n = 1, ..., \frac{N}{2} \right\}. \tag{3.10}$$

The wave numbers are symmetric around 0 and $k = 0$ is not in the set. The fermion wave numbers will be denoted with a k and the boson wavenumbers, that come with periodic boundary conditions, are denoted with a q. The transformed action for the resonator is:

$$S_R = \frac{g\sqrt{\beta N}}{2}\left(\phi^*_{0,0} + \phi_{0,0}\right) + \sum_q \sum_{i\omega_m} \bar{\phi}_{q,\omega_m}\left(-i\omega_m + \omega_R\right)\phi_{q,i\omega_m}. \tag{3.11}$$

That last term comes from $\sum_r \int d\tau e^{-iqr + i\omega_m\tau} = \beta N \delta_q \delta_{i\omega_m}$. For the fermion chain we get:

$$S_0 \;=\; \sum_k \sum_{i\omega_n} \Bigg\{ \bar{\psi}_{k,i\omega_n}\left(-i\omega_n + 2h - 2J\cos k\right)\psi_{k,i\omega_n} \;+\; iJ\sin k\left(\bar{\psi}_{-k,-i\omega_n}\bar{\psi}_{k,i\omega_n} + \psi_{-k,-i\omega_n}\psi_{k,i\omega_n}\right)\Bigg\}. \tag{3.12}$$

To see where the $i\sin k$ term comes from we use that for Grassman fields $\psi\psi' = \frac{\psi\psi' + \psi\psi'}{2} = \frac{\psi\psi' - \psi'\psi}{2}$ giving:

$$\sum_k \psi_k \psi_{-k} e^{ik} = \frac{1}{2}\sum_k \left(\psi_k\psi_{-k} - \psi_{-k}\psi_k\right)e^{ik} \tag{3.13}$$

$$= \frac{1}{2}\sum_k \psi_k\psi_{-k}\left(e^{ik} - e^{-ik}\right) \tag{3.14}$$

$$= i\sum_k \sin k \psi_k \psi_{-k}, \tag{3.15}$$

and similar for the $\bar{\psi}_k$ term. The interaction term turns out to be:

$$S_{int} = -\frac{g}{\sqrt{\beta N}} \sum_{k,k'} \sum_{i\omega_n, i\omega_{n'}} \left[\phi_{k-k', i\omega_n - i\omega_{n'}} + \phi^*_{k'-k, i\omega_{n'} - i\omega_n}\right]\bar{\psi}_{k,i\omega_n}\psi_{k',i\omega_{n'}}, \tag{3.16}$$

and the interaction is seen to conserve energy and momentum. To make notation easier we will define the vector $k = (k, i\omega_n)$ and then remember that a sum $\sum_k = \sum_k \sum_{i\omega_n}$. Preparing for a Bogoliubov transformation we rewrite the action:

$$S_0 = \sum_k \Bigg\{ \frac{1}{2}\bar{\psi}_k\left(-i\omega_n + 2h - 2J\cos k\right)\psi_k$$

$$- \frac{1}{2}\psi_k\left(-i\omega_n + 2h - 2J\cos k\right)\bar{\psi}_k + iJ\sin k\left(\psi_{-k}\psi_k - \bar{\psi}_k\bar{\psi}_{-k}\right)\Bigg\}, \tag{3.17}$$

an upon letting $k \to -k$ meaning $k, i\omega_n \to -k, -i\omega_n$, we get:

$$= \sum_k \Bigg\{ \frac{1}{2}\bar{\psi}_k \left(-i\omega_n + 2h - 2J\cos k\right)\psi_k - \frac{1}{2}\psi_{-k}\left(i\omega_n + 2h - 2J\cos k\right)\bar{\psi}_{-k} + iJ\sin k\left(\psi_{-k}\psi_k - \bar{\psi}_k\bar{\psi}_{-k}\right)\Bigg\}, \tag{3.18}$$

which is written on matrix form as:

$$S_0 = \sum_k \left(\bar{\psi}_k, \psi_{-k}\right) \begin{pmatrix} \frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right) & -iJ\sin k \\ iJ\sin k & -\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) \end{pmatrix} \begin{pmatrix} \psi_k \\ \bar{\psi}_{-k} \end{pmatrix} \tag{3.19}$$

$$= \sum_{k,k'} \Psi_k^\dagger \mathcal{G}_{0,k}^{-1}\delta_{k,k'}\Psi_{k'}, \tag{3.20}$$

where we defined the spinors:

$$\Psi_k^\dagger = \left(\bar{\psi}_k, \psi_{-k}\right) \qquad \Psi_k = \begin{pmatrix} \psi_k \\ \bar{\psi}_{-k} \end{pmatrix}, \tag{3.21}$$

and the operator $\mathcal{G}_{0,k}^{-1}\delta_{k,k'}$ that is diagonal in k and $i\omega_n$-space but not in spinor space. The interaction is also put on matrix form:

$$S_{int} = -\frac{g}{2\sqrt{\beta N}} \sum_{k,k'} \left[\phi_{k-k'} + \phi^*_{k'-k}\right]\left(\bar{\psi}_k\psi_{k'} - \psi_{k'}\bar{\psi}_k\right)$$

$$= -\frac{g}{2\sqrt{\beta N}} \sum_{k,k'} \left(\left[\phi_{k-k'} + \phi^*_{k'-k}\right]\bar{\psi}_k\psi_{k'} - \left[\phi_{k'-k} + \phi^*_{k-k'}\right]\psi_{-k'}\bar{\psi}_{-k}\right)$$

$$= -\frac{g}{2\sqrt{\beta N}} \sum_{k,k'} \left(\left[\phi_{k-k'} + \phi^*_{k'-k}\right]\bar{\psi}_k\psi_{k'} - \left[\phi_{k-k'} + \phi^*_{k'-k}\right]\psi_{-k}\bar{\psi}_{-k'}\right)$$

$$= \sum_{k,k'} \left(\bar{\psi}_k, \psi_{-k}\right) \begin{pmatrix} -\frac{g}{2\sqrt{\beta N}}\left[\phi_{k-k'} + \phi^*_{k'-k}\right] & 0 \\ 0 & +\frac{g}{2\sqrt{\beta N}}\left[\phi_{k-k'} + \phi^*_{k'-k}\right] \end{pmatrix} \begin{pmatrix} \psi_{k'} \\ \bar{\psi}_{-k'} \end{pmatrix} \tag{3.22}$$

$$= \sum_{k,k'} \Psi_k^\dagger \Gamma_{k-k'}\Psi_{k'}, \tag{3.23}$$

where in the second line we took $k, k' \to -k, -k'$ and third line we did $k \to k', k' \to k$, and we then defined the matrix $\Gamma_{k-k'}$ that is not diagonal in k,$i\omega_n$. Explicitly we have the matrix:

$$\Gamma_{k-k'} = -\frac{g}{2\sqrt{\beta N}}\left[\phi_{k-k'} + \phi^*_{k'-k}\right]\sigma^z, \tag{3.24}$$

where the σ^z operates on the spinor space. The action is quadratic in the Grassmann fields and the gassian integral can be performed:

$$\int \mathcal{D}\left[\bar{\psi}, \psi\right] e^{-S_0 - S_{int}} = \int \mathcal{D}\left[\bar{\psi}, \psi\right] e^{-\sum_{k,k}\Psi_k^\dagger\left(\mathcal{G}_{0,k}^{-1}\delta_{k,k'} + \Gamma_{k-k'}\right)\Psi_{k'}} \tag{3.25}$$

$$= \int \mathcal{D}\left[\bar{\psi}, \psi\right] e^{-\Psi^\dagger\hat{\mathcal{G}}^{-1}\Psi} \tag{3.26}$$

$$= \mathrm{Det}\left(\hat{\mathcal{G}}_F^{-1}\right), \tag{3.27}$$

where $\hat{\mathcal{G}}_F^{-1}$ lives in k, $i\omega_n$ and spinor space, so the determinant is taken over all three spaces. Written in k,$i\omega_n$-space the inverse Green's function is:

$$\hat{\mathcal{G}}^{-1}_{F,kk',i\omega_n i\omega_{n'}} = \mathcal{G}^{-1}_{0,k}\delta_{k,k'}\delta_{n,n'} + \Gamma_{k-k'}, \tag{3.28}$$

which is a two by two matrix in spinor space. We denote the spinor space by the indices σ where eg:

$$\hat{\mathcal{G}}^{-1}_{F,kk',i\omega_n i\omega_{n'},12} = iJ\sin k\delta_{k,k'}\delta_{n,n'}, \tag{3.29}$$

$$\hat{\mathcal{G}}^{-1}_{F,kk',i\omega_n i\omega_{n'},11} = \frac{1}{2}\left(-i\omega_n + 2h - J\cos k\right)\delta_{k,k'}\delta_{n,n'} - \frac{g}{2\sqrt{\beta N}}\left[\phi_{k-k',i\omega_n - i\omega_{n'}} + \phi^*_{k'-k,i\omega_{n'}-i\omega_n}\right]. \tag{3.30}$$

The determinant is re-exponentiated: [1]

$$\text{Det}\left(\hat{\mathcal{G}}_F^{-1}\right) = e^{\ln\left(\text{Det}\hat{\mathcal{G}}_F^{-1}\right)} \tag{3.31}$$

$$= e^{\text{tr}\left(\ln\hat{\mathcal{G}}_F^{-1}\right)}, \tag{3.32}$$

giving the partition function:

$$\mathcal{Z} = \int \mathcal{D}\left[\phi^*,\phi\right] e^{-S_R + \text{tr}\left(\ln\hat{\mathcal{G}}_F^{-1}\right)} \tag{3.33}$$

$$= \int \mathcal{D}\left[\phi^*,\phi\right] e^{-g\frac{\sqrt{\beta N}}{2}(\phi_0^* + \phi_0) - \sum_q \phi_q^*(-i\omega_m + \omega_R)\phi_q + \text{tr}\left(\ln\hat{\mathcal{G}}_F^{-1}\right)}, \tag{3.34}$$

with the bosonic action from integrating out the fermions:

$$S_{eff}[\phi^*,\phi] = \frac{g\sqrt{\beta N}}{2}\left(\phi_0^* + \phi_0\right) + \sum_q \phi_q^*\left(-i\omega_m + \omega_R\right)\phi_q - \text{tr}\left(\ln\hat{\mathcal{G}}_F^{-1}\right). \tag{3.35}$$

We seek a stationary phase saddle point solutions to the action in terms of the boson fields ϕ. We require:

$$\frac{\delta S_{eff}[\phi^*,\phi]}{\delta\phi_q} = 0. \tag{3.36}$$

Differentiating the trace gives[10]:

$$\frac{\delta}{\delta\phi_q}\text{tr}\left(\ln\hat{\mathcal{G}}_F^{-1}\right) = \text{tr}\left[\hat{\mathcal{G}}_F\left(\frac{\delta}{\delta\phi_q}\hat{\mathcal{G}}_F^{-1}\right)\right]. \tag{3.37}$$

We use that an element $\left[\hat{\mathcal{G}}_F\left(\frac{\delta}{\delta\phi_q}\hat{\mathcal{G}}_F^{-1}\right)\right]_{kk',\sigma_1\sigma_2} = \sum_{k''}\sum_{\sigma_3}\mathcal{G}_{F,kk'',\sigma_1\sigma_3}\left(\frac{\delta}{\delta\phi_q}\hat{\mathcal{G}}^{-1}_{F,k''k',\sigma_3\sigma_2}\right)$, and remember that $k = (k, i\omega_n)$. With this in mind the trace is found as:

$$= \text{tr}\left[\hat{\mathcal{G}}_F\left(\frac{\delta}{\delta\phi_q}\hat{\mathcal{G}}_F^{-1}\right)\right] = \sum_{k,k'}\sum_{\sigma_1,\sigma_2}\mathcal{G}_{F,kk',\sigma_1\sigma_2}\left(\frac{\delta}{\delta\phi_q}\hat{\mathcal{G}}^{-1}_{F,k'k,\sigma_2\sigma_1}\right). \tag{3.38}$$

The goal is now to evaluate the functional differential. The matrix element to be differentiated is:

$$\hat{\mathcal{G}}^{-1}_{F,k'k,\sigma_2\sigma_1} = \left[\mathcal{G}^{-1}_{0,k}\delta_{k',k} + \Gamma_{k'-k}\right]_{\sigma_1,\sigma_2}. \tag{3.39}$$

[1] $\ln(\text{Det A}) = \ln\prod_n \lambda_n = \sum_n \ln\lambda_n = \text{tr}\ln A$

There is no dependence on the bosonic ϕ_q in $\mathcal{G}_0^{-1}$ so that term will be zero when differentiating. Γ is diagonal in spinor space so the differentiation yields:

$$\frac{\delta}{\delta\phi_q}\hat{\mathcal{G}}^{-1}_{F,k'k,\sigma_2\sigma_1} = \frac{\delta}{\delta\phi_q}\Gamma_{k'-k,\sigma_1\sigma_2} \tag{3.40}$$

$$= \frac{\delta}{\delta\phi_q}\frac{-g}{2\sqrt{\beta N}}\left[\phi_{k'-k} + \phi^*_{k-k'}\right]\sigma^z_{\sigma_1\sigma_2}, \tag{3.41}$$

which gives us that:

$$\frac{\delta}{\delta\phi_q}\hat{\mathcal{G}}^{-1}_{F,k'k,\sigma_2\sigma_1} = -\frac{g}{2\sqrt{\beta N}}\delta_{q,k'-k}\sigma^z_{\sigma_1\sigma_2}. \tag{3.42}$$

As a small check we see that q contains a bosonic frequency and that $k'-k$ will also be bosonic. Plugging in to (3.38) and carrying out the trace we find:

$$\frac{\delta}{\delta\phi_q}\mathrm{tr}\left(\ln\hat{\mathcal{G}}_F^{-1}\right) = \sum_{k,k'}\sum_{\sigma_1,\sigma_2}\mathcal{G}_{F,kk',\sigma_1\sigma_2}\frac{-g}{2\sqrt{\beta N}}\delta_{q,k'-k}\sigma^z_{\sigma_1\sigma_2} \tag{3.43}$$

$$= -\frac{g}{2\sqrt{\beta N}}\sum_k\left[\mathcal{G}_{F,11}(k,q+k) - \mathcal{G}_{F,22}(k,q+k)\right]. \tag{3.44}$$

The 11 and 22 represents respectively the $\bar{\psi}_k\psi_{-k}$ and $\psi_k\bar{\psi}_{-k}$ products in the action. Combining everything in the saddle point equation (3.36), gives:

$$\frac{\delta S_{eff}[\phi^*,\phi]}{\delta\phi_q} = 0$$

$$\Rightarrow \phi^*_q\left(-i\omega_m + \omega_R\right) + \frac{g\sqrt{\beta N}}{2}\delta_q = -\frac{g}{2\sqrt{\beta N}}\sum_k\left[\mathcal{G}_{F,11}(k,q+k) - \mathcal{G}_{F,22}(k,q+k)\right], \tag{3.45}$$

where for the second term on the LHS it was used that $\frac{\delta\phi_0}{\delta\phi_q} = \delta_q$.

Saddle point guess: $\phi_q = 0$

We guess a solution where the field is uniformly zero such that $\phi_q = 0\,\forall\, q$. We check if the guess satisfies the saddle point equation. Since $\phi^*_q = (\phi_q)^* = 0$ we only need to show that:

$$\frac{g}{2\sqrt{\beta N}}\sum_k\left[\mathcal{G}_{F,11}(k,q+k) - \mathcal{G}_{F,22}(k,q+k)\right]_{\phi_q=0} = -\frac{g\sqrt{\beta N}}{2}\delta_q, \tag{3.46}$$

for all q. For checking the units it is used that N is just a number with no units such that the units do match up. As another check we see that before dividing with g on both sides we could take the limit of no interaction, $g\to 0$ and see that the saddle point equation is then satisfied for $\phi_q = 0\,\forall\, q$ as both sides are equal zero. This is expected since for $g=0$ we have free bosons where the lowest energy configuration is uniformly zero. Since $\phi_q = \phi^*_q = 0$ we have $\mathcal{G}^{-1}_{int} = 0$, giving:

$$\mathcal{G}_F^{-1}|_{\phi_q=0} = \mathcal{G}_0^{-1}. \tag{3.47}$$

$\mathcal{G}_0^{-1}$ is diagonal in k, implying that $\mathcal{G}_{0,kk'}$ also is diagonal in k. Therefore:

$$\mathcal{G}_0^{-1} = \delta_k \mathcal{G}_0^{-1}(k). \tag{3.48}$$

Since $\mathcal{G}_{0,k}^{-1}$ is a two by two matrix as seen in (3.20), we find the inverse. The determinant of $\mathcal{G}_{0,k}^{-1}$ in spinor space is found as:

$$\begin{aligned}
\text{Det}G_0^{-1}(k) &= \text{Det}\begin{pmatrix} \frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right) & -iJ\sin k \\ iJ\sin k & -\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) \end{pmatrix} && (3.49)\\
&= -\frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right)\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) + \left(iJ\sin k\right)^2 && (3.50)\\
&= -\frac{1}{4}\left[\left(2h - 2J\cos k\right)^2 - \left(i\omega_n\right)^2\right] - \frac{1}{4}\left(2J\sin k\right)^2 && (3.51)\\
&= -\frac{1}{4}\left[\left(2h - 2J\cos k\right)^2 + \left(2J\sin k\right)^2 - \left(i\omega_n\right)^2\right] && (3.52)\\
&= \frac{(i\omega_n)^2 - \epsilon_k^2}{4}, && (3.53)
\end{aligned}$$

where $\epsilon_k = \sqrt{\left(2J\sin k\right)^2 + \left(2h - 2J\cos k\right)^2}$ is the energy of the Jordan-Wigner fermion as expected. The inverse of the matrix is thus:

$$G_{0,k} = \frac{4}{(i\omega_n)^2 - \epsilon_k^2}\begin{pmatrix} -\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) & +iJ\sin k \\ -iJ\sin k & \frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right) \end{pmatrix}. \tag{3.54}$$

Collecting everything gives:

$$\mathcal{G}_F(k,k')|_{\phi_q=0} = \frac{4}{(i\omega_n)^2 - \epsilon_k^2}\begin{pmatrix} -\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) & +iJ\sin k \\ -iJ\sin k & \frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right) \end{pmatrix}\delta_{k,k'}. \tag{3.55}$$

Inserting the result into the saddle point equation (3.46) gives:

$$\begin{aligned}
-\frac{g\sqrt{\beta N}}{2}\delta_q &= \frac{g}{2\sqrt{\beta N}}\sum_k \left[\mathcal{G}_{F,11}(k,q+k) - \mathcal{G}_{F,22}(k,q+k)\right]\\
&= \frac{g}{2\sqrt{\beta N}}\sum_k \frac{4}{(i\omega_n)^2 - \epsilon_k^2}\left[-\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) - \frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right)\right]\delta_{k,q+k}\\
&= \frac{g}{2\sqrt{\beta N}}\sum_k \frac{4}{\epsilon_k^2 - (i\omega_n)^2}\left(2h - 2J\cos k\right)\delta_q, && (3.56)
\end{aligned}$$

from using the diagonal elements, 11 and 22 in $\mathcal{G}_F|_{\phi_q=0}$. The Dirac delta gives zero for all $q \neq 0$ so for all $\phi_{q\neq 0} = 0$ the action is minimized, as both sides are equal zero. We conclude that $\phi_q = 0 \quad q \neq 0$ satisfies the saddle point equation. However, $q = 0$ is included in the resonator q-space since the resonators have periodic boundary conditions in real space. For $q = 0$ and $g \neq 0$ (since we divide by g) we get:

$$-\frac{N}{T} = \sum_k \frac{4}{-(i\omega_n)^2 + \epsilon_k^2}\left(2h - 2J\cos k\right), \tag{3.57}$$

with k being the fermionic two-vector for the resonator wavenumber and Matsubara frequency. In the special case that $h = 0$ we can carry out the sum on RHS:

$$
\begin{aligned}
&4\sum_{i\omega_n}\sum_k \frac{2h - 2J\cos k}{-(i\omega_n)^2 + (2J\sin k)^2 + (2h - 2J\cos k)^2}\Bigg|_{h=0} \\
=& -4\sum_{i\omega_n}\sum_k \frac{2J\cos k}{-(i\omega_n)^2 + 4J^2(\sin^2 k + \cos^2 k)} \\
=& 4\sum_{i\omega_n} \frac{2J}{(i\omega_n)^2 - 4J^2} \sum_k \cos k = 0,
\end{aligned} \tag{3.58}
$$

and it is seen that even for no transverse field the uniformly zero boson field is no solution to the saddle point equation. In the case of non-zero, h, we carry out the Matsubara sum, using that the poles of $f(z) = (z^2 - \epsilon^2)^{-1}$ are in $z = \pm\epsilon$ and that the Fermi function $\frac{1}{e^{\beta z}+1} = n_F(z)$ has poles in $z = i\omega_n = i\frac{2n+1}{\beta}\pi$. Following [11] the following integral over the entire complex plane is written down:

$$
I = \oint_{\mathcal{C}_\infty} \frac{dz}{2\pi i} f(z) n_F(z) e^{\tau z}. \tag{3.59}
$$

The $e^{\tau z}$ factor is for convergence and $\beta > \tau > 0$. For the integral taken over the entire complex plane it is found that:

$$
n_F(z)e^{\tau z} = \frac{e^{\tau z}}{e^{\beta z}+1} \propto \begin{cases} e^{(\tau-\beta)\mathrm{Re}[z]} & \mathrm{Re}[z] > 0 \\ e^{\tau \mathrm{Re}[z]} & \mathrm{Re}[z] < 0 \end{cases} \to 0 \quad \mathrm{Re}[z] \to \infty, \tag{3.60}
$$

giving that the integral contour goes to zero along the edge. This gives from the residue theorem that the integral is zero such that:

$$
\begin{aligned}
I &= 0 \\
&= \sum \mathrm{Res}\left[f(z) n_F(z) e^{\tau z}\right] \\
&= \sum_{i\omega_n} \operatorname*{Res}_{z=i\omega_n}\left[n_F(z)\right] f(i\omega_n) e^{i\tau\omega_n} + \sum_{z_j=\pm\epsilon} \operatorname*{Res}_{z=z_j}\left[f(z)\right] n_F(z_j) e^{\tau z}.
\end{aligned} \tag{3.61}
$$

The poles of $f(z)$ are simple so they are found to be $\operatorname*{Res}_{z=\pm\epsilon} f(z) = \frac{1}{\pm 2\epsilon}$. For $n_F(z)$ the residue gives:

$$
\operatorname*{Res}_{z=i\omega_n} n_F(z) = -\frac{1}{\beta}. \tag{3.62}
$$

This is shown from the residue for a simple pole: $\lim_{z\to i\omega_n} \frac{z - i\omega_n}{e^{\beta z}+1} = \lim_{\delta\to 0} \frac{\delta}{e^{i\beta\omega_n} e^{\beta\delta}+1}$, where it is used that $e^{i\beta\omega_n} = 1$ to get $= \lim_{\delta\to 0} \frac{\delta}{-\beta\delta + \mathcal{O}(\delta^2)} = \frac{1}{-\beta}$. It results in the formula:

$$
\sum_{i\omega_n} f(i\omega_n) = \beta \sum_j \operatorname*{Res}_{z=z_j}\left[f(z_j)\right] n_F(z_j). \tag{3.63}
$$

The Matsubara sum then evaluated as:

$$
0 = \frac{1}{-\beta} \sum_{i\omega_n} \frac{1}{-(i\omega_n)^2 + \epsilon^2} e^{i\tau\omega_n} + \frac{1}{2\epsilon} n_F(\epsilon) e^{\epsilon\tau} - \frac{1}{2\epsilon} n_F(-\epsilon) e^{-\epsilon\tau}. \tag{3.64}
$$

The limit of $\tau \to 0$ is then taken since the integral has converged, giving:

$$\begin{aligned}\sum_{i\omega_n} \frac{1}{i\omega_n^2 + \epsilon^2} &= \frac{\beta}{2\epsilon} n_F(\epsilon) - \frac{\beta}{2\epsilon} n_F(-\epsilon) \\ &= \frac{\beta}{2\epsilon}\big(n_F(\epsilon) - n_F(-\epsilon)\big) \\ &= \frac{\beta}{\epsilon}\left(n_F(\epsilon) - \frac{1}{2}\right). \end{aligned} \tag{3.65}$$

This can be reduced:

$$\begin{aligned}\frac{1}{e^{\beta z}+1} - \frac{1}{2} &= \frac{1}{2}\frac{1-e^{\beta z}}{1+e^{\beta z}} \\ &= -\frac{1}{2}\tanh\frac{\beta z}{2}.\end{aligned}$$

The Matsubara sum is therefore evaluated to:

$$\sum_{i\omega_n} \frac{1}{(i\omega_n)^2 - \epsilon^2} = -\frac{\beta}{2\epsilon}\tanh\left(\frac{\beta\epsilon}{2}\right). \tag{3.66}$$

Inserting into (3.57) gives the saddle point equation:

$$\begin{aligned}&4\sum_k (2h - 2J\cos k)\sum_{i\omega_n}\frac{1}{-(i\omega_n)^2 + \epsilon_k^2} \\ =&\frac{2}{T}\sum_k \frac{2h - 2J\cos k}{\epsilon_k}\tanh\left(\frac{\beta\epsilon_k}{2}\right) = -\frac{N}{T}.\end{aligned} \tag{3.67}$$

This requires carrying out the integral over fermion k-space. It can be solved in the special case of having only two coupled DQDs in which case $N = 2$ and $k = \pm\frac{\pi}{2}$ according to (3.10). Since $\sin\pm\frac{\pi}{2} = \pm 1$ and $\cos\pm\frac{\pi}{2} = 0$ we have $\epsilon\left(\pm\frac{\pi}{2}\right) = 2\sqrt{J^2 + h^2}$ and evaluate the sum:

$$\begin{aligned}&\frac{2}{T}\left[\frac{2h - 2J\cos\left(-\frac{\pi}{2}\right)}{\epsilon\left(-\frac{\pi}{2}\right)}\tanh\frac{\epsilon\left(-\frac{\pi}{2}\right)}{2T} + \frac{2h - 2J\cos\frac{\pi}{2}}{\epsilon\left(+\frac{\pi}{2}\right)}\tanh\frac{\epsilon\left(+\frac{\pi}{2}\right)}{2T}\right] \\ =&\frac{4}{T}\left(\frac{h}{\sqrt{J^2+h^2}}\tanh\frac{\sqrt{J^2+h^2}}{2T}\right).\end{aligned} \tag{3.68}$$

For $N = 2$ and $\phi_0 = 0$ we thus have the saddle point equation for $q = 0$:

$$\begin{aligned}\frac{4}{T}\left(\frac{h}{\sqrt{J^2+h^2}}\tanh\frac{\sqrt{J^2+h^2}}{2T}\right) &= -\frac{2}{T} \\ \Rightarrow \frac{2h}{\sqrt{J^2+h^2}}\tanh\frac{\sqrt{J^2+h^2}}{2T} &= -1.\end{aligned} \tag{3.69}$$

We study this in the limit of $T \to 0$. Then $\tanh\frac{\sqrt{J^2+h^2}}{2T} \to 1$ and we get:

$$\frac{h}{J} = \pm\sqrt{\frac{1}{3}}. \tag{3.70}$$

For the special case of $N = 2$ and $T \to 0$ and $h/J = \sqrt{1/3}$, then $\phi = 0$ is a solution to the saddle point equation. It is also seen that for $g \neq 0$ but $h = 0$, then $\phi_q = 0$ can not be a solution. Turning off the transverse field will not make $\phi_q = 0$ a solution.

Since the ϕ_0 that corresponds to $q, i\omega_m = 0$, so uniform in space, did not generally satisfy the saddle point equation we will need a uniform field to satisfy the saddle point equation. We have already showed that $\phi_q = 0$ for $q \neq 0$ satisfies the saddle point equation, therefore we guess a new solution where only the uniform field is finite. This is also motivated by the transverse field being uniform, and we could expect a uniform boson field as a reaction to a uniform transverse field.

Saddle point guess: $\phi_{k-k'} = \phi_0 \delta_{k,k'}$

The new guess at a solution based on the previous findings is:

$$\phi_{k-k'} = \phi_0 \delta_{k,k'}. \tag{3.71}$$

We remember that the delta function is for the fermionic both wavenumber and Matsubara frequence. We check if this guess satisfies the saddle point equation (3.45). We insert into (3.39) to find $\mathcal{G}_F|_{\phi_q=\phi_0\delta_q}$ to find the Green's function evaluated at the new guess at a solution. It was already shown that $\phi_q = 0\,,\, q \neq 0$ satisfies the saddle point equation so we only check for $q = 0$:

$$\begin{aligned}\Gamma(k,k')|_{\phi_q=\phi_0\delta_q} &= -\frac{g}{2\sqrt{\beta N}}\left[\phi_{k-k'} + \phi^*_{k'-k}\right]\sigma^z|_{\phi_q=\phi_0\delta_q} \\ &= -\frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]\sigma^z \delta_{k,k'}.\end{aligned} \tag{3.72}$$

Γ is now diagonal in k-space. Since $\mathcal{G}_0^{-1}$ is also diagonal in k the full $\mathcal{G}_F$ will also be diagonal in k-space and we therefore only have to invert the spinor-space matrix. Inserting into the saddle point equation (3.45) for the bosonic $q = 0$ and using that $\mathcal{G}_F^{-1}(k,k') = \mathcal{G}_F^{-1}(k)\delta_{k,k'}$ we get:

$$\phi_0^*(-i\omega_0 + \omega_R) + \frac{g\sqrt{\beta N}}{2} = -\frac{g}{2\sqrt{\beta N}}\sum_k \left[\mathcal{G}_{F,11}(k, q+k) - \mathcal{G}_{F,22}(k, q+k)\right]|_{q=0} \tag{3.73}$$

$$= -\frac{g}{2\sqrt{\beta N}}\sum_k \left[\mathcal{G}_{F,11}(k) - \mathcal{G}_{F,22}(k)\right]. \tag{3.74}$$

The units still match up since $[\phi_q] = \sqrt{\frac{1}{Energy}}$ since $[N] = 1$, and all terms have unit $\sqrt{\text{Energy}}$. Since $\mathcal{G}_F^{-1}$ was diagonal in k-space we invert it to find $\mathcal{G}_F$:

$$\begin{aligned}&\mathcal{G}_F(k)|_{\phi_q=\phi_0\delta_q} = \left[\mathcal{G}_0^{-1}(k) + \Gamma(k,k)\right]^{-1} \\ &= \begin{pmatrix} \frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right) - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right] & -iJ\sin k \\ iJ\sin k & -\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) + \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right] \end{pmatrix}^{-1} \\ &= \frac{1}{D(k)}\begin{pmatrix} -\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) + \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right] & iJ\sin k \\ -iJ\sin k & \frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right) - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right] \end{pmatrix},\end{aligned} \tag{3.75}$$

where $D(k)$ is the determinant of the matrix. The matrix elements in the saddle point equation are then:

$$\begin{aligned}\mathcal{G}_{F,11}(k) - \mathcal{G}_{F,22}(k) =& \frac{1}{D(k)}\Bigg[-\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) + \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right] \\ & - \left(\frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right) - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]\right)\Bigg] \\ =& -\frac{2h - 2J\cos k - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]}{D(k)}. \end{aligned} \tag{3.76}$$

The determinant is given by:

$$\begin{aligned} D_k =& \left[\frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right) - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]\right] \\ & \times \left[-\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right) + \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]\right] + (iJ\sin k)^2 \\ =& \frac{(i\omega_n)^2}{4} - \frac{1}{4}\left[2h - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right] - 2J\cos k\right]^2 - \frac{1}{4}(2J\sin k)^2 \\ =& \frac{(i\omega_n)^2 - \xi_k^2}{4}, \end{aligned} \tag{3.77}$$

with:

$$\xi_q^2 = \left[2h - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right] - 2J\cos k\right]^2 + (2J\sin k)^2, \tag{3.78}$$

which is recognized as the Jordan-Wigner fermion energy with a shifted transverse field. Since the poles of D^R gives us the spectrum of the TFI chain we define $\xi_k > 0$ (we could equally as well have chosen < 0) since this will not change the spectrum, which will still have a positive and negative solution. Getting back to the saddle point equation (3.74) we have:

$$\phi_0^*(-i\omega_0 + \omega_R) + \frac{g\sqrt{\beta N}}{2} = \frac{g}{2\sqrt{\beta N}}\sum_k \frac{2h - 2J\cos k - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]}{D(k)} \tag{3.79}$$

$$= \frac{2g}{\sqrt{\beta N}}\sum_{k,i\omega_n} \frac{2h - 2J\cos k - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]}{(i\omega_n)^2 - \xi_k^2}. \tag{3.80}$$

Using (3.66) we carry out the Matsubara sum and get the equation:

$$\phi_0^*(-i\omega_0 + \omega_R) + \frac{g\sqrt{\beta N}}{2} = -g\sqrt{\frac{\beta}{N}}\sum_k \frac{2h - 2J\cos k - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]}{\xi_k}\tanh\left(\frac{\beta\xi_k}{2}\right). \tag{3.81}$$

For the bosonic Matsubara frequencies we have $\omega_0 = 0$ which leaves:

$$\phi_0^* = -\frac{g}{\omega_R}\sqrt{\frac{\beta}{N}}\sum_k \frac{2h - 2J\cos k - \frac{g}{2\sqrt{\beta N}}\left[\phi_0 + \phi_0^*\right]}{\xi_k}\tanh\left(\frac{\beta\xi_k}{2}\right) - \frac{g\sqrt{\beta N}}{2\omega_R}. \tag{3.82}$$

The RHS is real, and we conclude that $\phi_0 \in \mathbb{R}$ giving:

$$\phi_0 = -\frac{g}{\omega_R}\sqrt{\frac{\beta}{N}}\sum_k \frac{2h - 2J\cos k - \frac{g}{\sqrt{\beta N}}\phi_0}{\xi_k}\tanh\left(\frac{\beta\xi_k}{2}\right) - \frac{g\sqrt{\beta N}}{2\omega_R}. \tag{3.83}$$

It is seen that for ϕ_0 very large the term inside the sum on the RHS goes to a constant:

$$\lim_{\phi_0\to\pm\infty} \frac{2h - 2J\cos k + \frac{g}{\sqrt{\beta N}}\phi_0}{\sqrt{\left(2h + \frac{g}{\sqrt{\beta N}}\phi_0 - 2J\cos k\right)^2 + (2J\sin k)^2}} = \mp 1, \tag{3.84}$$

and the tanh tends to 1. Therefore for $\phi_0 \to \pm\infty$ the right hand side goes to:

$$\lim_{\phi_0 \to \pm\infty} RHS = \pm \frac{g}{\hbar\omega_R}\sqrt{\frac{\beta}{N}}\sum_k -\frac{g\sqrt{\beta N}}{2\omega_R}$$
$$= \pm\frac{g\sqrt{\beta N}}{\omega_R} - \frac{g\sqrt{\beta N}}{2\omega_R}. \tag{3.85}$$

We therefore know that there will always be a solution to ϕ_0, since at some ϕ_0 the left and right side will cross. The LHS goes from $-\infty$ to ∞ while RHS goes from $-\frac{3g}{2\omega_R}\sqrt{\beta N}$ to $\frac{g}{2\omega_R}\sqrt{\beta N}$. A graphical solution to the saddle point equation is shown in 3.83. The saddle point equation (3.83) is written in Matsubara frequency and k-space. In real space and imaginary time we have: $\phi(r,\tau) = \frac{1}{\sqrt{\beta N}}\sum_{q,i\omega_m}\phi_0\delta_q\delta_m e^{iqr - i\omega_m\tau} \Rightarrow \phi_0^{RS} = \frac{\phi_0}{\sqrt{\beta N}}$. A uniform resonator field is equivalent to the single resonator coupling to all DQD examined in the first part. Examining the saddle point equation in real space and imaginary time gives:

$$\phi_0^{RS} = -\frac{g}{\omega_R}\frac{1}{N}\sum_k \frac{2h - 2J\cos k - g\phi_0^{RS}}{\sqrt{(2h - 2J\cos k - g\phi_0^{RS})^2 + (2J\sin k)^2}}\tanh\left(\frac{\xi_k}{2T}\right) - \frac{g}{2\omega_R}. \tag{3.86}$$

Going to zero temperature where $\tanh\left(\frac{\xi_k}{2T}\right) \to 1$ recovers the saddle point equation from the classical description of the single resonator in (2.113) up to the constant term that comes from describing the fermions with Grassman numbers.

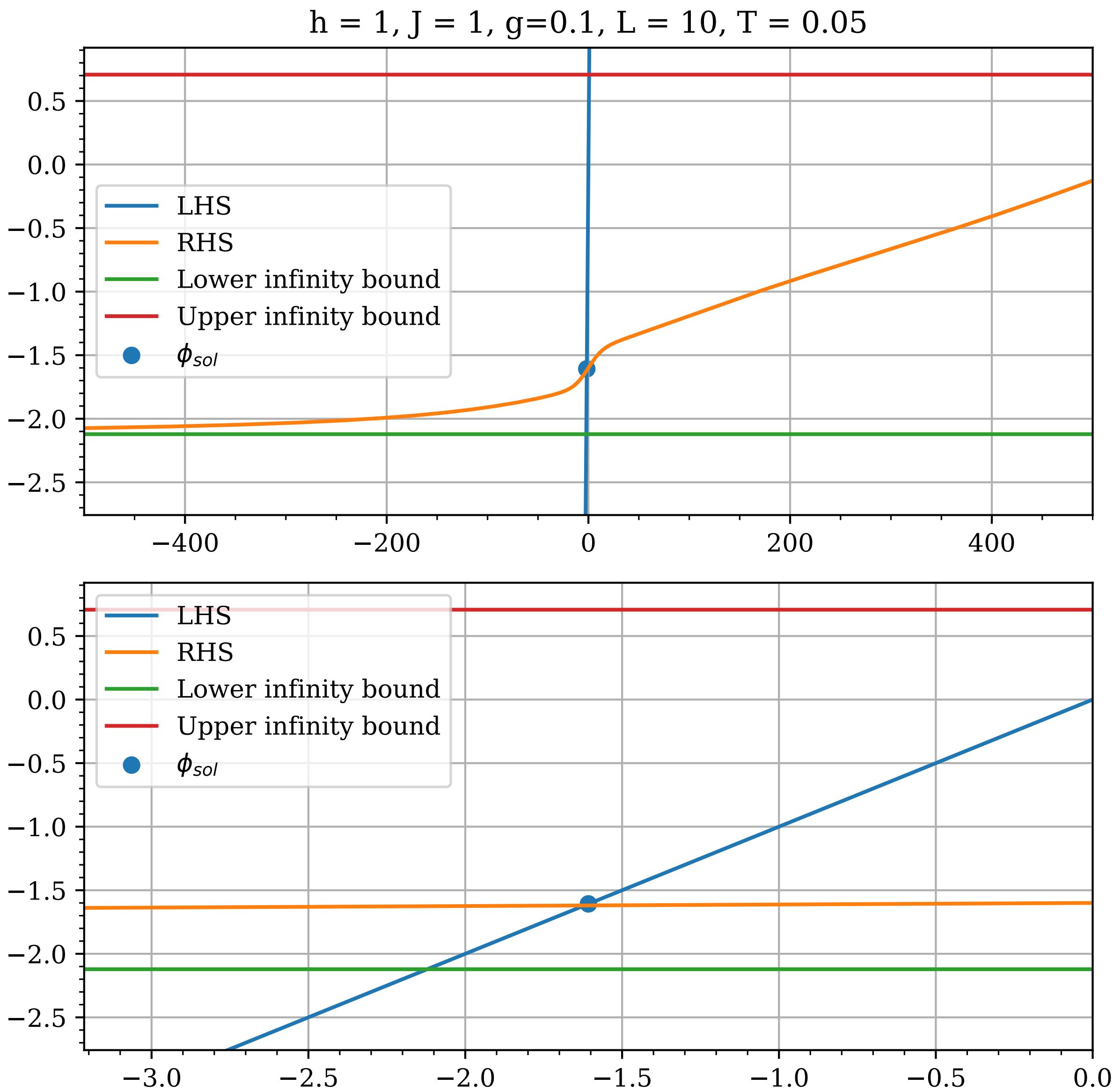

Figure 3.2: Saddle point equation with all parameters being in terms of ω_R except for L. The LHS looks vertical due to the range of the x-axis.

In the case that $T \to 0$ and in the weak coupling limit the saddle point equation can be solved in the fine tuned case of $h = J$ With $T \to 0$ it follows that $\tanh \frac{\xi_k}{2T} \to 1$. The saddle point equation (3.86) to lowest order in g expanded around $g = 0$ is then:

$$\phi_0^{RS} = -\frac{g}{\omega_R}\frac{1}{N}\sum_k \frac{2h - 2J\cos k}{\sqrt{(2h - 2J\cos k)^2 + (2J\sin k)^2}} \tanh\left(\frac{\xi_k}{2T}\right) - \frac{g}{2\omega_R} + \mathcal{O}(g^2). \tag{3.87}$$

This is still a problem to solve since the integrand is elliptical. We therefore consider the fine-tuned limit of $h = J$ giving:

$$\lim_{T\to 0} \phi_0^{RS} \overset{h=J}{=} -\frac{g}{\omega_R}\frac{1}{N}\sum_k \frac{2J(1-\cos k)}{2J\sqrt{2}\sqrt{1-\cos k}} - \frac{g}{2\omega_R}$$
$$= -\frac{g}{\omega_R}\frac{1}{\sqrt{2}N}\sum_k \sqrt{1-\cos k} - \frac{g}{2\omega_R}. \tag{3.88}$$

To evaluate the sum we assume it is smooth enough to take the limit of an integral where:

$$\sum_k \sqrt{1-\cos k} \to \frac{L}{2\pi}\int_{-\pi}^{\pi} dk\sqrt{1-\cos k} = \frac{N}{2\pi}\sqrt{2}\cdot 4. \tag{3.89}$$

We then have:

$$\phi_0^{RS} = -\frac{g}{\omega_R}\left(\frac{2}{\pi}+\frac{1}{2}\right). \tag{3.90}$$

In the Matsubara frequency and k-space the saddle point solutions is:

$$\phi_0 = -\frac{g}{\omega_R}\left(\frac{2}{\pi}+\frac{1}{2}\right)\sqrt{\beta N}. \tag{3.91}$$

The solution is marked in fig 3.2 as ϕ_{sol}, showing that it agrees well for small g. With a solution to the saddle point equation we can estimate the shift of h. It is seen from (3.86) that the shift in h, at fine tuned $h = J$, will be $\delta h = -\frac{g}{2}\phi_0^{RS} \approx g\frac{g}{2\omega_R}$. We can also estimate the change in the voltage of the resonator using[12]:

$$\hat{V} = \sqrt{\frac{\hbar\omega_R}{C}}\left(\hat{a}+\hat{a}^\dagger\right)$$
$$\Rightarrow \delta\langle V\rangle = -2\sqrt{\frac{\hbar\omega_R}{C}}\frac{g}{\omega_R}, \tag{3.92}$$

with C the capacitance of the superconducting resonator. We can thus calculate the expected shift in the voltage from the saddle point solution.

To recap, we found a saddle point equation for the imaginary time field integral and showed that the equation is similar to the classical equation (2.113). We have also shown that a solutions exists where a constant uniform field will minimize the effective action and we have solved for this constant field in the fine tuned case of $T \to 0$ and $J = h$ to lowest order in the coupling constant. In the next section we will expand the action around this saddle point solution and explore fluctuations in the resonator field.

3.2 Resonator field variations

Knowing that we have a minimum for the action at some uniform field (which we might only be able to find numerically) we can expand around this field, ϕ_0, to explore the action of non-uniform field fluctuations. To do so we start by defining the field over the two momentum q:

$$\phi_q = \phi_0\delta_q + \delta\phi_q, \tag{3.93}$$

which is the real field that satisfies the saddle point equation and a non-uniform field. Inserting into the action (3.35) gives:

$$S = g\sqrt{\beta N}\big(\phi_0 + \frac{\delta\phi_0 + \delta\phi_0^*}{2}\big) + \sum_q \left(\phi_0\delta_q + \delta\phi_q\right)^* \left(-i\omega_m + \omega_R\right)\left(\phi_0\delta_q + \delta\phi_q\right)$$
$$- \operatorname{tr}\ln\left(\mathcal{G}_0^{-1} - \frac{g}{2\sqrt{\beta N}}\big([\phi_0 + \delta\phi]^* + [\phi_0 + \delta\phi]\big)\sigma^z\right). \quad (3.94)$$

Collecting terms that depend on $\delta\phi_q$ gives:

$$= g\sqrt{\beta N}\phi_0 + 2\omega_R\phi_0^2 + g\sqrt{\beta N}\frac{\delta\phi_0 + \delta\phi_0^*}{2} + \omega_R\phi_0\big(\delta\phi_0 + \delta\phi_0^*\big)$$
$$+ \sum_q \delta\phi_q^* \left(-i\omega_m + \omega_R\right)\delta\phi_q - \operatorname{tr}\ln\left(\left(\mathcal{G}_0^{-1} + \Gamma_0\right)\left(1 + \left[\mathcal{G}_0^{-1} + \Gamma_0\right]^{-1}\delta\Gamma\right)\right)$$

$$= g\sqrt{\beta N}\phi_0 + 2\omega_R\phi_0^2 - \operatorname{tr}\ln\left(\mathcal{G}_0^{-1} + \Gamma_0\right) + \left(\frac{g\sqrt{\beta N}}{2} + \omega_R\phi_0\right)\left(\delta\phi_0 + \delta\phi_0^*\right)$$
$$+ \sum_q \delta\phi_q^* \left(-i\omega_m + \omega_R\right)\delta\phi_q - \operatorname{tr}\ln\left(1 + \left[\mathcal{G}_0^{-1} + \Gamma_0\right]^{-1}\delta\Gamma\right), \quad (3.95)$$

and the first three terms are defined as S_0 and the rest is the action for the fluctuations in the resonator field, $S[\delta\phi]$. It was used that $\phi_0 + \delta\phi$ is a vector in $q, i\omega_m$-space where $[\phi_0 + \delta\phi]_q = \phi_0\delta_q + \delta\phi_q$. It is also used that Γ_0 is defined as Γ from (3.24) evaluated in $\phi_0\delta_{k-k'}$ such that from inserting into (3.24) we get:

$$\left(\mathcal{G}_0^{-1} + \Gamma_0\right)(k, k') = \mathcal{G}_0^{-1}(k)\delta_{k,k'} - \frac{g\phi_0}{\sqrt{\beta N}}\delta_{k,k'}\sigma^z, \quad (3.96)$$

and likewise for $\delta\Gamma$ we use Γ evaluated in $\delta\phi_{k-k'}$:

$$\delta\Gamma(k, k') = \delta\Gamma(k - k') = -\frac{g}{2\sqrt{\beta N}}\left[\delta\phi_{k-k'} + \delta\phi_{k'-k}^*\right]\sigma^z, \quad (3.97)$$

which comes from the perturbation of the field. We thus have the action from the uniform field:

$$S_0 = 2\phi_0\omega_R + g\sqrt{\beta N}\phi_0 - \operatorname{tr}\ln\left(\mathcal{G}_0^{-1} + \Gamma_0\right), \quad (3.98)$$

and the action that depends on the fluctuations in ϕ:

$$S[\delta\phi] = \sum_q \delta\phi_q^* \left(-i\omega_m + \omega_m\right)\delta\phi_q - \operatorname{tr}\ln\left(1 + \left[\mathcal{G}_0^{-1} + \Gamma_0\right]^{-1}\delta\Gamma\right) + \left(\frac{g\sqrt{\beta N}}{2} + \omega_R\phi_0\right)\left(\delta\phi_0 + \delta\phi_0^*\right)$$
$$= \sum_q \delta\phi_q^* \left(-i\omega_m + \omega_R\right)\delta\phi_q - \operatorname{tr}\ln\left(1 + \mathcal{G}_F^0\delta\Gamma\right) + \left(\frac{g\sqrt{\beta N}}{2} + \omega_R\phi_0\right)\big(\delta\phi_0 + \delta\phi_0^*\big), \quad (3.99)$$

with the matrix $\Gamma_0 = -\frac{g\phi_0}{\sqrt{\beta N}}\delta_q\sigma^z$. From transforming the field $\phi \to \phi_0 + \delta\phi$, the integral measure is transformed as $\mathcal{D}[\phi] \to \mathcal{D}[\delta\phi]$ since the Jacobian of adding a constant (the saddle point solution) is zero. The partition function is then:

$$\mathcal{Z} = \int \mathcal{D}[\phi]e^{-S[\phi]} \to \int \mathcal{D}[\delta\phi]e^{-S_0 - S[\delta\phi]}$$
$$= \mathcal{Z}_0 \int \mathcal{D}[\delta\phi]e^{-S[\delta\phi]}. \quad (3.100)$$

We have expanded around a uniform field - we have not made any approximations yet. To get further in the analysis we assume the fluctuations around the uniform field to be small. The goal is now to expand the action to the lowest order in the field fluctuations. The logarithm can be expanded around $\delta\phi_q = 0$ due to the trade. The expansion gives $\mathrm{tr}\ln(1+\hat{A}\hat{B}) = -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}(\hat{A}\hat{B})^n$[10] such that:

$$S[\delta\phi] = \sum_q \delta\phi_q^*\left(-i\omega_m + \omega_R\right)\delta\phi_q + \left(\frac{g\sqrt{\beta N}}{2} + \omega_R\phi_0\right)\left(\delta\phi_0 + \delta\phi_0^*\right) + \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}(\mathcal{G}_F^0\delta\Gamma)^n. \tag{3.101}$$

Evaluated in the uniform field, the Green's function $\mathcal{G}_F^0$ is diagonal in $k, i\omega_n$-space:

$$\begin{aligned}\left[\mathcal{G}_F^0(k,k')\right]^{-1} &= \left[\mathcal{G}_0^{-1}(k,k') + \Gamma_0(k,k')\right] \\ &= \left[\mathcal{G}_0^{-1}(k) - \frac{g\phi_0}{\sqrt{\beta N}}\sigma^z\right]\delta_{k,k} \\ &= \left[\mathcal{G}_F^0(k)\right]^{-1}\delta_{k,k'},\end{aligned} \tag{3.102}$$

where as usual $\mathcal{G}_0(k)$ is an operator in spinor-space. As the matrix is diagonal in k-space and we invert it in spinor-space. Inserting the matrix forms of $\mathcal{G}_0^{-1}(k)$ from (3.20) and $\Gamma_0(k)$ gives:

$$\begin{aligned}\mathcal{G}_F^0(k,k') &= \delta_{k,k'}\begin{pmatrix} g_p(k) - \frac{g\phi_0}{\sqrt{\beta N}} & -iJ\sin k \\ iJ\sin k & g_h(k) + \frac{g\phi_0}{\sqrt{\beta N}}\end{pmatrix}^{-1} \\ &= \frac{\delta_{k,k'}}{D_k}\begin{pmatrix} g_h(k) + \frac{g\phi_0}{\sqrt{\beta N}} & iJ\sin k \\ -iJ\sin k & g_p(k) - \frac{g\phi_0}{\sqrt{\beta N}}\end{pmatrix},\end{aligned} \tag{3.103}$$

where:

$$D_k = \left[\left(g_p(k) - \frac{g\phi_0}{\sqrt{\beta N}}\right)\left(g_h(k) + \frac{g\phi_0}{\sqrt{\beta N}}\right)\right] - J^2\sin^2 k, \tag{3.104}$$

and $g_p(k) = \frac{1}{2}\left(-i\omega_n + 2h - 2J\cos k\right)$ and $g_h(k) = -\frac{1}{2}\left(i\omega_n + 2h - 2J\cos k\right)$. The uniform field shifts the transverse field h, so we define the new field $h' = h - \frac{g\phi_0}{\sqrt{\beta N}}$ and the shifted particle and hole functions $g_p' = g_p(h')$ and $g_h' = g_h(h')$ that are functions of the shifted field. This simplifies the expressions as:

$$\mathcal{G}_F^0(k,k') = \frac{\delta_{k,k'}}{D_k}\begin{pmatrix} g_h'(k) & iJ\sin k \\ -iJ\sin k & g_p'(k)\end{pmatrix}, \tag{3.105}$$

with

$$\begin{aligned} D_k &= g_p'(k)g_h'(k) - J^2\sin^2 k \\ &= -\frac{1}{4}\left(-i\omega_n + 2h' - 2J\cos k\right)\left(i\omega_n + 2h' - 2J\cos k\right) - \frac{1}{4}\left(2J\sin k\right)^2 \\ &= -\frac{(i\omega_n)^2 - \xi_k^2}{4},\end{aligned} \tag{3.106}$$

with ξ_k the same as (3.78), but with the shifted field h'. We are now ready to find the first order in the expansion in $\delta\phi_q$, so we calculate:

$$\begin{aligned}\mathrm{tr}\ \mathcal{G}_F^0\delta\Gamma &= \sum_{k,k'}\mathrm{tr}_\sigma\mathcal{G}_F^0(k,k')\delta\Gamma(k',k) \\ &= -\mathrm{tr}_\sigma\sum_{k,k'}\frac{\delta_{k,k'}}{(i\omega_n)^2 - \xi_k^2}\begin{pmatrix} g_h'(k) & iJ\sin k \\ -iJ\sin k & g_p'(k)\end{pmatrix}\frac{-g}{2\sqrt{\beta N}}X_{k-k'}\sigma^z,\end{aligned} \tag{3.107}$$

where we defined the field:

$$X_q = \delta\phi_q + \delta\phi^*_{-q}. \tag{3.108}$$

Notice that $X_q^* = X_{-q}$. Evaluating the Dirac delta gives:

$$\begin{aligned}
&= \mathrm{tr}_\sigma \sum_k \frac{1}{(i\omega_n)^2 - \xi_k^2} \begin{pmatrix} g_h'(k) & iJ\sin k \\ -iJ\sin k & g_p'(k) \end{pmatrix} \frac{g}{2\sqrt{\beta N}} X_0 \sigma^z \\
&= -X_0 \frac{g}{2\sqrt{\beta N}} \sum_{k,i\omega_n} \frac{2h' - 2J\cos k}{(i\omega_n)^2 - \xi_k^2}.
\end{aligned} \tag{3.109}$$

This is recognized as the saddle point equation from (3.79) giving that:

$$\mathrm{tr}\ \mathcal{G}_F^0 \delta\Gamma = -X_0 \left(\phi_0\omega_R + \frac{g\sqrt{\beta N}}{2}\right). \tag{3.110}$$

Inserting back into the action (3.101), the linear terms cancel as expected, since we are expanding around a minimum. We go on and look for a second order contribution to the action. For that we need to evaluate the following trace:

$$\frac{1}{2}\mathrm{tr}\left(\mathcal{G}_F^0\delta\Gamma\mathcal{G}_F^0\delta\Gamma\right). \tag{3.111}$$

We carry out the trace:

$$\begin{aligned}
\frac{1}{2}\mathrm{tr}_\sigma \sum_k \left[\mathcal{G}_F^0\delta\Gamma\mathcal{G}_F^0\delta\Gamma\right]_{k,k} &= \frac{1}{2}\mathrm{tr}_\sigma \sum_k \sum_{k_1,k_2,k_3} \mathcal{G}_F^0(k,k_1)\delta\Gamma(k_1,k_2)\mathcal{G}_F^0(k_2,k_3)\delta\Gamma(k_3,k) \\
&= \frac{1}{2}\mathrm{tr}_\sigma \sum_k \sum_{k_1,k_2,k_3} \mathcal{G}_F^0(k)\delta_{k,k_1}\frac{-g}{2\sqrt{\beta N}}X_{k_1-k_2}\sigma^z\mathcal{G}_F^0(k_2)\delta_{k_2,k_3}\frac{-g}{2\sqrt{\beta N}}X_{k_3-k}\sigma^z \\
&= \frac{g^2}{8\beta N}\sum_{k,k'} \mathrm{tr}_\sigma \mathcal{G}_F^0(k)X_{k-k'}\sigma^z\mathcal{G}_F^0(k')X_{k'-k}\sigma^z.
\end{aligned} \tag{3.112}$$

Using that $X_{k-k'}X_{k'-k} = |X_{k-k'}|^2$ gives:

$$\begin{aligned}
&= \frac{g^2}{8\beta N}\sum_{k,k'} |X_{k-k'}|^2 \mathrm{tr}_\sigma \mathcal{G}_F^0(k)\sigma^z\mathcal{G}_F^0(k')\sigma^z \\
&= \frac{g^2}{8\beta N}\sum_{k,k'} |X_{k-k'}|^2 \mathrm{tr}_\sigma \frac{1}{D_k}\begin{pmatrix} g_h'(k) & iJ\sin k \\ -iJ\sin k & g_p'(k) \end{pmatrix}\sigma^z \frac{1}{D_k'}\begin{pmatrix} g_h'(k') & iJ\sin k' \\ -iJ\sin k' & g_p'(k') \end{pmatrix}\sigma^z \\
&= \frac{g^2}{8\beta N}\sum_{k,k'} |X_{k-k'}|^2 \mathrm{tr}_\sigma \frac{1}{D_k}\begin{pmatrix} g_h'(k) & -iJ\sin k \\ -iJ\sin k & -g_p'(k) \end{pmatrix} \frac{1}{D_k'}\begin{pmatrix} g_h'(k') & -iJ\sin k' \\ -iJ\sin k' & -g_p'(k') \end{pmatrix}.
\end{aligned} \tag{3.113}$$

Evaluating the trace over spinor space gives:

$$\begin{aligned}
&= \frac{g^2}{8\beta N}\sum_{k,k'} |X_{k-k'}|^2 \frac{1}{D_k D_k'}\left\{\left[g_h'(k)g_h'(k') - J^2\sin k\sin k'\right] + \left[-J^2\sin k\sin k' + \left(-g_p'(k)\right)\left(-g_p'(k')\right)\right]\right\} \\
&= \frac{g^2}{8\beta N}\sum_{k,k'} |X_{k-k'}|^2 \frac{1}{D_k D_k'}\left[g_p'(k)g_p'(k') + g_h'(k)g_h'(k') - 2J^2\sin k\sin k'\right],
\end{aligned} \tag{3.114}$$

and by carrying out $g_p'(k)g_p'(k') + g_h'(k)g_h'(k')$ we get:

$$
\begin{aligned}
&= \frac{g^2}{8\beta N}\sum_{k,k'}|X_{k-k'}|^2\frac{1}{D_k D_k'}\left[\frac{(i\omega_n)(i\omega_{n'})}{2} + \frac{1}{2}(2h' - 2J\cos k)(2h' - 2J\cos k') - \frac{1}{2}2J\sin k\, 2J\sin k'\right] \\
&= \frac{g^2}{16\beta N}\sum_{k,k'}|X_{k-k'}|^2\frac{1}{D_k D_k'}\left[(i\omega_n)(i\omega_{n'}) + (2h' - 2J\cos k)(2h' - 2J\cos k') - 2J\sin k\, 2J\sin k'\right].
\end{aligned}
\tag{3.115}
$$

We define the bosonic two-momentum $k - k' = q$ and sum over k and q. Since k and k' are fermionic Matsubara frequencies, q will be a bosonic frequency as needed. Again we need to remember that the sum is over the frequencies too, so shifting $k' = k - q$ also shifts $i\omega_{n'} = i\omega_n - i\omega_m$:

$$
\begin{aligned}
= \frac{g^2}{16\beta N}\sum_{k,q}|X_q|^2\frac{1}{D_k D_{k-q}}\Big[& i\omega_n\left(i\omega_n - i\omega_m\right) + (2h' - 2J\cos k)(2h' - 2J\cos(k-q)) \\
& - 2J\sin k\, 2J\sin(k-q)\Big].
\end{aligned}
\tag{3.116}
$$

The denominator will be denoted as:

$$
\begin{aligned}
\Lambda_{k,q} &= i\omega_n\left(i\omega_n - i\omega_m\right) + (2h' - 2J\cos k)(2h' - 2J\cos(k-q)) - 2J\sin k\, 2J\sin(k-q) \\
&= i\omega_n\left(i\omega_n - i\omega_m\right) + f(k,q),
\end{aligned}
\tag{3.117}
$$

with $f(k,q)$ defined as:

$$
f(k,q) = (2h' - 2J\cos k)(2h' - 2J\cos(k-q)) - 2J\sin k\, 2J\sin(k-q). \tag{3.118}
$$

We look at the physics of this additional term in the action. We had from (3.92) that the voltages at the resonators were proportional to the position coordinate through $V_r = \sqrt{\frac{\hbar\omega_R}{C}}X_r$ for the single photon mode ω_R in each resonator. We define the polarization bubble:

$$
\Pi_q = \frac{g^2}{16\beta N}\sum_k \frac{\Lambda_{k,q}}{D_k D_{k-q}}. \tag{3.119}
$$

The the second order correction is then given as:

$$
\frac{g^2}{16\beta N}\sum_q X_q^*\left\{\sum_k \frac{\Lambda_{k,q}}{D_k D_{k-q}}\right\}X_q = \sum_q X_q^*\Pi_q X_q, \tag{3.120}
$$

which after a Fourier transformation becomes:

$$
\frac{1}{N\sqrt{N}}\sum_{r,s,x} e^{iqr}\delta X_r e^{isq}\Pi_s e^{-ixq}\delta X_x = \frac{C}{\hbar\omega_R}\frac{1}{\sqrt{N}}\sum_{r,x}\delta V_r \Pi_{x-r}\delta V_x. \tag{3.121}
$$

We have thus found that to second order in g, the voltages of the resonators interact non-locally through Π_q. The term Π_q consists of two Jordan-Wigner fermion propagators with a shifted transverse field. Inserting into the expansion of the effective action for the density variations we get to second order in the perturbation:

$$
\begin{aligned}
S[\delta\phi] &= \sum_q \delta\phi_q^*\left(-i\omega_m + \omega_R\right)\delta\phi_q + \frac{g^2}{16\beta N}\sum_{k,q}|(\delta\phi_q + \delta\phi_{-q}^*)|^2\frac{\Lambda_{k,q}}{D_k D_{k-q}} + \mathcal{O}\left(\delta\phi^3\right) \\
&= \sum_q\left\{\delta\phi_q^*\left(-i\omega_m + \omega_R\right)\delta\phi_q + X_q^*\Pi_q X_q\right\} + \mathcal{O}\left(\delta\phi^3\right).
\end{aligned}
\tag{3.122}
$$

To put the action on matrix form we define the vector:

$$\Phi_q^\dagger = (\, \delta\phi_q^* \,,\, \delta\phi_{-q} \,). \tag{3.123}$$

We also define:

$$h_q^0 = -i\omega_m + \omega_R, \tag{3.124}$$

to ease notation. We will use the inversion symmetries of $\Pi_q(i\omega_m)$ that are shown in the section 3.3:

$$\Pi_q(i\omega_m) = \Pi_{-q}(i\omega_m) \qquad \Pi_q(i\omega_m) = \Pi_{-q}(-i\omega_m) \qquad \Pi_q(i\omega_m)^* = \Pi_q(-i\omega_m) = \Pi_q(i\omega_m). \tag{3.125}$$

The action is rewritten as (since the ϕ_0 was absorbed into h, the delta is dropped $\delta\phi_q = \phi_q$):

$$\begin{aligned}
S[\phi] &= \sum_q \left\{ \phi_q^* h_q \phi_q + \left(\phi_q + \phi_{-q}^*\right) \Pi_q(i\omega_m) \left(\phi_q^* + \phi_{-q}\right) \right\} \\
&= \sum_{q>0} \Big\{ \phi_q^* h_q \phi_q + \phi_{-q}^* h_{-q} \phi_{-q} + \phi_q^* \Pi_q \phi_q + \phi_{-q}^* \Pi_{-q} \phi_{-q} \\
&+ \phi_q \Pi_q \phi_{-q} + \phi_{-q} \Pi_{-q} \phi_q + \phi_{-q}^* \Pi_q \phi_q^* + \phi_q^* \Pi_{-q} \phi_{-q}^* + \phi_{-q}^* \Pi_q \phi_{-q} + \phi_q^* \Pi_{-q} \phi_q \Big\} \\
&= \sum_{q>0} \Big\{ \phi_q^* h_q \phi_q + \phi_{-q}^* h_{-q} \phi_{-q} + 2\phi_q^* \Pi_q \phi_q + 2\phi_q \Pi_q \phi_{-q} + 2\phi_{-q}^* \Pi_q \phi_q^* + 2\phi_{-q}^* \Pi_q \phi_{-q} \Big\},
\end{aligned} \tag{3.126}$$

allowing us to write the action on matrix form:

$$\begin{aligned}
S[\Phi] &= \sum_{q>0} \begin{pmatrix} \delta\phi_q^* & \delta\phi_{-q} \end{pmatrix} \begin{pmatrix} h_q^0 + 2\Pi_q & 2\Pi_q \\ 2\Pi_q & h_{-q}^0 + 2\Pi_q \end{pmatrix} \begin{pmatrix} \delta\phi_q \\ \delta\phi_{-q}^* \end{pmatrix} \\
&= \sum_{q>0} \Phi_q^\dagger \left((\mathcal{G}^Q)^{-1} \right)_q \Phi_q.
\end{aligned} \tag{3.127}$$

The matrix $\mathcal{G}_q^Q$ is not normal, meaning $A^\dagger A \neq AA^\dagger$, and it can therefore not be diagonalized by a unitary transformation. With the action on matrix form, we find the integral measure from changing variable from the complex number $\delta\phi$ to the complex vector Φ. Before the change the integral measure is:

$$\mathcal{D}[\phi^\dagger, \phi] = \lim_{N\to\infty} \prod_m^N \left(\prod_{q=-k_F}^{k_F} d\phi_{q,m}^* d\phi_{q,m} \right), \tag{3.128}$$

where $d\phi^* d\phi = d\mathrm{Im}[\phi]\, d\mathrm{Re}[\phi]$ for integration over the complex plane. For each Matsubara n there is an integration over the coherent state with momenta $q \in [-k_F, k_F]$. An integral measure for integrating a complex vector over the complex plane is written as $d(v^\dagger, v) = \prod_i v_i^* v_i$. The Φ_q integral measure is thus:

$$d(\Phi_q^\dagger, \Phi_q) = d\delta\phi_q^* d\delta\phi_q d\delta\phi_{-q}^* d\delta\phi_{-q}. \tag{3.129}$$

Since the complex numbers all commute and going from $\phi \to \delta\phi$ only adds a number ϕ_0 with unit Jacobian, the path integral measure from (3.128) becomes:

$$
\begin{aligned}
\mathcal{D}[\phi^\dagger, \phi] &= \lim_{N\to\infty} \prod_m^N \left(\prod_{q>0}^{k_F} d\phi^*_{q,m} d\phi_{q,m} d\phi^*_{-q,-m} d\phi_{-q,-m} \right) \\
&= \lim_{N\to\infty} \prod_m^N \left(\prod_{q>0}^{k_F} d(\Phi^\dagger_{q,m}, \Phi_{q,m}) \right) \\
&= \mathcal{D}\left[\Phi\right].
\end{aligned} \tag{3.130}
$$

Since the action is quadratic, the Green's function for the fluctuations can be evaluated with Wick's theorem. We define the fluctuation Green's function:

$$
\mathcal{G}^\delta(q, i\omega_m) = -\langle \delta\phi_q \delta\phi^*_q \rangle. \tag{3.131}
$$

Using Wick's theorem this is found as:

$$
\langle \delta\phi^*_q \delta\phi_q \rangle = \mathcal{G}^Q_{q,++}, \tag{3.132}
$$

where the + indicate the positive momentum of the spinor. The 2×2 matrix $\mathcal{G}^Q_q$ is found by inverting (3.127):

$$
\begin{aligned}
\mathcal{G}^Q_q &= \frac{1}{(h^0_q + 2\Pi_q)(h^0_{-q} + 2\Pi_q) - 4\Pi_q^2} \begin{pmatrix} h^0_{-q} + 2\Pi_q & -2\Pi_q \\ -2\Pi_q & h^0_q + 2\Pi_q \end{pmatrix} \\
&= -\frac{1}{(i\omega_m)^2 - \omega_R^2 - 4\omega_R\Pi_q} \begin{pmatrix} i\omega_m + \omega_R + 2\Pi_q & -2\Pi_q \\ -2\Pi_q & -i\omega_m + \omega_R + 2\Pi_q \end{pmatrix}.
\end{aligned} \tag{3.133}
$$

The fluctuation Green's function is read off from the matrix elements:

$$
\begin{aligned}
\mathcal{G}^\delta(q, i\omega_m) = -\mathcal{G}^Q_{q,++} &= \frac{h^0_{-q} + 2\Pi_q}{(i\omega_m)^2 - \omega_R^2 - 4\omega_R\Pi_q} \\
&= \frac{i\omega_m + \omega_R + 2\Pi_q}{(i\omega_m)^2 - \omega_R^2\left(1 + \frac{4}{\omega_R}\Pi_q\right)} \\
&= \frac{i\omega_m + \omega_R + 2\Pi_q}{\left(i\omega_m - \omega_R\sqrt{1 + \frac{4}{\omega_R}\Pi_q}\right)\left(i\omega_m + \omega_R\sqrt{1 + \frac{4}{\omega_R}\Pi_q}\right)}
\end{aligned} \tag{3.134}
$$

$\langle \delta\phi^*_{-q} \delta\phi_{-q} \rangle$ is the same whether we invert the q in (3.134) or read off the matrix element $\mathcal{G}^Q_{q,--}$, using that $\Pi_q = \Pi_{-q}$.

Since $\Pi_q(i\omega_m)$ is real, the imaginary time Green's function $\mathcal{G}^\delta(i\omega_m)$ is analytic in the set of all Matsubara frequencies, $\{i\omega_m | m \in \mathbb{Z}\}$. We can analytically continue to the function $\mathcal{G}^\delta(z)$ where z is in the upper half of the complex plane such that $\mathcal{G}^\delta(z) = \mathcal{G}^\delta(i\omega_m)$ for $z \in \{i\omega_m | m \in \mathbb{Z}^+\}$. With the Lehmann representation is can be shown that the continuation $z \to \omega + i\eta$ gives the retarded Green's function for $\eta > 0$. The spectrum of the resonator field is found from the poles of the retarded Green's function, as shown in appendix D.

It is also possible to find the the Green's function for the resonator position coordinate. The position coordinate operator was defined as $X_q = \phi_q + \phi^*_{-q}$ where $\phi_q = \delta\phi_q$. It is defined similar to the normal phonon operator. The position coordinate average is found as:

$$\begin{aligned}
\langle X_q X_q^* \rangle &= \langle X_q X_{-q} \rangle \\
&= \left\langle \left(\phi_q + \phi^*_{-q}\right)\left(\phi^*_q + \phi_{-q}\right)\right\rangle \\
&= \langle \phi^*_q \phi_q \rangle + \langle \phi_{-q}\phi_q \rangle + \langle \phi^*_q \phi^*_{-q} \rangle + \langle \phi_{-q}\phi^*_{-q} \rangle, \qquad (3.135)
\end{aligned}$$

which is read off as one of each matrix element from (3.133). The position coordinate Green's function becomes:

$$\begin{aligned}
\mathcal{D}(q) &= -\langle X_q X_q^* \rangle \\
&= \frac{2\omega_R}{(i\omega_m)^2 - \omega_R^2\left(1 + \frac{4}{\omega_R}\Pi_q\right)}, \qquad (3.136)
\end{aligned}$$

and in the case of $g \to 0$ where we the resonators are non-interacting, we find:

$$\lim_{g\to 0} \mathcal{D}(q) = \frac{2\omega_R}{(i\omega_m)^2 - \omega_R^2}, \qquad (3.137)$$

which is the same as the free optical phonon Green's function with energy ω_R.

3.3 The polarization function

In this section we evaluate the Matsubara sum in the polarization function (3.119). The denominator was written as (3.117) and had a term depending on the Matsubara frequency and one depending on the wavenumber:

$$\Lambda_{k,q} = i\omega_n\left(i\omega_n - i\omega_m\right) + f(k,q). \qquad (3.138)$$

The denominator was:

$$D_k D_{k-q} = \frac{(i\omega_n)^2 - \xi_k^2}{4} \frac{\left(i\omega_n - i\omega_m\right)^2 - \xi_{k-q}^2}{4}. \qquad (3.139)$$

From (3.119) we defined $\Pi_q = \frac{g^2}{16\beta N}\sum_k \frac{\Lambda_{k,q}}{D_k D_{k-q}}$ giving:

$$\Pi_q(i\omega_m) = \frac{g^2}{\beta N}\sum_{k,i\omega_n} \frac{i\omega_n\left(i\omega_n - i\omega_m\right) + f(k,q)}{\left((i\omega_n)^2 - \xi_k^2\right)\left(\left(i\omega_n - i\omega_m\right)^2 - \xi_{k-q}^2\right)} \qquad (3.140)$$

From changing the dummy indices of the sum $(k, i\omega_n)$ to $(-k, -i\omega_n)$ and then using that $\xi_{-k+q} = \xi_{k-q}$ and $f(-k,-q) = f(k,q)$, it is seen that $\Pi_{-q}(-i\omega_m) = \Pi_q(i\omega_m)$ showing that the polarization function is symmetric in the two-vector $q \to -q$. It also has to obey the symmetry in only the momentum $\Pi_{-q}(i\omega_m) = \Pi_q(i\omega_m)$ as can be seen from changing the dummy index $k \to -k$ and using $f(-k,-q) = f(k,q)$. The integrand is rewritten as a function of a complex number:

$$\frac{f(k,q) + z\left(z - i\omega_m\right)}{\left(\xi_k + z\right)\left(\xi_k - z\right)\left(\xi_{k-q} + z - i\omega_m\right)\left(\xi_{k-q} - z + i\omega_m\right)} = g(z), \qquad (3.141)$$

that has four simple poles in $z_p = \pm\xi_k$, $i\omega_m \pm \xi_{k-q}$. Following the same procedure as in (3.59) we use:

$$\frac{1}{\beta}\sum_{i\omega_n} f(i\omega_n)e^{i\omega_n\tau} = \sum_{z_p}\operatorname*{Res}_{z=z_p}\left[f(z)\right] n_F(z_p)e^{\tau z_p}. \tag{3.142}$$

The residues are then evaluated one at a time:

①

$$\begin{aligned}\operatorname*{Res}_{z=\xi_k}[g(z)] &= \lim_{z\to\xi_k}\frac{f(k,q)+z(z-i\omega_m)}{(\xi_k+z)(\xi_k-z)\left(\xi_{k-q}^2-(z-i\omega_m)^2\right)}(z-\xi_k)\\ &= -\frac{f(k,q)+\xi_k\left(\xi_k-i\omega_m\right)}{2\xi_k\left(\xi_{k-q}^2-(\xi_k-i\omega_m)^2\right)}.\end{aligned} \tag{3.143}$$

②

$$\begin{aligned}\operatorname*{Res}_{z=-\xi_k}[g(z)] &= \lim_{z\to-\xi_k}\frac{f(k,q)+z(z-i\omega_m)}{(\xi_k+z)(\xi_k-z)\left(\xi_{k-q}^2-(z-i\omega_m)^2\right)}(z+\xi_k)\\ &= \frac{f(k,q)+\xi_k\left(\xi_k+i\omega_m\right)}{2\xi_k\left(\xi_{k-q}^2-(\xi_k+i\omega_m)^2\right)}.\end{aligned} \tag{3.144}$$

③

$$\begin{aligned}\operatorname*{Res}_{z=i\omega_m+\xi_{k-q}}[g(z)] &= \lim_{z\to i\omega_m+\xi_{k-q}}\frac{f(k,q)+z\left(z-i\omega_m\right)}{(\xi_k^2-z^2)\left[\xi_{k-q}+(z-i\omega_m)\right]\left[\xi_{k-q}-(z-i\omega_m)\right]}(z-(i\omega_m+\xi_{k-q}))\\ &= -\frac{f(k,q)+\xi_{k-q}\left(\xi_{k-q}+i\omega_m\right)}{2\xi_{k-q}\left(\xi_k^2-(\xi_{k-q}+i\omega_m)^2\right)}.\end{aligned} \tag{3.145}$$

④

$$\begin{aligned}\operatorname*{Res}_{z=i\omega_m-\xi_{k-q}}[g(z)] &= \lim_{z\to i\omega_m-\xi_{k-q}}\frac{f(k,q)+z\left(z-i\omega_m\right)}{(\xi_k^2-z^2)\left[\xi_{k-q}+(z-i\omega_m)\right]\left[\xi_{k-q}-(z-i\omega_m)\right]}(z-(i\omega_m-\xi_{k-q}))\\ &= \frac{f(k,q)+\xi_{k-q}\left(\xi_{k-q}-i\omega_m\right)}{2\xi_{k-q}\left(\xi_k^2-(\xi_{k-q}-i\omega_m)^2\right)}.\end{aligned} \tag{3.146}$$

Inserting the residues into (3.63) gives the polarization function:

$$\begin{aligned}\Pi_q(i\omega_m) = \frac{g^2}{2N}\sum_k\Bigg\{ &-\frac{f(k,q)+\xi_k\left(\xi_k-i\omega_m\right)}{\xi_{k-q}^2-(\xi_k-i\omega_m)^2}\frac{n_F(\xi_k)}{\xi_k}\\ &+\frac{f(k,q)+\xi_k\left(\xi_k+i\omega_m\right)}{\xi_{k-q}^2-(\xi_k+i\omega_m)^2}\frac{n_F(-\xi_k)}{\xi_k}\\ &-\frac{f(k,q)+\xi_{k-q}\left(\xi_{k-q}+i\omega_m\right)}{\xi_k^2-(\xi_{k-q}+i\omega_m)^2}\frac{n_F(\xi_{k-q})}{\xi_{k-q}}\\ &+\frac{f(k,q)+\xi_{k-q}\left(\xi_{k-q}-i\omega_m\right)}{\xi_k^2-(\xi_{k-q}-i\omega_m)^2}\frac{n_F(-\xi_{k-q})}{\xi_{k-q}}\Bigg\}.\end{aligned} \tag{3.147}$$

The expression is reduced by first shifting the third and fourth term by $k \to q-k$ and then using $f(q-k,q) = f(k,q)$:

$$\Pi_q(i\omega_m) = \frac{g^2}{2N}\sum_k \tanh\frac{\xi_k}{2T}\left(\frac{f(k,q)+\xi_k(\xi_k-i\omega_m)}{\xi_{k-q}^2-(\xi_k-i\omega_m)^2}+\frac{f(k,q)+\xi_k(\xi_k+i\omega_m)}{\xi_{k-q}^2-(\xi_k+i\omega_m)^2}\right). \tag{3.148}$$

We verify that the symmetries observed earlier still hold true. The symmetry $\Pi_{-q}(i\omega_n) = \Pi_q(i\omega_m)$ holds as we can change the dummy index $k \to -k$ and use that $f(-k,-q) = f(k,q)$ and $\xi(-k) = \xi(k)$. The two momentum inversion symmetry is also still obeyed. To see that, we shift the dummy index $k \to k-q$ and use that $f(k-q,-q) = f(k,q)$, then $\xi_k \to \xi_{k-q}$ and $\xi_{k+q} \to \xi_k$, and from there it is verified that $\Pi_q(i\omega_m) = \Pi_{-q}(-i\omega_m)$. In this form $\Pi_q(-i\omega_m) = \Pi_q(i\omega_m)$ is easily seen. As only the Matsubara frequencies are imaginary it gives that $\Pi_q(i\omega_m)^* = \Pi_q(-i\omega_m) = \Pi_q(i\omega_m)$.

Plotting the polarization

To better understand the behavior of Π_q we plot $\Pi_q(i\omega_m \to \omega + i\eta)$ for different parameter regimes. With the analytic continuation a finite η is needed to keep the poles of the polarization function from the real axis and in the lower half of the complex plane.

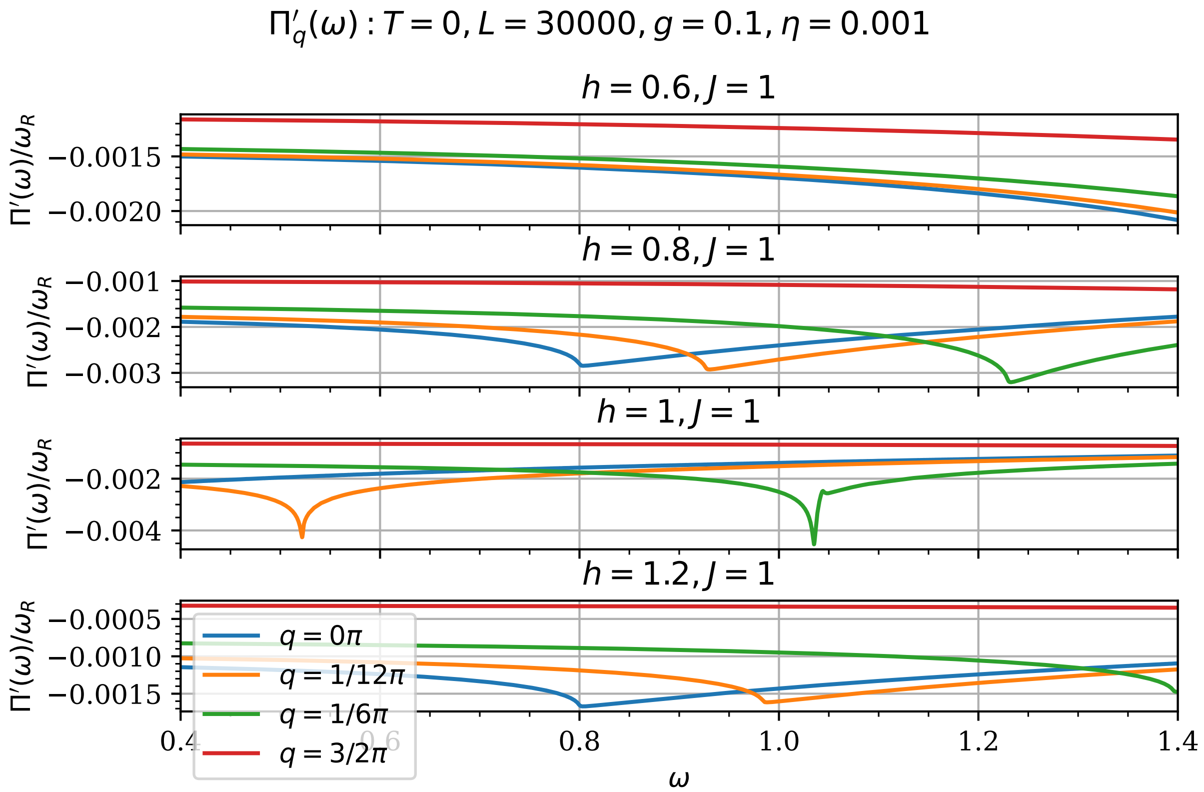

Figure 3.3: Plot of the real part of the polarization function, $\Pi'_q(\omega)$.

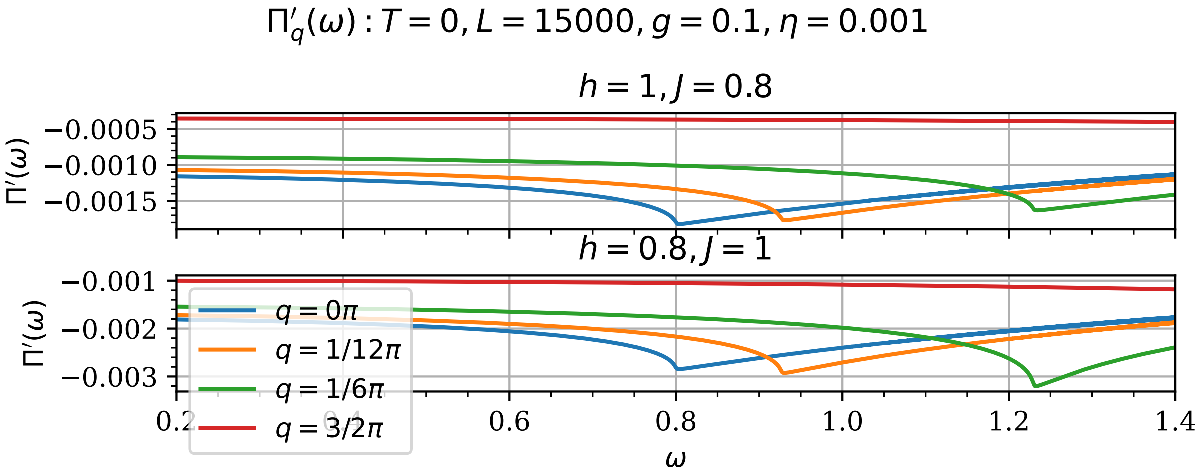

Figure 3.4: Plot comparing the real part of the polarization function, $\Pi'_q(\omega)$, in the ferromagnetic and paramagnetic regimes. All energies are in terms of ω_R

We learn from figure 3.4 that the real part of the polarization function is larger by almost a factor of two in the ferromagnetic regime when $J > h$ compared to the paramagnetic regime of $h > J$. The J originally came from the capacitance that made the DQDs interact, so there is more interaction in the ferromagnetic regime. The polarization function is smallest in the minimum energy of two Jordan-Wigner fermions. This can be seen in the second plot where the minimum energy of two fermions, with difference in wavenumber $q = 0$, is $4|h - J|$ which is $0.8\omega_R$ for $\frac{h}{\omega_R} = 1$ and $\frac{J}{\omega_R} = 0.8$.

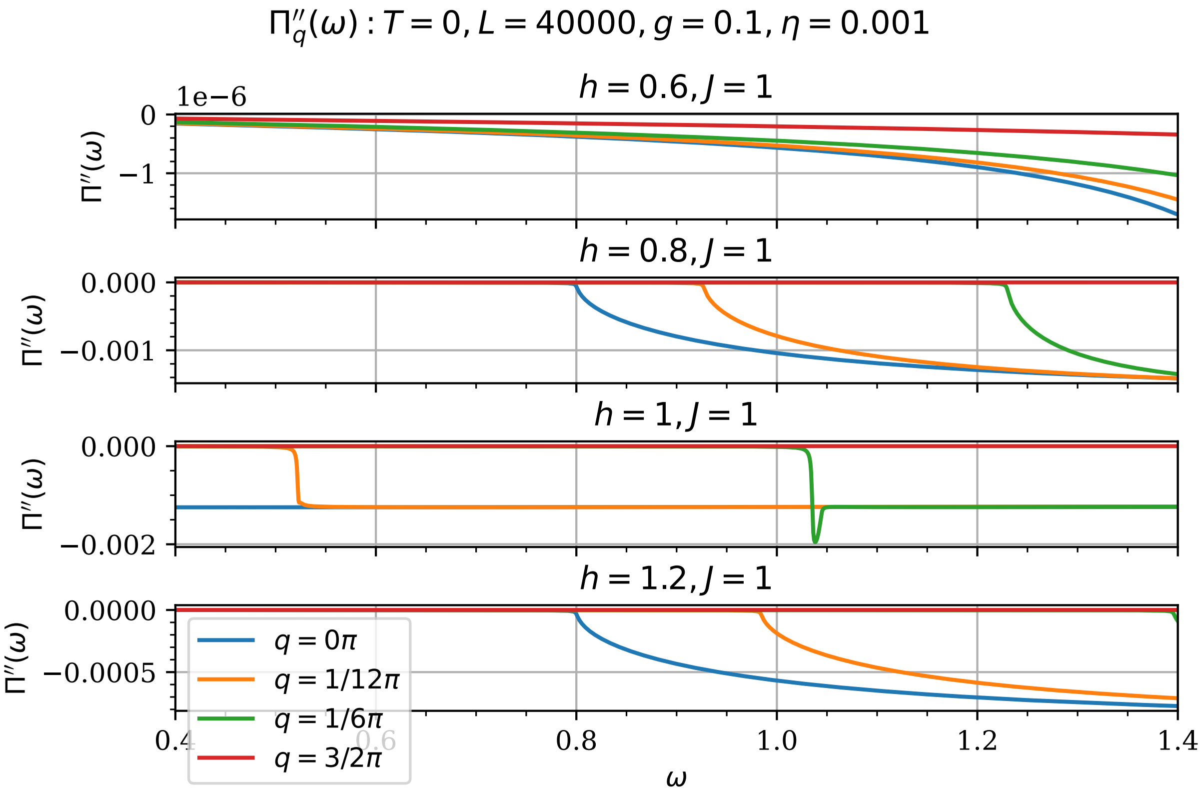

Figure 3.5: Plot of the imaginary part of the polarization function $\Pi''_q(\omega)$. Energies are in terms of ω_R.

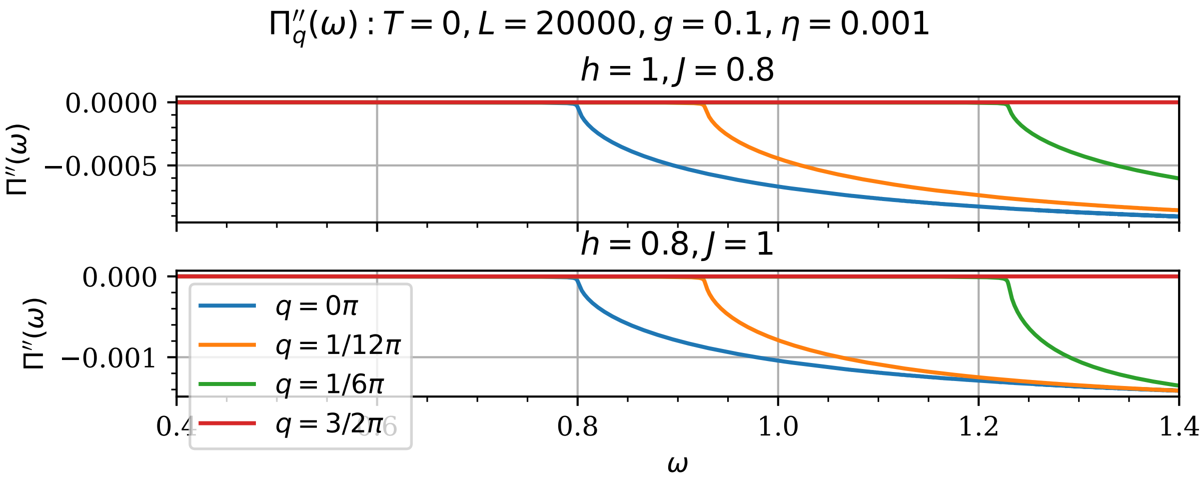

Figure 3.6: Plot comparing the imaginary part of the polarization function, $\Pi''_q(\omega)$, in the ferromagnetic and paramagnetic regimes. Energies are in terms of ω_R

It can be seen that the imaginary part is vanishing until the frequency becomes comparable to the energy two Jordan-Wigner fermions. In figure 3.6 it is seen that also the imaginary part is larger by around a factor of two in the ferromagnetic regime $J > h$.

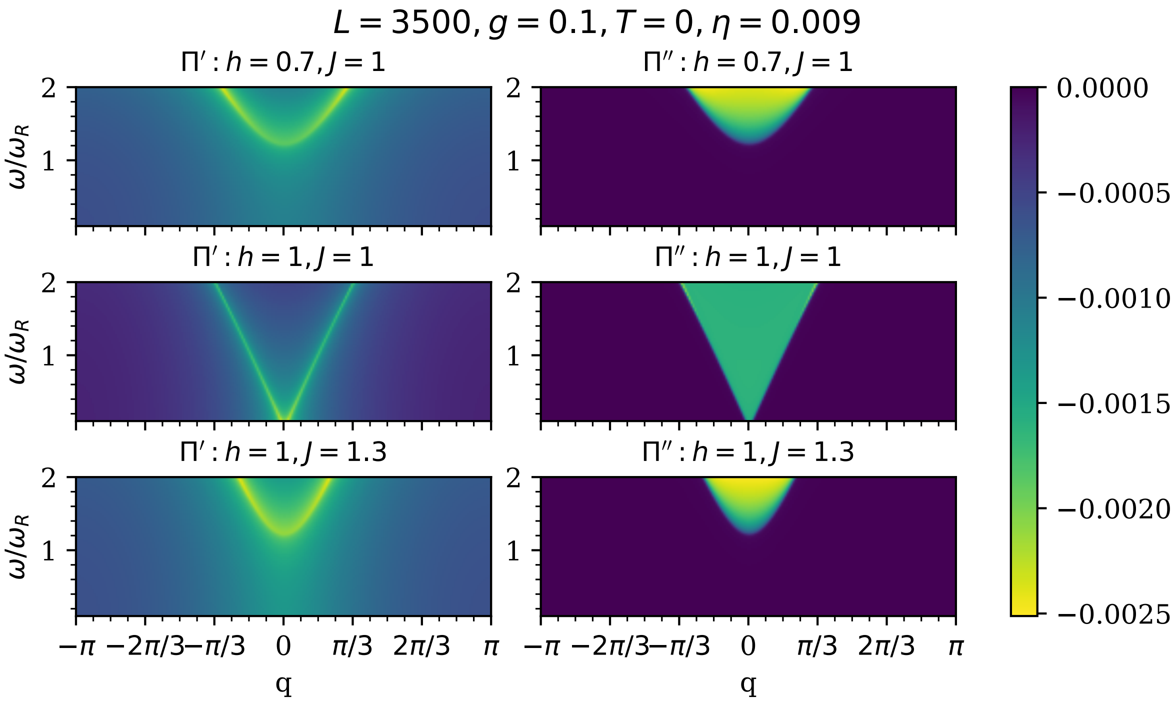

Figure 3.7: Plot of $\Pi''_q(\omega)$ in terms of ω_R

The limit of $\Pi_q|_{J=0}$

In (3.78) we defined $\xi_k = +\sqrt{(2h - 2J\cos k)^2 + (J\sin k)^2}$. We can calculate $\Pi_q(i\omega_n)$ in the limit of no interaction between DQDs, $J = 0$, where the system becomes N independent resonators. From (3.78) it is seen that $\xi_k|_{J=0} = 2h$. From (3.117) it is found that $f(k,q)|_{J=0} = (2h)^2$. Inserting into (3.147) gives:

$$\begin{aligned}
\Pi_q(i\omega_m)|_{J=0} =& 4g^2\Bigg\{-\frac{4h^2 + 2h\,(2h - i\omega_m)}{4h^-(2h - i\omega_m)^2}\frac{n_F(2h)}{2h} + \frac{4h^2 + 2h\,(2h + i\omega_m)}{4h^2 - (2h + i\omega_m)^2}\frac{n_F(-2h)}{2h} \\
&- \frac{4h^2 + 2h\,(2h + i\omega_m)}{4h^2 - (2h + i\omega_m)^2}\frac{n_F(2h)}{2h} + \frac{4h^2 + 2h\,(2h - i\omega_m)}{4h^2 - (2h - i\omega_m)^2}\frac{n_F(-2h)}{2h}\Bigg\} \\
&= 4g^2\big(n_F(-2h) - n_F(2h)\big)\left(\frac{4h - i\omega_m}{4h^2 - (2h - i\omega_m)^2} + \frac{4h + i\omega_m}{4h^2 - (2h + i\omega_m)^2}\right) \\
&= 4g^2\big(1 - 2n_F(2h)\big)\left(\frac{4h - i\omega_m}{(4h - i\omega_m)(i\omega_m)} - \frac{4h + i\omega_m}{(4h + i\omega_m)(i\omega_m)}\right) = 0
\end{aligned} \tag{3.149}$$

The Green's function (3.133) gives in this limit with $\Pi_q = 0$:

$$\begin{aligned}
\mathcal{G}^Q(q, i\omega_m) &= \frac{1}{\omega_R^2 - (i\omega_m)^2}\begin{pmatrix} \omega_R + i\omega_m & 0 \\ 0 & \omega_R - i\omega_m \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\omega_R - i\omega_m} & 0 \\ 0 & \frac{1}{\omega_R - (-i\omega_m}). \end{pmatrix}
\end{aligned} \tag{3.150}$$

In terms of the resonator fields we get then even simpler form from Wick's theorem:

$$
\begin{aligned}
-\langle \phi_q(i\omega_m)\phi_q^*(i\omega_m)\rangle &= \frac{1}{i\omega_m - \omega_R} \\
&\to \frac{1}{\omega - \omega_R + i\eta}.
\end{aligned} \tag{3.151}
$$

Without DQD interaction, and thus no longitudinal field, the system goes back to free optical photons with optical photon dispersion relations with frequency ω_R. The poles are in the negative half of the complex plane as they should be for a retarded Green's function.

3.4 Results

The spectrum of the resonator field is found from the poles of single particle retarded Green's function. The spectral function can be found from the imaginary part of the Green's function. To find the retarded Green's function we analytically continue $i\omega_m \to \omega + i\eta$. The continuation gives $\Pi_q \to \Pi^R(q,\omega) = \Pi' + i\Pi''$, and the single particle retarded Green's function for the resonator field (3.134) is found from:

$$
G_\delta^R(q,\omega+i\eta) = \frac{\omega + i\eta + \omega_R + 2\Pi^R(q,\omega+i\eta)}{\left(\omega + i\eta - \omega_R\sqrt{1+\frac{4}{\omega_R}\Pi^R(q,\omega+i\eta)}\right)\left(\omega + i\eta + \omega_R\sqrt{1+\frac{4}{\omega_R}\Pi^R(q,\omega+i\eta)}\right)}. \tag{3.152}
$$

The positive η is needed to keep the poles from the real axis and avoid divergences that would occur due to the infinitely narrow fermion density of states when evaluating the Green's function numerically.

3.4.1 Weak coupling dispersion relations

Since $\Pi^R \propto g^2$ we assume that g is a small parameter and that Π_q^R is sufficiently well behaved, such that the denominator can be expanded in g. Expanding the square root gives:

$$
\begin{aligned}
\omega_R\sqrt{1+\frac{4}{\omega_R}(\Pi' + i\Pi'')} &= \omega_R\left(1 + \frac{1}{2}\frac{4}{\omega_R}(\Pi' + i\Pi'')\right) + \mathcal{O}(g^4) \\
&= \omega_R + \Sigma_q(\omega) + i\Gamma_q(\omega),
\end{aligned} \tag{3.153}
$$

where we identified the self energy $\Sigma_q(\omega) = 2\Pi'$ and the inverse life time $\Gamma_q(\omega) = 2\Pi''$. For weak coupling the retarded Green's function is:

$$
G_\delta^R(q,\omega) = -\frac{\omega + \omega_R + \Sigma_q(\omega) + i\big(\Gamma_q(\omega) + \eta\big)}{[\omega_R + \Sigma_q(\omega) - \omega + i(\Gamma_q(\omega) - \eta)]\,[\omega_R + \Sigma_q(\omega) + \omega + i(\Gamma_q(\omega) + \eta)]}. \tag{3.154}
$$

The energy of the resonator modes are shifted by the self energy:

$$
\pm\omega(q) = \omega_R + \Sigma(q,\omega), \tag{3.155}
$$

while possibly also resulting in a particle life time $\Gamma^{-1}(q,\omega)$. The spectrum is found by numerically solving for the $\omega(q)$ that satisfies (3.155). The Newton-Raphson method was used as the root finder, and the dispersion relations are shown for different parameter regimes in figures 3.8 and 3.9.

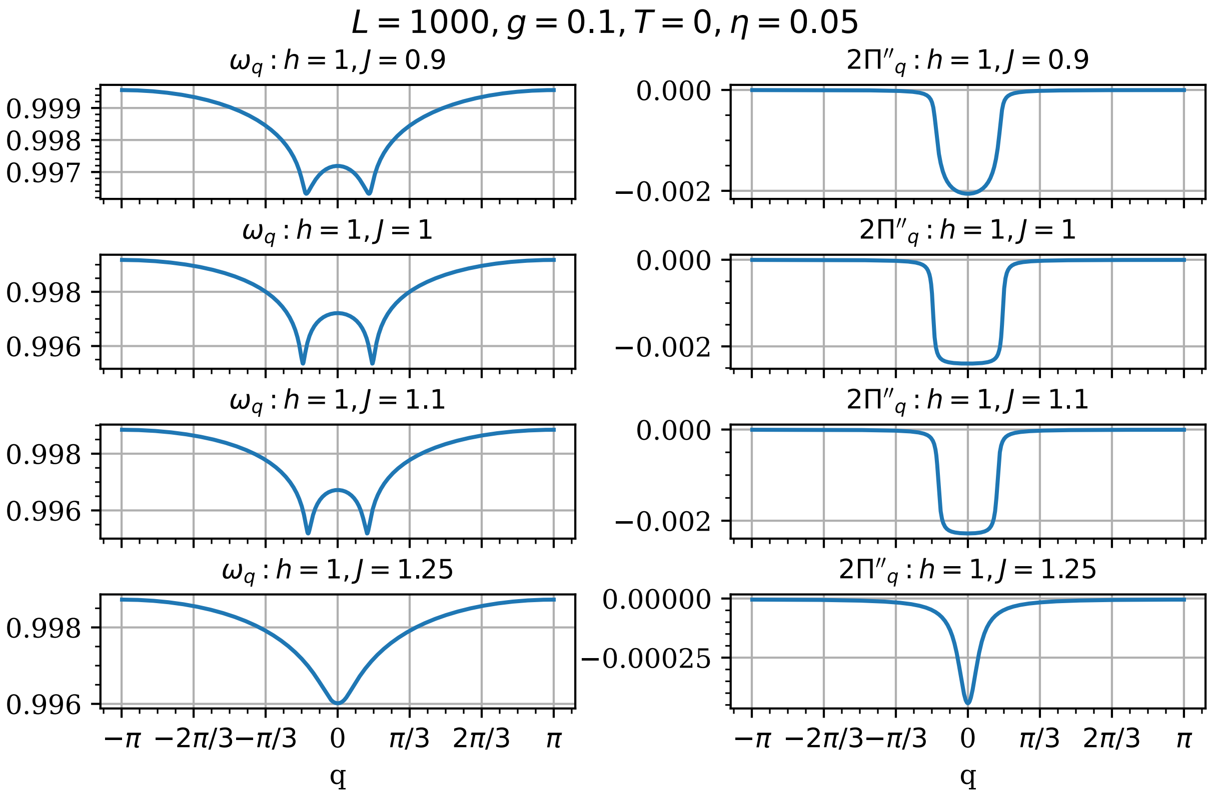

Figure 3.8: Dispersion relation and the corresponding $2\Pi''(\omega_q)$

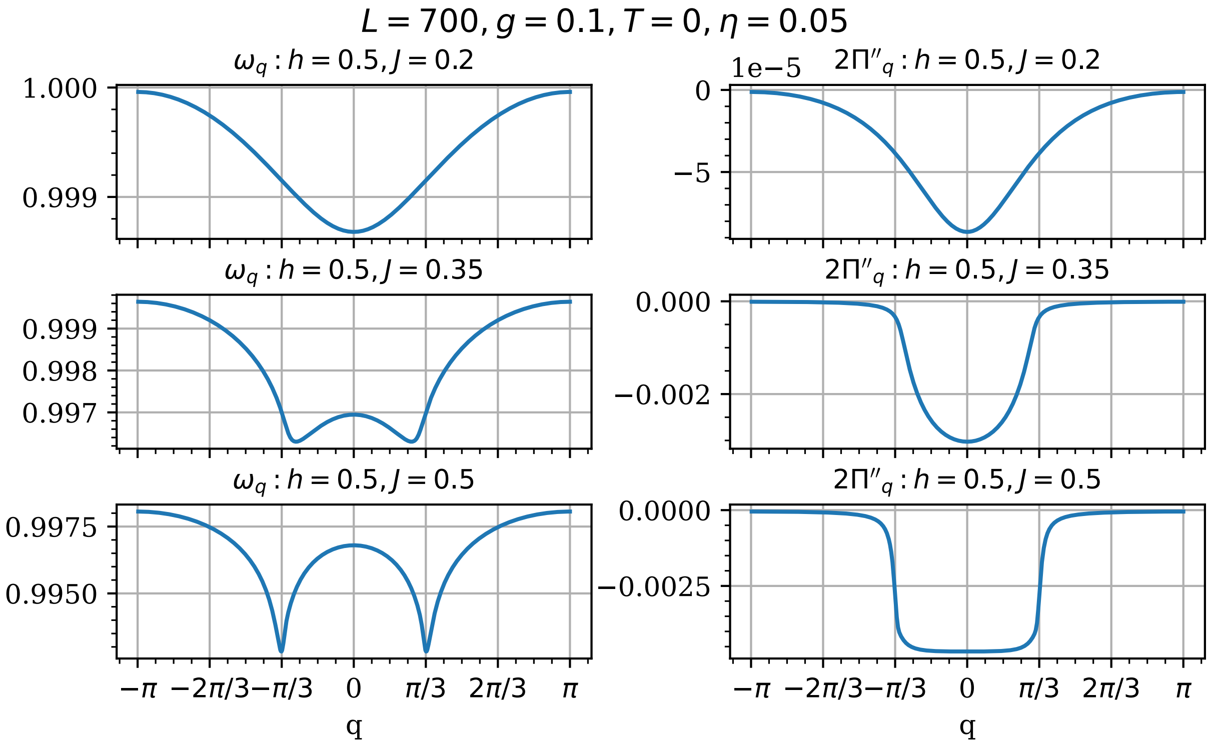

Figure 3.9: Dispersion relation and the corresponding $2\Pi''(\omega_q)$

We look to estimate the wavenumber, q, at which the modes start acquiring an imaginary part. As the resonator modes interact with the fermions through the polarization function, which is largest close to the poles of the two fermion propagators, we look to solve:

$$\xi_{k-q}^2 = (\xi_k \pm \omega)^2 . \tag{3.156}$$

Setting $k = 0$ and solving gives:

$$\begin{aligned} \xi_q^2 &= (\xi_0 \pm \omega)^2 \\ \cos q &= 1 - \frac{\omega}{8hJ} \pm \frac{|h-J|\omega}{4hJ}. \end{aligned} \tag{3.157}$$

For the parameter regime of $1 = h = J \approx \omega$ it gives $\cos q = 7/8 \Rightarrow q \approx \pi/6$, which matches the numerical result. At $h = J = 1/2$ the wavenumber is found as $\cos q = 1/2 \Rightarrow q \approx \pi/3$.

From the dispersion relation we find the group velocity. The group velocity is found from $v_g(q) = \partial_q \omega_q$ and numerically it is approximated by $v_g \approx \frac{\omega_{i+1} - \omega_i}{q_{i+1} - q_i}$.

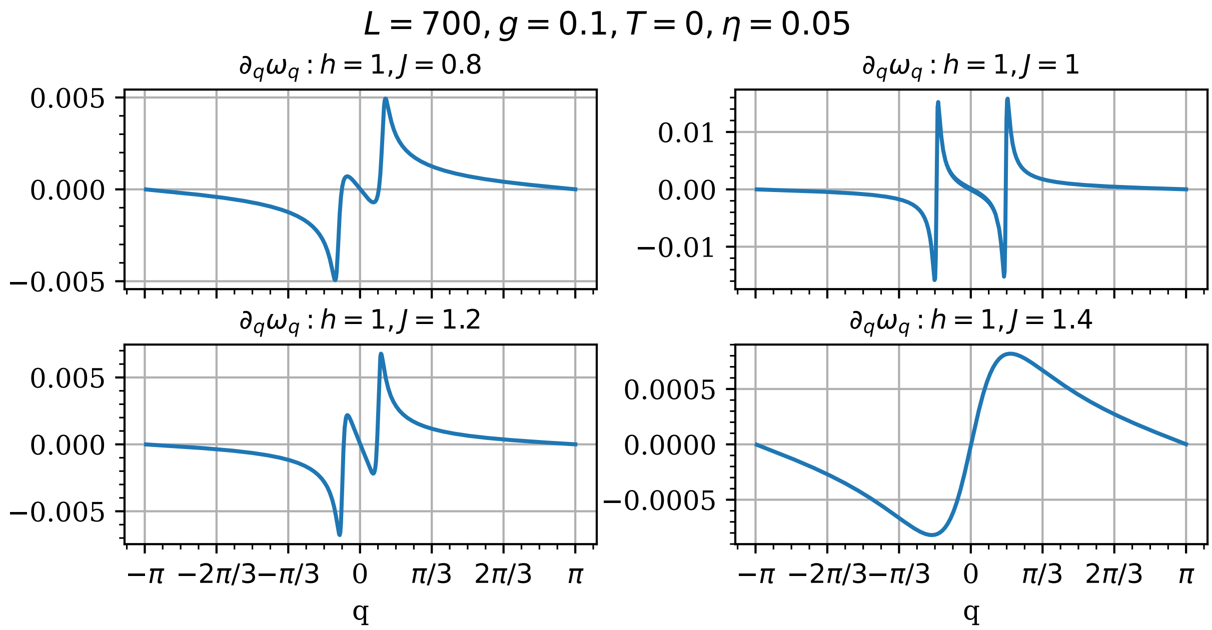

Figure 3.10: Group velocity

The group velocity is seen to be largest near the quantum critical point of $h = J$. By using realistic parameters from appendix A we can estimate the resonator wave group velocity. The units of the dispersion relation were in terms of ω_R and the wavenumber, q, is in terms of the distance between DQDs, a, which was set to one. The group velocity is thus in units of $a \times \omega_R$. Taking $h \approx J$ and q just large enough that the modes are barely damped, the group velocity is in the order of $10^{-3} a \times \omega_R$. From [13] we take $a \approx 100$nm and from appendix A we take $f_R \approx 6$GHz $\Rightarrow \omega_R \approx 40$GHz. This results in a group velocity $v_g \approx 4$m/s with the interaction strength $g = \omega_R/10$.

We have thus found the dispersion relation for the collective resonator waves to second order in g. From the dispersion relation we found the group velocity of the resonator waves in the system. The group velocity was estimated with our chosen parameters. From the imaginary part of the polarization function it was found that interaction between the Jordan-Wigner fermions and the resonators cause a decay of long-wavelength resonator modes.

3.4.2 Spectral function

The spectral function is defined as:

$$A^\delta(q,\omega) = -2\text{Im}G^R_\delta(q,\omega). \tag{3.158}$$

From this one can find the density of states as:

$$\langle \phi^*_q \phi_q \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A^\delta(q,\omega) n_F(\omega). \tag{3.159}$$

The spectral can therefore be thought of as the energy resolution of a resonator with wavenumber q. It gives an indication of how well the excitation created by adding a resonator mode with wavenumber q can be described by a free, non-interacting particle.

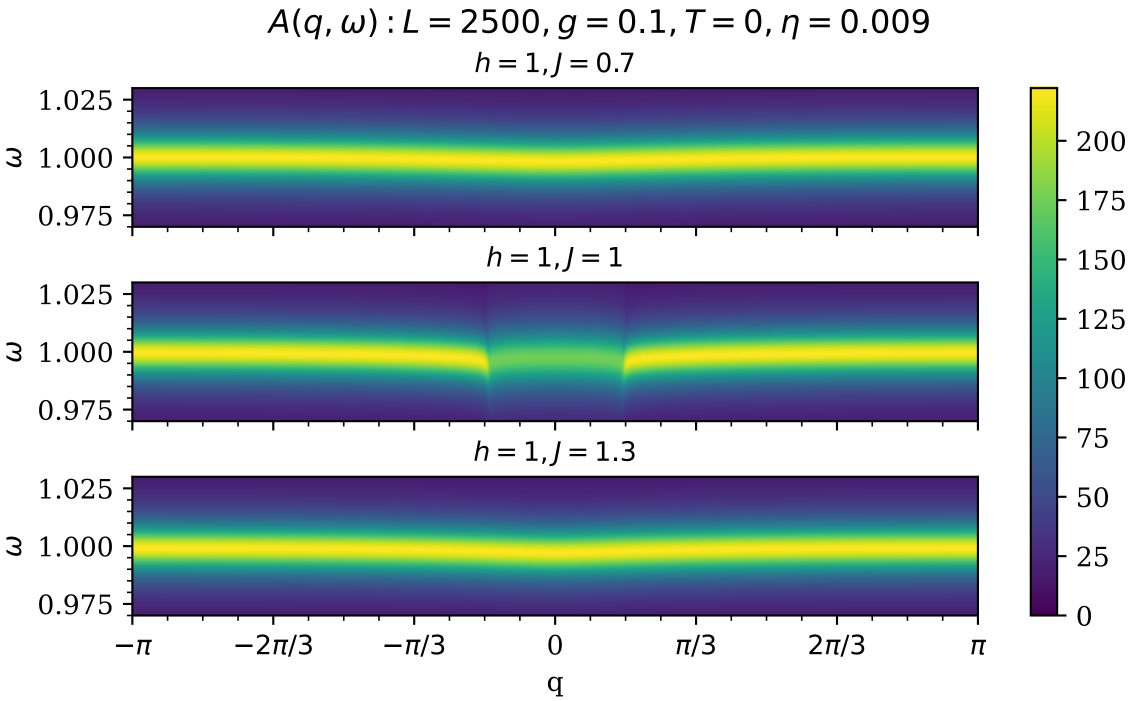

Figure 3.11: Spectral function for the resonator

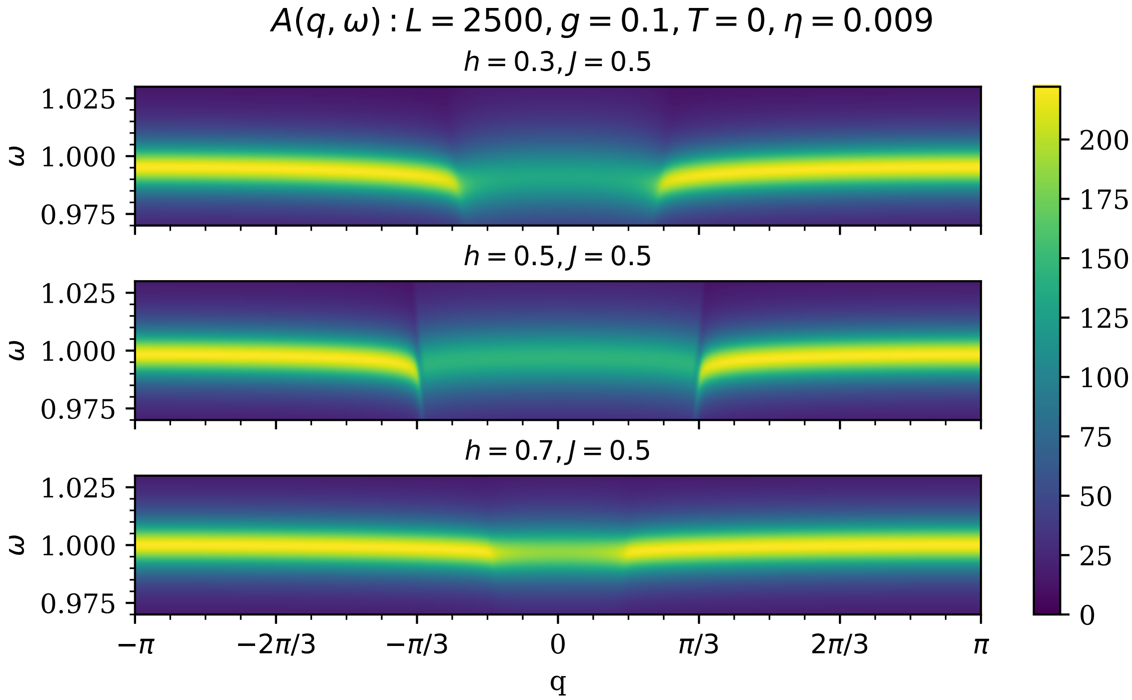

Figure 3.12: Spectral function for the resonator states

It is seen from the spectral function that the resonator states are not delta functions anymore but acquire a finite lifetime. The broadening of the spectral function is most pronounced in the fine tuned regime for long wavelengths.

In the case of no interaction we expect the spectral function to become a delta function and it is found that:

$$\begin{aligned} A|_{g=0} &= -2\text{Im}\frac{\omega + \omega_R + i\eta}{(\omega + i\eta)^2 - \omega_R^2} \\ &= \frac{1}{2}\frac{(\omega + \omega_R)^2 + \eta^2}{\omega^2}\frac{\eta}{\left(\frac{\omega^2 - \omega_R^2 - \eta^2}{2\omega}\right)^2 + \eta^2}, \end{aligned} \tag{3.160}$$

and by now letting $\eta \to 0$ and evaluating the delta function we get:

$$A|_{g=0}(\omega) = 2\pi\delta(\omega - \omega_R), \tag{3.161}$$

as expected, where we are left with the non-interacting optical photons.

Chapter 4

Conclusion and outlook

4.1 Summary

We will now summarize and conclude on the results obtained in the thesis. Motivated by dynamics in resonators coupled to spin chains we started from a chain of double quantum dots. We coupled a single resonator to the hopping elements in the DQDs. The physical DQD chain was transformed into the spin system. We could then transform the spin model into a model quadratic in spinless Jordan-Wigner fermions. To solve the system we integrated out the bosons by assuming that they were in a coherent state. Then we performed a Bogoliubov transformation. We found that the classical resonator would shift the transverse field and we also saw that the fermions would shift the resonator state. We found a saddle point equation for the classical minimum of the resonator potential for the fermion groundstate.

Going to the imaginary time path integral formalism we did not have to make assumptions about the resonator states. We could also couple many resonators to the DQD chain. With the path integral we integrated out the fermions and found an effective action for the coherent resonator states. We found that a uniform field would always exist as a solution to the resulting saddle point equation and we solved it in the fine tuned limit for weak interaction. The field that solves the saddle point equation gave a shift in resonator voltage.

To understand the dynamics of the resonator chain we expanded around the uniform field. We found an effective action for the resonator field fluctuations and expanded to lowest order in the coupling. We studied the resulting polarization function for the resonator position coordinates. From the Green's function we found the dispersion relation and dampening of the fluctuation, that came from the interaction with the Jordan-Wigner fermions. From the dispersion relations we found the group velocity of the resonator field fluctuations.

4.2 Perspectives

A natural extension of the work done in the second part of the thesis is to work on higher order interaction terms in the density wave action. With higher order terms we could hope to find more interesting dynamics from the nonlinear terms in the equations of motion.

Originally the coupling of the resonator(s) to the DQD hopping term was chosen such that the Jordan-Wigner transformation could diagonalize the fermion system. That required the DQD detuning to be zero. It could be interesting to couple to the DQD detuning according to (2.20) instead of the hopping. We could not use a Jordan-Wigner transformation, but we could possibly apply a Holstein-Primakoff transformation, transforming the spins into bosons. We would then have a system of two different types of bosons interacting, closer to the original model used by Davydov.

It would also have been interesting to study the dynamics from the point of view of the spins that were obtained from the DQDs. To do so we should integrate out the bosons instead of the fermions. Integrating out the bosons would lead to a non-quadratic model as shown in appendix E.

Bibliography

[1] A. Davydov, "The theory of contraction of proteins under their excitation", *Journal of Theoretical Biology*, vol. 38, no. 3, pp. 559–569, 1973, ISSN: 0022-5193. DOI: https://doi.org/10.1016/0022-5193(73)90256-7. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0022519373902567.

[2] T. N. De Silva and P. Bolt, *Bio-energy transport as a phonon dressed vibrational exciton in protein molecules*, 2019. DOI: 10.48550/ARXIV.1903.11581. [Online]. Available: https://arxiv.org/abs/1903.11581.

[3] T. Bonsen, P. Harvey-Collard, M. Russ, *et al.*, *Probing the jaynes-cummings ladder with spin circuit quantum electrodynamics*, 2022. DOI: 10.48550/ARXIV.2203.05668. [Online]. Available: https://arxiv.org/abs/2203.05668.

[4] P.-O. Löwdin, "A note on the quantum-mechanical perturbation theory", *The Journal of Chemical Physics*, vol. 19, no. 11, pp. 1396–1401, 1951. DOI: 10.1063/1.1748067. eprint: https://doi.org/10.1063/1.1748067. [Online]. Available: https://doi.org/10.1063/1.1748067.

[5] A. C. Scott, "Dynamics of davydov solitons", *Phys. Rev. A*, vol. 26, pp. 578–595, 1 Jul. 1982. DOI: 10.1103/PhysRevA.26.578. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevA.26.578.

[6] D. D. Georgiev and J. F. Glazebrook, *Physica A: Statistical Mechanics and its Applications*, vol. 517, pp. 257–269, Mar. 2019. DOI: 10.1016/j.physa.2018.11.026.

[7] W. C. Kerr and P. S. Lomdahl, "Quantum-mechanical derivation of the equations of motion for davydov solitons", *Phys. Rev. B*, vol. 35, pp. 3629–3632, 7 Mar. 1987. DOI: 10.1103/PhysRevB.35.3629. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevB.35.3629.

[8] Y. Y. Atas and E. Bogomolny, "Quantum ising model in transverse and longitudinal fields: Chaotic wave functions", *Journal of Physics A: Mathematical and Theoretical*, vol. 50, no. 38, p. 385 102, Aug. 2017. DOI: 10.1088/1751-8121/aa81f6. [Online]. Available: https://doi.org/10.1088/1751-8121/aa81f6.

[9] G. B. Mbeng, A. Russomanno, and G. E. Santoro, *The quantum ising chain for beginners*, 2020. DOI: 10.48550/ARXIV.2009.09208. [Online]. Available: https://arxiv.org/abs/2009.09208.

[10] A. Altland and B. Simons, *Condensed matter field theory*. Cambridge University Press, 2013.

[11] H. Bruus and K. Flensberg, *Many-body quantum theory in Condensed matter physics: An introduction*. Oxford University Press, 2020.

[12] L. Childress, A. S. Sørensen, and M. D. Lukin, "Mesoscopic cavity quantum electrodynamics with quantum dots", *Phys. Rev. A*, vol. 69, p. 042 302, 4 Apr. 2004. DOI: 10.1103/PhysRevA.69.042302. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevA.69.042302.

[13] D. R. Ward, D. Kim, D. E. Savage, *et al.*, "State-conditional coherent charge qubit oscillations in a si/sige quadruple quantum dot", *npj Quantum Information*, vol. 2, Oct. 2016. DOI: 10.1038/npjqi.2016.32.

[14] N. Samkharadze, G. Zheng, N. Kalhor, *et al.*, "Strong spin-photon coupling in silicon", *Science*, vol. 359, no. 6380, pp. 1123–1127, 2018. DOI: 10.1126/science.aar4054. eprint: https://www.science.org/doi/pdf/10.1126/science.aar4054. [Online]. Available: https://www.science.org/doi/abs/10.1126/science.aar4054.

[15] H. Toida, T. Nakajima, and S. Komiyama, "Vacuum rabi splitting in a semiconductor circuit qed system", *Phys. Rev. Lett.*, vol. 110, p. 066 802, 6 Feb. 2013. DOI: 10.1103/PhysRevLett.110.066802. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevLett.110.066802.

[16] D. Ferraro, G. M. Andolina, M. Campisi, V. Pellegrini, and M. Polini, "Quantum supercapacitors", *Phys. Rev. B*, vol. 100, p. 075 433, 7 Aug. 2019. DOI: 10.1103/PhysRevB.100.075433. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevB.100.075433.

[17] C. C. Gerry and P. L. Knight, *Introductory Quantum Optics*. Cambridge University Press, 2008.

[18] R. Feynmann, *Feynmann lectures*. [Online]. Available: https://www.feynmanlectures.caltech.edu/II_15.html.

[19] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, "Circuit quantum electrodynamics", *Rev. Mod. Phys.*, vol. 93, p. 025 005, 2 May 2021. DOI: 10.1103/RevModPhys.93.025005. [Online]. Available: https://link.aps.org/doi/10.1103/RevModPhys.93.025005.

[20] D. j. Griffiths, *Introduction to quantum machanics*. Pearson, 2014.

[21] W. G. van der Wiel, S. De Franceschi, J. M. Elzerman, T. Fujisawa, S. Tarucha, and L. P. Kouwenhoven, "Electron transport through double quantum dots", *Rev. Mod. Phys.*, vol. 75, pp. 1–22, 1 Dec. 2002. DOI: 10.1103/RevModPhys.75.1. [Online]. Available: https://link.aps.org/doi/10.1103/RevModPhys.75.1.

[22] C. J. van Diepen, P. T. Eendebak, B. T. Buijtendorp, *et al.*, "Automated tuning of inter-dot tunnel coupling in double quantum dots", *Applied Physics Letters*, vol. 113, no. 3, p. 033 101, 2018. DOI: 10.1063/1.5031034. eprint: https://doi.org/10.1063/1.5031034. [Online]. Available: https://doi.org/10.1063/1.5031034.

[23] N. S. Lai, W. H. Lim, C. H. Yang, *et al.*, "Pauli spin blockade in a highly tunable silicon double quantum dot", *Scientific Reports*, vol. 1, no. 1, 2011. DOI: 10.1038/srep00110.

[24] N. Samkharadze, G. Zheng, N. Kalhor, *et al.*, "Strong spin-photon coupling in silicon", *Science*, vol. 359, no. 6380, pp. 1123–1127, 2018. DOI: 10.1126/science.aar4054. [Online]. Available: https://www.science.org/doi/abs/10.1126/science.aar4054.

[25] P.-Q. Jin, M. Marthaler, A. Shnirman, and G. Schön, "Strong coupling of spin qubits to a transmission line resonator", *Phys. Rev. Lett.*, vol. 108, p. 190 506, 19 May 2012. DOI: 10.1103/PhysRevLett.108.190506. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevLett.108.190506.

Appendix A

Experimental parameters

$$\hbar\omega_r$$

In the work of [3] the bare resonator frequency $f_R = \omega_R/2\pi$ is given as ~ 6.9GHz. In the work on spin-photon coupling of [14] it is given that $f_R \sim 6$GHz. The bare resonator frequecy in energy is found as $\hbar \times 6\text{GHz} \sim 0.025$meV when using that $\hbar \times 1\text{GHz} \sim 6.6 \cdot 10^{-4}$meV. Another experiment studying vacuum Rabi splitting in a DQD coupled to a co-planar resonator has $f_R \sim 8.3$GHz which amounts to ~ 0.034meV [15].

$$t_h = h$$

The hopping term, $2t_h/h$, is given for the double dots in [14] to lie in the interval of $7.8 - 14.6$GHz. In [3] the hopping strength is given as $2t_h/h = 12.0$GHz. This gives the hopping t_h to be in the interval of $0.015 - 0.03$meV. In the experiment from Tokyo, $t_h/2\pi \sim 1.5$GHz or only $6 \cdot 10^{-6}$meV. t_h could experimentally be tuned around $\frac{t_h}{\hbar\omega_R} \sim 1$

$$g$$

An estimate on a modern strong charge-photon coupling is 200MHz. This is usually coupling to the detuning and not the tunneling. This is $\sim 8 \cdot 10^{-4}$meV or in terms of the resonator frequency $\frac{g}{\hbar\omega_R} \sim 1/30$. In terms of these quantities the coupling could be said to be in the weak coupling regime. We have to estimate the constant shift in h from the resonators to estimate h' that is used in most of the thesis, but the correction is on the order of g^2.

$$U = 4J$$

The Coulomb potential between adjacent quantum dots is proportional to the longitudinal field, J, of the TFI. In the work of [16] the Coulomb potential, U/h, is estimated as up to 30GHz which would result in $J \sim 7.5$GHz, again giving $J \sim h \sim \hbar\omega_R$ making the fine tuned regime not unreasonable. In the works of [13] they state that inter-DQD capacitive couplings in GaAs DQD lie in the range of $25 - 125$ μeV which would give J in the range of $12 - 31$ μeV much the same range as $t_h = h$.

Parameter	symbol	value [μeV]
Bare resonator freq	$\hbar\omega_R$	$25 - 35$
DQD-Resonator coupling	g	~ 0.8
DQD tunnel coupling	h	$6 - 30$
Inter-DQD potential	U	$25 - 125$

Appendix B

Coherent states

B.1 Coherent states from displacement operator

The displacement operator is defined as:

$$\hat{D}\left(\alpha\right) = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}. \tag{B.1}$$

With commutation relations $[a, a^{\dagger}] = 1$ it can be used from the Baker–Campbell–Hausdorff formula that:

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \tag{B.2}$$

such that:

$$\hat{D}(\alpha) = e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^{\dagger}}e^{\alpha^{*}\hat{a}}. \tag{B.3}$$

Applying the displacement operator onto a vacuum state gives:

$$\hat{D}(\alpha)\left|0\right\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^{\dagger}}\left|0\right\rangle \tag{B.4}$$

$$= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle, \tag{B.5}$$

which is a normalized coherent state with eigenvalue α. We see that the coherent states are vacuum states displaced by α.

To see why we call it a displacement operator we define $\alpha = \sqrt{\frac{m\omega}{2\hbar}}x \in \mathbb{R}$. For a harmonic oscillator $\hat{a} - \hat{a}^{\dagger} = i\sqrt{\frac{2}{\hbar m\omega}}\hat{p}$, which gives:

$$\hat{D}(\alpha) = e^{-ix\hat{p}} = \hat{T}(x), \tag{B.6}$$

and we recover the real space translation operator.

B.1.1 Displacement identity

Defining the two operators $A = \alpha a^{\dagger} - a^{*}a$ and $B = \beta a^{\dagger} - \beta^{*}a$ we have:

$$D(\alpha)D(\beta) = e^{A}e^{B}. \tag{B.7}$$

Since we have $[A, B] = -2\text{Im}(\alpha\beta^*) \in \mathbb{C}$ we use the Zessenhaus expansion to get:

$$\begin{aligned} D(\alpha)D(\beta) &= e^{A+B}e^{-\frac{1}{2}[A,B]} \\ &= e^{\text{Im}(\alpha\beta^*)}e^{(\alpha+\beta)a-(\alpha+\beta)^*a^\dagger}, \end{aligned}$$

such that:

$$D(\alpha)D(\beta) = e^{\text{Im}(\alpha\beta^*)}D(\alpha+\beta). \tag{B.8}$$

Up to a phase two displacements, $D(\alpha)$ and $D(\beta)$, are thus equal to the displacement $D(\alpha+\beta)$.

B.2 Generating coherent states from coupling classical and quantum

A coherent state can be generated by a classical current [17]. A classical current density is given by:

$$\vec{j}(\mathbf{r}, t) \tag{B.9}$$

An electromagnetic field is given by the vector potential $\mathbf{A}(\mathbf{r}, t)$. A single quantized mode of the field is given in the interaction picture by:

$$\mathbf{A}(\mathbf{r}, t) = \hat{e}\sqrt{\frac{\hbar}{2\omega\epsilon_0 V}}\left[\hat{a}e^{ikr-i\omega t} + \hat{a}^\dagger e^{-ikr+i\omega t}\right]. \tag{B.10}$$

Classical electromagnetic theory gives the interaction potential as:

$$V(t) = \int \mathbf{j}(\mathbf{r}, t)\mathbf{A}(\mathbf{r}, t)d\mathbf{r}.^1 \tag{B.11}$$

Inserting the quantized field mode gives:

$$\begin{aligned} V(t) &= \sqrt{\frac{\hbar}{2\omega\epsilon_0 V}}\left(\hat{a}\hat{e}\cdot\int \mathbf{j}(\mathbf{r}, t)e^{ikr}d\mathbf{r}e^{-i\omega t} + \hat{a}^\dagger\hat{e}\cdot\int \mathbf{j}(\mathbf{r}, t)e^{-ikr}d\mathbf{r}e^{i\omega t}\right) && \text{(B.12)} \\ &= \sqrt{\frac{\hbar}{2\omega\epsilon_0 V}}\left(\hat{a}\hat{e}\cdot\mathbf{j}(\mathbf{k}, t)e^{-i\omega t} + \hat{a}^\dagger\hat{e}\cdot\mathbf{j}^*(\mathbf{k}, t)e^{i\omega t}\right), && \text{(B.13)} \end{aligned}$$

using the Fourier transform $\int f(r)e^{ikr}dr = f(k)$. [2] Staying in the interaction picture we have a time evolution operator that depends on time. For small time steps we get:

$$\begin{aligned} \hat{U}(t+\delta t, t) &\approx e^{-i\frac{V(t)\delta t}{\hbar}} \\ &= e^{-\frac{i}{\hbar}\sqrt{\frac{\hbar}{2\omega\epsilon_0 V}}\left(\hat{a}\hat{e}\cdot\mathbf{j}(\mathbf{k},t)e^{-i\omega t}+\hat{a}^\dagger\hat{e}\cdot\mathbf{j}^*(\mathbf{k},t)e^{i\omega t}\right)\delta t} \\ &= e^{u(t)\hat{a}^\dagger-(u^*(t)\hat{a})\delta t} \\ &= \hat{D}\left(u(t)\delta t\right), \end{aligned} \tag{B.14}$$

[1] This is only true for static fields [18]

[2] Since the volume, V is only for the field we do not want to integrate over that volume but all of space and do not use V for the Fourier transform.

where we defined:

$$u(t) = -\frac{i}{\hbar}\sqrt{\frac{\hbar}{2\omega\epsilon_0 V}}\hat{e}\cdot\mathbf{j}^*(\mathbf{k},t)e^{i\omega t}, \tag{B.15}$$

and used the definition (B.1). We apply these small time evolutions in a time ordered product to get the time evolution over a finite time interval:

$$U(T,0) = \lim_{\delta t\to 0}\hat{\mathbb{T}}\prod_{l=0}^{\frac{T}{\delta t}} D(u(t_l)\delta t), \tag{B.16}$$

where the time $t_l = l\delta t$. We use (B.8) to get the time ordered product of time evolutions:

$$\begin{aligned} U(T,0) &= \lim_{\delta t\to 0} e^{i\Phi} D(\sum_{l=0}^{T/\delta t} u(t_l)\delta t) \\ &= e^{i\Phi} D(\alpha(T)), \end{aligned} \tag{B.17}$$

where the phase $\Phi \in \mathbb{C}$ is the overall phase accumulated from adding the displacement according to (B.8). The α is:

$$\begin{aligned} \alpha(T) &= \lim_{\delta t\to 0}\sum_{l=0}^{T/\delta t} u(t_l)\delta t \\ &\to \int_0^T u(t)dt, \end{aligned} \tag{B.18}$$

where the upper limit is T since the final time step $t_{T/\delta t} = \frac{T}{\delta t}\delta t = T$. If we let the state evolve from the infinite past to the infinite future, we get:

$$\begin{aligned} \alpha(\infty,-\infty) &\to \int_{-\infty}^{\infty} u(t)dt \\ &= -\frac{i}{\hbar}\sqrt{\frac{\hbar}{2\omega\epsilon_0 V}}\hat{e}\cdot\int_{-\infty}^{\infty}\mathbf{j}^*(\mathbf{k},t)e^{i\omega' t}dt \\ &= -\frac{i}{\hbar}\sqrt{\frac{\hbar}{2\omega\epsilon_0 V}}\hat{e}\cdot\mathbf{j}^*(\mathbf{k},\omega) = \alpha(\mathbf{k},\omega). \end{aligned} \tag{B.19}$$

Since it is now expanded in the plane wave basis $\int f(r,t)e^{ikr-i\omega t}dtdr = f(k,\omega)$ we get the $j^*(k,\omega)$ in the equation. We see that the time evolution operator is a displacement operator in the quantity $\alpha(T)$ that depends on the current with an overall phase.

If we had an electromagnetic field (eg a superconducting resonator) in the vacuum state, we could then turn on the classical current (eg a feedline) and evolve the vacuum state with the time-evolution operator. From (B.5) that results in the coherent state:

$$\begin{aligned} |\psi(T)\rangle &= U(T)\,|0\rangle \\ &= e^{-\frac{|\alpha(T)|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n(T)}{\sqrt{n!}}\,|n\rangle\,. \end{aligned} \tag{B.20}$$

Where we ignored the overall phase acquired from the displacement operators. We see that having an electromagnetic field coupled to a classical current results in a time evolution operator that evolves the vacuum state of the electromagnetic field into a coherent state.

We also see that for the case of (B.19) (but also in general) we have that $\bar{n} = |\alpha|^2 \propto \frac{|\mathbf{j}(\omega)|^2}{\omega}$. We see that the average photon number in the eletromagnetic field is proportional to the square of the classical current density.

Appendix C

Resonators and quantum dot

C.1 Classical circuits

How do we describe circuits when they are miniaturized and the signal wavelengths are on the order of the system size?

C.1.1 Lumped elements model

When calculating currents, voltages ect in circuits one usually starts from a "lumped element" model as described in fig C.1.

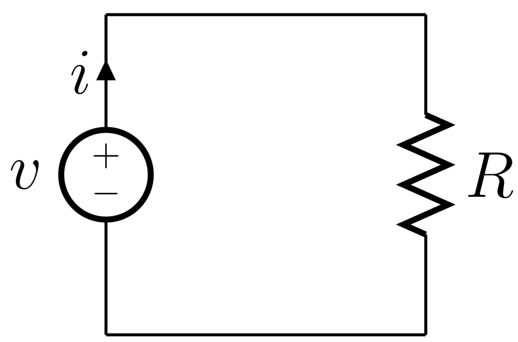

Figure C.1: A simple "lumped element" model showing a voltage source and a resistor.

In the lumped circuit model one imagines the wires as having no resistance and that the resistance in the circuit loop is concentrated in the resonator "lump". This is of course a simplification where the spatially distributed circuit parameters (resistance, capacitance, ..) are described in a "topology" of "a topology consisting of discrete entities that approximate the behavior of the distributed system under certain assumptions" (Wikipedia). All mentions about distance and position in this circuit are ignored.

The lumped element model is valid when $L_c \ll \lambda$ where L_c is the circuits characteristic wavelength and λ is the operating wavelength, eg for a uniform current of $\lambda \to \infty$. When the spatial variations in the signal can not be ignore, the lumped elements model breaks down and we have to go to the "distributed elements" model.

C.1.2 Distributed elements model

To consider spatial variations we take as an example the conventional co-axial cable consisting of a center conductor and a ground conductor separated by a dielectric as described in figure C.2.

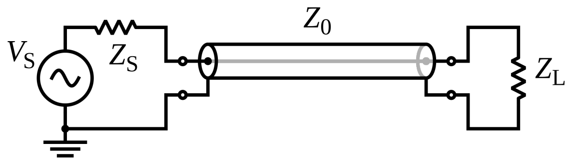

Figure C.2: Schematic representation of co-axial cable

In this system we have an electric and magnetic field between the center conductor and ground plane from the current. This is modeled as a capacitance between the two (there is an electric field in a capacitor but no current) and a conductance in the center conductor (giving a magnetic field). There is also a resistance in the system that we imagine as a resistance in the center conductor. There are certain boundary conditions we have to think about at the ends. To simplify this example we imagine the co-axial cable being made perfectly such that capacitance, resistance and inductance are evenly distributed in the system. An example of a distributed model of such a system is a lossy transmission line described by a telegrapher's type equation as seen in figure C.3.

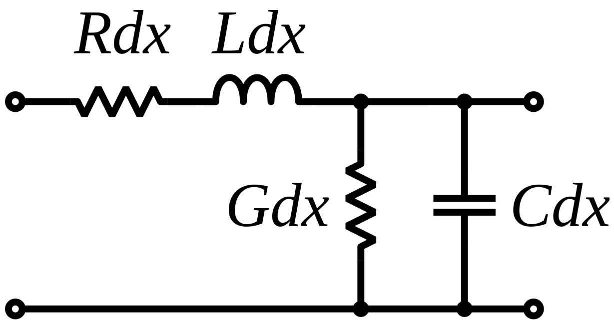

Figure C.3: Local element of a lossy transmission line.

Here we model the transmission line as N connected local lumped element models where the resistance is $R\,dx$, $dx = L/N$ such that the total resistance is $R \times L$. The same for the rest of the components. But taking the limit of $N \to \infty$ with L constant we get to an integral from Kirchoff's equations that we can solve for the signal and thus have an equation that includes spatial variations. The solution then naturally depends on the boundary conditions, whether for example the center conductor is capacitatively coupled or grounded in the ends.

C.2 Strip line or $\lambda/2$ resonators

In circuit QED, microwave resonators are superconducting 2D strips with a microwave field confined in the plane. The boundary conditions set on this plane lead to the discretization of the electromagnetic into distinct harmonic modes where each mode can be thought of as an independent harmonic oscillator. In the 2D superconducting strip resonator the modes are actual charge waves from the dissipation-less plasma waves [19].

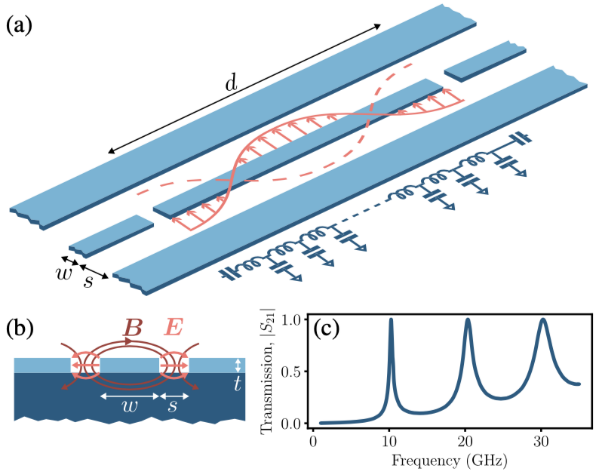

Figure C.4: Strip line resonator. A center conductor of length d and width w that is capacitively coupled to and input and output port for signals. The center conductor is separated from a ground by a distance s such that the resonator can be compared to a classical "squashed" coaxial cable. Figure (b) shows the electric and magnetic fields that are modelled as the capacitances and inductances in the distributed component model.

In figure C.4 we see that the resonator can be thought of as a "squashed" conventional co-axial cable where, just like in the co-axial cable, the E and B-fields are confined to a region between the center conductor and ground plane. The dimensions of the center conductor, dielectric substrate (that everything is deposited on), and gaps are chosen such that the field(s) are concentrated between the center conductor and ground plane.

We consider a resonator of length $d = 1$cm with a signal in the microwave regime. This means that we have $\lambda \approx d$ and we need to consider spatial variations of the electric and magnetic field, as illustrated in figure C.4. To do so we consider the following classical distributed element model:

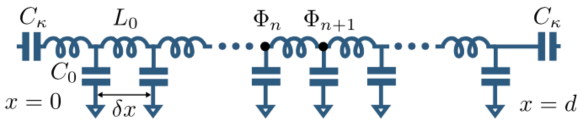

Figure C.5: Distributed element model of superconducting strip line resonator

In the model there is no resistances since we imagine having a superconducting center conductor (thus the charges are cooper pairs with charge $2e$). To solve a system like this we want to write down a Hamiltonian in terms of some conjugate variables. We follow the approach of [19]. The energy of a capacitor is:

$$E_C = \frac{Q^2}{2C}, \tag{C.1}$$

where Q is the charge on the capacitor and C is the capacitance. The energy of the inductor (thought of as circular loop) is given by $E_I = \frac{I^2}{2L}$. Using the relationship between the magnetic flux and current in a loop we have that $\Phi = IL$, using this we get the energy of the conductor in terms of the magnetic flux Φ as:

$$E_I = \frac{\Phi^2}{2L}, \tag{C.2}$$

where L is the inductance [20]. We then enforce charge conservation at each node such that $Q_n(t) = \int_{-\infty}^{t} I_n(t')dt'$, with I_n the current through node n. From Faraday's law we get the relationship between voltage and flux as $\Phi_n(t) = \int_{-\infty}^{t} V_n(t')dt'$, with V_n the voltage to ground at node n, where it is assumed that the charge and flux is zero at the infinite past. We can then rewrite:

$$\begin{aligned} &\partial_t Q_n(t) = C\partial_t^2 \Phi_n(t) \\ &\Rightarrow Q_n(t) = C\partial_t \Phi_n(t). \end{aligned} \tag{C.3}$$

We now imagine the energy of the capacitor as the kinetic energy since it depends on the time derivative of the flux (We could have also defined it in terms of the charge variable). We can now write down the Lagrangian:

$$L[\Phi, \dot{\Phi}] = \sum_{n=0}^{N} \left\{ \frac{C_0}{2}\dot{\Phi}_n^2 - \frac{(\Phi_{n+1} - \Phi_n)^2}{2L_0} \right\}, \tag{C.4}$$

where $\Phi_{n+1} - \Phi_n$ is the flux difference between node $n+1$ and n which is then the flux in the inductor. We can then find the conjugate momentum to the flux coordinate as:

$$\begin{aligned} \pi_n &= \frac{\delta L}{\delta \dot{\Phi}_n} \\ &= C_0 \dot{\Phi}_n = Q_n(t), \end{aligned} \tag{C.5}$$

where we found that the charge and flux at each node are conjugate variables. This gives a very physical picture since the inductor converts charge to flux so they are intuitively conjugate variables. We can now write down the classical Hamiltonian for the distributed elements circuit in figure C.5 in terms of the conjugate variables Φ_n and $\pi_n = Q_n$ at each node:

$$\begin{aligned} H &= \sum_{n=0}^{N-1} \pi_n \dot{\Phi}_n - L \\ &= \sum_{n=0}^{N-1} \left\{ \frac{Q_n^2}{2C_0} + \frac{(\Phi_{n+1} - \Phi_n)^2}{2L_0} \right\}. \end{aligned} \tag{C.6}$$

We now want to get a continuous model, so we let the voltage and charge at each node be fields over x such that $Q_n = Q(x_n)\delta x$ and $\Phi_n = \Phi(x_n)$, where $Q(x)$ is the charge density $q'(x)$ at position x. We then also need to take $C_0 = c\delta x$ and $L_0 = l\delta x$, where l and c are now inductance and capacitance per unit length (L/d and C/d, since we assumed uniform system) such that integrating from 0 to d gives the total capacitance and inductance. Lastly we also get $\pi(x_n) = Q(x_n) = c\dot{\Phi}(x_n)$. We can then expand the flux field in a Taylor expansion:

$$\Phi(x_{n+1}) = \Phi(x_n + \delta x) = \Phi(x_n) + \partial_x \Phi(x_n)\delta x + \mathcal{O}(\delta x^2). \tag{C.7}$$

Inserting into the Hamiltonian and remembering we are taking a limit of $N \to \infty$ we have:

$$\begin{aligned} H &= \sum_{n=0}^{N-1} \left\{ \frac{Q^2(x_n)}{2c} + \frac{[\partial_x \Phi(x_n)]^2}{2l} \right\} \delta x \\ &\to \int_0^d \left\{ \frac{Q^2(x)}{2c} + \frac{[\partial_x \Phi(x)]^2}{2l} \right\} dx. \end{aligned} \tag{C.8}$$

By using Hamilton's equations:

$$\frac{\partial \pi(x)}{\partial t} = -\frac{\partial \mathcal{H}}{\partial \Phi(x)}, \tag{C.9}$$

along with integration by parts:

$$\int_0^d [\partial_x \Phi(x)]^2 \, dx = \left[\Phi(x)\partial_x^2 \Phi(x)\right]_0^d - \int_0^d \Phi(x)\partial_x^2 \Phi(x) dx, \tag{C.10}$$

we get the equations of motion for the flux:

$$\begin{aligned} \frac{\partial \pi(x)}{\partial_t} &= c\partial_t^2 \Phi(x,t) \\ &= \frac{1}{2l}\partial_x^2 \Phi(x,t), \end{aligned}$$

which gives the equation for the field:

$$v^2 \partial_x^2 \Phi(x,t) - \partial_t^2 \Phi(x,t) = 0, \tag{C.11}$$

where $v = \sqrt{\frac{1}{2cl}}$. We assume separable solutions such that:

$$\Phi(x,t) = \sum_i u_i(x)\Phi_i(t), \tag{C.12}$$

we get that $\partial_t^2 \Phi_i(t) = -\omega_m^2 \Phi_i(t)$ and $v^2 \partial_x^2 u_i(x) = -\omega_m^2 u_i(x)$. We then solve the spatial part with:

$$u_i(x) = A_i \cos(k_i x + \theta_i). \tag{C.13}$$

The boundary conditions were such that the resonator is capacitively coupled in both ends giving that the current $I(x = 0, d) = 0$. The charge at node n was described by $Q_n = \int_{-\infty}^{t} I(t')dt'$ and we then got the continuous charge density $Q(x) = \partial_x q(x)$ giving us the charge conservation again like:

$$Q(x,t) = \int_{-\infty}^{t} \frac{\partial I(x,t)}{\partial x} dt', \tag{C.14}$$

such that $I'(x,t) = \dot{Q}(x,t) = c\partial_t^2 \Phi(x,t)$. From the equation of motion we then get the following equation for the current:

$$\begin{aligned}\partial_x I(x,t) =& c\partial_t^2 \Phi(x,t) \\ =& cv^2 \partial_x^2 \Phi(x,t) \\ \Rightarrow I(x,t) =& \frac{1}{2l} \partial_x \Phi(x,t).\end{aligned} \tag{C.15}$$

Inserting the boundary points we get:

$$I(x = 0, d) = \frac{1}{2l} \partial \Phi(x)|_{x=0,d} = 0, \tag{C.16}$$

giving us for all eigenmodes of the system that:

$$\begin{aligned}\sin(k_n x + \theta)|_{x=0,d} = 0 \\ \Rightarrow k_n = \frac{\pi}{d} n.\end{aligned} \tag{C.17}$$

From requiring normalized spatial modes we get:

$$\begin{aligned}\frac{1}{d} \int_0^d A_n \cos(k_n x) A_m \cos(k_m x) dx = \delta_{nm} \\ \Rightarrow A_n = \sqrt{1/2}\end{aligned} \tag{C.18}$$

From the equation $v^2 \partial_x^2 u_n(x) = -\omega_n^2 u_n(x)$ we get that $\omega_n = v k_n$. We can then write the solution to the classical system as:

$$\begin{aligned}\Phi(x,t) &= \sum_n u_n(x) \Phi_n(t) \\ &= \frac{1}{\sqrt{2}} \sum_n \cos(k_n x) \cos(v k_n t).\end{aligned} \tag{C.19}$$

By using that Q and Φ are conjugate variables we see that:

$$\begin{aligned}Q(x,t) &= c\dot{\Phi}(x,t) \\ &= -\sum_n \frac{1}{\sqrt{2}} \cos(k_n x) \omega_n \sin(\omega_n t) \\ &= \sum_n u_n(x) Q_n(t)\end{aligned} \tag{C.20}$$

At this point we are in position to answer the question of how the voltage and current looks. From Faraday's law we had that $\partial_t \Phi(x,t) = V(x,t)$ such that:

$$
\begin{aligned}
V(x,t) &= -\frac{1}{\sqrt{2}} \sum_n \omega_n \cos(k_n x) \sin(v k_n t) \\
&= \frac{1}{c} Q(x,t),
\end{aligned} \tag{C.21}
$$

which intuitively we think of the charge density Q times d, coming from $c = C/d$, such that it actually just says that voltage is the charge divided by the capacitance. For the current we had from (C.15):

$$
I(x,t) = -\frac{1}{2l} \frac{1}{\sqrt{2}} \sum_n k_n \sin(k_n x) \cos(v k_n t), \tag{C.22}
$$

and we see that for all modes we have that when the current is maximum in time or position, the voltage will be lowest and vice versa. Generally the voltage will be largest at each end of the resonator where $\cos k_n x|_{x=0,d} = 1$. If we insert the solution back in to the Hamiltonian and use that the spatial functions are orthogonal, and $Q = c\partial_t \Phi \Rightarrow Q_n = c\dot{\Phi}_n$ from multiplying each side with $u_n(x)$ and integrating we get:

$$
\begin{aligned}
H &= \sum_n \left\{ \frac{d\, Q_n^2}{2c} + \frac{d\, k_n^2 \Phi_n^2}{2l} \right\} \\
&= \sum_n \left\{ \frac{(d\, Q_n)^2}{2C} + \frac{d\, \omega_n^2 2cl \Phi_n^2}{2l} \right\} \\
&= \sum_n \left\{ \frac{q_n^2}{2C} + \frac{1}{2} (\sqrt{2}\omega_n)^2 C \Phi_n^2 \right\},
\end{aligned} \tag{C.23}
$$

where the charge density eigenmode, Q_n has been redefined to the charge eigenmode $d\, Q_n = q_n$, the length of the resonator times the charge density mode. Promoting q_n and Φ_n to non-commuting quantum operators we define the raising and lowering operator a_n and $a_n^\dagger$ (that should depend on time):

$$
\hat{q}_n = \sqrt{\frac{\hbar}{2}} \omega_n C \left(a_n^\dagger + a_n \right), \qquad \hat{\Phi}_n = i \sqrt{\frac{\hbar}{2}} \frac{1}{2\omega_n C} \left(a_n^\dagger - a_n \right), \tag{C.24}
$$

where $[a_n, a_m^\dagger] = \delta_{n,m}$. That gives the Harmonic oscillator with:

$$
H = \sum_n \hbar \omega_n \left(a_n^\dagger a_n + \frac{1}{2} \right). \tag{C.25}
$$

We could have just as easily chosen:

$$
\hat{\Phi}_n = \sqrt{\frac{\hbar}{2}} \frac{1}{2\omega_n C} \left(a_n^\dagger + a_n \right), \qquad \hat{q}_n = i \sqrt{\frac{\hbar}{2}} \omega_n C \left(a_n^\dagger - a_n \right), \tag{C.26}
$$

and this changes whether the voltage operator is given by $a^\dagger + a$ or $a^\dagger - a$ since $\hat{V}_n(x) = u_n(x) \hat{q}_n / C$.

C.3 Quantum dots and resonators

In semiconductors, quantum dots can be made by depleting a 2DEG such that the electrons get localized to a confined space. From figure C.6 we imagine closing of the barrier defined by gate 1 and 3 to have an "isolated" DQD. By adjusting how negative gate 2 is we adjust the depletion of the space between the two dots and thus we can adjust the interdot hopping.

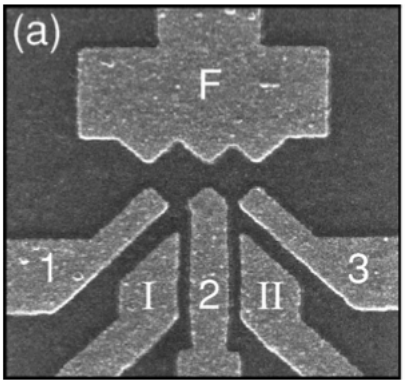

Figure C.6: SEM micrograph of a double dot defined by metallic gates (light gray areas). Metal gates are deposited on top of a GaAs/AlGaAs heterostructure with a 2DEG 100 nm below the surface (van der Vaart et al., 1995). Applying a negative voltage to all gates depletes the 2DEG underneath them and forms two quantum dots. Current can flow from the large electron reservoir on the left via the three tunnel barriers induced by the gate pairs 1-F, 2-F, and 3-F to the reservoir on the right. The transmission of each tunnel barrier can be controlled individually by the voltage on gates 1, 2, or 3. [21]

Other examples of devices:
Many dot array where many barriers adjust tunneling between the dots [22].
Plunger gates and barrier gates [23].

C.3.1 Coupling a resonator to a double quantum dot

When coupling a resonator to a double quantum double dot we do it by using the voltage in the resonator to adjust the electro-chemical potenatial on the electro-statically defined dot through gates. In figure C.7 the strip line resonator is coupled in the end (where the voltage of the lowest energy eigenmode is the largest, (C.21)) to plunger gates that adjust the detuning of the double quantum dot.

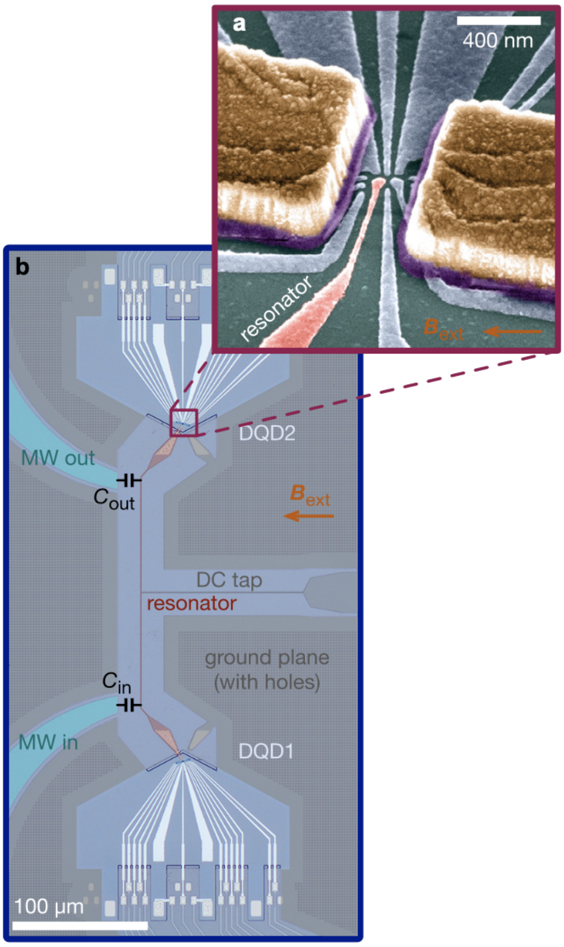

Figure C.7: SEM image of a driven resonator coupled capacitively to a double quantum dot. The strip line resonator is capacitively coupled in the ends to a classical feed line as shown in figure b where the ground plane is also seen. The coupling to the quantum double dot is from capacitively adjusting the electro-chemical potential. [3]

A way to define the double quantum dots is with the help of a barrier gate that adjust the electrostatically defined barrier between the dots. This is seen in [24] where the barrier gate is used to adjust t_h, the hopping constant, between 8 to 15 GHz. The setup is show in figure C.8. In panel B in the top a classical feedline is used to drive a hanger style resonator that is connected to the gates colored blue and red in panel C. Along with that the plunger gates RP and LP are used to adjust the chemical potentials in the dots and create a detuning.

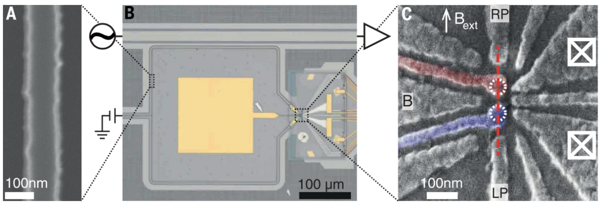

Figure C.8: Setup of hanger style resonator coupled to detuning of DQD

C.3.2 Coupling to the barrier

The proposition to couple the hopping is based on the idea that instead of coupling the gates capacitively to the resonator, such that they adjust the chemical potentials, the gate(s) are placed on top of barrier between the dots. The plunger gates are still used to adjust the dot chemical potentials, and the barrier gate and the resonator adjust the barrier between the dots and thus couple to the hopping constant. Coupling a resonator to the tunnel barrier has been proposed similarly in [25] where the resonator couples such that it adjusts the barrier. This is illustrated in figure C.9.

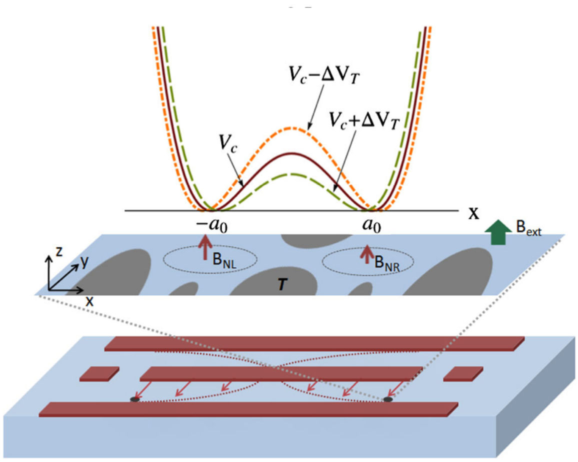

Figure C.9: A qubit formed in a double quantum dot, each dot containing one electron, is placed at a maximum of the electric field inside a superconducting transmission line resonator. The resonator electric field couples to the interdot tunnel gate T, which modifies the tunnel barrier height. [25]

Appendix D

Green's functions

This section gives and overview of the use of Green's functions in many-body quantum physics[11]. Initially we work with the second quantization operator formalism where state vectors live in a Fock-space. The operators commute or anti-commute depending on their statistics. The retarded Green's function for a many-body system is defined as:

$$G^R(r,t;r,t') = -i\theta(t-t')\left\langle [\hat{\Psi}(x,t), \hat{\Psi}^\dagger(x',t')] \right\rangle, \tag{D.1}$$

where the average is the thermal average and the commutator depends on the statistics. We can insert a complete set in a general a-basis and get:

$$G^R(a,t;b,t') = -i\theta(t-t')\left\langle \left[\hat{c}_a(t), \hat{c}_b^\dagger(t')\right] \right\rangle. \tag{D.2}$$

When the Hamiltonian is quadratic, the eigenbasis can be used to show that (D.2) is equal to $-i\theta(t-t')\langle n| e^{-iH(t-t')} |n'\rangle$, and we therefore regard it as a propagator. We also define the lesser and greater Green's functions:

$$G^>(x,t;x',t') = -i\left\langle \hat{\Psi}(x,t)\hat{\Psi}^\dagger(x',t') \right\rangle, \tag{D.3}$$

$$G^<(x,t;x',t') = -i\left\langle \hat{\Psi}^\dagger(x,t)\hat{\Psi}(x',t') \right\rangle. \tag{D.4}$$

The lesser and greater Green's functions have an interpretation of inserting either a hole or a particle at some position and time and then removing them again at another time and position to see how the system changed.

D.1 Single particle Green's functions

Why are these single particle Green's functions interesting? One reason is that they are used in linear response theory to understand how a system reacts to a perturbation. Another reason is that they reveal properties about the spectrum or the density of states of the system. A way to study the properties of

Green's functions is with the Lehmann representation, where a complete set of eigenvectors are inserted and the thermal average is taken over the eigenvectors too. In a general basis we would have for a time-independent Hamiltonian, that the retarded Green's function between the same state at different times is given in the frequency domain by:

$$G^R(a,\omega) = \frac{1}{Z}\sum_{n,m} \frac{\langle n| c_a |m\rangle \langle m| c_a^\dagger |n\rangle}{\omega + i\eta + E_n - E_m} \left(e^{-\beta E_n} + e^{-\beta E_m}\right), \tag{D.5}$$

and we see that in the case of $\eta \to 0$, the poles of the retarded Green's function reveal the spectrum of the system. This is seen since $\omega = E_m - E_n$ will result in a pole. Here n, m correspond to eigenstates of the full Hamiltonian.

Another interesting quantity is the spectral function which reveals the density of states of the system. It is defined as:

$$A(a,\omega) = -2\text{Im}G^R(a,\omega), \tag{D.6}$$

and it can be shown that:

$$\langle c_a^\dagger c_a \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(a,\omega) n_F(\omega), \tag{D.7}$$

meaning that the spectral function reveals how well the excitation $|a\rangle$ corresponds to a free non-interacting state of the system. If $|a\rangle$ corresponds to an eigenmode of the system then $A(a,\omega)$ would just project down onto $n_F(E_a)$ such that $A(a,\omega) = 2\pi\delta(\omega - E_a)$, with E_a and eigenenergy.

D.2 Imaginary time Green's functions

Introducing the Matsubara imaginary time is a powerful way of calculating correlation functions. With the imaginary time τ we have for time independent Hamiltonians that $\hat{a}(\tau) = e^{\hat{H}\tau}\hat{a}e^{-\hat{H}\tau}$. The imaginary time correlation function is defined as:

$$\mathcal{C}_{AB}(\tau,\tau') = -\langle \mathcal{T}_\tau \hat{A}(\tau)\hat{B}(\tau')\rangle, \tag{D.8}$$

and it can be shown from the cyclic property of the trace that $\mathcal{C}_{AB}(\tau,\tau') = \mathcal{C}_{AB}(\tau - \tau')$. $\mathcal{T}_\tau$ is the imaginary time ordering operator which orders the operators in imaginary time. One of the reasons why it is so powerful is that it allows perturbation theory and the retarded correlation function can be found with an analytic continuation. A Fourier transformation shows that $C^R_{AB}(\omega) = \mathcal{C}_{AB}(i\omega_n \to \omega + i\eta)$ where η is a positive number. The single particle Green's functions that we are interested in are then defined as:

$$\mathcal{G}(a,\tau;b,\tau') = -\langle \mathcal{T}_\tau \hat{c}_a(\tau)\hat{c}_b^\dagger(\tau)\rangle, \tag{D.9}$$

and again $G^R(ab,\omega) = \mathcal{G}(ab, i\omega_n \to \omega + i\eta)$.

D.3 Green's functions from path integrals

Moving away from the operator formalism and to the coherent state path integral formalism[10] we can describe the partition function, $\mathcal{Z}$ of the system with the functional path integral:

$$\mathcal{Z} = \int \mathcal{D}[\psi] e^{-S[\psi]}, \tag{D.10}$$

where ψ is a quantum field that depends on some quantum number and the imaginary time τ, or when Fourier transformed, the Matsubara frequency $i\omega_n$. The action has the form of:

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \left[\bar{\psi} (\partial_\tau - \mu) \psi + H(\bar{\psi}, \psi) \right], \tag{D.11}$$

where ψ can depend on τ and some other quantum number. Naturally the partition function is of interest in statistical mechanics since it reveals the thermodynamic properties of the system. We can calculate the free energy $F = -k_B T \ln \mathcal{Z}$ which is easily found for a quadratic action, S.

The path integral can also reveal microscopic properties of the system. If the action can be written on quadratic form with complex vectors $\vec{\phi}$:

$$S = \vec{\phi}^\dagger \mathcal{G}^{-1} \vec{\phi}, \tag{D.12}$$

then it becomes easy to find expectation values with the use of Wick's theorem, which states that:

$$\langle \phi_n^* \phi_m \rangle = \mathcal{G}_{n,m}, \tag{D.13}$$

where ϕ_n is a quantum field. These averages are useful for calculating response functions or Green's functions.

Appendix E

Integrating out the resonators

Starting from the action for the bosons and their interaction with the fermions (3.16) and (3.11):

$$S = \frac{g\sqrt{\beta N}}{2}\left(\phi_0^* + \phi_0\right) + \sum_q \phi_q^*\left(-i\omega_m + \omega_R\right)\phi_q - \frac{g}{\sqrt{\beta N}}\sum_{k,k'}\left[\phi_{k-k'} + \phi_{k'-k}^*\right]\bar{\psi}_k\psi_{k'}, \tag{E.1}$$

where the k's again represent fermionic two vectors with the wavenumber and Matsubara frequency. We rewrite the interaction by shifting $k = q + k'$ and summing over the bosonic q:

$$S_{int} = -\frac{g}{\sqrt{\beta N}}\sum_{k,q}\left[\phi_q + \phi_{-q}^*\right]\bar{\psi}_{q+k}\psi_k. \tag{E.2}$$

We define the bosonic density operator from the fermion fields:

$$\bar{\rho}_q = \sum_k \bar{\psi}_{q+k}\psi_k. \tag{E.3}$$

We will use that:

$$\begin{aligned} \rho_q &= \sum_k \bar{\psi}_k\psi_{q+k} \\ (k = k' - q) \qquad &= \sum_{k'} \bar{\psi}_{k'-q}\psi_{k'} \\ &= \bar{\rho}_{-q}. \end{aligned} \tag{E.4}$$

The action can be rewritten in terms of the density fields:

$$\begin{aligned} S_{int} &= -\frac{g}{\sqrt{\beta N}}\sum_q \bar{\rho}_q\phi_q - \frac{g}{\sqrt{\beta N}}\sum_q \phi_{-q}^*\bar{\rho}_q \\ &= -\frac{g}{\sqrt{\beta N}}\sum_q \bar{\rho}_q\phi_q - \frac{g}{\sqrt{\beta N}}\sum_q \phi_q^*\bar{\rho}_{-q} \\ &= -\frac{g}{\sqrt{\beta N}}\sum_q \bar{\rho}_q\phi_q - \frac{g}{\sqrt{\beta N}}\sum_q \phi_q^*\rho_q. \end{aligned} \tag{E.5}$$

Since the q-space is discrete we represent the fields as vectors and write the action in terms of vector products:

$$S = \vec{\phi}^\dagger\mathcal{G}_0^{-1}\vec{\phi} + \frac{g}{\sqrt{\beta N}}\left(\beta L\frac{\delta_q}{2} - \rho\right)^\dagger \cdot \vec{\phi} + \frac{g}{\sqrt{\beta N}}\vec{\phi}^\dagger \cdot \left(\beta N\frac{\delta_q}{2} - \rho\right). \tag{E.6}$$

The path integral is Gaussian and can be evaluated[10]:

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\left[\vec{\phi}^{\dagger}, \vec{\phi}\right] \exp\left(S\right) \\
&= \left(\det \mathcal{G}_0^{-1}\right)^{-1} e^{\frac{g^2}{\beta N}\left(\beta N \frac{\delta_q}{2} - \rho^{\dagger}\right)\mathcal{G}_0\left(\beta N \frac{\delta_q}{2} - \rho\right)} \\
&= \left(\det \mathcal{G}_0^{-1}\right)^{-1} e^{\frac{g^2}{\beta N}\left(\beta N \frac{\delta_q}{2} - \rho^{\dagger}\right)\mathcal{G}_0\left(\beta N \frac{\delta_q}{2} - \rho\right)}.
\end{aligned} \tag{E.7}$$

We multiply out the exponent:

$$\begin{aligned}
&\frac{g^2}{\beta N}\left(\rho^{\dagger} - \beta N \frac{\delta_q}{2}\right)\mathcal{G}_0\left(\rho - \beta N \frac{\delta_q}{2}\right) \\
=&\frac{g^2}{\beta N}\sum_{q,q'}\left[\bar{\rho}_q \mathcal{G}_0(q,q')\rho_{q'} - \beta N \bar{\rho}_q \mathcal{G}_0(q,q')\frac{\delta_{q'}}{2} - \beta N \frac{\delta_q}{2}\mathcal{G}_0(q,q')\rho_q + \beta^2 N^2 \frac{\delta_q}{2}\mathcal{G}_0(q,q')\frac{\delta_{q'}}{2}\right].
\end{aligned} \tag{E.8}$$

We use that the free Green's function is diagonal in q-space $\mathcal{G}_0(q,q') = \frac{\delta_{q,q'}}{-i\omega_m + \omega_R}$ such that the first term becomes:

$$\begin{aligned}
\sum_{q,q'} \frac{\bar{\rho}_q \delta_{q,q'} \rho_{q'}}{-i\omega_m + \omega_R} &= \sum_{k,k'}\sum_q \bar{\psi}_{k+q}\psi_k \frac{1}{-i\omega_m + \omega_R}\bar{\psi}_{k'}\psi_{k'+q} \\
&= \sum_{k,k'}\sum_q \bar{\psi}_{k+q}\bar{\psi}_{k'-q}\frac{1}{-i\omega_m + \omega_R}\psi_{k'}\psi_k.
\end{aligned} \tag{E.9}$$

We make the action symmetric in q:

$$\begin{aligned}
&= \frac{1}{2}\sum_{k,k'}\sum_q \bar{\psi}_{k+q}\bar{\psi}_{k'-q}\frac{1}{-i\omega_m + \omega_R}\psi_{k'}\psi_k + \frac{1}{2}\sum_{k,k'}\sum_q \bar{\psi}_{k'+q}\bar{\psi}_{k-q}\frac{1}{-i\omega_m - \omega_R}\psi_k\psi_{k'} \\
&= \frac{1}{2}\sum_{k,k'}\sum_q \bar{\psi}_{k+q}\bar{\psi}_{k'-q}\left(\frac{1}{i\omega_m - \omega_R} - \frac{1}{i\omega_m + \omega_R}\right)\psi_{k'}\psi_k \\
&= \sum_{k,k'}\sum_q \bar{\psi}_{k+q}\bar{\psi}_{k'-q}\frac{\omega_R}{(i\omega_m)^2 - \omega_R^2}\psi_{k'}\psi_k.
\end{aligned} \tag{E.10}$$

The second and third term becomes:

$$\begin{aligned}
&-\frac{\beta N}{2}\sum_{q,q'}\left[\frac{\bar{\rho}_q \delta_{q,q'}\delta_{q'}}{-i\omega_m + \omega_R} + \frac{\delta_q \delta_{q,q'}\rho_{q'}}{-i\omega_m + \omega_R}\right] \\
=&-\frac{\beta N}{2}\left[\frac{\bar{\rho}_0}{\omega_R} + \frac{\rho_0}{\omega_R}\right] \\
=&-\frac{\beta N}{\omega_R}\sum_k \bar{\psi}_k\psi_k,
\end{aligned} \tag{E.11}$$

where we used that $\bar{\rho}_0 = \rho_0 = \sum_k \bar{\psi}_k\psi_k$. The last term becomes a constant:

$$\beta^2 N^2 \frac{\delta_q}{2}\mathcal{G}_0(q,q')\frac{\delta_{q'}}{2} = \frac{\beta^2 N^2}{4\omega_R}, \tag{E.12}$$

which is just a constant we can ignore when looking at the fermions. We then have that:

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\left[\vec{\phi}^{\dagger}, \vec{\phi}\right] \exp\left(S\right) \\
&= \left(\det \mathcal{G}_0^{-1}\right)^{-1} e^{\frac{\beta^2 L^2}{4\omega_R}} e^{\frac{g^2}{\beta L}\sum_k\left[-\frac{\beta L}{\omega_R}\bar{\psi}_k\psi_k + \sum_{q,k'}\bar{\psi}_{k+q}\bar{\psi}_{k'-q}\frac{\omega_R}{(i\omega_m)^2 - \omega_R^2}\psi_{k'}\psi_k\right]}
\end{aligned} \tag{E.13}$$

Adding the free fermionic action for the Jordan-Wigner fermions (3.58), results in the effective action for the fermions:

$$S_{eff} = \sum_k \Bigg\{ \bar{\psi}_k \left(-i\omega_n + 2h + \frac{g^2}{\omega_R} - 2J\cos k \right) \psi_k + iJ\sin k \left(\bar{\psi}_{-k}\bar{\psi}_k + \psi_{-k}\psi_k \right) - \frac{g^2}{\beta L} \sum_{q,k'} \bar{\psi}_{k+q}\bar{\psi}_{k'-q} \frac{\omega_R}{(i\omega_m)^2 - \omega_R^2} \psi_{k'}\psi_k \Bigg\}. \quad \text{(E.14)}$$

The effect of the bosons in equillibrium is to shift the transverse field by $\frac{g^2}{2\omega}$, similar to what was found in (3.90), and introducing a quartic interaction term. The sign of the interaction vertex depends on the sign of the denominator $(i\omega_m)^2 - \omega_R^2$. The interaction can thus change between being attractive and repulsive.

Appendix F

Analytic continuation of $\Pi_q(i\omega_m)$

We now want to do an analytical continuation of the imaginary time Green's function, $\Pi_q(i\omega_q)$, to find the retarded Green's function. Since we want the retarded Green's function which shares poles with the Matsubara Green's function in the upper half of the complex plane, we continue such that $i\omega_q \to \omega + i\eta$. We will calculate the real and imaginary part of this function. To do this we notice that all the Fermi-functions are real, just like $f(k, q)$.

We therefore want to explore the real and imaginary parts of terms on form (suppressing the subscripts):

$$\frac{f/\epsilon + \epsilon \pm \omega \pm i\eta}{\xi^2 - (\epsilon \pm \omega \pm i\eta)^2}, \tag{F.1}$$

since all the denominators are on this form after the analytical continuation. To break it up into real and imaginary parts, we want to calculate:

$$g(\pm\omega \pm i\eta) = \frac{f/\epsilon + \epsilon \pm \omega \pm i\eta}{\xi^2 - (\epsilon \pm \omega \pm i\eta)^2} \frac{\xi^2 - (\epsilon \pm \omega \mp i\eta)^2}{\xi^2 - (\epsilon \pm \omega \mp i\eta)^2}, \tag{F.2}$$

First we find the nominator:

$$\begin{aligned}&(f/\epsilon + \epsilon \pm \omega \pm i\eta) \left(\xi^2 - (\epsilon \pm \omega)^2 + \eta^2 \pm 2i\eta(\epsilon \pm \omega)\right) \\ &\quad = (f/\epsilon + \epsilon \pm \omega) \left(\xi^2 - (\epsilon \pm \omega)^2 + \eta^2\right) - 2\eta^2(\epsilon \pm \omega) \pm i\eta \left(\xi^2 + (\epsilon \pm \omega)^2 + 2f/\epsilon(\epsilon \pm \omega) + \eta^2\right).\end{aligned} \tag{F.3}$$

Next the denominator:

$$\left(\xi^2 - (\epsilon \pm \omega \pm i\eta)^2\right) \left(\xi^2 - (\epsilon \pm \omega \pm i\eta)^2\right) = \left(\xi^2 - (\epsilon \pm \omega)^2 + \eta^2\right)^2 + 4(\epsilon \pm \omega)^2\eta^2. \tag{F.4}$$

From this we then find the real part as:

$$\mathrm{Re}[g(\pm\omega \pm i\eta)] = \frac{(f/\epsilon + \epsilon \pm \omega) \left(\xi^2 - (\epsilon \pm \omega)^2 + \eta^2\right) - 2\eta^2(\epsilon \pm \omega)}{\left(\xi^2 - (\epsilon \pm \omega)^2 + \eta^2\right)^2 + 4(\epsilon \pm \omega)^2\eta^2}. \tag{F.5}$$

We now want to take the limit $\eta \to 0$ since we can choose η as close to zero as we want as long as $\omega + i\eta$ is in the upper half of the complex plane.

$$\lim_{\eta \to 0} \mathrm{Re}[g(\omega + i\eta)] = \frac{f/\epsilon + \epsilon \pm \omega}{\xi^2 - (\epsilon \pm \omega)^2}. \tag{F.6}$$

Now for the imaginary part we find that:

$$\mathrm{Im}[g(\pm\omega \pm i\eta)] = \pm\eta \frac{\xi^2 + (\epsilon \pm \omega)^2 + 2f/\epsilon(\epsilon \pm \omega) + \eta^2}{\left(\xi^2 - (\epsilon \pm \omega)^2 + \eta^2\right)^2 + 4(\epsilon \pm \omega)^2\eta^2}. \tag{F.7}$$

Again taking the limit we can neglect the last term in the nominator and get:

$$\begin{aligned} &= \pm\left(\xi^2 + (\epsilon \pm \omega)^2 + 2f/\epsilon(\epsilon \pm \omega)\right) \frac{\eta}{\left(\xi^2 - (\epsilon \pm \omega)^2\right)^2 + 4(\epsilon \pm \omega)^2\eta^2} \\ &= \pm\frac{\xi^2 + (\epsilon \pm \omega)^2 + 2f/\epsilon(\epsilon \pm \omega)}{4(\epsilon \pm \omega)^2} \frac{\eta}{\left(\frac{\xi^2 - (\epsilon\pm\omega)^2}{2(\epsilon\pm\omega)}\right)^2 + \eta^2}. \end{aligned} \tag{F.8}$$

The imaginary part now resembles a Lorentzian and we use that $\lim_{\eta\to 0} \frac{\eta}{x^2+\eta^2} = \pi\delta(x)$, such that:

$$= \pm\pi\frac{\xi^2 + (\epsilon \pm \omega)^2 + 2f/\epsilon(\epsilon \pm \omega)}{4(\epsilon \pm \omega)^2}\delta\left(\frac{\xi^2 - (\epsilon \pm \omega)^2}{2(\epsilon \pm \omega)}\right), \tag{F.9}$$

then using the delta function in composition with a function [1], $\delta(g(x)) = \sum_k \frac{\delta(x-\alpha_k)}{|f'(\alpha_k|}$, where α_k is a root to the smooth function $g(x)$ in the domain that $g(x)$ is defined on. The roots in our case are $\epsilon \pm \omega = \pm\xi$. Differentiating the argument of the delta function, under the assumption that q in ξ or ϵ does not depend on ω, we get:

$$\begin{aligned} \left|\partial_\omega \frac{\xi^2 - (\epsilon \pm \omega)^2}{2(\epsilon \pm \omega)}\right|_{\epsilon\pm\omega=\pm\xi} &= \left|\mp\frac{\xi^2 - (\epsilon \mp \omega)^2}{2(\epsilon \mp \omega)^2} \mp 1\right|_{\epsilon\pm\omega=\pm\xi} \\ &= 1, \end{aligned} \tag{F.10}$$

and we get:

$$\mathrm{Im}[g(\pm\omega \pm i\eta)] \to \pm\frac{\pi}{2}\left[\left(1 - \frac{f}{\epsilon\xi}\right)\delta(\omega \pm \epsilon + \xi) + \left(1 + \frac{f}{\epsilon\xi}\right)\delta(\omega \pm \epsilon - \xi)\right]. \tag{F.11}$$

With this in mind we can immediately write down the real part of Π_q by simply taking $i\omega_q \to \omega$:

$$\begin{aligned} \mathrm{Re}\Pi_q(\omega) = \frac{4g^2}{L}\sum_k \Bigg\{ &- \frac{f(k,q) + \xi_k\,(\xi_k - \omega)}{\xi_{k-q}^2 - (\xi_k - \omega)^2}\frac{n_F(\xi_k)}{\xi_k} \\ &+ \frac{f(k,q) + \xi_k\,(\xi_k + \omega)}{\xi_{k-q}^2 - (\xi_k + \omega)^2}\frac{n_F(-\xi_k)}{\xi_k} \\ &- \frac{f(k,q) + \xi_{k-q}\,(\xi_{k-q} + \omega)}{\xi_k^2 - (\xi_{k-q} + \omega)^2}\frac{n_F(\xi_{k-q})}{\xi_{k-q}} \\ &+ \frac{f(k,q) + \xi_{k-q}\,(\xi_{k-q} - \omega)}{\xi_k^2 - (\xi_{k-q} - \omega)^2}\frac{n_F(-\xi_{k-q})}{\xi_{k-q}} \Bigg\}. \end{aligned} \tag{F.12}$$

We can reduce this by first shifting the the third and fourth term by $k \to q - k$ and using $f(q-k,q) = f(k,q)$:

$$\mathrm{Re}\Pi_q(\omega) = \frac{4g^2}{L}\sum_k \left(\tanh\frac{\xi_k}{2T}\right)\left(\frac{\frac{f(k,q)}{\xi_k} + \xi_k - \omega}{\xi_{k-q}^2 - (\xi_k - \omega)^2} + \frac{\frac{f(k,q)}{\xi_k} + \xi_k + \omega}{\xi_{k-q}^2 - (\xi_k + \omega)^2}\right). \tag{F.13}$$

[1] https://en.wikipedia.org/wiki/Dirac_delta_function#Composition_with_a_function

The poles lie in $\{\xi_k+\xi_{k-q}, -\xi_k-\xi_{k-q}, \xi_k-\xi_{k-q}, \xi_k+\xi_{k-q}\}$. The imaginary part we find as:

$$
\begin{aligned}
\mathrm{Im}\Pi_q(\omega) = \frac{2\pi g^2}{L}\sum_k \Bigg\{ &\delta(\omega-\xi_k+\xi_{k-q})\left(n_F(\xi_k)\left(1-\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)-n_F(\xi_{k-q})\left(1+\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)\right)\\
&+\delta(\omega-\xi_k-\xi_{k-q})\left(n_F(\xi_k)-n_F(-\xi_{k-q})\right)\left(1+\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)\\
&+\delta(\omega+\xi_k+\xi_{k-q})\left(n_F(-\xi_k)-n_F(\xi_{k-q})\right)\left(1-\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)\\
&+\delta(\omega+\xi_k-\xi_{k-q})\left(n_F(-\xi_k)\left(1+\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)-n_F(-\xi_{k-q})\left(1-\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)\right)\Bigg\}. \qquad \text{(F.14)}
\end{aligned}
$$

In the last line we can let the dummy index $k\to -k+q$ and use that $f(-k+q,q)=f(k,q)$ such that we get:

$$
\begin{aligned}
\mathrm{Im}\Pi_q(\omega) = \frac{2\pi g^2}{L}\sum_k \Bigg\{ &\delta(\omega-\xi_k+\xi_{k-q})\left(\tanh\frac{\xi_{k-q}}{2T}\left(1+\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)-\tanh\frac{\xi_k}{2T}\left(1-\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)\right)\\
&+\delta(\omega-\xi_k-\xi_{k-q})\left(n_F(\xi_k)-n_F(-\xi_{k-q})\right)\left(1+\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)\\
&+\delta(\omega+\xi_k+\xi_{k-q})\left(n_F(-\xi_k)-n_F(\xi_{k-q})\right)\left(1-\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)\Bigg\}, \qquad \text{(F.15)}
\end{aligned}
$$

where it was used that $n_F(-x)-n_F(x)=\tanh\frac{x}{2T}$. We can look at the imaginary part in a few regimes. For $T\to\infty$ when $n_F\to\frac{1}{2}$ and $\tanh\frac{x}{T}\to 0$ we have the imaginary part goes to zero. For $\omega>0$ and $T\to 0$ we see that only the first and second term should contribute since $\xi_k>0$.

We look at $\left(\frac{f(k,q)}{\xi_k\xi_{k-q}}-1\right)$. We can show that $\left|\frac{f(k,q)}{\xi_k\xi_{k-q}}\right|\leq 1$. To do so we start by defining $\alpha_k=(2h-2J\cos k)$ and $\beta_k=2J\sin k$ to make notation easier. That gives that $\xi_k=\sqrt{\alpha_k^2+\beta_k^2}$ and $f(k,q)=\alpha_k\alpha_{k-q}-\beta_k\beta k-q$. Then we have:

$$
\begin{aligned}
\left(\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)^2 &= \frac{(\alpha_k\alpha_{k-q})^2+(\beta_k\beta_{k-q})^2-2(\alpha_k\alpha_{k-q})(\beta_k\beta_{k-q})}{(\alpha_k\alpha_{k-q})^2+(\beta_k\beta_{k-q})^2+(\alpha_k\beta_{k-q})^2+(\beta_k\alpha_{k-q})^2}\\
\Rightarrow \left(\frac{f(k,q)}{\xi_k\xi_{k-q}}\right)^2-1 &= -\frac{(\alpha_k\beta_{k-q})^2+(\beta_k\alpha_{k-q})^2+2(\alpha_k\beta_{k-q})(\beta_k\alpha_{k-q})}{(\alpha_k^2+\beta_k^2)(\alpha_{k-q}^2+\beta_{k-q}^2)}\\
&= -\frac{(\alpha_k\beta_{k-q}+\beta_k\alpha_{k-q})^2}{(\alpha_k^2+\beta_k^2)(\alpha_{k-q}^2+\beta_{k-q}^2)}\leq 0\\
\Rightarrow \left|\frac{f(k,q)}{\xi_k\xi_{k-q}}\right| &\leq 1, \qquad \text{(F.16)}
\end{aligned}
$$

where in the second equality we used that both nominator and denominator are squared. We thus know that $(1\pm\frac{f}{\xi_k\xi_{k-q}})\geq 0$.

Resolving the delta functions analytically is not possible in general due to the elliptical functions. The roots to an argument like $\omega+\xi_k+\xi_{k-q}$ will be denoted $k_0^{++}(\omega)$ owing to the two plus signs and the resulting root being a function of the frequency. Assuming that we can find all the roots numerically or with some approximation or limit, we can resolve the delta functions:

$$\delta(\omega+\xi_k \pm \xi_{k-q}) = \sum_{k_0^{+\pm}} \frac{\delta(k-k_0^{+\pm})}{|\partial_k(\xi_k \pm \xi_{k-q})|_{k=k_0^{+\pm}}}$$
$$= \frac{1}{hJ}\sum_{k_0^{+\pm}} \frac{\delta(k-k_0^{+\pm})}{\left|\frac{\sin k_0^{+\pm}}{\xi_{k_0^{+\pm}}} \pm \frac{\sin(k_0^{+\pm}-q)}{\xi_{k_0^{+\pm}-q}}\right|}. \tag{F.17}$$

Following this recipe we go to continuum limit and rewrite (F.15) as:

$$\mathrm{Im}\Pi_q(\omega) = \frac{g^2}{hJ}\Bigg\{\sum_{k_0^{-+}} \frac{\tanh\frac{\xi_{k_0^{-+}-q}}{2T}\left(1+\frac{f(k_0^{-+},q)}{\xi_{k_0^{-+}}\xi_{k_0^{-+}-q}}\right) - \tanh\frac{\xi_{k_0^{-+}}}{2T}\left(1-\frac{f(k_0^{-+},q)}{\xi_{k_0^{-+}}\xi_{k_0^{-+}-q}}\right)}{\left|\frac{\sin k_0^{-+}}{\xi_{k_0^{-+}}} - \frac{\sin(k_0^{-+}-q)}{\xi_{k_0^{-+}-q}}\right|}$$
$$+\sum_{k_0^{--}} \frac{n_F(\xi_{k_0^{--}}) - n_F(-\xi_{k_0^{--}-q})}{\left|\frac{\sin k_0^{--}}{\xi_{k_0^{--}}} + \frac{\sin(k_0^{--}-q)}{\xi_{k_0^{--}-q}}\right|}\left(1+\frac{f(k_0^{--},q)}{\xi_{k_0^{--}}\xi_{k_0^{--}-q}}\right)$$
$$+\sum_{k_0^{++}} \frac{n_F(-\xi_{k_0^{++}}) - n_F(\xi_{k_0^{++}-q})}{\left|\frac{\sin k_0^{++}}{\xi_{k_0^{++}}} + \frac{\sin(k_0^{++}-q)}{\xi_{k_0^{++}-q}}\right|}\left(1-\frac{f(k_0^{++},q)}{\xi_{k_0^{++}}\xi_{k_0^{++}-q}}\right)\Bigg\}. \tag{F.18}$$

And the problem is boiled down to finding the roots of the arguments of the delta functions.

Spectral function

We start by finding the imaginary part of the retarded Green's function (3.134). Before we can do that we split up $\Pi^R(q,\omega)$ into the real and imaginary part $\Pi^R(q,\omega) = \Pi' + i\Pi''$ where the q,ω is implied:

$$2i\mathrm{Im}\left[\frac{\omega+\omega_R+2\Pi'+i(\eta+2\Pi'')}{\omega_R^2+4\omega_R\Pi'+i4\omega_R\Pi''-(\omega+i\eta)^2}\right]$$
$$= \frac{\omega+\omega_R+2\Pi'+i(\eta+2\Pi'')}{\omega_R^2+4\omega_R\Pi'+i4\omega_R\Pi''-(\omega+i\eta)^2} - \frac{\omega+\omega_R+2\Pi'-i(\eta+2\Pi'')}{\omega_R^2+4\omega_R\Pi'-i4\omega_R\Pi''-(\omega-i\eta)^2}$$
$$= \frac{①-②}{③}, \tag{F.19}$$

where we now find ①,②,③. We will expand $(\omega+i\eta)^2 = \omega^2 - 2i\omega\eta$ implying that $\eta \ll 1$:

$$③ = \left(\left[\omega_R^2+4\omega_R\Pi'-\omega^2\right] + i\left[4\omega_R\Pi''-2\omega\eta\right]\right)\left(\left[\omega_R^2+4\omega_R\Pi'-\omega^2\right] - i\left[4\omega_R\Pi''-2\omega\eta\right]\right)$$
$$= \left(\omega_R^2+4\omega_R\Pi'-\omega^2\right)^2 + \left(4\omega_R\Pi''-2\omega\eta\right)^2. \tag{F.20}$$

Then for the nominators:

$$① = \left([\omega+\omega_R+2\Pi'] + i[\eta+2\Pi'']\right)\left(\left[\omega_R^2+4\omega_R\Pi'-\omega^2\right] - i\left[4\omega_R\Pi''-2\omega\eta\right]\right)$$
$$= (\omega+\omega_R+2\Pi')\left(\omega_R^2+4\omega_R\Pi'-\omega^2\right) + (\eta+2\Pi'')(4\omega_R\Pi''-2\omega\eta)$$
$$+ i\left([\eta+2\Pi'']\left[\omega_R^2+4\omega_R\Pi'-\omega^2\right] - [\omega+\omega_R+2\Pi'][4\omega_R\Pi''-2\omega\eta]\right), \tag{F.21}$$

and likewise:

$$
\begin{aligned}
\textcircled{2} &= \left([\omega + \omega_R + 2\Pi'] - i\,[\eta + 2\Pi'']\right)\left(\left[\omega_R^2 + 4\omega_R\Pi' - \omega^2\right] + i\,[4\omega_R\Pi'' - 2\omega\eta]\right) \\
&= (\omega + \omega_R + 2\Pi')\left(\omega_R^2 + 4\omega_R\Pi' - \omega^2\right) + (\eta + 2\Pi'')\,(4\omega_R\Pi'' - 2\omega\eta) \\
&\quad - i\left([\eta + 2\Pi'']\left[\omega_R^2 + 4\omega_R\Pi' - \omega^2\right] - [\omega + \omega_R + 2\Pi']\,[4\omega_R\Pi'' - 2\omega\eta]\right), \qquad \text{(F.22)}
\end{aligned}
$$

and we find:

$$
2i\mathrm{Im}G_\delta^R(q,\omega) = -2i\frac{[\eta + 2\Pi'']\left[\omega_R^2 + 4\omega_R\Pi' - \omega^2\right] - [\omega + \omega_R + 2\Pi']\,[4\omega_R\Pi'' - 2\omega\eta]}{\left(\omega_R^2 + 4\omega_R\Pi' - \omega^2\right)^2 + \left(4\omega_R\Pi'' - 2\omega\eta\right)^2}, \qquad \text{(F.23)}
$$

and we now get the result:

$$
\begin{aligned}
A^\delta(q,\omega) &= -2\mathrm{Im}G_\delta^R(q,\omega) \\
&= 2\frac{[\eta + 2\Pi'']\left[\omega_R^2 + 4\omega_R\Pi' - \omega^2\right] - [\omega + \omega_R + 2\Pi']\,[4\omega_R\Pi'' - 2\omega\eta]}{\left(\omega_R^2 + 4\omega_R\Pi' - \omega^2\right)^2 + \left(4\omega_R\Pi'' - 2\omega\eta\right)^2} \\
&= 2\frac{\left(\omega + \omega_R\right)^2\eta + 4(\omega + \omega_R)\Pi'\eta - 2(\omega + \omega_R)^2\Pi''}{\left(\omega_R^2 + 4\omega_R\Pi' - \omega^2\right)^2 + \left(4\omega_R\Pi'' - 2\omega\eta\right)^2}. \qquad \text{(F.24)}
\end{aligned}
$$