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Master of Science in Physics

Uses of Killing Vectors and Tensors in General Relativity

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Abstract

Symmetries in General Relativity and their connection to conserved quantities and particle dynamics are studied through Killing vectors, conformal Killing vectors and Killing tensors. These objects are first studied on simple metrics such as the 2-sphere and flat space. Using a stereo graphic projection the complex structure of the 2-sphere is unveiled and is used to calculate the Killing vectors, conformal Killing vectors and Killing tensors. Then these objects are applied to study the symmetries on the Schwarzschild metric, which are characterised by four Killing vectors.

Furthermore, we also study how these Killing vectors allow the geodesic equation to be cast into a first-order form. Which is then perturbed in Newton's constant G to calculate the scattering angles for a test particle in the Schwarzschild geometry up to order G^3. Finally, the symmetries in the Kerr metric are examined where, in addition to two Killing vectors, an extra "hidden" symmetry is found from a Killing tensor. This "hidden" symmetry produces the Carter constant, which is used to cast the geodesic equation into a first-order form. The geodesic equation is then perturbed in G and the spin of the black hole a to compute scattering angles below order G^2a^2

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Chapter 1

Introduction

Symmetries have always been objects of interest, whether in art or mathematics or the sciences. The importance of the study of symmetries experienced an increase thanks to Emmy Noether's influential theorem, which tied the symmetry of a Lagrangian to the conserved quantities of the corresponding physical system. Conserved quantities allow one to make predictive statements about the behaviour of a physical system, allowing us to determine the state of the system in the past or the future. The study of symmetries is currently of great interest in modern physics, whether studying gauge theories built from gauge symmetries to explore particle physics or using Super-symmetry to try and extend the Standard Model of particle physics to come up with new physics. Symmetries are also currently being studied in Einstein's theory of General Relativity in order to explore particle dynamics in curved spaces. Particle dynamics for black hole binaries have become very relevant ever since the detection of gravitational waves at the Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2016.

We can model black hole binaries where the mass of one black hole is much larger than the other; the less massive black hole acts as a test particle orbiting the larger black hole. These orbits can then be used to construct gravitational waves, which we can then detect here on Earth. Gravitational wave astronomy can provide significant insights into astrophysical phenomena than what can be achieved from traditional electromagnetic observations.

In addition to constraining particle dynamics, conserved quantities also reduce the order of the differential equations. An example of this is particle motion in a potential $V(x)$. Newton's laws state that we must solve a second-order differential equation:

$$m\frac{d^2x}{dt^2} = -\frac{\partial V(x)}{\partial x}$$

However, with a conserved quantity such as energy E, we can construct a first-order differential equation instead:

$$\frac{m\left(\frac{dx}{dt}\right)^2}{2} = -V(x) + E \rightarrow \frac{dx}{dt} = \sqrt{\frac{2(E - V(x))}{m}}$$

which simplifies the problem and reduces computational difficulty since, in general, first-order equations are easier to solve than second-order. The presence of symmetry usually simplifies a lot of the difficulties one faces when studying curved space-times.

This thesis will focus on how symmetries are represented in general relativity and what uses they have. As a result, we study Killing vector fields and Killing tensor fields which are directly linked with the presence of symmetries and so-called "hidden" symmetries. In the second chapter (chapter 2), we study Killing and conformal Killing vectors and how they are found, we explore the symmetries of the Euclidean/Minkowski metric and the 2-sphere metric using Killing and conformal Killing vectors. Then we investigate Killing tensors and their uses in the third chapter (chapter 3); we also review Hamilton-Jacobi theory and its connections to Killing tensors. From chapter four (chapter 4), we investigate the symmetries of the non-spinning Schwarzschild metric using Killing vectors and tensors. Then we perturb the geodesic equations in the Schwarzschild background to compute scattering angles for a test particle. We then generalise to the Kerr metric in chapter five (chapter 5), where we find a hidden symmetry which leads to the celebrated Carter's constant; using this additional symmetry we compute scattering angles in the Kerr metric. Finally, we present our findings and conclude the thesis in chapter six (chapter 6).

Chapter 2

Killing and Conformal Killing Vectors

Before we begin we quickly review the notion of vectors and tensors on Manifolds.

2.1 Vectors and Manifolds

Vectors and vector fields are very useful objects to define and use in physics, from classical to quantum mechanics. Therefore we would also like to define and use them on curved space-time, but we run into the trouble of having to define what a vector is in curved space-time, and what is it defined on? We introduce the concept of smooth differentiable manifolds to model curved space-time. These are spaces that are locally flat but globally may have non-trivial curved structure [1, 2]. A Manifold is defined as the set of points which themselves are contained in a set of smooth "patches". These smooth patches locally look and act like flat Euclidean space we are familiar with, the patches are "sewn" together such that the intersections are also smooth. This implies we can define infinitely differentiable functions upon the manifold. The main idea is that in a small neigbourhood around a point on the manifold is locally flat, while globally we may have curvature.

In classical physics vectors were usually defined as the displacement between two points using a co-ordinate system, this leads to the classical definition of a vector as an object with magnitude and a direction. However this definition only works well in flat space with no curvature. Gravitational physics according to Einstein's theory occurs in curved space-times as general relativity states that mass and energy curve space-time [1]. Therefore we need to amend the definition of a vector.

As stated before we can define a smooth function $\mathfrak{f}$ on the manifold. These functions take points on the manifold as inputs and then maps these points to a number. To parameterise these functions we have another function γ which takes a number λ as input and maps this number λ to a point on the manifold. Then the corresponding point on the manifold can be mapped to a number by another function. These functions define curves on the manifold [1] and can be parameterised by λ. In general relativity a vector is defined as the directional derivative operator v that acts on a smooth (infinitely differentiable) function $f(\lambda)$ passing through a point p shown in figure 2.1. The function is parameterised by λ [3]:

$$\frac{df}{d\lambda}$$

If we have a co-ordinate system x^μ describing the patch around p then using the chain rule:

$$\frac{df}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} \tag{2.1}$$

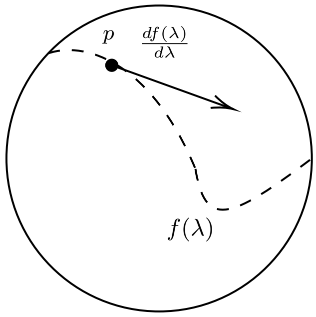


Figure 2.1: A smooth function $f(\lambda)$ drawn on a curved surface such as a sphere.

The object we identify as the tangent vector V is:

$$V = \frac{d}{d\lambda} = v^{\mu}\frac{\partial}{\partial x^{\mu}} \tag{2.2}$$

where V is the object referred to as the vector and $v^{\mu} = \frac{\partial x^{\mu}}{\partial \lambda}$ are the components of this vector [1]. Partial derivatives along the direction of the co-ordinates x^{μ} are a natural choice for a vector basis. This basis is called the co-ordinate basis. An important object is the scalar field which assigns a real or complex number on each point of the manifold. A vector field assigns each point on the manifold a tangent vector. Then we consider dual vectors $\omega = \omega_{\mu}dx^{\mu}$ which act on vectors to map them to real numbers, this is analogous to the "bra and ket" notation for quantum states, bra which is the dual vector acts on the ket which is a vector to yield a real number. The gradient of the smooth function $f(\lambda)$ is a dual vector: df.

$$df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}$$

which yields a number. The basis of dual vectors or "one-forms" are: dx^{μ}.

When we apply a change of co-ordinates from $x^{\mu} \rightarrow x^{\mu'}$ the components of vectors and one forms change but the tangent vector or dual vector is still the same since they are co-ordinate independent objects. Vector components v^{μ} and one-form components ω_{μ} transform as the following:

$$v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} v^{\mu} \tag{2.3}$$

$$\omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_{\mu} \tag{2.4}$$

A tensor is a multi-linear map which acts on vectors and dual vectors to yield numbers. A tensor T may be expressed in co-ordinate basis as:

$$T = T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m}\left(\frac{\partial}{\partial x^{\mu_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\mu_n}} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_m}\right) \tag{2.5}$$

here $\otimes$ denotes the tensor product. The rank of a tensor is described as the number of vectors and one-forms the tensors acts on, the above is a tensor of rank (n, m) which indicates how many upper and lower indices the tensor has. Under co-ordinate change $x^{\mu} \rightarrow x^{\mu'}$ its components transforms as:

$$T^{\mu_1' \ldots \mu_n'}_{\nu_1' \ldots \nu_m'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \ldots \frac{\partial x^{\mu_n'}}{\partial x^{\mu_n}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \ldots \frac{\partial x^{\nu_m}}{\partial x^{\nu_m'}} T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m} \tag{2.6}$$

This is the tensor transformation law. A very important tensor is the metric g which is a symmetric rank $(0, 2)$ tensor. It is used to describe line elements in curved space ds^2:

$$ds^2 = g = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{2.7}$$

the metric is positive definite in Riemannian geometry where all metric components are positive. For Lorentzian geometry we have the signature $(-1, +1, .., +1)$ for metric components. Where one component of the metric in D-dimensional space is negative. This tensor allows us to take a generalised dot product between two vectors v^μ and l^μ:

$$v^\mu l^\nu g_{\mu\nu} = v^\mu l_\mu$$

which yields a scalar. This operation is called a contraction and can be applied to tensors as well. The inverse metric can also be defined as $g^{\mu\nu}$ with the property:

$$g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu$$

Taking partial derivatives of tensors and vectors is unfortunately not co-ordinate independent. As a result we need to define the covariant derivative D_μ whose action on a generic tensor is given as:

$$D_\rho \left(T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m}\right) = \frac{\partial}{\partial x^\rho}\left(T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m}\right) + \Gamma^{\mu_1}_{\rho\lambda} T^{\lambda \ldots \mu_n}_{\nu_1 \ldots \nu_m} + \ldots + \Gamma^{\mu_n}_{\rho\lambda} T^{\mu_1 \ldots \lambda}_{\nu_1 \ldots \nu_m} - \Gamma^{\lambda}_{\rho\nu_1} T^{\mu_1 \ldots \mu_n}_{\lambda \ldots \nu_m} \ldots - \Gamma^{\lambda}_{\rho\nu_m} T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \lambda} \tag{2.8}$$

the $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols which are determined by the metric:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\rho} + \frac{\partial g_{\sigma\rho}}{\partial x^\nu} - \frac{\partial g_{\rho\nu}}{\partial x^\sigma}\right) \tag{2.9}$$

The action of the covariant derivative on the metric is 0:

$$D_\rho g_{\mu\nu} = 0 \tag{2.10}$$

The covariant derivative along a curve $f(\lambda)$ parameterised by λ is given as:

$$\frac{D}{d\lambda} = \frac{dx^\rho}{d\lambda} D_\rho \tag{2.11}$$

Finally we review the motion of freely falling particles in general relativity which are described by the geodesic equation:

$$\frac{d^2 x^\mu(\tau)}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu(\tau)}{d\tau} \frac{dx^\rho(\tau)}{d\tau} = 0 \tag{2.12}$$

for the geodesic path $x^\mu(\tau)$ parameterised by the proper time τ. For $x^\mu(\tau)$ we define the tangent four-velocity differentiated with respect to proper time τ: $U^\mu = \frac{dx^\mu(\tau)}{d\tau}$. To find the acceleration on this geodesic curve we take the covariant derivative along the curve:

$$\frac{D}{d\tau} U^\mu = \frac{dx^\rho(\tau)}{d\tau} D_\rho(U^\mu) = U^\rho \frac{\partial U^\mu}{\partial x^\rho} + \Gamma^\mu_{\rho\alpha} U^\alpha U^\rho = 0$$

This simplifies to:

$$\frac{D}{d\tau} U^\mu = \frac{dU^\mu}{d\tau} + \Gamma^\mu_{\rho\alpha} U^\alpha U^\rho \rightarrow \frac{d^2 x^\mu(\tau)}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu(\tau)}{d\tau} \frac{dx^\rho(\tau)}{d\tau} = 0$$

which is the geodesic equation hence it is equal to zero on a geodesic. We can now begin to think about symmetries of curved space, for example how can we describe the spherical symmetry possessed by a sphere? Continuous symmetries such as these are classified by Killing vectors which obey Killing's equation.

2.2 Killing vectors

Killing's equation is defined as follows [2] :

$$D_\mu K_\nu + D_\nu K_\mu = 0 \tag{2.13}$$

where K_μ is the Killing vector field. There are several ways of obtaining Killings equation, but the one presented involves the Lie derivative $\mathcal{L}_K$, as it makes concrete the link between the Killing vector fields and their effect of preserving the metric tensor $g_{\mu\nu}$. The Lie derivative measures the rate of change of a given tensor due to a vector field $K = K^\mu \frac{\partial}{\partial x^\mu}$. We introduce the Lie derivative axiomatically by first considering the action of $\mathcal{L}_K$ on a smooth function f which is defined as [3]:

$$\mathcal{L}_K (f) = K^\mu \frac{\partial f}{\partial x^\mu} \tag{2.14}$$

for an arbitrary tensor $T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m}$:

$$\mathcal{L}_K T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} = K^\rho \partial_\rho T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} - (\partial_\rho K^{\mu_1}) T^{\rho \dots \mu_n}_{\nu_1 \dots \nu_m} - (\partial_\rho K^{\mu_2}) T^{\mu_1 \rho \dots \mu_n}_{\nu_1 \dots \nu_m} - \dots + (\partial_{\nu_1} K^\rho) T^{\mu_1 \dots \mu_n}_{\rho \dots \nu_m} + (\partial_{\nu_2} K^\rho) T^{\mu_1 \dots \mu_n}_{\nu_1 \rho .. \nu_m} + \dots \tag{2.15}$$

as a result the lie derivative acting on another vector field U yields the Lie bracket of two vector fields.

$$\mathcal{L}_K U^\mu = K^\rho \partial_\rho (U^\mu) - U^\rho \partial_\rho (K^\mu) = [K, U]^\mu \tag{2.16}$$

which results in another vector field. The action on a differential form is also relevant for our analysis:

$$\mathcal{L}_K \mathrm{d}x^\mu = \mathrm{d}\mathcal{L}_K (x^\mu) = \mathrm{d}(K^\nu \partial_\nu x^\mu) = \mathrm{d}(K^\mu) = (\partial_\nu K^\mu) dx^\nu \tag{2.17}$$

Which allows the $\mathcal{L}_K$ to act on the co-ordinate inside the differential form. Before continuing let us note down useful properties of the Lie derivative that will be used. Lie derivatives have properties that are similar to what we expect with ordinary or covariant derivatives such as linearity:

$$\mathcal{L}_K (aT + bS) = a\mathcal{L}_K (T) + b\mathcal{L}_K (S) \tag{2.18}$$

Where T and S are tensors and a, b constants. The commutator of Lie derivatives w.r.t different vector fields X, Y is the Lie derivative of the vector field generated by the Lie bracket of X, Y:

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$$

One way of deriving equation Killing's equation in 2.13 is done by acting with the Lie derivative with respect to K on the metric, and requiring it to be 0, as this would imply tensor was not changed along this vector field:

$$\mathcal{L}_K g_{\mu\nu} = 0, \tag{2.19}$$

Where $g_{\mu\nu}$ is the metric. By using the action of $\mathcal{L}_K$ on a generic tensor we get:

$$\mathcal{L}_K g_{\mu\nu} = K^\rho D_\rho (g_{\mu\nu}) + g_{\nu\rho} D_\mu (K^\rho) + g_{\mu\rho} D_\nu (K^\rho) = D_\mu K_\nu + D_\nu K_\mu = 0 \tag{2.20}$$

Hence we derive Killing's equation by demanding we have a vector field whose action preserves the metric. Furthermore eq.2.19 tells us the Lie bracket of two Killing vector yields another Killing vector:

$$\mathcal{L}_{[X,Y]} g_{\mu\nu} = \mathcal{L}_X \mathcal{L}_Y (g_{\mu\nu}) - \mathcal{L}_Y \mathcal{L}_X (g_{\mu\nu}) = 0 \tag{2.21}$$

which is satisfied if X, Y are Killing vectors according to 2.20. For Killing vectors the Lie bracket acts as a commutator and builds a group of isometries which could be thought of as distance preserving transformations. Formally speaking an isometry is defined as a diffeomorphism that acts on the metric to give back the identical metric, more details are given in page 438 of [2].

The physical consequence of this is that we can construct conserved quantities along a geodesic curve $x^\mu(\tau)$ with Killing vectors. For a Killing vector field K^μ we contract it with the 4 velocity on the geodesic parameterised by proper time τ: $U^\mu = \frac{dx^\mu(\tau)}{d\tau}$ to construct the quantity $K^\mu U_\mu = K_\mu U^\mu$. We take the covariant derivative along this path with respect to τ to see how it varies along the geodesic:

$$\begin{aligned} \frac{d}{d\tau}(K_\mu U^\mu) &= \frac{dx^\rho}{d\tau}\frac{\partial}{\partial x^\rho}(K_\mu U^\mu) \\ = \frac{dx^\rho}{d\tau}D_\rho(K_\mu U^\mu) = U^\rho D_\rho(K_\mu U^\mu) &= U^\rho(U^\mu D_\rho K_\mu + K_\mu D_\rho U^\mu) = 0 \end{aligned} \tag{2.22}$$

the second term vanishes as it is the covariant acceleration which leads to the geodesic equation 2.12 which is 0. The above equation indicates $\frac{d(K_\mu U^\mu)}{d\tau} = 0$ when:

$$\frac{d}{d\tau}(K_\mu U^\mu) = \frac{U^\rho U^\mu}{2}(D_\mu K_\rho + D_\rho K_\mu) + 0 = 0 \tag{2.23}$$

we symmetrised the surviving term in the second line of 2.22 as the indices μ and ρ are symmetric under exchange, which leads to Killings equation which also equals 0. Hence the quantity $K_\mu U^\mu$ is conserved along a geodesic.

Solving Killing's equation for an arbitrary metric in favourable circumstances will give us several distinct independent Killing vectors which form a basis for all Killing vectors. Technically this means we have an infinite number of Killing vectors for some metrics as we can always take linear combinations of Killing vectors with constant coefficients to produce another Killing vector. However these linear combinations are not physically interesting for us as they correspond to linear combinations of conserved quantities which do not provide any new insight that we did not have before. Hence we want linearly independent Killing vectors which cannot be decomposed into other Killing vectors. Solving Killing's equation gives us exactly that, the solutions of Killings equation with distinct constants correspond to distinct Killing vectors.

Since we have explored how Killing vectors arise from the metric and their connection to the symmetries present on the manifold, one naturally wonders whether there is a connection between these Killing vectors and the curvature of the manifold itself? There is indeed a relation between the Killing vectors and the Riemann curvature tensor $R^\rho_{\sigma\mu\nu}$. We derive this by differentiating the Killing vector equation 2.13 again with with another covariant derivative :

$$D_\rho D_\mu K_\nu + D_\rho D_\nu K_\mu = 0 \tag{2.24}$$

We now have to commute the covariant derivatives, but doing so results in adding a Riemann tensor to our equations. We see this from the definition of the Riemann tensor from covariant derivatives of vector fields:

$$D_\kappa D_\nu K_\mu - D_\nu D_\kappa K_\mu = -K_\sigma R^\sigma_{\mu\nu\kappa} \tag{2.25}$$

which is found in page 140 of [4]. Using this definition we switch indices on eq.2.24:

$$D_\mu D_\rho K_\nu - K_\sigma R^\sigma_{\nu\mu\rho} + D_\rho D_\nu K_\mu = 0$$

then switch indices on the third term:

$$D_\mu D_\rho K_\nu - K_\sigma R^\sigma_{\nu\mu\rho} + D_\nu D_\rho K_\mu - K_\sigma R^\sigma_{\mu\nu\rho} = 0$$

the indices of the covariant derivative and the Killing vector on the third term are switched which also yields a minus sign:

$$-D_\mu D_\nu K_\rho - D_\nu D_\mu K_\rho = K_\sigma R^\sigma_{\nu\mu\rho} + K_\sigma R^\sigma_{\mu\nu\rho}$$

Switching the covariant derivative indices using eq.12 2.25 for the second term:

$$-D_\mu D_\nu K_\rho - \left(D_\mu D_\nu K_\rho - K_\sigma R^\sigma_{\rho\mu\nu}\right) = K_\sigma R^\sigma_{\nu\mu\rho} + K_\sigma R^\sigma_{\mu\nu\rho} \tag{2.26}$$

rearranging:

$$-2D_\mu D_\nu K_\rho = K_\sigma \left[-R^\sigma_{\rho\mu\nu} + R^\sigma_{\nu\mu\rho} + R^\sigma_{\mu\nu\rho}\right] \tag{2.27}$$

We need to remember another identity involving the Riemann curvature tensor, namely the algebraic Bianchi identity:

$$R^\sigma_{\mu\nu\rho} + R^\sigma_{\nu\rho\mu} + R^\sigma_{\rho\mu\nu} = 0 \tag{2.28}$$

Utilizing this on the third term on the r.h.s of eq.2.27 and focusing only on the r.h.s yields:

$$K_\sigma \left[-2R^\sigma_{\rho\mu\nu} - R^\sigma_{\nu\rho\mu} + R^\sigma_{\nu\mu\rho}\right] \tag{2.29}$$

Then using the fact that the Riemann tensor is anti-symmetric in its last two indices the third term is the minus of the second, so the r.h.s becomes:

$$-2K_\sigma \left[R^\sigma_{\rho\mu\nu} + R^\sigma_{\nu\rho\mu}\right] \tag{2.30}$$

Finally we apply Bianchis identity one more time on the first term and we are only left with one term:

$$+2K_\sigma \left[R^\sigma_{\mu\nu\rho}\right] \tag{2.31}$$

Now we equate both sides and derive the relation between a Killing vector and the Riemann tensor:

$$D_\mu D_\nu K_\rho = -K_\sigma R^\sigma_{\mu\nu\rho} \tag{2.32}$$

This a very valuable identity as it can be used to determine how many Killing vectors exist on a given metric. We now proceed to derive Killing vectors on a 2-sphere

2.2.1 Killing vectors on 2-sphere

The metric of a 2-sphere is defined as:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 = d\theta \otimes d\theta + \sin^2\theta(d\phi \otimes d\phi) \tag{2.33}$$

Where ϕ denotes the azimuthal angle ranging from 0 to 2π, likewise θ is the polar angle from 0 to π. Acting with the Lie derivative $\mathcal{L}_K$ with respect to Killing vector field K^μ should yield 0 therefore:

$$\mathcal{L}_K ds^2 = 0 \tag{2.34}$$

we notice the metric is independent of the azimuthal co-ordinate ϕ. Therefore a Killing vector of the 2-sphere must be ∂_ϕ. On the contrary we see ∂_θ cannot be a Killing vector as the metric is clearly dependent on the θ. The most general form of the Killing vector must be a linear combination of ∂_θ and ∂_ϕ which is given as: $K = K^\phi\partial_\phi + K^\theta\partial_\theta$. Clearly K^ϕ and K^θ cannot be simple constants and must instead be multi-variable functions of θ and ϕ. We expect to extract all possible Killing vectors contained by the metric from this form. We now act with $\mathcal{L}_K$ on $g_{\mu\nu}dx^\mu dx^\nu$, utilising the action of the Lie derivative on a function and a differential form we obtain:

$$\begin{aligned} \mathcal{L}_K(g_{\mu\nu}dx^\mu dx^\nu) &= \mathcal{L}_K(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin(\theta)^2(\mathrm{d}\phi \otimes \mathrm{d}\phi)) \\ = dK^\theta \otimes d\theta + d\theta \otimes dK^\theta &+ \sin^2(\theta)(dK^\phi \otimes d\phi + d\phi \otimes dK^\phi) + 2\sin(\theta)\cos(\theta)(d\phi \otimes d\phi) \end{aligned} \tag{2.35}$$

Decomposing the differentials of the functions K^μ into their respective basis one forms dx^μ yields the following:

$$\begin{aligned}
&(\frac{\partial K^\theta}{\partial \theta}d\theta + \frac{\partial K^\theta}{\partial \phi}d\phi) \otimes d\theta + d\theta \otimes (\frac{\partial K^\theta}{\partial \theta}d\theta + \frac{\partial K^\theta}{\partial \phi}d\phi) \\
&+ \sin^2\theta((\frac{\partial K^\phi}{\partial \theta}d\theta + \frac{\partial K^\phi}{\partial \phi}d\phi) \otimes d\phi + d\phi \otimes (\frac{\partial K^\phi}{\partial \theta}d\theta + \frac{\partial K^\phi}{\partial \phi}d\phi)) \\
&+ 2\sin\theta\cos\theta K^\theta(d\phi \otimes d\phi) = 0
\end{aligned} \tag{2.36}$$

Simplifying this expressions and collecting everything under the bases of $d\phi \otimes d\phi, d\theta \otimes d\theta, d\theta \otimes d\phi, d\phi \otimes d\theta$ then setting them equal to 0 yields system of partial differential equations. The system of coupled partial differential equations are as follow for the components K^θ, K^ϕ of the Killing vector field K :

$$\begin{aligned}
&\frac{\partial K^\theta}{\partial \theta} = 0 \\
&\frac{\partial K^\theta}{\partial \phi} + \sin^2\theta\frac{\partial K^\phi}{\partial \theta} = 0 \\
&\sin^2\theta\frac{\partial K^\phi}{\partial \phi} + \sin\theta\cos\theta K^\theta = 0
\end{aligned} \tag{2.37}$$

this system of equations is solved in the appendix. Solving this system yields the following for the Killing vector fields of the sphere:

$$\begin{aligned}
K^\theta &= A\cos\phi + B\sin\phi \\
K^\phi &= \cot\theta(-A\sin\phi + B\cos\phi) + C
\end{aligned} \tag{2.38}$$

The three possible independent Killing Vectors on a 2-sphere are:

$$Z = \frac{\partial}{\partial \phi} \tag{2.39}$$

$$X = \sin\phi\frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi}, \quad Y = \cos\phi\frac{\partial}{\partial \theta} - \cot\theta\sin\phi\frac{\partial}{\partial \phi} \tag{2.40}$$

The metric can be acted upon by these vectors and one can see that it does yield 0, thereby preserving the metric. The Killing vector Z is the easiest to visualise on the 2-sphere in figure 2.2. The figure represents the vector field rotating the sphere about the z-axis. The other fields X and Y have peculiar behaviour about the poles of the sphere at $\theta = \{0, \pi\}$ where the ϕ component of the vector field diverges as it becomes ambiguous how to define a tangent vector in the ϕ direction. This is simply due to a co-ordinate singularity and is nothing geometrical, it is a consequence of using the θ, ϕ co-ordinate system. The fields X and Y contain "stationary" points where the vector field vanishes, for X this occurs at: $\theta = \frac{\pi}{2}$and $\phi = \{0, \pi\}$ and for Y : $\theta = \frac{\pi}{2}$ and $\phi = \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. For each respective vector field we see that all the other points tangent vectors rotate about the point. In fact when the two sphere is embedded into flat space the stationary points map directly onto the x-axis and y-axis. Hence the vector fields X, Y represent the generators of rotation about these axes. Another way to see this is to notice when $\phi \to \phi + \frac{\pi}{2}$ gets shifted by $\frac{\pi}{2}$ the K.V fields X and Y interchange into each other with a sign change.i.e: $X \to Y$ $Y \to -X$.

Hence these vector fields are denoted respectively for the corresponding rotation they represent. Furthermore these vectors generate the SO(3) group symmetry:

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [X, Z] = -Y \tag{2.41}$$

As a result we see the expected rotational symmetry that is associated with the 2-sphere. The generators of symmetry as simply rotations along the x,y,z axes in spherical co-ordinates.

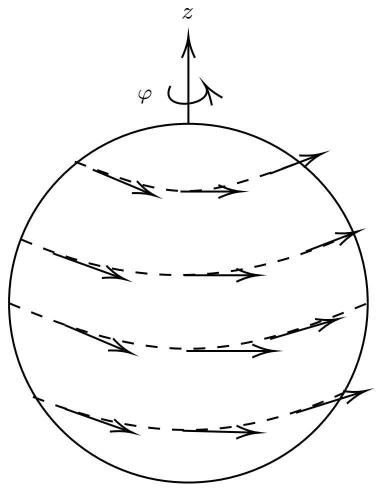


Figure 2.2: The Z Killing vector field on the 2-sphere.

These vector fields form a basis of Killing vectors as more can be constructed by simply taking linear combinations of the vectors X, Y, Z. But in terms of our physical interest in them these are trivial as they do not give rise to any other new conserved quantities as they are just combinations of the previous conserved quantities. We look for non-trivial and distinct conserved quantities that cannot be reduced to other conserved quantities.

2.2.2 Killing Vectors in flat space

A more familiar space is the flat Euclidean or Minkowski space. They are manifolds lacking any curvature whatsoever, meaning the curvature invariants such as the Riemann tensor $R^{\rho}_{\mu\nu\alpha}$ vanish at every point on the manifold [1]. The metric for these spaces are defined as:

$$ds^2 = dx_0^2 \pm dx_1^2 + dx_2^2 + ... + dx_{D-1}^2 \tag{2.42}$$

The sign change in between dx_0 and dx_1 determines whether we are in Euclidean (+) or Minkowski (-) space. Through physical intuition we know the symmetries of flat space as the set of translations along the direction of each of the D co-ordinates, as translating every point in flat space by a constant vector yields back flat space. The associated Killing vector field for translations must correspond to the generators of these translations:

$$P_\mu = \frac{\partial}{\partial x^\mu} \tag{2.43}$$

A basis for the translation Killing vector would be translations along each of the D co-ordinates. As one can imagine a constant vector field pointing in the same direction spanning the entire manifold, the action of this field would simply translate all the points by a constant vector is equivalent to translating the origin of a co-ordinate system to some other point. The other set of symmetries are rotations in a plane, as rotating every point by an arbitrary with reference to a plane simply yields back a similar flat space. Hence, the Killing vectors for rotations are the generators of rotation according to the notation in [5]:

$$L_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \tag{2.44}$$

which is the rotation with respect to the plane formed by co-ordinates x^μ and x^ν. Hence, we have calculated all the independent Killing vectors in flat space by simply considering the possible symmetries present. Hence in D-dimensional flat/Minkowski space one has D translation Killing vector in D directions and $\frac{D(D-1)}{2}$ rotation Killing vectors. As a result we have $D + \frac{D(D-1)}{2} = \frac{D(D+1)}{2}$ Killing vectors.

One can also solve Killings equation in flat space where the covariant derivatives reduce to partial derivatives:

$$\partial_\mu K_\nu + \partial_\nu K_\mu = 0 \tag{2.45}$$

the relation with curvature is already known:

$$D_\mu D_\nu K_\rho = -R^\lambda_{\nu\rho\mu} K_\lambda \tag{2.46}$$

in flat space the curvature is 0. As a result the right hand side vanishes and the covariant derivatives become partial derivatives:

$$\partial_\mu \partial_\nu K_\rho = 0 \tag{2.47}$$

The solution of which is linear:

$$K_\rho = P_\rho + \omega_{\rho\lambda} x^\lambda \tag{2.48}$$

Where P_ρ is any constant vector and $\omega_{\rho\lambda}$ a constant coefficient tensor. The constraints on $\omega_{\rho\lambda}$ is found by substituting the linear solution back into Killing's equation for flat space eq.2.45 , which indicates $\omega_{\rho\lambda} = -\omega_{\lambda\rho}$. This yields the Killing vectors we derived by intuition. Taking the Lie bracket of the Killing vectors with each other yields the Poincaré Algebra:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\rho, L_{\mu\nu}] &= \eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu \\ [L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} \end{aligned} \tag{2.49}$$

these are the set of commutation relations between the generators of translation and rotation in D-dimensional Euclidean and Minkowski space.

2.2.3 Conformal Killing Vectors

We have explored Killing vectors and seen how they give rise to conserved quantities. And seen how they are deeply related to the idea of symmetry or more specifically isometries which are distance preserving transformations. As a result we associate Killing vectors with isometries. Now one can ask themselves whether there are other type of vectors that represent some other symmetry that a metric may have, such as an conformal symmetry. Conformal transformations are local scale transformations that modify the metric with a positive multiplicative scale factor function which is dependent on the co-ordinates $\lambda(x)$:

$$g_{\mu\nu} \xrightarrow{Conformal \quad transformation} \lambda(x) g_{\mu\nu} \tag{2.50}$$

a direct result of this is that angles are always preserved under these types of transformations. An example of a conformal transformation is scaling the metric by some constant, meaning we scale up our manifold by some factor. Conformal transformations and symmetry are of great importance in various fields of physics, ranging from string theories to condensed matter physics. Electromagnetism is a conformally invariant theory, hence one can use conformal mappings to simplify situations in electromagnetism. Since this is considered a symmetry one may ask is there some sort of a vector that is associated with this? This question naturally leads us to our first generalisation of a Killing vector, namely the conformal Killing vector.

A conformal Killing vector can be defined as a vector field whose Lie derivative acts on the metric to alter the metric by a co-ordinate dependent function $\lambda(x)$:

$$\mathcal{L}_K g = \lambda(x) g, \tag{2.51}$$

where K is now a conformal Killing vector. The action of the Lie derivative on the left hand side of eq.2.51 yields the same terms as in Killing's equation :

$$D_\mu K_\nu + D_\nu K_\mu = \lambda(x) g_{\mu\nu} \tag{2.52}$$

The function $\lambda(x)$ can be found by taking the trace of this equation and rearranging which yields:

$$\lambda(x) = \frac{2D_\rho K^\rho}{Dim} \tag{2.53}$$

Where Dim denotes the number of dimensions of the manifold on which K is defined. Then the conformal Killing equation becomes:

$$D_\mu K_\nu + D_\nu K_\mu = \frac{2}{Dim}(D_\rho K^\rho)g_{\mu\nu} \tag{2.54}$$

One can think of Killing vectors as a subset of conformal Killing vectors with the added condition that the covariant divergence of the Killing vector field must vanish. In addition to representing the conformal symmetry of a manifold the existence of conformal Killing vector also provides conserved quantities for massless particles along null geodesics. To see this we repeat the argument for the conserved quantity of a Killing vector. We contract a conformal Killing vector K^μ with the tangent velocity along a null geodesic, one has to use the affine parameter λ instead of the proper time τ as it becomes ill defined. So $U^\mu = \frac{dx^\mu}{d\lambda}$:

$$\begin{aligned}\frac{D}{d\lambda}(K_\mu U^\mu) = \frac{U^\rho U^\mu}{2}(D_\mu K_\rho + D_\rho K_\mu) = \frac{U^\rho U^\mu}{2}(\frac{2}{Dim}(D_\alpha K^\alpha)g_{\mu\rho})\\ = \frac{U^\mu U_\mu}{Dim}(D \cdot K) = 0\end{aligned} \tag{2.55}$$

Where the $\cdot$ represents the covariant divergence between vectors. The final term vanishes since we are on a null geodesic hence $U \cdot U = 0$. Hence these quantities are conserved along null geodesics [2]. Conformal Killing vectors generate the conformal symmetry group of a manifold. Conformal symmetry plays an important role string theory and Field Theory.
We now consider the conformal Killing vector in D-dimensional Euclidean/Minkowski space, 2.54 simplifies to:

$$\partial_\mu K_\nu + \partial_\nu K_\mu = \frac{2}{D}(\partial_\rho K^\rho)\eta_{\mu\nu} \tag{2.56}$$

The approach taken to solve this equation is to derive a condition from eq.2.56 that truncates the power series expansion of the conformal Killing vector to a quadratic. Similar to how the relation between the curvature and Killing vectors in eq.2.32 was derived we differentiate eq.2.56 and repeatedly use eq.2.56 to switch indices to find the desired condition.
We define the d'Alembert (box) operator : $\partial^\mu \partial_\mu = \Box$ which appears when we act with ∂^μ on equation 2.56:

$$\begin{aligned}\Box K_\nu + \partial_\nu(\partial \cdot K) &= \frac{2}{D}\partial_\nu(\partial \cdot K)\\ \Box K_\nu &= (\frac{2-D}{D})\partial_\nu(\partial \cdot K)\end{aligned} \tag{2.57}$$

Applying ∂^ν and rearranging yields:

$$\Box(\partial \cdot K) = 0 \tag{2.58}$$

Now we utilise this relation by applying the box operator on eq.2.56:

$$\Box\partial_\mu K_\nu + \Box\partial_\nu K_\mu = \frac{2}{D}\Box(\partial \cdot K)\eta_{\mu\nu} \tag{2.59}$$

the right hand side vanishes, then using the second line of 2.57 the above simplifies to:

$$\partial_\mu \partial_\nu(\partial \cdot K) = 0 \tag{2.60}$$

We now multiply by this with the metric with new indices $\rho, \sigma; \eta_{\rho,\sigma}$ and utilising the CKV equation's r.h.s to substitute in the left hand side, leading to one of the final steps before the result:

$$\partial_\mu \partial_\nu (\partial_\rho K_\sigma + \partial_\sigma K_\rho) = \partial_\mu \partial_\nu \partial_\rho K_\sigma + \partial_\mu \partial_\nu \partial_\sigma K_\rho = 0 \tag{2.61}$$

We now use the conformal Killing vector equation 2.56 and eq.2.60 to switch indices to get the desired equation, we switch $\nu \leftrightarrow \sigma$ yielding:

$$\partial_\mu \partial_\rho (\frac{2}{D}(\partial \cdot K)\eta_{\nu\sigma} - \partial_\sigma K_\nu) + \partial_\mu \partial_\nu \partial_\sigma K_\rho = -\partial_\mu \partial_\rho \partial_\sigma K_\nu + \partial_\mu \partial_\nu \partial_\sigma K_\rho = 0 \tag{2.62}$$

The divergence term is killed off due to 2.60. We act one more time on the first term again this time switching $\rho \leftrightarrow \nu$, we neglect the divergence term due to eq.2.60:

$$\partial_\mu \partial_\sigma \partial_\nu K_\rho + \partial_\mu \partial_\nu \partial_\sigma K_\rho = 0 \rightarrow \partial_\mu \partial_\nu \partial_\sigma K_\rho = 0 \tag{2.63}$$

Which is the condition that allows us to truncate our power series expansion to a quadratic, hence:

$$K_\mu = C_\mu + C_{\mu\rho} x^\rho + C_{\mu\rho\nu} x^\rho x^\nu \tag{2.64}$$

Where the $C_{12\ldots i}$ are constant coefficient tensors. To find the behaviour of these coefficients we substitute the solution into the conformal Killing vector equation, the purely constant term C_μ can be any constant with no restriction, similar to the constant in the Killing vector case this corresponds to translations along co-ordinate directions [5]. As outlined in [5] the linear term gains a constraint:

$$C_{\mu\nu} + C_{\nu\mu} = \frac{2}{dim} c^\rho_\rho \eta_{\mu\nu} \tag{2.65}$$

The right hand side is non-zero only when $\mu = \nu$ or otherwise the trace part of $C_{\mu\nu}$ is non-zero. $\mu \neq \nu$ reveals it has to be anti symmetric in exchange of indices. We recognise this as $C_{\mu\nu}$ having an anti-symmetric part and a trace part, hence $C_{\mu\nu}$ can be expressed as:

$$C_{\mu\nu} = \frac{2}{dim} c^\rho_\rho \eta_{\mu\nu} + m_{\mu\nu} \tag{2.66}$$

where $m_{\mu\nu} = -m_{\nu\mu}$. Immediately we recognise the linear term of the conformal Killing vector contains the rotational Killing vectors associated with the anti-symmetric part, reinforcing the idea that conformal Killing vector are generalisations of the Killing vectors. Interpreting this geometrically the conformal group in flat space contains the Poincaré group. As translations and rotations do indeed preserve angles.

The trace part which is proportional to the metric term corresponds to the dilation operator which scales the metric by a constant factor. The coefficient multiplying $\eta_{\mu\nu}$ is a constant which we set to: $\frac{2}{dim} c^\rho_\rho = 1$. The dilation operator is defined as (ignoring the irrelevant antisymmetric part):

$$D = C^\mu_\nu x^\nu \frac{\partial}{\partial x^\mu} = \delta^\mu_\nu x^\nu \frac{\partial}{\partial x^\mu} = x^\mu \frac{\partial}{\partial x^\mu} \tag{2.67}$$

It could be visualised as a radial vector field "pushing" out all the points from the origin of our co ordinate system. We acquire dilation specifically by requiring the divergence of the conformal Killing vector to be a constant.

With the exception of the dilation vector field we have already encountered the rest which are just Euclidean/Minkowski Killing vector fields. The remaining quadratic term yields the special conformal Vector fields which scale the metric by a function dependent on co-ordinates. In this case the divergence of the conformal Killing vector is now a function of co-ordinates x^μ. These fields are well studied in flat space due to their application in String theory and conformal field theories.

To find the specific form of the scale transformations it is necessary to derive another equation from eq.2.56 by differentiating with another partial derivative:

$$\partial_\rho \partial_\mu K_\nu + \partial_\rho \partial_\nu K_\mu = \eta_{\mu\nu} \partial_\rho f \tag{2.68}$$

Where $f = \frac{2}{D}(\partial_\rho K^\rho)$. Switching $\rho \leftrightarrow \nu$ for the first term and $\rho \leftrightarrow \mu$ for the second on the l.h.s of the equation yields the desired result after collecting the double derivative terms and rearranging:

$$2\partial_\mu \partial_\nu K_\rho = \eta_{\mu\rho} \partial_\nu f + \eta_{\rho\nu} \partial_\mu f - \eta_{\mu\nu} \partial_\rho f \tag{2.69}$$

which places a constraint solely on the quadratic term of the conformal Killing vector solution in eq.2.64. This yields the part of the conformal Killing vector which is associated with "special conformal transformations" :

$$C_{\rho\nu\mu} = \frac{1}{D}(\eta_{\mu\rho} C^\lambda_{\lambda\nu} + \eta_{\rho\nu} C^\lambda_{\lambda\mu} - \eta_{\mu\nu} C^\lambda_{\lambda\rho}) \tag{2.70}$$

Hence the generators of the special conformal transformations are:

$$C = C^{\rho\nu\mu} x_\nu x_\mu \partial_\rho = \frac{1}{D}(\eta^{\mu\rho} C^{\lambda\nu}_\lambda + \eta^{\rho\nu} C^{\lambda\mu}_\lambda - \eta^{\mu\nu} C^{\lambda\rho}_\lambda) x_\nu x_\mu \partial_\rho \tag{2.71}$$

Setting $\frac{1}{D} C^\lambda_{\lambda\nu} = 1$.

$$C_\rho = 2x_\rho x^\nu \partial_\nu - x^2 \partial_\rho \tag{2.72}$$

which generate the special conformal transformations in D dimensional Minkowski or flat space, where $b = \frac{1}{D} C^\lambda_{\lambda\nu}$.

Now we summarise the conformal Killing vectors we have derived for Euclidean/Minkowski space and present it as follows from [5]:

Translations	$P_\mu = -i\frac{\partial}{\partial x^\mu}$
Rotations	$L_{\mu\nu} = i(x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu})$
Dilation	$D = -i(x^\mu \frac{\partial}{\partial x^\mu})$
Special Conformal Transformation	$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

constructing the Lie brackets of the conformal Killing vectors with each other forms the conformal group algebra of D-dimensional Euclidean/Minkowski space which contains the Poincaré group.

With these conformal Killing vectors we can Lie bracket them with each other to form the conformal group, which contains the Poincaré group:

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})
\end{aligned} \tag{2.73}$$

[5] An interesting case occurs in two dimensional flat/Minkowski space which has important applications in string theory. In two-dimensional Euclidean space the conformal Killing equation 2.54 reduces to:

$$\partial_\mu K_\nu + \partial_\nu K_\mu = (\partial_\rho K^\rho) \eta_{\mu\nu} \tag{2.74}$$

we make a co-ordinate transformation into the complex plane on the Euclidean space: $Z = \mathcal{X} + i\mathcal{Y}, \bar{Z} = \mathcal{X} - i\mathcal{Y}$ for flat space. We utilise these co-ordinate transformations to make the metric off-diagonal:

$$ds^2 = d\mathcal{X}^2 + d\mathcal{Y}^2 \to dZd\bar{Z}, \quad \text{with} \quad g_{Z\bar{Z}} = g_{\bar{Z}Z} = \frac{1}{2} \tag{2.75}$$

as a result our conformal Killing equation yields three equations:

$$\begin{aligned} \partial_Z K_Z &= 0 \\ \partial_{\bar{Z}} K_{\bar{Z}} &= 0 \\ \partial_Z K_{\bar{Z}} + \partial_{\bar{Z}} K_Z &= \partial_Z K_{\bar{Z}} + \partial_{\bar{Z}} K_Z \end{aligned} \tag{2.76}$$

The third equation is identically satisfied leaving few restrictions on the form of the CKV. The first two equations can be expressed further as the following by pulling out the metric:

$$\begin{aligned} \partial_Z K^{\bar{Z}} &= 0 \\ \partial_{\bar{Z}} K^Z &= 0 \end{aligned} \tag{2.77}$$

Implying that K^Z and $K^{\bar{Z}}$ are functions purely dependent on their respective co-ordinates, hence they can be any purely holomorphic or anti-holomorphic functions in complex space. As a consequence we can expand in a power series which shows us locally, there are an infinite number of conformal Killing vectors in two-dimensional flat space:

$$\begin{aligned} K^Z &= C_0 + C_1 Z + C_2 Z^2 + ... = \Sigma_{i=0}^{\infty} C_i Z^i \\ K^{\bar{Z}} &= \tilde{C}_0 + \tilde{C}_1 \bar{Z} + \tilde{C}_2 \bar{Z}^2 + ... = \Sigma_{i=0}^{\infty} \tilde{C}_i \bar{Z}^i \end{aligned} \tag{2.78}$$

Where $C_i, \tilde{C}_i$ are constants. Hence, there´s an infinite number of conformal Killing vectors in Euclidean space:

$$\begin{aligned} K^Z \frac{\partial}{\partial Z} &= C_0 \frac{\partial}{\partial Z} + C_1 Z \frac{\partial}{\partial Z} + C_2 Z^2 \frac{\partial}{\partial Z} + ... \\ K^{\bar{Z}} \frac{\partial}{\partial \bar{Z}} &= \tilde{C}_0 \frac{\partial}{\partial \bar{Z}} + \tilde{C}_1 \bar{Z} \frac{\partial}{\partial \bar{Z}} + \tilde{C}_2 \bar{Z}^2 \frac{\partial}{\partial \bar{Z}} + ... \end{aligned} \tag{2.79}$$

This result is also true in Minkowski space as well.

The algebra formed by taking the Lie bracket of these conformal Killing vectors is called the Witt algebra. Defining $L_n = -Z^{n+1}\partial_Z, \tilde{L}_n = -\bar{Z}^{n+1}\partial_{\bar{Z}}$:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} \\ [\tilde{L}_n, \tilde{L}_m] &= (n-m)\tilde{L}_{n+m} \\ [L_n, \tilde{L}_m] &= 0 \end{aligned} \tag{2.80}$$

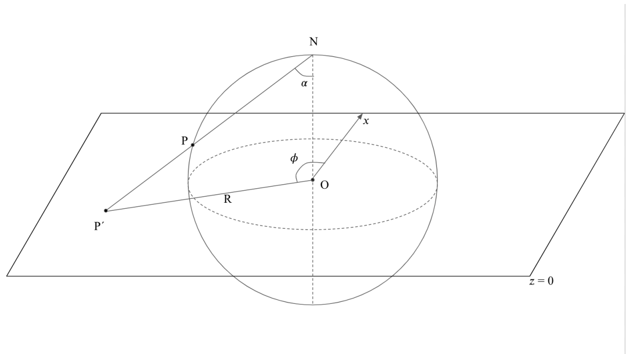
The algebra consists of a copy of left moving algebra with a right moving algebra of holomorphic and anti-holomorphic vectors with $L_0, \tilde{L}_0$ acting as the identity elements for their respective groups [5]. Furthermore there is a sub-algebra within the two between the vectors of L_0, L_{-1}, L_1:

$$\begin{aligned} [L_1, L_0] &= L_1 \\ [L_1, L_{-1}] &= 2L_0 \\ [L_{-1}, L_0] &= -L_{-1} \end{aligned} \tag{2.81}$$

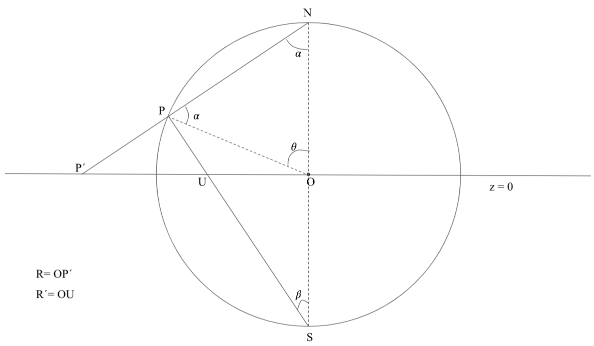
,and likewise for the $\tilde{L}_{-1,0,1}$ which contains the same commutation relations. We can interpret the physical action of these generators by comparing them to the D-dimensional conformal Killing vector case. Since $L_{-1} = -\frac{\partial}{\partial Z}$ this corresponds to a translation in the Z direction on the plane. $L_0 = -Z\frac{\partial}{\partial Z}$, the linear terms correspond to rotations and dilations. And finally the $L_1 = -Z^2\frac{\partial}{\partial Z}$ corresponds to the special conformal transformations in Z [5]. Witt algebra appears in the context of generators of the conformal symmetry on the world sheet in string theory.

2.2.4 Projection of 2-sphere onto the complex plane

The simplest non-trivial manifold after flat space are spheres. Previously we solved for the Killing vectors of the 2-sphere using the Lie derivative as it was more demonstrative of the geometric idea of symmetry. We now solve Killing's equation and the Conformal Killing equation for the 2-sphere directly. This involves using covariant derivatives which involve Christoffel symbols $\Gamma^{\mu}_{\nu\rho}$,in order to simplify these calculations we project the 2-sphere onto a plane using Stereographic projection. Furthermore we will see the projection of the sphere onto the complex plane simplifies the forms of the Killing and conformal Killing vectors and reveals the conformal structure of the sphere. The mapping involves projecting straight lines from one of the poles of the sphere to a plane centred at the equator of the sphere as shown in figure 2.3 where they intersect with the points on the sphere. The projection maps all points (except the pole where the projections originate from) onto a unique point on the plane. Hence we need two planes from the North and South poles such that all points on the 2-sphere are mapped. This is a common occurrence in differential geometry where in general a manifold cannot be covered by a single patch but requires two or more patches [1]. In our case the 2-sphere requires two patches to completely cover the manifold as each patch excludes on point: either the North or the South pole depending on which patch it is.



(a) Describes how points on 2-sphere manifold are projected onto a flat plane at $z = 0$



(b) Depicts the triangle created by points NOP from which the radius on the plane is acquired. R denotes the radius from the centre of the projection to the sphere point mapped from the North pole, whereas point R' denotes the radius of the same point mapped from the South pole

Figure 2.3: Stereographic projection

We begin by considering the triangle NOP' in figure 2.3. The projection angle α as function of the polar angle θ is just: $\alpha = \frac{\pi-\theta}{2}$. Hence:

$$\tan(\alpha) = \tan(\frac{\pi-\theta}{2}) = \cot(\frac{\theta}{2}) = \frac{\sin\theta}{1-\cos\theta} = R \tag{2.82}$$

From which we find the radial co-ordinate R of the point P mapped from the sphere. And trivially the ϕ angle is unchanged on the plane. Hence our co-ordinates become $(R, \phi) =$

$(\cot(\frac{\theta}{2}), \phi)$ from the north pole.

Alternatively mapping P from the South pole means we have to consider triangle NSP. We find $\beta = \frac{\theta}{2}$ which allows R' to be determined as:

$$\tan(\frac{\theta}{2}) = R' \tag{2.83}$$

now the ϕ angle maps to $-\phi$ on the plane from S since one of the co-ordinate axes on the plane facing S must pointing in the opposite direction compared to the axes on the plane facing N. The azimuthal angle for the plane facing S is labelled ϕ'. The co-ordinates are $(R', \phi') = (\tan(\frac{\theta}{2}), -\phi)$ as the co-ordinate systems on the two sides of the $z = 0$ plane differ in orientation, since one of the axes has to be reversed in order to match the orientation of the axes on the other side. Since P can be mapped from both the North and the South poles we can define a relation between the R and R', namely $R' = \frac{1}{R}$. This relation acts as a global condition that translates the points between the two patches. Furthermore we see that both the North pole and the South pole are mapped to the origin of the planes created by the opposing pole, so the North pole becomes the origin of the $(R', -\phi)$ and vice versa. Therefore no points are excluded in this stereographic projection. With this consistency condition we proceed by considering only the mapping from the North pole.

Using the North pole projection the metric of the 2-sphere is transformed to the following using $(\theta, \phi) = (2\text{arctan}(\frac{1}{R}), \phi)$:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \rightarrow \frac{4}{(R^2+1)^2}\left(dR^2 + R^2 d\phi^2\right)$$

We can further represent the (R, ϕ) in terms of cartesian co-ordinates on the plane $(\mathcal{X}, \mathcal{Y}) = (R\cos\phi, R\sin\phi)$ which changes the metric to the cartesian form:

$$ds^2 = \frac{4}{(\mathcal{X}^2 + \mathcal{Y}^2 + 1)^2}\left(d\mathcal{X}^2 + d\mathcal{Y}^2\right)$$

The 2-sphere expressed in this form shows that it is a conformally flat metric with a conformal factor of $\frac{2}{X^2+Y^2+1}$. From this cartesian form of the 2-sphere we make the transformation into complex co-ordinates Z and $\bar{Z}$ as previously defined in 2.2.3. This transforms the metric into simplified form which expresses the complex structure of the 2-sphere:

$$ds^2 = \frac{4}{(Z\bar{Z}+1)^2}(dZd\bar{Z}) \tag{2.84}$$

however, with this projection we are neglecting the North pole which has to covered by a complex patch formed by the South pole projection. We repeat the derivation for the South pole projection co-ordinates $(R', \phi') = (\tan(\frac{\theta}{2}), -\phi)$ which casts the 2-sphere metric as:

$$ds^2 = d\theta^2 + \sin^2(\theta) d\phi^2 \rightarrow \frac{4}{((R')^2+1)^2}\left(d(R')^2 + (R')^2 d\phi^2\right) \tag{2.85}$$

this can be converted into the Cartesian form and then we make the transformation into another complex plane parameterised by the complex co-ordinates $W = \mathcal{X} - i\mathcal{Y}$ and $\bar{W} = \mathcal{X} + i\mathcal{Y}$. This expresses the 2-sphere metric using the South pole projection as:

$$ds^2 = \frac{4}{(W\bar{W}+1)^2}(d\bar{W}dW) \tag{2.86}$$

which looks identical to the metric in the Z complex patch. As a result any tensor or vector that is defined on the 2-sphere exists simultaneously on both of these complex patches under the stereographic projection. This leads to the global condition for the projection 2.2.4 which we multiply by the phase $e^{i\phi}$ to derive:

$$Re^{i\phi} = \frac{1}{R'e^{-i\phi}} \rightarrow Z = \frac{1}{R'e^{i\phi'}} = \frac{1}{W} \tag{2.87}$$

this is the condition the 2-sphere satisfies as a complex manifold.

As we will see this complex form of the 2-sphere is easier to work in. Firstly we compute the Christoffel symbols of this form of the metric. We know for the original 2-sphere metric 2.33 to acquire the three non-zero Christoffel symbols namely:

$$\Gamma^{\theta}_{\phi\phi} = -\sin(\theta)\cos(\theta) \;\Big|\; \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot(\theta)$$

which are trigonometric functions of only θ. Now we see the Christoffel symbols are simpler in the complex projection of the 2-sphere, using the complex form of the metric 2.84 where our co-ordinates are $Z, \bar{Z}$, the only non-zero Christoffel symbols in this metric are:

$$\Gamma^{Z}_{ZZ} = \tfrac{-2\bar{Z}}{Z\bar{Z}+1} \;\Big|\; \Gamma^{\bar{Z}}_{\bar{Z}\bar{Z}} = \tfrac{-2Z}{Z\bar{Z}+1}$$

While these are now multi variable functions of $Z, \bar{Z}$ we note that we have one less Christoffel symbol compared to the original case. The two Christoffel symbols are either functions only of Z or $\bar{Z}$, this helps us when constructing the Killing's equations. The Killing equation is as follows when writing out each component separately:

$$\begin{aligned} D_Z K_Z &= 0 \\ D_{\bar{Z}} K_{\bar{Z}} &= 0 \\ D_Z K_{\bar{Z}} + D_{\bar{Z}} K_Z &= 0 \end{aligned} \tag{2.88}$$

Raising the Killing vectors indices by factorising out the metric completely (since the covariant derivative commutes with it) leads to the solutions :

$$\begin{aligned} D_Z(g_{Z\bar{Z}}K^{\bar{Z}}) = 0 \rightarrow D_Z(K^{\bar{Z}}) &= \frac{\partial K^{\bar{Z}}}{\partial Z} = 0 \\ D_{\bar{Z}} K^Z &= \frac{\partial K^Z}{\partial \bar{Z}} = 0 \\ D_Z K^Z + D_{\bar{Z}} K^{\bar{Z}} &= 0 \end{aligned} \tag{2.89}$$

As there are no "mixed" indices Christoffel symbols with both Z and $\bar{Z}$, the covariant derivatives in the first two equations reduce to partial derivatives. Which state that the holomorphic Killing vector K^Z is a function purely of Z and likewise the anti-holomorphic Killing vector is a function only of $\bar{Z}$. This situation is reminiscent of the two-dimensional conformal Killing vectors in Euclidean space, where our solution consists of an infinite number of vectors. However, in contrast there is still the third equation to be considered which provides relations between the holomorphic and anti holomorphic Killing vectors.

Secondly these Killing vectors have to exist on the other complex patch parameterised by W. As a result the global condition 2.2.4 we defined earlier becomes relevant. This will help us trim the infinite number of Killing vectors to a finite few.

From the first two equations in 2.89 we can expand the purely holomorphic and anti-holomorphic Killing vectors as a power series in their respective co-ordinates.

$$\begin{aligned} K^Z &= C_0 + C_1 Z + C_2 Z^2 + ... = \sum_{k=0}^{\infty} C_i Z^k \rightarrow K = K^Z \frac{\partial}{\partial Z} = \left(\sum_{k=0}^{\infty} C_i Z^k\right) \frac{\partial}{\partial Z} \\ K^{\bar{Z}} &= \tilde{C}_0 + \tilde{C}_1 \bar{Z} + \tilde{C}_2 \bar{Z}^2 + ... = \sum_{k=0}^{\infty} \tilde{C}_i \bar{Z}^k \rightarrow \overline{K} = K^{\bar{Z}} \frac{\partial}{\partial \overline{Z}} = \left(\sum_{k=0}^{\infty} \tilde{C}_i \overline{Z}^k\right) \frac{\partial}{\partial \overline{Z}} \end{aligned} \tag{2.90}$$

we apply the relation 2.2.4 to transition into W complex patch for the K^Z vector:

$$K^Z = \left(\sum_{k=0}^{\infty} C_i Z^k\right) \frac{\partial W}{\partial Z}\frac{\partial}{\partial W} = \left(\sum_{k=0}^{\infty} C_i Z^k\right)\left(-\frac{1}{Z^2}\frac{\partial}{\partial W}\right) = -\left[\frac{C_0}{Z^2} + \frac{C_1}{Z} + C_2 + C_3 Z + ...\right]\frac{\partial}{\partial W}$$
$$= -\left[C_0 W^2 + C_1 W + C_2 + \frac{C_3}{W} + ...\right]\frac{\partial}{\partial W} \tag{2.91}$$

these vectors represent finite non-diverging vectors on the 2-sphere therefore they should exist on both patches without any poles or divergences. Vectors with coefficient $C_{i>2}$ have poles at $W = 0$. Therefore we discard these vectors and retain the ones which do not contain poles. Only the vectors with coefficients C_0, C_1 and C_2 do not have poles at $W = 0$ therefore we find only three valid holomorphic vectors:

$$K = \left(C_0 + C_1 Z + C_2 Z^2\right)\frac{\partial}{\partial Z} \tag{2.92}$$

repeating the exact argument for the anti-holomorphic vectors yields the three valid vectors:

$$\bar{K} = \left(\tilde{C}_0 + \tilde{C}_1 \bar{Z} + \tilde{C}_2 \bar{Z}^2\right)\frac{\partial}{\partial \bar{Z}} \tag{2.93}$$

Then we input the truncated power series expansions for K^Z and $K^{\bar{Z}}$ into the the third equation in 2.89 which "mixes up" the purely holomorphic and anti-holomorphic parts. The third equation in 2.89 expanded out is:

$$\begin{aligned} \frac{\partial K^Z}{\partial Z} + \Gamma^Z_{ZZ} K^Z + \frac{\partial K^{\bar{Z}}}{\partial \bar{Z}} + \Gamma^{\bar{Z}}_{\bar{Z}\bar{Z}} k^{\bar{Z}} = 0 \\ \frac{\partial K^Z}{\partial Z} + \frac{\partial K^{\bar{Z}}}{\partial \bar{Z}} - \frac{2}{Z\bar{Z}+1}\left(ZK^{\bar{Z}} + \bar{Z}K^Z\right) = 0 \end{aligned} \tag{2.94}$$

Inputting in the vectors $K^Z, K^{\bar{Z}}$

$$C_1 + 2ZC_2 + \tilde{C}_1 + 2\bar{Z}\tilde{C}_2 - \frac{2}{Z\bar{Z}+1}\left(Z\tilde{C}_0 + Z\bar{Z}\tilde{C}_1 + Z\bar{Z}^2\tilde{C}_2 + \bar{Z}C_0 + \bar{Z}ZC_1 + \bar{Z}Z^2C_2\right) = 0$$

The coefficients must be fixed such that the terms cancel out and the equality is satisfied. Immediately one can set $C_1 = -\tilde{C}_1$, then getting rid of the denominator on the second term and simplifying we get the equation:

$$(\tilde{C}_2\bar{Z} + C_2 Z)Z\bar{Z} + (\tilde{C}_2 - C_0)\bar{Z} + (C_2 - \tilde{C}_0)Z - \tilde{C}_2\bar{Z}^2 Z - C_2\bar{Z}Z^2 = 0$$

From which we find relations between the holomorphic and anti holomorphic coefficients:

$$\begin{aligned} C_1 &= -\tilde{C}_1 \\ \tilde{C}_2 &= C_0 \\ C_2 &= \tilde{C}_0 \end{aligned} \tag{2.95}$$

since C_i and $\tilde{C}_i$ are complex coefficients these are solutions to the complex Killing equations in 2.89. However, the Killing vector fields defined on the 2-sphere are real valued vector fields. Therefore, we must further institute a reality condition that gives us real Killing vectors. The reality condition is:

$$\bar{K^Z} = K^{\bar{Z}} \tag{2.96}$$

which ensures the Killing vectors are real. This provides the relation $\bar{C}_i = \tilde{C}_i$. As a result C_1 is purely imaginary since:

$$C_1 = -\tilde{C}_1 = -\bar{C}_1$$

likewise for the other two:

$$\begin{aligned}\tilde{C}_2 &= \bar{C}_2 = C_0\\ C_2 &= \tilde{C}_0 = \bar{C}_0\end{aligned}$$

splitting these coefficients into the imaginary and real parts: $C_i = \mathfrak{Re}[C_i] + i\mathfrak{Im}[C_i]$. The above two equations yield the following:

$$\begin{aligned}\mathfrak{Re}[C_2] - i\mathfrak{Im}[C_2] &= \mathfrak{Re}[C_0] + i\mathfrak{Im}[C_0]\\ \mathfrak{Re}[C_2] + i\mathfrak{Im}[C_2] &= \mathfrak{Re}[C_0] - i\mathfrak{Im}[C_0]\end{aligned}$$

where $\mathfrak{Re}[C_i]$ and $\mathfrak{Im}[C_i]$ denote the real and imaginary parts of C_i. We deduce: $\mathfrak{Re}[C_2] = \mathfrak{Re}[C_0]$ and $\mathfrak{Im}[C_2] = -\mathfrak{Im}[C_0]$.

This allows us to find the three distinct Killing vectors for the sphere:

$$\begin{aligned}&K^Z\frac{\partial}{\partial Z} + K^{\bar{Z}}\frac{\partial}{\partial \bar{Z}} = C_0\left(\frac{\partial}{\partial Z} + \bar{Z}^2\frac{\partial}{\partial \bar{Z}}\right) + C_2\left(\frac{\partial}{\partial \bar{Z}} + Z^2\frac{\partial}{\partial Z}\right) + C_1\left(Z\frac{\partial}{\partial Z} - \bar{Z}\frac{\partial}{\partial \bar{Z}}\right)\\ &= \mathfrak{Re}[C_0]\left(\frac{\partial}{\partial Z} + \bar{Z}^2\frac{\partial}{\partial \bar{Z}} + \frac{\partial}{\partial \bar{Z}} + Z^2\frac{\partial}{\partial Z}\right) + i\mathfrak{Im}[C_0]\left(\frac{\partial}{\partial Z} + \bar{Z}^2\frac{\partial}{\partial \bar{Z}} - \frac{\partial}{\partial \bar{Z}} - Z^2\frac{\partial}{\partial Z}\right)\\ &\qquad + i\mathfrak{Im}[C_1]\left(Z\frac{\partial}{\partial Z} - \bar{Z}\frac{\partial}{\partial \bar{Z}}\right)\end{aligned} \tag{2.97}$$

By inverting the stereographic projections from the North pole patch in 2.2.4 while expressing R and ϕ in terms of $Z, \bar{Z}$ we find:

$$\begin{aligned}\theta &= 2\text{arctan}\left(\frac{1}{\sqrt{Z\bar{Z}}}\right)\\ \phi &= \frac{\ln\left[\frac{Z}{\bar{Z}}\right]}{2i}\end{aligned} \tag{2.98}$$

The Killing vector with the $\mathfrak{Im}[C_1]$ coefficient corresponds to the $\frac{\partial}{\partial\phi}$ rotation generator, whereas $\mathfrak{Re}[C_0]$ coefficient vector corresponds to the Y Killing vector and $\mathfrak{Im}[C_0]$ corresponds to the X Killing vector we derived in 2.40. We denote them by their respective θ, ϕ Killing vector and express them in Witt algebra generators:

$$\begin{aligned}\mathfrak{Re}[C_0] \to \hat{A}_Y &= -\left(\frac{\partial}{\partial Z} + \bar{Z}^2\frac{\partial}{\partial \bar{Z}} + \frac{\partial}{\partial \bar{Z}} + Z^2\frac{\partial}{\partial Z}\right) = L_{-1} + \tilde{L}_1 + \tilde{L}_{-1} + L_1\\ \mathfrak{Im}[C_1] \to \hat{A}_Z &= -i\left(Z\frac{\partial}{\partial Z} - \bar{Z}\frac{\partial}{\partial \bar{Z}}\right) = i\left(L_0 - \tilde{L}_0\right)\\ \mathfrak{Im}[C_0] \to \hat{A}_X &= -i\left(\frac{\partial}{\partial Z} + \bar{Z}^2\frac{\partial}{\partial \bar{Z}} - \frac{\partial}{\partial \bar{Z}} - Z^2\frac{\partial}{\partial Z}\right) = i\left(L_{-1} + \tilde{L}_1 - \tilde{L}_{-1} - L_1\right)\end{aligned} \tag{2.99}$$

where the sub-script label for $\hat{A}_Z$ is the $\frac{\partial}{\partial\phi}$ Killing vector in eq.2.40, not the complex coordinate Z. We can take the Lie bracket of these to find the following:

$$\begin{aligned}\left[\hat{A}_X, \hat{A}_Y\right] &= 4i\hat{A}_Z\\ \left[\hat{A}_X, \hat{A}_Z\right] &= \hat{A}_Y\\ \left[\hat{A}_Z, \hat{A}_Y\right] &= i\hat{A}_X\end{aligned} \tag{2.100}$$

2.2.5 Conformal Killing Vectors

Solving the conformal Killing equation in the complex projection is even simpler as the third equation which "mixes" the holomorphic and anti-holomorphic vector fields disappears. Therefore the conformal Killing equations on the 2-sphere complex projection patch of Z are 2.54:

$$\begin{aligned} D_Z K_Z &= 0 \\ D_{\bar{Z}} K_{\bar{Z}} &= 0 \\ g_{Z\bar{Z}}\left(D_Z K^Z + D_{\bar{Z}} K^{\bar{Z}}\right) &= g_{Z\bar{Z}}\left(D_Z K^Z + D_{\bar{Z}} K^{\bar{Z}}\right) \end{aligned} \tag{2.101}$$

The third equation is identically satisfied and we do not mix K^Z and $K^{\bar{Z}}$. We have six distinct complex conformal Killing Vectors which form a basis of conformal generators:

$$K^Z\frac{\partial}{\partial Z} + K^{\bar{Z}}\frac{\partial}{\partial \bar{Z}} = C_0\frac{\partial}{\partial Z} + \tilde{C}_0\frac{\partial}{\partial \bar{Z}} + C_1 Z\frac{\partial}{\partial Z} + \tilde{C}_1\bar{Z}\frac{\partial}{\partial \bar{Z}} + C_2 Z^2\frac{\partial}{\partial Z} + \tilde{C}_2\bar{Z}^2\frac{\partial}{\partial \bar{Z}} \tag{2.102}$$

These are just generators of the Witt algebra we discovered in 2.81 when studying the two dimensional Euclidean space conformal Killing vectors. So we label them as such matching the co-efficient of complex conformal Killing vector to the corresponding Witt generator: $C_0 \to L_{-1}, \tilde{C}_0 \to \tilde{L}_{-1}, C_1 \to L_0, \tilde{C}_1 \to \tilde{L}_0, C_2 \to L_1, \tilde{C}_2 \to \tilde{L}_1$. The Lie bracket structure is the same as before in 2.81, repeating it here [5]:

$$\begin{aligned} [L_1, L_0] &= L_1 \\ [L_1, L_{-1}] &= 2L_0 \\ [L_{-1}, L_0] &= -L_{-1} \end{aligned} \tag{2.103}$$

And similarly for $\tilde{L}_i$:

$$\begin{aligned} [\tilde{L}_1, \tilde{L}_0] &= \tilde{L}_1 \\ [\tilde{L}_1, \tilde{L}_{-1}] &= 2\tilde{L}_0 \\ [\tilde{L}_{-1}, \tilde{L}_0] &= -\tilde{L}_{-1} \end{aligned} \tag{2.104}$$

The conformal group from the complex conformal Killing vectors on the 2-sphere consists of a right moving and left moving copy of SL(2,$\mathcal{C}$) sub algebras which commute with each other. The SL(2,$\mathcal{C}$) group is the set of all two by two matrices with determinant 1. In fact this algebra is familiar to anyone who has studied Special Relativity as this is the SO(1,3) algebra. We know the SO(1,3) algebra splits into two sub commuting su(2) algebras similar to our conformal algebra on the 2-sphere [6]. Does this have any physical meaning or is it just a nice mathematical coincidence? There is indeed a physical interpretation of this, given by Roger Penrose in [7].

Now we apply the reality conditions upon the complex conformal Killing vectors to find their forms in θ, ϕ co-ordinates. Since $\bar{C}_i = \tilde{C}_i$:

$$\begin{aligned} K^Z\frac{\partial}{\partial Z} + K^{\bar{Z}}\frac{\partial}{\partial \bar{Z}} &= C_0\frac{\partial}{\partial Z} + \tilde{C}_0\frac{\partial}{\partial \bar{Z}} + C_1 Z\frac{\partial}{\partial Z} + \tilde{C}_1\bar{Z}\frac{\partial}{\partial \bar{Z}} + C_2 Z^2\frac{\partial}{\partial Z} + \tilde{C}_2\bar{Z}^2\frac{\partial}{\partial \bar{Z}} \\ &= \mathfrak{Re}[C_0]\left(\frac{\partial}{\partial Z} + \frac{\partial}{\partial \bar{Z}}\right) + i\mathfrak{Im}[C_0]\left(\frac{\partial}{\partial Z} - \frac{\partial}{\partial \bar{Z}}\right) + \mathfrak{Re}[C_1]\left(Z\frac{\partial}{\partial Z} + \bar{Z}\frac{\partial}{\partial \bar{Z}}\right) \\ &+ i\mathfrak{Im}[C_1]\left(Z\frac{\partial}{\partial Z} - \bar{Z}\frac{\partial}{\partial \bar{Z}}\right) + \mathfrak{Re}[C_2]\left(Z^2\frac{\partial}{\partial Z} + \bar{Z}^2\frac{\partial}{\partial \bar{Z}}\right) + i\mathfrak{Im}[C_2]\left(Z^2\frac{\partial}{\partial Z} - \bar{Z}^2\frac{\partial}{\partial \bar{Z}}\right) \end{aligned} \tag{2.105}$$

Complex : $(Z,\bar{Z})$	Spherical : (θ,ϕ)
L_{-1} : -$\frac{\partial}{\partial Z}$	-$\frac{1}{2}e^{-i\phi}\left[(1-\cos(\theta))\frac{\partial}{\partial\theta}+i\tan(\frac{\theta}{2})\frac{\partial}{\partial\phi}\right]$
$\tilde{L}_{-1}$: -$\frac{\partial}{\partial\bar{Z}}$	-$\frac{1}{2}e^{i\phi}\left[(1-\cos(\theta))\frac{\partial}{\partial\theta}-i\tan(\frac{\theta}{2})\frac{\partial}{\partial\phi}\right]$
L_0 : -$Z\frac{\partial}{\partial Z}$	-$\frac{1}{2}\left[\sin(\theta)\frac{\partial}{\partial\theta}+i\frac{\partial}{\partial\phi}\right]$
$\tilde{L}_0$: -$\bar{Z}\frac{\partial}{\partial\bar{Z}}$	-$\frac{1}{2}\left[\sin(\theta)\frac{\partial}{\partial\theta}-i\frac{\partial}{\partial\phi}\right]$
L_1 : -$Z^2\frac{\partial}{\partial Z}$	-$\frac{1}{2}e^{i\phi}\left[(1+\cos(\theta))\frac{\partial}{\partial\theta}+i\cot(\frac{\theta}{2})\frac{\partial}{\partial\phi}\right]$
$\tilde{L}_1$: -$\bar{Z}^2\frac{\partial}{\partial\bar{Z}}$	-$\frac{1}{2}e^{-i\phi}\left[(1+\cos(\theta))\frac{\partial}{\partial\theta}-i\cot(\frac{\theta}{2})\frac{\partial}{\partial\phi}\right]$

Table 2.1: The complex conformal Killing vectors of the 2-sphere in complex and spherical form. Note that $\tilde{L}_i$ is just the complex conjugate of L_i. We still have to apply the reality conditions

We now transform these real conformal Killing vectors back into spherical co-ordinates (θ,ϕ) using table.2.1:

$$\frac{\partial}{\partial Z}+\frac{\partial}{\partial\bar{Z}}=(1-\cos(\theta))\cos(\phi)\frac{\partial}{\partial\theta}+\tan(\frac{\theta}{2})\sin(\phi)\frac{\partial}{\partial\phi} \tag{2.106}$$

$$i\left(\frac{\partial}{\partial Z}-\frac{\partial}{\partial\bar{Z}}\right)=(1-\cos(\theta))\sin(\phi))\frac{\partial}{\partial\theta}-\tan(\frac{\theta}{2})\cos(\phi)\frac{\partial}{\partial\phi} \tag{2.107}$$

$$Z\frac{\partial}{\partial Z}+\bar{Z}\frac{\partial}{\partial\bar{Z}}=\sin(\theta)\frac{\partial}{\partial\theta} \tag{2.108}$$

$$i\left(Z\frac{\partial}{\partial Z}-\bar{Z}\frac{\partial}{\partial\bar{Z}}\right)=-\frac{\partial}{\partial\phi} \tag{2.109}$$

$$Z^2\frac{\partial}{\partial Z}+\bar{Z}^2\frac{\partial}{\partial\bar{Z}}=(1+\cos(\theta))\cos(\phi)\frac{\partial}{\partial\theta}-\cot(\frac{\theta}{2})\sin(\phi)\frac{\partial}{\partial\phi} \tag{2.110}$$

$$i\left(Z^2\frac{\partial}{\partial Z}-\bar{Z}^2\frac{\partial}{\partial\bar{Z}}\right)=-\left((1+\cos(\theta))\sin(\phi)\frac{\partial}{\partial\theta}+\cot(\frac{\theta}{2})\cos(\phi)\frac{\partial}{\partial\phi}\right) \tag{2.111}$$

we note that we can construct the X and Y Killing vectors just from taking linear combinations of conformal Killing vectors. Which indicates that applying these combinations of conformal Killing vectors has a null effect on the metric, in essence when paired together the conformal Killing vectors scale the metric in a way such that the overall effect “cancels” out for the metric.

Thanks to the complex projection our Killing vectors and conformal Killing vectors are of a much simpler form than what they are in the standard θ,ϕ co-ordinates. Since they are just polynomials in the complex co-ordinates compared to the complicated trigonometric functions we had in θ,ϕ co-ordinates. This simplicity allows us to easily find the structure of the Lie groups of the isometries and conformal transformations. Furthermore the stereo graphic projection also reveals the complex structure of the 2-sphere. [8]

2.3 Counting of Killing vectors and maximally symmetric spaces

We have investigated Killing vectors for the simple metrics of the flat/Minkowski space and the 2-sphere. We found there are $\frac{D(D+1)}{2}$ in flat/Minkowski space and three Killing vectors on the 2-sphere. The number of Killing vectors for the 2-sphere corresponds to the 2-dimensional Euclidean/Minkowski space, these are one of the most symmetric spaces one can imagine. However most metrics of physical interest do not share this high degree of symmetry and as a result will not have as many Killing vectors, in fact it is rare that a generic metric contains a Killing vector [1]. So this raises the question, for some given metric what are the maximum number of possible independent Killing vectors? and how many independent ones are there? The answer to the first question also leads us to the idea of maximally symmetric spaces [4]. But before we delve into that we focus on counting the maximum possible Killing vectors for a D-dimensional space.

One can actually guess the answer to this by considering the most symmetric space we know: the Euclidean/Minkowski space. Which contains $\frac{D(D+1)}{2}$ independent Killing vectors, since this is the most possible symmetric space with the most isometries it should have a corresponding number of K.Vs, and as every other space is either equivalent or less symmetric than this [1]. Therefore we can imagine this creates an upper bound on the possible number of K.V in a D-dimensional space. To prove this is the maximal number of K.V possible on a D-dimensional metric we use the curvature relation 2.32 and Killings equation, which are restated here:

$$D_\mu D_\nu K_\rho = -K_\sigma R^\sigma_{\mu\nu\rho}$$
$$D_\mu K_\nu + D_\nu K_\mu = 0$$

The curvature equation links the covariant derivative of a Killing vector back to itself contracted with the Riemann tensor. Therefore any higher order derivatives can always be expressed as either terms proportional to K_μ the Killing vector or $D_\mu K_\nu$ the covariant derivative of the Killing vector using the equations above. Hence if one were to Taylor expand the Killing vector around a point X^μ on the manifold and the neighbourhood of the point $X^\mu - x^\mu$, hence the expansion series would become:

$$K^{\mathrm{n}}_\mu(x) = A^\rho_\mu(x;X)K^{\mathrm{n}}_\rho(X) + B^{\rho\lambda}_\mu(x;X)D_\lambda K^{\mathrm{n}}_\rho(X) \tag{2.112}$$

Where n denotes the n possible Killing vectors of a metric. The functions $A^\rho_\mu(x;X)$ and $B^{\rho\lambda}_\mu$ are only dependent on the co-ordinates X^μ, x^μ and the metric $g_{\mu\nu}$ [4]. We distinguish between different Killing vectors by their values of $K_\mu(X)$ and $D_\rho K_\mu(X)$. From this equation a Killing vector is uniquely determined by the data from $K_\mu(X)$ and $D_\rho K_\mu(X)$, in D-dimensional space there are $D + \frac{D(D-1)}{2} = \frac{D(D+1)}{2}$ entries one has to specify. The data of a Killing vector can be represented as a vector in $\frac{D(D+1)}{2}$ dimensional space, and in a vector space of $\frac{D(D+1)}{2}$ dimensions there can only be $\frac{D(D+1)}{2}$ independent vectors. Therefore the maximum number of Killing vectors on a manifold of dimension D is $\frac{D(D+1)}{2}$. Spaces which contain the maximum number of Killing vectors are called Maximally Symmetric spaces. Maximally symmetric spaces are important in cosmology as they provide Isotropic and Homogeneous spaces to model the universe. It can be shown that maximally symmetric spaces have a special form of the Riemann tensor [1]:

$$R_{\rho\sigma\mu\nu} = \frac{R}{D(D-1)}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \tag{2.113}$$

Where R is the Ricci scalar and D denotes the dimension of the manifold. Usually a metric might not contain the maximum number of Killing vectors. So how does one count the possible number of Killing vectors then? One achieves this by using a variant of the Killing-curvature relation and repeatedly differentiating it. As shown in [9] this generates a series of algebraic equations for the Killing vector, its derivative and the Riemann tensor which can be calculated. From solving these equations we can eliminate the number of independent Killing vector for a space time and yield an upper bound on the number. The paper applies this method for the Kerr metric and properly predicts two Killing vectors for the Kerr space time.

Chapter 3

Killing Tensors and their applications

3.1 Killing Tensors

Isometries form an important group of symmetries that can be used to analyse manifolds in the context of physics, as they lead directly to conserved quantities which leads to integrability. A physical system is called integrable if it has the same number of conserved quantities as the number of degrees of freedom, this allows the system to be solved analytically. The physical behaviour of an integrable systems phase space is restricted to a sub-manifold of the total phase space. However many metrics of physical interest generally do not contain enough isometries/K.V to allow for integrability, the aforementioned Kerr metric only has two Killing vectors for 4 degrees of freedom, another constraint can be found from the metric but we still lack one conserved quantity. An important generalisation of a Killing vector is the Killing tensor, defined as follows:

$$D_{(\mu}K_{\nu_1 \ldots \nu_m)} = 0 \tag{3.1}$$

Where () of the indices denotes the symmetrisation operator for indices. Killing tensors are also symmetric under the exchange of their indices. The number of indices of a Killing tensor denotes the rank or the "valence" of the tensor. When $m = 1$ we recover the notion of the Killing vector. Similar to the Killing vector we can show that the quantity $J = K_{\nu_1 \ldots \nu_m} U^{\nu_1} \ldots U^{\nu_m}$ which is the Killing tensor contracted with the tangent velocity along a geodesic curve is conserved along the geodesic. The demonstration is reminiscent of the Killing vector case. Differentiating this J along the geodesic w.r.t proper time τ:

$$\begin{aligned}\frac{D}{d\tau}(K_{\nu_1 \ldots \nu_m} U^{\nu_1} \ldots U^{\nu_m}) &= U^\rho \left(U^{\nu_1} \ldots U^{\nu_m} D_\rho K_{\nu_1 \ldots \nu_m} + m K_{\nu_1 \ldots \nu_m} U^{\nu_1} \ldots D_\rho U^{\nu_m}\right) \\ &= U^\rho U^{\nu_1} \ldots U^{\nu_m} D_{(\rho} K_{\nu_1 \ldots \nu_m)} = 0\end{aligned}$$

Where in the first line the we used that the covariant derivative of tangent velocity U^μ is 0, then in the second line since all U_i^μ are identical we symmetrise all the indices of the Killing tensor and the derivative term which then becomes the Killing tensor equation. Hence all products of Killing tensors with tangent velocities are conserved on geodesics. Therefore we see Killing tensors have the potential to provide the "missing" conserved quantities for integrability in certain metrics which may lack the required number of independent Killing vectors, provided these tensors exist of course. However we have rather arbitrarily introduced these tensors and the Killing tensor equation. And unfortunately Killing tensors do not have a clear geometric interpretation unlike the Killing vectors. Hence why they are also named "Hidden symmetries". The symmetry does not generate a diffeomorphism as a Killing vector does. So it is in some sense hidden and cannot be easily seen from viewing the manifold directly, instead one discovers it by analysing the particle motion [10].

The Killing tensors of physical interest are of rank - 2, the next direct generalisation of the Killing vector. These are the objects that either provide the missing conserved quantities for integrability, or allow for the separation of variables for the Hamilton-Jacobi and Klein-Gordon equations. Therefore we will focus on these types of Killing tensors in this thesis. The Killing tensor equation for rank-2 becomes:

$$\begin{aligned} D_{(\mu}K_{\nu\rho)} &= 0 \\ \frac{2}{3!}\left(D_\mu K_{\nu\rho} + D_\rho K_{\mu\nu} + D_\nu K_{\rho\mu}\right) &= 0 \\ D_\mu K_{\nu\rho} + D_\rho K_{\mu\nu} + D_\nu K_{\rho\mu} &= 0 \end{aligned} \tag{3.2}$$

Where in the 2nd line we use the fact that Killing tensors are symmetric tensors. Each space-time will always have one trivial rank-2 Killing tensor in the form of the metric $g_{\mu\nu}$ as this satisfies eq.3.2. Indeed the metric does provide a conserved quantity in the form of a normalisation for the tangent velocities equivalent to the mass of the particle. Hence the metric $g_{\mu\nu}$ is a trivial rank-2 Killing tensor.

One way to construct rank-2 Killing tensors is by symmetrising the tensor product of two Killing vectors J and L:

$$\begin{aligned} K_{\mu\nu} &= J_{(\mu}L_{\nu)} \\ K_{\mu\nu} &= \frac{1}{2}\left(J_\mu L_\nu + J_\nu L_\mu\right) \\ K &= J \odot L \end{aligned}$$

[3]Where the symbol $\odot$ denotes the symmetrised tensor product. We can substitute this into the Killing tensor equation to check if it works:

$$D_{(\mu}K_{\nu\rho)} = D_{(\mu}J_{(\nu}L_{\rho))} = \frac{1}{2}D_{(\mu}(J_\nu L_\rho + J_\rho L_\nu)_{)} = 0$$

Using the Killing equation $D_{(\mu}J_{\nu)} = 0$ and $D_{(\mu}L_{\nu)} = 0$ repeatedly and using the product rule as well since the covariant derivative acts on both J_μ and L_μ. This proves that the symmetrized products of Killing vectors do form Killing tensors. However these tensors are not physically relevent as they do not produce new conserved quantities and instead just repackage multiplicative products of already known conserved quantities. They do not provide any new symmetries which tells us nothing new about the manifold. We call these type of Killing tensors reducible Killing tensors. We are interested in irreducible Killing tensors which cannot be broken down into smaller or other Killing tensors. We don't want our Killing tensor to be factorised into other smaller Killing tensors or vectors. Irreducible Killing tensors provide new conserved quantities which help build towards the integrability of a system. Unlike the metric or reducible Killing tensors finding an irreducible Killing tensor is a challenge as it usually involves solving a complicated system of coupled partial differential equations. These are usually very tedious and difficult to solve. However, there exists special methods for certain cases. Such as in [11] where the existence of a so called "hypersurface" orthogonal Killing vector allows the Killing tensor to be reduced into three components. These components have associated partial differential equations which can be solved. In this thesis we will either straight forwardly solve the Killing tensor equation by a power series expansion or by using *Maple* which can generate and solve the Killing tensor equation.

Vector fields had the anti-symmetric Lie bracket which produced another vector field by taking the Lie derivative with respect to a vector field, which we used to generate Killing vectors by taking the Lie bracket of two other Killing vectors. For tensors there is a generalisation of this operation called the Schouten-Nijenhuis brackets. Unlike Lie brackets which require vector fields the Schouten-Nijenhuis require tensors of any rank as input. For tensors A of rank p and B of rank q the Schouten-Nijenhuis bracket generates a tensor of rank-$p+q-1$, the operation is defined as:

$$[A,B]^{j_1\cdots j_{p+q-1}} = pA^{i(j_1\cdots j_{p-1}}D_i B^{j_p\cdots j_{p+q-1})} - qB^{i(j_1\cdots j_{q-1}}D_i A^{j_q\cdots j_{p+q-1})} \tag{3.3}$$

[12] It can be shown that the Schouten-Nijenhuis bracket of two Killing tensors generate another Killing tensor of some other rank, similar to the Killing vector case and Lie brackets. In fact if one of the input tensors is a vector field the Schouten-Nijenhuis bracket simplifies to the Lie derivative w.r.t the vector field. For both vectors it reduces to the Lie bracket of two vectors [10]. Finally there is also a relation between rank-2 Killing tensors and the Riemann curvature tensor similar to the Killing vector case, for a rank-2 Killing tensor T_{ab}:

$$D_r D_s T_{ab} - D_b D_a T_{rs} = 2R^p_{rs(a}T_{b)p} - 2R^p_{ba(r}T_{s)p} \tag{3.4}$$

The proof for this equation is given in reference [13].

To understand these Killing tensors we start by analysing them in the simple spaces, namely Euclidean/Minkowski space and later the 2-sphere.

3.1.1 Flat Space Killing tensor of rank-2

Similar to our analysis of Killing vectors we will start in Euclidean/Minkowski space. As we know Euclidean space is a maximally symmetric space with a sufficient number of conserved quantities to strongly constrain particle dynamics. From the Killing vectors in D-dimensional Euclidean space we have $\frac{D(D+1)}{2}$ number of isometries leading to $\frac{D(D+1)}{2}$ conserved quantities. Now we solve Killing tensor equation to see if we can find more distinct conserved quantities. One can guess probably not as we already have more than enough conserved quantities required for integrability. The rank-2 Killing tensor equation reduces to partial derivatives:

$$\partial_\mu K_{\nu\rho} + \partial_\rho K_{\mu\nu} + \partial_\nu K_{\rho\mu} = 0 \tag{3.5}$$

We can solve this by expanding the Killing tensor in a power series of the co-ordinates x^μ and constant tensors $K^{(i)}_{\mu\nu\alpha_1\ldots\alpha_i}$:

$$K_{\mu\nu} = K^{(0)}_{\mu\nu} + K^{(1)}_{\mu\nu\alpha_1}x^{\alpha_1} + K^{(2)}_{\mu\nu\alpha_1\alpha_2}x^{\alpha_1}x^{\alpha_2} + \ldots = \sum_{i=0}^{\infty} K^{(i)}_{\mu\nu\alpha_1\ldots\alpha_i}x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_i} \tag{3.6}$$

The superscript (i) denotes the power of the expansion term. The indices α_i are symmetric under exchangle with each other as the co-ordinates x^α_i and x^α_j commute. Since these are summation indices we can rename them however we want. Now we input this expansion into the rank - 2 Killing tensor equation. The 0th term is unconstrained as it disappears and can be any constant rank-2 tensor , we can form Killing tensors such as these from symmetrizing the D translation Killing vectors with each other. The first term that is linear in x^μ gets a constraint:

$$K^{(1)}_{\nu\rho\mu} + K^{(1)}_{\mu\nu\rho} + K^{(1)}_{\rho\mu\nu} = K^{(1)}_{(\mu\nu\rho)} = 0$$

This term contains the Killing tensor that can be formed by symmetrizing a translation Killing veector with a rotation vector, then the coefficients would be linear in the co-ordinates similar to the term $K^{(1)}_{\mu\nu\rho}$. Then looking at the term with (2) superscript which is quadratic in the co-ordinates receives a similar constraint, however we have an extra dummy index that contracts the surviving co-ordinate:

$$\left[K^{(2)}_{\nu\rho\mu\alpha_1} + K^{(2)}_{\mu\nu\rho\alpha_1} + K^{(2)}_{\rho\mu\nu\alpha_1}\right]x^{\alpha_1} = K^{(2)}_{(\mu\nu\rho)\alpha_1}x^{\alpha_1} = 0$$

We get the above equation by using the product rule,then using that indices in positions (1,2) and (3,4) are symmetric under exchange, and then renaming the dummy indices. This term should contain the symmetrization of rotation K.v with rotation K.v as they yield the quadratic polynomial terms this tensor is contracted with.

Then inputting the 3rd term in yields the following after using the product rule and renaming indices:

$$\begin{aligned}\left[K^{(3)}_{\nu\rho\mu\alpha_1\alpha_2}+K^{(3)}_{\mu\nu\rho\alpha_1\alpha_2}+K^{(3)}_{\rho\mu\nu\alpha_1\alpha_2}\right]x^{\alpha_1}x^{\alpha_2}=K^{(3)}_{(\mu\nu\rho)\alpha_1\alpha_2}x^{\alpha_1}x^{\alpha_2}=0\\ \rightarrow K^{(3)}_{\nu\rho\mu\alpha_1\alpha_2}+K^{(3)}_{\mu\nu\rho\alpha_1\alpha_2}+K^{(3)}_{\rho\mu\nu\alpha_1\alpha_2}=K_{(\mu\nu\rho)\alpha_1\alpha_2}=0\end{aligned} \tag{3.7}$$

If this is non-zero then it would correspond to a cubic polynomial Killing tensor which would be irreducible since we cannot form it from K.v. However we now show using this equation that it would in fact be 0, and as a consequence the power series terminates from the cubic terms and on wards. To show this we remind ourselves that we have the following symmetries in the indices of term 3:

$$\begin{aligned}K^{(3)}_{[\mu\nu]\rho\alpha_1\alpha_2}&=0\\ K^{(3)}_{\mu\nu\underline{\rho\alpha_1\alpha_2}}&-\text{Symmetric}\end{aligned}$$

The underlined indices are completely symmetric under exchange. Starting from eq.3.7 but rearranging the symmetric indices such that α_1 is in the third position. Then we use 2nd line of eq.3.7 to permute the first 3 indices :

$$\begin{aligned}K^{(3)}_{\mu\nu\alpha_1\rho\alpha_2}+K^{(3)}_{\rho\mu\alpha_1\nu\alpha_2}+K^{(3)}_{\nu\rho\alpha_1\mu\alpha_2}&=0\\ -K^{(3)}_{\alpha_1\mu\nu\rho\alpha_2}-K^{(3)}_{\nu\alpha_1\mu\rho\alpha_2}-K^{(3)}_{\alpha_1\rho\mu\nu\alpha_2}-K^{(3)}_{\mu\alpha_1\rho\nu\alpha_2}-K^{(3)}_{\alpha_1\nu\rho\mu\alpha_2}-K^{(3)}_{\rho\alpha_1\nu\mu\alpha_2}&=0\\ -2\left(K^{(3)}_{\alpha_1\mu\nu\rho\alpha_2}+K^{(3)}_{\alpha_1\nu\rho\mu\alpha_2}+K^{(3)}_{\alpha_1\rho\mu\nu\alpha_2}\right)&=0\end{aligned}$$

Where in the third lines we gathered all the similar terms together by just exchanging indices in the last 3 indices. From which we extract:

$$K^{(3)}_{\alpha_1\mu\nu\rho\alpha_2}+K^{(3)}_{\alpha_1\nu\rho\mu\alpha_2}+K^{(3)}_{\alpha_1\rho\mu\nu\alpha_2}=0 \tag{3.8}$$

Using eq.3.7 again we now permute the middle term with $\alpha_1\nu\rho\mu\alpha_2$ to see one of the permuted terms cancel out the $K^{(3)}_{\alpha_1\rho\mu\nu\alpha_2}$ term:

$$\begin{aligned}K^{(3)}_{\alpha_1\mu\nu\rho\alpha_2}-K^{(3)}_{\rho\alpha_1\nu\mu\alpha_2}-K^{(3)}_{\nu\rho\alpha_1\mu\alpha_2}+K^{(3)}_{\alpha_1\rho\mu\nu\alpha_2}&=0\\ K^{(3)}_{\alpha_1\mu\nu\rho\alpha_2}&=K^{(3)}_{\nu\rho\alpha_1\mu\alpha_2}\end{aligned}$$

$$K^{(3)}_{\alpha_1\mu\nu\rho\alpha_2}=K^{(3)}_{\nu\rho\alpha_1\mu\alpha_2} \tag{3.9}$$

Using symmetry in the first 2 indices we cancel out the third term with the permuted term. More importantly we extract eq.3.9 which states that we can switch the first two indices with the 3,4 indices. Beginning from eq.3.8 again we now rearrange such that the α_2 index is in the third position:

$$K^{(3)}_{\alpha_1\mu\alpha_2\nu\rho}+K^{(3)}_{\alpha_1\nu\alpha_2\rho\mu}+K^{(3)}_{\alpha_1\rho\alpha_2\mu\nu}=0$$

We permute the indices of all 3 terms so that the first two indices are α_1,α_2.

$$\begin{aligned}-K^{(3)}_{\alpha_2\alpha_1\mu\nu\rho}-K^{(3)}_{\mu\alpha_2\alpha_1\nu\rho}-K^{(3)}_{\alpha_2\alpha_1\nu\rho\mu}-K^{(3)}_{\nu\alpha_2\alpha_1\rho\mu}-K^{(3)}_{\alpha_2\alpha_1\rho\mu\nu}-K^{(3)}_{\rho\alpha_2\alpha_1\mu\nu}&=0\\ 3K^{(3)}_{\alpha_2\alpha_1\mu\nu\rho}+K^{(3)}_{\mu\alpha_2\alpha_1\nu\rho}+K^{(3)}_{\nu\alpha_2\alpha_1\rho\mu}+K^{(3)}_{\rho\alpha_2\alpha_1\mu\nu}&=0\end{aligned}$$

We almost have the result, now using eq.3.9 which is the indices switching identity onto the last three terms of the second line of the above expression, before we use it we use the symmetry in the last 3 indices to send α_1 completely to the right most position for all 3 terms.

$$3K^{(3)}_{\alpha_2\alpha_1\mu\nu\rho} + K^{(3)}_{\nu\rho\mu\alpha_2\alpha_1} + K^{(3)}_{\rho\mu\nu\alpha_2\alpha_1} + K^{(3)}_{\mu\nu\rho\alpha_2\alpha_1} = 0$$
$$3K^{(3)}_{\alpha_2\alpha_1\mu\nu\rho} + K^{(3)}_{(\mu\nu\rho)\alpha_2\alpha_1} = 0$$

So the collected second term disappears leaving:

$$K^{(3)}_{\alpha_1\alpha_2\mu\nu\rho} = 0 \tag{3.10}$$

As a result of this derivation there are no possible cubic polynomial rank-2 Killing tensor in flat space. As a result of the power series terminates as all other higher order tensor coefficients $K^i_{\mu\nu\alpha_1\ldots\alpha_i} = 0$ because they all have the same symmetry as $K^{(3)}_{\mu\nu\alpha_1\alpha_2\alpha_3}$ in the first 5 indices as a result the other extra indices can just be kept constant while one re does the derivation to show the coefficient is equal to 0. So our power series expansion becomes:

$$K^{\mu\nu} = K^{(0)\mu\nu} + K^{(1)\mu\nu}_{\alpha_1} x^{\alpha_1} + K^{(2)\mu\nu}_{\alpha_1\alpha_2} x^{\alpha_1} x^{\alpha_2} \tag{3.11}$$

Where we just raised the indices. Lets concretely find the Killing tensors for 3-dimensional Euclidean space, so $D = 3$. The Killing tensors are:

$$\begin{aligned} K^0 &= K^{(0)\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \\ K^1 &= K^{(1)\mu\nu}_{\alpha_1} x^{\alpha_1} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \\ K^2 &= K^{(2)\mu\nu}_{\alpha_1\alpha_2} x^{\alpha_1} x^{\alpha_2} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \end{aligned} \tag{3.12}$$

We expand these tensors out and then using the relations we found for the coefficients gather all like terms under one distinct coefficient, as a result this sum of tensor products gets partitioned into different groups which are the different Killing tensors. The first term has no constraints so it is just various partial derivative tensor products:

$$\frac{\partial}{\partial x^\mu} \odot \frac{\partial}{\partial x^\nu}$$

For $K^{(1)}$ we now have a constraint for the coefficient, namely:

$$K^{(1)}_{\nu\rho\mu} = -K^{(1)}_{\mu\nu\rho} - K^{(1)}_{\rho\mu\nu}$$

Expanding out $K^{(1)}$ we get various tangent vector products with polynomial coefficients in the cartesian co-ordinates, due to the symmetry in the first two indices and the constraint relation we gather all these terms under a few coefficients. Doing this generally by not using numbers but indices such as ρ, μ and ν we find the Killing tensor in a general form:

$$x^\rho \frac{\partial}{\partial x^\mu} \odot \frac{\partial}{\partial x^\nu} - x^\mu \frac{\partial}{\partial x^\rho} \odot \frac{\partial}{\partial x^\nu}$$

Which is simply a translational Killing vector $\frac{\partial}{\partial x^\nu}$ product $\odot$ with a rotational Killing vector $L_{\rho\mu} = x^\rho \frac{\partial}{\partial x^\mu} - x^\mu \frac{\partial}{\partial x^\rho}$. This is not necessarily interesting.

Now we look at the quadratic coefficient Killing tensor $K^{(2)}$ to see if we get anything irreducible, however repeating the same procedure as before we get the general form of the Killing tensor as:

$$x^\beta x^\rho \frac{\partial}{\partial x^\alpha} \odot \frac{\partial}{\partial x^\nu} - x^\alpha x^\rho \frac{\partial}{\partial x^\beta} \odot \frac{\partial}{\partial x^\nu} + x^\alpha x^\nu \frac{\partial}{\partial x^\beta} \odot \frac{\partial}{\partial x^\rho} - x^\nu x^\beta \frac{\partial}{\partial x^\alpha} \odot \frac{\partial}{\partial x^\rho}$$

When we factorise this we recognise this just as the $\odot$ product between two rotational Killing vectors : $L_{\beta\alpha} = x_\beta \frac{\partial}{\partial x^\alpha} - x_\alpha \frac{\partial}{\partial x^\beta}$ and $L_{\rho\nu} = x_\rho \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\rho}$. Which is also not interesting. So for rank-2 there are no irreducible Killing tensors. One could have guessed this by the fact that flat Euclidean space already possesses all possible isometries due to the virtue of being a maximally symmetric space. Now one can go further and investigate the rank-3 Killing tensors to see if the hidden symmetries are hidden away in higher rank Killing tensors. However it can be shown for rank-3 the polynomial coefficient cannot be higher than a cubic, the proof is shown in the appendix as it is tedious but the derivation is similar to the rank-2 case. We expand this Killing tensor in a power series then using the rank-3 Killing tensor equation find relations between the coefficients, using these relations we can show the constant tensor associated with the cubic polynomial is 0 which terminates the series expansion at cubic coefficients. So for now it seems as there are no irreducible (rank-2) Killing tensor in flat space, now lets see if this statement is true in a slightly more complicated metric, the 2-sphere.

3.1.2 Rank-2 Killing tensors on the 2-sphere

Now we have to deal with covariant derivatives and Christoffel symbols which will complicate the coupled differential equations we have to solve in (ϕ, θ) co-ordinates. However we have seen how the complex stereo graphic projection in $(Z, \bar{Z})$ co-ordinates simplifies the calculations and the vector fields. So we use the projection to solve the rank-2 Killing's equation. The rank-2 Killing equation on curved space is :

$$D_\mu K_{\nu\rho} + D_\rho K_{\mu\nu} + D_\nu K_{\rho\mu} = 0$$

Decomposed into Christoffel symbols and using the symmetry $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ it becomes:

$$\partial_\mu K_{\nu\rho} + \partial_\rho K_{\mu\nu} + \partial_\nu K_{\rho\mu} - 2\left(\Gamma^\alpha_{\mu\nu} K_{\alpha\rho} + \Gamma^\alpha_{\mu\rho} K_{\alpha\nu} + \Gamma^\alpha_{\rho\nu} K_{\mu\alpha} = 0\right)$$

Remembering the complex stereo graphic projection changes the metric to the form:

$$ds^2 = \frac{4}{(Z\bar{Z}+1)^2} dZ d\bar{Z}$$

with the non-zero Christoffel symbols being:

$$\Gamma^Z_{ZZ} = \frac{-2\bar{Z}}{Z\bar{Z}+1}$$
$$\Gamma^{\bar{Z}}_{\bar{Z}\bar{Z}} = \frac{-2Z}{Z\bar{Z}+1}$$

Since we only have two co-ordinates $(Z, \bar{Z})$ the number of Killing field components we have to solve are 3: $K_{Z\bar{Z}}, K_{\bar{Z}\bar{Z}}, K_{ZZ}$ since $K_{Z\bar{Z}} = K_{\bar{Z}Z}$ is symmetric. We now find all the equations we have to solve,starting with all the same co-ordinate $\mu = \rho = \nu = Z$:

$$D_Z K_{ZZ} = 0$$
$$D_Z K^{\bar{Z}\bar{Z}} = 0 \rightarrow \partial_Z K^{\bar{Z}\bar{Z}} = 0$$

Where we factorized out $g^2_{Z\bar{Z}}$ out from the Killing tensor field so that the indices are raised. Likewise for $\mu = \rho = \nu = \bar{Z}$:

$$D_{\bar{Z}} K_{\bar{Z}\bar{Z}} = 0$$
$$D_{\bar{Z}} K^{ZZ} = 0 \rightarrow \partial_{\bar{Z}} K^{ZZ} = 0$$

Hence the Killing field component $K^{\bar{Z}\bar{Z}}$and K^{ZZ} are purely holomorphic or anti-holomorphic. Similar to the case of the Killing vector we can expand these as a power series in their respective co-ordinates. However we do have to remember to institute the global consistency condition, this will be done after deriving the Killing tensors. So $K^{ZZ}, K^{\bar{Z}\bar{Z}}$ are:

$$
\begin{aligned}
K^{ZZ} &= \sum_{a=0}^{\infty} c_a Z^a \\
K^{\bar{Z}\bar{Z}} &= \sum_{a=0}^{\infty} d_a \bar{Z}^a
\end{aligned}
\tag{3.13}
$$

Where c_a, d_a are complex coefficients of the power series. The global constraint for the 2-sphere 2.2.4 has to be instituted so we retain the tensor fields which do not have poles in either complex patch, for K^{ZZ}:

$$
K^{ZZ}\frac{\partial}{\partial Z}\otimes\frac{\partial}{\partial Z} = \sum_{a=0}^{\infty} c_a Z^a \frac{\partial}{\partial Z}\otimes\frac{\partial}{\partial Z}
$$

This tensor field has to be pole-less under change of co-ordinates to the W complex patch projected from the South pole: $Z = \frac{1}{W}$:

$$
\sum_{a=0}^{\infty} c_a Z^a \left(\frac{1}{Z^4}\frac{\partial}{\partial W}\otimes\frac{\partial}{\partial W}\right) = \sum_{a=0}^{\infty} c_a \left((\frac{1}{W})^{a-4}\right)
$$

Similar to the Killing vector case we now have a bigger power in the denominator, this allows more terms to valid. The tensors should not have poles as $\to 0$. Meaning only $a = 0, 1, 2, 3, 4$ are valid. Repeating this argument for the anti-holomorphic tensor field we find the 10 valid tensor fields:

$$
K^{ZZ} = c_0 + c_1 Z + c_2 Z^2 + c_3 Z^3 + c_4 Z^4 \tag{3.14}
$$

$$
K^{\bar{Z}\bar{Z}} = d_0 + d_1 \bar{Z} + d_2 \bar{Z}^2 + d_3 \bar{Z}^3 + d_4 \bar{Z}^4 \tag{3.15}
$$

Now we just need to find the $K^{Z\bar{Z}}$ tensor field. The associated equations for this tensor are when $\mu = \nu = Z$ and $\rho = \bar{Z}$ and the opposite $\mu = \nu = \bar{Z}$ and $\rho = Z$:, then 3.2 becomes two equations with similar structure:

$$
\begin{aligned}
2D_Z K_{Z\bar{Z}} + D_{\bar{Z}} K_{ZZ} = 0 \to 2D_Z K^{\bar{Z}Z} + D_{\bar{Z}} K^{\bar{Z}\bar{Z}} &= 0 \\
2D_{\bar{Z}} K^{Z\bar{Z}} + D_Z K^{ZZ} &= 0
\end{aligned}
\tag{3.16}
$$

$K^{Z\bar{Z}}$ is a function of both co-ordinates but it is connected to both K^{ZZ} and $K^{\bar{Z}\bar{Z}}$ through these equations, as a result we can expect the final result to connect the complex coefficients c_a and d_a so that our Killing tensors are composed of both holomorphic and anti-holomorphic parts. Expanding the covariant derivatives into partial derivatives and Christoffel symbols the two equations become:

$$
2(Z\bar{Z}+1)\frac{\partial K^{\bar{Z}Z}}{\partial Z} - 4\bar{Z}K^{\bar{Z}Z} = -(Z\bar{Z}+1)\frac{\partial K^{\bar{Z}\bar{Z}}}{\partial \bar{Z}} + 4ZK^{\bar{Z}\bar{Z}} \tag{3.17}
$$

$$
2(Z\bar{Z}+1)\frac{\partial K^{\bar{Z}Z}}{\partial \bar{Z}} - 4ZK^{\bar{Z}Z} = -(Z\bar{Z}+1)\frac{\partial K^{ZZ}}{\partial Z} + 4\bar{Z}K^{ZZ} \tag{3.18}
$$

The factor of $Z\bar{Z}+1$ present in both sides comes from the denominator of the Christoffel symbols which we multiply out. Since $K^{\bar{Z}\bar{Z}}$ and K^{ZZ} are both functions which can be expressed simple polynomials we assume $K^{\bar{Z}Z}$ should be a function of $(Z, \bar{Z})$ in simple polynomials. So we expand out $K^{Z\bar{Z}} = f(Z, \bar{Z})$ in a multivariate power series, we rename $K^{Z\bar{Z}}$ for ease:

$$K^{Z\bar{Z}} = f(Z, \bar{Z}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{i,j} Z^i \bar{Z}^j \tag{3.19}$$

Where the terms $b_{i,j}$ are constant complex coefficients for each distinct power of $Z^i\bar{Z}^j$.We input this series into both equations above and then try to match both the left hand side and the right hand side in powers of $Z^i\bar{Z}^j$ which leads to relations between the different coefficients. Starting with the second equation of eq.3.16:

$$2(Z\bar{Z}+1)\frac{\partial K^{\bar{Z}Z}}{\partial \bar{Z}} - 4ZK^{\bar{Z}Z} = -(Z\bar{Z}+1)\frac{\partial K^{ZZ}}{\partial Z} + 4\bar{Z}K^{ZZ}$$

We insert in the expansion term by term starting with $\frac{\partial K^{\bar{Z}Z}}{\partial \bar{Z}}$:

$$\frac{\partial f}{\partial \bar{Z}} = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} j(b_{ij} Z^i \bar{Z}^{j-1}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{i,j+1}(j+1) Z^i \bar{Z}^j$$

Where we rename and shift the indices in the summation such i, j still start at 0. Then we multiply the coefficient $2(1 + Z\bar{Z})$ to the derivative:

$$2(Z\bar{Z}+1)\frac{\partial K^{\bar{Z}Z}}{\partial \bar{Z}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 2(j+1)\left[b_{i,j+1} Z^{i+1}\bar{Z}^{j+1} + b_{i,j+1} Z^i \bar{Z}^j\right]$$

Then the $4Zf(Z, \bar{Z})$ term:

$$4Zf(Z, \bar{Z}) = 4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{i,j} Z^{i+1}\bar{Z}^j$$

Both these terms represent the L.H.S of the second equation in 3.18. The right hand side is slightly simpler. Starting with $\frac{\partial K^{ZZ}}{\partial Z}$, inputting the holomorphic power series expansion:

$$\frac{\partial K^{ZZ}}{\partial Z} = \sum_{a=0}^{\infty}(a+1)c_{a+1}Z^a$$

Which becomes the following after multiplying the $(Z\bar{Z}+1)$ factor:

$$(Z\bar{Z}+1)\frac{\partial K^{ZZ}}{\partial Z} = \sum_{a=0}^{\infty}(a+1)c_{a+1}\left[Z^{a+1}\bar{Z} + Z^a\right]$$

finally the $4\bar{Z}K^{ZZ}$ term becomes:

$$4\bar{Z}K^{ZZ} = \sum_{a=0}^{\infty} 4c_a \bar{Z} Z^a$$

We now have the right hand side of 3.18. Equating the left and right hand sides results in:

$$\sum_{i,j=0}^{\infty}\left(2(j+1)b_{i,j+1}\left[Z^{i+1}\bar{Z}^{j+1} + Z^i\bar{Z}^j\right] - 4b_{i,j}Z^{i+1}\bar{Z}^j\right) = \sum_{a=0}^{\infty}\left(-c_{a+1}(a+1)\left[Z^{a+1}\bar{Z} + Z^a\right] + 4c_a\bar{Z}Z^a\right) \tag{3.20}$$

Power	L.H.S index	R.H.S index	Equation
constant	$i = j = 0$	$a = 0$	$b_{0,1} = -\frac{c_1}{2}$
Z	$(i = 1, j = 0), (i = 0, j = 0)$	$a = 1$	$c_2 = 2b_{0,0} - b_{1,1}$
$\bar{Z}$	$i = 0, j = 1$	$a = 0$	$b_{0,2} = c_0$
Z^2	$(i = 2, j = 0), (i = 1, j = 0)$	$a = 2$	$-3c_3 = 2b_{2,1} - 4b_{1,0}$
$Z\bar{Z}$	$(i = j = 0), (i = j = 1), (i = 0, j = 1)$	$a = 0, a = 1$	$c_1 = 2b_{1,2}$
$\bar{Z}^2$	$(i = 0, j = 2)$	None	$b_{0,3} = 0$
Z^3	$(i = 3, j = 0), (i = 2, j = 0)$	$a = 3$	$2c_4 = -b_{3,1} + 2b_{2,0}$
$Z^2\bar{Z}$	$(i = 1, j = 0), (i = 2, j = 1), (i = j = 1)$	$a = 1, a = 2$	$c_2 = 2b_{2,2} - b_{1,1} \rightarrow b_{0,0} = b_{2,2}$
$Z\bar{Z}^2$	$(i = 0, j = 1), (i = 1, j = 2), (i = 0, j = 2)$	None	$b_{1,3} = 0$
$\bar{Z}^3$	$i = 0, j = 3$	None	$b_{0,4} = 0$
Z^4	$(i = 4, j = 0), (i = 3, j = 0)$	$a = 4$	$-5c_5 = 2b_{4,1} - 4b_{3,0}$
$Z^3\bar{Z}$	$(i = 2, j = 0), (i = 3, j = 1), (i = 2, j = 1)$	$a = 2, a = 3$	$c_3 = -2b_{2,1} + 4b_{3,2}$
$Z^2\bar{Z}^2$	$(i = j = 1), (i = j = 2), (i = 1, j = 2)$	None	$b_{2,3} = 0$
$Z\bar{Z}^3$	$(i = 0, j = 2), (i = 1, j = 3), (i = 0, j = 3)$	None	$b_{1,4} = 0$
$\bar{Z}^4$	$(i = 0, j = 4)$	None	$b_{0,5} = 0$

Table 3.1: Depicts all the equations that are derived by matching powers of $Z^i\bar{Z}^j$ on both sides of eq.3.20 . Also shows the relevant values of indices which generate the power

Immediately one can see several coefficients of specific i, j powers will automatically be set to 0 since the same power is absent on the R.H.S since we only get increasing powers in Z.Now we expand this equation out on both sides and match the various powers of $Z^i\bar{Z}^j$ which yields a system of equations: for $a = 0, i = j = 0$ which are the constant terms by matching we get:

$$b_{0,1} = -\frac{c_1}{2}$$

For power of Z:R.H.S :$a = 1$ $i = 1, j = 0$ and $i = j = 0$ for L.H.S and $a = 0$for R.H.S:

$$c_2 = 2b_{0,0} - b_{1,1}$$

For $\bar{Z}$:R.H.S: $a = 0$: $i = 0, j = 1$:

$$b_{0,2} = c_0$$

and onwards to higher powers until Z^4 and $\bar{Z}^4$, we summarise these equations in a table: Table.2 shows how the coefficients of the K^{ZZ} and $K^{\bar{Z}Z}$ are matched. Using the information from table.2 we can simplify our relations to.

$$\begin{aligned}
b_{0,1} &= -\frac{c_1}{2} \\
c_2 &= 2b_{0,0} - b_{1,1} \\
c_2 = 2b_{2,2} - b_{1,1} b_{0,2} &= c_0 \\
-3c_3 &= -2b_{2,1} - 4b_{10} \\
c_3 &= -2b_{2,1} + 4b_{3,2} \\
b_{1,2} = \frac{1}{2}c_1 &= -b_{0,1} \\
b_{2,0} &= c_4 \\
b_{0,0} &= b_{2,2} \\
b_{0,3} &= 0 \\
b_{1,0} &= -b_{2,1} \\
b_{1,3} = b_{0,4} = b_{2,3} = b_{1,4} &= 0
\end{aligned} \tag{3.21}$$

Where b_{00} is the constant which is unconstrained other than the fact it is related to c_2. We still have to consider the other equation, we then repeat the same entire process for the first equation in 3.18 and using the expansion for $K^{\bar{Z}\bar{Z}}$ we find the form of the first equation in 3.18 as:

$$\sum_{k,j=0}^{\infty}\left(2(k+1)b_{k+1,j}\left[Z^{k+1}\bar{Z}^{j+1}+Z^{k}\bar{Z}^{j}\right]-4b_{k,j}Z^{k}\bar{Z}^{j+1}\right)=\sum_{a=0}^{\infty}\left(-d_{a+1}(a+1)\left[Z\bar{Z}^{a+1}+\bar{Z}^{a}\right]4d_{a}Z\bar{Z}^{a}\right) \tag{3.22}$$

Which we notice is the exact same form as equation as 3.20 except we switch $K^{ZZ}\to K^{\bar{Z}\bar{Z}}$ such that the coefficients change $c_i\to d_i$ and we switch the positions of the indices in $b_{ij}\to b_{ji}$, and importantly the co-ordinates switch from $Z\leftrightarrow\bar{Z}$. So we get the exact same equations as before but with the aforementioned changes, namely $c_i\to d_i$ and $b_{ij}\to b_{ji}$, using these rules we write down the equations between coefficients:

$$\begin{aligned}
b_{1,0}&=-\frac{d_1}{2}\\
d_2&=2b_{0,0}-b_{1,1}\\
d_2&=2b_{2,2}-b_{1,1}\\
b_{2,0}&=d_0=c_4\\
-3d_3&=2b_{1,2}-4b_{10}\\
d_3&=-2b_{1,2}+4b_{2,3}\\
b_{2,1}&=\frac{1}{2}d_1=-b_{1,0}\\
b_{0,0}&=b_{2,2}\\
b_{3,0}&=0\\
b_{0,1}&=-b_{1,2}\\
b_{3,1}=b_{4,0}=b_{3,2}=b_{4,1}&=0
\end{aligned} \tag{3.23}$$

From these set of equations we can combine the previous set of equations to find the following results between coefficients:

$$\begin{aligned}
b_{2,0}&=d_0=c_4\\
b_{0,2}&=c_0=d_4\\
c_3=-2b_{2,1}&=2b_{1,0}=-d_1\\
d_3=-2b_{1,2}&=2b_{0,1}=-c_1\\
c_2=d_2&=2b_{0,0}-b_{1,1}
\end{aligned} \tag{3.24}$$

All other $b_{i,j}$ are set to 0 due to the consistency condition and the fact that $K^{ZZ},K^{\bar{Z}\bar{Z}}$ are also truncated to quartic terms. From the last equation and using $b_{1,1}=2b_{2,2}-d_2$we can express $f(Z,\bar{Z})$ as:

$$f(Z,\bar{Z})=b_{0,0}+b_{0,1}\bar{Z}+b_{1,0}Z+...=\frac{b_{11}}{2}(1+2Z\bar{Z}+Z^2\bar{Z}^2)+\frac{d_2}{2}(1+Z^2\bar{Z}^2)+...$$

The $b_{1,1}$ coefficient corresponds to the metric of the 2-sphere in raised indices: $g^{Z\bar{Z}}=\frac{1+2Z\bar{Z}+Z^2\bar{Z}^2}{2}$. Now we can expand out the Killing tensor as before in flat space by contracting them with the $\partial_Z,\partial_{\bar{Z}}$ basis and using the relations between coefficients to gather all the terms under the same coefficient:

$$
\begin{aligned}
&K^{\mu\nu}\left(\frac{\partial}{\partial Z^{\mu}}\otimes\frac{\partial}{\partial Z^{\nu}}\right) \\
&= c_0\left(\frac{\partial}{\partial Z}\otimes\frac{\partial}{\partial Z}+\bar{Z}^4\frac{\partial}{\partial \bar{Z}}\otimes\frac{\partial}{\partial \bar{Z}}+2\bar{Z}^2\frac{\partial}{\partial \bar{Z}}\odot\frac{\partial}{\partial Z}\right)+d_3\left(-Z\frac{\partial}{\partial Z}\otimes\frac{\partial}{\partial Z}+\bar{Z}^3\frac{\partial}{\partial \bar{Z}}\otimes\frac{\partial}{\partial \bar{Z}}+(\bar{Z}-\bar{Z}^2Z)\frac{\partial}{\partial \bar{Z}}\odot\frac{\partial}{\partial Z}\right) \\
&+c_3\left(Z^3\frac{\partial}{\partial Z}\otimes\frac{\partial}{\partial Z}-\bar{Z}\frac{\partial}{\partial \bar{Z}}\otimes\frac{\partial}{\partial \bar{Z}}+(Z-\bar{Z}Z^2)\frac{\partial}{\partial \bar{Z}}\odot\frac{\partial}{\partial Z}\right)+c_4\left(Z^4\frac{\partial}{\partial Z}\otimes\frac{\partial}{\partial Z}+\frac{\partial}{\partial \bar{Z}}\otimes\frac{\partial}{\partial \bar{Z}}+2Z^2\frac{\partial}{\partial \bar{Z}}\odot\frac{\partial}{\partial Z}\right) \\
&c_2\left(Z^2\frac{\partial}{\partial Z}\otimes\frac{\partial}{\partial Z}+\bar{Z}^2\frac{\partial}{\partial \bar{Z}}\otimes\frac{\partial}{\partial \bar{Z}}+(1+\bar{Z}^2Z^2)\frac{\partial}{\partial \bar{Z}}\odot\frac{\partial}{\partial Z}\right)+b_{11}\left((1+2Z\bar{Z}+Z^2\bar{Z}^2)\frac{\partial}{\partial \bar{Z}}\otimes\frac{\partial}{\partial \bar{Z}}\right)
\end{aligned}
\tag{3.25}
$$

where $Z^{\mu}=(Z,\bar{Z})$ and we are using the symmetrised tensor dot product notation: $A\odot B=\frac{1}{2}(A\otimes B+B\otimes A)$. b_{11} coefficient corresponds to the metric of the 2-sphere. Furthermore the last line could be expressed in a different way by using $b_{1,1}=b_{2,2}-d_2$ which yields:

$$
d_2\left(Z^2\frac{\partial}{\partial Z}\otimes\frac{\partial}{\partial Z}+\bar{Z}^2\frac{\partial}{\partial \bar{Z}}\otimes\frac{\partial}{\partial \bar{Z}}-2(Z\bar{Z})\frac{\partial}{\partial \bar{Z}}\odot\frac{\partial}{\partial Z}\right)+2b_{22}\left((1+2Z\bar{Z}+Z^2\bar{Z}^2)\frac{\partial}{\partial \bar{Z}}\otimes\frac{\partial}{\partial \bar{Z}}\right)
$$

Which yield all possible rank-2 Killing tensors on the 2-sphere. From looking at the 2-sphere Killing vectors 2.99 we see that every Killing tensor derived could easily be constructed from the Killing vectors. Using notation from 2.99:

$$
\begin{aligned}
\hat{C}_2\odot\hat{C}_2 &\to c_4\\
\hat{C}_1\odot\hat{C}_1 &\to d_2\\
\hat{C}_0\odot\hat{C}_0 &\to c_0\\
\hat{C}_2\odot\hat{C}_1 &\to c_3\\
\hat{C}_2\odot\hat{C}_0 &\to c_2\\
\hat{C}_1\odot\hat{C}_2 &\to d_3
\end{aligned}
$$

And finally b_{11} is simply the metric which could be constructed out of the Killing vector products. Hence, we see that there are no irreducible rank-2 Killing tensors on the 2-sphere. There are no hidden symmetries present on the 2-sphere. And it seems very unlikely that there are potential irreducible Killing tensor in higher orders. We could have expected this fact as well, as we know that the 2-sphere is a maximally symmetric space with constant curvatuure, so it is similar to flat space in that it contains the maximal set of isometries.

Now we have derived rank-2 Killing tensors for two simple metrics and found out there are no irreducible Killing tensor. One may wonder if this is a property of spaces such as these that they possess no irreducible Killing tensors. It turns out there is a proof that states spaces of constant curvature lack irreducible Killing tensors by G.Thompson in [12]. The paper proves for flat space that there exists no Killing tensor that cannot be decomposed into symmetrized products of Killing vectors. Meaning any rank Killing tensor can be decomposed into symmetric Killing vector products, the proof is indirectly extended to Killing tensors on spaces of constant curvature such as the 2-sphere. However physically interesting spaces such as the Kerr black hole metric does in fact contain a rank-2 Killing tensor which forms the Carter constant which allows for integrability around black holes. We will look at this constant and also use it to calculate scattering angles around the Kerr black hole. This concludes the brief discuss on rank-2 Killing tensors. Now we briefly review and introduce Hamiltonian mechanic to curved spaces. This will provide a background to how Killing tensors appear when considering the dynamics of particles in curved spaces.

3.2 Hamiltonian Mechanics and Hamilton-Jacobi theory

Starting with the familiar Lagrangian approach we define the Action of a massive free particle in curved space with metric $g_{\mu\nu}$ between two proper time points τ_1, τ_2 as:

$$\mathcal{S} = \frac{1}{2}\int_{\tau_1}^{\tau_2} \mathcal{L} d\tau = \frac{1}{2}\int_{\tau_1}^{\tau_2} g_{\mu\nu} U^\mu U^\nu d\tau \tag{3.26}$$

Where the Lagrangian $\mathcal{L}$ (Not lie derivative) is identified as:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} U^\mu U^\nu \tag{3.27}$$

Following the derivation laid out in [14, 15] we vary the path inside the action and demand this difference $\delta\mathcal{S} = 0$, this is how Euler-Lagrange equations are derived by finding constraints such that $\delta\mathcal{S} = 0$. For our massive free particle we know it obeys the geodesic equation $\frac{DU^\nu}{d\tau} = 0$ (remembering $U^\mu = \frac{dx^\mu}{d\tau}$)which obeys the Euler-Lagrange equations, so varying the action should produce the geodesic equation inside the integral along with any other boundary terms that vanish:

$$\delta\mathcal{S} = \int_{\tau_1}^{\tau_2} \left(-\delta x^\mu g_{\mu\nu} \frac{DU^\nu}{D\tau} + \frac{d}{d\tau}(\delta x^\mu p_m) \right) d\tau$$

where $p_\mu = g_{\mu\nu} U^\nu$. The proper derivation requires using integration by parts and etc, it is a very standard derivation in text books such as page 107 of [1] (although he drops the 2nd term due to boundary conditions), so we will not recreate the total derivation. The paper referenced derives it by using Euler-Lagrange equations but both derivations in text book or the paper are similar. From this we can set $\delta\mathcal{S} = 0$ by instituting the geodesic equation for the first term and for the second term $\frac{d}{d\tau}(\delta x^\mu p_m)$ by first integrating it and using that we choose our deviation such that $\delta x^\mu = 0$ at the boundary points τ_1, τ_2. This is a standard way to derive the geodesic equation. However, for our purposes we consider what happens if the boundary condition does not automatically set the 2nd term to 0. In this case let $\delta x^\mu = K^\mu$ which implies we make a co-ordinate transformation where the difference is this vector field K^μ, then we have:

$$\delta\mathcal{S} = \frac{1}{2}\int_{\tau_1}^{\tau_2} \frac{d}{d\tau}(K^\mu p_m) d\tau$$

Which is non-zero at boundary points τ_1, τ_2.This can still be equal to 0 if the remaining term is equal to 0. We actually recognise this to be 0 if K^μ is a Killing vector as this what we derived in chapter 1 when we proved this quantity is conserved on geodesics. We can generalise and say if we expandδx^μ in a series $\delta x^\mu = K^\mu_{(0)} + K^{\mu\nu_1}_{(1)} p_{\nu_1} + K^{\mu\nu_1\nu_2}_{(2)} p_{\nu_1} p_{\nu_2} + ...$. All of these satisfy $\delta\mathcal{S} = 0$ if $K_i^{\mu\nu_1 ... \nu_n}$ are Killing tensors obeying Killing tensor equation, as then they would correspond to Killing tensors contracted with velocity which we know are conserved quantities on geodesics. Now we see how Killing tensors arise naturally in the context of particle dynamics in curved spaces using Lagrangian's. Now we switch to introduce Hamilton mechanics onto curved spaces.

Using:

$$\mathcal{H} = U^\alpha \frac{\partial\mathcal{L}}{\partial U^\alpha} - \mathcal{L} = \frac{1}{2} g_{\mu\nu} U^\mu U^\nu = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \tag{3.28}$$

[16] The canonical momentum is defined as:

$$p_\mu = \frac{\partial\mathcal{L}}{\partial U^\mu} = g_{\mu\nu} U^\nu \tag{3.29}$$

Which is what we had before [17]. Now remembering the relevant Hamilton's equations for $\dot{p}_\mu$ and $\dot{x}^\mu$ from non-relativistic mechanics generalised to general relativity:

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial \mathcal{H}}{\partial p_\mu} = g^{\mu\nu} p_\nu \\ \dot{p}_\mu = -\frac{\partial \mathcal{H}}{\partial x^\mu} &= -\frac{1}{2}\frac{\partial g^{\alpha\nu}}{\partial x^\mu} p_\alpha p_\nu \end{aligned} \tag{3.30}$$

[17, 10]. The geodesic equation becomes in momenta:

$$p^\rho D_\rho p^\mu = 0 \tag{3.31}$$

using the canonical momentum definition. These are just U^μ but we express them in momenta. Our Hamiltonian is not time dependent as a result it is conserved, we let it equal to a constant value:

$$\mathcal{H} = -\frac{1}{2} m^2 \tag{3.32}$$

Where m is the mass of the particle. This leads to the relation:

$$g^{\mu\nu} p_\mu p_\nu = -m^2 \tag{3.33}$$

Which the reference states normalises the momenta. However, the other way to look at this is that we recall that the metric is a rank-2 Killing tensor. As a result for any space-time we already have a conserved quantity which is provided by the metric $g_{\mu\nu}$. So the above relation could be thought of as the first constant of motion in our system. Then with other Killing vectors and irreducible K. tensors we could constrain the particle dynamics more.

Now we will briefly introduce and derive the Hamilton-Jacobi equation through Canonical transformations. Hamilton-Jacobi has many uses in many areas of physics, especially in quantum mechanics as it provides a way of analysing semi-classical behaviour from quantum mechanics. In general relativity it is the standard tool used to study particle motion on manifolds with curvature. The Hamilton-Jacobi equation was used by Brandon Carter in 1967 to discover the Carter constant for the Kerr metric which originates from an irreducible rank-2 Killing tensor. We now reproduce a derivation laid out by Herbert Goldstein in his book Classical Mechanics 3rd edition [17], the derivation is classical but can be easily generalised to curved space.

If an arbitrary Hamiltonian $\mathcal{H}(x^\mu, p_\mu)$ has a specific co-ordinate absent say x^i in the Hamiltonian. Then the associated canonical momentum is conserved:

$$-\dot{p}_i = \frac{\partial \mathcal{H}}{\partial x^i} = 0 \rightarrow p_i = const.$$

Then the co-ordinate x^i is called a cyclic co-ordinate [17]. This is analogous to the Killing vector $\frac{\partial}{\partial x^i}$ is present when the co-ordinate x^i is not present in the metric. Casting the form in this form where co-ordinates are cyclic is very beneficial as it reduces a lot of calculations. The form of the Hamiltonian depends on which co-ordinate we choose to cast it in. A concrete example is the classical central potential problem, when the Hamiltonian is expressed in spherical co-ordinates r, ϕ it is cyclic in ϕ, as opposed to expressing in Cartesian's x, y.

It is useful to find the right co-ordinate transform that will make some co-ordinate in the Hamiltonian and also ensures the Hamiltonian equations are preserved under this co-ordinate transform, these transformations are called Canonical transformations. We introduce this in the context of classical mechanics and then generalise the final result. Starting from q_i, p_i which are classical generalised co-ordinates for momentum p_i and co-ordinates q_i. These co-ordinates obey the Hamilton's equation for the Hamiltonian H .We make a transformation to a new set of co-ordinates Q_i and momenta P_i which are functions of p_i, q_i:

$$\begin{aligned} Q_i &= Q_i(p, q, t) \\ P_i &= P_i(q, p, t) \end{aligned}$$

Which are transformations in phase-space. Require these new co-ordinates to be canonical such that there is another function $K(Q,P,t)$ which acts as a Hamiltonian:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}$$
$$\dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

Both K and H are Hamiltonians that are derived from their Lagrangians using the Legendre transformations. As a result both of them obey the following:

$$\delta \int_{t_1}^{t_2} L(q,p,t)dt = \delta \int_{t_1}^{t_2} (p_i\dot{q}_i - H(q,p,t))dt = 0$$
$$\delta \int_{t_1}^{t_2} L(Q,P,t)dt = \delta \int_{t_1}^{t_2} (P_i\dot{Q}_i - H(Q,P,t))dt = 0$$

Where t_1, t_2 are the classical boundary points of the variation .Which is satisfied when:

$$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF}{dt} \tag{3.34}$$

The function F is dependent on both the old and new canonical co-ordinates,it is a term that originates due to the fact we could have a term present that disappears due to boundary conditions during the variation. This called a generating function and it acts as " bridge" between old and new canonical co-ordinates as given in page 371 of [17]. F can take 4 different combinations of old and new co-ordinates, but for our purposes we choose the so called type-2 function where $F_2(q,P,t)$is a function of old co-ordinates q and new momenta P. Furthermore expressing $F = F_2(q,P,t) - Q_iP_i$ and submitting this into eq.3.34 yields:

$$p_i\dot{q}_i - H = -Q_i\dot{P}_i - K + \frac{dF_2}{dt}$$

Now $F_2(q,P,t)$ is expanded out in chain rule in terms of its arguments:

$$\frac{dF_2(q,P,t)}{dt} = \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i$$

We input this into the previous equation and see that it is satisfied when:

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} \\ Q_i &= \frac{\partial F_2}{\partial P_i} \end{aligned} \tag{3.35}$$

So now the relation becomes :

$$K = H + \frac{\partial F_2}{\partial t} \tag{3.36}$$

We will now use this Canonical transformation to derive Hamilton-Jacobi equation. For a system with n momenta p_i and n co-ordinates q_i, and if the Hamiltonian is conserved we could make a canonical transformation to co-ordinates for co-ordinates (p,q) to their initial values at a certain time q_0, p_0 at $t = 0$. This simply yields the solution of the system:

$$\begin{aligned} q &= q(q_0,p_0,t) \\ p &= p(q_0,p_0,t) \end{aligned} \tag{3.37}$$

The new Hamiltonian $K = 0$ such that the new momenta are constants as we expect:

$$\begin{aligned} \frac{\partial K}{\partial P_i} &= \dot{Q}_i = 0 \\ -\frac{\partial K}{\partial Q_i} &= \dot{P}_i = 0 \to P_i = \alpha_i \end{aligned} \tag{3.38}$$

Where P_i, Q_i are the new constant momenta and co-ordinates. And α_i is a constant. Then the equation linking K and H together becomes:

$$0 = H + \frac{\partial F_2}{\partial t} \tag{3.39}$$

Then using the fact $p_i = \frac{\partial F_2}{\partial q_i}$ to change the old co-ordinates and then rename $F_2 = S$ we get Hamilton-Jacobi equation:

$$H(q_1, ..., q_n; \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}; t) + \frac{\partial S}{\partial t} = 0 \tag{3.40}$$

Where we now refer to S as Hamilton's principal function. If our Hamiltonian is not explicitly dependent on time then we split the principal function in two components $S = W(q, \alpha) - Et$ [17]. Where E is a constant. The function $W(q, \alpha)$ is called Hamilton's characteristic function. This leads to the time-independent Hamilton-Jacobi equation:

$$H(q_i, \frac{\partial W}{\partial q_i}) = \alpha_1 \tag{3.41}$$

With equations:

$$p_i = \frac{\partial W}{\partial q_i} \tag{3.42}$$

The usefulness of Hamilton-Jacobi theory occurs when we can consider separable solutions for S. If S can be split into a piece that is only dependent on one co-ordinate q_k and another piece which is completely absent of that co-ordinate then we get:

$$S(q_1, .., q_n; \alpha; t) = S_1(q_1; \alpha; t) + S'(q_2, ..., q_n; \alpha; t)$$

Which will split the Hamilton-Jacobi equation into 2 sub Hamilton-Jacobi equations, one with S_1 and another with S'. We can generalise this to say we have separability in all co-ordinates so that S becomes completely separable:

$$S = \Sigma_i S_i(q_i; \alpha; t) \tag{3.43}$$

Where the principal function splits into several sub functions where some of the sub functions are completely dependent on only one specific co-ordinate q_i. This splits the Hamilton-Jacobi into smaller H-J equation:

$$H_i(q_j, \frac{\partial S_j}{\partial q_j}; \alpha; t) = -\frac{\partial S_j}{\partial t} \tag{3.44}$$

Which are n Hamilton-Jacobi(H-J) equations for co-ordinates q_j. For the time dependent case which we usually deal with :

$$S_i(q_j; \alpha; t) = W_i(q_j; \alpha) - \alpha_i t$$

So the time independent H-J equations become:

$$H_i(q_i; \frac{\partial W_i}{\partial q_i}; \alpha) = \alpha_i \tag{3.45}$$

Where α_i are separation constants which could be constants of motion such as energy, momentum or angular momentum. As a result solving the system becomes very easy. Goldstein then states the solution is reduced to "quadratures". We are now done with Goldstein's derivation of Hamilton-Jacobi theory and its separability. Now we generalise this to curved spaces. Firstly the classical time parameter t becomes the proper time τ:

$$H(q^i; \frac{\partial S}{\partial q^i}, \tau) + \frac{\partial S}{\partial \tau} = 0 \tag{3.46}$$

Where q^i are the co-ordinates of the co-ordinates system being used to describe the metric and the momenta are $p_i = \frac{\partial S}{\partial q^i}$, and $S(q)$ is the principal function that is only dependent on co-ordinates q(we use q to represent all the q^i) we also suppress the dependence of S on α_i as they are just constants so we depict them purely as a function of co-ordinates. Our free particle Hamiltonians are time independent so we can split the principal function into the characteristic and time dependent part $S(q,\tau) = W(q) + \frac{1}{2}m^2\tau$. Concurrently the proper time τ independent equation becomes:

$$H(q, \frac{\partial W}{\partial q}) = -\frac{1}{2}m^2 \tag{3.47}$$

Where m is the mass of the particle, plus we drop the subscripts on p, q but these are still q_i, p_i. We can now consider separability if Hamilton-Jacobi equation in curved space. The necessary condition for separability for is the existence of a rank-2 Killing tensor.

The Hamiltonian for a free particle in curved space is given as before: $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = -\frac{1}{2}m^2$. Since $p_\mu = \frac{\partial W}{\partial q^\mu}$ the Hamilton-Jacobi equation can be written as:

$$g^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu} + m^2 = 0 \tag{3.48}$$

We will now see how a rank-2 Killing tensor is required for separability. There are also several external conditions on the metric and characteristic function $W(q)$ have to satisfy:

$$\begin{aligned} W(q^\mu) &= W_x(x^1, ..., x^k) + W_y(x^{k+1}, .., k^n) \\ g^{\mu\nu} &= \frac{X^{\mu\nu}(x) + Y^{\mu\nu}(y)}{f_x - f_y}, X^{y\nu} = 0, Y^{x\nu} = 0 \\ & \qquad\qquad\qquad\qquad \frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x} = 0 \end{aligned} \tag{3.49}$$

Firstly we split our co-ordinates q^μ into three sets x^i, y^a and cyclic co-ordinates z^c which are co-ordinates associated with Killing vectors, we ignore the cyclic co-ordinates. This leads to the first condition, where we can split the characteristic function into two functions that are only dependent on either co-ordinates x or y. Then the metric must also have a special separable form as shown in condition 2 where it can be expressed as a sum of two rank-2 tensors which are again only dependent on x or y and have a special structure. Then we have function f_x, f_y in the denominator which are functions of x and y respectively. Condition 2 with the special tensor structure on $X^{\mu\nu}$ and $Y^{\mu\nu}$ is required for the eq.3.48 to separate into condition 1 [18]. If our metric follows these conditions then we can cast eq.3.48 into the form:

$$X^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu} + m^2 f_x = -Y^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu} + m^2 f_y \tag{3.50}$$

The left hand side is purely a function of co-ordinates x^μ and the right hand side is purely a function of co-ordinates y^μ, so they must be equal to a constant of integration I:

$$I = X^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu} + m^2 f_x = -Y^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu} + m^2 f_y$$

then also using $m^2 = -g^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu}$

$$I = -\left(\frac{f_y X^{\mu\nu} + f_x Y^{\mu\nu}}{f_x - f_y}\right)\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu} = K^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu} \tag{3.51}$$

[19] Since we know a constant of motion must be sourced by a Killing tensor or a Killing vector contracted with momenta. In this case since the right hand side is composed of rank-2 tensors, I has to be sourced by a rank-2 Killing tensor $K^{\mu\nu}$.

If we know a rank-2 Killing tensor exists it can lead to separation of variables.

We see now how Killing tensors play a role in separating Hamilton-Jacobi equations, which significantly simplify our analyses of particle dynamics. Additionally, we could use irreducible Killing tensors to construct additional conserved quantities which may lead to integrability, we will see this explicitly for the Kerr metric in chapter 5. Before we end this chapter however, there is one more use of the rank-2 Killing tensors. Killing tensors also play a role in the separation of the Klein-Gordon equation.

Klein-Gordon equation separation conditions

The derivation followed is presented in detail for the reference [20]. The Klein-Gordon equation in curved space-time is defined as the following:

$$\Box\Phi(x) = g^{\mu\nu}D_\mu D_\nu \Phi(x) = m^2\Phi(x) \tag{3.52}$$

Where m is the mass of the particle and $\Box$ is the Klein-Gordon operator. The Klein-Gordon equation dictates the behaviour of a massive scalar field $\Phi(x)$. To solve this equation by separation of variables one requires the existence of a rank-2 irreducible Killing tensor $K^{\mu\nu}$. Then the Klein-Gordon equation is separable if:

$$D_\rho\,[R, K]^\rho_\mu = 0 \tag{3.53}$$

Where $R_{\mu\nu}$ is the Ricci tensor. As mentioned the in depth reason why this is needed is found in : [20]. As stated in the paper this condition is already satisfied in the Kerr, Kerr-Newman metrics.

So now we have seen how Killing tensors are used in cases of separability. This thesis only covers a small portion of this subject as the literature is vast and is still an active area of research. Having explored simple metrics such as the 2-sphere and flat space now we start to analyse physically interesting black hole metrics such as the Schwarzschild and the Kerr metrics. Before we end this chapter we briefly mention another generalisation of the Killing tensor. Namely, the conformal generalisation of the Killing tensor. Which is called the Conformal Killing tensor, these objects provide conserved quantities for massless particles, and also play a role in looking at symmetries in Laplacians and Dirac operator in curved space [19, 21].

Chapter 4

Schwarzschild Metric

4.1 Schwarzschild Metric

The Schwarzschild metric describes the space-time curvature around a spherically symmetric mass distribution centered at the origin of co-ordinate system in spherical co-ordinates [1]. The metric is defined as the following in spherical co-ordinates (t, r, θ, ϕ), where t is co-ordinate time. The polar angle ϕ measured between $0 \leq \theta \leq \pi$, azimuthal θ between $0 \leq \phi \leq 2\pi$ and radial co-ordinate r from $0 \leq r \leq \infty$.:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.1}$$

Where G is Newton's constant and M is the mass of the spherically symmetric body(in our case the black hole). Immediately we recognise the 2-sphere metric inside the Schwarzschild metric, this is the part that actually provides the spherical symmetry. Additionally, we see the co-ordinate singularities that arise at $r = 2GM$ and at $r = 0$. Where the g_{rr} and g_{tt} metric components diverge. The singularity at $r = 2GM$ denotes the event horizon radius, this is not a true geometric singularity but rather a consequence of the co-ordinate system. This can be remedied by using Finkelstein-Eddington co-ordinates or Painlevé-Gullstrand which remove this singularity [22]. There is however a true geometric singularity at $r = 0$ where the Kretschmann scalar: $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}$ diverges to infinity, the Kretschmann scalar describes the curvature as it is just the Riemann tensor "squared". This metric is a vacuum solution of Einsteins field equations The Schwarzschild metric describes a stationary non-spinning black hole, with spherical symmetry as we will see. A simple diagram of the geometry is given figure 4.1. It is also considered one of the most symmetric solutions, and we will see why by studying its Killing vectors. When Newton's constant $G \to 0$ the metric reduces to Minkowski space in spherical co-ordinates (r, θ, ϕ) where r is the radial co-ordinate, θ is the polar angle measured from the z-axis and ϕ is the azimuthal angle which is measured

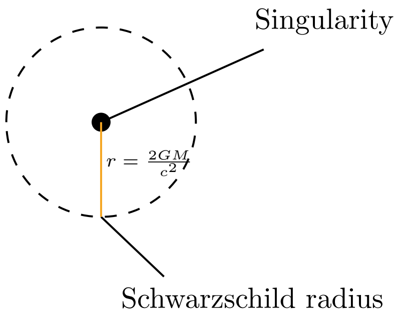


Figure 4.1: Cross-section of Schwarzschild geometry.

$\Gamma^t_{tr} = \frac{R_s}{2(r^2-R_s r)} = \Gamma^t_{rt}$	$\Gamma^r_{tt} = \frac{R_s(r-R_s)}{2r^3}$	$\Gamma^r_{rr} = \frac{R_s}{2(R_s r-r^2)}$
$\Gamma^r_{\theta\theta} = R_s - r$	$\Gamma^r_{\phi\phi} = (R_s - r)\sin^2(\theta)$	$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$
$\Gamma^\theta_{\phi\phi} = -\cos(\theta)\sin(\theta)$	$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot(\theta)$	$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}$

Table 4.1: Christoffel symbols for the Schwarzschild metric

counterclockwise from the x-axis. The metric becomes:

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{4.2}$$

this form of the metric is also acquired when $r \gg 1$, so the Schwarzschild metric is asymptotically flat.

4.1.1 Killing and Conformal Killing vectors of Schwarzschild

Immediately one can spot that this is a static metric in time due to the lack of t in the metric. As a result the first Killing vector is : $\frac{\partial}{\partial t}$. To find all the independent Killing vectors we have to solve Killings equations K_μ:

$$D_\mu K_\nu + D_\nu K_\mu = 0$$

The Christoffel symbols of this metric are given as follow, before defining the Schwarzschild radius $R_s = 2GM$: With the list of non-zero Christoffel symbols we can now generate all 10 coupled partial differential equations:

$$\partial_t K^t + \frac{R_s}{2(r^2 - R_s r)}K^r = 0 \tag{4.3}$$

$$\partial_r K^r + \frac{R_s}{2(R_s - r)r}K^r = 0 \tag{4.4}$$

$$\partial_\theta K^\theta + \frac{K^r}{r} = 0 \tag{4.5}$$

$$\partial_\phi K^\phi + \cot(\theta)K^\theta + \frac{K^r}{r} = 0 \tag{4.6}$$

$$r^2\partial_r K^\theta + \frac{\partial_\theta K^r}{1 - \frac{R_s}{r}} = 0 \tag{4.7}$$

$$\partial_t K^r - (1 - \frac{R_s}{r})^2\partial_r K^t = 0 \tag{4.8}$$

$$r^2\partial_t K^\theta - (1 - \frac{R_s}{r})\partial_\theta K^t = 0 \tag{4.9}$$

$$r^2\sin^2(\theta)\partial_r K^\phi + \frac{\partial_\phi K^r}{1 - \frac{R_s}{r}} = 0 \tag{4.10}$$

$$r^2\sin^2(\theta)\partial_t K^\phi - (1 - \frac{R_s}{r})\partial_\phi K^t = 0 \tag{4.11}$$

$$\sin^2(\theta)\partial_\theta K^\phi + \partial_\phi K^\theta = 0 \tag{4.12}$$

Where we use $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Using $Maple$ we find there are just the 4 Killing vectors, three of which are the 2-sphere Killing vectors and another along the time co-ordinate:

$$\begin{gathered} T = \frac{\partial}{\partial t}, Z = \frac{\partial}{\partial \phi}, \\ X = \sin\phi\frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi}, \quad Y = \cos\phi\frac{\partial}{\partial \theta} - \cot\theta\sin\phi\frac{\partial}{\partial \phi} \end{gathered} \tag{4.13}$$

We know X, Y, Z have the $SO(3)$ group structure through Lie brackets, but the Lie bracket of X, Y, Z with T leads to 0. The time killing vector T commutes with all of them. Now we consider the Conformal Killing vectors of Schwarzschild metric. 4 of the 10 equations get modified, namely the $\nu = \mu$ equations:

$$\begin{aligned}\partial_t K^t + \frac{R_s}{2(r^2 - R_s r)} K^r &= \lambda(x) \\ \partial_r K^r + \frac{R_s}{2(R_s - r)r} K^r &= \lambda(x) \\ \partial_\theta K^\theta + \frac{K^r}{r} &= \lambda(x) \\ \partial_\phi K^\phi + \cot(\theta) K^\theta + \frac{K^r}{r} &= \lambda(x) \\ \lambda(x) = \frac{1}{4}\left(\partial_t K^t + \partial_r K^r + \partial_\theta K^\theta + \partial_\phi K^\phi + \frac{2}{r} K^r + \cot(\theta) K^\theta\right)\end{aligned} \tag{4.14}$$

The other six equations remain the same. To find the solution of this system of equations we use Maple's *Differential Geometry* package which contains the *Conformal Killing Vector* command. This generates and solves the conformal Killing vectors for a given metric. In the case of the Schwarzschild metric using the command only yields the Killing vectors T, X, Y, Z, implying the metric contains no conformal Killing vectors. One has to wade through numerous complicated partial differential equations just to get these simple vector fields, the complexity will only rise as solving for the rank-2 Killing tensors means we will have to solve $\frac{4^3}{2} = 32$ equations. Therefore, there are many techniques and algorithms [23] that have been developed to find Killing tensors without solving the Killing tensor equations.

In terms of explicit symmetries the Schwarzschild metric only has rotational symmetry SO(3) and time symmetric since it is a static solution.

4.1.2 Killing tensors

The Killing tensor equation for rank-2 tensors is repeated here:

$$D_{(\mu} K_{\nu\rho)} = 0 \rightarrow D_\mu K_{\nu\rho} + D_\rho K_{\mu\nu} + D_\nu K_{\rho\mu} = 0$$

Since we have symmetry in the two indices of the Killing tensor the number of equations one has to solve is $\frac{4^3}{2} = 32$ since we have 3 indices which could take identical values. These equations are complicated partial differential equations with various Christoffel symbols shown in table.4.1. From reference: [24] we know there exist no irreducible rank-2 Killing tensors As a result all possible rank-2 Killing tensors can be found by symmetrising the Killing vectors and the metric itself.

$$\begin{aligned}
&K^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}\right)= \\
&C_1\left(-\frac{1}{1-\frac{2GM}{r}}\frac{\partial}{\partial t}\otimes\frac{\partial}{\partial t}+(1-\frac{2GM}{r})\frac{\partial}{\partial r}\otimes\frac{\partial}{\partial r}+\frac{1}{r^2}(\frac{\partial}{\partial\theta}\otimes\frac{\partial}{\partial\theta}+\frac{1}{\sin^2(\theta)}\frac{\partial}{\partial\phi}\otimes\frac{\partial}{\partial\phi})\right) \\
&+C_2\left(\frac{\partial}{\partial t}\otimes\frac{\partial}{\partial t}\right)+C_3\left(\frac{\partial}{\partial f}\otimes\frac{\partial}{\partial f}\right)+C_4\left(\frac{\partial}{\partial t}\odot\frac{\partial}{\partial f}\right)+C_5\left(\sin\phi\frac{\partial}{\partial t}\odot\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial t}\odot\frac{\partial}{\partial\phi}\right) \\
&+C_6\left(\sin\phi\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\phi}\right)+C_7\left(\cos\phi\frac{\partial}{\partial t}\odot\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial t}\odot\frac{\partial}{\partial\phi}\right) \\
&+C_8\left(\cos\phi\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\phi}\right) \\
&+C_9\left(\sin^2(\phi)\frac{\partial}{\partial\theta}\odot\frac{\partial}{\partial\theta}+2\sin(\phi)\cos(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\theta}+\cot^2(\theta)\cos^2(\phi)\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\phi}\right) \\
&+C_{10}\left(\cos^2(\phi)\frac{\partial}{\partial\theta}\odot\frac{\partial}{\partial\theta}-2\sin(\phi)\cos(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\theta}+\cot^2(\theta)\sin^2(\phi)\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\phi}\right) \\
&C_{11}\left(\sin(\phi)\cos(\phi)\frac{\partial}{\partial\theta}\odot\frac{\partial}{\partial\theta}+\cot(\theta)(\cos^2(\phi)-\sin^2(\phi))\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\theta}-\cot^2(\theta)\sin(\phi)\cos(\phi)\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\phi}\right)
\end{aligned} \tag{4.15}$$

These are all just trivial reducible Killing tensors which can be formed from the symmetrised tensor products of Killing vectors. With the exception of the first Killing tensor which is just the metric. To check we can use *Maple* to generate and solve for the Killing tensors with their *Killing Tensors* command from the same package as *Killing Vectors*. When used to solve for the Schwarzschild metric, the output is just symmetrized Killing vector products in linear combinations with each other. As a result there exists no hidden symmetries in the Schwarzschild metric. But if we are analysing the motion of a test particle this would not matter as we already have more than the required number of conserved quantities for integrability. We can cast the square angular momentum Killing tensors into a simpler form by simply summing them to get the total angular momentum Killing tensor:

$$X\odot X+Y\odot Y+Z\odot Z=\frac{\partial}{\partial\theta}\odot\frac{\partial}{\partial\theta}+\frac{1}{\sin^2(\theta)}\frac{\partial}{\partial\phi}\odot\frac{\partial}{\partial\phi}$$

4.1.3 Geodesic motion around the Schwarzschild black-hole

We form conserved quantities by contracting the Killing vector or tensor fields with momentum. For $T^\mu=(-1,0,0,0)$ we can form the energy E of the particle.:

$$E=T^\mu p_\mu=-p_t=\dot{t}(1-\frac{2GM}{r})$$

Where the $\dot{x}^\mu=\frac{dx^\mu}{d\tau}$ and using $p_\nu=g_{\mu\nu}\dot{x}^\mu$.With the Z Killing vector the angular momentum can be formed:

$$L_\phi=Z^\mu p_\mu=p_\mu=r^2\sin^2(\theta)\dot{\phi}$$

We can go on and form the other angular momentum with the rotation Killing vectors X and Z to analyse arbitrary orbits not constrained to the xy plane:

$$\begin{aligned}L_X&=X^\mu p_\mu=\sin(\phi)p_\theta+\cot(\theta)\cos(\phi)p_\phi\\L_Y&=Y^\mu p_\mu=\cos(\phi)p_\theta-\cot(\theta)\sin(\phi)p_\phi\end{aligned}$$

The total angular momentum J would then be:

$$J=L_X^2+L_Y^2+L_Z^2=J^{\mu\nu}p_\mu p_\nu=p_\theta^2+\frac{L_\phi^2}{\sin^2(\theta)} \tag{4.16}$$

However, the Schwarzschild metric has spherical symmetry and as a result for any arbitrary orbit not constrained in xy plane we can just rotate the co-ordinate system such that they are constrained to the xy plane. Due to spherical symmetry we will only analyse hyperbolic orbits in the $\theta = \frac{\pi}{2}$ plane since we can rotate these orbits into non planar orbits due to spherical symmetry. In the $\theta = \frac{\pi}{2}$ plane the $L_X = L_Y = 0$, only L_ϕ is non-zero. Particle motion is also constrained to the $\theta = \frac{\pi}{2}$ plane, as a result $p_\theta = 0$. We will not focus on the time co-ordinate as it is not relevant when considering scattering, the proper time τ parameter is enough. So we have 2 co-ordinates to solve for r and ϕ with 2 constants of motion E and L_ϕ. Along with the "normalisation" of momenta equation from the metric:

$$\begin{aligned} g^{\mu\nu}p_\mu p_\nu = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu &= -m^2 \\ -\left(1-\frac{2GM}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{2GM}{r}\right)} + r^2\sin(\frac{\pi}{2})^2\dot{\phi}^2 &= -m^2 \\ -E^2 + \dot{r}^2 + \frac{L_\phi^2}{r^2}\left(1-\frac{2GM}{r}\right) &= -m^2\left(1-\frac{2GM}{r}\right) \\ \left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(\frac{L_\phi^2}{r^2}+m^2\right)&\left(1-\frac{2GM}{r}\right) \end{aligned} \tag{4.17}$$

The θ components disappear as it remains constant. In the 2nd line we use the definition of the conserved quantities for energy and angular momentum, then we multiplied the equation with $1 - \frac{2GM}{r}$ and rearranged to find the radial equation of motion. Once we find the radial co-ordinate as a function of τ we can find the angle ϕ from the definition of angular momentum:

$$\frac{d\phi}{d\tau} = \frac{L_\phi}{r(\tau)^2} \tag{4.18}$$

Solving these two differential equations yields the orbits around a Schwarzschild black hole. These yield the individual co-ordinates of a orbit, such as r and ϕ as a function of τ.

The solution to the geodesic equation contains bound orbits to unbound ones, the type we are interested in are unbound scattering orbits, where the test particle passes far away from the black hole. These unbound orbits allow us to treat this test particle and black hole system as a scattering process, similar to Rutherford's experiment. The Schwarzschild geodesics can be used to approximate the interaction between two black holes with large mass ratios.

Scattering angles have already been calculated to very high precision in the reference: [25] the approach used in the reference relies on the Hamilton-Jacobi formalism.

4.2 Scattering angles in Schwarzschild background

4.2.1 Scattering set up and Minkowski trajectory

We will now calculate the scattering angles in the Schwarzschild background. We will calculate unbound geodesic orbits with an initial condition that mimics the scattering situation for the test particle. The test particle will approach from infinity with an impact parameter distance b from the black hole and will get scattered due to the black hole curvature. The motion of the particle is constrained to this trajectory, where it will begin its journey infinitely far away from the black hole. The particle will only graze the black hole and will only feel the weak field effect of the black hole. The situation is described in figure 4.2.

The test particle approaches the black hole from infinity with an impact distance of b, where b is very large. As it approaches it is bent or scattered by some angle and then it escapes to infinity again. This situation is shown in the bottom figure of 4.2. We are interested in the angle of scattering caused by the presence of the black hole, defined as the difference between the approaching angle $\phi(-\infty)$ and the final angle $\phi(\infty)$. From solving the first order form of the geodesics we can describe this orbit in terms of the spherical co-ordinates r and ϕ, the time co-ordinate t is not required for our analysis. We consider the weak field effects

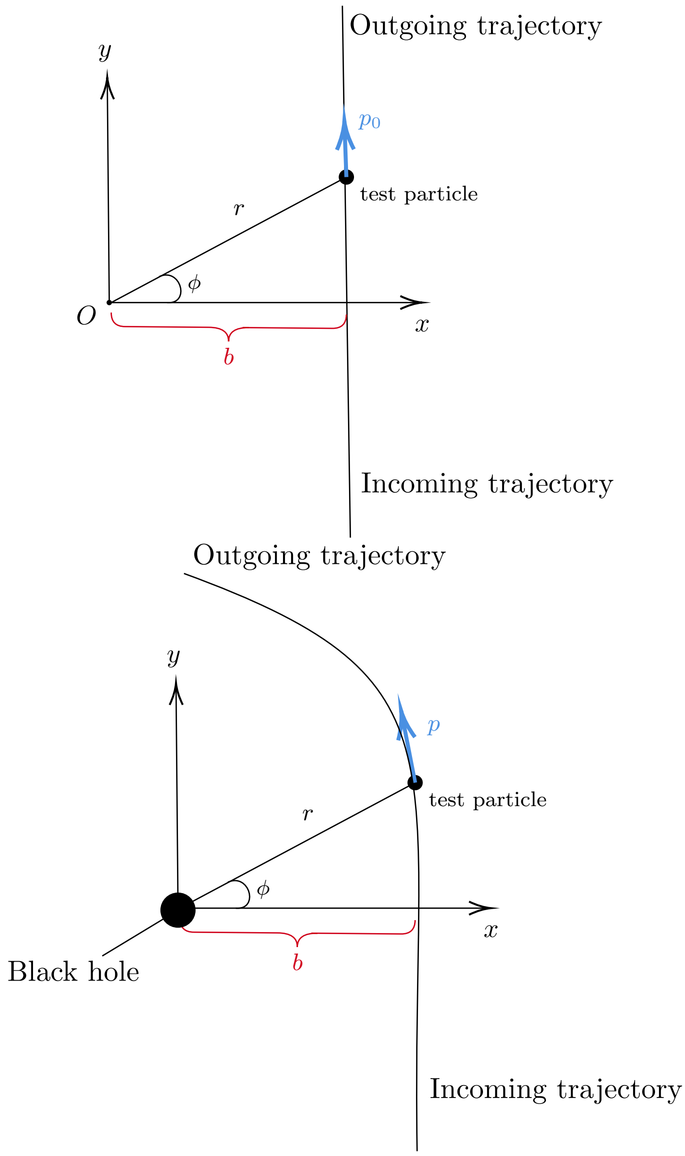


Figure 4.2: The test particle is scattered from its initial straight line trajectory, the particle initially approaches the black hole with a constant x distance which is b the impact parameter. The top figure shows the scattering trajectory in the absence of a black hole. The bottom depicts the scattering trajectory in the presence of a Schwarzschild black hole. In the bottom figure we drop the 0 sub-script on p to indicate it is no longer the Minkowski momentum near the black hole

of gravity on our test particle where $G \ll 1$. We treat the terms dependent on G in the first order form of the geodesics as perturbations to straight line motion in Minkowski space which occurs in the absence of the black hole. We are interested in the particle trajectory since in scattering we want the initial and final states only.

To find these scattering angles in the presence of weak gravity we perturb the geodesic equations in terms of G. As $G \to 0$ the geodesic equations reduce to describing straight line motion in Minkowski space on the $\theta = \frac{\pi}{2}$ plane.

The motion is described in polar co-ordinates r and ϕ constrained to the $\theta = \frac{\pi}{2}$ plane. The co-ordinate system we are using is the spherical co-ordinate system with radial r, azimuthal ϕ and polar θ co-ordinates, the co-ordinate system is centred on the black hole, hence (r, θ, ϕ) are all measured with respect to the black hole. It is easier to work in Cartesian co-ordinates than spherical co-ordinates when analysing straight line motion, co-ordinates $x = r(\tau)\cos[\phi(\tau)]$ and $y = r(\tau)\sin[\phi(\tau)]$ can be formed. This acts as a check as well to see if our solutions behave as expected. The straight line geodesic equations are:

$$\left(\frac{dr_0}{d\tau}\right)^2 = p_0^2 - \frac{L_\phi^2}{r_0^2} \tag{4.19}$$

$$\frac{d\phi_0}{d\tau} = \frac{L_\phi}{r_0^2} \tag{4.20}$$

where $p_0^2 = E^2 - m^2$ denotes the magnitude of 3-momentum in Minkowski space. We add a 0 sub-script to the radial and azimuthal co-ordinates r and ϕ to indicate these geodesics occur in Minkowski space. The solution to the radial equation 4.20 are:

$$r_0(\tau) = \frac{\sqrt{L_\phi^2 + p_0^4(\tau + C_r^0)^2}}{p_0} \tag{4.21}$$

The C_r^0 is a constant of integration that describes when the test particle begins its motion on the trajectory, we can safely set $C_r^0 = 0$ as this has no influence on the trajectory of the particle. Then the ϕ_0 equation is solved by integration:

$$\phi_0(\tau) = \tan^{-1}\left(\frac{p_0^2\tau}{L_\phi}\right) + C_\phi^0 \tag{4.22}$$

C_ϕ^0 is the associated integration constant for ϕ_0. We cannot ignore this constant as it influences the trajectory. These two equations should describe the test particle just travelling in a straight line in Minkowski space. The particle starts infinitely far away when $\tau = -\infty$ and ends its trajectory at infinity again when $\tau = \infty$.

r_0 has the right form since taking $L_\phi \to 0$ leads to a linear equation which describes the particle on a straight line trajectory through the origin. While $\phi_0 = \pi + C_\phi^0$ which is consistent with the trajectory passing through the origin with an angle ϕ_0 from the x-axis. To concretely see they do describe a straight line we construct the Cartesian co-ordinates x and y from r_0 and ϕ_0:

$$x(\tau) = r_0(\tau)\cos(\phi_0(\tau)) = \cos(C_\phi^0)\frac{L_\phi}{p_0} - p_0\sin(C_\phi^0)\tau \tag{4.23}$$

$$y(\tau) = r_0(\tau)\sin(\phi_0(\tau)) = \sin(C_\phi^0)\frac{L_\phi}{p_0} + p_0\cos(C_\phi^0)\tau \tag{4.24}$$

$x(\tau)$ and $y(\tau)$ are both linear and parameterise an arbitrary straight line. Eliminating the proper time and momentum $p_0\tau$ from the two equations expresses $x(\tau)$ in terms of $y(\tau)$:

$$y = -\cot(C_\phi^0)x + \frac{L_\phi}{p_0}\frac{1}{\sin(C_\phi^0)} \to x = \frac{L_\phi}{p_0}\frac{1}{\cos(C_\phi^0)} - \tan(C_\phi^0)y \tag{4.25}$$

Which is a negative gradient linear equation of y in terms of x and vice versa. This describes an arbitrary straight line trajectory, on which we need to impose the scattering boundary condition. The most convenient trajectory is when the geodesic lies parallel to one of the Cartesian axis x or y. We choose the trajectory to travel parallel to the y-axis, so the x co-ordinate of the geodesic is constant, which we label as the impact parameter b. Which is possible when the ϕ_0 constant of integration is $C_\phi^0 = 0$:

$$x = \frac{L_\phi}{p_0} = b \tag{4.26}$$

this defines a relation between the angular momentum L_ϕ , the momentum p_0 and the impact parameter b. The angle ϕ_0 becomes:

$$\phi_0(\tau) = \tan^{-1}(\frac{p_0^2\tau}{L_\phi}) \tag{4.27}$$

which spans from $-\frac{\pi}{2}$ when $\phi_0(-\infty)$ to $\frac{\pi}{2}$ at $\phi_0(\infty)$.

This yields the desired scattering trajectory in the absence of a black hole. Now we consider what happens when there is a Schwarzschild black hole present at the centre of the co-ordinate system. Since we only account for the weak effects of the black hole b must be a very large parameter implying the L_ϕ of the particle is far larger than its linear momentum p_0. The weak effects of the black hole will bend the straight line trajectory of the particle as it passes by the black hole and then escapes to infinity. We will not consider the particle passing through or anywhere near the event horizon. The bending of the trajectory changes the value of $\phi(\infty)$ which will be the scattering angle as shown in the figure.4.2. The scattering angle due to the black hole will be calculated perturbatively in Newton's constant G, the gravitational effects of the black hole will add small corrections to our trajectory co-ordinates $r(\tau)$ and especially $\phi(\tau)$, these corrections will be proportional to powers of the perturbation parameter G.

Perturbation theory

Perturbation theory is the method of approximately solving a differential equation by comparing it with a similar differential equation which already has a known solution. Say we want to solve the following differential equation parameterised by a parameter t:

$$F(y(t), \frac{dy(t)}{dt}, t) + \lambda G(y(t), \frac{dy(t)}{dt}, t) = 0 \tag{4.28}$$

Where $F(y(t), \frac{dy(t)}{dt}, t)$ and $G(y(t), \frac{dy(t)}{dt}, t)$ are some functions of the solution $y(t)$,its derivative with respect to t and t itself and λ is a dimensionless parameter. Now we assume we know the solution to:

$$F(y(t), \frac{dy(t)}{dt}, t) = 0 \tag{4.29}$$

which we denote as $y_0(t)$. If $\lambda \ll 1$ then $G(y(t), \frac{dy(t)}{dt}, t)$ is nearly negligible and we nearly reduce equation 4.28 to the solvable eq.4.29. Therefore the solution to 4.28 will be dominated by $y_0(t)$, the $y_0(t)$ solution acquires small corrections that come from the "perturbing" term $G(y(t), \frac{dy(t)}{dt}, t)$. As a result we say eq.4.29 is perturbed by the $G(y(t), \frac{dy(t)}{dt}, t)$ term, the perturbative parameter λ controls the strength of the perturbation. Under these conditions we can solve eq.4.28 approximately by a power series solution in the parameter λ.:

$$y(t) = y_0(t) + \lambda y_1(t) + \lambda^2 y_2(t) + ... \tag{4.30}$$

This is the perturbative series solution for eq.4.28. Functions $y_i(t)$ are small corrections to the dominant solution $y_0(t)$, the coefficients of these corrections are λ^i. So higher order corrections have less impact on our solution but still provide more accuracy. We substitute the pertubative series into eq.4.28 and gather all the terms by order of λ. This generates a set of differential equations for $y_i(t)$ which we have to solve order by order in λ. Usually the previous order solutions have to be used in higher order corrections. Perturbation theory is widely used in quantum mechanics to calculate corrections to the energy levels of a system [26].

In our case the ordinary differential equations we need to solve are the radial and ϕ equations for Schwarzschild geodesics 4.17. When $G \to 0$ the differential equations reduces to Minkowski radial and ϕ geodesic differential equations whose solutions we already solved for in 4.22 and 4.21. So we perturb these Minkowski differential equations by the small parameter $G << 1$. The solution to Schwarzschild geodesic equations are given in a power series dependent on the small parameter G. Since G is not dimensionless by itself we divide it by the impact parameter b such that it is, the new dimensionless parameter we expand in is then: $\tilde{G} = \frac{GM}{b}$, since $c = 1$ it is absent. Since $b \gg 1$ then $\tilde{G} \ll 1$ .Then the perturbation series solution to 4.17 and 4.18 are:

$$r(\tau) \approx r_0(\tau) + \tilde{G}r_1(\tau) + \tilde{G}^2 r_2(\tau) + \tilde{G}^3 r_3(\tau) + ... \tag{4.31}$$

$$\phi(\tau) \approx \phi_0(\tau) + \tilde{G}\phi_1(\tau) + \tilde{G}^2\phi_2(\tau) + \tilde{G}^3\phi_3(\tau) + ... \tag{4.32}$$

The 0th order perturbations are the known Minkowski geodesics. These solutions get corrections proportional to powers of $\tilde{G}$ which represent the effects of the black hole and provide the bending of the trajectory. The higher the power of $\tilde{G}$ the more accurate the solution. We will only perturb till order $\tilde{G}^3$. We replace G with $\tilde{G}$ in 4.18 and 4.17, we also get rid of the powers in the denominators:

$$r^3 \left(\frac{dr}{d\tau}\right)^2 = r^3 p_0^2 + 2\tilde{G}bm^2r^2 - L_\phi^2 r + 2\tilde{G}bL_\phi^2 \tag{4.33}$$

$$r^2 \frac{d\phi}{d\sigma} = L_\phi \tag{4.34}$$

Substituting in the perturbative expansions for r and ϕ lead to a series of differential equations for the corrections $r_i(\tau)$ and $\phi_i(\tau)$. Before we do so however we make a co-ordinate substitution for the proper time variable τ. From the form of $r_0(\tau)$ in 4.21 we change $\tau = \frac{L_\phi}{p_0^2}\sinh(\nu)$, now ν is the hyperbolic proper time parameter. The Minkowski geodesics are much simpler now:

$$r_0(\nu) = \frac{\sqrt{L_\phi^2(1+\sinh(\nu)^2)}}{p_0} = \frac{L_\phi}{p_0}\cosh(\nu) = b\cosh(\nu) \tag{4.35}$$

$$\phi_0(\nu) = \arctan[\sinh(\nu)] \tag{4.36}$$

which makes the perturbation differential equations simpler since we will be dealing with hyperbolic functions as opposed to logarithmic functions that occur from using τ.

All r_i and ϕ_i perturbative corrections will be a function of ν and the series expansion is truncated to 3rd order in $\tilde{G}$:

$$r(\nu) \approx r_0(\nu) + \tilde{G}r_1(\nu) + \tilde{G}^2 r_2(\nu) + \tilde{G}^3 r_3(\nu) + \mathcal{O}(\tilde{G}^4) \tag{4.37}$$

$$\phi(\nu) \approx \phi_0(\nu) + \tilde{G}\phi_1(\nu) + \tilde{G}^2\phi_2(\nu) + \tilde{G}^3\phi_3(\nu) + \mathcal{O}(\tilde{G}^4) \tag{4.38}$$

the derivatives also have to be switched to the hyperbolic proper time ν:

$$d\tau = d(\frac{L_\phi}{p_0^2}\sinh(\nu)) = \frac{L_\phi}{p_0^2}\cosh(\nu)d\nu$$

which slightly changes 4.17 and 4.18:

$$r(\nu)^3\left(\frac{dr(\nu)}{d\nu}\right)^2 = \frac{L_\phi^2\cosh^2(v)\left(2b\tilde{G}m^2r(\nu)^2+2b\tilde{G}L_\phi^2-L_\phi^2r(\nu)+p_0^2r(\nu)^3\right)}{p_0^4} \tag{4.39}$$

$$r(\nu)^2\frac{d\phi(\nu)}{d\nu} = \frac{L_\phi^2}{p_0^2}\cosh(\nu) \tag{4.40}$$

Now we substitute in the perturbative series for $r(\nu)$ and $\phi(\nu)$ which are the same as 4.32 with τ replaced by ν. This leads to a large equation:

$$\begin{aligned}(r_0(\nu)+\tilde{G}r_1(\nu)+\tilde{G}^2r_2(\nu)+\tilde{G}^3r_3(\nu))^3&\left(\frac{d(r_0(\nu)+\tilde{G}r_1(\nu)+\tilde{G}^2r_2(\nu)+\tilde{G}^3r_3(\nu))}{d\nu}\right)^2\\ &=\frac{L_\phi^2\cosh^2(v)}{p_0^4}[2b\tilde{G}m^2(r_0(\nu)+\tilde{G}r_1(\nu)+\tilde{G}^2r_2(\nu)+\tilde{G}^3r_3(\nu))^2\\ +2b\tilde{G}L_\phi^2-L_\phi^2(r_0(\nu)+\tilde{G}r_1(\nu)+\tilde{G}^2r_2(\nu)+\tilde{G}^3r_3(\nu))&+p_0^2(r_0(\nu)+\tilde{G}r_1(\nu)+\tilde{G}^2r_2(\nu)+\tilde{G}^3r_3(\nu))^3]\end{aligned} \tag{4.41}$$

$$(r_0(\nu)+\tilde{G}r_1(\nu)+\tilde{G}^2r_2(\nu)+\tilde{G}^3r_3(\nu))^2\frac{d(r_0(\nu)+\tilde{G}r_1(\nu)+\tilde{G}^2r_2(\nu)+\tilde{G}^3r_3(\nu))}{d\nu} = \frac{L_\phi^2}{p_0^2}\cosh(\nu) \tag{4.42}$$

we let *Mathematica* expand this out and segregate each term by order of $\tilde{G}^n$, then the expressions associated with a specific power of $\tilde{G}^n$ on the left and right hand side of the equation have to match in order to satisfy the equation. This produces a series of differential equations that need to be solved to find the perturbative corrections r_i and ϕ_i. The zeroth order is the Minkowski geodesics were already solved in 4.22 and 4.21. For ease we introduce notation for differentiation by ν : $\frac{df(\nu)}{d\nu}=f'(\nu)$ .The first order corrections in $\tilde{G}$ are given as follow:

$$\begin{aligned}-\frac{2bL_\phi^4\cosh^2(\nu)}{p^4}-\frac{2bL_\phi^2m^2r_0(\nu)^2\cosh^2(v)}{p^4}+\frac{L_\phi^4r_1(\nu)\cosh^2(\nu)}{p^4}\\ -\frac{3L_\phi^2r_0(\nu)^2r_1(\nu)\cosh^2(\nu)}{p^2}+3r_0(\nu)^2r_1(\nu)r_0'(\nu)^2=-2r_0(\nu)^3r_0'(\nu)r_1'(\nu)\end{aligned} \tag{4.43}$$

for the radial r co-ordinate and:

$$2r_0(\nu)r_1(\nu)\phi_0'(\nu)+r_0^2\phi_1'(\nu)=0 \tag{4.44}$$

for the ϕ co-ordinate. Solving these equations will provide the first order corrections in $\tilde{G}$ to the Minkowski geodesic, from which the scattering angle can be extracted. Solving these couple of equations provides two constants of integration for r_1 and ϕ_1, which have to be set to some value. To set these constants we need to revisit the boundary condition. Previously when $G=0$ we were able to demand that $x(\tau)=b$, clearly now this condition cannot hold for all τ. As the particle begins at infinity it will initially not feel the effect of the black hole, as a result its motion will first follow that of Minkowski geodesics, then it will deviate as it approaches the black hole by weakly bending and then escaping to infinity. Therefore our boundary condition is that $x(-\infty)=r(-\infty)\cos(\phi(-\infty))=b$, as this incorporates the fact that the particle effectively starts in Minkowski space at an infinite distance away. This boundary condition will help set the constants of integration for the $\phi_i(\nu)$ perturbations. To first order this boundary condition becomes:

$$\begin{aligned}x(-\infty)=\left(r_0(-\infty)+\tilde{G}r_1(-\infty)\right)\cos(\phi_0(-\infty)+\tilde{G}\phi_1(-\infty))=b\\ r_0(-\infty)\cos(\phi_0(-\infty))+\tilde{G}\left[r_1(-\infty)\cos(\phi_0(-\infty))-r_0(-\infty)\sin(\phi_0(-\infty))\phi_1(-\infty)\right]=b\end{aligned} \tag{4.45}$$

Which is satisfied if the expression in the $\tilde{G}$ bracket disappears , which occurs if $\phi_1(-\infty) = 0$ since $\cos(\phi_0(-\infty)) = 0$. While this boundary condition seems simple enough, actually implementing the condition was quite non-trivial since the perturbations in r do not behave as they should, as we will see. The boundary condition only fixes the constant of integration for the ϕ angle corrections and provides no condition for the radial corrections. For the r corrections we can demand that the perturbations go to 0 at $\nu = \pm\infty$ as the particle is infinitely far away that it can be treated as if it was is in Minkowski space, so the corrections should disappear at $\nu = \pm\infty$: $r_i(\pm\infty) = 0$ where $i > 0$. However we will see that this does not occur.

We now solve the first order perturbation equations 4.43 and 4.44, just substituting in the equations for r_0 yields an inhomogeneous ordinary differential equation, which can be solved with *Mathematica* or the integrating factor method manually:

$$r_1(\nu) = \frac{C_r^1}{p_0^2}\tanh(\nu) - b(\frac{m^2}{p_0^2} + 1) + \frac{bm^2}{p_0^2}\nu\tanh(\nu) \tag{4.46}$$

$$\phi_1(\nu) = \frac{C_r^1 \text{sech}^2(\nu)}{b} + \frac{m^2\tanh(\nu)}{p^2} + \frac{m^2\nu\text{sech}^2(\nu)}{p^2} + 2\tanh(\nu) + C_\phi^1 \tag{4.47}$$

C_r^1 is the unfixed integration constant from the radial perturbation and C_ϕ^1 is the integration constant for angular perturbation. Immediately we notice discrepancies with the radial perturbation. Firstly we expected the perturbation to cease at $\nu = \pm\infty$, however r_1 diverges to infinity at the $\nu = \pm\infty$, which may suggest the perturbation series solution may have failed as they are not bounded as we expected. We expected the divergence to disappear as the particle is effectively in Minkowski space (at $\nu = \pm\infty$) so it returns to our original Minkowski geodesic meaning $r(|\nu|) \approx r_0(|\nu|)$ when $\nu >> 1$ meaning $r_1(|\nu|) \approx 0$. r_1 suggests otherwise, stating that the Minkowski trajectory still receives infinite corrections even when it is effectively in flat space, which is unphysical. Perhaps these infinite corrections occur due to the slight bending of the trajectory, since the particle trajectory is bent at an angle, maybe the discrepancy between the bent geodesic and the Minkowski geodesic $x = b$ causes this problem. However, if that were the case we would see this discrepancy in the ϕ_1 angle geodesic, this would be the scattering angle we are calculating, the bending discrepancy should be accounted by ϕ_1 not r_1 which is just the correction to the radial distance between the particle and the black hole.

Secondly we still have not fixed the integration constant C_r^1, which was originally supposed to set $r_1(-\infty) = 0$, but due to the divergence C_r^1 is unfixed. These two problems seem to suggest that there is something wrong with a perturbative approach to calculating these hyperbolic scattering geodesics. However, we can actually safely ignore the divergent behaviour due to fact that r_1 diverges much slower than r_0. The divergent term in $r_1(\nu)$ is $\frac{bm^2\nu\tanh(\nu)}{p_0^2}$ which diverges linearly in $\nu \gg 1$ as opposed to $r_0(\nu)$ which diverges exponentially in the same regime. As a result $r_1(\nu)$ will not overtake $r_0(\nu)$ at any point, as that would make the perturbation invalid. Since the boundary condition does not impose anything on C_r^1, we keep the unfixed constant in the calculation but it does not seem to contribute to the scattering angle as it is damped by a factor of $\text{sech}(\nu)^2$. The boundary condition $\phi_1(-\infty) = 0$ fixes C_ϕ^1:

$$\phi_1(-\infty) = C_\phi^1 - 2(\frac{m^2}{p_0^2} + 1) + \frac{m^2}{p_0^2} = 0$$

$$C_\phi^1 = \frac{2(m^2 + p_0^2) - m^2}{p_0^2} = \frac{2E^2 - m^2}{E^2 - m^2} \tag{4.48}$$

$$\phi_1(\nu) = \frac{C_r^1\text{sech}^2(\nu)}{b} + \frac{m^2\tanh(\nu)}{p^2} + \frac{\nu m^2\text{sech}^2(\nu)}{p^2} + \frac{m^2}{p^2} + 2\tanh(\nu) + 2$$

From which we extract the scattering angle by taking $\phi_1(\nu \to \infty)$:

$$\phi_1(\infty) = \frac{4E^2 - 2m^2}{E^2 - m^2} \tag{4.49}$$

The unfixed constant C_r^1 does not contribute at all to the scattering angle in ϕ, in fact all integration constants related to the radial perturbations have no impact on any of the scattering angles. The constants C_r^i act as a gauge choice, we simply choose to set these to 0: $C_r^i = 0$. We can think of them similar to C_r^0 which dictated when the motion of the particle began, which is inconsequential when only studying the trajectory and scattering angle. The above is still not the full scattering angle as it needs to be multiplied by the respective power of $\tilde{G}$:

$$\boxed{\Delta\phi_1 = \tilde{G}\phi_1(\infty) = \frac{GM(4E^2 - 2m^2)}{b(E^2 - m^2)} = \frac{2GM(v^2 + 1)}{bv^2}} \tag{4.50}$$

v is the velocity of the particle at infinity. The scattering angle $\Delta\phi_1$ does behave as we expected, intuitively a particle with high velocity will not be deflected as much. To check what we have been doing is sensible we can take the mass-less limit of this deflection angle where $m \to 0$:

$$\boxed{\Delta\phi_1^{massless} = 4\tilde{G} = \frac{4GM}{c^2 b}} \tag{4.51}$$

where we have reinserted c to show that this is the formula for the deflection of light by the Sun which was used to test general relativity in 1919. Additionally, the velocity form of the scattering angle $\Delta\phi_1$ matches with the literature value in reference [25]. Which indicates our calculations are indeed correct. We can also verify that $r_1(\nu)$ provides the bending trajectory we expected, as $r_1(\nu)$ reaches the minimum around $\nu = 0$ where r_1 provides the highest negative contribution to r_0, meaning it takes the particle closer to the black hole around $\nu = 0$. We now proceed to calculate higher order scattering angles up to $\tilde{G}^3$.

The boundary condition becomes:

$$\begin{aligned}
&\left(r_0 + \tilde{G}r_1 + \tilde{G}^2 r_2 + +\tilde{G}^3 r_3\right)\cos(\phi_0 + \tilde{G}\phi_1 + \tilde{G}^2\phi_2 + \tilde{G}^3\phi_3) \\
&= \tilde{G}^2\left(r_0\left(-\frac{1}{2}\phi_1^2\cos(\phi_0) - \phi_2\sin(\phi_0)\right) - \phi_1 r_1 \sin(\phi_0) + r_2\cos(\phi_0)\right) \\
&+\frac{1}{6}\tilde{G}^3\left(\phi_1^3 r_0\sin(\phi_0) - 3\phi_1^2 r_1\cos(\phi_0) - 6\phi_1\phi_2 r_0\cos(\phi_0) - 6\phi_1 r_2\sin(\phi_0) - 6\phi_2 r_1\sin(\phi_0) - 6\phi_3 r_0\sin(\phi_0) + 6r_3\cos(\phi_0)\right) \\
&+\tilde{G}(r_1\cos(\phi_0) - \phi_1 r_0\sin(\phi_0)) + r_0\cos(\phi_0) = b
\end{aligned} \tag{4.52}$$

Evaluated at $\nu = -\infty$, the condition is satisfied if all $\phi_i(-\infty) = 0$. This fixes all the integration constants for the ϕ perturbations, whereas the integration constants for the r perturbations do not contribute to the scattering angle since they act as a gauge choice, so they can be ignored:$C_r^i = 0$.

Now the perturbation series for r and ϕ truncated to third order in$\tilde{G}$ are substituted into the differential equations 4.17 and 4.18, then matching the left and right hand side order by order in $\tilde{G}$ yields a series of differential equations for r_2, r_3 and ϕ_2, ϕ_3. We solve these equations using *Mathematica* to find the r and ϕ corrections to third order in $\tilde{G}$, we summarise the scattering angles in table 4.2. The r and ϕ perturbations are lengthy and tedious so they are given in the appendix. We also use $p_0 = \gamma m v = \frac{mv}{1-v^2}$, where v is the velocity of the particle as it begins at $\nu = -\infty$. v is one of the two parameters which dictates the scattering angle of a particle, the other being the impact parameter b. v is a good parameter since it simultaneously take into account both the mass m and energy E of the particle. When $v \to 1$ we reach the massless limit. The derived scattering angles are identical to those derived in [25]. The method of perturbing the geodesic equations, even though contains several peculiarities does calculate the correct scattering angles. Interestingly from the appendix,

Order of $\tilde{G}$	Scattering angle
$\tilde{G}$	$\Delta\phi_1 = \frac{2GM(v^2+1)}{bv^2}$
$\tilde{G}^2$	$\Delta\phi_2 = \frac{3G^2M^2\pi}{4b^2}\left(\frac{4m^2}{p_0^2}+5\right) = \frac{3G^2M^2\pi}{4b^2}\left(\frac{4+v^2}{v^2}\right)$
$\tilde{G}^3$	$\Delta\phi_3 = \frac{G^3M^3}{b^3}\left(-\frac{2m^6}{3p_0^6}+\frac{8m^4}{p_0^4}+\frac{48m^2}{p_0^2}+\frac{128}{3}\right) = \frac{2G^3M^3}{3b^3}\frac{\left(5v^6+45v^4+15v^2-1\right)}{v^6}$

Table 4.2: Scattering angles for the test particle in the ϕ co-ordinate expressed in v.

$r_2(\nu)$ and $r_3(\nu)$ are not divergent and converge at $\nu = \pm\infty$. While the method works it is very tedious and inefficient when compared to the approach taken by [25], where they very simply just solve for the scattering angles correction from integration. On the other hand this method provides the actual trajectory of the test particle as it passes by the black hole, which maybe useful for studying gravitational waves. Scattering angles maybe useful when computing gravitational scattering amplitudes as it provides classical results which can act as a check for results derived from amplitude methods [25]. More interestingly one can construct a differential cross section from the scattering angle corrections, to various order in $\tilde{G}$. From this cross section we can Fourier transform it, analogous the Rutherford scattering, to find a potential. From this potential we can use the Lippmann-Schwinger equation to construct a quantum mechanical cross section. Then one can compare to see if the classical and quantum cross sections agree with each other [26].

In this chapter we analysed the Schwarzschild metric to see what symmetries it possesses, and in the process discovered it has four Killing vectors: T,X,Y and Z and no Conformal Killing vectors. Furthermore it contains no hidden symmetries in the form of a rank-2 Killing tensor, however we have enough constants of motions from the Killing vectors and the metric to cast the geodesic equation into a simple first order form. The first order form of the geodesic equation allowed us to calculate scattering angles around a Schwarzschild black hole, due to its spherical symmetry we could just calculate the scattering angles in the $\theta = \frac{\pi}{2}$ plane. In the next chapter we analyse a generalisation of the Schwarzschild metric where the black hole is spinning, this is described by the Kerr metric.

Chapter 5

Kerr metric

5.1 Geometry

The Kerr metric in Boyer-Lindquist co-ordinates is expressed as the following (We set $c = 1$):

$$ds^2 = -\frac{\Delta}{\rho^2}\left[dt - a\sin^2(\theta)d\phi\right]^2 + \frac{\sin^2(\theta)}{\rho^2}\left[(r^2+a^2)d\phi - adt\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$
$$= -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{4GMra\sin^2(\theta)}{\rho^2}dtd\phi + \frac{\left((r^2+a^2)^2 - \Delta a^2\sin^2(\theta)\right)}{\rho^2}d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{5.1}$$

$\rho^2 = r^2 + a^2\cos^2(\theta)$ and $\Delta = r^2 - 2GMr + a^2$. The metric describes the geometry around a rotating black hole. The rotation parameter $a = \frac{J}{M}$ describes the angular momentum (J) per unit mass of the black hole.

We now state some general properties of the Kerr metric. Unlike the Schwarzschild metric, the time and ϕ angle co-ordinates are "mixed" producing off-diagonal components in the metric. The dr^2 component diverges due to co-ordinate singularities occurring when $\Delta = 0$, these are two event horizons the Kerr black hole possesses at:

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2} \tag{5.2}$$

These are shown in the figure 5.1 which is reproduced from [1]. The black hole possesses a region called the Ergosphere where one experiences so called "frame-dragging" effects, where an observer cannot stay stationary within this region. They will be rotated alongside the spinning black hole. Finally, the Kerr has a curvature singularity at the centre similar to the Schwarzschild. But unlike the Schwarzschild case where one had a point singularity, Kerr possesses a ring singularity. This occurs at $\rho = r^2 + a^2\cos^2(\theta) = 0$.

An important inequality between the spin a of the black hole and mass M stems from the cosmic censorship hypothesis:

$$a \leq GM \tag{5.3}$$

otherwise there would be a naked singularity, which is forbidden under the Cosmic censorship hypothesis.

Finally, we examine some limits of the parameter a and G which will be used later. In the limit of $G \to 0$ the metric simplifies to flat space in ellipsoid co-ordinates [27]:

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2(\theta)}{r^2+a^2}\right)dr^2 + (r^2 + a^2\cos^2(\theta))^2 d\theta^2 + (r^2+a^2)\sin^2(\theta)d\phi^2 \tag{5.4}$$

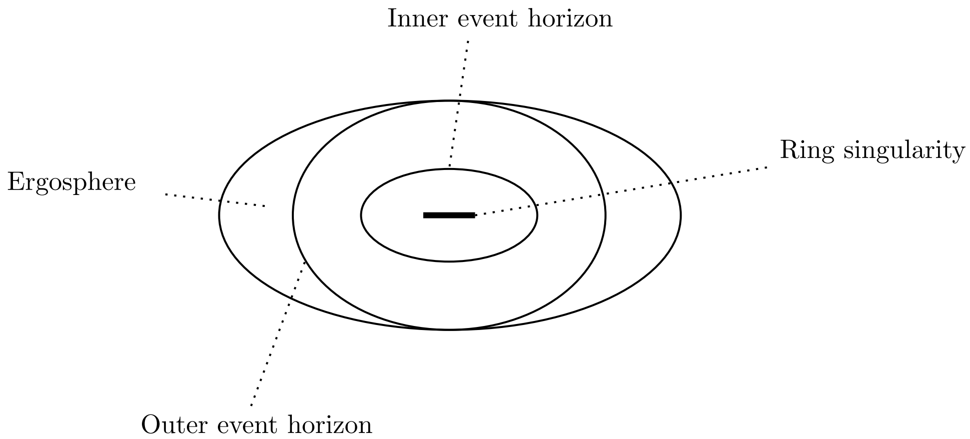


Figure 5.1: Cross-section of the Kerr black hole which depicts the event horizons and the ergo sphere. The diagram is adapted from page 264 of [1]

the ellipsoid co-ordinates are expressed in Cartesian co-ordinates as:

$$\begin{aligned} x &= \sqrt{r^2+a^2}\sin(\theta)\cos(\phi) \\ y &= \sqrt{r^2+a^2}\sin(\theta)\sin(\phi) \\ z &= r\cos(\theta) \end{aligned}$$

Additionally, taking $a \to 0$ reduces to Minkowski space in spherical co-ordinates. Then if we let $G \neq 0$ and take the spin $a \to 0$ the metric reduces to the Schwarzschild metric in standard (t, r, θ, ϕ) co-ordinates. We state these limits as they provide a useful check for the validity of the Kerr scattering angles. Now we explore the symmetries of the Kerr metric

5.2 Symmetries of the Kerr metric

We start with the possible isometries of Kerr, represented by Killing vectors.

Killing vectors

Unlike the Schwarzschild metric the Kerr metric does not contain the 2-sphere metric. As a result we will suffer the loss of spherical symmetry which we had in Schwarzchild metric. To see which Killing vectors survive from the Schwarzschild case we examine the components of the Kerr metric and immediately extract two independent Killing vector fields for co-ordinates ϕ and t. Since they are absent in the metric. As a result we can construct the following Killing vector fields :

$$T = \frac{\partial}{\partial t}, \quad Z = \frac{\partial}{\partial \phi} \tag{5.5}$$

The Z Killing vector shows the metric has rotational symmetry along the ϕ direction. T shows that the metric is not time dependent even though the black hole is spinning, this can be interpreted as the black hole spins constantly without varying its spin through time. From before the conserved quantity associated with T is $-E$ and L_ϕ for Z. To find more Killing vectors we have to solve Killing's equation for the Kerr metric. These lead to a set of 10 difficult coupled partial derivative equations. Instead one can show that the Kerr metric contains only two Killing vectors [9]. Meaning the only possible Killing vectors in Kerr are Z and T. These two Killing vectors provide two conserved quantities, namely the energy E and the angular momentum L_ϕ. Along with the metric which provides normalisation for momenta, we only have three conserved quantities for four co-ordinates. As a result we do not have integrability as we lack one conserved quantity. The final conserved quantity required for integrability comes from a rank-2 Killing tensor the Kerr metric possesses, this conserved quantity is called Carter's constant. We will now derive this constant and the associated Killing tensor.

The lack of spherical symmetry from the loss of X and Y Killing vectors in Kerr will have an effect when we analyse scattering. Namely, we have to consider non-equatorial $\theta \neq \frac{\pi}{2}$ scattering as a particle approaching the black hole with a $\theta \neq \frac{\pi}{2}$ will face a different geometry than one in the equatorial $\theta = \frac{\pi}{2}$ plane.

5.2.1 Deriving Carter's constant and Killing tensor

Now after deriving the Killing vectors by inspection we turn our attention to the most interesting property of the Kerr metric, the existence of a hidden symmetry in the form of Carter´s constant. This constant was found by Brandon Carter and presented in his paper: "Global structure of the Kerr family of gravitational fields" in 1968 [28]. The constant was found by separating the Hamilton-Jacobi equation we introduced in chapter 2. To find the rank-2 Killing tensor which yields Carter´s constant one could try to solve the 32 rather difficult coupled partial differential equations. However, instead we will follow Carters derivation of the constant by first separating the Hamilton-Jacobi equation in Boyer-Lindquist co-ordinate, then using the formula 3.51 given in chapter 2 to extract the Killing tensor. Before we begin we have to check the conditions laid out in 3.49. The conditions are satisfied since we can firstly divide the co-ordinates into cyclic: t, ϕ and the separable co-ordinates on which the metric depends on r, θ. The inverse metric for Kerr is :

$$g^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}} = -\frac{1}{\Delta\rho^2}\left[(r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \phi}\right]^2 + \frac{1}{\rho^2 \sin^2(\theta)}\left[\frac{\partial}{\partial \phi} + a\sin^2(\theta)\frac{\partial}{\partial t}\right]^2 + \frac{\Delta}{\rho^2}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial}{\partial \theta}\right)^2 \tag{5.6}$$

[22] which can be separated into a tensor $R^{\mu\nu}(r)$ which is only dependent on the r co-ordinate, and $\Theta^{\mu\nu}(\theta)$ which depends only on θ:

$$g^{\mu\nu} = \frac{\Theta^{\mu\nu} + R^{\mu\nu}}{f_\theta - f_r} \tag{5.7}$$

Comparing to the inverse metric we identify the components as:

$$\begin{aligned} R^{tt} &= \frac{(r^2+a^2)^2}{\Delta} \qquad \Theta^{tt} = a^2\sin^2(\theta) \\ R^{\phi\phi} &= -\frac{a^2}{\Delta} \quad \Theta^{\phi\phi} = \frac{1}{\sin^2(\theta)} \\ R^{rr} &= \Delta \quad \Theta^{rr} = 0 \\ R^{\theta\theta} &= 0 \quad \Theta^{\theta\theta} = 1 \\ R^{\phi t} &= \frac{-a(r^2+a^2)}{\Delta} \quad \Theta^{\phi t} = 2a \end{aligned} \tag{5.8}$$

along with identifying $\rho^2 = f_\theta - f_r$ as $f_\theta = a^2\cos^2(\theta)$ and $f_r = -r^2$. The conditions laid out for separation in 3.49 are satisfied, we now proceed to separation. The free particle Hamiltonian is just:

$$H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = -\frac{1}{2}m^2 \tag{5.9}$$

with m being the test particle mass. The proper time independent Hamilton-Jacobi equation is then:

$$g^{\mu\nu}\frac{\partial W}{\partial x^{\mu}}\frac{\partial W}{\partial x^{\nu}} = -m^2 \tag{5.10}$$

inputting this into the Hamilton-Jacobi equation yields:

$$-\frac{1}{\Delta\rho^2}\left[(r^2+a^2)\frac{\partial W}{\partial t}+a\frac{\partial W}{\partial\phi}\right]^2+\frac{1}{\rho^2\sin^2(\theta)}\left[\frac{\partial W}{\partial\phi}+a\sin^2(\theta)\frac{\partial W}{\partial t}\right]^2+\frac{\Delta}{\rho^2}\left(\frac{\partial W}{\partial r}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial W}{\partial\theta}\right)^2=-m^2 \tag{5.11}$$

Now we propose a separable ansatz using already known conserved quantities from Killing vectors : $W=-Et+L_\phi\phi+W_r(r)+W_\theta(\theta)$. Where $W_\theta(\theta)$ is a function only of θ and $W_r(r)$ is function only of r. Using this ansatz in the Hamilton-Jacobi equation and multiplying by ρ^2 on both sides yields the following:

$$-m^2(r^2+a^2\cos^2(\theta))=-\frac{1}{\Delta}\left[-(r^2+a^2)E+aL_\phi\right]^2+\frac{1}{\sin^2(\theta)}\left[L_\phi-Ea\sin^2(\theta)\right]^2+\Delta\left(\frac{\partial W_r(r)}{\partial r}\right)^2+\left(\frac{\partial W_\theta(\theta)}{\partial\theta}\right)^2 \tag{5.12}$$

The Hamilton-Jacobi equation becomes a function of only θ and r,furthermore the θ and r functions are completely separate and do not mix. As a result we move all θ dependence onto the left hand side, and all r dependence to the right hand side. Expanding everything out yields:

$$\begin{aligned}-m^2a^2\cos^2(\theta)-\frac{L_\phi^2}{\sin^2(\theta)}+2L_\phi Ea-E^2a^2\sin^2(\theta)-\left(\frac{\partial W_\theta(\theta)}{\partial\theta}\right)^2\\=m^2r^2-\frac{\left[(r^2+a^2)E-aL_\phi\right]^2}{\Delta}+\Delta\left(\frac{\partial W_r(r)}{\partial r}\right)^2\end{aligned} \tag{5.13}$$

Since the left hand side is purely a function of θ and the right hand side is purely a function of r both sides have to equal a constant $-K$. Therefore we have found a new conserved quantity K called Carter´s constant. From the θ dependent side of 5.13 it is defined as:

$$K=m^2a^2\cos^2(\theta)+\frac{\left(L_\phi-aE\sin^2(\theta)\right)^2}{\sin^2(\theta)}+p_\theta^2 \tag{5.14}$$

where we also used the definition of momentum from 3.42 for the θ component. We can now find the Killing tensor that produces the conserved quantity using the formula from 3.51, this irreducible Carter Killing tensor $K^{\mu\nu}$ will be:

$$K^{\mu\nu}=-\left(\frac{f_r\Theta^{\mu\nu}+f_\theta R^{\mu\nu}}{\rho^2}\right) \tag{5.15}$$

the components can be calculated from 5.8. The Carter Killing tensor is:

$$\begin{aligned}&K^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}\\&=\left(\frac{a^2\left(a^2+r^2\right)^2\cos^2(\theta)}{\Delta\rho^2}+\frac{a^2r^2\sin^2(\theta)}{\rho^2}\right)\frac{\partial}{\partial t}\otimes\frac{\partial}{\partial t}-\left(\frac{a^2\cos^2(\theta)\Delta}{\rho^2}\right)\frac{\partial}{\partial r}\otimes\frac{\partial}{\partial r}\\&+2\left(\frac{r^2a+\frac{a^3\cos^2(\theta)(r^2+a^2)}{\Delta}}{\rho^2}\right)\frac{\partial}{\partial t}\odot\frac{\partial}{\partial\phi}+\left(\frac{r^2}{\rho^2}\right)\frac{\partial}{\partial\theta}\otimes\frac{\partial}{\partial\theta}+\left(\frac{\frac{r^2}{\sin^2(\theta)}+\frac{a^4\cos^2(\theta)}{\Delta}}{\rho^2}\right)\frac{\partial}{\partial\phi}\otimes\frac{\partial}{\partial\phi}\end{aligned} \tag{5.16}$$

This constant of motion is split into an explicitly θ dependent constant and a square of the energy and angular momentum quantities. The constant is defined as $\mathcal{C}$:

$$\mathcal{C}=p_\theta^2+\cos^2(\theta)\left[a^2(m^2-E^2)+\frac{L_\phi^2}{\sin^2(\theta)}\right] \tag{5.17}$$

which is just:

$$\mathcal{C} + (L_\phi - aE)^2 = K \tag{5.18}$$

$\mathcal{C}$ is also referred to as Carter´s constant, however K is usually preferred since $K \geq 0$. We will instead use $\mathcal{C}$ when we analyse the perturbations of the geodesic equations. From the definition of K in 5.14 we see it acts as the total angular momentum of the system similar to J in the Schwarzschild metric 4.16, however Carter's constant also contains the energy E of the particle so it is not an exact replacement for J. Especially $\mathcal{C}$ reduces to $\mathcal{C}_{a\to 0} = p_\theta^2 + L_\phi \text{cot}^2(\theta)$ which is just the sum of angular momentum of $L_x^2 + L_y^2$, the conserved quantities which are formed from Killing vectors X and Y which disappear in the Kerr metric. So Carter's constant $\mathcal{C}$ acts as a replacement for the missing angular momentum. We can now use this conserved quantities to find the geodesic motion of a test particle in Kerr geometry.

5.3 Scattering around Kerr black holes

With K we now have four conserved quantities in the Kerr geometry, which we will use to analyse the geodesic motion of a test particle. Similar to the Schwarzschild case we can now cast the geodesic equation into a first order form:

$$\begin{aligned}
\rho^2 \frac{dr}{d\tau} &= \sqrt{(E(r^2+a^2) - aL_\phi)^2 - \Delta(m^2 r^2 + (L_\phi - aE)^2 + \mathcal{C})} \\
\rho^2 \frac{d\theta}{d\tau} &= \sqrt{\mathcal{C} - \cos^2(\theta)\left(a^2(m^2 - E^2) + \frac{L_\phi}{\sin^2(\theta)}\right)} \\
\rho^2 \frac{d\phi}{d\tau} &= -(aE - \frac{L_\phi}{\sin^2(\theta)}) + (\frac{a}{\Delta})(E(r^2+a^2) - aL_\phi) \\
\rho^2 \frac{dt}{d\tau} &= -a(aE\text{sin}^2(\theta) - L_\phi) + (r^2+a^2)\frac{E(r^2+a^2) - aL_\phi}{\Delta}
\end{aligned} \tag{5.19}$$

these equations can be derived 5.13 by just rearranging for p_θ or p_r and using $p_\mu = g_{\mu\nu}\frac{dx^\nu}{d\tau}$ as it was done by Carter in [28]. An alternative derivation is given in the appendix from the Carter canonical form of the metric given in [10]. Since $\rho^2 = r^2 + a^2\cos^2(\theta)$ these are coupled differential equations. Geodesic motion is not constrained to a plane such as in the Schwarzschild case. Compared to the Schwarzschild geodesics we now have terms which depend on the angular momentum parameter of the black hole a. Taking the limit $a \to 0$ these geodesics revert to the Schwarzschild geodesics. Taking both G and a to 0 yields Minkowski geodesics. We can now perturb these geodesic equations to find the scattering trajectories to calculate scattering angles. Due to these two parameters we attempt a perturbation series solution in both a and G. a has dimensions of length, as a result we define a dimensionless parameter $\tilde{a} = \frac{a}{b}$ where b was the impact parameter from before. The first case considered is scattering in the $\theta = \frac{\pi}{2}$ plane.

5.3.1 Equatorial scattering

Similar to the Schwarzschild case the particle motion will be constrained to the equatorial plane when θ is set to $\frac{\pi}{2}$, in terms of cartesian co-ordinate system centered on the black hole, the particle is constrained to xy plane and $z = 0$. As a result we can neglect the θ angle again. Figure 4.2 describes the situation except now there is a spinning Kerr black hole at the centre instead of a stationary Schwarzschild black hole. Additionally, Carter´s constant $\mathcal{C}$ also disappears which intuitively makes sense when we consider $\mathcal{C}$ as a replacement for the L_X^2 and L_Y^2 square angular momenta, since we only have ϕ component of angular momentum in the equatorial plane. The set up of the scattering is actually identical to the Schwarzschild case,except now we also consider the spin effects of the black hole, which are represented by a. The simplified geodesics we need to consider are:

$$r^4\frac{dr}{d\tau} = (E(r^2+a^2) - aL_\phi)^2 - \Delta(m^2r^2 + (L_\phi - aE)^2) \tag{5.20}$$

$$r^2\frac{d\phi}{d\tau} = -(aE - L_\phi) + (\frac{a}{\Delta})(E(r^2+a^2) - aL_\phi) \tag{5.21}$$

$\rho^2 = r^2$ due to $\theta = \frac{\pi}{2}$. Which are rearranged into the following form:

$$r^4\frac{dr}{d\tau} = (E(r^2+\tilde{a}^2b^2) - \tilde{a}bL_\phi)^2 - \Delta(m^2r^2 + (L_\phi - \tilde{a}bE)^2) \tag{5.22}$$

$$r^2(r^2 - 2\tilde{G}b + \tilde{a}^2b^2)\frac{d\phi}{d\tau} = -(\tilde{a}E - L_\phi)(r^2 - 2\tilde{G}b + \tilde{a}^2b^2) + (\tilde{a})(E(r^2+a^2) - aL_\phi) \tag{5.23}$$

where we also replace the parameters G and a by their dimensionless counterparts. We also work in the same hyperbolic proper time parameter we introduced in chapter 3: $\tau = \frac{L_\phi}{p_{00}^2}\sinh(\nu)$, $p_{00} = E^2 - m^2$ is also used. We denote the asymptotic momentum p_{00} since we are perturbing in two parameters. Now we propose a perturbation series solution in both $\tilde{a}$ and $\tilde{G}$ for the co-ordinates r and ϕ:

$$r(\nu) \approx r_{00}(\nu) + \tilde{G}r_{10}(\nu) + \tilde{a}r_{01} + \tilde{G}\tilde{a}r_{11}(\nu) + \tilde{G}^2r_{20}(\nu) + \tilde{a}^2r_{02} + \tilde{G}^2\tilde{a}r_{21}(\nu) + \tilde{G}\tilde{a}^2r_{12}(\nu) \tag{5.24}$$

$$\phi(\nu) \approx \phi_{00}(\nu) + \tilde{G}\phi_{10}(\nu) + \tilde{a}\phi_{01} + \tilde{G}\tilde{a}\phi_{11}(\nu) + \tilde{G}^2\phi_{20}(\nu) + \tilde{a}^2\phi_{02} + \tilde{G}^2\tilde{a}\phi_{21}(\nu) + \tilde{G}\tilde{a}^2\phi_{12}(\nu) \tag{5.25}$$

The indices of co-ordinates r_{ij} and ϕ_{ij} correspond to order of perturbation in $\tilde{G}^i$ and in $\tilde{a}^j$. The perturbation series is truncated to order $\mathcal{O}(\tilde{G}^2\tilde{a}^2)$. The boundary condition is also identical to the Schwarzschild case, as before we want the Cartesian co-ordinate $x = r\cos\phi\sin\theta = const. = b$ at $\nu = -\infty$, in the equatorial plane θ is constant and equal to $\frac{\pi}{2}$. So the boundary condition reduces to: $x = r\cos\phi = b$ at $\nu = -\infty$ which is the same boundary condition as Schwarzschild. We substitute in the perturbation series for r and ϕ for the boundary condition:

$$\begin{aligned} &\Big(r_{00}(\nu) + \tilde{G}r_{10}(\nu) + \tilde{a}r_{01} + \tilde{G}\tilde{a}r_{11}(\nu) + \tilde{G}^2r_{20}(\nu) + \tilde{a}^2r_{02} + \tilde{G}^2\tilde{a}r_{21}(\nu) + \tilde{G}\tilde{a}^2r_{12}(\nu)\Big) \\ &\times\Big(\cos(\phi_{00}(\nu) + \tilde{G}\phi_{10}(\nu) + \tilde{a}\phi_{01} + \tilde{G}\tilde{a}\phi_{11}(\nu) + \tilde{G}^2\phi_{20}(\nu) + \tilde{a}^2\phi_{02} + \tilde{G}^2\tilde{a}\phi_{21}(\nu) + \tilde{G}\tilde{a}^2\phi_{12}(\nu))\Big) = b \end{aligned} \tag{5.26}$$

at $\nu = -\infty$. Expanding this out we find the same Schwarzschild boundary condition for the ϕ perturbations, that they must vanish at $\nu = -\infty$: $\phi_{ij}(-\infty) = 0$. And we are free to set the C^r_{ij} integration constants which correspond to the radial perturbations to 0 as in the Schwarzschild case.

The perturbations are calculated by substituting the perturbation series 5.23 into the differential equations 5.23. Then matching each term order by order in $\tilde{G}^i\tilde{a}^j$ on the left and right hand side to a series of differential equations for the perturbations, which have to be solved. The 0th order corresponds to the Minkowski perturbation where the particle is not scattered. Before we start solving these ordinary differential equations, the perturbative corrections in $\tilde{G}^i$ only exclude all a dependence, as a result these correspond to Schwarzschild perturbations we have already solved for in chapter 4, which allows for less work to be done. Additionally, purely $\tilde{a}^j$ corrections correspond to setting $G = 0$ which reduces the Kerr metric to Minkowski space in ellipsoid co-ordinates. As a result these perturbations do not give any corrections to the scattering angle. However these are calculated just as a check to ensure no mistakes. The non trivial perturbations that do provide corrections from the spin of the

Order of $\tilde{G}\tilde{a}$	Scattering angle
$\tilde{G}\tilde{a}$	$\Delta\phi_{11} = -\frac{4GMa}{b^2v}$
$\tilde{G}^2\tilde{a}$	$\Delta\phi_{21} = -\frac{2\pi G^2M^2a\left(3v^2+2\right)}{b^3v^3}$
$\tilde{G}\tilde{a}^2$	$\Delta\phi_{12} = \frac{GMa^2}{b^3}\left(\frac{2}{v^2}+2\right)$
$\tilde{a}$	$\Delta\phi_{01} = 0$
$\tilde{a}^2$	$\Delta\phi_{02} = 0$
$\tilde{G}$	$\Delta\phi_{10} = \frac{2GM(v^2+1)}{bv^2}$
$\tilde{G}^2$	$\Delta\phi_{20} = \frac{3G^2M^2\pi}{4b^2}\left(\frac{4+v^2}{v^2}\right)$

Table 5.1: The scattering angles as a function of the velocity v of the particle at an infinite distance away from the black hole

black hole are the mixed terms $\tilde{G}^i\tilde{a}^j$. We solve the following differential equations for up to $\mathcal{O}(\tilde{G}^2\tilde{a}^2)$. The equations for the corrections r_{ij} and ϕ_{ij} are given in the appendix as they become cumbersome. The scattering angles corresponding to powers of $\tilde{G}^i\tilde{a}^j$ are given in table 5.1 where we also repeat the Schwarzschild scattering. From table 5.1 we see that the spin dominated contribution $\tilde{G}\tilde{a}^2$ decreases the scattering angle. It would be interesting to see whether the contributions with $j > i$ in $\tilde{G}^i\tilde{a}^j$ contributions all act this way. The purely $\tilde{a}^j$ terms indeed contribute no corrections to the scattering angle as we expected. Though we do start getting non-zero perturbations to the radial corrections in $\tilde{a}^i$ as seen in the appendix for r_{02}. While this is unexpected this may be explained by the fact the metric is expressed in ellipsoid co-ordinates as opposed to the spherical co-ordinates we are using. Finally we compare the results of these scattering angles with those shown in table.2 of the reference [25]. The authors of the paper calculate scattering angles using a Hamilton-Jacobi approach and perturbing in order of $\tilde{G}$ only. As a result they calculate the scattering angles already summed up in all orders of $\tilde{a}$. Therefore, our calculated scattering angles correspond to the $a \ll 1$ limit. The authors express the scattering angles in the table as factors of $\frac{G^nM^n}{v^{2n}(b^2-a^2)^{\frac{3n-1}{2}}}$. The term in order of G^2 in the reference is $\frac{\pi G^2M^2\left(v^4\left(b^2-a^2\right)^{5/2}+(a-bv)^3\left(-4a^2v+3ab+b^2v\right)\right)}{2a^2v^4(b^2-a^2)^{5/2}}$, expanding this to order a^2 yields:

$$G^2M^2\left(\frac{3\pi a^2}{2b^4v^4}+\frac{27\pi a^2}{2b^4v^2}+\frac{45\pi a^2}{16b^4}-\frac{4\pi a}{b^3v^3}-\frac{6\pi a}{b^3v}+\frac{3\pi}{b^2v^2}+\frac{3\pi}{4b^2}\right) \tag{5.27}$$

When expanded out the scattering angle contains the scattering angles we derived in order of $\tilde{G}^2$ and $\tilde{G}^2\tilde{a}$. This shows while the method of perturbing the geodesic equations still functions we only calculate the partial scattering angles as opposed to all orders in a as given in table 2 of [25]. In order to recreate these scattering angles we will have to calculate an infinite number of perturbations in powers of $\tilde{a}$ for each power of $\tilde{G}$.

We have been analysing the geodesics of these black holes to simulate the scattering of two black holes with large mass ratios. Having the smaller black hole scatter through the equatorial plane is a narrow case of what would generally happen. Therefore we have to consider scattering not constrained to the equatorial plane, the test particle now approaches the Kerr black hole with a non-zero z distance. Since we lose spherical symmetry we cannot simply orient our co-ordinate system such that the particle is in the equatorial plane. Hence the θ component cannot be ignored and it will evolve as a function of ν. We consider this case of non-equatorial scattering in the next section.

5.3.2 Non-equatorial scattering

Since $z \neq 0$ at $\nu = -\infty$, our test particle will travel a more complicated trajectory. As seen in the form of the geodesic equations which now become coupled by a factor $\rho^2 = r^2 + a^2\cos^2(\theta)$, which is present in all the relevant first order differential equations for r, θ and ϕ(we ignore t co-ordinate as before):

$$\begin{aligned}\rho^4\left(\frac{dr}{d\tau}\right)^2 &= (E(r^2+a^2)-aL_\phi)^2-\Delta(m^2r^2+(L_\phi-aE)^2+\mathcal{C})\\ \rho^4\left(\frac{d\theta}{d\tau}\right)^2 &= \mathcal{C}-\cos^2(\theta)\left(a^2(m^2-E^2)+\frac{L_\phi}{\sin^2(\theta)}\right)\\ \rho^2\frac{d\phi}{d\tau} &= -(aE-\frac{L_\phi}{\sin^2(\theta)})+(\frac{a}{\Delta})(E(r^2+a^2)-aL_\phi)\end{aligned} \tag{5.28}$$

These three coupled differential equations have to perturbed in both $\tilde{G}$ and $\tilde{a}$. The problem arises when the r differential equation which was previously uncoupled with θ and was purely a function of r. The other differential equation required for the trajectory the ϕ co-ordinate was just dependent on r, solving for r meant that ϕ was just a matter of integration. But now r is coupled with θ which means in theory we have to solve coupled differential equations which is a harder task than before. However utilising a perturbation series solution in fact simplifies the problem, since the coupled $a^2\cos^2(\theta)$ has a power of $\tilde{a}^2$, it will not appear in the first order perturbation equations for r. The coupling behaviour for r with θ arises at order $\tilde{G}^i\tilde{a}^2$, but this will only introduce the already solved lower order θ corrections. Hence a perturbative series solution in $\tilde{G}$ and $\tilde{a}$ decouples the differential equations. Before we begin we cast the θ equation into a more appropriate form for perturbation theory. Since θ occurs inside a trigonometric function we define a new co-ordinate $Y(\nu)=\cos(\theta(\tau))$, and we multiply both sides of the 2nd equation in 5.28 with $\sin^2(\theta(\tau))$ and absorb the $\sin(\theta(\nu))$ inside the differential where it becomes a $\cos(\theta(\tau))$. Then all $\sin^2(\theta(\tau))=1-Y(\tau)^2$. Additionally, the ϕ differential equation is rearranged as well in 5.28. Finally, we also make the hyperbolic proper time co-ordinate substitution $\tau=\frac{\sqrt{L_\phi^2+\mathcal{C}}}{p_{00}^2}\sinh(\nu)$ which is now slightly different:

$$\begin{aligned}\rho^4\left(\frac{dr}{d\nu}\right)^2 &= \left((E(r^2+a^2)-aL_\phi)^2-\Delta(m^2r^2+(L_\phi-aE)^2+\mathcal{C})\right)\frac{\cosh(\nu)\sqrt{\mathcal{C}+L_\phi^2}}{p_{00}^2}\\ \rho^4\left(\frac{dY(\nu)}{d\nu}\right)^2 &= \left(\mathcal{C}(1-Y(\nu)^2)-Y^2(\nu)\left(a^2(m^2-E^2)(1-Y(\nu)^2)+L_\phi\right)\right)\frac{\cosh(\nu)\sqrt{\mathcal{C}+L_\phi^2}}{p_{00}^2}\\ \rho^2(1-Y(\nu)^2)\Delta\frac{d\phi}{d\nu} &= \left(-(aE(1-Y(\nu)^2)\Delta-L_\phi\Delta)+(a(1-Y(\nu)^2))(E(r^2+a^2)-aL_\phi)\right)\frac{\cosh(\nu)\sqrt{\mathcal{C}+L_\phi^2}}{p_{00}^2}\end{aligned} \tag{5.29}$$

the free particle travelling on a scattering trajectory in Minkowski metric will be different. The situation is shown in figure.5.2. Previously the trajectory was confined to the xy plane with $z=0$. Now the particle trajectory is $x(-\infty)=b$ and $z(-\infty)=r(-\infty)Y(-\infty)=const.=h$ where h is the height of the particle from the black hole. This trajectory introduces additional angular momenta in the form of Carter's constant $\mathcal{C}$ which reduces to sum of $L_x^2+L_y^2$, hence the Minkowski co-ordinate trajectory becomes slightly different:

$$\begin{aligned}r_{00} &= \frac{\sqrt{\mathcal{C}+L_\phi^2}\cosh(\nu)}{p_{00}}\\ Y_{00} &= \frac{\sqrt{\mathcal{C}}\sin\left((L_\phi^2+\mathcal{C})\left(-\frac{2\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)}{L_\phi^2+\mathcal{C}}+C_\theta^{00}\right)\right)}{L_\phi^2+\mathcal{C}}\\ \phi_{00} &= \tan^{-1}\left(\frac{\sqrt{C+L^2}\sinh(v)}{L}\right)+C_\phi^{00}\end{aligned} \tag{5.30}$$

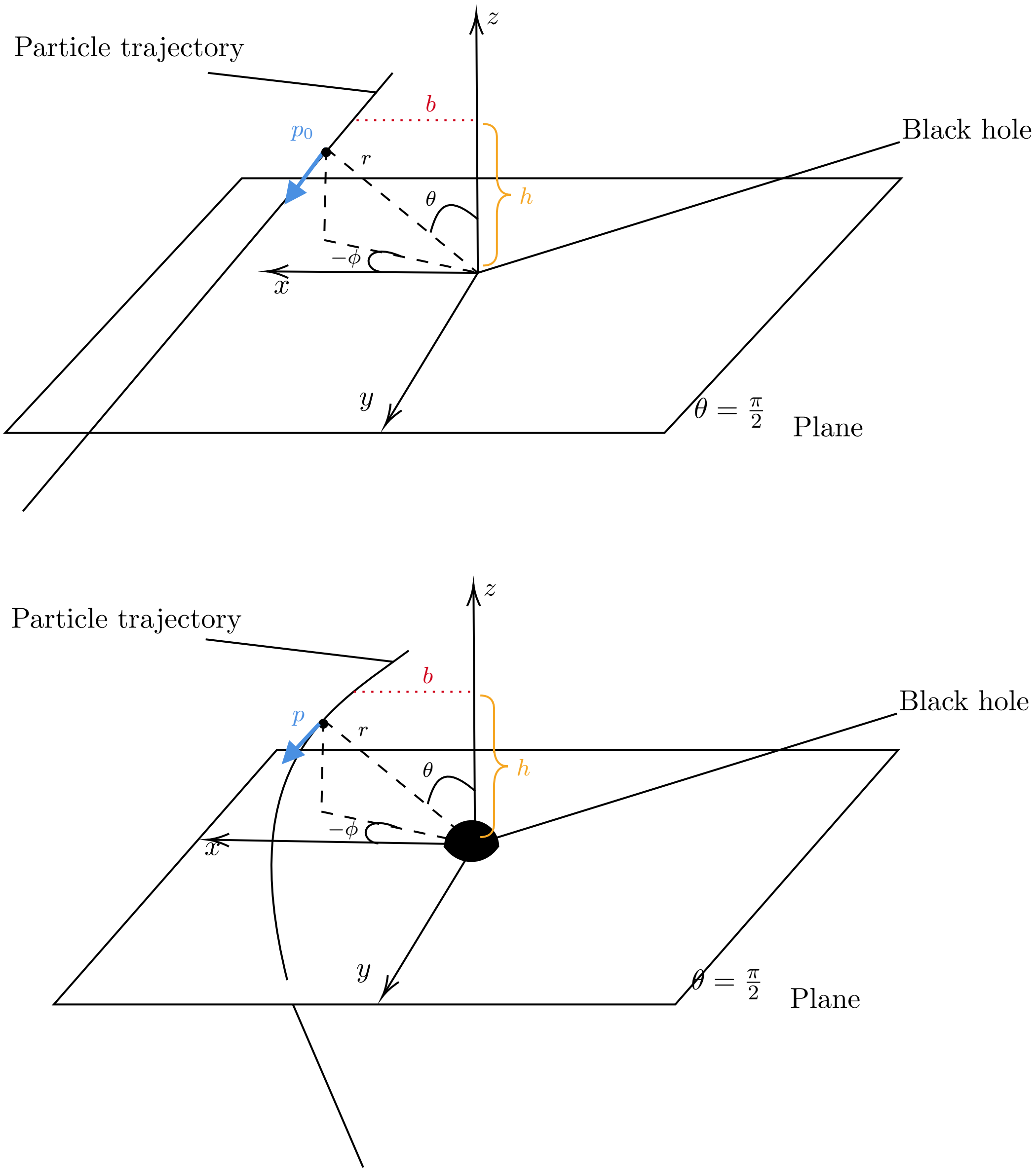


Figure 5.2: The non equatorial scattering case where the test particle approaches from infinity with a constant distance b from the black hole in the x co-ordinate and distance h in the z co-ordinate. The bottom figure is exaggerated to show the trajectory crossing the $\theta = \frac{\pi}{2}$ plane which it only does when it is infinitely far away. The above figure shows the trajectory in the absence of a black hole. The bottom figure shows the curved trajectory due to the Kerr black hole.

the integration constant for r_{00} is set to $C_r^{00} = 0$. Now we propose a perturbative series solution for co-ordinates r, Y and ϕ in terms of the same dimensionless parameters as before $\tilde{a}, \tilde{G}$:

$$\begin{aligned}
r(\nu) &\approx r_{00}(\nu) + \tilde{G}r_{10}(\nu) + \tilde{a}r_{01} + \tilde{G}\tilde{a}r_{11}(\nu) + \tilde{G}^2 r_{20}(\nu) + \tilde{a}^2 r_{02} + \tilde{G}^2\tilde{a}r_{21}(\nu) + \tilde{G}\tilde{a}^2 r_{12}(\nu) \\
Y(\nu) &\approx Y_{00}(\nu) + \tilde{G}Y_{10}(\nu) + \tilde{a}Y_{01} + \tilde{G}\tilde{a}Y_{11}(\nu) + \tilde{G}^2 Y_{20}(\nu) + \tilde{a}^2 Y_{02} + \tilde{G}^2\tilde{a}Y_{21}(\nu) + \tilde{G}\tilde{a}^2 Y_{12}(\nu) \\
\phi(\nu) &\approx \phi_{00}(\nu) + \tilde{G}\phi_{10}(\nu) + \tilde{a}\phi_{01} + \tilde{G}\tilde{a}\phi_{11}(\nu) + \tilde{G}^2\phi_{20}(\nu) + \tilde{a}^2\phi_{02} + \tilde{G}^2\tilde{a}\phi_{21}(\nu) + \tilde{G}\tilde{a}^2\phi_{12}(\nu)
\end{aligned} \tag{5.31}$$

The integration constants have to be fixed by boundary conditions $z(-\infty) = h$ and $x(-\infty) = b$.

Order of $\tilde{G}$ and $\tilde{a}$	Scattering angle in ϕ	Scattering angle in θ
$\tilde{G}$	$\Delta Y_{10} = -\frac{2GMh\left(v^2+1\right)}{v^2(b^2+h^2)}$	$\Delta\phi_{10} = \frac{2bGM\left(v^2+1\right)}{v^2(b^2+h^2)}$
$\tilde{a}$	$\Delta Y_{01} = 0$	$\Delta\phi_{01} = 0$
$\tilde{G}\tilde{a}$	$\Delta Y_{11} = \frac{8abGhM}{v(b^2+h^2)^2}$	$\Delta\phi_{11} = -\frac{4aGM\left(b^2-h^2\right)}{v(b^2+h^2)^2}$
$\tilde{a}^2$	$\Delta Y_{02} = 0$	$\Delta\phi_{20} = 0$
$\tilde{G}\tilde{a}^2$	$\Delta Y_{12} = \frac{2a^2GhM\left(v^2+1\right)\left(h^2-3b^2\right)}{v^2(b^2+h^2)^3}$	$\Delta\phi_{12} = \frac{2a^2bGM\left(v^2+1\right)\left(b^2-3h^2\right)}{v^2(b^2+h^2)^3}$
$\tilde{G}^2$	$\Delta Y_{20} = -\frac{3\pi G^2hm^3M^2\left(v^2+4\right)}{4v^2(m^2(b^2+h^2))^{3/2}}$	$\Delta\phi_{20} = \frac{3\pi bG^2M^2\left(v^2+4\right)}{4v^2(b^2+h^2)^{3/2}}$
$\tilde{G}^2\tilde{a}$	$\Delta Y_{21} = \frac{3\pi abG^2hM^2\left(3v^2+2\right)}{v^3(b^2+h^2)^{5/2}}$	$\Delta\phi_{21} = -\frac{\pi aG^2M^2\left(3v^2+2\right)\left(2b^2-h^2\right)}{v^3(b^2+h^2)^{5/2}}$

Table 5.2: The Kerr scattering angles in θ and ϕ in order of $\tilde{G}^i\tilde{a}^j$. The angles are expressed in velocity v

For the z boundary condition the constant $C^0_\theta = \frac{\pi}{2(L_\phi^2+\mathcal{C})}$ satisfies the boundary condition where $z(-\infty) = h = \frac{\sqrt{\mathcal{C}}}{p_{00}}$. This yields a relation between the Carter's constant and the height of approach: $\mathcal{C} = (p_{00}h)^2$. As a result the $x(-\infty) = r(-\infty)\cos(-\infty)\sin(-\infty) = r_{00}(-\infty)\cos(\phi_{00}(-\infty)) = b$ which is the same boundary condition as before, as a result the relation $\frac{L_\phi}{p_{00}} = b$ still holds from before. Therefore $C^0_\phi = 0$ as before. The boundary condition is the same as before for the equatorial plane case for ϕ, we introduce a similar condition for θ to fix the respective integration constants in the perturbations. Therefore the boundary conditions for the ϕ_{ij} corrections are the same as before: $\phi_{ij}(-\infty) = 0$ which fix the C^{ij}_ϕ associated with ϕ perturbations integration constants. On the other hand the integration constants C^{ij}_θ for θ perturbations are fixed by $z(-\infty) = r(-\infty)Y(-\infty) = h$. Submitting in the perturbation series for both Y and r in the boundary condition yields:

$$\begin{aligned} z(\nu) = &\Big(r_{00}(\nu) + \tilde{G}r_{10}(\nu) + \tilde{a}r_{01} + \tilde{G}\tilde{a}r_{11}(\nu) + \tilde{G}^2r_{20}(\nu) + \tilde{a}^2r_{02} + \tilde{G}^2\tilde{a}r_{21}(\nu) + \tilde{G}\tilde{a}^2r_{12}(\nu)\Big) \\ \times &\Big(Y_{00}(\nu) + \tilde{G}Y_{10}(\nu) + \tilde{a}Y_{01} + \tilde{G}\tilde{a}Y_{11}(\nu) + \tilde{G}^2Y_{20}(\nu) + \tilde{a}^2Y_{02} + \tilde{G}^2\tilde{a}Y_{21}(\nu) + \tilde{G}\tilde{a}^2Y_{12}(\nu)\Big) = h \end{aligned} \tag{5.32}$$

evaluated at $\nu = -\infty$. Expanding this out one can see the condition is satisfied when all $Y_{ij}(-\infty) = 0$. The C^{ij}_r integration constants for $r_{ij}(\nu)$ perturbations, are set to $C^{ij}_r = 0$ as before.

We insert the perturbation series for r, Y and ϕ in 5.31 into the differential equations, gather all terms by powers of $\tilde{G}^i\tilde{a}^j$ and solve the associated ordinary differential equations. Letting *Mathematica* solve these equations yield the perturbative corrections to the three co-ordinates which are given in the appendix. The respective boundary conditions are applied upon the perturbations which fix the integration constants. After the scattering angles are calculated from these angle perturbations by taking $\nu = \infty$, $\Delta\phi_{ij} = \tilde{G}^i\tilde{a}^j\phi_{ij}(\infty)$ likewise $\Delta Y_{ij} = \tilde{G}^i\tilde{a}^jY_{ij}(\infty)$. The scattering angles for ϕ and θ are tallied in table 5.2.

We were not able to find a reference to compare our results. Therefore, we check if our results are sensible under. As expected the pure $\tilde{a}^j$ contributions provided nothing as the particle is in Minkowski space, even though the perturbative corrections in $\tilde{a}^j$ for r and θ were non-zero at times. This might be an artefact of the co-ordinate system, since Minkowski space is expressed in ellipsoid co-ordinates in $\tilde{a}$ perturbations and we are still using spherical co-ordinates to describe the trajectory. Since we utilised the conditions $\mathcal{C} = (hp_{00})^2$ and $L_\phi = bp_{00}$ we expressed the scattering angles in terms of these parameters. Every scattering angle has a denominator with some power of $h^2 + b^2$, this is distance of closest approach to the origin for the Minkowski geodesic. The θ scattering angles are directly proportional to h which is consistent as these θ scattering angles have to disappear when $h \to 0$ since the particle is once again constrained to the equatorial plane and we retrieve the equatorial plane scattering. Taking $h \to 0$ for the ϕ scattering angles reproduces the equatorial Kerr scattering angles in table 5.1. Our calculated scattering angles for both θ and ϕ behave as expected. The θ scattering angles displayed in table.5.2 are corrections to $\cos(\theta)$, as a result the actual scattering angle in θ:

$$\Delta\theta(\infty) = \cos^{-1}(Y_{00}(\infty) + \tilde{G}Y_{10}(\infty) + \tilde{G}\tilde{a}Y_{11}(\infty) + \tilde{G}^2 Y_{20}(\infty) + ...) \tag{5.33}$$

The purely $\tilde{G}^i$ perturbations for θ are negative, meaning the value of the argument inside the $\cos^{-1}$ is negative which means $\theta(-\infty) \geq \frac{\pi}{2}$. This is expected as the test particle is deflected towards the black hole which increases the polar angle θ (see figure), this is the contribution from the non-spinning part of the black hole. Interestingly however the spin and gravity mixed terms $\tilde{G}^i\tilde{a}^j$ are all positive, implying they will decrease the scattering angle. The spin of the black hole seems to slightly negate the effect of gravity on the θ scattering angle. This effect is of course relatively very small compared to the biggest contribution to the θ scattering angle which is $\Delta\theta_0$. We also note that the ϕ_{11} scattering angle interestingly disappears completely when $b = h$. Furthermore we have only calculated a few corrections to the θ scattering angle therefore we cannot say for certain that the mixed "spin and gravitaional" terms $\tilde{G}^i\tilde{a}^j$ will always be positive, perhaps they might become negative for higher orders. For ϕ perturbative corrections we have the same behaviour as with the equatorial scattering, mixed terms dominated by $\tilde{G}^i\tilde{a}^j$ where $i > j$ are negative and increase scattering angle, whereas $\tilde{G}^i\tilde{a}^j$ terms with $i < j$ are positive. To make more concrete observations about the behaviour of the scattering angles in θ or ϕ we need higher order calculations.This task becomes difficult when one considers the increasingly long expressions for the perturbations given in appendix.

Chapter 6

Discussion and Conclusion

Throughout this thesis, we have explored the idea of continuous symmetries on curved spaces through objects such as Killing vectors and Killing tensors. Beginning with the Killing vector fields, which represented isometries. Then we examined their connection to conserved quantities on geodesics.

These objects were first explored in simple spaces such as the Minkowski metric and the 2-sphere metric. Then the Killing vector was generalised to the conformal Killing vectors to study conformal symmetry. These vectors were studied on Minkowski space, and the special case of two-dimensional Minkowski was analysed to see how the conformal killing vectors formed the Witt algebra. Then the 2-sphere case was analysed, utilising a stereographic projection onto two complex patches, which simplified both the calculations and the forms of the Killing and Conformal Killing vectors, as well as revealing the complex structure of the 2-sphere. Then we analysed the higher dimensional tensor generalisation of Killing vectors which were Killing tensors, and these were symmetric tensors which, when contracted with momenta, provided quantities that were conserved on geodesics. The Killing tensors do not stem from a geometric origin, such as with Killing vectors. As a result, we label these "hidden symmetries".

Killing tensors are explored on Minkowski and the 2-sphere metric, and it was shown that there exists no irreducible rank-2 Killing tensors that cannot be formed from symmetrising the available Killing vectors. This lack of irreducible Killing tensors for spaces of constant curvature was proved by G.Thompson in [12]. The role of the rank-2 Killing tensors in separating the Hamilton-Jacobi equation was explored, and we briefly discuss the separability conditions for the Klein-Gordon equation.

After this, we begin to apply our knowledge of the Killing vectors and tensors to analyse physically interesting metrics such as the Schwarzschild and Kerr black holes. It was found that the Schwarzschild geometry contains four Killing vectors; one along the time coordinate, which makes the metric static and the other three were the rotational Killing vectors which give the Schwarzschild metric its spherical symmetry. However, there were no irreducible rank-2 Killing tensors, but it was still possible to cast the geodesic equation into a first-order form due to two Killing vectors and the metric.

The geodesic equations were then perturbed in Newton's constant G to calculate the scattering angles of a test particle. Finally, the Kerr metric was investigated, which possessed only two Killing vectors; the time t and ϕ coordinate vectors due to the loss of spherical symmetry caused by the spin of the black hole. However, the geometry does possess a "hidden symmetry" through an irreducible rank-2 Killing tensor which gives rise to Carter's constant. This constant of motion provides the necessary number of conserved quantities for integrability, allowing the geodesic equation to be cast into a first-order form. Next, these geodesic equations were perturbed in G and the spin of the black hole a to find scattering angles for ϕ in the equatorial case and for ϕ and θ in the non-equatorial case. These scattering angles were compared with literature values and were found to be in agreement in the $a \ll 1$ limit.

6.1 Future Work

We have only scratched the surface of these topics, as there are many ways to continue further research. The thesis was primarily focused on Killing tensors, but there are more fundamental objects, such as Killing-Yano tensors. These objects can be used to produce Killing tensors. For example, for the Kerr-NUT Ads metric, the conformal generalisation of Killing-Yano is a very important object that can be used to generate all other symmetries present in the metric [10]. Furthermore, Killing-Yano tensors can be used to study spinning particles with supersymmetry.

The geodesics were studied to model the two-body black hole binary system, where one black hole was much more massive than the other. This was done to compute scattering angles from classical general relativity to compare with the results produced from utilising quantum field theory amplitude techniques. Currently amplitude techniques are being used to model gravitational waves from black hole binaries [29]. The scattering angles computed from using general relativity may provide a check for those computed from amplitudes.

The more interesting problem is binary black hole scattering involving two spinning black holes, which would involve the test particle containing its own angular momentum. As a result, it will not travel in the same geodesics as before; the equations of motion are now determined by the Mathisson-Papapetrou-Dixon equations. It would be interesting to see how the scattering angles would change when the test particle has spin. Hyperbolic orbits for spinning test particles were already examined for Schwarzschild metrics in reference [30]. Therefore investigating the change in the dynamics of the test particle when it contains angular momentum would be a very relevant and interesting problem to investigate as part of future research.

Chapter 7

Appendix

7.1 Appendix-A: Solving the system of partial differential equations for 2-sphere Killing vectors

The system of partial differential equations to solve for the Killing vectors are as follows:

$$\frac{\partial K^\theta}{\partial \theta} = 0 \tag{7.1}$$

$$\frac{\partial K^\theta}{\partial \phi} + \sin^2\theta \frac{\partial K^\phi}{\partial \theta} = 0 \tag{7.2}$$

$$\sin^2\theta \frac{\partial K^\phi}{\partial \phi} + \sin\theta\cos\theta K^\theta = 0 \tag{7.3}$$

from equation 7.1 we deduce that K^θ is purely a function of ϕ:

$$K^\theta = f(\phi) \tag{7.4}$$

where $f(\phi)$ is a function of ϕ. We substitute this into eq.7.2 and differentiate by $\frac{\partial}{\partial \phi}$ to get:

$$\frac{\partial^2 f}{\partial \phi^2} = -\sin^2(\theta) \frac{\partial^2 K^\phi}{\partial \phi \partial \theta} \tag{7.5}$$

dividing both sides by $\sin^2(\theta)$ for equation 7.3 yields:

$$\frac{\partial K^\phi}{\partial \phi} = -\cot(\theta) K^\theta \tag{7.6}$$

we substitute this into equation 7.5 to eliminate K^ϕ, such that we get a differential equation for $K^\theta = f(\phi)$:

$$\frac{\partial^2 f}{\partial \phi^2} = -\sin^2(\theta) \frac{\partial}{\partial \theta} \left(-\cot(\theta) f(\phi)\right) = -f(\phi) \tag{7.7}$$

which is a simple harmonic oscillator differential equation. Hence the equation for $f(\phi)$ is:

$$f(\phi) = A\cos(\phi) + B\sin(\phi) \tag{7.8}$$

where A and B are constants. Now we assume separability for the vector field K^ϕ, such that it becomes $K^\phi = a(\theta)b(\phi)$. Inputting this form of K^ϕ into equation 7.2 and differentiating by $\frac{\partial}{\partial \phi}$ yields:

$$\begin{aligned} &\frac{\partial^2 f(\phi)}{\partial \phi^2} + \sin^2(\theta) \frac{\partial a(\theta)}{\partial \theta} \frac{\partial b(\phi)}{\partial \phi} \\ &\frac{\partial b}{\partial \phi} = -\frac{\partial^2 f}{\partial \phi^2} \frac{1}{\sin^2(\theta) \frac{\partial a}{\partial \theta}} \end{aligned} \tag{7.9}$$

which we substitute into equation 7.3 such that it becomes:

$$-a(\theta)\frac{\partial^2 f}{\partial\phi^2}\frac{1}{\sin(\theta)\frac{\partial a}{\partial\theta}}+\cos(\theta)f=0 \tag{7.10}$$

the f dependent terms cancel out with a minus sign which makes the equation:

$$\frac{\partial a}{\partial\theta}=-\frac{a(\theta)}{\sin(\theta)\cos(\theta)} \tag{7.11}$$

solving this for $a(\theta)$ yields: $a(\theta)=\cot(\theta)$. This allows us to solve for $b(\phi)$ whose differential equation is given in the 2nd line of 7.9. Inputting $a(\theta)$ reduces this equation to:

$$\frac{\partial b(\phi)}{\partial\phi}=-f=-A\cos(\phi)-B\sin(\phi) \tag{7.12}$$

hence $K^\phi=\cot(\theta)(-A\sin(\phi)+B\cos(\phi))$. As a result we have solved for the 2-sphere Killing vector components. The two integration constants A and B segregate the two different Killing vectors. Along with the $\frac{\partial}{\partial\phi}$ Killing vector the 2-sphere Killing vectors are:

$$K=K^\mu\frac{\partial}{\partial x^\mu}=A\left(\cos(\phi)\frac{\partial}{\partial\theta}-\sin(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right)+B\left(\sin(\phi)\frac{\partial}{\partial\theta}+\cos(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right)+C\left(\frac{\partial}{\partial\phi}\right) \tag{7.13}$$

C is also a constant.

7.2 Appendix-B: Proof that rank-3 flat space Killing tensors contain only cubic polynomials as the highest polynomial power.

The proof follows the same idea as to what was done in Chapter 2 when finding the rank-2 tensors. The rank-3 Killing tensor is expanded in a power series, which is then substituted into the Killing's equation for rank-3. This yields symmetries for the indices of the constant coefficient tensors in the power series. Then we use these symmetries to show that one of the higher rank coefficient tensor vanishes, which terminates the series and bounds the highest rank of the polynomial to a cubic.

The rank-3 Killing equation is:

$$\partial_{(\mu}K_{\nu\rho\beta)}=0\rightarrow\partial_\mu K_{\nu\rho\beta}+\partial_\beta K_{\mu\nu\rho}+\partial_\rho K_{\beta\mu\nu}+\partial_\nu K_{\rho\beta\mu}=0 \tag{7.14}$$

$K_{\mu\nu\rho}$ is completely symmetric under the exchange of indices. Now we find the solution to this equation by expanding in power series:

$$K_{\mu\nu\rho}=K^{(0)}_{\mu\nu\rho}+K^{(1)}_{\mu\nu\rho\alpha_1}x^{\alpha_1}+K^{(2)}_{\mu\nu\rho\alpha_1\alpha_2}x^{\alpha_1}x^{\alpha_2}+K^{(3)}_{\mu\nu\rho\alpha_1\alpha_2\alpha_3}x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}+K^{(4)}_{\mu\nu\rho\alpha_1\alpha_2\alpha_3\alpha_4}x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}x^{\alpha_4}+... \tag{7.15}$$

In rank-2 case the cubic polynomial term disappeared, so here we expect the quartic polynomial term $K^{(4)}_{\mu\nu\rho\alpha_1\alpha_2\alpha_3\alpha_4}$ to disappear. Furthermore, it is symmetric in the first three and last four indices. We substitute this term into the rank-3 Killing's equation, which leads to the following after renaming the indices:

$$\begin{gathered}\left(K^{(4)}_{\nu\rho\beta\mu\alpha_1\alpha_2\alpha_3}+K^{(4)}_{\mu\nu\rho\beta\alpha_1\alpha_2\alpha_3}+K^{(4)}_{\mu\nu\beta\rho\alpha_1\alpha_2\alpha_3}+K^{(4)}_{\mu\beta\rho\nu\alpha_1\alpha_2\alpha_3}\right)x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}=0\\ K^{(4)}_{\nu\rho\beta\mu\alpha_1\alpha_2\alpha_3}=-K^{(4)}_{\mu\nu\rho\beta\alpha_1\alpha_2\alpha_3}-K^{(4)}_{\mu\nu\beta\rho\alpha_1\alpha_2\alpha_3}-K^{(4)}_{\mu\beta\rho\nu\alpha_1\alpha_2\alpha_3}\end{gathered} \tag{7.16}$$

which institutes this cyclic relation on the first four indices. The aim now is to get these summation indices α_i into the position of the first three indices of $K^4_{\dots}$. Starting from:

$$K^{(4)}_{\mu\nu\rho\alpha_1\beta\alpha_2\alpha_3} + K^{(4)}_{\beta\mu\nu\alpha_1\rho\alpha_2\alpha_3} + K^{(4)}_{\rho\beta\mu\alpha_1\nu\alpha_2\alpha_3} + K^{(4)}_{\nu\rho\beta\alpha_1\mu\alpha_2\alpha_3} = 0 \tag{7.17}$$

we apply 7.16 on each term of this equation to push α_1 into the first three index positions. This yields twelve terms which we sum up using the symmetries in the first 3 indices and divide by -2 to get the following:

$$K^{(4)}_{\alpha_1\mu\nu\rho\beta\alpha_2\alpha_3} + K^{(4)}_{\rho\alpha_1\mu\nu\beta\alpha_2\alpha_3} + K^{(4)}_{\nu\rho\alpha_1\mu\beta\alpha_2\alpha_3} + K^{(4)}_{\alpha_1\beta\mu\nu\rho\alpha_2\alpha_3} + K^{(4)}_{\nu\alpha_1\beta\mu\rho\alpha_2\alpha_3} + K^{(4)}_{\alpha_1\rho\beta\mu\nu\alpha_2\alpha_3} \tag{7.18}$$

we use 7.16 on the first term then simplifying and sending some terms to the right hand side yields.:

$$K^{(4)}_{\mu\nu\rho\alpha_1\beta\alpha_2\alpha_3} = K^{(4)}_{\mu\alpha_1\beta\nu\rho\alpha_2\alpha_3} + K^{(4)}_{\alpha_1\nu\beta\mu\rho\alpha_2\alpha_3} + K^{(4)}_{\alpha_1\beta\rho\mu\nu\alpha_2\alpha_3} \tag{7.19}$$

this switches the two of the indices in the first three positions with indices 4 and 5 or α_1, β get exchanged into the first three index positions containing μ, ν, ρ. Then revisiting 7.18 we move α_2 index to the fourth position for each term, then we apply 7.16 on each term to push α_2 into the first 3 index positions. We get eighteen terms which we simplify using symmetry in the first 3 indices, then the equation reduces to:

$$\begin{aligned} 3\left(K^{(4)}_{\alpha_1\alpha_2\mu\nu\rho\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\nu\mu\rho\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\rho\mu\nu\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\beta\mu\nu\rho\alpha_3}\right) + K^{(4)}_{\rho\alpha_2\mu\alpha_1\nu\beta\alpha_3} + K^{(4)}_{\alpha_2\nu\mu\alpha_1\rho\beta\alpha_3} \\ + K^{(4)}_{\alpha_2\mu\beta\alpha_1\nu\rho\alpha_3} + K^{(4)}_{\alpha_2\nu\rho\alpha_1\mu\beta\alpha_3} + K^{(4)}_{\alpha_2\beta\nu\alpha_1\mu\rho\alpha_3} + K^{(4)}_{\alpha_2\rho\beta\alpha_1\mu\nu\alpha_3} = 0 \end{aligned} \tag{7.20}$$

We notice the 5th, 6th and 7th terms can be combined into one term using 7.19:

$$\begin{aligned} 3\left(K^{(4)}_{\alpha_1\alpha_2\mu\nu\rho\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\nu\mu\rho\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\rho\mu\nu\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\beta\mu\nu\rho\alpha_3}\right) + K^{(4)}_{\rho\nu\beta\alpha_2\mu\alpha_1\alpha_3} \\ + K^{(4)}_{\alpha_2\nu\rho\alpha_1\mu\beta\alpha_3} + K^{(4)}_{\alpha_2\beta\nu\alpha_1\mu\rho\alpha_3} + K^{(4)}_{\alpha_2\rho\beta\alpha_1\mu\nu\alpha_3} = 0 \end{aligned} \tag{7.21}$$

then applying 7.16 for the 5th term (1st term outside bracket) permutes α_2 through the first 3 index positions which cancels out the 6th, 7th and 8th terms leaving only.:

$$K^{(4)}_{\alpha_1\alpha_2\mu\nu\rho\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\nu\mu\rho\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\rho\mu\nu\beta\alpha_3} + K^{(4)}_{\alpha_1\alpha_2\beta\mu\nu\rho\alpha_3} = 0 \tag{7.22}$$

Finally, we need to permute α_3 through the first 3 index positions. For each term in the above equation we shift α_3 to the 4th index position and then apply 7.16 on each term and multiply by -1 to find twelve terms:

$$\begin{aligned} &K^{(4)}_{\alpha_3\alpha_2\alpha_1\mu\nu\rho\beta} + K^{(4)}_{\mu\alpha_3\alpha_2\alpha_1\nu\rho\beta} + K^{(4)}_{\alpha_1\mu\alpha_3\alpha_2\nu\rho\beta} \\ +&K^{(4)}_{\alpha_3\alpha_2\alpha_1\nu\mu\rho\beta} + K^{(4)}_{\nu\alpha_3\alpha_2\alpha_1\mu\rho\beta} + K^{(4)}_{\alpha_1\nu\alpha_3\alpha_2\mu\rho\beta} \\ +&K^{(4)}_{\alpha_3\alpha_2\alpha_1\rho\mu\nu\beta} + K^{(4)}_{\rho\alpha_3\alpha_2\alpha_1\mu\nu\beta} + K^{(4)}_{\alpha_1\rho\alpha_3\alpha_2\mu\nu\beta} \\ +&K^{(4)}_{\alpha_3\alpha_2\alpha_1\beta\mu\nu\rho} + K^{(4)}_{\beta\alpha_3\alpha_2\alpha_1\mu\nu\rho} + K^{(4)}_{\alpha_1\beta\alpha_3\alpha_2\mu\nu\rho} = 0 \end{aligned}$$

The first column is identical so it adds up to give a factor of 4, the second and third columns disappear due to 7.22. Hence, we obtain the desired result:

$$K^{(4)}_{\alpha_3\alpha_2\alpha_1\mu\nu\rho\beta} = 0 \tag{7.23}$$

this ensures that the power series terminates from the quartic polynomial terms. As a result the highest polynomial power that flat rank-3 Killing tensors can have is a cubic polynomial.

7.3 Appendix-C: Schwarzschild perturbation equations up to order $\tilde{G}^3$

Here are the following Schwarzschild radial perturbations by order of $\tilde{G}$.

$\tilde{G}$:

$$\begin{aligned} r_1(\nu) = \frac{\operatorname{sech}^3(v)}{4bp^4}&[2b^2m^4v^2 + 2b^2m^4v\sinh(2v) + b^2m^4 - 2b^2m^2p^2 \\ +b^2\left(m^4 - 2m^2p^2 - 3p^4\right)\cosh(2v) &- 3b^2p^4\sinh(v)\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right) \\ &-3b^2p^4\sinh(3v)\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right) \\ &-3b^2p^4] \end{aligned} \tag{7.24}$$

$\tilde{G}^2$:

$$\begin{aligned} r_2(\nu) = \frac{\tanh(v)\operatorname{csch}\left(\frac{v}{2}\right)\operatorname{sech}\left(\frac{v}{2}\right)\operatorname{sech}^2(v)}{8bp^4}&[2b^2m^4v^2 + 2b^2m^4v\sinh(2v) + b^2m^4 - 2b^2m^2p^2 \\ &+b^2\left(m^4 - 2m^2p^2 - 3p^4\right)\cosh(2v) \\ -3b^2p^4\sinh(v)\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right) &- 3b^2p^4\sinh(3v)\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right) - 3b^2p^4] \end{aligned} \tag{7.25}$$

$\tilde{G}^3$:

$$\begin{aligned} r_3(\nu) = -\frac{b}{2p^6}&[m^6v^2(v\tanh(v) - 3)\operatorname{sech}^4(v) + m^6 - 2m^4p^2 + 6m^2p^4v\operatorname{sech}^3(v)\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right) \\ &+6m^2p^4\tanh(v)\operatorname{sech}(v)\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right) \\ &+3m^2p^4v\tanh(v)\left(\log\left(1 - e^{-2v}\right) - \log\left(e^{-v} + 1\right) - \log(\sinh(v) - \cosh(v) + 1)\right) \\ &+8m^2p^4 + \operatorname{sech}^2(v)\left(m^6\left(v^2 - 1\right) + 2m^4p^2 - m^4v\left(m^2 + 2p^2\right)\tanh(v) - 8p^6\right) + 16p^6] \end{aligned} \tag{7.26}$$

These are presented directly from *Mathematica* after solving the associated differential equation. As a result these can possibly be simplified further.

Now we present the ϕ angle perturbations after instituting the boundary condition $\phi_i(-\infty) = 0$: $\tilde{G}$:

$$\phi_1(\nu) = \frac{m^2\tanh(v)}{p^2} + \frac{m^2v\operatorname{sech}^2(v)}{p^2} + \frac{m^2}{p^2} + 2\tanh(v) + 2 \tag{7.27}$$

$\tilde{G}^2$:

$$\begin{aligned} \phi_2(\nu) = -\frac{m^2v\operatorname{sech}^3(v)\left(bm^2v\tanh(v) - 2b\left(m^2 + p^2\right)\right)}{bp^4} &- \frac{\operatorname{sech}^4(v)\left(3b\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right) + 3b\cosh(2v)\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right)\right)}{2b} \\ -\frac{23m^4\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right)}{12p^4} &+ \frac{2m^4\tan^{-1}(\sinh(v) + \cosh(v))}{3p^4} + \frac{m^2\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right)}{p^2} \\ +\frac{3}{4}\left(\frac{4m^2}{p^2} + 5\right)\tanh(v)\operatorname{sech}(v) &+ \frac{5}{16}\left(\frac{2m^4}{p^4} + \frac{8m^2}{p^2} + 6\right)\tan^{-1}(\sinh(v)) \\ +\pi\left(-\frac{m^4}{6p^4} + \frac{3m^2}{2p^2} + \frac{15}{8}\right) &+ \frac{15}{4}\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right) \end{aligned} \tag{7.28}$$

$\tilde{G}^3$:

$$\begin{aligned}\phi_3(\nu) = \frac{1}{6b^3p^6}[&-8b^3m^6v^3\text{sech}^6(v) - b^3\left(2m^6 - 24m^4p^2 - 144m^2p^4 - 128p^6\right)\tanh(v)\\ &-2b^3\left(2m^6 - 9m^4p^2 - 24m^2p^4 - 8p^6\right)\tanh(v)\text{sech}^2(v)\\ &-36b^2p^4\text{sech}^3(v)\tan^{-1}\left(\tanh\left(\frac{v}{2}\right)\right)\left(b\left(m^2+p^2\right) - bm^2v\tanh(v)\right)\\ &+6bm^2v\text{sech}^4(v)\left(b^2\left(m^4\left(v^2+2\right)+8m^2p^2+6p^4\right) - b^2m^2v\left(4m^2+3p^2\right)\tanh(v)\right)]\\ &-\frac{m^6}{3p^6} + \frac{4m^4}{p^4} + \frac{24m^2}{p^2} + \frac{64}{3}\end{aligned} \tag{7.29}$$

7.4 Appendix-D: Derivation of first order geodesics for Kerr metric

Here we derive the first order form of the geodesic equations for the Kerr metric in the "Carter Canonical" form, which is acquired by a change of co-ordinates involving θ and t:

$$y = a\cos(\theta), \psi = \frac{\phi}{a},$$
$$v = t - a\phi$$

The trigonometric $\cos(\theta)$ is suppressed in the new co-ordinate y, ϕ is scaled by the spin parameter a and most distinctly the time co-ordinate t is translate by a factor of ϕ scaled by a. Whereas the r and θ co-ordinates are unchanged. The authors undertake this co-ordinate transformation because of its usefulness when studying higher dimensional black holes. For our purposes it casts the metric and conserved quantities into a more manageable form without trigonometric functions:

$$ds^2 = \frac{1}{\Sigma}\left[-\Delta_r(dv + y^2d\psi)^2 + \Delta_y(dv - r^2d\psi)^2\right] + \Sigma\left[\frac{dr^2}{\Delta_r} + \frac{dy^2}{\Delta_y}\right] \tag{7.30}$$

Where $\Delta_r = r^2 - 2GM + a^2$, $\Delta_y = a^2 - y^2$ and $\Sigma = r^2 + y^2$. We only use this form to derive the first order form of the geodesic equations and nothing else. The energy killing vector still remains the same: $\frac{\partial}{\partial t} = \frac{\partial}{\partial v}$. The angular momentum changes to a combination of energy and angular momentum: $\frac{\partial}{\partial \phi} = \frac{1}{a}\frac{\partial}{\partial \psi} - a\frac{\partial}{\partial v}$:

$$E \to -E = \left(\frac{\Delta_y - \Delta_r}{\Sigma}\right)\dot{v} - \left(\frac{\Delta_r y^2 + \Delta_y r^2}{\Sigma}\right)\dot{\psi} \tag{7.31}$$

$$L_\psi = aL_\phi - a^2E = \left(\frac{-y^4\Delta_r + r^4\Delta_y}{\Sigma}\right)\dot{\psi} - \left(\frac{\Delta_r y^2 + \Delta_y r^2}{\Sigma}\right)\dot{v} \tag{7.32}$$

finally the Carter constant changes as:

$$K = k_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{\Sigma r^2}{\Delta_y}\dot{y}^2 - \frac{\Sigma y^2}{\Delta_r}\dot{r}^2 + (\frac{y^2\Delta_r + r^2\Delta_y}{\Sigma})\dot{v}^2 + (\frac{y^6\Delta_r + r^6\Delta_y}{\Sigma})\dot{\psi}^2 + 2\dot{\psi}\dot{v}(\frac{y^4\Delta_r + r^4\Delta_y}{\Sigma}) \tag{7.33}$$

The final required equation is the "normalisation" of the momenta or the velocities, from the metric Killing tensor which leads to the mass of the particle:

$$-m^2 = \frac{\Sigma}{\Delta_y}\dot{y}^2 + \frac{\Sigma}{\Delta_r}\dot{r}^2 + \frac{\Delta_y - \Delta_r}{\Sigma}\dot{v}^2 + (\frac{-y^4\Delta_r + r^4\Delta_y}{\Sigma})\dot{\psi}^2 - 2(\frac{\Delta_r y^2 + \Delta_y r^2}{\Sigma})\dot{\psi}\dot{v} \tag{7.34}$$

$\dot{\psi}$ and $\dot{v}$ can be worked out from eq.7.31 and 7.32. Isolating $\dot{\psi}$ first:

$$\dot{\psi} = \frac{\Sigma}{\Delta_r y^2 + \Delta_y r^2}\left[E + (\frac{\Delta_y - \Delta_r}{\Sigma})\dot{v}\right]$$

Which is then inputted into eq.7.32, which becomes the following after simplifying everything:

$$L_\psi = \frac{-y^4\Delta_r + r^4\Delta_y}{\Delta_r y^2 + \Delta_y r^2}\left[(\frac{\Delta_y - \Delta_r}{\Sigma})\dot{v} - (\frac{r^2\Delta_y + \Delta_r y^2}{\Sigma})\dot{v}\right]$$

Which after simplifying and solving for $\dot{v}$ becomes our first equation in the time variable which we will not use:

$$\Sigma\dot{v} = \frac{r^2(Er^2 - L_\psi)}{\Delta_r} - \frac{y^2(Ey^2 + L_\psi)}{\Delta_y} \tag{7.35}$$

Then using this equation we isolate $\dot{v}$ and just input this into the equation 7.31:

$$-\Sigma E = (\Delta_y - \Delta_r)\left[\frac{1}{\Sigma}(\frac{r^2(Er^2 - L_\psi)}{\Delta_r} - \frac{y^2(Ey^2 + L_\psi)}{\Delta_y})\right] - (\Delta_r y^2 + \Delta_y r^2)\dot{\psi}$$

Which we solve for $\dot{\psi}$:

$$\Sigma\dot{\psi} = \frac{Er^2 - L_\psi}{\Delta_r} + \frac{Ey^2 + L_\psi}{\Delta_y} \tag{7.36}$$

Which becomes our second differential equation for the $\psi = \frac{\phi}{a}$ angle. Now we input both of these into equations 7.34 and 7.33 and take a linear combination such that we isolate y, after simplifying the equation becomes:

$$\Delta_y(K - m^2y^2) = \Sigma^2\dot{y}^2 + \frac{(r^2+y^2)^2}{\Sigma^2}\left(Ey^2 + L_\psi\right)^2 = \Sigma^2\dot{y}^2 + \left(Ey^2 + L_\psi\right)^2$$

Where we simply solve for $\Sigma^2\dot{y}$:

$$\Sigma\dot{y} = \sqrt{\Delta_y(K - m^2y^2) - (Ey^2 + L_\psi)^2} \tag{7.37}$$

Repeating the same procedure but taking a combination of $\Delta_r(K + m^2r^2)$ and simplifying and rearranging leads to the differential equation for r:

$$\Sigma\dot{r} = \sqrt{(Er^2 - L_\psi)^2 - \Delta_r(K + m^2r^2)} \tag{7.38}$$

So we sum up all the derived equations:

$$\begin{aligned}
\Sigma\dot{r} &= \sqrt{(Er^2 - L_\psi)^2 - \Delta_r(K + m^2r^2)} \\
\Sigma\dot{y} &= \sqrt{\Delta_y(K - m^2y^2) - (Ey^2 + L_\psi)^2} \\
\Sigma\dot{\psi} &= \frac{Er^2 - L_\psi}{\Delta_r} + \frac{Ey^2 + L_\psi}{\Delta_y} \\
\Sigma\dot{v} &= \frac{r^2(Er^2 - L_\psi)}{\Delta_r} - \frac{y^2(Ey^2 + L_\psi)}{\Delta_y}
\end{aligned} \tag{7.39}$$

If we now switch back to the original Boyer-Lindquist co-ordinates and ensure there are no factors of a on the left hand side of the equations we find the same form of the geodesics equations as seen in page 899 of [22] or in Carter's paper [28]:

$$\begin{aligned}
\rho^2\frac{dr}{d\lambda} &= \sqrt{(E(r^2+a^2) - aL_\phi)^2 - \Delta_r(m^2r^2 + (L_\phi - aE)^2 + C)} \\
\rho^2\frac{d\theta}{d\lambda} &= \sqrt{\mathcal{C} - \cos^2(\theta)(a^2(m^2 - E^2) + \frac{L_\phi}{\sin^2(\theta)})} \\
\rho^2\frac{d\phi}{d\lambda} &= -(aE - \frac{L_\phi}{\sin^2(\theta)}) + (\frac{a}{\Delta_r})(E(r^2+a^2) - aL_\phi) \\
\rho^2\frac{dt}{d\lambda} &= -a(aE\sin^2(\theta) - L_\phi) + (r^2+a^2)\frac{E(r^2+a^2) - aL_\phi}{\Delta_r}
\end{aligned} \tag{7.40}$$

Where K is split into $\mathcal{C}$ which is defined as : $\mathcal{C} = p_\theta^2 + \cos^2(\theta)(a^2(m^2 - E^2) + \frac{L_\phi^2}{\sin^2(\theta)})$

7.5 Appendix-E: Perturbation equations for Schwarzschild and Kerr in $\theta = \frac{\pi}{2}$ plane

The radial differential equations for the perturbation are displayed in one column of the table whereas the solution to the equation is shown in the 2nd column. The Schwarzschild perturbations are combined with the Kerr perturbations(r_{i0} are Schwarzschild perturbations whereas r_{0i} are flat space perturbations) Remember we also set the radial integration constants to 0 as they do not matter:

$$
\begin{aligned}
r_0(\nu) &= b\cosh(\nu)\\
r_{10}(\nu) &= \frac{bm^2\nu\tanh(\nu)}{p_{00}^2} - \frac{b\left(m^2+p_{00}^2\right)}{p_{00}^2}\\
r_{20}(\nu) &= \frac{1}{4p_{00}^4}(\text{sech}^3(\nu)\left(p_{00}^4\sinh(\nu)\left(-3b\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)\right)\right.\\
&+2bm^4\nu^2+2bm^4\nu\sinh(2\nu)+bm^4-2bm^2p_{00}^2+b\left(m^4-2m^2p_{00}^2-3p_{00}^4\right)\cosh(2\nu)\\
&-3bp_{00}^4\sinh(3\nu)\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)-3bp_{00}^4)\\
r_{11}(\nu) &= \frac{\text{sech}^3(\nu)\left(3b^2p_{00}\sqrt{m^2+p_{00}^2}\cosh(\nu)+b^2p_{00}\sqrt{m^2+p_{00}^2}\cosh(3\nu)\right)}{2bp_{00}^2}\\
r_{12}(\nu) &= \frac{b\left(-2\left(m^2+2p_{00}^2\right)+m^2\tanh^2(\nu)+m^2\nu\tanh(\nu)\text{sech}^2(\nu)\right)}{2p_{00}^2}\\
r_{21}(\nu) &= \frac{\text{sech}^3(\nu)}{4p_{00}^3}(\left(p_{00}^2\sinh(\nu)\left[12b\sqrt{m^2+p_{00}^2}\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)\right]\right.\\
&+4b\sqrt{m^2+p_{00}^2}\left(m^2+3p_{00}^2\right)\cosh(2\nu)+12bp_{00}^2\sqrt{m^2+p_{00}^2}\sinh(3\nu)\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)\\
&+12bp_{00}^2\sqrt{m^2+p_{00}^2}+4bm^2\sqrt{m^2+p_{00}^2})\\
r_{01} &= 0\\
r_{02} &= -\frac{1}{2}b\text{sech}(\nu)
\end{aligned}
\tag{7.41}
$$

$$
\begin{aligned}
\phi_{00} &= \tan^{-1}(\sinh(\nu)) \\
\phi_{01} &= 0 \\
\phi_{02} &= 0 \\
\phi_{10} &= \frac{m^2\tanh(\nu)}{p_{00}^2} + \frac{m^2\nu\text{sech}^2(\nu)}{p_{00}^2} + \frac{m^2}{p_{00}^2} + 2\tanh(\nu) + 2 \\
\phi_{20} &= \frac{1}{24bp_{00}^4}(-12p_{00}^4\text{sech}^4(\nu)(3b\tan^{-1}(\tanh\left(\frac{\nu}{2}\right)) + 3b\cosh(2\nu)\tan^{-1}(\tanh(\frac{\nu}{2}))) \\
&-46bm^4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right) \\
&+15bm^4\tan^{-1}(\sinh(\nu)) + 16bm^4\tan^{-1}(\sinh(\nu)+\cosh(\nu)) - 4\pi bm^4 + 24bm^2p_{00}^2\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right) \\
&+60bm^2p_{00}^2\tan^{-1}(\sinh(\nu)) \\
&-24bm^2\nu\text{sech}^3(\nu)\left(m^2\nu\tanh(\nu) - 2\left(m^2+p_{00}^2\right)\right) + 18bp_{00}^2\left(4m^2+5p_{00}^2\right)\tanh(\nu)\text{sech}(\nu) + 36\pi bm^2p_{00}^2 \\
&+90bp_{00}^4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right) \\
&+45bp_{00}^4\tan^{-1}(\sinh(\nu)) + 45\pi bp_{00}^4) \\
\phi_{11} &= -\frac{2\sqrt{m^2+p_{00}^2}(\tanh(\nu)+1)}{p_{00}} \\
\phi_{12} &= \text{sech}^2(\nu)\left(-\tanh(\nu)\right) + \frac{\left(m^2+2p_{00}^2\right)\left(\tanh(\nu)+1\right)}{p_{00}^2} \\
\phi_{21} &= -\frac{1}{4bp_{00}^3}(\text{sech}^2(\nu)\Bigg(\sinh(\nu)\Bigg(8b\sqrt{m^2+p_{00}^2}\left(2m^2+5p_{00}^2\right) \\
&-3C_r^{11}m^2p_{00}\nu - 4C_r^{21}p_{00}^3 \\
&+2\tan^{-1}(\sinh(\nu)+\cosh(\nu))\Bigg(8b\sqrt{m^2+p_{00}^2}\left(2m^2+5p_{00}^2\right) \\
&-3C_r^{11}m^2p_{00}\nu - 2\text{sech}^3(\nu)\Bigg(-4bm^2\nu\sqrt{m^2+p_{00}^2} \\
&+3C_r^{11}m^2p_{00} + 4C_r^{11}p_{00}^3 - 48bp_{00}^2\sqrt{m^2+p_{00}^2}\text{sech}^2(\nu)\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right) \\
&+3iC_r^{11}m^2p_{00}\text{Li}_2(-i(\cosh(\nu)+\sinh(\nu))) \\
&-3iC_r^{11}m^2p_{00}\text{Li}_2(i(\cosh(\nu)+\sinh(\nu))) - 3C_r^{11}m^2p_{00}\text{sech}(\nu) + 6C_r^{11}m^2p_{00}\nu\tanh(\nu)\text{sech}^3(\nu))
\end{aligned} \tag{7.42}
$$

For the perturbation ϕ_{21} we include some radial integration constants C_r^{ij} to demonstrate that these constants indeed do not contribute to the scattering angle at $\phi_{21}(\pm\infty)$, as they are heavily damped by factors of $\text{sech}(\nu)$. We also see how complicated the perturbations become if $C_r^{ij} \neq 0$ as they produce di-logarithms in the perturbations.

7.6 Appendix-F: Perturbation equations for Kerr not constrained to $\theta = \frac{\pi}{2}$ plane

Here we record the perturbative corrections for r,θ and ϕ. Starting with r:

$$
\begin{aligned}
r_{00} &= \sqrt{b^2+h^2}\cosh(\nu) \\
r_{10} &= -\frac{b\left(m^2(-\nu)\tanh(\nu)+m^2+p_{00}^2\right)}{p_{00}^2} \\
r_{01} &= 0 \\
r_{11} &= \frac{2b^3\sqrt{m^2+p_{00}^2}}{p_{00}(b^2+h^2)} \\
r_{02} &= -\frac{b^2\mathrm{sech}(\nu)\left(b^2+h^2\tanh^2(\nu)\right)}{2\left(b^2+h^2\right)^{3/2}} \\
r_{20} &= \frac{b^2\mathrm{sech}^3(\nu)}{4p_{00}^4\sqrt{b^2+h^2}}[2m^4\nu^2+2m^4\nu\sinh(2\nu)+m^4-2m^2p_{00}^2+\left(m^4-2m^2p_{00}^2-3p_{00}^4\right)\cosh(2\nu) \\
&-3p_{00}^4\sinh(\nu)\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)-3p_{00}^4\sinh(3\nu)\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)-3p_{00}^4] \\
r_{12} &= \frac{b^3}{2p_{00}^2\left(b^2+h^2\right)^2}[\tanh^2(\nu)\left(b^2m^2+h^2\left(m^2+4p_{00}^2\right)\right) \\
&-\tanh(\nu)\mathrm{sech}^2(\nu)\left(-b^2m^2\nu+h^2\left(4p_{00}^2-m^2(\nu-2)\right)+h^2\left(3m^2+4p_{00}^2\right)\tanh(\nu)\right) \\
&-2b^2\left(m^2+2p_{00}^2\right)-3h^2m^2\nu\tanh(\nu)\mathrm{sech}^4(\nu)] \\
r_{21} &= \frac{2b^4\sqrt{m^2+p_{00}^2}\mathrm{sech}(\nu)\left(m^2+3p_{00}^2\sinh(\nu)\tan^{-1}(\sinh(\nu))+3p_{00}^2\right)}{p_{00}^3\left(b^2+h^2\right)^{3/2}}
\end{aligned}
\tag{7.43}
$$

The θ perturbative corrections:

$$
\begin{aligned}
Y_{00} &= \sqrt{\frac{h^2}{b^2+h^2}}\mathrm{sech}(\nu) \\
Y_{10} &= -\frac{b\sqrt{\frac{h^2}{b^2+h^2}}\tanh(\nu)\left(\left(m^2+2p_{00}^2\right)(\tanh(\nu)+1)+m^2\nu\,\mathrm{sech}^2(\nu)\right)}{p_{00}^2\sqrt{b^2+h^2}} \\
Y_{01} &= 0 \\
Y_{11} &= \frac{4b^3\sqrt{\frac{h^2}{b^2+h^2}}\sqrt{m^2+p_{00}^2}\tanh(\nu)(\tanh(\nu)+1)}{p_{00}\left(b^2+h^2\right)^{3/2}} \\
Y_{02} &= \frac{b^2\sqrt{\frac{h^2}{b^2+h^2}}\mathrm{sech}^5(\nu)\left(\left(b^2+h^2\right)\cosh(2\nu)+b^2-h^2\right)}{4\left(b^2+h^2\right)^2} \\
Y_{20} &= -\frac{b^2\left(\frac{h^2}{b^2+h^2}\right)^{3/2}}{32h^2p_{00}^4}[12p_{00}^2\left(4m^2+5p_{00}^2\right)\tanh(\nu)\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)+\pi\right) \\
&+\mathrm{sech}^5(\nu)(32m^4\nu^2+16m^4\nu+4m^4\sinh(4\nu)+4m^4+32m^2p_{00}^2\nu+16m^2p_{00}^2\sinh(4\nu) \\
&+4m^2p_{00}^2+8\cosh(2\nu)\left(m^4\left(-2\nu^2+2\nu+1\right)+4m^2p_{00}^2(\nu+1)+4p_{00}^4\right) \\
&+\left(4m^4+28m^2p_{00}^2+31p_{00}^4\right)\cosh(4\nu)+16p_{00}^4\sinh(4\nu)+p_{00}^4) \\
&+16\tanh(\nu)\mathrm{sech}^3(\nu)\left(m^4(6\nu+1)+4m^2p_{00}^2(2\nu+1)+4p_{00}^4\right)-96p_{00}^4\tanh(\nu)\mathrm{sech}^2(\nu)\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)] \\
Y_{12} &= -\frac{b^3h\sqrt{\frac{1}{b^2+h^2}}\tanh(\nu)\mathrm{sech}^6(\nu)(\sinh(\nu)+\cosh(\nu))}{16p_{00}^2\left(b^2+h^2\right)^{5/2}}[2\cosh(\nu)(b^2\left(m^2(9\nu+19)+37p_{00}^2\right) \\
&-h^2\left(m^2(11\nu+16)+29p_{00}^2\right))+\cosh(3\nu)\left(b^2\left(2m^2(3\nu+8)+33p_{00}^2\right)+h^2\left(m^2(6\nu+1)-3p_{00}^2\right)\right) \\
&+2b^2m^2\sinh(\nu)-6b^2m^2\nu\sinh(\nu)+3b^2m^2\sinh(3\nu)-6b^2m^2\nu\sinh(3\nu) \\
&+b^2m^2\sinh(5\nu)+2b^2m^2\cosh(5\nu)+2b^2p_{00}^2\sinh(\nu) \\
&+3b^2p_{00}^2\sinh(3\nu)+b^2p_{00}^2\sinh(5\nu)+5b^2p_{00}^2\cosh(5\nu) \\
&-4h^2m^2\sinh(\nu)+34h^2m^2\nu\sinh(\nu)-4h^2m^2\sinh(3\nu) \\
&-6h^2m^2\nu\sinh(3\nu)-h^2m^2\cosh(5\nu) \\
&-2h^2p_{00}^2\sinh(\nu)-h^2p_{00}^2\sinh(3\nu)+h^2p_{00}^2\sinh(5\nu)-3h^2p_{00}^2\cosh(5\nu)] \\
Y_{21} &= \frac{b^4h\left(\frac{1}{b^2+h^2}\right)^{5/2}\sqrt{m^2+p_{00}^2}\mathrm{sech}^3(\nu)}{8p_{00}^3}[32\tanh(\nu)\left(\left(m^2+2p_{00}^2\right)\cosh(2\nu)+2m^2\nu+m^2+2p_{00}^2\right) \\
&+9\left(2m^2+5p_{00}^2\right)\sinh(\nu)\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)+\pi\right) \\
&+\cosh(3\nu)\mathrm{sech}(\nu)\left(3\left(2m^2+5p_{00}^2\right)\sinh(\nu)\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)+\pi\right)+28m^2+62p_{00}^2\right) \\
&+32m^2\nu+4m^2-48p_{00}^2\sinh(\nu)\tan^{-1}(\sinh(\nu))+2p_{00}^2]
\end{aligned}
\tag{7.44}
$$

The ϕ perturbative corrections:

$$
\begin{aligned}
{}_{00}hi_{00} &= \tan^{-1}\left(\frac{\sqrt{b^2+h^2}\sinh(\nu)}{b}\right)\\
{}_{00}hi_{10} &= \frac{2b^2(\sinh(\nu)+\cosh(\nu))\left(\cosh(\nu)\left(m^2(\nu+1)+2p_{00}^2\right)-m^2\nu\sinh(\nu)\right)}{p_{00}^2\left((b^2+h^2)\cosh(2\nu)+b^2-h^2\right)}\\
{}_{00}hi_{01} &= 0\\
{}_{00}hi_{11} &= -\frac{2b^2\sqrt{m^2+p_{00}^2}(\tanh(\nu)+1)\left(\left(b^2-h^2\right)\cosh(2\nu)+b^2+h^2\right)}{p_{00}\left(b^2+h^2\right)\left((b^2+h^2)\cosh(2\nu)+b^2-h^2\right)}\\
{}_{00}hi_{02} &= 0\\
{}_{00}hi_{20} &= \frac{b^3}{16p_{00}^4\sqrt{b^2+h^2}\left((b^2+h^2)\cosh(2\nu)+b^2-h^2\right)^2}\left[32\cosh(\nu)\left(4b^2m^2\nu\left(m^2+p_{00}^2\right)+h^2\left(\left(m^2+2p_{00}^2\right)^2-4m^2p_{00}^2\nu\right)\right)\right.\\
&+3p_{00}^2\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)\left(p_{00}^2\left(7b^2+3h^2\right)-4m^2\left(h^2-3b^2\right)\right)+\pi\left(3b^2-h^2\right)\left(4m^2+5p_{00}^2\right)\right)\\
&+12p_{00}^2\cosh(2\nu)\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)\left(p_{00}^2\left(3b^2-2h^2\right)+4b^2m^2\right)+\pi b^2\left(4m^2+5p_{00}^2\right)\right)\\
&+3p_{00}^2\left(b^2+h^2\right)\left(4m^2+5p_{00}^2\right)\cosh(4\nu)\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)+\pi\right)\\
&-4\sinh(\nu)\left(b^2\left(16m^4\nu^2-12m^2p_{00}^2-15p_{00}^4\right)+h^2\left(8m^4(2\nu(\nu+2)-1)+4m^2p_{00}^2(16\nu+1)+13p_{00}^4\right)\right)\\
&-4\sinh(3\nu)\left(h^2\left(8m^4+20m^2p_{00}^2+17p_{00}^4\right)-3b^2p_{00}^2\left(4m^2+5p_{00}^2\right)\right)\\
&\left.-32h^2\left(m^2+2p_{00}^2\right)^2\cosh(3\nu)+128h^2m^2p_{00}^2\nu\,\mathrm{sech}(\nu)\right]\\
{}_{00}hi_{21} &= -\frac{b^3\sqrt{m^2+p_{00}^2}\,\mathrm{sech}^3(\nu)}{32p_{00}^3\left(b^2+h^2\right)^{3/2}\left((b^2+h^2)\cosh(2\nu)+b^2-h^2\right)^2}\left[-64b^2h^2\left(m^2+2p_{00}^2\right)\cosh(6\nu)\right.\\
&+32\left(3b^4m^2\nu+4b^2h^2\left(m^2(\nu+1)+2p_{00}^2\right)-3h^4m^2\nu\right)\\
&+4\sinh(2\nu)\left(10b^4\left(2m^2+5p_{00}^2\right)-b^2h^2\left(m^2(64\nu+22)+47p_{00}^2\right)-5h^4\left(2m^2+5p_{00}^2\right)\right)\\
&+16\sinh(4\nu)(2b^4\left(2m^2+5p_{00}^2\right)-8b^2h^2\left(m^2(\nu+1)+2p_{00}^2\right)\\
&+h^4\left(2m^2+5p_{00}^2\right))\\
&+4\sinh(6\nu)(2m^2\left(2b^4-7b^2h^2-h^4\right)\\
&+p_{00}^2\left(10b^4-27b^2h^2-5h^4\right))\\
&+64\cosh(2\nu)\left(2m^2\nu\left(b^4+h^4\right)+b^2h^2\left(m^2+2p_{00}^2\right)\right)\\
&+32\cosh(4\nu)(b^4m^2\nu-4b^2h^2\left(m^2(\nu+1)+2p_{00}^2\right)\\
&-h^4m^2\nu)+\left(2b^4+b^2h^2-h^4\right)\left(2m^2+5p_{00}^2\right)\cosh(7\nu)(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)\\
&+\pi)+\cosh(\nu)(96b^2p_{00}^2\left(h^2-5b^2\right)\tan^{-1}(\sinh(\nu))\\
&+\left(70b^4-5b^2h^2-3h^4\right)\left(2m^2+5p_{00}^2\right)\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)+\pi\right))\\
&+\cosh(3\nu)(\left(42b^4+b^2h^2+3h^4\right)\left(2m^2+5p_{00}^2\right)\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)+\pi\right)-48b^2p_{00}^2\left(5b^2+h^2\right)\tan^{-1}(\sinh(\nu)))\\
&+\cosh(5\nu)(\left(14b^4+3b^2h^2+h^4\right)\left(2m^2+5p_{00}^2\right)\left(4\tan^{-1}\left(\tanh\left(\frac{\nu}{2}\right)\right)\right.\\
&\left.+\pi-48b^2p_{00}^2\left(b^2+h^2\right)\tan^{-1}(\sinh(\nu)))\right]\\
{}_{00}hi_{12} &= \frac{b^4(\tanh(\nu)+1)\mathrm{sech}^2(\nu)}{4p_{00}^2\left(b^2+h^2\right)^2\left((b^2+h^2)\cosh(2\nu)+b^2-h^2\right)}\left[4\cosh(2\nu)(b^2\left(m^2+2p_{00}^2\right)\right.\\
&-h^2\left(m^2+3p_{00}^2\right))+\cosh(4\nu)(b^2\left(m^2+3p_{00}^2\right)\\
&-h^2\left(2m^2+5p_{00}^2\right))+3b^2m^2-2b^2p_{00}^2\sinh(2\nu)\\
&-b^2p_{00}^2\sinh(4\nu)+5b^2p_{00}^2-2h^2m^2\sinh(2\nu)\\
&-h^2m^2\sinh(4\nu)-2h^2m^2\\
&\left.+2h^2p_{00}^2\sinh(2\nu)-h^2p_{00}^2\sinh(4\nu)+h^2p_{00}^2\right]
\end{aligned}
\tag{7.45}
$$

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