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Master's Thesis

Many-body Physics of Fock Parafermions A Field Theoretical Approach Through Bosonisation

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Abstract

Research into a type of quasiparticles known as anyons has been an active branch of research ever since they were hypothesised to exist in two-dimensional materials in 1977. It was quickly shown that the modes of the fractional quantum Hall effect exhibit anyonic properties, when the effect was discovered experimentally in 1982 [1]. A one-dimensional anyonic quasiparticle called the parafermion, which first emerged in the area of condensed matter physics in the study of the vector Potts model, displays anyonic properties by having its exchange statistics obey a \mathbb{Z}_p -symmetry for some $p \in \mathbb{N}$ [2]. This exchange statistics makes the parafermions very interesting to study from a theoretical standpoint. In 2014 [3], it was furthermore shown that parafermions admit for a Fock space representation, which in turn permits these Fock parafermions to be described by particle number degrees of freedom. This realisation allows for the construction of Fock parafermions tight-binding models, which has been commenced to be studied over the last couple of years [4, 5].

This thesis seeks to expand upon the literature by developing a field theoretical description of the general \mathbb{Z}_p -symmetric Fock parafermions through bosonisation. Of special note is that the resulting description depends on $p - 1$ dual bosonic fields. The bosonisation description is applied to the tight-binding model of the \mathbb{Z}_3 -symmetric Fock parafermions for both single and pair-hopping terms, and it is shown how the model is a sum of a chiral Luttinger liquid and a Luttinger liquid up to second order Taylor expansion. Furthermore, the two-point correlation functions of the Fock parafermions are calculated and it is shown that requiring the model to consist of $p - 1$ pairs of dual field renders the correct predictions of the two-point correlation functions as compared to earlier numerical calculations done by Rossini et al. [4].

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Contents

Introduction	1
1 Parafermions	3
1.1 Weyl Algebra and parafermions	3
1.2 Fock parafermions	5
1.3 Physical Models Where Parafermions Arise	9
2 Chiral Luttinger Liquids	13
2.1 Lagrangian Dynamics	13
2.2 Partition Functions	15
2.3 Two-point Correlation Functions	16
2.4 The Chiral Luttinger Field ψ	18
3 Fock Parafermion Tight Binding Model	19
3.1 The Model	19
3.2 Nearest Neighbour Model of \mathbb{Z}_3 Fock Parafermions	21
4 Bosonisation of Fock Parafermions	24
4.1 Field Theoretical Bosonisation of the Fock Parafermions	25
4.2 Constructive Bosonisation of the Fock Parafermions	26
4.3 Bosonisation of One-dimensional \mathbb{Z}_3 Fock Parafermions	30
5 Field Theoretical Limit of \mathbb{Z}_3 Fock Parafermion Nearest Neighbour Hopping Hamiltonian	33
5.1 Single Hopping Hamiltonian	34
5.2 Pair Hopping Hamiltonian	40
5.3 Combined Single and Pair Hopping Hamiltonian	44
6 Conclusion and Outlook	54
Bibliography	56
A Bosonisation of Fermions and Bosons and the Luttinger Liquid	58
B Chiral Luttinger Liquid Two Point Correlation Functions Calculations	62
B.1 High Energy Limit	65
C \mathbb{Z}_3 Fock Parafermion Nearest-Neighbour Hopping Hamiltonian Calculations	66
D Various Results	69
D.1 Cumulant Expansion	69
D.2 Derivation of the $(d + 1)$ -dimensional QFT Euler-Lagrange Equation	69
D.3 Level Spacing Statistics	69

Introduction

Conventional quantum mechanics dictate that two forms of fundamental particles exist, bosons and fermions [6]. The feature that sets these types of particles apart is the way that the wavefunction of two interchanged indistinguishable particles behaves. As it is well known, the exchange of two identical bosons will not change the global wavefunction, whereas fermions, on the other hand, will have their global wavefunction gain a negative sign upon the interchange. The difference in exchange statistics has far reaching consequences: one of the biggest is the Pauli exclusion principle that states only one fermion can be in each state, whereas bosons do not experience such constraint. If we generalise what happens, we can consider the exchange of the particles as gaining a phase to the global wavefunction. The question arises whether this phase has to be real, or if any complex phase would be allowed. Anyons are exactly such particles, where a complex phase is gained upon exchange of two indistinguishable particles. The exchange is explicitly done by braiding the particles around each other in a two-dimensional plane in a clockwise rotation, such that:

$$\psi(x_1, x_2) = e^{i\theta} \psi(x_2, x_1) \quad (\text{clockwise rotation}) \quad (1)$$

Notice that there is a notion of chirality built into the system: if the particles were braided in the anti-clockwise direction we would gain a phase of $e^{-i\theta}$. From now on we assume all the rotations are made in a clockwise direction, unless otherwise stated. It is evident that the fermions and bosons can be explained as special cases of the anyons, where the phase factor is $\theta = \pi$ and $\theta = 2\pi$, respectively. As such the notion of an anyon is more general.

Exchanging the particles twice in the clockwise direction makes the global wavefunction gain the overall phase factor of $e^{i2\theta}$. This observation makes it impossible for anyons to exist in three or more dimensions, but they can arise in two-dimensional systems [1]. In 1982, two experimentalists, Tsui and Störmer, discovered a phenomena in a two-dimensional material subjected to a strong magnetic field, which would be known as the fractional quantum Hall effect, since a plateau in the Hall conductance is observed when the filling factor ν is a nice ratio [7]. By the "filling factor" is meant the ratio of electrons in the system to the total magnetic flux quanta. Laughlin discovered that the process can be described by a condensation of the two-dimensional electron gas into a fractally charged excited liquid state at the edges [8]. These edge states were noted by Halperin to be anyonic quasiparticles with fractional charge [9]. The phenomena is still a major topic of research in condensed matter physics to this day [10].

In this thesis we will study a special subspecies of the anyons called parafermions. What sets the particles aside from the general anyons is that they arise in one-dimensional systems with the phase achieved by the exchange of parafermions is a divisor of 2π , ie. $\theta \mid 2\pi$. Furthermore, the phase does not depend on the direction of the braiding, but only on the initial relative position of the parafermions. If we denote the phase $\theta = 2\pi i/p$, the parafermions display a \mathbb{Z}_p -symmetry upon exchange. We distinguish between parafermions of different classes by denoting them \mathbb{Z}_p parafermions after their symmetry. Note that the charge of a \mathbb{Z}_p parafermion is fractional at $2e/p$, where e is the elementary charge of an electron [5]. The parafermions arise as modes or degrees of freedom of the clock model, which is a generalisation of the Ising model with p spin states [2]

The parafermions do not readily allow for a second quantised description [3]. This calls for the creation of a Fock space version of the parafermions, which are called Fock parafermions. Exactly such description was constructed by Cobanera and Ortiz in 2014 [3]. Compared to the parafermions, the Fock parafermions display a particle degree of freedom, which makes it possible to consider the particles by themselves and to count them.

As we will see, Majorana fermions are a special example of the parafermions, which display a \mathbb{Z}_2 -symmetry. The Fock space representation of the Majorana fermions are the conventional fermions [11]. Thus, one way of thinking about the Fock parafermions is that they are to parafermions what fermions are to Majoranas. This realisation leads us to conjecture that the Fock parafermions are in fact abelian particles (as the fermions are), as compared to the non-abelian parafermions (Majoranas) [12].

Non-abelian anyonic particles, in particular the Majorana fermions, have shown to be promising candidates for realising topological quantum computing due to their braiding statistics - something that has been the subject of heavy research in the recent years [13]. Parafermions show promising theoretical candidates for quantum computation due to their non-abelian nature [14]. Note that the Fock parafermions, which will be our main point of interest in this thesis, are not suitable for this purpose, since they do not seem to display non-abelian properties [12]. We refer to the literature for a study of the braiding statistics of the parafermions and their possible application in quantum computing [3].

Since the Fock parafermions can be considered as particle degrees of freedom, it makes sense to talk about them in terms of a tight-binding model, where Fock parafermions are effectively bound to some sites in a lattice and hopping between sites is allowed. The main work of this thesis will be to treat such a model of Fock parafermions and to develop a field theoretical approach of describing the system through bosonisation. Notice that the treatment of this system is complicated by the fact that it cannot be solved directly through integration or by the way of an ansatz similar to Bethe ansatz [4]. However, earlier works suggest that the system is a conformal field theory that allows for a bosonised description [4, 5].

Outline of Thesis

Below we will outline the thesis and account for some of the main conclusions in each chapter.

Chapter 1 The theoretical foundation of parafermions is accounted for on the basis of the article by Cobanera and Ortiz [3]. Fock parafermions are introduced, and it is shown how the Fock parafermions relate to a type of bosonic particle through a Fradkin-Kadanoff transformation. Some physical models that give rise to parafermions are introduced.

Chapter 2 The chiral Luttinger liquid is introduced and two-point correlation functions of the fields that constitute the model are calculated.

Chapter 3 The Fock parafermion tight-binding nearest-neighbour hopping model is introduced. Earlier work has treated this model numerically and we go over the main points of these articles [4, 5]. Of special note is that the analysis shows that the \mathbb{Z}_3 Fock parafermions tight-binding model will require two bosonic modes to be described.

Chapter 4 The Fock parafermions are described in the language of bosonisation. As we will see, the bosonisation of \mathbb{Z}_p Fock parafermions requires $p - 1$ sets of dual bosonic fields.

Chapter 5 The field theoretical description of the nearest-neighbour hopping Hamiltonian of the \mathbb{Z}_3 Fock parafermions is studied. The newly-defined bosonised description of the Fock parafermions is utilised in the tight-binding model. It is shown that the bosonisation of the Fock parafermions yield a chiral Luttinger liquid description up to the second order. Two-point correlation functions of the Fock parafermion operators are calculated, and it is shown that our description predicts the correct dynamics of the system when one type of tight-binding hopping is allowed for the \mathbb{Z}_3 Fock parafermions. However, when both single and pair hopping is allowed simultaneously our analysis falls short.

Chapter 6 Conclusion and outlook.

Chapter 1

Parafermions

In this chapter we will introduce and study the mathematical foundations of the group of non-abelian anyonic quasiparticles called parafermions that we presented in the introduction. These quasiparticles can be understood as a generalisation of the Majorana fermions whose commutation displays a \mathbb{Z}_p -symmetry (rather than a \mathbb{Z}_2 -symmetry in the case of the Majoranas).

The original description of the parafermions does not translate to a second-quantisation description, which is necessary for the particles to exist independently [12]. In 2014, Cobanera and Ortiz constructed a mathematical description to describe these pseudoparticles over a Fock space [3]. In order to realise the importance of this description, note that these Fock parafermions allow for a second quantised description to exist. Such a description in turn allows for the conservation of particle numbers and tight binding models [4, 5]. Compared to the regular parafermions, the Fock parafermions are not said to be non-abelian, however.

In the end of this chapter, we will discuss a couple of systems in which the parafermions arise and recount for potential candidates for a physical realisation of the parafermions.

1.1 Weyl Algebra and parafermions

In this section we will see how the Weyl algebra naturally arises as an answer to the problem of constructing a finite dimensional representations of the quantum mechanical operators through a discretisation of the space. This is not obvious, since Heisenberg's canonical commutation relations $[X_i, p_j] = i\hbar\delta_{i,j}\mathbb{1}$, does not allow for finite dimensional matrix representations [3, p. 33]. To be able to describe quantum mechanics by finite dimensional representations Weyl suggested considering a discretised form of quantum mechanics by studying the algebra generated by the translations in position and momentum space in (1+1)-dimensions:

$$V_{i,\mathbf{x}} = e^{i\mathbf{x}p_i/\hbar} \quad U_{j,\mathbf{p}} = e^{i\mathbf{p}X_j/\hbar} \quad (1.1)$$

For $i, j \in \{1, \dots, M\}$, where M is the number of particles. Note, the commutation relation of these unitary operators is:

$$[V_{i,\mathbf{x}}, U_{j,\mathbf{p}}] = e^{i\mathbf{x}\mathbf{p}/\hbar}\delta_{i,j} \quad (1.2)$$

Now we introduce a (fundamental) lattice spacing a making \mathbf{x} and \mathbf{p} discrete:

$$\mathbf{x}_m = ma \quad \mathbf{p}_n = \frac{2\pi\hbar n}{pa} \quad (1.3)$$

With $m, n \in \mathbb{Z}$. In this case we may redefine the (fundamental) translation operators:

$$V_i = e^{iap_i/\hbar} \quad U_j = e^{i2\pi X_j/pa} \quad (1.4)$$

Note that we have $V_i^m = V_{i,\mathbf{x}_m}$ and $U_j^n = U_{j,\mathbf{p}_n}$. Commuting the two types of operators yields a non-standard commutation relation:

$$V_i U_j = \omega^{\delta_{i,j}} U_j V_i \quad V_i^\dagger U_j = \omega^{-\delta_{i,j}} U_j V_i^\dagger \quad \omega = e^{2\pi i/p} \quad (1.5)$$

These commutation relations admit the construction of a finite dimensional representation, $W_p(M)$, of the operators V_i and U_i in a Hilbert space of dimension p^M over the complex numbers, \mathbb{C} , such that:

$$\begin{aligned} V_i &= \mathbb{1} \otimes \cdots \otimes V \otimes \cdots \otimes \mathbb{1} & (i\text{'th position}) \\ U_j &= \mathbb{1} \otimes \cdots \otimes U \otimes \cdots \otimes \mathbb{1} & (j\text{'th position}) \end{aligned} \quad (1.6)$$

For $i, j \in \{1, \dots, M\}$. Where the Hermitian matrices U and V are called the Weyl generators with finite period p , and are defined as the $p \times p$ -matrices:

$$V = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \omega^{p-1} \end{pmatrix} \quad (1.7)$$

One of the simplest cases of our system, where $p = 2$ gives rise to a system where $\omega = -1$, such that $V = \sigma^x$ and $U = \sigma^z$, with $\sigma^{x/z}$ being the pauli matrices defined in the standard way. This fact renders two observations: First of all, we notice how this fact implicitly shows that the system with $p = 2$ should be related to the system consisting of indistinguishable fermions. Second of all, the $p = 2$ case also hints at us an important physical interpretation of the Weyl generators V and U as a pair of spin operators: U is unitary and represents an observable, which corresponds to the spin of a \mathbb{Z}_p -symmetric spin system, while V is a 'ladder' operator that makes it possible to jump between the different spin states.

1.1.1 Weyl parafermions

Recall that the Jordan-Wigner transformation maps the fermionic spin operators to the creation and annihilations operators of the fermion [15]. By a Jordan-Wigner-like transformation of our generalised spin operators, U and V , it should thus be possible to define a family of pseudoparticles, which we will denote (Weyl) parafermions. This allows us to define the (Weyl) parafermion operators:

$$\Gamma_i \equiv V_i \left(\prod_{j=1}^{i-1} U_j \right) \quad \Delta_i \equiv V_i U_i \left(\prod_{j=1}^{i-1} U_j \right) = \Gamma_i U_i \quad (1.8)$$

They define an alternative set of generators of the Weyl algebra, $W_p(M)$, since we may describe the original Weyl operators in terms of the parafermionic operators:

$$U_i = \Gamma_i^\dagger \Delta_i \quad V_i = \Gamma_i \left(\prod_{j=1}^{i-1} \Delta_j^\dagger \Gamma_j \right) = \Gamma_i \prod_{j=1}^{i-1} U_j^\dagger \quad (1.9)$$

These newly defined operators commute as:

$$\begin{aligned} \Gamma_i \Gamma_j &= \omega \Gamma_j \Gamma_i & \Delta_i \Delta_j &= \omega \Delta_j \Delta_i & \text{for } i < j \\ \Gamma_i \Delta_j &= \omega \Delta_j \Gamma_i & & & \text{for } i \leq j \end{aligned} \quad (1.10)$$

It is important to note that the commutation of the operators would look different were $i > j$. In this case we would find the commutation relations to be the following, $\Gamma_i \Gamma_j = \omega^{-1} \Gamma_j \Gamma_i$, $\Delta_i \Delta_j = \omega^{-1} \Delta_j \Delta_i$, and $\Gamma_i \Delta_j = \omega^{-1} \Delta_j \Gamma_i$. Powers of our operators at the same site tell us:

$$(\Gamma_i)^p = (\Delta_i)^p = \mathbb{1} \quad (\Gamma_i)^\dagger = (\Gamma_i)^{-1} \quad (\Delta_i)^\dagger = (\Delta_i)^{-1} \quad (1.11)$$

The algebra created by our parafermion operators is known as the generalised Clifford algebra [16]. Specifically for $p = 2$, the algebra generated parafermionic operators is that of the Majorana fermions: Recast $a_i = \Gamma_i$ and $b_i = i\Delta_i$, which are two operators that will construct the well known Clifford algebra:

$$\{a_i, a_j\} = 2\delta_{i,j} \quad \{b_i, b_j\} = 2\delta_{i,j} \quad \{a_i, b_j\} = 0 \quad (1.12)$$

a_i and b_i are the Majorana fermions when regarded as quantum degrees of freedom [3].

Weyl parafermions in Spatial Dimensions Bigger than One

To define the parafermions in spatial dimensions, $N > 1$, we need to define some sort of lexicographical order that each site in the constructed N -dimensional hypercubic lattice goes into. For example for two dimensions, $N = 2$, we could define the order:

$$(1, 1) \rightarrow (1, 2) \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (3, 2) \rightarrow \dots \quad (1.13)$$

This way of counting in a zig-zag pattern is rather special since it allows for a lexicographical order of non-finite system sizes. Now the parafermionic operators are defined as:

$$\Gamma_{\mathbf{r}} = V_{\mathbf{r}} \prod_{\mathbf{x} < \mathbf{r}} U_{\mathbf{x}} \quad \Delta_{\mathbf{r}} = V_{\mathbf{r}} U_{\mathbf{r}} \prod_{\mathbf{x} < \mathbf{r}} U_{\mathbf{x}} \quad (1.14)$$

This construction still obeys the generalised Clifford algebra structure that we have defined earlier. Thus the operators $\Gamma_{\mathbf{r}}$ and $\Delta_{\mathbf{r}}$ will be parafermions when regarded as degrees of freedom of the system.

1.2 Fock parafermions

In the previous section it was shown how the $p = 2$ case of the Weyl parafermions are the Majorana fermions. The Majorana fermions by themselves cannot exist, but appear as quasiparticle edge modes in a Kitaev chain, due to the collective interactions of the super conducting electrons in the material. If fermions were the underlying particles that gave rise to the $p = 2$ case of the Weyl parafermions, it is natural to ask if particles with a Fock description (be they quasiparticles or regular particles) exist for the general p case. This question is answered by the introduction of the Fock parafermions. These particles are defined to have a Fock space representation, which in term means that they are defined through second quantisation. That the Fock parafermions have a second-quantisation description tells us that they can exist individually and be used in models like tight-binding models.

We define the Fock space of indistinguishable, independent particles in terms of the following pieces of information:

1. The state space of a single particle \mathcal{H} and its available orbitals;
2. A rule for multiplying N single-particle states to generate an N -body state with the correct exchange and exclusion statistics.

Let the single-particle state space \mathcal{H} be spanned by an orthonormal basis $\{\phi_1, \dots, \phi_M\}$. Assume that each of these Fock parafermions are independent particles. Specify a many-body state, where n_i Fock parafermions occupy the ϕ_i orbital, then the occupation numbers may be organised into a unique ordered list denoting the many-body state, $|n_1, \dots, n_i, \dots, n_M\rangle$. For ease of notation we use the following notation $|n_i\rangle = |0, \dots, 0, n_i, 0 \dots, 0\rangle$. The general structure of the Fock space is:

$$\mathcal{F}_p(M) = \bigoplus_{N=0} \text{span} \left\{ |n_1, \dots, n_i, \dots, n_M\rangle \left| \sum_{i=1}^M n_i = N \right. \right\} \quad (1.15)$$

This Fock space also being a Hilbert space allows for the construction of an inner product, which is physically motivated to be orthogonal for states with distinct orbital occupations, defined as:

$$\langle n_1, \dots, n_M | n'_1, \dots, n'_M \rangle = \prod_{i=1}^M \delta_{n_i, n'_i} \quad (1.16)$$

The present description of the Fock space contains no information on the exchange statistics of the Fock parafermions. To encode this information in the Fock space we define an associative multiplication, \times , which physically describes the process of adding Fock parafermions to any fixed orbital:

$$\underbrace{|n_i = 1\rangle \times \dots \times |n_i = 1\rangle}_{m \text{ times}} \equiv |n_i = m\rangle \quad (1.17)$$

We may also denote this by an exponent $(|n_i = 1\rangle)^m = |n_i = m\rangle$. In the case of fermions, we recall that only one particle per orbital may exist. This motivates the definition of an exclusion parameter $n_E \geq 2$, as the smallest integer such that:

$$(|n_i = 1\rangle)^{n_E} = |n_i = n_E\rangle = 0 \quad (1.18)$$

The parafermions do not commute under exchange. Hence to capture the exchange statistics of the parafermions, the multiplication of two different orbital must obey the following property, for $i < j$:

$$|n_i, n_j\rangle \equiv |n_i\rangle \times |n_j\rangle = \omega^{n_i n_j} |n_j\rangle \times |n_i\rangle \quad (1.19)$$

Where $\omega \equiv e^{i2\pi/p}$. We note that any state in Fock space may be generated by multiplying single-particle states:

$$|n_1, \dots, n_M\rangle = (|1, 0, \dots, 0\rangle)^{n_1} \times \dots \times (|0, \dots, 0, 1\rangle)^{n_M} \quad (1.20)$$

Define the Fock space vacuum, $|0\rangle$:

$$|0\rangle \equiv |n_1 = 0, \dots, n_M = 0\rangle = (|1, 0, \dots, 0\rangle)^0 \times \dots \times (|0, \dots, 0, 1\rangle)^0 \quad (1.21)$$

And note that this state must be the multiplicative identity:

$$|n_1, \dots, n_M\rangle = |0\rangle \times |n_1, \dots, n_M\rangle = |n_1, \dots, n_M\rangle \times |0\rangle \quad (1.22)$$

We assume if a state $|\psi\rangle$ commutes with any other state $|\phi\rangle \in \mathcal{F}_p(M)$, then $|\psi\rangle$ must be a scalar multiple of the Fock vacuum. With this piece of information, we try and determine the exclusion parameter. We observe that:

$$|n_i = p\rangle \times |n_j\rangle = \begin{cases} \omega^p |n_j\rangle \times |n_i = p\rangle = |n_j\rangle \times |n_i = p\rangle & \text{for } i < j \\ \omega^{-p} |n_j\rangle \times |n_i = p\rangle = |n_j\rangle \times |n_i = p\rangle & \text{for } i > j \end{cases} \quad (1.23)$$

While for $i = j$ the same trivially holds. To sum up $|n_i = p\rangle \times |n_j\rangle = |n_j\rangle \times |n_i = p\rangle$, which is true for all $|n_j\rangle$, thus as per our assumption $|n_i = p\rangle = \alpha_i |0\rangle$ for $i \in \{1, \dots, M\}$. These states contain p particles, so the inner product with the Fock vacuum is:

$$\alpha_i = \langle 0 | \alpha_i | 0 \rangle = \langle 0 | n_i = p \rangle = \delta_{p,0} = 0 \quad (1.24)$$

In other words $|n_i = p\rangle = 0$. It turns out that this is the lowest such exponent that exists, hence $n_E = p$. The dimension of the Fock space is thus:

$$\dim_{\mathbb{C}}(\mathcal{F}_p(M)) = p^M \quad (1.25)$$

In fact the $\mathcal{F}_p(M)$ algebra is the p -Grassmann algebra with M generators [3]. To distinguish between different types of Fock parafermions we will denote \mathbb{Z}_p Fock parafermions as the Fock parafermions for which p is the smallest integer such that (eq. 1.30) is obeyed. We will further more denote p as the order of the Fock parafermions.

1.2.1 Second Quantisation of the Fock parafermions

Having defined a Fock space over which the parafermions may exist, we wish to define a pair of creation and annihilation operators that will allow us to capture the physics of the Fock parafermions. Fundamentally, we want to define the creation operator F_i^\dagger as adding one Fock parafermion to any given state in the i 'th orbital. This may be done by multiplication from the left of any given state by the state $|n_i = 1\rangle$. The creation operator becomes:

$$F_i^\dagger |n_1, \dots, n_i, \dots, n_M\rangle = \omega^{-\sum_{j < i} n_j} |n_1, \dots, n_i + 1, \dots, n_M\rangle \quad (1.26)$$

The annihilation operator, F_i , is similarly defined:

$$F_i |n_1, \dots, n_i, \dots, n_M\rangle = \omega^{\sum_{j < i} n_j} |n_1, \dots, n_i - 1, \dots, n_M\rangle \quad (1.27)$$

By application of these definitions, we get for $i < j$ the relations:

$$F_i^\dagger F_j^\dagger = \omega F_j^\dagger F_i^\dagger \quad F_i F_j = \omega F_j F_i \quad F_i^\dagger F_j = \omega^{-1} F_j F_i^\dagger \quad F_i F_j^\dagger = \omega^{-1} F_j^\dagger F_i \quad (1.28)$$

The first two commutation relations can be expressed in terms of each other through a Hermitian conjugation, and the same is true for the two last commutation relations. Thus the four commutation relations can be condensed to two commutation relations for general $i \neq j$:

$$F_i F_j = \omega^{\text{sgn}(i-j)} F_j F_i \quad F_i^\dagger F_j = \omega^{-\text{sgn}(i-j)} F_j^\dagger F_i \quad (1.29)$$

Applying the creation operator to vacuum p times yields $F_i^{\dagger p} |0\rangle = |n_i = p\rangle = 0$, whereby it must be true that:

$$F_i^{\dagger p} = 0 \quad F_i^p = 0 \quad (1.30)$$

Furthermore, it is noted how the Fock parafermions will obey the $p - 1$ relations:

$$(F_i^{\dagger})^m (F_i)^m + (F_i)^{p-m} (F_i^{\dagger})^{p-m} = \mathbb{1} \quad m \in \{1, \dots, p-1\} \quad (1.31)$$

Which stems from the fact that $(F_i^{\dagger})^m (F_i)^m |n_1, \dots, n_i, \dots, n_M\rangle = |n_1, \dots, n_i, \dots, n_M\rangle$, if $n_i - m \geq 0$ and zero otherwise, while the second term $(F_i)^{p-m} (F_i^{\dagger})^{p-m} |n_1, \dots, n_i, \dots, n_M\rangle = |n_1, \dots, n_i, \dots, n_M\rangle$, if $p - m + n_i < p$ and zero otherwise. Hence, if $n_i \geq m$ the first term of (eq. 1.31) will be the identity while the second term is zero, and if $n_i < m$ the first term is zero, while the second term is the identity.

If we let $p = 2$, we see how our relations (Eqs. 1.28-1.31) simplify to the standard fermionic algebra with $F_i^{\dagger} F_j + F_j F_i^{\dagger} = \delta_{i,j}$ and $(F_i^{\dagger})^2 = (F_i)^2 = 0$. Therefore, the Fock parafermion operators with $p = 2$ obey the anti-commutation relations of the standard fermions, which is exactly what we would expect. In other words, the $p = 2$ case of the Fock parafermions yields the system of the standard fermions.

In closing, we define the number operator such that:

$$N_i |n_1, \dots, n_i, \dots, n_M\rangle \equiv n_i |n_1, \dots, n_i, \dots, n_M\rangle \quad (1.32)$$

As we noted earlier $(F_i^{\dagger})^m (F_i)^m |n_1, \dots, n_i, \dots, n_M\rangle = |n_1, \dots, n_i, \dots, n_M\rangle$ if $m \leq n_i$ and zero otherwise. Hence, we can count how many Fock parafermions are in the i 'th orbit by adding together the number of non-zero $(F_i^{\dagger})^m (F_i)^m$. Mathematically this is expressed as the equation:

$$N_i = \sum_{m=1}^{p-1} F_i^{\dagger m} F_i^m \quad (1.33)$$

This operator commutes with the Fock parafermions like so:

$$[N_i, F_i] = -F_i \quad [N_i, F_i^{\dagger}] = F_i^{\dagger} \quad (1.34)$$

These commutation relations witness that the operator N_i is a generator of the $U(1)$ Lie algebra over the Fock parafermions [17]. Thus on the defined Fock space, the operator, N_i , generates a $U(1)$ -symmetry. This underlines the important point about the nature of the Fock parafermions that $U(1)$ -symmetric systems conserve the number of Fock parafermions as is the case with fermions and bosons.

1.2.2 Fock vs Weyl Representation of the parafermions

Relating the Fock parafermions to the Weyl parafermions is not as straightforward as one would suggest. However Cobanera and Ortiz [3, p. 27] found out that the following definition yields two sets of operators that simultaneously obey the right commutation relations of Weyl and Fock parafermions (Eqs. 1.10, 1.11 1.28, 1.30, 1.31):

$$\Gamma_i = F_i + F_i^{\dagger p-1} \quad \Delta_i = \left(F_i + F_i^{\dagger p-1} \right) \omega^{N_i} \quad (1.35)$$

The number operator in the ω exponent can be rewritten as:

$$\omega^{N_i} = \mathbb{1} + (\omega - 1) \sum_{m=1}^{p-1} \omega^{m-1} F_i^{\dagger m} F_i^m \quad (1.36)$$

Inserting this result in (eq. 1.35) and solving the resulting system of equations for the Fock parafermions yields:

$$\begin{aligned} F_i &= \frac{p-1}{p} \Gamma_i - \frac{1}{p} \sum_{m=1}^{p-1} \omega^{m(m+1)/2} \Gamma_i^{m+1} \Delta_i^{\dagger m} \\ F_i^{\dagger} &= \frac{p-1}{p} \Gamma_i^{\dagger} - \frac{1}{p} \sum_{m=1}^{p-1} \omega^{-m(m+1)/2} \Delta_i^m \Gamma_i^{\dagger m+1} \end{aligned} \quad (1.37)$$

This result is not readily available to us, but will nonetheless be easily derived when the main result of the subsequent section is derived.

1.2.3 Fradkin-Kadanoff Transformation (Weyl Hard-core Bosons)

Weyl hard-core bosons as we will define them are related to the Fock parafermions by a Jordan-Wigner-like transformation generalisation that is called the Fradkin-Kadanoff transformation. To see how this transformation arises, express the Weyl generators in terms of Weyl parafermions:

$$U_i = \Gamma_i^\dagger \Delta_i = \left(F_i^\dagger + F_i^{p-1} \right) \left(F_i + F_i^{\dagger p-1} \right) \omega^{N_i} = \omega^{N_i} \quad (1.38)$$

This result is rather interesting as it shows that the unitary operator, U_i , is equal to the exchange statistics factor, ω , to the power of number of particles at site i . Furthermore:

$$V_i = \Gamma_i \prod_{j=1}^{i-1} U_j^\dagger = \left(F_i + F_i^{\dagger p-1} \right) \prod_{j=1}^{i-1} U_j^\dagger = B_i + B_i^{\dagger p-1} \quad (1.39)$$

Where we have defined the bosonic-like creation and annihilation operators, $B_i^\dagger = F_i^\dagger \prod_{j<i} U_j$ and $B_i = F_i \prod_{j<i} U_j^\dagger$, that we will call the Weyl hard-core bosons. To see that these operators, B_i^\dagger and B_i , are in fact bosonic-like, we consider the commutation:

$$0 = [V_j, V_j] = [B_i, B_j] + [B_i, B_j^\dagger] B_j^{\dagger p-2} + B_i^{\dagger p-2} [B_i^\dagger, B_j] + B_i^{\dagger p-2} [B_i^\dagger, B_j^\dagger] B_j^{\dagger p-2} \quad (1.40)$$

Notice that the equation above is always zero, whereby each of the terms must also yield zero. Hence it is noted that the operators obey the hard-core boson commutation relations:

$$[B_i, B_j] = 0 \quad [B_i, B_j^\dagger] = 0 \quad [B_i^\dagger, B_j^\dagger] = 0 \quad (1.41)$$

And by insertion of the definition of the B_i 's in terms of F_i 's we see that $B_i^\dagger B_i = F_i^\dagger F_i$, which renders:

$$B_i^p = B_i^{\dagger p} = 0 \quad (B_i^\dagger)^m (B_i)^m + (B_i)^{p-m} (B_i^\dagger)^{p-m} = \mathbb{1} \quad \text{for } m \in \{1, \dots, p-1\} \quad (1.42)$$

We have thus arrived at a generalised version of the Jordan-Wigner transformation, also in the literature called the Fradkin-Kadanoff transformation [3, 5], which maps from the Fock parafermions to the Weyl hard-core bosons :

$$F_i = B_i \prod_{j=1}^{i-1} U_j \quad F_i^\dagger = B_i^\dagger \prod_{j=1}^{i-1} U_j^\dagger \quad (1.43)$$

Due to the lemma that $B_i^\dagger B_i = F_i^\dagger F_i$, the number operator in terms of the hard-core boson operators is:

$$N_i = \sum_{m=1}^{p-1} (B_i^\dagger)^m (B_i)^m \quad (1.44)$$

Notice that $V_i^\dagger B_i = B_i^\dagger B_i + B_i^p = B_i^\dagger B_i = F_i^\dagger F_i$. This operator will yield the identity, $\mathbb{1}$, as long as $n_i \geq 1$. In other words, B_i is the inverse of the Hermitian matrix V_i^\dagger , which is V_i , except for in the state where $n_i = 0$. In the matrix representation defined earlier the Weyl hard-core boson operator is thus:

$$B_i = \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{i-1} \otimes B \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1} \quad B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (1.45)$$

Notice how the matrix B is exactly the same as the matrix representation V , with the only difference being the 1 turned into a 0 in the bottom right corner. The similarity between B_i and V_i will make B_i inherit the commutation relations with U_i :

$$B_i U_j = \omega^{\delta_{i,j}} U_j B_i \quad B_i^\dagger U_j = \omega^{-\delta_{i,j}} U_j B_i^\dagger \quad \omega = e^{2\pi i/p} \quad (1.46)$$

Notice that the sum of all the powers of the roots of unity in the complex plane is 0, hence excluding 1 itself we find that $\sum_{m=1}^{p-1} \omega^m = -1$. Thus:

$$\sum_{m=1}^{p-1} (U_i)^m = \sum_{m=1}^{p-1} (U_i^\dagger)^m = \begin{pmatrix} (p-1) & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \quad (1.47)$$

This observation is used to isolate the first column of V_i from the other columns, such that we may remove it to get B_i . What it means is that the expression below must be true:

$$B_i = \frac{p-1}{p} V_i - \frac{1}{p} \sum_{m=1}^{p-1} V_i (U_i^\dagger)^m \quad (1.48)$$

One may check that the above equation is true in the matrix representation that we have defined. This result is used in conjunction with the definition of the Weyl parafermions (eq. 1.8) and the Fradkin-Kadanoff transformation (eq. 1.43) to derive (eq. 1.37).

1.3 Physical Models Where Parafermions Arise

Our treatment of the Weyl- and Fock parafermions have so far been purely mathematical and hypothetical. It is not trivial that such particles as discussed should arise in systems in the real world. However, as we will see in this chapter, proposals for the experimental realisation of these quasiparticles have been made. The most promising method is to use the edges of the Laughlin and Read-Rezayi states of the fractional quantum Hall effect [18, 19]. Therefore, this work is not purely theoretical, but also of practical importance.

1.3.1 Vector Potts Model

The vector Potts or clock model model is not technically physically realisable, but is an interesting system that arises in theoretical condensed matter physics. This model is a generalisation of the Ising and Heisenberg models, where the regular two spin states at each site are expanded to a total of n spin states. The possible values of the spin, U_j , at site j , by $1, \omega, \omega^2, \dots, \omega^{n-1}$, where $\omega = e^{2\pi i/n}$. The Potts model has been given its nickname the clock model due to the resemblance of the spin states to the hands of a clock. That this model gives rise to edge modes that are expressed by parafermions was first derived by Fendley in 2012 [2].

In one dimension the vector Potts model with nearest neighbour interaction on a lattice consisting of sites $1, \dots, L$ is:

$$H_{\text{vp}}[h_i, j_i] = -\frac{1}{2} \left(\sum_{i=1}^L h_i U_i + \sum_{i=1}^{L-1} J_i V_{i+1}^\dagger V_i \right) + \text{H.c.} \quad (1.49)$$

The first term can be understood as the coupling of the spin states U_i to an external magnetic field \vec{h} , and the second term is the interaction of the spins at adjacent sites, which seeks to align the spins for $J > 0$ and anti-align them when $J < 0$. As is evident, the model bears a close resemblance to the Ising model, but with an arbitrary number of spin states rather than only two spin states.

We note that by insertion of the definitions we have $V_{i+1}^\dagger V_i = \Gamma_{i+1}^\dagger \Delta_i$ and $U_i = \Gamma_i^\dagger \Delta_i$. Hence we note that we may write the Hamiltonian in terms of the Weyl parafermionic operators through the Jordan-Wigner-like transformation (eq. 1.8):

$$H_{\text{vp}} = -\frac{1}{2} \left(\sum_{i=1}^M h_i \Gamma_i^\dagger \Delta_i + \sum_{i=1}^{M-1} J_i \Gamma_{i+1}^\dagger \Delta_i \right) + \text{H.c.} \quad (1.50)$$

The vector Potts model may be defined in terms of the parafermions, but the physical interpretation of these operators eludes us. We will try and remedy that here. First, we note that the operator $Q_p =$

$\prod_{j=1}^M U_j$ is a global \mathbb{Z}_p symmetry of the system. Let $r(i) = L + 1 - i$, be the reflection through the midpoint of the system, and define the "dual" mapping, ϕ_d :

$$\begin{aligned} U_1 &\xrightarrow{\phi_d} V_L^\dagger & U_i &\xrightarrow{\phi_d} V_{r(i)}^\dagger V_{r(i)+1} \quad \text{for } i \in \{2, \dots, L\} \\ V_L &\xrightarrow{\phi_d} U_1 & V_j &\xrightarrow{\phi_d} U_{r(j)} \quad \text{for } j \in \{1, \dots, L-1\} \end{aligned} \quad (1.51)$$

The mapping ϕ_d is an isomorphism which guarantees the structure of the algebra is conserved even after the mapping. This guarantees that the transformed operators also obey the relations required for a set of Weyl generators. If we map the vector potts hamiltonian (eq. 1.49) under this transformation we get:

$$H_{\text{vp}}[h_i, J_i] \xrightarrow{\phi_d} -\frac{1}{2} \left(h_1 V_{r(1)} + \sum_{i=2}^L h_i V_{r(i)}^\dagger V_{r(i)+1} + \sum_{i=1}^{L-1} J_i U_{r(i)} \right) + \text{H.c.} = H_{\text{vp}}[h_i^*, J_i^*] \quad (1.52)$$

By defining the dual couplings $h_i^* \equiv J_{r(i)}$ and $J_i^* \equiv h_{r(i)}$. In other words, any pair of vector Potts Hamiltonian associated to the points $\{h_i, J_i\}$ and $\{h_i^*, J_i^*\}$ is isospectral - the model is in other words self-dual. The dual Hamiltonian however is not globally symmetric under the action of Q_p :

$$Q_p = \prod_{j=1}^L U_j \xrightarrow{\phi_d} V_{r(1)} \prod_{j=2}^L V_{r(j)}^\dagger V_{r(j)+1} = \left(\prod_{j=1}^{L-1} V_{r(j)} V_{r(j)}^\dagger \right) V_{r(M)}^\dagger = V_1^\dagger \quad (1.53)$$

This means that the dual Hamiltonian is symmetric in $Q_p \mapsto V_1^\dagger$ only on the boundary. Such symmetry that is globally symmetric and becomes a local symmetry on the boundary under a dual mapping, is denoted as a holographic symmetry. The existence of such a symmetry makes V_1^\dagger an edge mode [20].

1.3.2 Fractional Quantum Hall Edge Modes

The Laughlin states, a type of quasi particle states that arise in the fractional quantum Hall effect, display a very interesting behaviour similar to the parafermions [21, pp. 480-484]. The Laughlin states with filling factor $\nu = 1/p$, have quasi-particle operators $e^{i\phi(x)}$ that create right-moving charge e/p excitations, for which:

$$e^{i\phi(x)} e^{i\phi(x')} = e^{i\phi(x')} e^{i\phi(x)} e^{i\frac{\pi}{p} \text{sgn}(x' - x)} \quad (1.54)$$

This commutation looks nearly identical to the commutation relation of the parafermions, which seems to suggest we can use Laughlin states to physically realise parafermions. Clarke et al. [19] used this fact to propose an experimental setup for the realisation of the parafermions. We will go through this construction here.

Recall that the fractional quantum Hall effect arises when a two dimensional electron gas is subjected to a strong magnetic field at low energies. The electron gas is confined to two dimensions by the construction of an effective quantum well which is then doped at different filling factors. The Laughlin states (or the fractional quantum Hall effect) are created when the doping is such that the filling factor (which are the number of electrons to magnetic flux quanta in the quantum well) take on a nice rational number [22].

The setup proposed by Clarke et al. [19] works as follows: the Laughlin edge states run along the edges of the quantum well, so a series of steps is taken to localise them in a type of particle, which will be our parafermions. To create a localised edge mode, a gap is needed to be created. The gap is constructed by running two counter-propagating Laughlin states through a setup of superconductors and spin-orbit-coupled insulators in the region between two quantum wells (see fig. 1.1). The desired gap can appear through (1) tunneling of electrons across the junction of the setup, (2) facilitating electrons from each edge into Cooper pairs. In the latter case a pairing gap opens at the interface between superconductor and spin-orbit-coupling insulator via the proximity effect with ordinary s -wave superconductors. A tunnelling gap can arise from the spin orbit induced back-scattering between the edge states, either through the quantum wells or the insulator.

Assume the electrons only tunnel through the interface by the spin-orbit-coupled insulator. We may consider the momenta of the right-/left-moving e/p charged quasiparticles, $\phi_{R/L}$, and their commutation relations:

$$[\phi_{R/L}(x), \phi_{R/L}(x')] = \pm i \frac{\pi}{p} \text{sgn}(x - x') \quad [\phi_L(x), \phi_R(x')] = i \frac{\pi}{p} \quad (1.55)$$

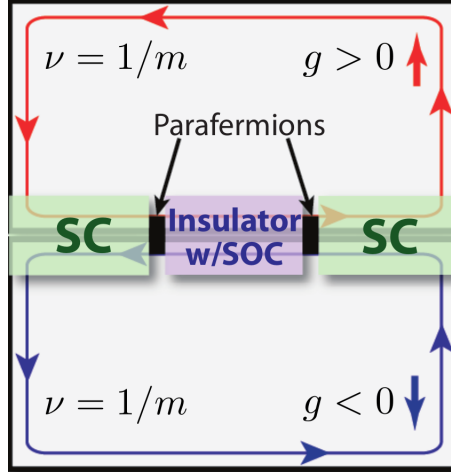


Figure 1.1: Proposed experimental setup for realisation of parafermions. Two counter-propagating Laughlin states are made to run through a series of superconducting and spin-orbit-coupling insulator regions. Back scattering in the intermediate region localises the Laughlin states such that the anyonic parafermions are formed. Figure is borrowed from Clarke et al. [19].

The electron operators are proportional to some vertex operator, $\psi_{R/L} \sim e^{i\phi_{R/L}}$, where the $\phi_{R/L}$ fields are bosonic. By the theory of bosonisation¹, we can split up the chiral bosonic fields as $\phi_{R/L} = \varphi \pm \theta$ with $\rho = \partial_x \theta / \pi$, where φ and θ are bosonic fields. The commutation relation of the newly defined bosonic fields is from (eq. 1.55).

$$[\varphi(x), \theta(x')] = i \frac{\pi}{p} \Theta(x - x') \quad (1.56)$$

Which is exactly what one would expect from the bosonic fields (see appendix A). The Hamiltonian of the interface is assumed to be $H = H_0 + H_1$, where H_0 is the Luttinger liquid Hamiltonian:

$$H_0 = \frac{pv}{2\pi} \int dx [(\partial_x \varphi)^2 + (\partial_x \theta)^2] \quad (1.57)$$

H_1 is the Hamiltonian of the superconductor and the spin-orbit-coupled insulator combined into one:

$$H_1 = \int dx \left(\overbrace{\Delta(x) \psi_R \psi_L}^{\text{superconductor}} + \overbrace{\mathcal{M}(x) \psi_R^\dagger \psi_L}^{\text{SOC-insulator}} + \text{H.c.} \right) \sim \int dx (-\Delta(x) \cos(2p\varphi(x)) - \mathcal{M}(x) \cos(2p\theta(x))) \quad (1.58)$$

The $p = 1$ -case has previously been extensively studied, and leads to the formation of Majorana fermions at the edges of the insulator.

For the $p > 1$ -cases we assume the terms, $\Delta(x)$ and $\mathcal{M}(x)$, are so big in the superconductor and spin-orbit insulator, respectively, that they will dominate in each of these regions, but will otherwise be negligible. This results in the fields ϕ and θ being pinned down in the superconducting and spin-orbit insulator domain, respectively. For example, in the superconductor the minimal Hamiltonian will be achieved when the term, $-\Delta(x) \cos(2p\varphi(x))$, is minimal, which renders $\varphi \in \{0, \frac{\pi}{m}, 2\frac{\pi}{m}, \dots, (2m-1)\frac{\pi}{m}\}$.

Define \hat{n} as a integer valued operator, such that in our experimental setup, $\varphi(x < x_1) = \frac{\pi}{m} \hat{n}_\varphi^{(1)}$ and $\varphi(x > x_2 + \ell) = \frac{\pi}{m} \hat{n}_\varphi^{(2)}$ in the left and right superconductors, while $\varphi(x_1 + \ell < x < x_2) = \frac{\pi}{m} \hat{n}_\theta$ in the spin-orbit-coupled insulator. Note by the commutation relation of φ and θ we get:

$$[\hat{n}_\varphi^{(1)}, \hat{n}_\theta] = 0 \quad [\hat{n}_\varphi^{(2)}, \hat{n}_\theta] = i \frac{m}{\pi} \quad (1.59)$$

Assuming the fields are pinned as prescribed in the superconductors and insulators, then at low energies of the system we can focus on the regions $(x_i, x_i + \ell)$ between the superconductors and insulators, where

¹See appendix A.

$\Delta(x)$ and $\mathcal{M}(x)$ both vanish. The effective Hamiltonian of the system is thus only displaying an gapless bosonic mode in the these regions:

$$H_{\text{eff}} = \frac{mv}{2\pi} \sum_{i=1}^2 \int_{x_i}^{x_i+\ell} dx [(\partial_x \varphi)^2 + (\partial_x \theta)^2] \quad (1.60)$$

The following operators $\alpha_j, j = 1, 2,$, commute with this effective Hamiltonian, and represent, therefore, zero-modes that are bound to the domain wall in the system (black parts of fig 1.1):

$$\alpha_j = e^{i\frac{\pi}{m}(\hat{n}_\varphi^{(j)} + \hat{n}_\theta)} \int_{x_i}^{x_i+\ell} dx \left[e^{-i\frac{\pi}{m}(\hat{n}_\varphi^{(j)} + \hat{n}_\theta)} e^{i(\varphi+\theta)} + e^{-i\frac{\pi}{m}(\hat{n}_\varphi^{(j)} - \hat{n}_\theta)} e^{i(\varphi-\theta)} + \text{H.c.} \right] \quad (1.61)$$

In the ground state limit, the integral in the above expression evaluates to a constant. Hence the operators α_j become:

$$\alpha_j \sim e^{i\frac{\pi}{m}(\hat{n}_\varphi^{(j)} + \hat{n}_\theta)} \quad (1.62)$$

These two operators are (Weyl) parafermions, since they obey the relations (provable by use of the BCH formula and our definitions and commutators):

$$(\alpha_j)^{2m} = 1 \quad \alpha_j \alpha_{j'} = \alpha_{j'} \alpha_j e^{i\frac{\pi}{m} \text{sgn}(j'-j)} \quad (1.63)$$

Notice that the constructed fermion is of even order with $p = 2m$. To get odd orders of parafermions more steps need to be taken. The construction of a parafermion like the \mathbb{Z}_3 parafermion cannot be made by the simple description that has ben outlined here. The \mathbb{Z}_3 parafermion has to be constructed through a Read-Rezayi edge state rather than a Laughlin edge state [18].

The astute reader might say that this proposed setup seems impossible, since it requires a quantum well with a strong magnetic field to be right next to a superconducting region, which famously does not like such fields, and a magnetic field is not something that can just be isolated to exist in local regions. The practicality of the setup was consequently dubious until 2020 when Gül et.al. [23] constructed a heterostructure device that displays superconducting behaviour at the edge of a fractional quantum Hall edge.

Chapter 2

Chiral Luttinger Liquids

Before proceeding to the main point of this thesis, which is to treat the Fock parafermions through bosonisation, we will study a system which seems unrelated at first, but turns out to be of great importance in our future analysis. The system in question is the one-dimensional chiral Luttinger liquid. This system arises as a hydrodynamic description of the edge excitations of the fractional quantum Hall effect, the derivation of which is made in [24] and [21, pp. 609-615]. In this chapter we will however derive the system differently in a rather unorthodox way.

Consider the Tomonga-Luttinger liquid in one dimension defined on the system of size L . The Hamiltonian of such system, which is derived in appendix A, looks like:

$$H_{LL} = \frac{\hbar v}{2\pi} \int_0^L dx K (\partial_x \varphi)^2 + \frac{(\partial_x \theta)^2}{K} = \frac{\hbar v}{4\pi} \int_0^L dx \left(\sqrt{K} \partial_x \varphi + \frac{\partial_x \theta}{\sqrt{K}} \right)^2 + \left(\sqrt{K} \partial_x \varphi - \frac{\partial_x \theta}{\sqrt{K}} \right)^2 \quad (2.1)$$

Where we in the last equality have split up the Hamiltonian of the system in two parts. Our idea is to consider each of the part by themselves. As we will see, each of these parts corresponds to one of the dispersions of the Luttinger liquid that has velocity $\pm v$. The model is dubbed the chiral Luttinger liquid for this reason. We let $\epsilon = \hbar v K / 4\pi$ and $\kappa = 1/K$ and consequently define the chiral Luttinger liquid on a system of size L :

$$H_{CLL} = \epsilon \int_0^L dx (\partial_x \varphi + \kappa \partial_x \theta)^2 \quad (2.2)$$

Where $\epsilon > 0$ and κ are real numbers. Note that we should impose the restraint $\epsilon > 0$, since $\epsilon < 0$ will favour a system where $\partial_x \varphi \rightarrow \pm\infty$ and $\partial_x \theta \rightarrow \pm\infty$, which will be unphysical, while $\epsilon = 0$ is a pathological case of the system which will not correspond to any real system. The two fields φ and θ are a pair of dual bosonised fields, which commute as:

$$\left[\frac{\partial_x \theta}{\pi}(x), \varphi(y) \right] = -i\delta(y-x) \quad (2.3)$$

Note that the chiral Luttinger liquid is not Lorentz invariant, because of the mixing between the dual fields $\partial_x \varphi$ and $\partial_x \theta$. This makes it impossible to go the standard way of finding a canonical diagonal representation of the Hamiltonian that readily yields eigenvectors and such of the system. Instead, we will study this system by deriving its correlation functions using the Feynman path integral formulation. To do so we must first determine the Lagrangian of the system through a Legendre transformation of the Hamiltonian.

2.1 Lagrangian Dynamics

2.1.1 φ -Field Lagrangian

Our first step in studying the Lagrangian of our system in the field φ , is to define the canonical momentum field, $\Pi_\varphi(x)$, such that it obeys the standard canonical commutation relation with the field $\varphi(x)$, ie.:

$$[\varphi(x), \Pi(y)] = i\hbar\delta(x-y) \quad (2.4)$$

By the definition, (eq. 2.3), the following operator obeys exactly this relationship:

$$\Pi_\varphi(x) = \frac{\hbar}{\pi} \partial_x \theta(x) \quad (2.5)$$

Substituting $\partial_x \theta$ in the Hamiltonian by the canonical momentum, Π_φ , we find the time derivative of the φ -fields by Hamilton's equation:

$$\partial_t \varphi = \frac{\partial}{\partial \Pi_\varphi} \mathcal{H}_{\text{CCL}} = \frac{2\pi\epsilon\kappa}{\hbar} \left(\partial_x \varphi + \frac{\pi\kappa}{\hbar} \Pi_\varphi \right) \quad (2.6)$$

These two equations make it possible to conduct a Legendre transformation of the Hamiltonian density, getting the Lagrangian density of the system in terms of the φ -field only:

$$\mathcal{L}_{\text{CCL}} = \Pi_\varphi \partial_t \varphi - \mathcal{H}_{\text{CCL}} = \frac{1}{4\epsilon} \left(\frac{\hbar}{\pi\kappa} \right)^2 (\partial_t \varphi)^2 - \frac{\hbar}{\pi\kappa} \partial_x \varphi \partial_t \varphi \quad (2.7)$$

The quantum field theoretical Euler-Lagrange equations in $(1+1)$ -dimensions¹ give us the equations of motions of the φ -field:

$$\partial_t \left(\frac{1}{2\epsilon} \frac{\hbar}{\pi\kappa} \partial_t \varphi - 2\partial_x \varphi \right) = 0 \quad (2.8)$$

In this partial differential equation we carry out the integration over time. Assuming the integration constant is zero, we find the dispersion relation:

$$\partial_t \varphi = v_\varphi \partial_x \varphi \quad v_\varphi = \frac{4\pi\epsilon\kappa}{\hbar} \quad (2.9)$$

Only one direction is present in the dispersion relation, which is why we called the chiral Luttinger liquid *chiral*. With this velocity defined, the Lagrangian density can take on a simpler form:

$$\mathcal{L}_{\text{CCL}}(\varphi) = \frac{\hbar}{\pi\kappa} \left(\frac{1}{v_\varphi} (\partial_t \varphi)^2 - \partial_x \varphi \partial_t \varphi \right) \quad (2.10)$$

Comparison to the Regular Luttinger Liquid and Considerations for $\kappa = 0$

That the dispersion relation of the chiral Luttinger liquid only has one direction is in contrast to the case of the regular Luttinger liquid, where the dispersion is in both directions with velocities $\pm v$. The chiral Luttinger liquid is in fact a more general case of the standard Luttinger liquid. To see why this is the case, note that we may construct a Luttinger liquid with velocities $\pm v$, from two chiral Luttinger liquids each with velocity $+v$ and $-v$, respectively:

$$H_{\text{CCL},-v} + H_{\text{CCL},+v} = \epsilon \int dx (\partial_x \varphi - \kappa \partial_x \theta)^2 + (\partial_x \varphi + \kappa \partial_x \theta)^2 = \epsilon \int dx (\partial_x \varphi)^2 + \kappa^2 (\partial_x \theta)^2 = H_{\text{LL}} \quad (2.11)$$

In our analysis we have so far assumed $\kappa \neq 0$. Letting² $\kappa \rightarrow 0$ the chiral Luttinger liquid becomes:

$$H_{\text{CCL}} \xrightarrow{\kappa \rightarrow 0} \epsilon \int dx (\partial_x \varphi)^2 \quad (2.12)$$

This Hamiltonian is that of the Luttinger liquid with $K \rightarrow \infty$. This is a zero dispersion mode since the canonical momentum of the φ -field, Π_φ , will not appear in the Hamiltonian. For this reason, Hamilton's equations will render $\partial_t \varphi = 0$. The same will be true for the θ -field in this limit.

2.1.2 θ -Field Lagrangian

We want to construct the canonical momentum to the θ -field, Π_θ . By the fundamental theorem of calculus, the derivative of an integral of a function is the function itself. Using this theorem and partial integration, it is noted that:

$$\begin{aligned} [\varphi, \partial_x \theta] &= \partial_x \int dx [\varphi, \partial_x \theta] = \partial_x \int dx (\varphi \partial_x \theta - (\partial_x \theta) \varphi) = \partial_x \left([\varphi \theta - \theta \varphi]_{-\infty}^{\infty} - \int dx ((\partial_x \varphi) \theta - \theta \partial_x \varphi) \right) \\ &= -[\partial_x \varphi, \theta] = [\theta, \partial_x \varphi] \end{aligned} \quad (2.13)$$

¹See appendix D.2.

²The exact limit $\kappa \rightarrow 0$ cannot be taken directly, however, as $\epsilon \rightarrow \infty$

Where we assume that the derivative of the fields φ and θ die off as $x \rightarrow \infty$. By invoking the commutation relation of the fields (eq. 2.3), we determine the canonical momentum to the θ -field, $\Pi_\theta = \frac{\hbar}{\pi} \partial_x \varphi$. Hamilton's equations are then used to determine the temporal derivative $\partial_t \theta = \frac{2\pi\epsilon}{\hbar} (\kappa \partial_x \theta + \frac{\pi}{\hbar} \Pi_\theta)$. Whereby, it is possible to determine the Lagrangian through a Legendre transformation:

$$\mathcal{L}_{\text{CLL}}(\theta) = \frac{\hbar\kappa}{\pi} \left(\frac{1}{v_\theta} (\partial_t \theta)^2 - \partial_x \theta \partial_t \theta \right) \quad (2.14)$$

Where we have used the Euler-Lagrange equation to determine the dispersion relation and velocity, v_θ :

$$\partial_t \theta = v_\theta \partial_x \theta \quad v_\theta = \frac{4\pi\epsilon\kappa}{\hbar} \quad (2.15)$$

It is observed that the two dual fields θ and φ have exactly the same dispersion with $v = v_\varphi = v_\theta$.

2.2 Partition Functions

The Lagrangians of the chiral Luttinger liquid in the two fields φ and θ (Eqs. 2.10 and 2.14) proved very similar. This will render calculations of the two-point correlation functions very easy to infer for one of the fields, say θ , provided the other field's correlation functions have been calculated, say that of φ . To do this we introduce a small abstraction as the constant α , and denote the velocity $v = v_\theta = v_\varphi$ such that:

$$\mathcal{L}_{\text{CLL}} = \alpha \hbar \left(\frac{1}{v} (\partial_t \varphi)^2 - \partial_x \varphi \partial_t \varphi \right) \quad (2.16)$$

Where $\alpha = 1/\pi\kappa$ for the φ -field. The calculation of the two-point correlation function of the θ -fields will be exactly the same as the one for the θ -field, but letting $\alpha = \kappa/\pi$.

We rewrite the Lagrangian in imaginary time by letting $t \rightarrow \tau = it$. This substitution makes the partial derivatives $\partial_t = \frac{\partial \tau}{\partial t} \partial_\tau = i \partial_\tau$. The Lagrangian in terms of this new imaginary time coordinate is:

$$\mathcal{L}_{\text{CLL}} = -\alpha \hbar \left(i \partial_x \varphi \partial_\tau \varphi + \frac{1}{v_\varphi} (\partial_\tau \varphi)^2 \right) \quad (2.17)$$

Cf. the Feynman path integral approach, we can define the partition function over the Lagrangian, $\mathcal{L}(\varphi)$, by:

$$Z \equiv \int D\varphi \exp \left[-\frac{1}{\hbar} \int dx d\tau \mathcal{L}(\varphi) \right] \quad (2.18)$$

To calculate this partition function, we define the Fourier transformation of the φ -field:

$$\hat{\varphi}(k, \omega) = \int dx d\tau \varphi(x, \tau) e^{-i(kx - \omega\tau)} \quad \varphi(x, \tau) = \frac{1}{2\pi} \int dk d\omega \hat{\varphi}(k, \omega) e^{i(kx - \omega\tau)} \quad (2.19)$$

By these definitions, we see that the partial derivatives on the φ -fields can be written as:

$$\begin{aligned} \partial_x \varphi(x, \tau) &= \frac{i}{2\pi} \int dk d\omega k \cdot \hat{\varphi}(k, \omega) e^{i(kx - \omega\tau)} \\ \partial_\tau \varphi(x, \tau) &= \frac{-i}{2\pi} \int dk d\omega \omega \cdot \hat{\varphi}(k, \omega) e^{i(kx - \omega\tau)} \end{aligned} \quad (2.20)$$

By insertion of these definitions in the integral, (eq. 2.17), we find:

$$\begin{aligned} \int dx d\tau \mathcal{L}_{\text{CLL}}(x, \tau) &= -\alpha \hbar i \cdot \frac{1}{(2\pi)^2} \int dx d\tau dk d\omega dk' d\omega' k\omega' \cdot \hat{\varphi}(k, \omega) \hat{\varphi}(k', \omega') e^{i((k+k')x - (\omega+\omega')\tau)} \\ &\quad - \frac{\alpha \hbar}{v_\varphi} \cdot \frac{(-1)}{(2\pi)^2} \int dx d\tau dk d\omega dk' d\omega' \omega\omega' \cdot \hat{\varphi}(k, \omega) \hat{\varphi}(k', \omega') e^{i((k+k')x - (\omega+\omega')\tau)} \end{aligned} \quad (2.21)$$

We may carry out the integration over the τ and x , realising the integration will yield a Dirac-delta function [25]:

$$\int dx dt e^{i((k+k')x - (\omega+\omega')t)} = (2\pi)^2 \delta(k+k') \delta(\omega+\omega') \quad (2.22)$$

The delta functions make the integration over k' and ω' simple, such that we can effectively just replace $k' = -k$ and $\omega' = -\omega$. All in all we find:

$$\int dx d\tau \mathcal{L}_{\text{CLL}}(x, \tau) = \alpha \hbar \int dk d\omega \left[\hat{\varphi}(-k, -\omega) \left(\frac{\omega^2}{v_\varphi} - ik\omega \right) \hat{\varphi}(k, \omega) \right] \equiv \int dk d\omega \hat{\mathcal{L}}_{\text{CLL}}(k, \omega) \quad (2.23)$$

The partition function of the chiral Luttinger liquid (eq. 2.16) is:

$$Z_{\text{CLL}} = \int D\varphi \exp \left[-\alpha \int dk d\omega \hat{\varphi}(-k, -\omega) \left(\frac{\omega^2}{v_\varphi} - ik\omega \right) \hat{\varphi}(k, \omega) \right] \quad (2.24)$$

2.3 Two-point Correlation Functions

In this section we calculate the two point-correlation functions of the φ field first, and then extend the calculations such that the two-point correlation functions of the θ fields can be found. We may do this since the Lagrangian of the system in the two fields are very similar. We now define some small generating function, $\eta(x, \tau)$:

$$Z(\eta) \equiv \int D\varphi \exp \left[\int dk d\omega \left(-\frac{1}{\hbar} \cdot \hat{\mathcal{L}}_{\text{CLL}}(k, \omega) + \eta(-k, -\omega) \hat{\varphi}(k, \omega) \right) \right] \quad (2.25)$$

This partition function will evaluate as [26]:

$$Z(\eta) = Z_{\text{CLL}} \cdot \exp \left[\int dk d\omega \eta(-k, -\omega) G(k, \omega) \eta(k, \omega) \right] \quad (2.26)$$

Where we have defined the Greens functions, which can easily be calculated in momentum space as:

$$G(k, \omega) = \alpha^{-1} \left(\frac{\omega^2}{v_\varphi} - ik\omega \right)^{-1} \quad (2.27)$$

The reason why we defined this partition function in terms of some generating function is that we can easily infer the two-point correlation functions from this function. Cf. [27, p. 15] the two-point correlation function is:

$$\langle \varphi(k, \omega) \varphi(-k, -\omega) \rangle = \lim_{\eta \rightarrow 0} \partial_{\eta(k, \omega)} \partial_{\eta(-k, -\omega)} \frac{Z(\eta)}{Z_{\text{CLL}}} = G(k, \omega) \quad (2.28)$$

Using an inverse Fourier transformation we can thus determine the real-space correlation functions:

$$\langle \varphi(x, \tau) \varphi(x', \tau') \rangle = G(x, \tau; x', \tau') = \frac{1}{(2\pi)^2} \int dk d\omega e^{i(k(x-x') - \omega(\tau - \tau'))} G(k, \omega) \quad (2.29)$$

The explicit calculation of the real space correlation function is a bit long, and is therefore relegated to appendix B. At its heart, the calculation makes use of the Cauchy integral formula to evaluate (eq. 2.29). It is noted that the velocity plays a big role on the domain for which the integral will converge and be non-zero. The results for the two-point correlation functions are enumerated below for varying values of v . Note that we have defined $\xi \equiv k(x - x')/\omega - i(\tau - \tau') = (x - x')/v + (t - t')$.

Case $v > 0$

Other than the sign of the velocity we also need to take into account the sign of $(x - x')$. If $(x - x') > 0$, we find the two-point correlation function:

$$\langle \varphi(x, t) \varphi(x', t') \rangle = -\frac{\kappa}{2} \int_{-\infty}^0 d\omega \frac{e^{\omega\xi}}{\omega} = \begin{cases} \text{sgn}(\kappa) \cdot \infty & \text{for } \xi < 0 \text{ and } (x - x') > 0 \\ -\frac{\kappa}{2} (\log(|\xi| \Lambda_{\min}) + \gamma) & \text{for } \xi > 0 \text{ and } (x - x') > 0 \end{cases} \quad (2.30)$$

Where we have defined some cut-off frequency Λ_{\min} , and γ is the Euler-Mascheroni constant. While the case $(x - x') < 0$ renders the correlation function:

$$\langle \varphi(x, t) \varphi(x', t') \rangle = \frac{\kappa}{2} \int_0^{\infty} d\omega \frac{e^{\omega\xi}}{\omega} = \begin{cases} -\frac{\kappa}{2} (\log(|\xi| \Lambda_{\min}) + \gamma) & \text{for } \xi < 0 \text{ and } (x - x') < 0 \\ \text{sgn}(\kappa) \cdot \infty & \text{for } \xi > 0 \text{ and } (x - x') < 0 \end{cases} \quad (2.31)$$

Case $v < 0$

Other than the sign of the velocity we also need to take into account the sign of $(x - x')$. If $(x - x') > 0$, we find the two-point correlation function:

$$\langle \varphi(x, t) \varphi(x', t') \rangle = -\frac{\kappa}{2} \int_0^\infty d\omega \frac{e^{\omega\xi}}{\omega} = \begin{cases} \frac{\kappa}{2} (\log(|\xi| \Lambda_{\min}) + \gamma) & \text{for } \xi < 0 \text{ and } (x - x') > 0 \\ -\text{sgn}(\kappa) \cdot \infty & \text{for } \xi > 0 \text{ and } (x - x') > 0 \end{cases} \quad (2.32)$$

While the case $(x - x') < 0$ renders the correlation function:

$$\langle \varphi(x, t) \varphi(x', t') \rangle = \frac{\kappa}{2} \int_{-\infty}^0 d\omega \frac{e^{\omega\xi}}{\omega} = \begin{cases} -\text{sgn}(\kappa) \cdot \infty & \text{for } \xi < 0 \text{ and } (x - x') < 0 \\ \frac{\kappa}{2} (\log(|\xi| \Lambda_{\min}) + \gamma) & \text{for } \xi > 0 \text{ and } (x - x') < 0 \end{cases} \quad (2.33)$$

Case $v = 0$

$v \rightarrow 0$, means either that the Hamiltonian approaches the limit:

$$H_{\text{CLL}} \xrightarrow{\kappa \rightarrow 0} \epsilon \int dx (\partial_x \varphi)^2 \quad (2.34)$$

Or that the entire thing is zero due to $\epsilon \rightarrow 0$. Both cases will however yield $\langle \varphi(x, t) \varphi(x', t') \rangle = \text{const}$, as noted earlier and evidenced by the calculation in appendix B.

Summary: General $v \neq 0$ and Equal Time Correlation Function

Note that $v \propto \epsilon \kappa$, cf. (Eqs. 2.9 and 2.15). However, earlier it was argued that we must require $\epsilon > 0$ for the system to be physically sound. This is due to the fact that if $\epsilon < 0$ the state that will minimise the energy of the Hamiltonian is the state for which $\partial_x \varphi \rightarrow \infty$ and $\partial_x \theta \rightarrow \infty$, which is rather an unphysical system. This will make the sign of v be the sign of κ . One can quickly check that the results above, where the two-point correlation function $\langle \varphi(x, t) \varphi(x', t') \rangle = \infty$, corresponds to the impossible case where the sign of v and κ are different, and would as such not be attainable. This underlines our intuition that such a system should be unphysical.

On the other hand, if $x - x' > v(t - t')$ (like in the case of equal time correlation function $t = t'$) the correlation function is:

$$\langle \varphi(x, t) \varphi(x', t') \rangle = -\frac{|\kappa|}{2} (\log(|\xi| \Lambda_{\min}) + \gamma) \quad (2.35)$$

If we define the system over some one-dimensional lattice of length L and with lattice spacing a , a natural definition of our cut-off frequencies $\Lambda_{\min} \propto 2\pi L^{-1}$ and $\Lambda_{\max} \propto 2\pi a^{-1}$ arise due to the periodicity of the unit cell in momentum space. Notice that we want the unit of the cut-off frequencies to be on the energy scale, whereby we may let $\Lambda_{\min} = 2\pi |v|/L$ and $\Lambda_{\max} = 2\pi |v|/a$. Hereby the equal time correlation functions of the Chiral Luttinger liquid on the one-dimensional lattice are:

$$\langle \varphi(x, t) \varphi(x', t) \rangle = -\frac{|\kappa|}{2} \log |x - x'| + \text{const} \quad (2.36)$$

At small $|x - x'| \approx a$ the system approaches a high energy description which is also treated in the appendix. Here it is shown that the resulting two-point correlation functions are rapidly exponentially decreasing, which makes them effectively zero at the long-range orders. We find that:

$$\langle \varphi(x, t) \varphi(x', t) \rangle \propto -\frac{|\kappa|}{2} \cdot e^{-\frac{|x-x'|}{a}} \quad \text{for } |x - x'| \approx a \quad (2.37)$$

2.3.1 Correlation Functions in the θ -fields

To get the correlation functions of the θ -fields, one simply substitutes $\kappa \rightarrow 1/\kappa$, in the calculations that were just made, due to the similarity between (eq. 2.10) and (eq. 2.14). Therefore, the equal time correlation functions with $t = t'$ are:

$$\langle \theta(x, t) \theta(x', t) \rangle = -\frac{1}{2|\kappa|} \log |x - x'| + \text{const} \quad (2.38)$$

If we have $x = x'$ or if $\kappa = 0$, we define:

$$\langle \theta(x, t)\theta(x, t) \rangle = \text{const} \quad (2.39)$$

2.4 The Chiral Luttinger Field ψ

In this section we will quickly go over an alternative approach of representing the chiral Luttinger liquid. This is done by defining the chiral field, ψ :

$$\psi \equiv (\varphi + \kappa\theta) \quad (2.40)$$

This definition will make it such that $[\partial_x \psi(x), \psi(x')] = -i2\pi\kappa\delta(x-x')$. Note that hereby we find that the canonical momentum is $\Pi_\psi = \frac{\hbar}{2\pi\kappa}\partial_x \psi$. The Hamiltonian density of the chiral Luttinger liquid is thus:

$$\mathcal{H} = \epsilon \cdot (\partial_x \psi)^2 = \epsilon \cdot \left(\frac{2\pi\kappa}{\hbar} \right)^2 (\Pi_\psi)^2 \quad (2.41)$$

From the Hamilton equations we find that the time derivative $\partial_t \psi = \frac{4\pi\kappa\epsilon}{\hbar}\partial_x \psi$, which makes the velocity $v_\psi = \frac{4\pi\kappa\epsilon}{\hbar}$ - exactly the same velocity as those of the two fields φ and θ . A Legendre transformation yields the Lagrangian:

$$\mathcal{L} = \frac{\hbar}{4\pi\kappa} \left(\partial_t \psi \partial_x \psi - \frac{1}{v_\psi} (\partial_t \psi)^2 \right) \quad (2.42)$$

The Lagrangian is on the same form as we studied in previous sections, where we let the abstraction constant $\alpha = -(4\kappa)^{-1}$. The calculation of the two-point correlation functions saw that this size was proportional to the absolute value of half of this value, ie. $\langle \psi\psi \rangle \propto |\alpha|/2$. The two-point correlation functions are thus:

$$\langle \psi(x, t)\psi(x', t) \rangle = -2|\kappa| \log \left| \frac{(x-x') - v_\psi(t-t')}{a} \right| \quad (2.43)$$

Notice that this value is different from the two-point correlation function of the normalised chiral field where with $[\partial'_x \psi(x), \psi'(x')] = -i\pi\kappa\delta(x-x')$ by a factor of 2.

Chapter 3

Fock Parafermion Tight Binding Model

3.1 The Model

In the rest of this thesis we will analyse the nearest neighbour tight-binding model of the \mathbb{Z}_p Fock parafermions. It can be considered as a generalisation of the Hubbard model for \mathbb{Z}_p Fock parafermions. We define the model in one dimension but it is possible to expand this to multiple spatial dimensions by the introduction of some lexicological order that we touched upon in section 1.1.1.

The model assumes a one-dimensional chain of length L of tight-binding sites, with equilateral lattice spacing a , to which \mathbb{Z}_p Fock parafermions can bind. The nomenclature, tight-binding sites, are meant to denote sites where it is assumed that the overlap of the wave functions of the Fock parafermions of adjacent sites is small such that it is effectively ignored, and the Fock parafermions at each site can be treated by themselves. A system such as this is achieved by a deep potential well at each lattice site, which will effectively bind the Fock parafermions to them.

Up to $p - 1$ Fock parafermions can be bound to each site of the lattice. This stems from the fact that p is the smallest integer such that $(F_i^\dagger)^p = 0$, cf. (eq. 1.30). This means $(F_i^\dagger)^{p-1} |0\rangle \neq 0$, and we may interpret F_i^\dagger as creating the Fock parafermions at site i by the second quantisation.

Occasionally, a \mathbb{Z}_p Fock parafermion may jump from one site, i , to another, j , by quantum tunneling. This action is described in the language of second quantisation as the operator $F_j^\dagger F_i$. Nothing prevents us from only allowing more than one Fock parafermion to make the jump, in fact m Fock parafermions can make the jump, which is described by $(F_j^\dagger)^m (F_i)^m$. Each hop of m Fock parafermions from site i to j has an energy associated with them, which we will denote $t_{m,i,j}$. In general the Hamiltonian describing the physics of this Fock parafermion tight-binding model is:

$$H_{\text{TB}} = - \sum_{i,j=1}^L \sum_{m=1}^{p-1} t_{m,i,j} (F_j^\dagger)^m (F_i)^m + \text{H.c.} \quad (3.1)$$

The locality of the wave function makes nearest-neighbour hopping much more likely than hopping over distances that are bigger than this. These other types hopping are thus ignored. We derive at this by assuming the potential at each site is a quantum well, which makes the shape of the wave function sinusoidal on the inside of the well but exponentially decreasing at the edges and outside the well. Furthermore, if we assume translational invariance, it can be assumed that no site is favoured over another such that $t_{m,i,i+1} = t_{m,j,j+1} = t_m$ for all $i, j \in [1, L]$. This gives us the model:

$$H_{\text{NN}} = - \sum_{m=1}^{p-1} t_m \sum_{i=1}^L (F_{i+1}^\dagger)^m (F_i)^m + \text{H.c.} \quad (3.2)$$

Arguably t_m should be a real number bigger than 0 to make the Hamiltonian an observable and facilitate the hopping. The total number of parafermions in the system will be:

$$N = \sum_{i=1}^L N_i = \sum_{i=1}^L \sum_{m=1}^{p-1} (F_i^\dagger)^m (F_i)^m \quad (3.3)$$

This operator is very clearly a symmetry of the system, since it will commute with each of the terms in the nearest-neighbour Hamiltonian. Intuitively the number of particles should also be conserved in the model with Hamiltonian (eq. 3.2). It is in fact a $U(1)$ -symmetry, since it generates the algebra for such a symmetry as described in (eq. 1.34).

To this model might be added other terms as we please to describe our system. One such term could be the chemical potential:

$$H_\mu = -\mu N = -\mu \sum_{i=1}^L \sum_{m=1}^{p-1} (F_i^\dagger)^m (F_i)^m \quad (3.4)$$

Another term is the on-site interaction energy, which will seek to spread out the Fock parafermions such that they are not all concentrated on the same sites. The term is constructed such that each Fock parafermion at site i interacts each of the other Fock parafermions at site i with energy U (which is N_i choose 2):

$$H_U = U \sum_{i=1}^L \frac{N_i(N_i - 1)}{2} = \frac{U}{2} \sum_{i=1}^L \sum_{m=2}^{p-1} \left((m-1)(F_i^\dagger)^m (F_i)^m + \sum_{n=m}^{p-1} (F_i^\dagger)^n (F_i)^n \right) \quad (3.5)$$

With the help of some algebra this interaction term has been written out. However, doing so yields no further insight.

Particle-Hole-like Symmetry

In this section we prove that a form of particle-hole symmetry exists, which makes it possible to study the system for filling factors $n = N/L \leq \frac{p-1}{2}$ only. The following argumentation is based on a generalisation of the argumentation by Rossini et al. [4]. Define the transformation $T : F_{L/2+j} \rightarrow F_{L/2-j}^\dagger$, which both takes the Hermitian conjugate of the Fock parafermion operators and mirror them in the midpoint. The translation conserves the generalised Clifford algebra of the Fock parafermions. For example, since for $0 < i < j < L/2$:

$$F_{L/2+i}^\dagger F_{L/2+j} \longrightarrow F_{L/2-i} F_{L/2-j}^\dagger = \omega F_{L/2-j}^\dagger F_{L/2-i} \longrightarrow \omega^{-1} F_{L/2+j} F_{L/2+i}^\dagger \quad (3.6)$$

Where we have used the commutation relations (eq. 1.28). This is exactly what one would expect. The rest of the commutation relations can be similarly calculated.

For real values of t_m , this transformation does not alter the tight binding or nearest-neighbour hopping Hamiltonians (eqs. 3.1 and 3.2), since all powers of $(F_j^\dagger)^m (F_i)^m$ are transformed into their complex conjugate mirrored around the midpoint $L/2$. Summing over all sites keeps the Hamiltonian intact. Hereby it is concluded that the transformation $T : F_{L/2+j} \rightarrow F_{L/2-j}^\dagger$ is a symmetry of the system. The total number operator N is however not preserved:

$$\begin{aligned} T : N &= \sum_{i=1}^L \sum_{m=1}^{p-1} (F_i^\dagger)^m (F_i)^m \longrightarrow \sum_{i=1}^L \sum_{m=1}^{p-1} (F_i)^m (F_i^\dagger)^m = \sum_{i=1}^L \sum_{m=1}^{p-1} (1 - (F_i^\dagger)^{p-m} (F_i)^{p-m}) \\ &= (p-1)L - N \end{aligned} \quad (3.7)$$

Where the second to last equality is per the general anti-commutation relations of the Fock parafermions (eq. 1.31). Furthermore, we have not bothered with the mirror inversion in the transformation step, since the sum includes all the terms and is left as is. In conclusion, the symmetry, T , shows a correspondence $T : n \rightarrow (p-1) - n$, which demonstrates that the system is symmetric in n around $(p-1)/2$.

Nearest Neighbour Hopping from Vector Potts

A nearest-neighbour Fock parafermion hopping term appears in the vector Potts model (eq. 1.49). Inserting (eq. 1.39) into the Hamiltonian we find:

$$\begin{aligned} H_{vp} &= -\frac{1}{2} \left(\sum_{i=1}^L h_i U_i + \sum_{i=1}^{L-1} J_i V_{i+1}^\dagger V_i \right) + \text{H.c.} \\ &= -\frac{1}{2} \left(\sum_{i=1}^L h_i U_i + \sum_{i=1}^{L-1} J_i \left(B_{i+1}^\dagger B_i + (B_i^\dagger)^{p-1} (B_{i+1})^{p-1} + (B_{i+1})^{p-1} B_i + B_{i+1}^\dagger (B_i^\dagger)^{p-1} \right) \right) + \text{H.c.} \end{aligned} \quad (3.8)$$

Numerous terms appear in this Hamiltonian. The first term, $h_i U_i$, is the coupling of the clock operator to an external magnetic field. We ignore this term but note that it may be used to favour a type of filling (single, double, triple, etc.) of Fock parafermions at a site, due to the relationship $\omega^{N_i} = U_i$. I.e. if some U_i is favoured it will correspond to favouring N_i Fock parafermions at site i .

The fourth and fifth terms, $J_i \left((B_{i+1})^{p-1} B_i + B_{i+1}^\dagger (B_i^\dagger)^{p-1} \right)$, do not conserve the number of particles.

This stems from the fact that the number operator, $N_i = \sum_{m=1}^{p-1} (F_i^\dagger)^m (F_i)^m = \sum_{m=1}^{p-1} (B_i^\dagger)^m (B_i)^m$, does not commute with the terms in question. The action of the operators are however constrained by the configuration of the system, since they only work on a site that either contains no particles or is full with $p-1$ particles. What the operators do is to completely fill/empty a site that is completely empty/full and create/remove a particle at an adjacent site. This is indeed quite an interesting and counter-intuitive pair of operators.

The second and third terms of the Hamiltonian, $J_i \left(B_{i+1}^\dagger B_i + (B_i^\dagger)^{p-1} (B_{i+1})^{p-1} \right)$ and $J_i (B_i^\dagger)^{p-1} (B_{i+1})^{p-1}$, are where the nearest-neighbour hopping arises. This model only allows for the hopping of a single particle, or $(p-1)$ particles from a full site, as opposed to all possible numbers of particles that our general nearest-neighbour model does. If we conduct a Fradkin-Kadanoff transformation of the hard-core bosonic operators we recognise the nearest-neighbour hopping Hamiltonian arises from the single hopping site (if we let $t_{1,i} = J_i \omega^{N_i}$):

$$H_{\text{vp,NN}} = -\frac{1}{2} \sum_{i=1}^{L-1} J_i \omega^{N_i} F_{i+1}^\dagger F_i + \text{H.c.} \quad (3.9)$$

Some of the physics of the vector Potts model is contained in the Fock parafermion nearest-neighbour hopping model, but as we have seen a big part of it is unaccounted for, and there exists no one-to-one correspondence between the two models.

3.2 Nearest Neighbour Model of \mathbb{Z}_3 Fock Parafermions

This nearest-neighbour hopping Hamiltonian of the \mathbb{Z}_3 Fock parafermions has been the subject of analysis in recent years [4, 5]. In this section we sum up the key takeaways of these two articles.

3.2.1 Review of Rossini et al. 2019 [4]

In [4] the nearest neighbour hopping model of a single Fock parafermion was treated by Davide Rossini et al. The Hamiltonian in question is:

$$H = -t \sum_{i=1}^L F_{i+1}^\dagger F_i + \text{H.c.} \quad (3.10)$$

Compared to previously studied anyonic models, the Hamiltonian is non-integrable through Bethe ansatz or similar. In fact that the model is non-integrable may be proved by considering the level spacing statistics of the model¹. The reason for this non-integrability behaviour of the system, is that the Fock parafermions are strongly interacting, whereby the many-body system cannot simply be reduced to a problem of single particle systems. This strong interaction stems from the fact that the commutation of two Fock parafermions depends on their lexicological order, and is as such either ω or ω^{-1} for $\omega = e^{2\pi i/p}$. If $p = 2$, which is the case for fermions, $\omega = \omega^{-1}$ and whereby the commutation of the particles do not depend on their lexicological order. This system can be solved, as is known from the Hubbard model [28].

Even though the system does not allow for a directly integrable solution, the low-energy description of Fock parafermions of order $p = 3$ is a conformal field theory with central charge $c = 1$ for $n < 1$. Rossini et al. derived this result by calculating the bipartite entanglement entropy of the system. In a one-dimensional conformal field theory Calabrese and Cardy showed that the bipartite entanglement entropy is of the form [29]:

$$S(\rho_\ell) = a + \frac{c}{6} \log \left(\frac{2L}{\pi} \sin \left(\frac{\pi \ell}{L} \right) \right) \quad (3.11)$$

¹See appendix D.3.

Where ℓ is the length of one of the bipartite sections and a is some constant. Rossini et al. observed that the entanglement entropy agrees with the Calabrese-Cardy formula for $c = 1$. In conformal field theory the central charge has a central position, as it indicates the types and numbers of free field theories contained in the system [30]. Another way of stating this, is that it measures the effective number of degrees of freedom of the system. The central charge for a free boson is $c = 1$, while for a free fermion it is $c = 1/2$. Hence the central charge $c = 1$ indicates that the system can be describe by a boson. One possible way of having this happen is if the system is a Luttinger liquid, which many systems can be described as through the process of bosonisation (see appendix A).

Anyonic correlation functions

Define the Fock parafermion correlation:

$$G_1(x, x+r) = \langle F_x^\dagger F_{x+r} \rangle \quad G_2(x, x+r) = \langle (F_x^\dagger)^2 (F_{x+r})^2 \rangle \quad \dots \quad (3.12)$$

In the gapless phase ($n < 1$ for $p = 3$) these exhibit a clear power law $r^{-\alpha_1}$ behaviour [4] (see figs. 5.1 and 5.2), which is what one would expect of a gapless bosonic system. It is provable that a low-energy Luttinger liquid description of our system based on [31] results in the correlation function:

$$|G_1(x, x+r)| \sim r^{-(\kappa^2 K + 1/K)/2} \quad (3.13)$$

Where K is the Luttinger parameter, which is $K = 1/\kappa$ for a free anyonic gas [31], where $\kappa = 2/p$. Hence the model dictates the power law $|G_1| \sim r^{-2/p}$, which suits very well what has been observed (see fig. 5.1). This description overestimates the exponent in the power law behaviour of $G_2(x, x+r)$ for $p = 3$, however, at $|G_2| \sim r^{-8/p}$ (see fig. 5.2).

In contrast to the bosonic case, the correlation functions $G_1(x, x+r)$ in the general anyonic systems are complex. A rough estimate can elucidate the expected behaviour: assume the one-dimensional anyonic gas is uniformly distributed. To compute $G_1(x, x+r)$ we need to move a Fock parafermionic operator $N \frac{r}{L} = \rho r$ times from x to $x+r$, where ρ is the density of the system. As such a factor of $\omega^{\rho r}$, would be expected to appear in the correlation function G_1 . In the calculation of G_2 two operators are moved for each step, hence it would be expected that $G_2 \propto \omega^{2\rho r}$.

The correlation functions in the gapped phase decay as $e^{-r/\xi}$, where ξ is of the order of a few lattice spacings. The correlation function is very short ranged, and thus effectively zero in the low-energy description.

An important concluding remark is that there is an effective strong interaction between the Fock parafermions, such that they describe more than just fractions of fermions. To see why this is the case let $p = 6$. Fermionic operators, c_j , may be constructed from the Fock parafermionic ones as $c_j = F_j^3$. Numerics show that the correlation function $G_3 \sim r^{-3}$ [4, p. 8]. This is clearly different from the decay of the free fermionic correlation function of r^{-1} . It is concluded that some strong interaction coming from the special commutation of the Fock parafermions, makes the model more than just fractionalised fermions.

3.2.2 Review of Mahyaeh et al. 2020 [5]

In [5], the system considered is the one-dimensional Hamiltonian of the \mathbb{Z}_3 Fock parafermions on a chain of length L with open boundary conditions:

$$H(g) = -t \sum_{j=1}^{L-1} \left((1-g) F_j^\dagger F_{j+1} + g (F_j^\dagger)^2 (F_{j+1})^2 \right) + \text{H.c.} \quad (3.14)$$

This model describes the interpolation of single particle hopping (at $g = 0$) and coherent pair hopping (at $g = 1$) of the Fock parafermions. The system displays four phases (see fig. 3.1). The phase diagram contains three gapless phases and a gapped phase. Two of the gapless phases (L and R) has a central charge $c = 1$, which should allow for a single bosonic mode to describe the system. The difference between these two phases is that the two-point correlation functions G_1 and G_2 display different power law behaviours. The last gapless phase (M) has a central charge $c = 2$, which is very interesting to note. This suggests that a system that describes this phase should consist of two bosonic modes. The gapped phase (G) shows that the gapped phase observed in [4] is extended to $g \leq 0.45$ at $n = 1$.

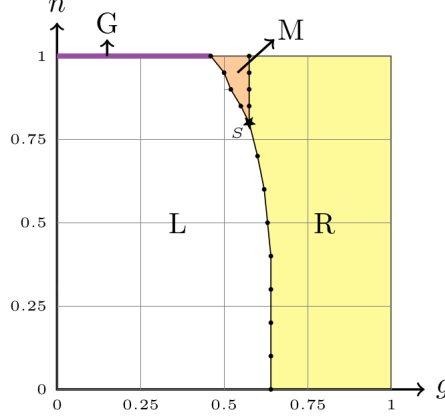


Figure 3.1: Phase diagram of the \mathbb{Z}_3 Fock parafermions. Four phases (R, L, G and M) arise as the filling factor n and the relative strength of the hopping of single Fock parafermions and pairs of Fock parafermions, g , are varied. Notably has the M phase a central charge of $c = 2$, while the phases R and L has $c = 1$ and the G phase has $c = 0$. Three of the phases meet in . Phase diagram is from the article [5].

Numerical calculations of the correlation functions G_1 and G_2 show that the number of single occupied sites dominate the L phase, while almost no sites exhibit double occupancy [5]. The converse is true for the R phase. Mahyaeh et al. use this observation to simplify the terms in the Hamiltonian. For example in the L phase they ignore the doubly-occupied sites and project the system onto the Hilbert space with at most one particle per site. In this space the matrices B and U become:

$$B \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad U \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \quad (3.15)$$

Recognise the $B_j = \sigma_j^+$. I.e. we are left with two (spin) states per site and may use a Jordan-Wigner transformation to map the system to spinless fermions, ψ_j . The fermions are linearised around the Fermi points $k = \pm k_F$, and cf. bosonisation:

$$\psi_j = \sqrt{a} (e^{ik_F x} \psi_+(x) + e^{-ik_F x} \psi_-(x)) \quad \psi_{\pm} = \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{\pi}(\pm\phi(x) - \theta(x))} \quad (3.16)$$

Where a is the lattice spacing constant, $x = ja$, and α is a momentum cut-off. Cf. the theory of bosonisation $\psi_{\pm} = \phi \pm \theta$, where $\phi(x)$ and θ are dual fields, such that $[\phi(x), \theta(x')] = i\pi\Theta(x' - x)$. Rescale these fields by the Luttinger parameter K , such that $\phi \mapsto \sqrt{K}\phi$ and $\theta \mapsto \theta/\sqrt{K}$, to account for the interactions of the fields.

Carrying out the calculation of the correlation function in the L phase, yields [5]:

$$|G_1(x, x+r)| \propto r^{-\left(\frac{1}{2K} + \frac{2}{9}K\right)} \left(1 + \cos 2k_F r \left(\frac{\alpha}{r}\right)^{\frac{2}{3}K}\right) \quad (3.17)$$

This is very similar to the result in [4], however this result also allows for the subtle oscillating behaviour that is seen in (fig. 5.1). Recall that the Luttinger parameter is $K = 3/2$. A downside to this approach is that the behaviour of $G_2(r)$ cannot be derived in the L phase by this bosonisation approach.

In the R phase, the system is projected to the Hilbert space that does not contain the single occupied states. This makes the calculation of G_2 virtually equivalent to the calculation of G_1 in the L phase, whereby G_2 is equivalent to the right hand side of (eq. 3.17) in the R phase.

The Luttinger parameter is set to be $K = 1$ in this case, since it is argued that the theory is free because only the doubly-occupied and free sites are allowed [5]. However, I will argue that this is up to debate since it contradicts the point that Rossini made about the effective strong interaction of the Fock parafermions [4]. It turns out $K = 1$ and $K = 3/2$, yield very similar two-point correlation function, G_2 , in the R phase. Hence making a distinction between these two cases is difficult. We will discuss this topic further down the line in section 5.2.

Chapter 4

Bosonisation of Fock Parafermions

In this chapter, we will try to define a field theoretical continuous version $(\psi^\dagger(x), \psi(x))$ of the discrete Fock parafermion creation and annihilation operators, F_i^\dagger, F_i . The operators should obey the generalised Clifford algebra defined by the following relationships:

$$\begin{aligned} F_i^{\dagger p} &= F_i^p = 0 \\ F_i F_j &= \omega^{\text{sgn}(i-j)} F_j F_i & F_i^\dagger F_j &= \omega^{-\text{sgn}(i-j)} F_j F_i^\dagger \end{aligned} \quad (4.1)$$

And:

$$(F_i^\dagger)^m (F_i)^m + (F_i)^{p-m} (F_i^\dagger)^{p-m} = \mathbb{1} \quad m \in [1, p-1] \quad (4.2)$$

Two types of bosonisation exist: fields theoretical bosonisation, and constructive bosonisation [32]. We will first employ a phenomenological, field theoretical approach of bosonising the Fock parafermions. This approach is effectively done through the generalisation of the standard bosonisation of the bosons and fermions. We will see that this approach describes the system well in some cases, but not in general. With that in mind, we derive another description for the bosonisation of the Fock parafermions that utilises a constructivist approach. It is seen that this latter description better describes the system, as it theoretically allows for phases with central charge $c > 1$ to arise. The constructivist approach to bosonise the Fock parafermions will in fact result in a theory of $p-1$ pairs of dual bosonic fields.

It was conjectured that Fock parafermions are abelian. The following section will thus treat the bosonisation of the Fock parafermions as abelian. For a non-abelian approach to bosonisation see [33].

4.0.1 Continuum (Field Theoretical) Limit

Before we proceed to describe the Fock parafermions in terms of bosonised fields, we note that the description arises in the continuum limit, where the physical fields of the system can be described as liquids. In the continuum limit we assume a model defined on some hyper-lattice with lattice spacing, a , lets the lattice spacing approach zero, $a \rightarrow 0$. In this limit we might consider our lattice a continuum.

In one spatial dimension we define the the lattice of length L , where each lattice site is spaced at a distance a apart. The total number of sites is then L/a . In the continuum limit, it is possible to define the derivative of some field f in terms of the difference of two adjacent points. We will use the following definition:

$$\partial_x f(x_i) \equiv \frac{1}{a} (f(x_i + a) - f(x_i)) = \frac{1}{a} (f(x_{i+1}) - f(x_i)) \quad (4.3)$$

Furthermore, note that in continuum limit, the sum over the sites turns into an integral:

$$\sum_{i=1}^{L/a} \longrightarrow \int_0^L \frac{dx}{a} \quad \text{for } a \rightarrow 0 \quad (4.4)$$

4.1 Field Theoretical Bosonisation of the Fock Parafermions

In the derivation of the bosonisation operators of the bosons and fermions¹, we saw how the description of the fermions could be constructed from the bosons by changing the sum over q such that $2q \rightarrow 2q + 1$. The addition of this +1-term in the exponent will cause the bosonised fields to anti-commute, which is expected of the fermionic operators. It is hypothesised that this approach can be used on the \mathbb{Z}_p Fock parafermions by letting $2q \rightarrow 2q + 2/p$:

$$\begin{aligned}\psi_p(x) &= e^{i\phi(x)} \sum_q e^{i2(q+\frac{1}{p})(\pi\rho_0x-\theta(x))} \sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}} \\ \psi_p^\dagger(x) &= \sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}} \sum_q e^{-i2(q+\frac{1}{p})(\pi\rho_0x-\theta(x))} e^{-i\phi(x)}\end{aligned}\tag{4.5}$$

Where we have defined the dual bosonic fields:

$$\begin{aligned}[\phi(x), \phi(x')] &= 0 & [\theta(x), \theta(x')] &= 0 \\ \left[\frac{\partial_x\theta(x)}{\pi}, \phi(x')\right] &= -i\delta(x' - x) & [\theta(x), \phi(x')] &= -i\pi\Theta(x - x')\end{aligned}\tag{4.6}$$

It is true that this definition of the fields obey the commutation relation, which is at heart of the physics of the Fock parafermion operators for $x \neq x'$:

$$\begin{aligned}\psi_{PF}(x)\psi_{PF}(x') &= \omega^{\text{sgn}(x-x')}\psi_{PF}(x')\psi_{PF}(x) \\ \psi_{PF}^\dagger(x)\psi_{PF}(x') &= \omega^{-\text{sgn}(x-x')}\psi_{PF}(x')\psi_{PF}^\dagger(x)\end{aligned}\tag{4.7}$$

The all important exchange statistics of the Fock parafermions is hereby obeyed by the proposed candidate. The same site relations are more difficult to calculate and are as such just assumed to also be obeyed.

Ties to Bosons and Fermions

We recognise that the \mathbb{Z}_p Fock parafermions condense to the fermions and bosons, when $p = 2$ and $p \rightarrow \infty$, respectively. This is indeed reflected in the bosonised parafermionic fields $\psi_p(x)$ and $\psi_p^\dagger(x)$, since the fermion fields arise as $p = 2$: $\psi_2(x) = \psi_F(x)$ and $\psi_2^\dagger(x) = \psi_F^\dagger(x)$. While the letting $p \rightarrow \infty$ results in the bosonic fields: $\psi_\infty(x) = \psi_B(x)$ and $\psi_\infty^\dagger(x) = \psi_B^\dagger(x)$.

We may ask ourselves what the case $p = 1$ corresponds to. The resulting commutation relations of this system tell us that it is bosonic. This system is however quite uninteresting, since in terms of the clock model it corresponds to a case where only one state is allowed per site. This system is obviously not displaying any forms of dynamics, because only one state is allowed.

Lowest Order Bosonisation Mode

Following the literature [34, 27], it is arguable that the bosonisation operator mode that has the biggest influence on the ground state of the system is the one for which $q = 0$. Recall that this approximation corresponds to letting the dispersion of the parafermions be linear. Furthermore at low densities the variation in the change in the density fluctuation will be minimal thus $\partial_x\theta/\pi \ll \rho_0$. Under these assumptions the bosonised field can be assumed to be:

$$\begin{aligned}\psi_p(x) &\simeq \sqrt{\rho_0} e^{i\phi(x)} e^{\frac{2i}{p}(\pi\rho_0x-\theta(x))} \\ \psi_p^\dagger(x) &\simeq \sqrt{\rho_0} e^{-\frac{2i}{p}(\pi\rho_0x-\theta(x))} e^{-i\phi(x)}\end{aligned}\tag{4.8}$$

Note that the ρ_0 is simply a constant that doesn't do much for our operators and can consequently be omitted in our analysis.

¹See appendix A.

4.1.1 Nearest Neighbour Hopping of a Single Fock Parafermion

The nearest neighbour hopping of these Fock parafermions between two sites a apart in a 1-dimensional lattice is described through the operator (in the lowest order bosonisation mode):

$$\begin{aligned}\psi_p^\dagger(x)\psi_p(x+a) &= e^{-\frac{2i}{p}(\pi\rho_0x-\theta(x))}e^{-i\phi(x)}e^{i\phi(x+a)}e^{\frac{2i}{p}(\pi\rho_0(x+a)-\theta(x+a))} \\ &= \exp\left(i(\phi(x+a)-\phi(x))-\frac{2i}{p}(\theta(x+a)-\theta(x))+\frac{\pi i}{p}(2\rho_0a-1)\right)\end{aligned}\quad (4.9)$$

The BCH-formula is used in collecting the exponents together, and all the terms except for the ones containing $\phi(x)$ and $\theta(x+a)$ commute as per our definitions.

In the continuum limit (as per section 4.0.1) with the lattice spacing $a \rightarrow 0$, the single hopping Hamiltonian is calculated to be:

$$H_1 = -t_1 \int_0^L \frac{dx}{a} \psi_p^\dagger(x)\psi_p(x+a) + \text{H.c.} = -2t_1 \int_0^L \frac{dx}{a} \cos\left(a\left(\partial_x\phi - \frac{2}{p}\partial_x\theta\right) + \frac{\pi}{p}(2n-1)\right) \quad (4.10)$$

By a Taylor expansion of the cosine in $a\left(\partial_x\phi - \frac{2}{p}\partial_x\theta\right)$ which we will assume small around zero, we find that the Hamiltonian gives rise to a second order term:

$$H_1^{(2)} = t_1 a \cos\left(\frac{\pi}{p}(2n-1)\right) \int_0^L dx \left(\partial_x\phi - \frac{2}{p}\partial_x\theta\right)^2 \quad (4.11)$$

This Hamiltonian we recognise as that of the chiral Luttinger liquid with fractional charge $\kappa = 2/p$. In previous section we showed that the equal-time two point correlation functions of this system (for $x \neq x'$) are:

$$\langle\phi(x)\phi(x')\rangle = -\frac{1}{p}\log|x-x'| \quad \langle\theta(x)\theta(x')\rangle = -\frac{p}{4}\log|x-x'| \quad (4.12)$$

Correlation Functions

The correlation functions of the hopping operator in the system H_1 are by the cumulant expansion:

$$\langle\psi_p^\dagger(x)\psi_p(x+r)\rangle_{H_1} = e^{\frac{\pi}{p}(2\rho_0r-1)} \cdot |r|^{-\frac{2}{p}} \quad (4.13)$$

In [4] it was observed how this system does indeed seem to display a power law behaviour equivalent to the one we just found $\left|\langle\psi_p^\dagger(x)\psi_p(x+r)\rangle_{H_1}\right| = |r|^{-\frac{2}{p}}$, for the cases $p = 3$ and $p = 6$. The correlation functions of the square of the hopping operators (corresponding to the hopping of a pair of Fock parafermions) yield an erroneous prediction, however, as:

$$\left|\langle(\psi_p^\dagger(x))^2(\psi_p(x+r))^2\rangle_{H_1}\right| = |r|^{-\frac{8}{p}} \quad (4.14)$$

For $p = 3$ this prediction is arguably not correct as it overshoots the the size of the power-law exponent (see fig. 5.2).

Our key takeaway from this analysis is that bosonising the Fock parafermions as per (eq. 4.5) is a good description in the single hopping Hamiltonian (eq. 4.10) at low amounts of filling of fock parafermions, where it can be assumed that powers of the Fock parafermionic operators are zero, $\langle(\psi_p^\dagger)^m(\psi_p)^m\rangle_{H_1} = 0$ for $m > 1$. When this is not the case, the description breaks down, since it does not give us an accurate prediction of the theory for the powers where $m > 1$.

4.2 Constructive Bosonisation of the Fock Parafermions

The idea behind the constructive bosonisation procedure is to rewrite a particle operator in terms of bosonic vertex operators multiplied by some factors that first of all, act as ladder operators changing the number of particles in the system; secondly, they ensure that the commutation relations of the original particles are obeyed [32, p. 14-15].

The Fradkin-Kadanoff transformation maps the Fock parafermion creation and annihilation operators, F_j^\dagger/F_j , onto creation and annihilation operators of hard-core bosons, B_j^\dagger/B_j . In doing so, the Fradkin-Kadanoff transformation automatically maps the system from a basis of \mathbb{Z}_p Fock parafermions to a basis of hard-core bosons that allow up to $p-1$ bosons to exist at each site. Recall the definition of the Fradkin-Kadanoff transformation over some lexicological order of the Fock parafermions:

$$F_j = B_j \prod_{l=1}^{j-1} U_l \quad F_j^\dagger = B_j^\dagger \prod_{l=1}^{j-1} U_l^\dagger \quad (4.15)$$

On the one-dimensional lattice of length L and lattice spacing a , the natural lexicological order is numerating the sites, $\{x_i\}$, from one end to the other. The hard-core bosons operators have the following matrix representation in the basis consisting of the number of hard-core bosons occupying each site, j , $\{|N_j = n\rangle \mid n \in [0, p-1]\}$:

$$B_j = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i-1} \otimes \overbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}}^p \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \quad (4.16)$$

$$U_j = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i-1} \otimes \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{p-1} \end{pmatrix} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \quad \omega = \exp\left(i\frac{2\pi}{p}\right)$$

Notice that the operator B_j works as a ladder operator between the state where N_j and $N_j - 1$ hard-core bosons occupy each site. Furthermore, the string, $\prod_{l < j} U_l$, will ensure that the commutation relations of the Fock parafermions are obeyed. The Klein factors are precisely this combination of ladder and 'commutation keeping' operators, hereby we identify the product, $B_j \prod_{l < j} U_l$, as the latter operators that simultaneously preserve the commutation relations of the Fock parafermions.

Now, a crucial ansatz for our bosonisation is stated. We assume that each eigenstate of the hard-core bosons are described by its own species of dual bosonic fields ϕ_k and θ_k , for $k \in [1, p-1]$. In the eigenbasis we defined earlier, k represents the number of particles, ie. $|N_j = k\rangle$. That ϕ_k and θ_k are dual is meant that:

$$[\partial_x \theta_k(x), \phi_{k'}(x')] = -i\pi \delta(x-x') \delta_{k,k'} \quad (4.17)$$

Notice that by integrating this commutation relationship over the variable x , the variable x' does not influence this integral at all, since it is defined over a different field than x is. Hence, the partial derivative of x can be removed from the commutation relation:

$$[\theta_k(x), \phi_{k'}(x')] = -i\pi \delta_{k,k'} \Theta(x-x') \quad , \quad \Theta(x-x') = \begin{cases} 1 & \text{for } x > x' \\ 0 & \text{for } x \leq x' \end{cases} \quad (4.18)$$

There is some leeway for determining the definition of the Θ -function in the point $x = 0$. The way we have defined it is the right way due to our definition of the continuum limit for the partial derivatives (eq. 4.3). The fields are moreover bosonic, which means that:

$$[\phi_k(x), \phi_{k'}(x')] = 0 \quad , \quad [\theta_k(x), \theta_{k'}(x')] = 0 \quad (4.19)$$

Furthermore, we assume that the eigenvector $|N_j = k\rangle$ is described by the vertex operator of the field ϕ_k :

$$|N_j = k\rangle \equiv e^{-i\phi_k(x_j)} |0\rangle \quad (4.20)$$

Where x_j denotes the spatial coordinate of the j 'th site. Notice that the bosons we describe have a hard core which means only one vertex operator can be on each site. Algebraically this means:

$$|N_j = k\rangle = e^{-i\phi_k(x_j)} |0\rangle \neq \left(e^{-i\phi_m(x_j)}\right) \left(e^{-i\phi_n(x_j)}\right) |0\rangle = 0 \quad (4.21)$$

Where $k = m + n$. A corollary to this result is that the square of any two bosonic vertex operators is zero. And it is also shown that we cannot go between different states by application of these vertex

operators only. The only two types of bosonic operators that can be put together to render something different from zero are the vertex operator and its hermitian conjugate, in this case the result is the identity, $e^{i\phi_m(x_j)}e^{-i\phi_m(x_j)}|0\rangle = |0\rangle$. With this in mind, we define the hard-core bosonic operators:

$$\begin{aligned} B_j &\equiv \exp(i\phi_1(x_j)) + \sum_{k=2}^{p-1} \exp(i(\phi_k(x_j) - \phi_{k-1}(x_j))) \\ B_j^\dagger &\equiv \exp(-i\phi_1(x_j)) + \sum_{k=2}^{p-1} \exp(-i(\phi_k(x_j) - \phi_{k-1}(x_j))) \end{aligned} \quad (4.22)$$

The latter operator creates a hard-core boson, as $B_j^\dagger|N_j = k\rangle = |N_j = k + 1\rangle$. The former operator on the other hand annihilates a hard-core boson, since $B_j|N_j = k\rangle = |N_j = k - 1\rangle$. These relationships are easily derived with (eq. 4.21) in mind. The m 'th power of these operators are:

$$\begin{aligned} (B_j)^m &= \exp(i\phi_m(x_j)) + \sum_{k=m+1}^{p-1} \exp(i(\phi_k(x_j) - \phi_{k-m}(x_j))) \\ (B_j^\dagger)^m &= \exp(-i\phi_m(x_j)) + \sum_{k=m+1}^{p-1} \exp(-i(\phi_k(x_j) - \phi_{k-m}(x_j))) \end{aligned} \quad (4.23)$$

It was earlier shown that the unitary operators $U_j = \omega^{N_j}$. We have so far distinguished between the number of hard-core bosons and Fock parafermions at a site i . This distinction is redundant, however, since the number of hard-core bosons is exactly equal to the number of Fock parafermions. This is proven by the fact that the form of the number operator is left invariant under the Fradkin-Kadanoff transformation:

$$N_j = \sum_{m=1}^{p-1} (F_j^\dagger)^m (F_j)^m = \sum_{m=1}^{p-1} (B_j^\dagger)^m (B_j)^m \quad (4.24)$$

The density operator of the bosonised field was defined through the θ_k -field which is dual to the ϕ_k -field used in the vertex operator that describes the eigenstates of the system, (eq. 4.20). To reiterate² the density operator of the k 'th type bosonised field is:

$$\hat{\rho}_k = \rho_k - \frac{\partial_x \theta_k}{\pi}(x_j) \quad (4.25)$$

Where ρ_k is the average of the number of sites that contain k Fock parafermions/hard-core bosons (divided by the length of the system L). From this equation it is understood that $\partial_x \theta_k$ defines the deviation of the density from the average ρ_k . We also introduce a rescaled version of the density operator, \hat{n}_j , which we will call the number density operator, \hat{n}_j :

$$\hat{n}_k \equiv a\hat{\rho}_k = n_k - a\frac{\partial_x \theta_k}{\pi}(x_j) \quad (4.26)$$

Where we have defined the average of the number density operator $n_k = a\rho_k$. The idea behind this definition is to get an operator that is able to count the number of parafermions at a site x_j . There is 1 Fock parafermions at the site x_j in the state $|N_j = 1\rangle$, 2 in the state $|N_j = 2\rangle$, etc. A natural definition for the total number density operator, n , which counts the number of Fock parafermions at any gives site, arises naturally:

$$\sum_{j=1}^{L/a} \sum_{k=1}^{p-1} k\hat{n}_k = \sum_{j=1}^{L/a} N_j = N \quad (4.27)$$

The total number of Fock parafermions, N , remains invariant in the nearest-neighbour hopping Hamiltonian, cf. (eq. 3.3), which makes the average number density $n = \sum k n_k$ a well-defined constant of the system. We have in the above made it explicitly clear that $\hat{\rho}$ and \hat{n} are operators by including the hat. This degree of explicitness is rather redundant and will be left out unless otherwise stated.

²See appendix A.

The last piece of the puzzle, in order to write the operator U_j in terms of our bosonic fields θ_k and $\phi_{k'}$, is to rewrite the derivative $a\partial_x\theta_k(x_j) = \theta_k(x_{j+1}) - \theta_k(x_j)$. The unitary operator U_j is thus:

$$\begin{aligned} U_j &\equiv \exp\left(\frac{2\pi i}{p}n - \frac{2i}{p}\sum_{k=1}^{p-1}k(\theta_k(x_{j+1}) - \theta_k(x_j))\right) \\ U_j^\dagger &\equiv \exp\left(-\frac{2\pi i}{p}n + \frac{2i}{p}\sum_{k=1}^{p-1}k(\theta_k(x_{j+1}) - \theta_k(x_j))\right) \end{aligned} \quad (4.28)$$

Note that ω is \mathbb{Z}_p -symmetric, meaning that the powers $\omega^{p+k} = \omega^k$. This makes the powers $(U_j)^m$ difficult to calculate explicitly:

$$(U_j)^m = \omega^{mN_j} = \omega^{\sum_{k=1}^{p-1}mkn_k - \frac{1}{\pi}\sum_{k=1}^{p-1}mk(\theta_k(x_{j+1}) - \theta_k(x_j))} \quad (4.29)$$

Due to the periodicity of ω the above cannot simply take out m from the sums over k . If we define $\overline{kj} \equiv kj \pmod{p}$, the powers of U_j are:

$$\begin{aligned} (U_j)^m &= \exp\left(\frac{2\pi i}{p}\sum_{k=1}^{p-1}\overline{mkn}_k - \frac{2i}{p}\sum_{k=1}^{p-1}\overline{mk} \cdot (\theta_k(x_{j+1}) - \theta_j(x_j))\right) \\ (U_j^\dagger)^m &= \exp\left(-\frac{2\pi i}{p}\sum_{k=1}^{p-1}\overline{mkn}_k + \frac{2i}{p}\sum_{k=1}^{p-1}\overline{mk} \cdot (\theta_k(x_{j+1}) - \theta_j(x_j))\right) \end{aligned} \quad (4.30)$$

Lastly, it is important to note that the right commutation relation is obeyed when U_j and B_i commute, (eq. 1.46). Namely:

$$(B_i)^m(U_j)^n = \omega^{\overline{mn}\delta_{i,j}}(U_j)^n(B_i)^m, \quad (B_i^\dagger)^m(U_j)^n = \omega^{-\overline{mn}\delta_{i,j}}(U_j)^n(B_i^\dagger)^m \quad (4.31)$$

This is quickly checked by having in mind that for a specific value of $k' \in [0, p-1]$:

$$\begin{aligned} &\left[\pm i(\phi_{k'}(x_j) - \phi_{k'-m}(x_j)), -\frac{2i}{p}\sum_{k=1}^{p-1}\overline{kn} \cdot (\theta_k(x_{j+1}) - \theta_k(x_j))\right] \\ &= \pm \frac{2}{p}\left(\overline{k'n}[\phi_{k'}(x_j), \theta_{k'}(x_{j+1})] - \overline{n(k'-m)}[\phi_{k'-m}(x_j), \theta_{k'-m}(x_{j+1})]\right) = \frac{2\pi i}{p}\overline{mn} \end{aligned} \quad (4.32)$$

4.2.1 Bosonised Fock Parafermions

The Fock parafermion operators, F_i , in the bosonised description are described through a Fradkin Kadanoff transformation of the operators B_i and U_i that was just studied. Notice that the string of U_i operators in the bosonised description simplifies, since the product changes into a telescopic sum in the exponent:

$$\begin{aligned} \prod_{l=1}^{j-1} U_l &\equiv \exp\left(\frac{2\pi i}{p}n(j-1) - \frac{2i}{p}\sum_{l=1}^{j-1}\sum_{k=1}^{p-1}k(\theta_k(x_{l+1}) - \theta_k(x_l))\right) \\ &= \exp\left(\frac{2\pi i}{p}n(j-1) - \frac{2i}{p}\sum_{k=1}^{p-1}k(\theta_k(x_j) - \theta_k(x_1))\right) \end{aligned} \quad (4.33)$$

The bosonised description of the Fock parafermions is then:

$$\begin{aligned} F_j &\equiv \left[e^{i\phi_1(x_j)} + \sum_{k=2}^{p-1}e^{i(\phi_k(x_j) - \phi_{k-1}(x_j))}\right] \exp\left(\frac{2\pi i}{p}n(j-1) - \frac{2i}{p}\sum_{k=1}^{p-1}k(\theta_k(x_j) - \theta_k(x_1))\right) \\ F_j^\dagger &\equiv \left[e^{-i\phi_1(x_j)} + \sum_{k=2}^{p-1}e^{-i(\phi_k(x_j) - \phi_{k-1}(x_j))}\right] \exp\left(-\frac{2\pi i}{p}n(j-1) + \frac{2i}{p}\sum_{k=1}^{p-1}k(\theta_k(x_j) - \theta_k(x_1))\right) \end{aligned} \quad (4.34)$$

This result is one of the main results of this thesis. Recall that the bosonised fields commute by the following relationship:

$$[\theta_k(x), \phi_{k'}(x')] = -i\pi\delta_{k,k'}\Theta(x-x') \quad \Theta(x-x') = \begin{cases} 1 & \text{for } x > x' \\ 0 & \text{for } x \leq x' \end{cases} \quad (4.35)$$

Powers of these Fock parafermion operators are:

$$(F_j)^m \equiv \left[e^{i\phi_m(x_j)} + \sum_{k=m+1}^{p-1} e^{i(\phi_k(x_j) - \phi_{k-m}(x_j))} \right] \times \exp \left(\frac{2\pi i}{p} (j-1) \sum_{k=1}^{p-1} \overline{mk} n_k - \frac{2i}{p} \sum_{k=1}^{p-1} \overline{mk} (\theta_k(x_j) - \theta_k(x_1)) \right) \quad (4.36)$$

Where the power $(F_j^\dagger)^m$ is found by the complex conjugate of this expression.

4.3 Bosonisation of One-dimensional \mathbb{Z}_3 Fock Parafermions

A framework for bosonising the Fock parafermions has been made in previous section. To see whether this framework is good or not we study its application to the one-dimensional \mathbb{Z}_3 Fock parafermions, and study what kinds of prediction this model yields.

The matrix representation of the Fradkin-Kadanoff transformed operators, B_i and U_i , of the \mathbb{Z}_3 Fock parafermions are defined as:

$$B_i = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i-1} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \quad (4.37)$$

$$U_i = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \quad \omega = \exp \left(i \frac{2\pi}{3} \right)$$

In the analysis to come the tensor product notation is dismissed for simplicity, but note it will still be implied. The basis of each of these 3×3 -matrices are the states containing $n \in \{0, 1, 2\}$ number of Fock parafermions occupying the site i , $\{ |N_i = n\rangle \mid n \in \{0, 1, 2\} \}$.

4.3.1 Hard-core Bosonic Operator B_i and its Powers

We derived the general form of the bosonisation of the B_j -operator of the \mathbb{Z}_p Fock parafermions. For the specific case of the \mathbb{Z}_3 Fock parafermions the non-zero powers of this operator is:

$$B_j = e^{i\phi_1(x_j)} + e^{i(\phi_2(x_j) - \phi_1(x_j))} \quad (4.38)$$

$$(B_j)^2 = e^{i\phi_2(x_j)}$$

Where $x_i = ia$ is the placement of the i 'th site in the lattice with spacing a . Notice that the cube of the operator B_j should simply yield zero:

$$(B_j)^3 = 0 \quad (4.39)$$

Identities of B_i

If we directly take powers of the B_i it would be reasonable to expect that the resulting equations give us a pair of relations that will act as constraints on the ϕ_j -fields. We note that:

$$(B_i)^n = \left(e^{i\phi_1} + e^{i(\phi_2 - \phi_1)} \right)^n = \left(e^{i(\phi_1 - \frac{1}{2}\phi_2)} + e^{i(\frac{1}{2}\phi_2 - \phi_1)} \right)^3 \left(e^{i\frac{1}{2}\phi_2} \right)^3 = 2^n \cos^n \left(\phi_1 - \frac{1}{2}\phi_2 \right) e^{i\frac{n}{2}\phi_2} \quad (4.40)$$

The square of the hard-core bosonic operator should obey the relation, $\psi_2 = e^{i\phi_2} = (B_i)^2$. This equation will yield a constraint on the ϕ_j -fields such that, $\cos \left(\phi_1 - \frac{1}{2}\phi_2 \right) = \pm \frac{1}{2}$, hence:

$$\phi_2 - 2\phi_1 \equiv \frac{n}{3}\pi \pmod{2\pi}, \quad n = 1, 2, 4, 5 \quad (4.41)$$

Another defining feature of the hard-core bosons are that they obey the relation $(B_i)^3 = 0$. From the equation above it is then implied:

$$\phi_2 - 2\phi_1 \equiv \pi \pmod{2\pi} \quad (4.42)$$

We cannot simultaneously obey both constraints on the ϕ_j -fields (Eqs. 4.42 and 4.41). Therefore, the above constraints of the vertex operators cannot be imposed from the get go. Note that the fact that the operators B_i are that of a hard-core boson, should hint to us that these same site relations cannot be imposed, due to the fact that hard-core bosons are strongly interacting at the same site. Hence this discussion is rather redundant.

4.3.2 Unitary Operator U_i and its Powers

The powers of the unitary operator U_i of the \mathbb{Z}_3 Fock parafermions on the one-dimensional lattice are:

$$\begin{aligned} U_i &= \omega^{n - \frac{1}{\pi} [(\theta_1(x_{i+1}) - \theta_1(x_i)) + 2(\theta_2(x_{i+1}) - \theta_2(x_i))]} \\ (U_i)^2 &= \omega^{2n_1 + n_2 - \frac{1}{\pi} [2(\theta_1(x_{i+1}) - \theta_1(x_i)) + (\theta_2(x_{i+1}) - \theta_2(x_i))]} \end{aligned} \quad (4.43)$$

Where we have defined the average number density, $n \equiv n_1 + 2n_2$. Before we proceed further in our analysis, we note that a seemingly equivalent way of defining the operator U_i exist: it seems that one would be able to define the operator through the difference, $U_i = \omega^{\hat{n}_1 - \hat{n}_2}$, rather than by $U_i = \omega^{\hat{n}_1 + 2\hat{n}_2}$. This alternative definition is sound by the earlier definition of the operator, (eq. 4.37), since $\omega^2 = \omega^{-1}$. However, by this definition of the operator U_i we will need to keep charge of each of the number of species of Fock parafermions, since the exponent will be $(n_1 - n_2) - \frac{a}{\pi}(\partial_x \theta_1 - \partial_x \theta_2)$. Compared to n , $n_1 - n_2$ is not a conserved quantity, hence this approach is not preferred.

For the square of U_i there are two seemingly equivalent definitions of the operator $(U_i)^2$ as well. The first way of finding the square is as in (eq. 4.43). The other way is by straight-forward squaring U_i :

$$(U_i)^2 = \omega^{2n - \frac{2}{\pi} [(\theta_1(x_{i+1}) - \theta_1(x_i)) + 2(\theta_2(x_{i+1}) - \theta_2(x_i))]} \quad (4.44)$$

The difference between these two operators stems from the fact that the number, ω , is \mathbb{Z}_3 symmetric, such that $\omega^4 = \omega$. Even though the operators are mathematically equivalent, the physical interpretation is quite different between the two of them: Compare simply doubling both of the fields θ_1 and θ_2 , versus changing the two fields into each other, $\theta_1 \leftrightarrow \theta_2$. As we will see the definition as given in (eq. 4.43), is the better choice.

Identities of U_i

Lastly, the unitary operator, U_i , should by definition obey the fact that its cube is the identity operator. Hence:

$$\mathbb{1} = (U_i)^3 = \omega^{3n - \frac{3}{\pi} [(\theta_1(x_{i+1}) - \theta_1(x_i)) + 2(\theta_2(x_{i+1}) - \theta_2(x_i))]} = e^{2\pi i n - i2 [(\theta_1(x_{i+1}) - \theta_1(x_i)) + 2(\theta_2(x_{i+1}) - \theta_2(x_i))]} \quad (4.45)$$

Which is obeyed when:

$$2\pi n - 2 [(\theta_1(x_{i+1}) - \theta_1(x_i)) + 2(\theta_2(x_{i+1}) - \theta_2(x_i))] \equiv 0 \pmod{2\pi} \quad (4.46)$$

From this constraint, we can define a stronger version, by summing over the sites x_i up to x_{j-1} and letting $[\theta_1(x_0) + 2\theta_2(x_0)] = 0$, which we may define, since it is only the dynamics of the system that matters to us. In total we find:

$$2\pi n j - 2 [\theta_1(x_j) + 2\theta_2(x_j)] \equiv 0 \pmod{2\pi} \quad (4.47)$$

4.3.3 Fock Parafermion Operators

We are now able to describe the \mathbb{Z}_3 Fock parafermion operators in our bosonisation description. Remember that the product of U_k operators from $k = 1$ to $k = j - 1$ is described as a telescoping sum over the θ -fields. Hence, the single occupation Fock parafermion becomes:

$$F_j = \exp\left(\frac{2\pi i}{3} n(j-1) - \frac{2i}{3} [(\theta_1(x_j) - \theta_1(x_1)) + 2(\theta_2(x_j) - \theta_2(x_1))]\right) \cdot \left(e^{i\phi_1(x_j)} + e^{i(\phi_2(x_j) - \phi_1(x_j))}\right) \quad (4.48)$$

By the same calculation we find that the double occupation Fock parafermion is:

$$(F_j)^2 = \exp\left(\frac{2\pi i}{3} (2n_1 + n_2)(j-1) - \frac{2i}{3} [2(\theta_1(x_j) - \theta_1(x_1)) + (\theta_2(x_j) - \theta_2(x_1))]\right) \cdot e^{i\phi_2(x_j)} \quad (4.49)$$

Fock Parafermion Hopping Terms

The operator describing the hopping of a single Fock parafermions from site j to site k is described by the normal ordered operator $F_k^\dagger F_j$. Inserting the bosonised form of our Fock parafermion operator we get:

$$F_k^\dagger F_j = e^{\frac{2\pi i}{3}n(j-k)} \left(e^{-i\phi_1(x_k)} + e^{-i(\phi_2(x_k)-\phi_1(x_k))} \right) e^{-\frac{2i}{3}[(\theta_1(x_j)-\theta_1(x_k))+2(\theta_2(x_j)-\theta_2(x_k))]} \\ \times \left(e^{i\phi_1(x_j)} + e^{i(\phi_2(x_j)-\phi_1(x_j))} \right) \quad (4.50)$$

Note, if $j = k$ the above expression will simply become 1 if the site x_j is occupied by one or two Fock parafermions, and zero if there is no occupancy, due to the vertex operator $e^{i\phi_k(x_j)}$ on the right hand side. This is exactly what we would expect. The nearest neighbour hopping, $j = k \pm 1$, is of special interest to us, as we will study it in the sections to come. Note, how the prefactor in (eq. 4.50) becomes $e^{\frac{2\pi i}{3}n}$ in the case of the nearest-neighbour hopping term.

Other than the hopping of a single Fock parafermion from site j to k , a pair of Fock parafermions can do the hopping together, signified by the operator $(F_k^\dagger)^2(F_j)^2$. We can simply write the pair hopping operator in its bosonised form as:

$$(F_k^\dagger)^2(F_j)^2 = e^{\frac{2\pi i}{3}(2n_1+n_2)(j-k)} e^{-i\phi_2(x_i)} e^{-\frac{2i}{3}[2(\theta_1(x_{i+1})-\theta_1(x_i))+(\theta_2(x_{i+1})-\theta_2(x_i))]} e^{i\phi_2(x_{i+1})} \quad (4.51)$$

Chapter 5

Field Theoretical Limit of \mathbb{Z}_3 Fock Parafermion Nearest Neighbour Hopping Hamiltonian

The system that is to be analysed in this chapter is the system of nearest neighbour hopping of \mathbb{Z}_3 Fock parafermions on a one-dimensional lattice with $l = L/a$ equally spaced sites of distance a . The Hamiltonian is:

$$H = - \sum_{i=1}^{l-1} t_1 F_i^\dagger F_{i+1} + t_2 (F_i^\dagger)^2 (F_{i+1})^2 + \text{H.c.} \quad (5.1)$$

The Fradkin-Kadanoff transformation of this Hamiltonian is:

$$H = - \sum_{i=1}^{l-1} t_1 B_i^\dagger U_i B_{i+1} + t_2 (B_i^\dagger)^2 (U_i)^2 (B_{i+1})^2 + \text{H.c.} \quad (5.2)$$

Symmetries of the Hamiltonian

By writing the Hamiltonian in terms of the vertex operators we have lost the symmetries of the system. We will reintroduce these symmetries by finding the right constraints to put on our fields. Consider the operator:

$$U = \prod_{i=1}^l U_i \quad , \quad U_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad (5.3)$$

This operator will in fact define a global \mathbb{Z}_3 symmetry of the Hamiltonian. It is \mathbb{Z}_3 , since:

$$(U_i)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \quad (U_i)^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} \quad (5.4)$$

While it's a symmetry of the Hamiltonian, since the operator commutes with each of the terms in the Hamiltonian. The commutation relations of each of the individual terms of the Hamiltonian with U are easily calculated keeping in mind $U_i B_i = \omega^{-1} B_i U_i$ and $U_i B_i^\dagger = \omega B_i^\dagger U_i$. The globally \mathbb{Z}_3 symmetric operator, U , tells us something important about the boundary terms of the θ_j -fields. Note, we can evaluate the product over the U_i operators as a telescoping sum over the θ_j -fields :

$$U = \prod_{j=1}^l U_j = \omega^{nl - \frac{1}{\pi} [\theta_1(x) + 2\theta_2(x)]_{x=0}^{x=L}} = \omega^{N - \frac{1}{\pi} [\theta_1(x) + 2\theta_2(x)]_{x=0}^{x=L}} \quad (5.5)$$

This operator is a symmetry of the system, which must make the sum in the exponent of ω a constant number. We earlier showed that N was a constant of our system. Therefore, the edge term of the θ_j -fields will also be a constant of the system:

$$\left[\theta_1(x) + 2\theta_2(x) \right]_{x=0}^{x=L} = \text{const} \quad (5.6)$$

5.1 Single Hopping Hamiltonian

Before proceeding to treat the entire Hamiltonian (eq. 5.1), we first study two extreme cases, where $t_2 = 0$ and $t_1 = 0$, by themselves. In this section we will analyse the t_1 -term of (eq. 5.1). This Hamiltonian corresponds to the hopping of single Fock parafermions between nearest neighbouring sites in the one-dimensional lattice:

$$H_1 = -t_1 \sum_{i=1}^{l-1} F_i^\dagger F_{i+1} + \text{H.c.} = -t_1 \sum_{i=1}^{l-1} B_i^\dagger U_i B_{i+1} + \text{H.c.} \quad (5.7)$$

5.1.1 Field Theoretical Limit

We will evaluate the field theoretical limit of the $F_i^\dagger F_{i+1}$ -operator as given in (eq. 4.50), see appendix C for this calculation. In conclusion we get four complex exponentials, to which adding their hermitian conjugate, will render cosine terms, assuming t_1 is real.

The lattice spacing a is then set to approach zero, letting the two points x_i and x_{i+1} approach one another. By defining the differential quotients as in (eq. 4.3) we may rewrite the Hamiltonian (eq. 5.7) in terms of the partial derivatives in x of the fields, ϕ_j and θ_j . In the continuum limit the sum becomes an integral, and for simplicity we may denote the x_i -variable x . Hence the expression for the Hamiltonian, which is derived in appendix C becomes:

$$\begin{aligned} H_1 = -2t_1 \int_0^L \frac{dx}{a} & \left[\cos \left[a\partial_x \phi_1 - \frac{2}{3}a (\partial_x \theta_1 + 2\partial_x \theta_2) + \frac{\pi}{3} (2n-1) \right] \right. \\ & + \cos \left[\phi_2 - 2\phi_1 + a\partial_x \phi_2 - a\partial_x \phi_1 - \frac{2}{3}a (\partial_x \theta_1 + 2\partial_x \theta_2) + \frac{\pi}{3} (2n-1) \right] \\ & + \cos \left[-\phi_2 + 2\phi_1 + a\partial_x \phi_1 - \frac{2}{3}a (\partial_x \theta_1 + 2\partial_x \theta_2) + \frac{\pi}{3} (2n-1) \right] \\ & \left. + \cos \left[a\partial_x \phi_2 - a\partial_x \phi_1 - \frac{2}{3}a (\partial_x \theta_1 + 2\partial_x \theta_2) + \frac{\pi}{3} (2n-1) \right] \right] \end{aligned} \quad (5.8)$$

Now, we let $a \rightarrow 0$, and only keep the most relevant terms in a . The two middle terms of (eq. 5.8) are very easily considered, since here we can simply let $a = 0$ to get the following:

$$\cos \left[\phi_2 - 2\phi_1 + \frac{\pi}{3} (2n-1) \right] + \cos \left[-(\phi_2 - 2\phi_1) + \frac{\pi}{3} (2n-1) \right] = 2 \cos \left[\phi_2 - 2\phi_1 \right] \cos \left[\frac{\pi}{3} (2n-1) \right] \quad (5.9)$$

I.e. we see that these two terms give rise to a sine-Gordon term. The two other terms, however, do not behave in this way as the arguments inside the cosine are proportional to the lattice spacing, a (bar the constant $-\pi/3$). To find the most relevant part of these terms, we Taylor-expand the cosine functions. For the function $\cos(kx + b)$ the Taylor expansion in x around a point x_0 is:

$$\cos(kx + b) = \cos(kx_0 + b) - \sin(kx_0 + b)k(x - x_0) - \frac{\cos(kx_0 + b)}{2}k^2(x - x_0)^2 + O(x^3) \quad (5.10)$$

This Taylor expansion of the first and fourth term of (eq. 5.8) is carried out in a around zero¹. The resulting Hamiltonian densities are proportional to the zeroth, first, second, etc. order in a . The Hamiltonian is approximated at the three lowest of such orders, such that $H_1 \approx H_1^{(0)} + H_1^{(1)} + H_1^{(2)}$. All the zeroth order terms of the Hamiltonian densities are gathered to yield a sine-Gordon term Hamiltonian:

$$H_1^{(0)} = -4t_1 \cos \left[\frac{\pi}{3} (2n-1) \right] \int_0^L \frac{dx}{a} \left(1 + \cos \left[\phi_2 - 2\phi_1 \right] \right) \quad (5.11)$$

When studying the dynamics of the system, we can ignore the constant term in this Hamiltonian as it will simply account for a constant energy. The physical interpretation of the cosine term in the integrand,

¹See appendix C.

is that it will seek to gap out the combination of fields, such that $\phi_2 - 2\phi_1 = 2\pi n$, for some $n \in \mathbb{Z}$. The first order terms to the Hamiltonian are:

$$\begin{aligned} H_1^{(1)} &= 2t_1 a \sin \left[\frac{\pi}{3} (2n - 1) \right] \int_0^L \frac{dx}{a} \left(\partial_x \phi_2 - \frac{4}{3} (\partial_x \theta_1 + 2\partial_x \theta_2) \right) \\ &= -2t_1 \sin \left[\frac{\pi}{3} (2n - 1) \right] \left(\left[\phi_2(x) \right]_{x=0}^{x=L} - \frac{4}{3} \left[\theta_1(x) + 2\theta_2(x) \right]_{x=0}^{x=L} \right) \\ &= t_1 \sin \left[\frac{\pi}{3} (2n - 1) \right] \left[\phi_2(x) \right]_{x=0}^{x=L} \end{aligned} \quad (5.12)$$

Which is dependent on the boundary terms of the bosonised fields. The θ_j -boundary terms in the Hamiltonian, (eq 5.12), sum to a constant cf. (eq.5.6) and can in consequence be omitted from the calculations. In the bulk we can by definition disregard the boundary terms of the ϕ_j -fields as well. This also works under the assumption of periodic boundary conditions, since here $\phi_j(0) = \phi_j(L)$, or if we consider the system in the thermodynamic limit where $L \rightarrow \infty$. Thus we may simply ignore the boundary terms of the Hamiltonian in the following analysis, where we limit ourselves to studying the bulk behaviour of the system. However, if we were to calculate the Hamiltonian for a finite system, we would indeed need to account for these edge contributions.

The second order part of the Hamiltonian is:

$$\begin{aligned} H_1^{(2)} &= 2 \frac{t_1 a^2}{2} \cos \left[\frac{\pi}{3} (2n - 1) \right] \int_0^x \frac{dx}{a} \left(\left(\partial_x \phi_1 - \frac{2}{3} a (\partial_x \theta_1 + 2\partial_x \theta_2) \right)^2 \right. \\ &\quad \left. + \left(\partial_x \phi_2 - \partial_x \phi_1 - \frac{2}{3} a (\partial_x \theta_1 + 2\partial_x \theta_2) \right)^2 \right) \\ &= 2t_1 a \cos \left[\frac{\pi}{3} (2n - 1) \right] \int_0^L dx \left(\left(\frac{1}{2} \partial_x \phi_2 - \frac{2}{3} (\partial_x \theta_1 + 2\partial_x \theta_2) \right)^2 + \left(\partial_x \phi_1 - \frac{1}{2} \partial_x \phi_2 \right)^2 \right) \end{aligned} \quad (5.13)$$

This second-order Hamiltonian gives rise to some dynamics, which will be the attention of our analysis.

5.1.2 Canonical Transformation of the Single Hopping Hamiltonian

Our first impulse to treat the second-order Hamiltonian that we have just derived, is to find a unitary canonical transformation that will diagonalise the Hamiltonian. This is not possible for this particular system, since it is not Lorentz invariant, which will make this particular approach impossible.

This doesn't hinder us from defining some canonical transformation such that the Hamiltonian becomes easier to deal with. Start by noting that the θ_j -operators always feature as the sum $\theta_1 + 2\theta_2$ in our Hamiltonian. This observation leads us to define the transformation:

$$\tilde{\theta}_1 = \theta_1 \quad \tilde{\phi}_1 = \phi_1 - \frac{1}{2} \phi_2 \quad \tilde{\theta}_2 = \theta_1 + 2\theta_2 \quad \tilde{\phi}_2 = \frac{1}{2} \phi_2 \quad (5.14)$$

The transformed field, $\tilde{\phi}_1$, is defined such that it will commute with the $\tilde{\theta}_2$ -field, which is what we want from a canonical transformation:

$$\left[\tilde{\theta}_2, \tilde{\phi}_1 \right] = \left[\theta_1, \phi_1 \right] + \left[\theta_1, -\frac{1}{2} \phi_2 \right] + \left[2\theta_2, \phi_1 \right] + \left[2\theta_2, -\frac{1}{2} \phi_2 \right] = \left[\theta_1, \phi_1 \right] - \left[\theta_2, \phi_2 \right] = 0 \quad (5.15)$$

We have a bit of leeway in defining the two other operators, while still keeping the transformation canonical, such that the commutation relations are preserved. It turns out the definition we have just used is the simplest way of defining these two remaining fields, $\tilde{\phi}_2$ and $\tilde{\theta}_1$.

Hereby the quadratic part of the Hamiltonian (eq. 5.13) becomes:

$$H_1^{(2)} = \tilde{t}_1 \int_0^L dx \left((\partial_x \tilde{\phi}_1)^2 + \left(\partial_x \tilde{\phi}_2 - \frac{2}{3} \partial_x \tilde{\theta}_2 \right)^2 \right) \quad (5.16)$$

Where we have defined:

$$\tilde{t}_1 = 2at_1 \cos \left[\frac{\pi}{3} (2n - 1) \right] \quad (5.17)$$

We recognise the Hamiltonian (eq. 5.16) as that of a chiral Luttinger liquid in $\tilde{\phi}_2$ and a massless free scalar in $\tilde{\phi}_1$. Note that the chiral Luttinger liquid as we previously discussed requires that $\tilde{t}_1 > 0$ for the system to be physical. We proved earlier that the system displays a mirror symmetry around $n = 1$, hence only one of these mirrored regions (say $0 \leq n \leq 1$) needs be considered. We note that $\tilde{t}_1 \geq t_1 a/2 > 0$ in this region, if $t_1 > 0$.

Some interaction term should be added to the Hamiltonian to account for the constraint on the $\tilde{\theta}_1$ that arose from the cube of the unitary matrix U_i (eq. 4.47). We keep in mind that $x_j = ja$. We pin the θ -fields with the addition of a cosine-term:

$$\mathcal{H}_{\text{int}}(\tilde{\theta}_1) \propto \cos\left(\frac{2\pi n x}{a} - 2\tilde{\theta}_2\right) \quad (5.18)$$

We also include the cosine-term in $\tilde{\phi}_1$ that arose from the first order Taylor expansion, hence the the bulk Hamiltonian of the system is:

$$H_1 = \int_0^L dx \tilde{t}_1 \left(\partial_x \tilde{\phi}_2 - \frac{2}{3} \partial_x \tilde{\theta}_2 \right)^2 - M_{\tilde{\theta}} \cos\left(\frac{2\pi n x}{a} - 2\tilde{\theta}_2\right) + \tilde{t}_1 \left((\partial_x \tilde{\phi}_1)^2 - \frac{2}{a^2} \cos(2\tilde{\phi}_1) \right) \quad (5.19)$$

We see that this bulk Hamiltonian consists of some second order terms and a couple of cosine terms. In the subsequent section we will analyse the second order part of the Hamiltonian. One of the points of interest in our analysis going forward will be to determine when the different terms in this Hamiltonian will be relevant. We will do this in one of the following sections by conducting a renormalisation group analysis on the system, where we will consider the cosine terms as a perturbation to the second order Hamiltonian.

5.1.3 Correlation Functions

Note that the second order Hamiltonian of our system (eq. 5.16) is the sum of two different systems, which we will denote as:

$$\mathcal{H}_1^{(2)} = \mathcal{H}_{\tilde{\phi}_1} + \mathcal{H}_{\tilde{\phi}_2} \quad \mathcal{H}_{\tilde{\phi}_1} = \tilde{t}_1 \left(\partial_x \tilde{\phi}_1 \right)^2 \quad \mathcal{H}_{\tilde{\phi}_2} = \tilde{t}_1 \left(\partial_x \tilde{\phi}_2 - \frac{2}{3} \partial_x \tilde{\theta}_2 \right)^2 \quad (5.20)$$

By finding the Lagrangian of the system from a Legendre transformation of its Hamiltonian, we notice we may define a diagonal Lagrangian matrix:

$$\underline{\mathcal{L}} = \begin{pmatrix} \mathcal{L}_{\tilde{\phi}_1} & 0 \\ 0 & \mathcal{L}_{\tilde{\phi}_2} \end{pmatrix} \quad (5.21)$$

If we define the vector $\vec{\phi} = (\tilde{\phi}_1 \quad \tilde{\phi}_2)^T$ and some small generating fields $\vec{\eta} = (\eta_1 \quad \eta_2)^T$, we may calculate the generating functional/partition function of the system by the Feynman path integral:

$$Z(\vec{\eta}) \equiv \int D\vec{\phi} \exp \left[\int dx dt \left(\frac{i}{\hbar} \cdot \vec{\phi}^T \underline{\mathcal{L}} \vec{\phi} + \vec{\eta}^T \vec{\phi} \right) \right] \quad (5.22)$$

Note that the non-mixing of the fields of species $\tilde{\phi}_1$ and $\tilde{\phi}_2$ in the Hamiltonian (eq. 5.16), and by extension the diagonality of the Lagrangian matrix $\underline{\mathcal{L}}$, will make the total Hamiltonian separable into two systems that can be solved independently. This will mean the mixed correlation function between fields of different species are zero, i.e.:

$$\langle \tilde{\phi}_1(x, t) \tilde{\phi}_2(x', t') \rangle = \langle \tilde{\phi}_2(x, t) \tilde{\phi}_1(x', t') \rangle = 0 \quad (5.23)$$

The two-point correlation functions of the chiral Luttinger liquid were calculated in section 2.3. In the low energy limit with $|x - x'| \gg a$ and for equal time $t = t'$, these two-point correlation functions are:

$$\begin{aligned} \langle \tilde{\phi}_1(x, t) \tilde{\phi}_1(x', t') \rangle &= \text{const} \\ \langle \tilde{\phi}_2(x, t) \tilde{\phi}_2(x', t') \rangle &= -\frac{1}{3} \log |x - x'| + \text{const} \end{aligned} \quad (5.24)$$

And:

$$\begin{aligned} \langle \tilde{\theta}_1(x, t) \tilde{\theta}_1(x', t') \rangle &= 0 \\ \langle \tilde{\theta}_2(x, t) \tilde{\theta}_2(x', t') \rangle &= -\frac{3}{4} \log |x - x'| + \text{const} \end{aligned} \quad (5.25)$$

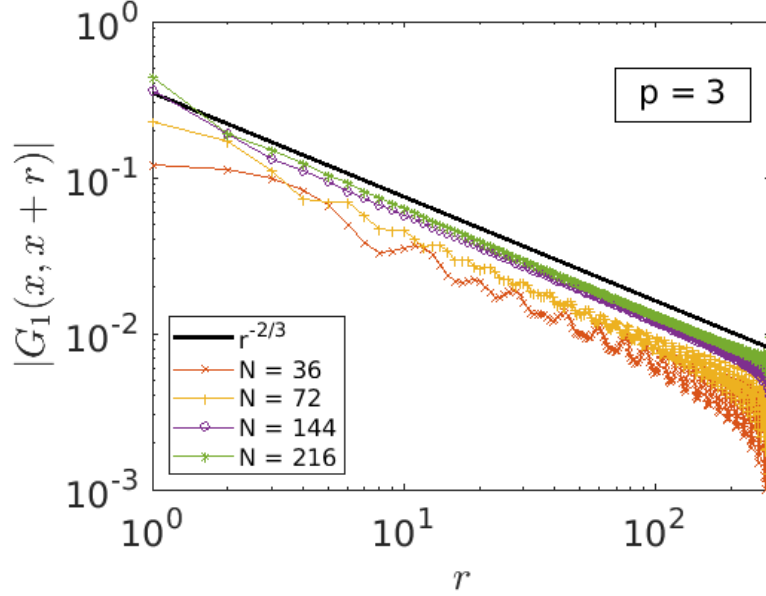


Figure 5.1: Numerical results of the two-point correlation function $|G_1(x, x+r)| = \langle F_x^\dagger F_{x+r} \rangle$ of the single hopping Hamiltonian (with $t_2 = 0$) versus the theoretical predictions made in the text with $G_1(x, x+r) \propto e^{\frac{\pi i}{3}(2\rho r - 1)} \cdot r^{-\frac{2}{3}}$. Numerical results and graph courtesy of Rossini et.al. [4].

Two-point Correlation Functions of the Fock Parafermion Operators

The two-point correlation functions of the bosonic fields ϕ_k and $\theta_{k'}$, do not readily yield any particular physical significance to us. Of bigger importance are the two-point correlation functions of the Fock parafermions, which are defined as:

$$G_1(x, x+r) \equiv \langle F_x^\dagger F_{x+r} \rangle \quad G_2(x, x+r) \equiv \langle (F_x^\dagger)^2 (F_{x+r})^2 \rangle \quad (5.26)$$

Where F_x is the Fock parafermion operator in the spatial coordinate x . The bosonised Fock parafermion operators to be inserted are defined in (eqs. 4.50 and 4.51). Notice that these operators are defined on the lattice. In the continuum we change $n \rightarrow \rho = N/L$, since the density should be used because $nj = nx_j/a = \rho x_j$, when going from the site number j to its position in spatial coordinates. In the tilde transformed fields (eq. 5.14), G_1 is:

$$G_1(x, x+r) = e^{\frac{2\pi i}{3}\rho r} \left\langle \left(e^{-i(\tilde{\phi}_2(x) + \tilde{\phi}_1(x))} + e^{-i(\tilde{\phi}_2(x) - \tilde{\phi}_1(x))} \right) e^{\frac{2i}{3}(\tilde{\theta}_2(x) - \tilde{\theta}_2(x+r))} \right. \\ \left. \times \left(e^{i(\tilde{\phi}_2(x+r) + \tilde{\phi}_1(x+r))} + e^{i(\tilde{\phi}_2(x+r) - \tilde{\phi}_1(x+r))} \right) \right\rangle$$

Invoking the BCH formula, the correlation function becomes:

$$G_1(x, x+r) = e^{\frac{\pi i}{3}(2\rho r - 1)} \left[\left\langle \exp \left(i \left(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x) + \tilde{\phi}_1(x+r) - \tilde{\phi}_1(x) + \frac{2}{3} (\tilde{\theta}_2(x) - \tilde{\theta}_2(x+r)) \right) \right) \right\rangle \right. \\ + \left\langle \exp \left(i \left(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x) + \tilde{\phi}_1(x+r) + \tilde{\phi}_1(x) + \frac{2}{3} (\tilde{\theta}_2(x) - \tilde{\theta}_2(x+r)) \right) \right) \right\rangle \\ + \left\langle \exp \left(i \left(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x) - \tilde{\phi}_1(x+r) - \tilde{\phi}_1(x) + \frac{2}{3} (\tilde{\theta}_2(x) - \tilde{\theta}_2(x+r)) \right) \right) \right\rangle \\ \left. + \left\langle \exp \left(i \left(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x) - \tilde{\phi}_1(x+r) + \tilde{\phi}_1(x) + \frac{2}{3} (\tilde{\theta}_2(x) - \tilde{\theta}_2(x+r)) \right) \right) \right\rangle \right]$$

Each of these four terms can be calculated by the cumulant expansion². For example we see:

$$\begin{aligned} & \left\langle \exp \left(i \left(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x) + \tilde{\phi}_1(x+r) - \tilde{\phi}_1(x) + \frac{2}{3} \left(\tilde{\theta}_2(x) - \tilde{\theta}_2(x+r) \right) \right) \right) \right\rangle \\ & \propto \exp \left(-\frac{1}{2} \left\langle \left(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x) \right)^2 \right\rangle - \frac{1}{2} \left\langle \left(\tilde{\phi}_1(x+r) - \tilde{\phi}_1(x) \right)^2 \right\rangle - \frac{1}{2} \left(\frac{2}{3} \right)^2 \left\langle \left(\tilde{\theta}_2(x+r) - \tilde{\theta}_2(x) \right)^2 \right\rangle \right) \end{aligned}$$

Note that the same site two-point correlation functions are ignored. Furthermore, N_i is a $U(1)$ -symmetry of the system, as discussed earlier. That such a symmetry exists guarantees that the one point correlation functions are all zero. By these same assumptions the two-point correlation functions of the square are easily calculated:

$$\begin{aligned} \left\langle \left(\tilde{\phi}_1(x+r) - \tilde{\phi}_1(x) \right)^2 \right\rangle &= -2 \left\langle \tilde{\phi}_1(x+r) \tilde{\phi}_1(x) \right\rangle = \text{const} \\ \left\langle \left(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x) \right)^2 \right\rangle &= -2 \left\langle \tilde{\phi}_2(x+r) \tilde{\phi}_2(x) \right\rangle = -2 \cdot \left(-\frac{1}{3} \log r \right) + \text{const} \\ \left\langle \left(\tilde{\theta}_2(x+r) - \tilde{\theta}_2(x) \right)^2 \right\rangle &= -2 \left\langle \tilde{\theta}_2(x+r) \tilde{\theta}_2(x) \right\rangle = -2 \left(-\frac{3}{4} \log r \right) + \text{const} \end{aligned}$$

This result is inserted into the correlation function, we get that the first term of G_1 is proportional to $r^{-\frac{2}{3}}$. The same is the case for the four other terms. In total we find that the one body density matrix G_1 is:

$$G_1(x, x+r) \propto e^{\frac{\pi i}{3}(2\rho r-1)} \cdot r^{-\frac{2}{3}} \quad (5.27)$$

This result is substantiated by the literature [4, 5]. For comparison to numerical results made in [4] see (fig. 5.1). Notice also that the complex factor in front of the right hand side agrees with the non rigorous prediction made in [4]. This complex oscillating factor could not be described by the former treatments, but was however predicted to be there [4, 5]. The result does not predict the weak oscillating behaviour that G_1 exhibit, which was one of the strengths of [5]. It is possible that some higher order terms could account for this oscillation.

The calculation of G_2 is very similar to that of G_1 , but before we can carry it out, one should note that the square of the Fock parafermion operator changes $n = n_1 + 2n_2 \rightarrow 2n_2 + n_1$, in the exponent (eq. 4.51). In general one cannot say what the relationship between n_1 and n_2 is, since F_i can change this. However, two extreme cases arise: $n_2 = 0$ and $n_1 = 0$. In the first case $n_1 = n$, while the latter case yields $n_2 = n/2$. In the following the first case is assumed, where $n_1 = n$ and $n_2 = 0$.

In the tilde transformed variables (eq. 5.14) we find $2\theta_1 + \theta_2 = \frac{3}{2}\tilde{\theta}_1 + \frac{1}{2}\tilde{\theta}_2$, hereby we may find the two-point correlation function, G_2 :

$$G_2(x, x+r) = e^{\frac{\pi i}{3}(4\rho r-1)} \left\langle \exp \left(2i \left(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x) \right) - \frac{i}{3} \left(\tilde{\theta}_2(x+r) - \tilde{\theta}_2(x) \right) - i \left(\tilde{\theta}_1(x+r) - \tilde{\theta}_1(x) \right) \right) \right\rangle$$

The cumulant expansion yields:

$$G_2(x, x+r) \propto e^{\frac{\pi i}{3}(4\rho r-1)} r^{-\frac{17}{12}} \quad (5.28)$$

At different filling factors the fraction of doubly-occupied states might increase, resulting in a change in the value of 4ρ in the above. In the extreme case, where all sites are doubly-occupied, the correlation function is $G_2(x, x+r) \propto e^{\frac{\pi i}{3}(\rho r-1)} r^{-\frac{17}{12}}$.

This correlation function does resemble the numerical results closely (see Figure 5.2). Compare this to the result, if the \mathbb{Z}_3 -symmetry of ω is ignored³ and $(U_i)^2$ is given by (eq. 4.44). Under these assumptions we find that $|G_2| \propto r^{-8/3}$. This corresponds to the prediction made in [4], which was based on the argument that the system is a single anyonic fluid. We also saw that the phenomenological bosonisation we defined earlier, where only one pair of dual bosonic fields was used, yielded this result.

The numerical results as seen in (fig. 5.2) do indeed favour our new prediction of $|G_2(x, x+r)| \sim r^{-\frac{17}{12}}$. Fundamentally, the change came from the \mathbb{Z}_3 -symmetry of our model and by introducing multiple

²See appendix D.1.

³See appendix C.

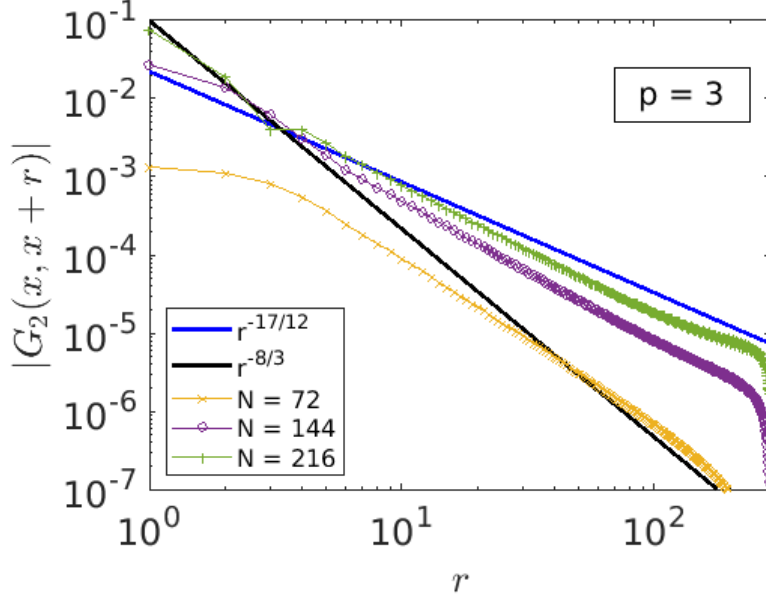


Figure 5.2: Numerical results of the two-point correlation function $|G_2(x, x+r)| = \langle (F_x^\dagger)^2 (F_{x+r})^2 \rangle$ versus the theoretical predictions. The system in question is the single hopping Hamiltonian with $t_2 = 0$. Theoretical predictions are $|G_2(x, x+r)| \propto r^{-8/3}$ and $|G_2(x, x+r)| \propto r^{-17/12}$ depending on the model and type of bosonisation used. Judging from the plot the latter prediction is better. Numerical results and graph courtesy of Rossini et.al. [4].

species of dual bosonic fields for the bosonisation. The latter part is important since squaring the U_i -operator takes the two θ -fields into each other, $\theta_1 \leftrightarrow \theta_2$, rather than just multiplying both of the by two in the exponent. I.e. the difference in the two descriptions is:

$$(U_i)^2 \sim \omega^{2\theta_1 + \theta_2} \quad \text{versus} \quad (U_i)^2 \sim \omega^{2\theta_1 + 4\theta_2} \quad (5.29)$$

These two descriptions are mathematically equivalent as they are portrayed, but their Taylor expansions are very different. This result underlines the importance of describing the system as multiple pairs of dual bosonic fields, rather than just as one pair of dual bosonic fields that most of the literature have done until now.

5.1.4 Phases of the Single Hopping Hamiltonian

We now return to consider the full bulk Hamiltonian with $t_2 = 0$ that includes the second order terms and the cosine terms (eq. 5.19). We do this by considering the cosine terms a perturbation to the chiral Luttinger liquid that we just treated. Note that we may only consider the system for fillings $0 \leq n \leq 1$, since a form of particle hole symmetry arises that will render the fillings $1 \leq n \leq 2$ mirroring the system at $0 \leq n \leq 1$.

Whether the $M_{\tilde{\theta}}$ -term becomes relevant depends on the value of n . If $n \neq 1$, $\cos\left(\frac{2\pi nx}{a} - 2\tilde{\theta}_2\right)$ will be fast oscillating as it will strongly fluctuate as x goes along the lattice. By the idea of the theory of the renormalisation group this term is thus non-relevant. If $n \approx 1$ the term begins to oscillate more slowly and will hence be able to become relevant. We will therefore calculate the scaling dimension of the $M_{\tilde{\theta}}$ -term for $n \approx 1$. We Taylor-expand around $n = 1$ to find:

$$\cos\left(\frac{2\pi nx}{a} - 2\tilde{\theta}_2\right) = \cos 2\tilde{\theta}_2 + \frac{(1-n)x}{a} \sin 2\tilde{\theta}_2 + O\left(\left(\frac{1-n}{a}\right)^2\right) \quad (5.30)$$

We note that this will blow up if $(1-n) > a$. Hence we need $(1-n) < a$, which means in the continuum limit only $n = 1$ will yield a term that is not non-relevant in the RG sense. Euler's formula tells us that $\cos(x) = \frac{1}{2}(\exp(ix) + \exp(-ix))$, and the scaling dimension of the vertex operator $e^{i2\tilde{\theta}_2(x)}$ is easily

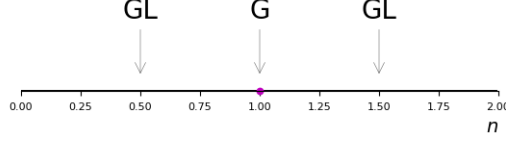


Figure 5.3: Phase diagram of the single hopping Hamiltonian. For $n \neq 1$ a gapless bosonic mode arises and the system is described by a chiral Luttinger liquid. This phase is denoted by GL in the above diagram. This bosonic mode is however gapped out when $n = 1$ by the $M_{\tilde{\theta}}$ -term in the bulk Hamiltonian (eq. 5.19). This phase is denoted by G in the above.

calculated:

$$\left\langle e^{i2\tilde{\theta}_2(x)} e^{-2i\tilde{\theta}_2(x')} \right\rangle = \exp\left(-\frac{2^2}{2} \cdot (-2 \langle \tilde{\theta}_2(x) \tilde{\theta}_2(x') \rangle)\right) = \exp\left(4 \left(-\frac{3}{4} \log|x-x'|\right)\right) = |x-x'|^{-3} \quad (5.31)$$

The scaling dimension of the cosine-term will be half of the negative value of this exponent. Thus the scaling dimension of the $M_{\tilde{\theta}}$ -term for $n = 1$ is $\Delta_{M_{\tilde{\theta}}} = 3/2$. Notice that the dimension of the system is $D = 2$, corresponding to one spatial and one temporal dimension. We thus see $\Delta_{M_{\tilde{\theta}}} < D$ and the $M_{\tilde{\theta}}$ -term is relevant for $n = 1$.

The cosine terms will try and pin the field, such that $2\tilde{\theta}_2 \approx 2\pi m$, for some $m \in \mathbb{Z}$. How strong this pinning is compared to the chiral Luttinger liquid part must be considered for us to be able to determine if the pinning will actually happen. Consider some small fluctuation about the pinned fields, we may write this as a linear term away from the pinning value, for some small δx : $\tilde{\theta}_2 \approx \pi m(1 + \delta x)$ and $\tilde{\phi}_2 \approx 2\pi m'(1 + \delta x)/3$. The Hamiltonian in these fields finds its minimum in these pinnings if $m = m'$:

$$\mathcal{H} \approx \tilde{t}_2 \left(\partial_x \tilde{\phi}_2 - \frac{2}{3} \partial_x \pi m(1 + \delta x) \right)^2 - M_{\tilde{\theta}} \cos(2\pi m \delta x) = \tilde{t}_2 \left(\partial_x \tilde{\phi}_2 - \frac{2}{3} \pi m \delta x \right)^2 - M_{\tilde{\theta}} \quad (5.32)$$

Minimising the energy will pin the other field $\tilde{\phi}_2 = \frac{2}{3} \pi m \delta x$. This corresponds to gapping out the free bosonic mode that corresponds to the dual fields $\tilde{\phi}_2$ and $\tilde{\theta}_2$ [12]. In general this "gapping out" will not happen if $n \neq 1$, since here the $M_{\tilde{\theta}}$ -term is irrelevant.

Lastly note that the term $-(2\tilde{t}_1/a^2) \cos 2\tilde{\phi}_1$ will always be relevant. The term has scaling dimension $\Delta_{\tilde{\phi}_1} = 0$, since the two-point correlation function of the $\tilde{\phi}_1$ -field is zero (eq. 5.24). It is evident that the energy will be minimised if the pinning happens, since $\tilde{\phi}_1 = 0$ yields a minimum of the energy. In other words, a gapping out of the free bosonic mode that corresponds to the fields $\tilde{\phi}_1$ and $\tilde{\theta}_1$ will always happen in the single hopping Hamiltonian (eq. 5.19).

Both of the cosine-terms, $-(2\tilde{t}_1/a^2) \cos 2\tilde{\phi}_1$ and $-M_{\tilde{\theta}} \cos\left(\frac{2\pi n x}{a} - 2\tilde{\theta}_2\right)$, can gap out bosonic mode simultaneously as the two terms commute [12], ie. $[\tilde{\phi}_1, \tilde{\theta}_2] = 0$. To sum up, consider (fig. 5.3) for a phase diagram. We saw that the term $-(2\tilde{t}_1/a^2) \cos 2\tilde{\phi}_1$ is relevant and will gap out the field $\tilde{\phi}_1$ for all n , which makes only one bosonic mode is present. At $n = 1$ the term $\cos\left(\frac{2\pi n x}{a} - 2\tilde{\theta}_2\right)$ gaps out the remaining bosonic mode, whereby the system becomes fully gapped.

5.2 Pair Hopping Hamiltonian

In this section we will analyse the t_2 -term of the nearest-neighbour hopping Hamiltonian (eq. 5.1) for \mathbb{Z}_3 parafermions. This Hamiltonian corresponds to the hopping of a pair of Fock parafermions between nearest-neighbouring sites in the one-dimensional lattice:

$$H_2 = -t_2 \sum_{i=1}^{l-1} (F_i^\dagger)^2 (F_{i+1})^2 + \text{H.c.} = -t_2 \sum_{i=1}^{l-1} (B_i^\dagger)^2 (U_i)^2 (B_{i+1})^2 + \text{H.c.} \quad (5.33)$$

The bosonised version of the pair hopping operator is found in (eq. 4.51). The BCH formula grants us the ability to gather everything in the same exponent:

$$(F_i^\dagger)^2 (F_{i+1})^2 = \exp \left[i(\phi_2(x_{i+1}) - \phi_2(x_i)) - \frac{2i}{3} [2(\theta_1(x_{i+1}) - \theta_1(x_i)) + (\theta_2(x_{i+1}) - \theta_2(x_i))] \right. \\ \left. - \frac{\pi i}{3} + \frac{2\pi i}{3}(2n_1 + n_2) \right]$$

Compared to the single hopping terms, the square of the Fock parafermion operators looks to be taking the θ -fields "into each other", such that $\theta_1 \rightarrow 2\theta_1$ and $2\theta_2 \rightarrow \theta_2$.

It must be mentioned that by n_1 and n_2 are meant the averages of these densities, rather than the exact densities at the sites. n_1 and n_2 can be understood as the fraction of sites that are occupied by a single Fock parafermion and a pair of Fock parafermions, respectively. It is reasonable to assume that these are somehow determined by the parameters of the system in particular t_2 and the filling factor n . This makes the latter part, $\frac{2\pi i}{3}(2n_1 + n_2)$, a number and as such a conserved quantity. Another observation that underlines this fact is that we can write the operator that counts the number of sites that are occupied by 2 Fock parafermions as $N_2 = \sum_i (F_i^\dagger)^2 (F_i)^2$. This operator is a symmetry of the Hamiltonian, and as such $n_2 = N_2/L$ is constant of the system. We can determine n_1 from n_2 , since $n_1 = n - 2n_2$, and hence n_1 must likewise be a constant of the system.

The system with the Hamiltonian, H_2 , cannot contain only one Fock parafermion per site. Only free sites or doubly-occupied sites are allowed. To see why this is the case, notice that for any site j :

$$H_2 |n_j = 1\rangle = t_2 \sum_{i=1}^{l-1} \left[(F_i^\dagger)^2 (F_{i+1})^2 + (F_{i+1}^\dagger)^2 (F_i)^2 \right] F_j^\dagger |0\rangle = 0 \quad (5.34)$$

In other words, the single occupied orbitals are eigenstates with eigenvalue zero of the Hamiltonian, H_2 . Notice furthermore, that the Hamiltonian does not allow for the single occupied sites to move around, whereby they will effectively act as partitions of the system. The choices for how we fill the sites with Fock parafermions are thus constrained to be either zero or two per site. This makes $n_1 = 0$ and $n_2 = n/2$, for any given filling $n \in [0, 2]$.

Continuum Limit and Taylor Expansion

We insert the bosonised description of $(F_i^\dagger)^2 (F_{i+1})^2$ in the Hamiltonian, H_2 , and let the lattice sites approach each other such that lattice constant $a \rightarrow 0$. In this limit, natural definitions of the spatial derivatives and integrals arise, as per section 4.0.1. The Hamiltonian of the t_2 -hopping term is:

$$H_2 = -2t_2 \int \frac{dx}{a} \cos \left[a \left(\partial_x \phi_2 - \frac{2}{3} (2\partial_x \theta_1 + \partial_x \theta_2) \right) + \frac{\pi}{3} (n-1) \right] \quad (5.35)$$

As before, we Taylor-expand the integrand, and terms of equal order in a are collected:

$$H_2^{(0)} = -\frac{2t_2 L}{a} \cos \left(\frac{\pi}{3} (n-1) \right) \\ H_2^{(1)} = 2t_2 \sin \left(\frac{\pi}{3} (n-1) \right) \left([\phi_2(x)]_{x=0}^{x=L} - \frac{2}{3} [2\theta_1(x) + \theta_2(x)]_{x=0}^{x=L} \right) \\ H_2^{(2)} = at_2 \cos \left(\frac{\pi}{3} (n-1) \right) \int_0^L dx \left(\partial_x \phi_2 - \frac{2}{3} (2\partial_x \theta_1 + \partial_x \theta_2) \right)^2 \quad (5.36)$$

The resulting system is very similar to the single hopping Hamiltonian H_1 . As was the case for single hopping Hamiltonian, we ignore the boundary terms that arise in $H_2^{(1)}$, which corresponds to the periodic boundary conditions or the thermodynamic limit $L \rightarrow \infty$. For a given set of parameters n and so on, the zeroth order Hamiltonian, $H_2^{(0)}$, is simply a constant and can be ignored in the dynamics of the system. The first non-trivial terms of the hamiltonian are the second order terms, $H_2^{(2)}$.

Notice, both the operator and the average of the single site filling will yield zero, i.e. $\hat{n}_1 = n_1 = 0$. This will in fact put a constraint our bosonic fields such that:

$$\partial_x \theta_1 \sim n_1 = 0 \quad \implies \quad \theta_1 = \text{const} \quad (5.37)$$

This constrain was lost on us, when we went to the bosonised description. However it may be reintroduced as a cosine-interaction term to our Hamiltonian. Without loss of generality we let $const = 0$. Hereby the full bulk Hamiltonian becomes:

$$H_2 = \hat{t}_2 \int_0^L dx \left(\partial_x \phi_2 - \frac{2}{3} (2\partial_x \theta_1 + \partial_x \theta_2) \right)^2 - M_2 \int_0^L dx \cos(\theta_1) \quad (5.38)$$

Where we have defined:

$$\hat{t}_2 = at_2 \cos\left(\frac{\pi}{3}(n-1)\right) \quad (5.39)$$

It is reasonable to assume the energy associated with the constraint on θ_1 , M_2 , is of the same order as the zero point energy. We saw that this was the case for the constraining of the fields $\phi_1 - \frac{1}{2}\phi_2$ in the single hopping case. Therefore it may be assumed that:

$$M_2 \approx \frac{2\hat{t}_2}{a^2} \quad (5.40)$$

The hopping energy, \hat{t}_2 is bigger than zero for all $n \in [0, 2]$. We define the following canonical transformation:

$$\hat{\phi} = \phi_2 \quad \hat{\theta} = 2\theta_1 + \theta_2 \quad (5.41)$$

The second order Hamiltonian is thus a chiral Luttinger liquid in these newly transformed fields with fractional charge $\kappa = -2/3$:

$$H_2^{(2)} = \hat{t}_2 \int_0^L dx \left(\partial_x \hat{\phi} - \frac{2}{3} \partial_x \hat{\theta}(x) \right)^2 \quad (5.42)$$

In fact defining this transformation is a little redundant since the \cos -interaction term in (eq. 5.38), will pin the field θ_1 , such that its spatial derivative disappears from the Hamiltonian entirely, and the resulting system is a chiral Luttinger liquid in the fields ϕ_2 and θ_2 exclusively. We construct these definitions to make our approach more general, which will come in hand later.

5.2.1 Correlation Functions

The equal-time two-point correlation functions of the bosonic fields $\hat{\phi}$ and $\hat{\theta}$ are:

$$\begin{aligned} \langle \hat{\phi}(x, t) \hat{\phi}(x', t') \rangle &= -\frac{1}{3} \log |x - x'| + const \\ \langle \hat{\theta}(x, t) \hat{\theta}(x', t') \rangle &= -\frac{3}{4} \log |x - x'| + const \end{aligned} \quad (5.43)$$

Since the Hamiltonian (eq. 5.42) is that of a chiral Luttinger liquid with fractional charge $\kappa = -2/3$.

Two-point Correlation Functions of the Fock Parafermions

Earlier it was argued that the lattice sites only contain zero or two Fock parafermions per site. The case where a single parafermion occupies a lattice site is simply not an eigenstate of the system. An important consequence of this fact is that a single Fock parafermion operator acting on the ground state of the system is zero, $F_i |\text{GS}\rangle = 0$. This in turn guarantees that:

$$G_1(x, x+r) = \langle F_x^\dagger F_{x+r} \rangle = 0 \quad (5.44)$$

This result is what is expected from the single Fock parafermion filling not being allowed. The result is furthermore substantiated by the phase diagram (fig. 3.1) [5]. The correlation function of the square of the Fock parafermion operators, is non-zero on the other hand:

$$G_2(x, x+r) = \langle (F_x^\dagger)^2 (F_{x+r})^2 \rangle = e^{\frac{\pi i}{3}(\rho r - 1)} \left\langle \exp \left(i \left(\hat{\phi}(x+r) - \hat{\phi}(x) \right) + \frac{2i}{3} \left(\hat{\theta}(x+r) - \hat{\theta}(x) \right) \right) \right\rangle \quad (5.45)$$

By the cumulant expansion we find:

$$G_2(x, x+r) \propto e^{\frac{\pi i}{3}(\rho r - 1)} r^{-\frac{2}{3}} \quad (5.46)$$

Noticeably, this result displays the same power law behaviour as the correlation function G_1 in the system H_1 . The marked difference between the two correlation functions is the different complex factor in front, $e^{\frac{\pi i}{3}(\rho r - 1)}$ versus $e^{\frac{\pi i}{3}(2\rho r - 1)}$.

Compared to previous treatments this correlation function is predicted to be $|G_2| \propto r^{-\frac{13}{18}}$, which numerical results substantiated [5]. This prediction is nevertheless very close to our prediction, since $2/3 = 12/18 \approx 13/18$. The prediction, $|G_2| \propto r^{-\frac{13}{18}}$, arises from our system too if we change the specific fractional charge $-2/3$ to be the general symbol κ . We find that the correlation function is:

$$G_2(x, x+r) = \left\langle e^{-i\hat{\phi}(x)} e^{\frac{4\pi i}{3}\rho r + i\kappa(\hat{\theta}(x) - \hat{\theta}(x+r))} e^{i\hat{\phi}(x+r)} \right\rangle \propto e^{\frac{\pi i}{3}(4\rho r + 1)} r^{-\left(\frac{1}{2\kappa} + \frac{2\kappa}{9}\right)} \quad (5.47)$$

In [5] it is assumed $\kappa = 1$, by arguing that the system we just considered can be mapped onto a non-interacting fermionic system. This assumption yields $G_2(x, x+r) \propto r^{-\frac{13}{18}}$. The validity of this assumption is however dubious, since the fractional charge $\kappa = -2/3$ in the second order Hamiltonian terms (Eqs. 5.13 and 5.36) has its roots in the definition of $\omega = e^{-i\pi\kappa}$, which described the commutation relation of the Fock parafermions. So it is not evident how $\kappa = 1$ can be allowed to explain our system, without ignoring the underlying statistics of the Fock parafermions.

5.2.2 Consideration for General \mathbb{Z}_p Fock Parafermions

The study of the \mathbb{Z}_3 Fock parafermions is put on a temporary hiatus, in favour of a small interim where we consider the Fock parafermion of some general order p . Note that the hopping of a pair of \mathbb{Z}_3 Fock parafermions is a special case of the hopping of $(p-1)$ \mathbb{Z}_p Fock parafermions. Recall that the general Hamiltonian of nearest-neighbour hopping of Fock parafermions of general p on a one-dimensional lattice of length L with lattice spacing a is:

$$H = - \sum_{i=1}^{L/a} \sum_{k=1}^{p-1} t_k (F_i^\dagger)^k (F_{i+1})^k + \text{H.c.} = - \sum_{i=1}^{L/a} \sum_{k=1}^{p-1} t_k (B_i^\dagger)^k (U_i)^k (B_{i+1})^k + \text{H.c.} \quad (5.48)$$

Let ρ denote the average density of Fock parafermions, and $n = a\rho$, be the average number of Fock parafermions per site. If all the hopping parameters except for the last one, t_{p-1} , are zero we end up with a system that is easier to consider. Note that the $(p-1)$ -power of the bosonic operator gives rise to a two-state system consisting of the state $|0\rangle$ and $|p-1\rangle$:

$$(B_i)^{p-1} = \exp(i\phi_{p-1}(x_i)) \quad (U_i)^{p-1} = \exp\left(\frac{2\pi i(p-1)}{p}n - \frac{2i}{p} \sum_{j=1}^{p-1} (p-j)(\theta_j(x_{i+1}) - \theta_j(x_i))\right) \quad (5.49)$$

The transformation $\phi \rightarrow \hat{\phi} = \phi_{p-1}(x_i)$ and $\theta \rightarrow \hat{\theta} = \sum_{j=1}^{p-1} (p-j)(\theta_j(x_{i+1}) - \theta_j(x_i))$ is canonical, with $[\partial_x \hat{\theta}(x), \hat{\phi}(x')] = -i\pi\delta(x-x')$. In the continuum limit the Hamiltonian becomes:

$$H_{p-1} = -2t_{p-1} \int_0^L \frac{dx}{a} \cos\left(\partial_x \hat{\phi} - \frac{2}{p} \partial_x \hat{\theta} + \frac{\pi}{p} (2(p-1)n - 1)\right) \quad (5.50)$$

As we have calculated many times before, the second order Taylor expansion of this system gives rise to a chiral Luttinger liquid with fractional charge $\kappa = \frac{2}{p}$ and energy term $\epsilon = t_{p-1}a \cos\left(\frac{\pi}{p} (2(p-1)n - 1)\right)$:

$$H_{p-1}^{(2)} = t_{p-1}a \cos\left(\frac{\pi}{p} (2(p-1)n - 1)\right) \int_0^L dx \left(\partial_x \hat{\phi} - \frac{2}{p} \partial_x \hat{\theta}\right)^2 \quad (5.51)$$

As we just calculated this system gives rise to the two-point correlation function:

$$\langle (F_x^\dagger)^{p-1} (F_{x+r})^{p-1} \rangle \propto e^{\frac{\pi i}{p}(2(p-1)\rho r - 1)} \cdot r^{-\frac{2}{p}} \quad (5.52)$$

The two-point correlation function of the Fock parafermionic operators, $\langle (F_x^\dagger)^k (F_{x+r})^k \rangle$, will be effectively zero (specifically exponentially decreasing with some small correlation length) for $k \neq p-1$, since $H_{p-1} |n_i\rangle = 0$ for all $i \in [0, L/a]$.

5.3 Combined Single and Pair Hopping Hamiltonian

Now that we have studied each of the systems for single and pairs of Fock parafermions hopping individually, we want to consider the system where both can happen simultaneously. The two Hamiltonians (eqs. 5.19 and 5.42) added together becomes up to the second order:

$$\begin{aligned}
H = \int_0^L dx \left[\tilde{t}_1 \left(\frac{1}{2} \partial_x \phi_2 - \frac{2}{3} (\partial_x \theta_1 + 2 \partial_x \theta_2) \right)^2 + \tilde{t}_1 \left(\partial_x \phi_1 - \frac{1}{2} \partial_x \phi_2 \right)^2 \right. \\
\left. + \tilde{t}_2 \left(\partial_x \phi_2 - \frac{2}{3} (2 \partial_x \theta_1 + \partial_x \theta_2) \right)^2 - M_\theta \cos \left(\frac{2\pi n x}{a} - 2(\theta_1 + 2\theta_2) \right) \right. \\
\left. - \frac{2}{a^2} (\tilde{t}_1 \cos(2\phi_1 - \phi_2) + \tilde{t}_2 \cos(\theta_1)) \right] \quad (5.53)
\end{aligned}$$

Where we have defined:

$$\tilde{t}_1 = 2at(1-g) \cos \left(\frac{\pi}{3} (2n-1) \right) \quad \tilde{t}_2 = atg \cos \left(\frac{\pi}{3} (4n_1 + 2n_2 - 1) \right) \quad (5.54)$$

Note that $g = t_2/(t_2 + t_1)$, which measures the relative strength of the hopping of a pair of Fock parafermions (compared to single and pair hopping). In most physical systems, hopping of a single particle is more likely than the hopping of multiple particles, hence it would be reasonable to assume g is rather small in physical systems.

We encounter a problem in our calculations, since we don't know the number of sites that are occupied by a single Fock parafermion, n_1 , and/or how many are occupied by a pair of Fock parafermions, n_2 . At least one of these is needed to be known to calculate \tilde{t}_2 . When only the pair hopping terms of the Hamiltonian were considered, it was shown that $n_1 = 0$, which simplified things greatly. We cannot do this in the general case however, when including the single hopping terms, which guarantees the single hopping sites are still eigenvectors of the Hamiltonian.

Even though we cannot definitively determine n_1 and n_2 based on analytical reasoning, we can nonetheless make a few remarks on it's behaviour. Remember that n_k tells us the fraction of sites that are occupied by k Fock parafermions. The total fraction cannot exceed one, hence $n_1 + n_2 \leq 1$, and the total number of Fock parafermions $n = n_1 + n_2$, as always. If g is big and close to 1, we have determined that $n_1 \approx 0$, and for small n and $g \approx 0$ the converse must be true with $n_2 \approx 0$. The difficult part is in determining the behaviour of the region in between the two extreme cases. The numerical calculations by Mahyaeh et.al. [5], showed that the intermediate region experienced a discontinuity and was split up in two phases, where one phase, L, was characterised by $n_2 \approx 0$, while in the other phase, R, $n_1 \approx 0$.

Assuming that the two phases L and R are extensions of the systems, for which $g = 0$ and $g = 1$, respectively, the former phase will be dominated by single hopping, while the latter is dominated by pair hopping of the Fock parafermions. The strength of the pair hopping in the R phase is then:

$$\tilde{t}_2 \approx atg \cos \left(\frac{\pi}{3} (n-1) \right) \quad (\text{in R phase}) \quad (5.55)$$

Recall that the zeroth order energy of the single⁴ and pair hopping terms are $H_1^{(0)} = -2L\tilde{t}_1/a^2$ and $H_2^{(0)} = -2L\tilde{t}_2/a^2$, respectively. If it is assumed only one type of hopping is allowed, the type of hopping that will happen is determined by which of these energies is the smallest - which means finding which is maximal between \tilde{t}_1 and \tilde{t}_2 . For a given n the critical g^* , where the \tilde{t}_1 in the L phase is equal to \tilde{t}_2 in the R phase, is:

$$g^* = \frac{2 \cos \left(\frac{\pi}{3} (2n-1) \right)}{2 \cos \left(\frac{\pi}{3} (2n-1) \right) + \cos \left(\frac{\pi}{3} (n-1) \right)} \quad (5.56)$$

Systems with $g < g^*$ are dominated by the single hopping, while systems with $g > g^*$ are dominated by pair hopping, as per our assumptions. The resulting phase diagram derived by this phenomenological approach is seen in (fig. 5.4). It shows a decent approximation of the phase diagram (fig. 3.1) calculated by Mahyaeh et.al. [5]. The differences that arise might come from the fact that $n_1 \neq 0$ and $n_2 \neq 0$ as is assumed. This will make changes to the curve of g^* , since the second term in the denominator in (eq. 5.56) will be $\cos \left(\frac{\pi}{3} (4n_1 + 2n_2 - 1) \right)$. A few points of criticism arise in this phenomenological approach.

⁴Excluding some term proportional $\int dx \cos(2\phi_1 - \phi_2)$

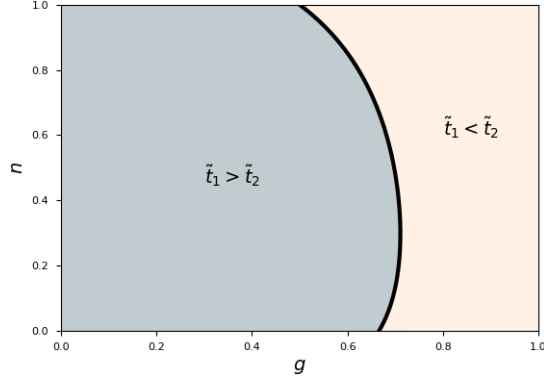


Figure 5.4: Phase diagram of the model distinguishing between the single and pair hopping of the Fock parafermions. The phase diagram is derived from phenomenological observations, and is as such not theoretically rigid. In the left region, phase L, the single Fock parafermion hopping (denoted by the hopping strength \tilde{t}_1) is dominant. In the right region, phase R, it is assumed $n_1 = 0$, and the region is defined such that the pair hopping (denoted by the hopping strength \tilde{t}_2) is dominant. The critical line g^* is defined in (eq. 5.56). At the ends $g^* = 2/3$ for $n = 0$ and $g^* = 1/2$ for $n = 1$.

First of all, this analysis doesn't show why it is that $n_1 = 0$ in the R phase and $n_2 = 0$ in the L phase. Second of all, we cannot account for the M phase in (fig. 3.1) that had central charge $c = 2$. Compared to (fig. 3.1), in (fig 5.4) the R and the M phases are fused together to one phase. The phase transition from the M to the R phases in (fig. 3.1) happens at a constant $g = 0.58$, so a possible explanation of this phase could be that single hopping will always be allowed whenever $g < 0.58$.

5.3.1 Analytical Description of Chiral Luttinger Liquid Fields

We analyse the second order terms of the Hamiltonian (eq. 5.53) and consider the cosine-terms as a perturbation to the resulting system at a later point, as was done in earlier chapters. We define three fields $\eta_j, j = 1, 2, 3$, such that:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{t}_1} \left(\frac{1}{2}\phi_2 - \frac{2}{3}(\theta_1 + 2\theta_2) \right) \\ \sqrt{\tilde{t}_1} \left(\phi_1 - \frac{1}{2}\phi_2 \right) \\ \sqrt{\tilde{t}_2} \left(\phi_2 - \frac{2}{3}(2\theta_1 + \theta_2) \right) \end{pmatrix} \quad (5.57)$$

It is true that $\tilde{t}_1 > 0$, for all $n \leq 1$, but we need to assume $\tilde{t}_2 > 0$ for all considered values of $n \leq 1$, whereby it will makes sense to take the square root of these values. In practice we do this by letting \tilde{t}_2 be defined as in (eq. 5.55). This is not strictly true, but we assume it is good enough to display to us the physics of the system.

The second order Hamiltonian of (eq. 5.53) is diagonal in $(\partial_x \eta_1 \quad \partial_x \eta_2 \quad \partial_x \eta_3)^T$. We define a fourth component η_4 , to make the system invertible, since we can thus go back and forth between our two descriptions. We define η_4 such that it commutes canonically with the other components: $[\partial_x \eta_4, \eta_i] = 0$ for $i = 1, 2, 3$. The following definition of η_4 obeys these relationships:

$$\eta_4 = \sqrt{\frac{4}{3}} \left(\frac{3}{4}\phi_1 + \frac{1}{2}\theta_1 + \theta_2 \right) \quad (5.58)$$

In summation we can express $\vec{\eta}$ as a linear transformation of the original fields:

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{1}{2}\sqrt{\tilde{t}_1} & -\frac{2}{3}\sqrt{\tilde{t}_1} & -\frac{4}{3}\sqrt{\tilde{t}_1} \\ \sqrt{\tilde{t}_1} & -\frac{1}{2}\sqrt{\tilde{t}_1} & 0 & 0 \\ 0 & \sqrt{\tilde{t}_2} & -\frac{4}{3}\sqrt{\tilde{t}_2} & -\frac{2}{3}\sqrt{\tilde{t}_2} \\ \sqrt{\frac{3}{4}} & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{4}{3}} \end{pmatrix}}_S \begin{pmatrix} \phi_1 \\ \phi_2 \\ \theta_1 \\ \theta_2 \end{pmatrix} \quad (5.59)$$

Whereby the Hamiltonian becomes:

$$\mathcal{H}^{(2)} = \partial_x \vec{\eta}^T \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}}_{\mathcal{T}} \partial_x \vec{\eta} = \sum_{i=1}^3 \partial_x \eta_i^T \partial_x \eta_i \quad (5.60)$$

We define some transformation of the system, $\vec{\eta} \rightarrow \vec{\Psi}$, such that the commutation relations of the resulting species of fields are $[\partial_x \Psi_i(x), \Psi_j(y)] = \pm i2\pi\delta(x-y)\delta_{i,j}$, which makes it possible to split up the system in simultaneously diagonalisable parts. Note that the sign in front of the commutation relation will determine the chirality of the field Ψ_i . We define the matrix W , which keeps track of the current commutation relations:

$$[\partial_x \eta_i(x), \eta_j(y)] = i2\pi\delta(x-y)W_{ij} \quad , \quad W = \frac{1}{2} \left(\begin{array}{ccc|c} \frac{4}{3}\tilde{t}_1 & 0 & \frac{5}{3}\sqrt{\tilde{t}_1\tilde{t}_2} & 0 \\ 0 & 0 & \sqrt{\tilde{t}_1\tilde{t}_2} & 0 \\ \frac{5}{3}\sqrt{\tilde{t}_1\tilde{t}_2} & \sqrt{\tilde{t}_1\tilde{t}_2} & \frac{4}{3}\tilde{t}_2 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array} \right) \quad (5.61)$$

This matrix is a block diagonal matrix, consisting of a 3×3 block, W' , and a 1×1 identity block. This block diagonal construction is the motivation for the way we defined the fourth component η_4 , since it does not influence our system, which simplifies things a lot. If we are able to diagonalise the matrix W' with an orthogonal matrix \mathcal{O} , this matrix will define the transformation $\vec{\eta} \rightarrow \vec{\Psi}$. We note that W' is a symmetric 3×3 -matrix, which guarantees it has 3 not necessarily distinct eigenvectors, λ_i , each with an eigenvector, \vec{v}_i , that make up an orthogonal basis [25]. The eigenvectors can be determined from the characteristic polynomial of the matrix:

$$-\lambda \left(\frac{2}{3}\tilde{t}_1 - \lambda \right) \left(\frac{2}{3}\tilde{t}_2 - \lambda \right) - \tilde{t}_1\tilde{t}_2 \left(\frac{1}{3}\tilde{t}_1 - \frac{17}{18}\lambda \right) = 0 \quad (5.62)$$

This equation is not easily solvable analytically. However we note a few things about the solutions. First of all the determinant $\det W' = -\frac{4}{3}(\tilde{t}_1)^2\tilde{t}_2 < 0$ which tells us that either exactly one or all of the eigenvalues are less than zero, since $\det W' = \lambda_1\lambda_2\lambda_3$, while the trace $\text{Tr } W' = \frac{4}{3}(\tilde{t}_1 + \tilde{t}_2) > 0$ tells us at least one eigenvalue is greater than zero, since $\text{Tr } W' = \lambda_1 + \lambda_2 + \lambda_3$. Hence we find that two eigenvalues (say λ_1 and λ_2) will be positive and one (say λ_3) will be negative. Numerical calculations of the eigenvalues supports these claims (fig. 5.5).

Denote the eigenvectors corresponding to the eigenvalue λ_i by v_i . These eigenvectors will be orthonormal and will make up the first block of the orthogonal matrix \mathcal{O} :

$$\mathcal{O} = \left(\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \quad , \quad \mathcal{O}^T \mathcal{O} = \mathbb{1} \quad (5.63)$$

Therefore we may define a diagonal matrix, Λ , consisting of the eigenvalues λ_i in its diagonal, such that:

$$\Lambda = \mathcal{O}^T W \mathcal{O} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & -1/2 \end{pmatrix} \quad (5.64)$$

The commutation relations are then described as:

$$i2\pi\delta(x-y)\Lambda = i2\pi\delta(x-y)\mathcal{O}^T W \mathcal{O} = \mathcal{O}^T [\partial_x \vec{\eta}(x), \vec{\eta}(y)] \mathcal{O} = [\mathcal{O}^T \partial_x \vec{\eta}(x), \mathcal{O}^T \vec{\eta}(y)] \quad (5.65)$$

I.e we find that the following vector will have the canonical commutation relations between its components, which are diagonalised and normalised:

$$\vec{\Psi} \equiv |\Lambda|^{-1/2} \mathcal{O}^T \vec{\eta} \quad , \quad \left[\partial_x \vec{\Psi}(x), \vec{\Psi}(y) \right] = i2\pi\delta(x-y) \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (5.66)$$

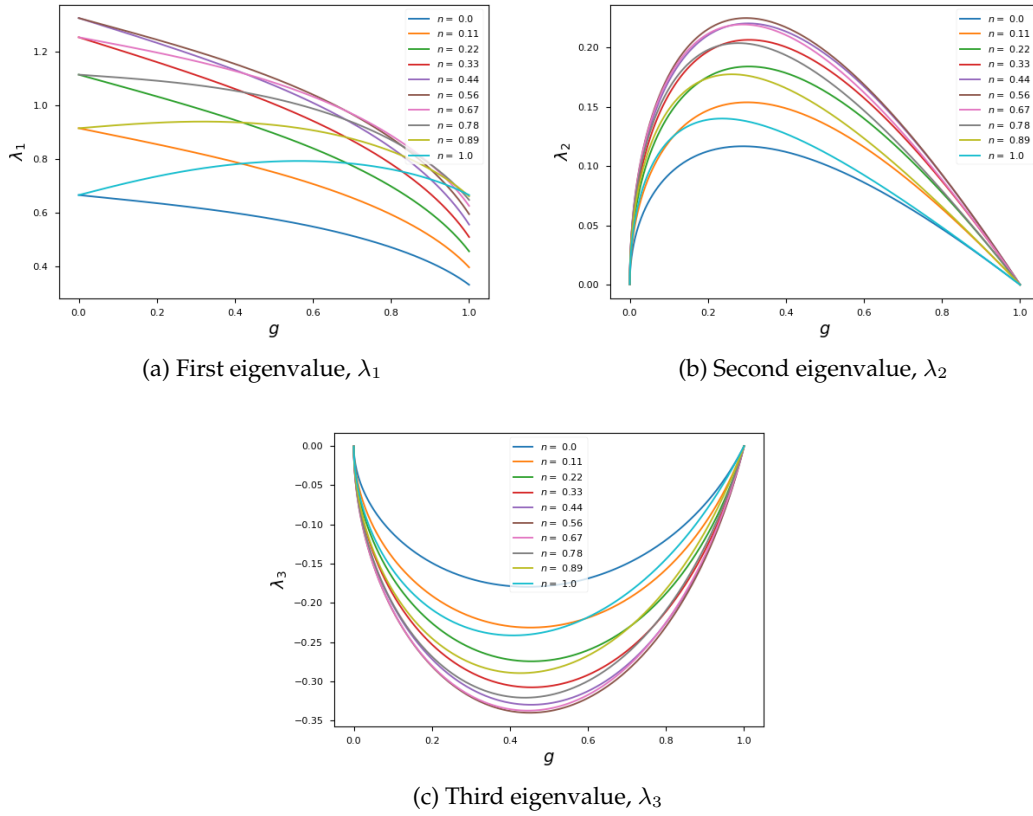


Figure 5.5: Numerical calculations for the first three eigenvalues of the matrix W . The last eigenvalue is simply -1 . W is defined as keeping track of the commutation relations $[\partial_x \eta_i(x), \eta_j(y)] = i\pi \delta(x-y) W_{ij}$. In the calculation of the eigenvalues it is assumed that $\tilde{t}_2 = atg \cos(\frac{\pi}{3}(n-1))$. By assuming the value of \tilde{t}_2 , the calculations are not exact, however it looks as if $\lambda_2 \approx -\lambda_3$. This is especially true for small g .

Note that we need to take the absolute value of the eigenvalues in our definition of $\vec{\Psi}$, since two of the eigenvalues of the matrix W are negative, which complicates things from just taking the square root directly. The Hamiltonian of the system is found to be:

$$\begin{aligned} \mathcal{H}^{(2)} &= (\partial_x \vec{\eta})^T \mathcal{T} \partial_x \vec{\eta} = (\partial_x \vec{\Psi})^T (|\Lambda|^{1/2})^T \mathcal{O} \mathcal{T} \mathcal{O}^T |\Lambda|^{1/2} \partial_x \vec{\Psi} = (\partial_x \vec{\Psi})^T \begin{pmatrix} |\lambda_1| & & & \\ & |\lambda_2| & & \\ & & |\lambda_3| & \\ & & & 0 \end{pmatrix} \partial_x \vec{\Psi} \\ &= \sum_{i=1}^3 |\lambda_i| (\partial_x \Psi_i)^2 \end{aligned} \quad (5.67)$$

The Hamiltonian is that of three chiral Luttinger liquids in the fields Ψ_i . The signs of the commutation relations, $[\partial_x \Psi_i(x), \Psi_j(x')] = \mp i 2\pi \delta(x-x') \delta_{i,j}$, will correspond to a (fractional) charge $\kappa = \pm 1$, for the negative and positive values of the commutations respectively. The sign of the fractional charge will determine the chirality of the field Ψ_i , since the velocity of the mode is proportional to the chirality (eq. 2.9 and 2.15). In other words, will the sign of the eigenvalues λ_i in fact determine the chirality of Ψ_i .

The eigenvalues λ_2 and λ_3 are zero in the extreme cases, where $g = 0$ and $g = 1$. The system is in this case a sole chiral Luttinger liquid in Ψ_1 . This agrees with our previous calculations, where we saw that a single chiral Luttinger liquid arises. In particular, if $g = 0$, the system corresponds to a Chiral Luttinger liquid in the field $\eta_1 = \frac{1}{2}\phi_2 - \frac{2}{3}(\theta_1 + 2\theta_2)$. If $g = 1$, the system simplifies to a chiral Luttinger liquid in $\eta_3 = \phi_2 - \frac{2}{3}(2\theta_1 + \theta_2)$.

In the general case the eigenvalues λ_2 and λ_3 are non-zero. And as such three chiral Luttinger liquids should arise in our model. This is more than expected, especially since our model started out as only described through two species of dual bosonic fields. However the first order calculations we just did,

Two-point correlation function	$g = 0$ or $t_2 = 0$	$g = 1$ or $t_1 = 1$
$\langle(\phi_1 - \phi_2/2)(x)(\phi_1 - \phi_2/2)(x')\rangle$	$\gamma_1 - \gamma_{12} + \frac{1}{4}\gamma_2 = 0$	
$\langle\phi_2(x)\phi_2(x')\rangle$	$\gamma_2 = 4/3$	$\gamma_2 = 1/3$
$\langle(\theta_1 + 2\theta_2)(x)(\theta_1 + 2\theta_2)(x')\rangle$	$\epsilon_1 + 4\epsilon_{12} + 4\epsilon_2 = 3/4$	
$\langle(2\theta_1 + \theta_2)(x)(2\theta_1 + \theta_2)(x')\rangle$	$4\epsilon_1 + 4\epsilon_{12} + \epsilon_2 = 3/16$	$4\epsilon_1 + 4\epsilon_{12} + \epsilon_2 = 3/4$

Figure 5.6: Two-point correlation functions in the extreme cases of single and pair hopping, and what their result translates to in terms of coefficients γ_{ij} and ϵ_{ij} . A successful theory should yield these results for $g = 0$ and $g = 0$, in our interpolated Hamiltonian (eq. 5.53).

showed interestingly that $\lambda_2 \approx -\lambda_3$. To realise why this is interesting one needs to remember that a normal Luttinger liquid can be constructed from two counter-propagating chiral Luttinger liquids. Let $\Psi_2 = \Phi - \Theta$ and $\Psi_3 = \Phi + \Theta$, where $[\partial_x \Theta(x), \Phi(x')] = -i\pi\delta(x - x')$. Crucially these two fields still commute by the canonical commutation relations, $[\partial_x \Psi_2, \Psi_3] = 0$. Their total Hamiltonian is:

$$\begin{aligned} \mathcal{H}_{\Psi_2, \Psi_3}^{(2)} &= (|\lambda_2| + |\lambda_3|)((\partial_x \Phi)^2 + (\partial_x \Theta)^2) - (|\lambda_2| - |\lambda_3|)(\partial_x \Phi \partial_x \Theta + \partial_x \Theta \partial_x \Phi) \\ &\approx (|\lambda_2| + |\lambda_3|)((\partial_x \Phi)^2 + (\partial_x \Theta)^2) \end{aligned} \quad (5.68)$$

The system is thus the sum of a chiral Luttinger liquid in $\Psi = \Psi_1$ and regular Luttinger liquid in Φ and Θ with Luttinger parameter $K = 1$:

$$\mathcal{H}^{(2)} = |\lambda_1|(\partial_x \Psi)^2 + (|\lambda_2| + |\lambda_3|)((\partial_x \Phi)^2 + (\partial_x \Theta)^2) \quad (5.69)$$

In conclusion, it is seen that the system is the sum of three chiral Luttinger liquids, which themselves show a notable resemblance to the sum of a chiral Luttinger liquid and a regular Luttinger liquid. It is thus argued that the free field part of the Hamiltonian, (eq. 5.53), is made up of two bosonic modes - the chiral Luttinger liquid and the regular Luttinger liquid.

5.3.2 Correlation Functions

The system is described by three chiral Luttinger liquids, Ψ_i , $i = 1, 2, 3$. The two-point correlation functions in Ψ_i were previously calculated and is, no matter the value of $|\lambda_i| > 0$:

$$\langle\Psi_i(x, t)\Psi_j(x', t')\rangle = -2 \cdot \log \left| \frac{(x - x') - v(t - t')}{a} \right| \delta_{i,j} \quad (5.70)$$

While $|\lambda_i| = 0$ renders the correlation function equal to zero. The original fields ϕ_j and θ_k can be expressed in terms of the fields Ψ_i through an inverse transformation of (eqs. 5.66 and 5.59):

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \theta_1 \\ \theta_2 \end{pmatrix} = \underbrace{\mathcal{S}^{-1} \mathcal{O} |\Lambda|^{1/2}}_{\mathcal{M}} \vec{\Psi} \quad (5.71)$$

The two-point correlation functions of the original fields are found by squaring each element in the matrix \mathcal{M} and multiplying the sum of the first three elements in each row. This statement makes more sense when it is written out. For example for ϕ_1 we have:

$$\begin{aligned} \langle\phi_1(x)\phi_1(x')\rangle &= \left\langle \left(\sum_i \mathcal{M}_{1i} \Psi_i(x) \right) \left(\sum_j \mathcal{M}_{1j} \Psi_j(x') \right) \right\rangle = \sum_{i,j} \mathcal{M}_{1i} \mathcal{M}_{1j} \langle\Psi_i(x)\Psi_j(x')\rangle \\ &\propto - \sum_{i=\{1,2,3\}} 2(\mathcal{M}_{1i})^2 \log |x - x'| \end{aligned} \quad (5.72)$$

We define the equal time two-point correlation functions of the fields ϕ_i and θ_i :

$$\langle\phi_i(x)\phi_j(x')\rangle \equiv -\gamma_{ij} \log |x - x'| \quad \langle\theta_i(x)\theta_j(x')\rangle \equiv -\epsilon_{ij} \log |x - x'| \quad (5.73)$$

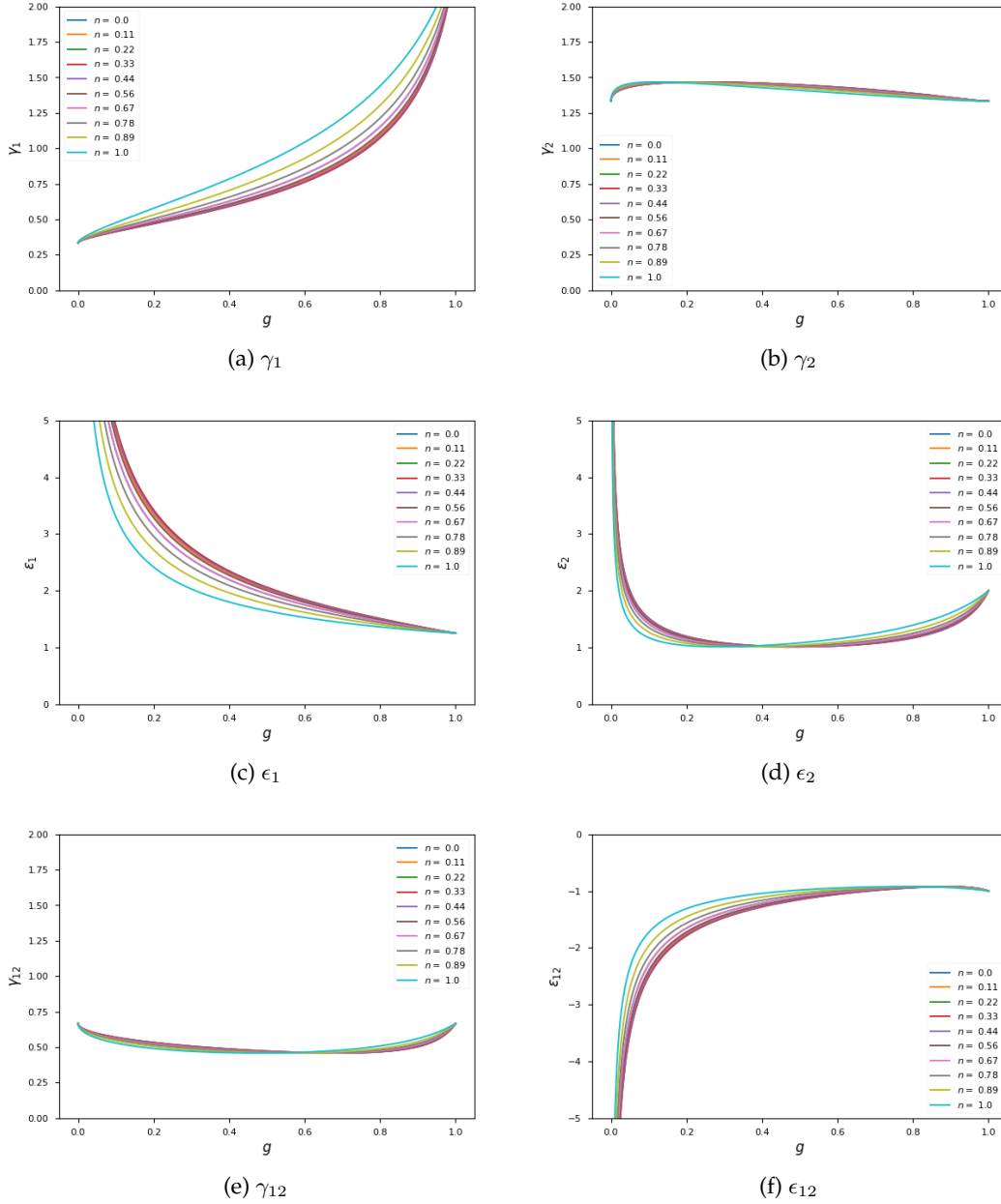


Figure 5.7: Numerical results of the coefficients of the two points correlation functions of the fields ϕ_i and θ_i . For $i = 1, 2$ we have defined $\langle \phi_i(x)\phi_i(x') \rangle = -\gamma_i \log |x - x'|$ and $\langle \theta_i(x)\theta_i(x') \rangle = -\epsilon_i \log |x - x'|$. A mixing of the fields makes it such that the cross terms, $\langle \phi_1(x)\phi_2(x') \rangle = -\gamma_{12} \log |x - x'|$ and $\langle \theta_1(x)\theta_2(x') \rangle = -\epsilon_{12} \log |x - x'|$, are non-zero. The plots show the values of the different coefficients for varying filling, n , and the relative strength of the hopping of a pair of Fock parafermions, $g = t_2/(t_1 + t_2)$.

Note that the two-point correlation functions of our original fields are not orthogonal, so the mixed correlation functions are generally non-zero. I.e. in general $\gamma_{12} = \gamma_{21} \neq 0$ and $\epsilon_{12} = \epsilon_{21} \neq 0$. Furthermore, for ease we denote $\gamma_{ii} = \gamma_i$ and $\epsilon_{ii} = \epsilon_i$. We study the predictions of these coefficients, γ_{ij} and ϵ_{ij} , for varying g and n . The calculations are made numerically using Python and the Numpy package. The resulting plots are shown in (fig. 5.7).

At the extremes $g = 0$ and $g = 1$, we have already calculated a few values that the coefficients, γ_{ij} and ϵ_{ij} , should obey. In (fig. 5.6) we sum up these values, and show how they translate to γ_{ij} 's and ϵ_{ij} 's. One of the predictions is that $\gamma_2 = 4/3$ at $g = 0$, while $\gamma_2 = 1/3$ at $g = 1$. This behaviour is however not what we see in the plot of γ_2 , (fig. 5.7b), where it is seen that $\gamma_2 = 4/3$ at both $g = 0$ and $g = 1$. The other two-point correlation functions follow the same tendency - the values obtained for small g

Term	Relevancy Criteria/Scaling Dimension
$M_\theta \cos\left(\frac{2\pi n x}{a} - 2(\theta_1 + 2\theta_2)\right)$	$n = 1$ and $\Delta_1 = 2(\epsilon_1 + 4\epsilon_2 + 4\epsilon_{12}) < 2$
$\frac{2\tilde{t}_2}{a^2} \cos(\theta_1)$	$\Delta_2 = \epsilon_1/2 < 2$
$\frac{2\tilde{t}_1}{a^2} \cos(2\phi_1 - \phi_2)$	$\Delta_3 = (4\gamma_1 + \gamma_2 - 4\gamma_{12})/2 < 2$

Figure 5.8: Criteria for the relevancy of the cosine-terms in the Hamiltonian (eq. 5.53). For the terms to be relevant in the RG sense their scaling dimensions need be lower than the dimension of the system, $D = 2$.

resemble the expected results, but the same is not true for the values at $g \approx 1$. This discrepancy might partly be explained by our approximation that $2n_1 + n_2 \approx n_2 = n/2$ in the coefficient \tilde{t}_2 . However this approximation was shown to be the case at large $g \approx 1$, hence the disagreement to the study of the pair hopping Hamiltonian would actually be expected to be less for large values of g .

The eigenvalues λ_2 and λ_3 both go to zero when $g \rightarrow 0$ and $g \rightarrow 1$. Notice how the fields Ψ_i are defined as the multiple of the inverse of the eigenvalues, cf. (eq. 5.66). This means that the fields Ψ_2 and Ψ_3 will inexpediently blow up at the ends, where $g \rightarrow 0$ and $g \rightarrow 1$. The blowing up of the fields makes numerical calculations less certain. This likely explains why we do not get the expected values of the power-laws for $g = 0$ and $g = 1$.

5.3.3 Interaction Terms

So far the analysis has not considered the cosine interaction terms in the Hamiltonian, (eq. 5.53). We saw in the study of the single hopping Hamiltonian that these terms, when relevant, can pin down a field, which will gap out a bosonic mode of the description.

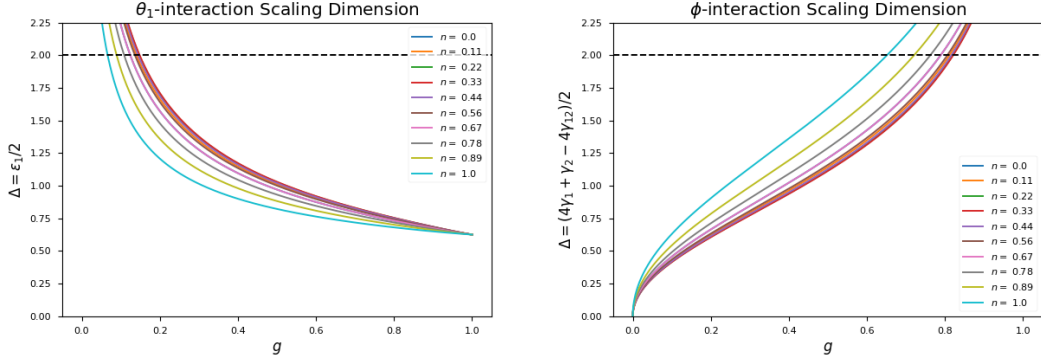
Two cosine terms can gap out two different bosonic modes, only if the terms commute. If the two terms do not commute the one that is the most relevant will "win" resulting in that mode being gapped out while the other runs freely [12]. The cosine term, $M_\theta \cos\left(\frac{2\pi n x}{a} - 2(\theta_1 + 2\theta_2)\right)$, commutes with the rest of the interaction terms, $\frac{2}{a^2}\tilde{t}_1 \cos(2\phi_1 - \phi_2)$ and $\frac{2}{a^2}\tilde{t}_2 \cos(\theta_1)$, and can as such simultaneously gap out the fields, $\theta_1 + 2\theta_2$, with either θ_1 or $2\phi_1 - \phi_2$ being gapped, depending on the relevancy of the cosine-terms. The two cosine-terms in θ_1 and $2\phi_1 - \phi_2$ cannot be simultaneously gapped however, since they do not commute. It is necessary to determine which term is most relevant to see which field will be gapped out. Notice that the cosine terms are proportional to \tilde{t}_1 and \tilde{t}_2 respectively, which hints as us why the explanation that was used comparing the strength of these hopping energies, (fig. 5.4), showed a good approximation of the phase diagram (fig. 3.1).

Whether a term in the Hamiltonian will be relevant or not, is determined by its scaling dimension (up to first order RG). A cosine-term, $\cos(\alpha\theta_1 \pm \beta\theta_2)$, will have the scaling dimension $\Delta = (\epsilon_1\alpha^2 + \epsilon_2\beta^2 \pm 2\epsilon_{12}\alpha\beta) / 2$, since:

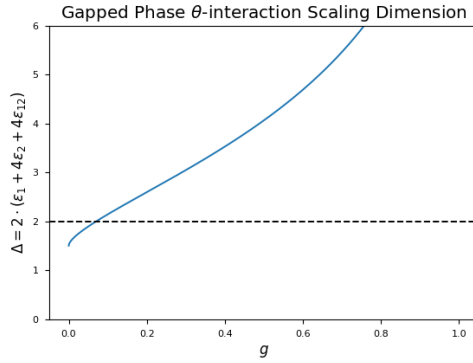
$$\begin{aligned} \langle e^{i(\alpha\theta_1(x) \pm \beta\theta_2(x))} e^{-i(\alpha\theta_1(x') \pm \beta\theta_2(x'))} \rangle &= \exp\left(-\frac{1}{2} \left\langle (\alpha(\theta_1(x) - \theta_1(x')) \pm \beta(\theta_2(x) - \theta_2(x')))^2 \right\rangle\right) \\ &\propto \exp\left(-\frac{1}{2} \left(-2\alpha^2 \langle \theta_1(x)\theta_1(x') \rangle - 2\beta^2 \langle \theta_2(x)\theta_2(x') \rangle \pm (-2\alpha\beta (\langle \theta_1(x)\theta_2(x') \rangle + \langle \theta_2(x)\theta_1(x') \rangle)) \right)\right) \\ &= |x - x'|^{-(\epsilon_1\alpha^2 + \epsilon_2\beta^2 \pm 2\epsilon_{12}\alpha\beta)} \end{aligned}$$

The exponent is double the scaling dimension of the cosine-term. A similar calculation can be done for a cosine-term in the ϕ -fields, $\cos(\alpha\phi_1 \pm \beta\phi_2)$, where $\Delta = (\gamma_1\alpha^2 + \gamma_2\beta^2 \pm 2\gamma_{12}\alpha\beta) / 2$. The cosine-terms are relevant (to the first order RG) if their scaling dimensions are less than that of the system, which in our case is $D = 2$. (Fig. 5.8) sums up when the different interaction terms of the Hamiltonian (eq. 5.53) are relevant by comparing their scaling dimension to the scaling dimension of the system, $\Delta < D = 2$. In (fig. 5.9) the actual numerical values of the scaling dimensions of the interaction terms are plotted.

The scaling dimensions of the interaction-terms, Δ_i for $i = 1, 2, 3$ (see fig. 5.8 for definitions), as they are in (fig. 5.9) show a couple of tendencies. First of all the scaling dimension $\Delta_1 < 2$ for $g < 0.1$, whereby $\theta_1 + 2\theta_2$ is gapped out at $n = 1$ for small value of g . This result agrees with the analysis of the single



(a) Scaling dimension, Δ_2 , of the term $\frac{2\tilde{t}_2}{a^2} \cos(\theta_1)$. (b) Scaling dimension, Δ_3 , of the term $\frac{2\tilde{t}_1}{a^2} \cos(2\phi_1 - \phi_2)$.



(c) Scaling dimension, Δ_1 , of the term $M_\theta \cos\left(\frac{2\pi n x}{a} - 2(\theta_1 + 2\theta_2)\right)$, for filling factor $n = 1$.

Figure 5.9: Scaling dimension of the cosine-interaction terms of the Hamiltonian (eq. 5.53). The black striped line signifies the limit for which the interaction is relevant in our model, and scaling dimensions above the line will render the term irrelevant.

hopping system that said such a gapping out of this field would take place. The scaling dimensions of the other cosine-terms, Δ_2 and Δ_3 , are both below $D = 2$ for values of $g \in [0.2, 0.8]$. However, over all the possible values of g , at least one of the terms is relevant. It is thus guaranteed that one bosonic mode will always be gapped out, according to this analysis. We assume that the bosonic mode that is gapped out is the mode that has the lowest scaling dimension. This creates a phase transition in the system where the sites may be occupied by a single Fock parafermion, $n_1 \neq 0$, when $2\phi_1 - \phi_2$ is gapped out, to one where single occupancy is not allowed, $n_1 = 0$, due to the gapping out of the θ_1 -field. This phase transition happens in this direction for increasing g for all filling factors, n . The point where $\Delta_2 = \Delta_3$ denotes the boundary of the phase transition. In (fig. 5.10) this phase transition is found numerically.

If the phase diagram (fig. 5.10) is compared to the phase diagram (fig. 3.1) made by Mahyaeh et al. [5], a few changes are immediately visible. First of all, we do not observe a phase where two bosonic modes are gapless, which was observed for n around 1 and suitable g in (fig. 3.1). Second, the fully gapped phase that arises if both of the interaction-terms, $\frac{2\tilde{t}_2}{a^2} \cos(\theta_1)$ and $\frac{2\tilde{t}_1}{a^2} \cos(2\phi_1 - \phi_2)$, are relevant, is much smaller and only observed for $n = 1$ in values $g < 0.1$. In Mahyaeh et al. this phase was observed for $g < 0.45$. Third of all, the phase transition from the ($n_1 \neq 0$)-phase to the ($n_1 \approx 0$)-phase takes place at too low g 's as compared to Mahyaeh et al. Note that to this last point we can move the phase transition boundary left and right by multiplying the argument of the cosine term $\frac{2\tilde{t}_2}{a^2} \cos(\theta_1)$, by some appropriate factor. It is not set in stone that this factor is 1, since we based the argumentation of this term on requiring the field θ_1 is pinned to 0 in the pair hopping Hamiltonian.

It should be kept in mind that our calculations are made up to the first order of renormalisation group theory, and furthermore utilises perturbation theory. Difference between our result and the numerical result by Mahyaeh et al. would be expected. With the exception of the exclusion of a phase where two

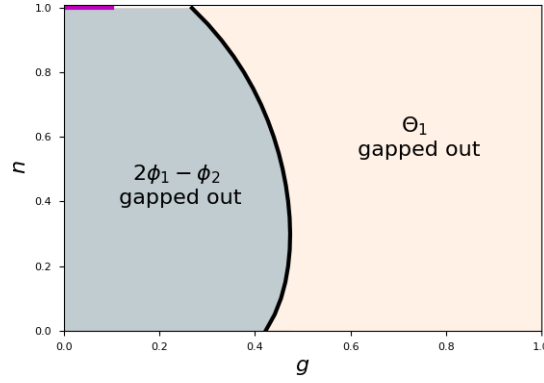


Figure 5.10: Phase diagram of the interpolated single and pair \mathbb{Z}_3 Fock parafermions hopping Hamiltonian (eq. 5.53), based on first order RG and scaling dimensions of the interaction fields. In the left region sites may be occupied by a single Fock parafermion. The phase transition changes this such that single occupancy is not allowed in the right region, ie. $n_1 = 0$. The purple region at the top of the plot, is the fully gapped region, where the remaining un-gapped bosonic field is gapped out by the term $M_\theta \cos\left(\frac{2\pi n x}{a} - 2(\theta_1 + 2\theta_2)\right)$.

bosonic modes exist, the phase diagram (fig. 5.10) portrays an overall structure similar to the earlier numerical calculations. It offers us a good qualitative explanation of the processes happening in the nearest-neighbour Fock parafermion hopping Hamiltonian: We see that one phase for which $n_1 = 0$ exists, which entails only hopping of pairs of Fock parafermions is happening in this region. For smaller g 's a phase exists that allows for both single and pair hopping, but gaps out the bosonic field $2\phi_1 - \phi_2$. Furthermore, a fully gapped phase appears for $n = 1$ at small g 's.

Our analysis encourages further work to be done on the system. Of interest in particular would be to study the phase transitions at higher orders of the renormalisation group, where other factors than the scaling dimensions plays a factor. It could also be fruitful to include the interaction term, $\frac{2}{a^2} \tilde{t}_2 \cos(\theta_1)$, in the calculation of the fields. Notice that the term comes from the realisation that $\partial_x \theta_1 = 0$ in the pair hopping system. If we assume it is small, the cosine term can be Taylor-expanded yielding the Hamiltonian:

$$\begin{aligned}
H' = \int_0^L dx \left[\tilde{t}_1 \left(\frac{1}{2} \partial_x \phi_2 - \frac{2}{3} (\partial_x \theta_1 + 2\partial_x \theta_2) \right)^2 + \tilde{t}_1 \left(\partial_x \phi_1 - \frac{1}{2} \partial_x \phi_2 \right)^2 \right. \\
+ \tilde{t}_2 \left(\partial_x \phi_2 - \frac{2}{3} (2\partial_x \theta_1 + \partial_x \theta_2) \right)^2 + \frac{\tilde{t}_2}{a^2} (\partial_x \theta_1)^2 \\
\left. - M_\theta \cos\left(\frac{2\pi n x}{a} - 2(\theta_1 + 2\theta_2)\right) - \frac{2\tilde{t}_1}{a^2} \cos(2\phi_1 - \phi_2) \right] \quad (5.74)
\end{aligned}$$

However, this Hamiltonian is valid in the right phase only, since this is the only region where we can assume $\partial_x \theta_1 = 0$.

Notice that we have assumed that \tilde{t}_2 ignored the single site occupation, since it would otherwise become negative for $n_1 > 5/8$. That a constant like \tilde{t}_2 that should not be negative, is negative, is often an indicator that the description has some problems. A possible explanation could be that in the Taylor expansion of (eq. 5.35) the assumption that the bosonic fields ϕ_k and θ_k are slow oscillating, so their partial derivatives may be expanded around zero, is not true. The interaction terms and the symmetries of the system will effectively pin down the fields as discussed multiple times already. The pinning might happens such that zero is not a good point to Taylor expand about. We try to see if this type of pinning can be the explanation for the shortcomings of our model. We assume that the combination of fields $a \left(\partial_x \phi_2 - \frac{2}{3} (2\partial_x \theta_1 + \partial_x \theta_2) \right)$ are pinned around some value δ . The second order Hamiltonian of the system is thus:

$$H_2'^{(2)} = \frac{t_2}{a} \cos\left(\frac{\pi}{3} (4n_1 + 2n_2 - 1) - \delta\right) \int_0^L dx \left(a \left(\partial_x \phi_2 - \frac{2}{3} (2\partial_x \theta_1 + \partial_x \theta_2) \right) - \delta \right)^2 \quad (5.75)$$

We can square out the integrand with respect to the value δ , but notice doing so will only yield some boundary terms that we will ignore and a constant, which can be ignored. Thus the only influence the introduction of δ has was changing the factor in front of the integral, such that:

$$\tilde{t}_2 \longrightarrow at_2 \cos\left(\frac{\pi}{3}(4n_1 + 2n_2 - 1) - \delta\right) \quad (5.76)$$

We have done the numerical calculations and found the phase diagram for different values of $\delta \in \{\pi, \pi/3, 2\pi/3, \dots\}$. None of these calculation yielded something that would be considered more correct with respect to the phase diagram (fig. 3.1), however.

Chapter 6

Conclusion and Outlook

In this thesis we effectively studied the anyonic particles called Fock parafermions, which are a type of emergent particles defined by Cobanera and Ortiz in 2014 [3], and derived field-theoretical description of these particles. This description was derived from a constructive bosonisation of the Weyl hard-core bosons, which are defined from the Fock parafermions through a Fradkin-Kadanoff transformation. The resulting bosonised description of the \mathbb{Z}_2 -symmetric Fock parafermion annihilation and creation operators at site j , F_j and F_j^\dagger , was shown to be:

$$\begin{aligned} F_j &\equiv \left[e^{i\phi_1(x_j)} + \sum_{k=2}^{p-1} e^{i(\phi_k(x_j) - \phi_{k-1}(x_j))} \right] \exp \left(\frac{2\pi i}{p} n(j-1) - \frac{2i}{p} \sum_{k=1}^{p-1} k (\theta_k(x_j) - \theta_k(x_1)) \right) \\ F_j^\dagger &\equiv \left[e^{-i\phi_1(x_j)} + \sum_{k=2}^{p-1} e^{-i(\phi_k(x_j) - \phi_{k-1}(x_j))} \right] \exp \left(-\frac{2\pi i}{p} n(j-1) + \frac{2i}{p} \sum_{k=1}^{p-1} k (\theta_k(x_j) - \theta_k(x_1)) \right) \end{aligned} \quad (6.1)$$

Where the dual bosonic fields, ϕ_k and θ_k , commute by the following relationship:

$$[\theta_k(x), \phi_{k'}(x')] = -i\pi\delta_{k,k'}\Theta(x-x') \quad \Theta(x-x') = \begin{cases} 1 & \text{for } x > x' \\ 0 & \text{for } x \leq x' \end{cases} \quad (6.2)$$

Of special note is that the description requires $p-1$ pairs of dual bosonic fields ϕ_k and θ_k . A description like this has not been previously considered. Even though, that more than just one bosonic field is required to describe the Fock parafermions is backed up by the numerical studies of the \mathbb{Z}_3 Fock parafermions [5], that found the system under the right conditions has a central charge $c=2$.

The newly-defined bosonised Fock parafermions were used to develop a low-energy field theoretical description of the nearest-neighbour hopping Hamiltonian of the \mathbb{Z}_3 Fock parafermions (eq. 5.1). The study of the Hamiltonian was split up in three cases: the study of the single hopping Hamiltonian, where $g=0$; the pair hopping Hamiltonian with $g=1$; and a general case where both types of hopping was allowed and the relative strength between them was varied by an interpolation variable $0 < g < 1$.

The field theoretical description of the the \mathbb{Z}_3 Fock parafermion nearest-neighbour hopping model predicted that the system could be described by a single chiral Luttinger liquid in the extreme cases, where either only single hopping or pair hopping is allowed. This system has not been studied in the literature as much as its older brother, the regular Luttinger liquid, but a connection of it to the edge modes of the fractional quantum Hall effect has been made [21]. Earlier suggestions for bosonisation pointed to the Luttinger liquid rather than this chiral version as a as a low energy field-theoretical description of the Fock parafermions [4]. That the system should be described by a chiral Luttinger liquid rather than a Luttinger liquid is likely owing to the fact that a chirality is inherent in the commutation relation of the Fock parafermions (eq. 1.28).

Furthermore, the two-point correlation functions of the chiral Luttinger liquids were calculated, and this result was used to calculate the two-point correlation functions of the Fock parafermions in the nearest-neighbour hopping Hamiltonian. The obtained values are summed up in (fig. 6.1) and the power law behaviours agree with earlier numerical calculations [4, 5]. For more see (figs. 5.1 and 5.2). It was

Single Hopping $g = 0$	Pair Hopping, $g = 1$
$G_1(x, x+r) \propto e^{\frac{\pi i}{3}(2\rho r-1)} \cdot r^{-\frac{2}{3}}$	$G_1(x, x+r) = 0$
$ G_2(x, x+r) \propto r^{-17/12}$	$G_2(x, x+r) \propto e^{\frac{\pi i}{3}(\rho r-1)} \cdot r^{-\frac{2}{3}}$

Figure 6.1: Two-point correlation functions of the \mathbb{Z}_3 Fock parafermions tight-binding model. $G_1(x, x+r) = \langle F_x^\dagger F_{x+r} \rangle$ and $G_2(x, x+r) \equiv \langle (F_x^\dagger)^2 (F_{x+r})^2 \rangle$. The obtained values agree with earlier numerical calculations [4, 5].

seen that the \mathbb{Z}_3 -symmetry of the \mathbb{Z}_3 Fock parafermions, coupled with the fact that the bosonisation was defined in terms of $(p-1)$ pairs of dual bosonic fields, were at the base of our successful predictions.

For general g the field-theoretical limit of the \mathbb{Z}_3 Fock Parafermion nearest-neighbour hopping Hamiltonian was calculated to be:

$$\begin{aligned}
H = \int_0^L dx \left[\tilde{t}_1 \left(\frac{1}{2} \partial_x \phi_2 - \frac{2}{3} (\partial_x \theta_1 + 2 \partial_x \theta_2) \right)^2 + \tilde{t}_1 \left(\partial_x \phi_1 - \frac{1}{2} \partial_x \phi_2 \right)^2 \right. \\
+ \tilde{t}_2 \left(\partial_x \phi_2 - \frac{2}{3} (2 \partial_x \theta_1 + \partial_x \theta_2) \right)^2 - M_\theta \cos \left(\frac{2\pi n x}{a} - 2(\theta_1 + 2\theta_2) \right) \\
\left. - \frac{2}{a^2} (\tilde{t}_1 \cos(2\phi_1 - \phi_2) + \tilde{t}_2 \cos(\theta_1)) \right] \quad (6.3)
\end{aligned}$$

Where we have defined:

$$\tilde{t}_1 = 2at(1-g) \cos \left(\frac{\pi}{3} (2n-1) \right) \quad \tilde{t}_2 = atg \cos \left(\frac{\pi}{3} (4n_1 + 2n_2 - 1) \right) \quad (6.4)$$

It was shown that this system gives rise to three chiral Luttinger liquids. Numerical calculations showed that two of the counter-propagating chiral Luttinger liquids could indeed be approximated by a regular Luttinger liquid. This constricts the number of bosonic modes of the system from three to two, which is what we expected the theory to describe.

Moreover, it was shown how the cosine terms in (eq. 6.3) gave rise to an interaction that effectively pins the fields in the argument of the cosines, when the cosine-terms themselves were relevant. A phase transition between two phases consisting of a single bosonic mode, where one phase allowed the tight-binding sites to be occupied by a single Fock parafermions to one that did not, was observed for increasing g . Furthermore, a fully gapped phase at $n=1$ and $g < 0.1$ was observed. The phase diagram derived by the model can be observed in (fig. 5.10). The description failed to capture a phase, where the two bosonic modes of the system would be simultaneously gapless, which earlier work prescribed would exist [5]. In consequence, further work is encouraged to study this system. A couple of points for further research which might remedy the shortcoming of our model are:

- Renormalisation group analysis for the combined single and pair hopping Hamiltonian (eq. 6.3). Including higher orders in the calculations could be the way to expand the derived phase diagram, such that it includes a phase whose central charge $c=2$, where two bosonic modes are allowed;
- Finding another way of transforming the interpolated Hamiltonian (eq. 6.3), such that it does not depend on some fields that blow up at the ends where $g \rightarrow 0$ and $g \rightarrow 1$;
- Finding a way of determining the ratio n_1/n_2 for specific values of g and n . The hopping parameter \tilde{t}_2 was shown to depend explicitly on these values. In the analysis we assumed $n_1=0$ for when calculating \tilde{t}_2 . This is however a gross simplification, which made the calculations of the interpolated Hamiltonian (eq. 6.3) less accurate.

Furthermore, we have not explicitly made the calculations that shows the Fock parafermions are abelian versions of the non-abelian parafermions. This would need to be done to render credence to the validity of our bosonisation, since we assumed abelian bosonisation could be used.

In conclusions, in this thesis we have discovered the consequences of the bosonisation of the \mathbb{Z}_3 Fock parafermions as a proof of concept. The bosonisation (eq. 6.1) is applicable also to general \mathbb{Z}_p Fock parafermions. Further work is encouraged for the study of the general case. One point of interest could be to study the case $p=2k$, since it bridges the gap to fractional fermions.

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Appendix A

Bosonisation of Fermions and Bosons and the Luttinger Liquid

In this appendix we will review the theory behind the bosonisation of bosons and fermions. This technique consists of a clever way of rewriting the creation and annihilation operators of some interacting particles mapping them to free scalar fields theories. This technique works most often only in the low energy limit, where fluctuations are not too pronounced, and is as such a field theoretical low energy approximation of the system.

There are two approaches to bosonisation: fields theoretical and constructive bosonisation [32, pp. 4-5]. In this appendix we will recount the field theoretical approach to bosonisation of the fermions and bosons. The derivation is based on chapter 3 of [34] the lecture notes [27].

Density Operator and Phenomelological Bosonisation

Consider a 1-dimensional chain of particles located at sites $x = x_j$ for $j \in [1 \dots l]$. The density operator of such chain can be explained by the operator:

$$\rho(x) = \sum_{j=1}^N \delta(x - x_j) \quad (\text{A.1})$$

To count the number of particles before the point x_j we introduce the field θ such that:

$$\theta(x_j) = 2\pi j \quad (\text{A.2})$$

This field has its definition expanded such that it is continuous in x . It is assumed to behave nicely such that it is monotonically increasing. We define the function:

$$f(x) = -i \left(e^{i\theta(x)} - 1 \right) \quad (\text{A.3})$$

Notice that this function has zeroes when $x = x_j$. The following identity:

$$\delta(f(x)) = \sum_{j \in \text{zeroes of } f} \frac{1}{|\partial_x f(x_j)|} \delta(x - x_j) = \sum_{j \in \text{zeroes of } f} \frac{1}{|\partial_x \theta(x_j)|} \delta(x - x_j) \quad (\text{A.4})$$

Notice that the function $f(x)$ is zero when $\theta(x) = 2\pi j$. We have also assumed $\theta(x)$ is monotonically increasing, and can thus omit taking the absolute value of its spatial derivative. In other words we may recast $\delta(x - x_j)$ in terms of the function $\theta(x)$. Hereby we find that:

$$\rho(x) = \sum_{j=1}^N \partial_x \theta(x) \delta(\theta(x) - 2\pi j) \quad (\text{A.5})$$

We solve this equation by Fourier transformation the summand. Poisson's integration formula guarantees that the sum stays the same:

$$\rho(x) = \frac{\partial_x \theta(x)}{2\pi} \sum_{q=-\infty}^{\infty} e^{iq\theta(x)} \quad (\text{A.6})$$

So far the function θ has kept track of all the particle in the field. But if the number of particles are N and the one-dimensional chain has length L , we may define $\rho_0 = N/L$, which is the average density. Now we can recast the definition of θ such that it only accounts for the derivation from this mean. In particular we let:

$$\theta(x) \longrightarrow 2\pi\rho_0 x - 2\theta(x) \quad (\text{A.7})$$

Whereby the density becomes:

$$\rho(x) = \left(\rho_0 - \frac{\partial_x \theta(x)}{\pi} \right) \sum_{q=-\infty}^{\infty} e^{i2q(\pi\rho_0 x - \theta(x))} \quad (\text{A.8})$$

Notice that the density operators of both bosons and fermions are bosonic. Hence we will find that $[\theta(x), \theta(x')] = 0$, in different $x \neq x'$. Summing over large distances compared to the average distance between the particles all other powers of $q \neq 0$ sum to zero. Thus, the part of the density outside the sum over q turns out to dominate in many applications.

We note that the density operator can be constructed from the single particle operators $\psi(x)$. Namely, $\psi^\dagger(x)\psi(x) = \rho(x)$. This construction is well known from quantum mechanics.

Bosons

We make an ansatz that we may write the bosonic operator as:

$$\psi_B^\dagger(x) = \sqrt{\rho(x)} e^{-i\phi(x)} \quad (\text{A.9})$$

Notice that this operator obeys that $\psi_B^\dagger(x)\psi_B(x) = \rho(x)$. For the operator to be bosonic we must require that it obeys the relationship:

$$[\psi_B(x), \psi_B^\dagger(x')] = \delta(x - x') \quad (\text{A.10})$$

If $x = x'$ we find that this equation gives us:

$$e^{i\phi(x)} \rho(x) e^{-i\phi(x)} - \rho(x) = \delta(x - x') \quad (\text{A.11})$$

Multiplying this equation by $e^{-i\phi(x)}$ from the left on both sides yields:

$$\left[\rho(x), e^{-i\phi(x)} \right] = e^{-i\phi(x)} \delta(x - x') \quad (\text{A.12})$$

We insert the definition of the density operator (eq. A.8) with only the slow oscillating parts, $q = 0$. Furthermore, using that $[A, e^{-iB}] = -i[A, B]e^{-iB}$, we find:

$$\left[\frac{\partial_x \theta(x)}{\pi}, \phi(x') \right] = -i\delta(x - x') \quad (\text{A.13})$$

Hence the two fields θ and ϕ obey this relationship. This relationship is very fundamental so rather than just being a coincidence we let it be one of the defining features of the bosonisation theory. Integrating over x (notice that x' will be a constant under this integration), will yield a commutation relationship:

$$[\theta(x'), \phi(x')] = -i\pi\Theta(x - x') \quad (\text{A.14})$$

Where $\Theta(x)$ is the Heaviside step function.

With this commutation relation in mind it can be shown that the fast oscillating terms of (eq. A.8) do not contribute anything to the commutation relation. The bosonised version of the bosonic field is expressed as:

$$\psi_B^\dagger(x) = \sqrt{\rho_0 - \frac{\partial_x \theta(x)}{\pi}} \sum_q e^{-i(2q)(\pi\rho_0 x - \theta(x))} e^{-i\phi(x)} \quad (\text{A.15})$$

Fermions

The fermionic operators, $\psi_F^\dagger(x)$, can be defined from the bosonic operators $\psi_B^\dagger(x)$ by changing the sum over q such that $2q \rightarrow 2q + 1$, hence:

$$\psi_F^\dagger(x) = \sqrt{\rho_0 - \frac{\partial_x \theta(x)}{\pi}} \sum_q e^{-i(2q+1)(\pi\rho_0 x - \theta(x))} e^{-i\phi(x)} \quad (\text{A.16})$$

The addition of this term in the exponent makes it such that the operators, $\psi_F^\dagger(x)$ and $\psi_F(x')$, anti-commute. This is quite easily derived from application of the commutation relations (eqs. A.13 and A.14) to the anti-commutator $\{\psi_F^\dagger(x), \psi_F(x')\}$. The calculations will be omitted here.

Notice that it is often expedient if we linearise this mode around the Fermi surface. Around the Fermi surface we have $\rho_0 = k_F/\pi$, where k_F is the fermi wavevector. Linearising effectively means that only the lowest orders (of oscillation) are allowed. Notice two such exist, one for $q = 0$ and one for $q = 1$. As such we may write:

$$\psi_F^\dagger(x) \approx N \left(e^{ik_F x} e^{-i(\phi(x) + \theta(x))} + e^{-ik_F x} e^{-i(\phi(x) - \theta(x))} \right) \quad (\text{A.17})$$

Where N is some normalisation constant. On the lattice with lattice constant a this is $N = \sqrt{1/2\pi a}$

Luttinger Liquids

The Luttinger liquid arises as a low energy-field theoretical description of the one-dimensional tight-binding model Hamiltonian with quartic density-density interactions:

$$H = -t \sum_r \left(c_{r+a}^\dagger c_r + \text{H.c.} \right) + U \sum_r c_{r+a}^\dagger c_{r+a} c_r^\dagger c_r \quad (\text{A.18})$$

Where c_r^\dagger and c_r are the creation and annihilation operators in the site r on the lattice of length L and lattice constant a . In the field theoretical description we assume the model becomes a continuum where we can describe the creation and annihilation operators in terms of the bosonised description that was just derived. In this section we will derive the Luttinger liquid description of the fermions, but it can likewise be done for one-dimensional chain of bosons or a p-wave superconductor. We define the density operator:

$$\rho(x) = \lim_{a \rightarrow 0} \psi_F^\dagger(x) \psi_F(x+a) \quad (\text{A.19})$$

Through a first order Taylor expansion of the fields:

$$\phi(r+a) \approx \phi(r) + a \partial_x \phi(r) \quad \theta(r+a) \approx \theta(r) + a \partial_x \theta(r) \quad (\text{A.20})$$

It can be shown that the density operator approximates as:

$$\rho(x) = -\frac{\partial_x \theta}{\pi} + \frac{i}{2\pi a} \left(e^{-2ik_F x} e^{2i\theta(x)} - e^{2ik_F x} e^{-2i\theta(x)} \right) \quad (\text{A.21})$$

The same Taylor expansion yields that the product of two density operators becomes:

$$\rho(x+a)\rho(x) \longrightarrow \frac{1}{\pi^2} \left[(1 - \cos(2k_F a)) (\partial_x \theta(x))^2 + \frac{1}{a} \sin(2k_F a) \partial_x \theta(x) + \frac{1}{2a^2} \cos(2k_F a) \right] \quad \text{for } a \rightarrow 0 \quad (\text{A.22})$$

We insert this result in the Hamiltonian $H_{\text{int}} = U \int dx (\rho(x))^2$. The last term can be ignored as it is a simple constant. The second term with integrate to give us a boundary term in θ , which we will dismiss. The interaction Hamiltonian is then:

$$H_{\text{int}} = U \int dx \frac{1}{\pi^2} \left[(1 - \cos(2k_F a)) (\partial_x \theta(x))^2 \right] \quad (\text{A.23})$$

We now direct our attention towards the the free part of the Hamiltonian. We rewrite the creation and annihilation operators in their bosonised description. The Taylor expansion that was used earlier yields:

$$\begin{aligned} \psi_F^\dagger(x+a) \psi_F^\dagger(x) &= \frac{1}{2\pi a} \left[2 \sin k_F a - ai \sin k_F a \partial_x \phi - a \cos k_F a \partial_x \theta - a^2 \sin k_F a (\partial_x \phi)^2 \right. \\ &\quad \left. - a^2 \sin k_F a (\partial_x \theta)^2 + 2i \cos k_F a \partial_x \phi \partial_x \theta \right] \end{aligned} \quad (\text{A.24})$$

Plus some fast-oscillating terms that will be ignored. We add the Hermitian conjugate to this expression, whereby the imaginary terms disappear. The free part of the Hamiltonian (eq. A.18) is thus:

$$H_0 = -t \int dx 2 \cdot \frac{-a \sin k_F a}{2\pi} ((\partial_x \phi)^2 + (\partial_x \theta)^2) = \int dx \frac{u}{2\pi} ((\partial_x \phi)^2 + (\partial_x \theta)^2) \quad (\text{A.25})$$

Where we have defined $u = 2ta \sin k_F a$. We have integrated out the $2 \sin k_F a$ and $a \cos k_F a \partial_x \theta$ -terms, which would yield a constant and a boundary term that can be ignored. The total Hamiltonian of the system is:

$$H = \frac{v_F}{2\pi} \int dx K (\partial_x \phi)^2 + \frac{1}{K} (\partial_x \theta)^2 \quad (\text{A.26})$$

Where we have defined the Luttinger parameters s.t.:

$$\begin{cases} v_F K &= u \\ \frac{v_F}{K} &= u + \frac{2U}{\pi} (1 - \cos 2k_F a) \end{cases} \implies \begin{cases} v_F &= \sqrt{u \left(u + \frac{2U}{\pi} (1 - \cos 2k_F a) \right)} \\ K &= \sqrt{\frac{u}{u + \frac{2U}{\pi} (1 - \cos 2k_F a)}} \end{cases} \quad (\text{A.27})$$

The Hamiltonian (eq. A.26) is the famous Luttinger liquid. It is very useful to describe a system as a Luttinger liquid since so many types of systems becomes Luttinger liquids in their lowest order description. Furthermore are correlation functions easy to calculate since the Luttinger liquid yields the correlation functions of the two fields:

$$\langle \phi(x) \phi(0) \rangle = -\frac{1}{2K} \ln |x| \quad , \quad \langle \theta(x) \theta(0) \rangle = -\frac{K}{2} \ln |x| \quad (\text{A.28})$$

Appendix B

Chiral Luttinger Liquid Two Point Correlation Functions Calculations

In this appendix we show the explicit calculations used in the derivation of the two point correlations function of the chiral Luttinger liquid. The correlation function (Eq. 2.29) is evaluated by noting that the $i\omega$ may be taken out of the denominator of the integrand/Green's function:

$$G(k, \omega) = \left(\frac{\omega^2}{v_\varphi} - ik\omega \right)^{-1} = -\frac{1}{i\omega} \cdot \frac{1}{\frac{i\omega}{v_\varphi} + k} \quad (\text{B.1})$$

Hereby the calculation of the correlation function splits up into two integrals, such that:

$$\langle \varphi(x, \tau) \varphi(x', \tau') \rangle = -\frac{1}{\alpha(2\pi)^2} \int d\omega \frac{1}{i\omega} e^{-i\omega(\tau-\tau')} \int dk e^{ik(x-x')} \frac{1}{\frac{i\omega}{v_\varphi} + k} \quad (\text{B.2})$$

Note that ω and v_φ both are real numbers. The integration over k can thus be carried out using Cauchy's Residue Theorem [25]. To use this theorems we must require that the function $e^{ik(x-x')}$ is analytical in $k \in \mathbb{C}$, which may be denoted as $k = k_1 + ik_2$, for $k_1, k_2 \in \mathbb{R}$. Thus:

$$e^{ik(x-x')} = e^{ik_1(x-x')} e^{-k_2(x-x')} \quad (\text{B.3})$$

This function will blow up and be non analytical for $k_2 \rightarrow \pm\infty$ if $k_2(x-x') < 0$. Hence for $(x-x') > 0$ we must require that $k_2 > 0$ as well, and we can choose a path in the upper half of the complex plane to integrate over. On the other hand will $(x-x') < 0$ require the imaginary part of k to be negative, hence a path in the lower part of the complex plane is to be chosen to be integrated over. Thus by choosing a closed contour path $C_{+/-}$ in the upper or lower half of the imaginary plane in the cases $(x-x') > 0$ or $(x-x') < 0$, respectively, and if the point $-i\omega/v_\varphi$ is contained in the closed contour, it will act as a simple pole of the function, rendering the integration:

$$\oint_{C_{+/-}} dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} = 2\pi i e^{\frac{\omega}{v_\varphi}(x-x')} \quad (\text{B.4})$$

Where we have assumed $v_\varphi \neq 0$. If the point $-i\omega/v_\varphi$ is not inside the closed contour on the other hand, the integral will evaluate to zero. Note that $\omega, v_\varphi \in \mathbb{R}$, so $-i\omega/v_\varphi$ will be in the upper half of the complex plane for $\omega/v_\varphi < 0$, while it will be in the lower half if $\omega/v_\varphi > 0$.

Assume we consider the case $(x-x') > 0$, so that we may construct the curve C_+ going from $-R$ to R in the real numbers and in a half circle from R back to $-R$, denoted by Γ_+ , for some $R > |\omega/v_\varphi|$. In this case the integral can be deconstructed as:

$$\oint_{C_+} dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} = \int_{-R}^R dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} + \int_{\Gamma_+} dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} \quad (\text{B.5})$$

The integral over the curve $C\Gamma_+$ can be rewritten as taking the integral over the curve with $k = Re^{i\theta}$ for $\theta \in [0, \pi]$. Substituting this value into the integral and by the triangle inequality we find:

$$\left| \int_{\Gamma_+} dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} \right| = \left| \int_0^\pi d\theta iRe^{i\theta} \frac{e^{iRe^{i\theta}(x-x')}}{Re^{i\theta} + i\frac{\omega}{v_\varphi}} \right| \leq \int_0^\pi d\theta \frac{Re^{-R(x-x')\Im(e^{i\theta})}}{\left| Re^{i\theta} + \frac{i\omega}{v_\varphi} \right|} \quad (\text{B.6})$$

Note that $\Im(e^{i\theta}) > 0$, since we are in the upper part of the complex plane. If we let $R \rightarrow \infty$, the term $Re^{i\theta}$ will dominate the denominator, hence:

$$\left| \int_{\Gamma_+} dk e^{ik(x-x')} \frac{1}{\frac{i\omega}{v_\varphi} - k} \right| \leq \int_0^\pi d\theta e^{-R(x-x')\Im(e^{i\theta})} \rightarrow 0 \text{ for } R \rightarrow \infty \quad (\text{B.7})$$

Thus this integral will evaluate to zero in this $R \rightarrow \infty$ limit. Hence for $(x - x') > 0$:

$$\int_{-\infty}^{\infty} dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} = \oint_{C_+} dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} = 2\pi i e^{\frac{\omega}{v_\varphi}(x-x')} \Theta\left(-\frac{\omega}{v_\varphi}\right) \quad (\text{B.8})$$

We have introduced the heavyside step function to make sure the pole of the function, $-i\omega/v_\varphi$, is contained inside of the closed curve C_+ . If $(x - x') < 0$, on the other hand, we may only integrate over the lower half of the complex plane, getting by the same steps as before:

$$\int_{-\infty}^{\infty} dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} = - \oint_{C_-} dk e^{ik(x-x')} \frac{1}{k + \frac{i\omega}{v_\varphi}} = -2\pi i e^{\frac{\omega}{v_\varphi}(x-x')} \Theta\left(\frac{\omega}{v_\varphi}\right) \quad (\text{B.9})$$

We can ignore the integration with value $\omega = 0$, since we can exclude a single point from an integral and not change the value, since the measure of a single point is zero [25].

Evidently the sign of the velocity has a great impact on the two point correlation functions, as its value determines the domain for the integration over ω . Note the fractional charge κ of the chiral Luttinger liquid determines the sign of the velocity, v , so it will be most telling to consider each of these cases by themselves.

However, before we proceed to calculating the exact correlation functions we observe that the correlation function will generally be on the form:

$$|\langle \varphi(x, \tau) \varphi(x', \tau') \rangle| = \frac{1}{2\pi\alpha} \int d\omega \frac{e^{\omega\left(\frac{x-x'}{v} - i(\tau-\tau')\right)}}{\omega} = \frac{1}{2\pi\alpha} \int d\omega \frac{e^{\omega\xi}}{\omega} \quad (\text{B.10})$$

Where we have defined:

$$\xi \equiv \frac{x-x'}{v} - i(\tau-\tau') = \frac{x-x'}{v} + (t-t') \quad (\text{B.11})$$

No matter the value of ξ , the integral in (Eq. B.10) will diverge when integrating over the both the domains $(0, \infty)$ and $(-\infty, 0)$. To get around this issue we introduce some cut-off frequencies, Λ_{\min} and Λ_{\max} , such that we may carry out the integration by parts:

$$\int_0^\infty d\omega \frac{e^{\omega\xi}}{\omega} \rightarrow \int_{\Lambda_{\min}}^{\Lambda_{\max}} d\omega \frac{e^{\omega\xi}}{\omega} = [e^{\omega\xi} \log \omega]_{\omega=\Lambda_{\min}}^{\omega=\Lambda_{\max}} - \xi \int_{\Lambda_{\min}}^{\Lambda_{\max}} d\omega e^{\omega\xi} \log \omega \quad (\text{B.12})$$

Two extreme cases arises for evaluating the integral on the right hand side. The first case is to let $\Lambda_{\min} \ll \Lambda_{\max}$. The second case lets $\Lambda_{\min} \approx \Lambda_{\max}$, which is to be considered for the renormalisation group in a later section. If $\Lambda_{\min} \ll \xi \ll \Lambda_{\max}$, the latter integral on the right hand side of (Eq. B.12) can be approximated as an integral from 0 to ∞ . If $\xi > 0$ the integral will diverge and go to ∞ . Hence we will only evaluate the integral for $\xi = -|\xi| < 0$:

$$\begin{aligned} \int_0^\infty d\omega e^{-\omega|\xi|} \log \omega &= \int_0^\infty \frac{du}{|\xi|} e^{-u} \log \frac{u}{|\xi|} = \frac{1}{|\xi|} \left(\int_0^\infty du e^{-u} \log u - \left(\int_0^\infty du e^{-u} \right) \log |\xi| \right) \\ &= \frac{1}{|\xi|} (-\gamma - \log |\xi|) \end{aligned} \quad (\text{B.13})$$

Where we have substituted $u = w |\xi|$ in the integral, and where γ is the Euler-Mascheroni constant. The first term of the right hand side of (Eq. B.12) is in this limit:

$$[e^{\omega\xi} \log \omega]_{\omega=\Lambda_{\min}}^{\omega=\Lambda_{\max}} = e^{\Lambda_{\max}\xi} \log \Lambda_{\max} - e^{\Lambda_{\min}\xi} \log \Lambda_{\min} \approx -\log \Lambda_{\min} \quad (\text{B.14})$$

Since letting $\Lambda_{\max} \rightarrow \infty$ for $\xi < 0$, the first term disappears due to l'Hôpital's rule:

$$e^{\Lambda_{\max}\xi} \log \Lambda_{\max} = \frac{\log \Lambda_{\max}}{e^{\Lambda_{\max}|\xi|}} \rightarrow \frac{\Lambda_{\max}^{-1}}{|\xi| e^{\Lambda_{\max}|\xi|}} = \frac{1}{\Lambda_{\max} |\xi| e^{\Lambda_{\max}|\xi|}} \rightarrow 0 \quad (\text{B.15})$$

And $\Lambda_{\min} \rightarrow 0$ will make $e^{\Lambda_{\min}\xi} \rightarrow 1$. Thus, for $\Lambda_{\min} \ll \xi \ll \Lambda_{\max}$:

$$\int_0^\infty d\omega \frac{e^{\omega\xi}}{\omega} \approx \begin{cases} -\log(|\xi| \Lambda_{\min}) - \gamma & \text{for } \xi < 0 \\ \infty & \text{for } \xi > 0 \end{cases} \quad (\text{B.16})$$

On the other hand the integral over the domain $(-\infty, 0)$ will become:

$$\int_{-\infty}^0 d\omega \frac{e^{\omega\xi}}{\omega} \rightarrow \int_{-\Lambda_{\max}}^{-\Lambda_{\min}} d\omega \frac{e^{\omega\xi}}{\omega} = -\int_{-\Lambda_{\min}}^{-\Lambda_{\max}} d\omega \frac{e^{\omega\xi}}{\omega} = -\int_{\Lambda_{\min}}^{\Lambda_{\max}} du \frac{e^{-u\xi}}{u} \quad (\text{B.17})$$

Where we have done the substitution $u = -\omega$. Carrying out the integration over u , by previous calculation we see, for $\Lambda_{\min} \ll \xi \ll \Lambda_{\max}$:

$$\int_{-\infty}^0 d\omega \frac{e^{\omega\xi}}{\omega} \approx \begin{cases} -\infty & \text{for } \xi < 0 \\ \log(|\xi| \Lambda_{\min}) + \gamma & \text{for } \xi > 0 \end{cases} \quad (\text{B.18})$$

Case $v = 0$

So far we have not considered the extreme case for which the system is $v = 0$. This instance can be achieved either by letting $\kappa = 0$ or $\epsilon = 0$. The case $\epsilon = 0$ is not very complicated, since this corresponds simply to letting everything, including the two point correlations functions, be zero. For $\kappa = 0$, the Hamiltonian becomes:

$$H_{\text{CLL},\kappa=0} = \epsilon \int dx (\partial_x \varphi)^2 \quad (\text{B.19})$$

As mentioned earlier, the absence of a dependence on the θ -field, will render the time derivative $\partial_t \varphi = 0$. The Lagrangian density is easily found through a Legendre transformation, since the time derivative $\partial_t \varphi = 0$:

$$\mathcal{L}_{\text{CLL},\kappa=0} = -\epsilon (\partial_x \varphi)^2 \quad (\text{B.20})$$

This Lagrangian yields the partition function:

$$Z_{\text{CLL},\kappa=0} = \int D\varphi \exp \left[\frac{\epsilon}{\hbar} \int dk d\omega \hat{\varphi}(-k, -\omega) k^2 \hat{\varphi}(k, \omega) \right] \quad (\text{B.21})$$

From this partition function we easily find the Green's function:

$$G_{\kappa=0}(k, \omega) = -\frac{\hbar}{\epsilon} \frac{1}{k^2} \quad (\text{B.22})$$

Taking the Fourier transformation over k and ω gives us the correlation functions in position space:

$$\langle \varphi(x, \tau) \varphi(x', \tau') \rangle_{\kappa=0} = -\frac{\hbar}{\epsilon} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega e^{-i\omega(\tau-\tau')} \int_{-\infty}^{\infty} dk \frac{e^{ik(x-x')}}{k^2} = -\frac{\hbar}{\epsilon} \frac{1}{2\pi} \delta(\tau - \tau') \int_{-\infty}^{\infty} dk \frac{e^{ik(x-x')}}{k^2} \quad (\text{B.23})$$

The integral can be rewritten as

$$\int_{-\infty}^{\infty} dk \frac{e^{ik(x-x')}}{k^2} = \int_0^\infty dk \frac{e^{ik(x-x')} + e^{-ik(x-x')}}{k^2} = 2 \int_0^\infty dk \frac{\cos(k(x-x'))}{k^2} \quad (\text{B.24})$$

This integral will blow up and diverge if we integrate from $x = 0$. Hence we introduce some cut-off frequencies, M_{\min} and M_{\max} to do the integration over. Integrating by parts yields:

$$\int_{M_{\min}}^{M_{\max}} dk \frac{\cos(k(x-x'))}{k^2} = - \left[\frac{\cos(k(x-x'))}{k} \right]_{k=M_{\min}}^{k=M_{\max}} - (x-x') \int_{M_{\min}}^{M_{\max}} dk \frac{\sin(k(x-x'))}{k} \quad (\text{B.25})$$

The latter integral converges in the limit $(M_{\min}, M_{\max}) \rightarrow (0, \infty)$ to $\frac{\pi}{2} \text{sgn}(x-x')$ [35, p. 110]. In this limit the first term will disappear for $k \rightarrow \infty$, but diverge for $k \rightarrow 0$. However for $k \rightarrow 0$ the cosine will approach 1. In total we get that the real space partition function for $\kappa = 0$:

$$\langle \varphi(x, t) \varphi(x', t') \rangle_{\kappa=0} = \frac{\hbar}{2\pi\epsilon} \left(\frac{2}{M_{\min}} + \pi |x-x'| \right) \delta(t-t') \quad (\text{B.26})$$

This result works for $x = x'$ too, hence we need not consider this case separately, as we did when considering the system for $\kappa \neq 0$. Note that the units of the above expression is in length, as \hbar has units energy times time, and $\delta(t-t')$ will have units one over time.

Defining the chiral Luttinger liquid on a lattice of size L and lattice spacing a , the natural definition of $M_{\min} = 1/L$ and $M_{\max} = 1/a$, whereby the equal time correlation functions are:

$$\langle \varphi(x, t) \varphi(x', t) \rangle_{\kappa=0} = \frac{\hbar}{2\pi\epsilon} (2L + \pi |x-x'|) \quad (\text{B.27})$$

In the thermodynamic limit, the latter term becomes irrelevant, hence the correlation is simply a constant:

$$\langle \varphi(x, t) \varphi(x', t) \rangle_{\kappa=0} = \frac{\hbar L}{\pi\epsilon} = \text{const} \quad (\text{B.28})$$

B.1 High Energy Limit

We have so far assumed $\Lambda_{\min} \gg |\xi| \gg \Lambda_{\max}$, which corresponds to a low energy approximation of the system. In the high energy limit we assume ξ is of the same order as Λ_{\max} . Furthermore are smaller values assumed to be negligible, which mwhich gives us the ability to let the cut-offs approach each other, such that $\Lambda_{\min} \approx \Lambda_{\max}$. We can define an infinitesimal dl such that:

$$\frac{\Lambda_{\max}}{\Lambda_{\min}} \approx 1 + dl \quad (\text{B.29})$$

Remember that through integration by parts:

$$\int_{\Lambda_{\min}}^{\Lambda_{\max}} d\omega \frac{e^{-\omega|\xi|}}{\omega} = \left[e^{-\omega|\xi|} \log \omega \right]_{\omega=\Lambda_{\min}}^{\omega=\Lambda_{\max}} + |\xi| \int_{\Lambda_{\min}}^{\Lambda_{\max}} d\omega e^{-\omega|\xi|} \log \omega \quad (\text{B.30})$$

The latter integral will approach zero in the limit $\Lambda_{\min} \approx \Lambda_{\max}$, since we can approximate the integral as:

$$\begin{aligned} \int_{\Lambda_{\min}}^{\Lambda_{\max}} d\omega e^{-\omega|\xi|} \log \omega &\approx \frac{\Lambda_{\max} - \Lambda_{\min}}{2} \left(e^{-\Lambda_{\max}|\xi|} \log \Lambda_{\max} - e^{-\Lambda_{\min}|\xi|} \log \Lambda_{\min} \right) \\ &\approx \frac{dl}{2} e^{-\Lambda_{\max}|\xi|} \log \frac{\Lambda_{\max}}{\Lambda_{\min}} \approx \frac{(dl)^2}{2} e^{-\Lambda_{\max}|\xi|} \end{aligned} \quad (\text{B.31})$$

This integral is infinitesimally small on the order of $(dl)^2$ and can thus be ignored. Since $\Lambda_{\min} \approx \Lambda_{\max}$, we may let $e^{-|\xi|\Lambda_{\min}} \approx e^{-|\xi|\Lambda_{\max}}$, whereby:

$$\int_{\Lambda_{\min}}^{\Lambda_{\max}} d\omega \frac{e^{-\omega|\xi|}}{\omega} \approx e^{-|\xi|\Lambda_{\max}} (\log \Lambda_{\max} - \log \Lambda_{\min}) \approx e^{-|\xi|\Lambda_{\max}} \log \frac{\Lambda_{\max}}{\Lambda_{\min}} \quad (\text{B.32})$$

Thus the correlation functions are:

$$\langle \varphi(x, t) \varphi(x', t') \rangle \approx -\frac{|\kappa|}{2} \cdot e^{-|\xi|\Lambda_{\max}} \log \frac{\Lambda_{\max}}{\Lambda_{\min}} \quad (\text{B.33})$$

And:

$$\langle \theta(x, t) \theta(x', t') \rangle \approx -\frac{1}{2|\kappa|} \cdot e^{-|\xi|\Lambda_{\max}} \log \frac{\Lambda_{\max}}{\Lambda_{\min}} \quad (\text{B.34})$$

What is important to note is that the correlation functions are exponentially decreasing, and very rapidly so, since Λ_{\max} in the exponent is a big number. Effectively the correlation functions are zero at the long-range orders that we will consider.

Appendix C

\mathbb{Z}_3 Fock Parafermion Nearest-Neighbour Hopping Hamiltonian Calculations

In this appendix we sum up a few of the calculations made in the derivation of the field theoretical description of the \mathbb{Z}_3 Fock parafermions nearest neighbour-hopping Hamiltonian.

Single Hopping Hamiltonian Continuum Limit Calculation

The Hamiltonian to evaluate:

$$H_1 = -t_1 \sum_{i=1}^{l-1} F_i^\dagger F_{i+1} + \text{H.c.} = -t_1 \sum_{i=1}^{l-1} B_i^\dagger U_i B_{i+1} + \text{H.c.} \quad (\text{C.1})$$

By our definitions we may calculate the single Fock parafermionic hopping of the Hamiltonian:

$$B_i^\dagger U_i B_{i+1} = \left(e^{-i\phi_1(x_i)} + e^{-i(\phi_2(x_i) - \phi_1(x_i))} \right) \omega^{-\frac{a}{\pi} (\partial_x \theta'_1(x_i) + 2\partial_x \theta'_2(x_i))} \left(e^{i\phi_1(x_{i+1})} + e^{i(\phi_2(x_{i+1}) - \phi_1(x_{i+1}))} \right) \quad (\text{C.2})$$

Where we have used the fact that the total number of parafermions is a conserved quantity, and will simply be a number. Hence n is a conserved quantity that can be absorbed in a pair of canonical, unitary transformed fields, θ'_j , by defining:

$$-a \left(\frac{\partial_x \theta'_1}{\pi} + 2 \frac{\partial_x \theta'_2}{\pi} \right) \equiv n - a \left(\frac{\partial_x \theta_1}{\pi} + 2 \frac{\partial_x \theta_2}{\pi} \right) \quad (\text{C.3})$$

By the BCH formula, we calculate the first terms of the expression above:

$$\begin{aligned} & \exp[-i\phi_1(x_i)] \exp \left[-ia \frac{2}{3} \left[\partial_x \theta'_1(x_i) + 2\partial_x \theta'_2(x_i) \right] \right] \exp[i\phi_1(x_{i+1})] \\ &= \exp \left(-i\phi_1(x_i) - i \frac{2a}{3} \left[\partial_x \theta'_1(x_i) + 2\partial_x \theta'_2(x_i) \right] + i\phi_1(x_{i+1}) + \frac{1}{2} \left[-i\phi_1(x_i), -i \frac{2}{3} a \partial_x \theta'_1(x_i) \right] \right) \\ &= \exp \left[i \left(\phi_1(x_{i+1}) - \phi_1(x_i) - \frac{2}{3} \left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i)) \right] - \frac{1}{3}\pi \right) \right] \end{aligned} \quad (\text{C.4})$$

Likewise, we calculate the other terms of the expression:

$$\begin{aligned} & \exp[-i\phi_1(x_i)] \exp \left[-i \frac{2}{3} \left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i)) \right] \right] \exp[i(\phi_2(x_{i+1}) - \phi_1(x_{i+1}))] \\ &= \exp \left[i \left(\phi_2(x_{i+1}) - \phi_1(x_{i+1}) - \phi_1(x_i) - \frac{2}{3} \left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i)) \right] - \frac{1}{3}\pi \right) \right] \end{aligned} \quad (\text{C.5})$$

And:

$$\begin{aligned}
& \exp[-i(\phi_2(x_i) - \phi_1(x_i))] \exp\left[-i\frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) - (\theta'_2(x_{i+1}) - \theta'_2(x_i))\right]\right] \exp[i\phi_1(x_{i+1})] \\
&= \exp\left(-i(\phi_2(x_i) - \phi_1(x_i)) - i\frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i))\right] + i\phi_1(x_{i+1})\right. \\
&\quad \left. + \frac{1}{2}\left[-i\phi_2(x_i), -i\frac{4}{3}\theta'_2(x_{i+1})\right] + \frac{1}{2}\left[i\phi_1(x_i), -i\frac{2}{3}\theta'_1(x_{i+1})\right]\right) \\
&= \exp\left[i\left(-\phi_2(x_{i+1}) + \phi_1(x_{i+1}) + \phi_1(x_i) - \frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i))\right] - \frac{1}{3}\pi\right)\right]\right]
\end{aligned} \tag{C.6}$$

And:

$$\begin{aligned}
& \exp[-i(\phi_2(x_i) - \phi_1(x_i))] \exp\left[-i\frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) - (\theta'_2(x_{i+1}) - \theta'_2(x_i))\right]\right] \exp[i(\phi_2(x_{i+1}) - \phi_1(x_{i+1}))] \\
&= \exp\left[i\left((\phi_2(x_{i+1}) - \phi_2(x_i)) - (\phi_1(x_{i+1}) - \phi_1(x_i)) - \frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i))\right] - \frac{1}{3}\pi\right)\right)\right]
\end{aligned} \tag{C.7}$$

Adding the hermitian conjugate to these expressions renders the term, $B_i^\dagger U_i B_{i+1} + \text{H.c.}$, equal to, assuming t_1 is real:

$$\begin{aligned}
& 2 \cos\left[\phi_1(x_{i+1}) - \phi_1(x_i) - \frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i))\right] - \frac{1}{3}\pi\right] \\
& + 2 \cos\left[\phi_2(x_{i+1}) - \phi_1(x_{i+1}) - \phi_1(x_i) - \frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i))\right] - \frac{1}{3}\pi\right] \\
& + 2 \cos\left[-\phi_2(x_i) + \phi_1(x_{i+1}) + \phi_1(x_i) - \frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i))\right] - \frac{1}{3}\pi\right] \\
& + 2 \cos\left[(\phi_2(x_{i+1}) - \phi_2(x_i)) - (\phi_1(x_{i+1}) - \phi_1(x_i)) - \frac{2}{3}\left[(\theta'_1(x_{i+1}) - \theta'_1(x_i)) + 2(\theta'_2(x_{i+1}) - \theta'_2(x_i))\right] - \frac{1}{3}\pi\right]
\end{aligned} \tag{C.8}$$

The single hopping Hamiltonian is thus:

$$\begin{aligned}
H_1 = -2t_1 \int_0^L \frac{dx}{a} & \left[\cos\left[a\partial_x\phi_1 - \frac{2}{3}a\left(\partial_x\theta'_1 + 2\partial_x\theta'_2\right) - \frac{1}{3}\pi\right] \right. \\
& + \cos\left[\phi_2 - 2\phi_1 + a\partial_x\phi_2 - a\partial_x\phi_1 - \frac{2}{3}a\left(\partial_x\theta'_1 + 2\partial_x\theta'_2\right) - \frac{1}{3}\pi\right] \\
& + \cos\left[-\phi_2 + 2\phi_1 + a\partial_x\phi_1 - \frac{2}{3}a\left(\partial_x\theta'_1 + 2\partial_x\theta'_2\right) - \frac{1}{3}\pi\right] \\
& \left. + \cos\left[a\partial_x\phi_2 - a\partial_x\phi_1 - \frac{2}{3}a\left(\partial_x\theta'_1 + 2\partial_x\theta'_2\right) - \frac{1}{3}\pi\right] \right]
\end{aligned} \tag{C.9}$$

In the second and third line we have used the approximation:

$$\phi_j(x_{i+1}) + \phi_j(x_i) = \phi_j(x_{i+1}) - (\phi_j(x_i) - \phi_j(x_i)) + \phi_j(x_i) \approx a\partial_x\phi(x_i) + 2\phi(x_i) \quad \text{for } j = 1, 2 \tag{C.10}$$

In the end, we need to substitute the θ'_j -fields that were considered by the θ_j -fields. Remember the θ'_j -fields came from having absorbed the constant number density, n , into the θ_j -fields, i.e. by the transformation:

$$-\frac{2}{3}a\left(\partial_x\theta'_1 + 2\partial_x\theta'_2\right) \longrightarrow -\frac{2}{3}a\left(\partial_x\theta_1 + 2\partial_x\theta_2\right) + \frac{2\pi}{3}n \tag{C.11}$$

Taylor Expansion of the Single Hopping Terms

The Taylor expansion of a general cosine-function in x around zero is:

$$\cos(kx + b) = \cos(b) - \sin(b)kx - \frac{\cos(b)}{2}k^2x^2 + O(x^3) \quad (\text{C.12})$$

Taylor expanding the first term in (Eq. 5.8) in a , we find:

$$\begin{aligned} & \cos \left[a\partial_x\phi_1 - \frac{2}{3}a(\partial_x\theta_1 + 2\partial_x\theta_2) + \frac{\pi}{3}(2n-1) \right] \\ & \approx \cos \left[\frac{\pi}{3}(2n-1) \right] - a \sin \left[\frac{\pi}{3}(2n-1) \right] \left(\partial_x\phi_1 - \frac{2}{3}(\partial_x\theta_1 + 2\partial_x\theta_2) \right) \\ & - \frac{a^2}{2} \cos \left[\frac{\pi}{3}(2n-1) \right] \left(\partial_x\phi_1 - \frac{2}{3}(\partial_x\theta_1 + 2\partial_x\theta_2) \right)^2 \end{aligned} \quad (\text{C.13})$$

While Taylor expanding the fourth term in (Eq. 5.8) in a , yields:

$$\begin{aligned} & \cos \left[a\partial_x\phi_1 - a\partial_x\phi_2 - \frac{2}{3}a(\partial_x\theta_1 + 2\partial_x\theta_2) + \frac{\pi}{3}(2n-1) \right] \\ & \approx \cos \left[\frac{\pi}{3}(2n-1) \right] - a \sin \left[\frac{\pi}{3}(2n-1) \right] \left(\partial_x\phi_2 - \partial_x\phi_1 - \frac{2}{3}(\partial_x\theta_1 + 2\partial_x\theta_2) \right) \\ & - \frac{a^2}{2} \cos \left[\frac{\pi}{3}(2n-1) \right] \left(\partial_x\phi_2 - \partial_x\phi_1 - \frac{2}{3}(\partial_x\theta_1 + 2\partial_x\theta_2) \right)^2 \end{aligned} \quad (\text{C.14})$$

Calculation of $G_2(x, x+r)$ Ignoring \mathbb{Z}_3 -Symmetry

We now consider the parafermionic correlation:

$$G_2(x, x+r) \equiv \langle (F_x^\dagger)^2 (F_{x+r})^2 \rangle = \left\langle (B_x^\dagger)^2 \left(\prod_{k=x}^{x+r-a} (U_k)^2 \right) (B_{x+r})^2 \right\rangle \quad (\text{C.15})$$

We ignore the \mathbb{Z}_3 -symmetry that makes $\omega^{4n_2} = \omega^{n_2}$. This is equivalent to naively squaring the operator U_k , such that:

$$\prod_{k=x}^{x+r-a} (U_k)^2 = e^{\frac{4\pi i}{3}\rho r} e^{-\frac{4i}{3}(\tilde{\theta}_2(x+r) - \tilde{\theta}_2(x))} \quad (\text{C.16})$$

Whereby the product:

$$\left\langle (B_x^\dagger)^2 \left(\prod_{k=x}^{x+r-1} (U_k)^2 \right) (B_{x+r})^2 \right\rangle = e^{\frac{\pi i}{3}(4\rho r - 4)} \left\langle e^{i2(\tilde{\phi}_2(x+r) - \tilde{\phi}_2(x)) - \frac{4i}{3}(\tilde{\theta}_2(x+r) - \tilde{\theta}_2(x))} \right\rangle \quad (\text{C.17})$$

The cummulant expansion yields:

$$\left\langle e^{i(\phi_2(x+r) - \phi_2(x))} \right\rangle \propto e^{-\frac{1}{2} \cdot (-2) \cdot 2^2 \cdot \left(-\frac{1}{3} \log|x+r-x|\right)} = r^{-\frac{4}{3}} \quad (\text{C.18})$$

And:

$$\left\langle e^{i(\theta_2(x+r) - \theta_2(x))} \right\rangle \propto e^{-\frac{1}{2} \cdot \left(\frac{4}{3}\right)^2 \cdot (-2) \cdot \left(-\frac{3}{4} \log|x+r-x|\right)} = r^{-\frac{4}{3}} \quad (\text{C.19})$$

Multiply these together, we notice get:

$$G_2(x, x+r) \propto e^{\frac{4\pi i}{3}(\rho r - 1)} r^{-\frac{8}{3}} \quad (\text{C.20})$$

This corresponds to the prediction made in [4], which was based on the argumentation of the system being an anyonic fluid which had its correlation function calculated in [31]. However the numerical results in [4] do not back up this prediction, where the result is closer to $|G_2(x, x+r)| \sim r^{-\frac{5}{3}}$.

Appendix D

Various Results

D.1 Cumulant Expansion

This rather short section simply states the cumulant expansion:

$$\langle e^{ikx} \rangle \approx e^{ik\langle x \rangle - \frac{k^2}{2}(\langle x^2 \rangle - \langle x \rangle^2)} \quad (\text{D.1})$$

D.2 Derivation of the $(d+1)$ -dimensional QFT Euler-Lagrange Equation

Consider a Lagrangian density in $(d+1)$ -dimensional space, such that $\mathcal{L}[\varphi, \partial_i \varphi]$, for $i \in [0, d]$, where 0 is the time dimension and $[1, d]$ are the spacial dimensions. The variation in the action, S is:

$$\delta S = \int d^{d+1}x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \sum_{i=0}^d \frac{\partial \mathcal{L}}{\partial (\partial_i \varphi)} \delta (\partial_i \varphi) \right) \quad (\text{D.2})$$

It is evident that:

$$\partial_i \left(\frac{\partial \mathcal{L}}{\partial (\partial_i \varphi)} \delta \varphi \right) = \partial_i \left(\frac{\partial \mathcal{L}}{\partial (\partial_i \varphi)} \right) \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_i \varphi)} \delta (\partial_i \varphi) \quad (\text{D.3})$$

Setting $\delta \varphi = 0$ at the spatial and temporal start and end points of the integration, integrating the above expression yield a left hand side of zero. Hence we see:

$$\delta S = \int d^{d+1}x \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \sum_{i=0}^d \partial_i \left(\frac{\partial \mathcal{L}}{\partial (\partial_i \varphi)} \right) \right) \delta \varphi \quad (\text{D.4})$$

Minimising the variation of the action for arbitrary $\delta \varphi$, we arrive at the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \sum_{i=0}^d \partial_i \left(\frac{\partial \mathcal{L}}{\partial (\partial_i \varphi)} \right) = 0 \quad (\text{D.5})$$

D.3 Level Spacing Statistics

Level spacing statistics is a technique to study the integrability of quantum systems by studying the statistics of the energy eigenstates of the Hamiltonian. The probability that the energy difference between to adjacent levels $s_n = E_{n+1} - E_n$ lies in a given interval $[s, s + ds]$ is considered. If the system is integrable the probability distribution will be Poissonian, ie. $P_P(s) \propto e^{-s}$. For non-integrable solutions the spectrum is conjectured to follow the rules of random matrix theory which leads to the probability distribution being a Wigner-Dyson surmise: $P_{WD}(s) \sim A s^\beta e^{-B s^2}$.

The value of β is most often determined by the symmetries of the system, where $\beta = 1$ is true for anti-unitary symmetries (time reversal invariance) and the probability distribution is called Gaussian

orthogonal ensemble, whereas for more general complex systems have $\beta = 2$ called Gaussian unitary ensemble. The term with s^β spreads out the probability distribution and is exactly what measures the level repulsion.

The article uses exact diagonalisation to simulate the energy levels of the system with open boundary conditions. And then show that they observe convergence to the Gaussian unitary ensemble fairly quickly for $p = 3$ and even quicker for $p = 6$, for a given number of particles N with varying the length of the system L . Furthermore they observe a crossover from a Gaussian unitary ensemble to a Gaussian orthogonal ensemble by increasing the tunneling strength of the hopping between neighbouring sites, quite possibly signifying a change in symmetry of the system, which they call for further work into.