FACULTY OF SCIENCE
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Master's Thesis

Continuous Wave Single Photon Transistor with Rydberg Atoms

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Abstract

In this thesis we develop a model of a single photon transistor, which consists of an ensemble of Rydberg atoms located inside a single-sided cavity, coupled to two driving fields. A 'signal' field incident on the ensemble can be reflected or lost, conditioned by the absence or presence of a 'control' field that is mapped to a collective Rydberg excitation, which leads to Rydberg blockade. The advantage of the current proposal compared to previous models, is that driving fields are continuously turned on throughout the entire protocol, leading to the continuous wave version of the single photon transistor, under impedance matching condition for a signal photon. Another advantage of our proposal, is that through Rydberg induced dephasing of the long lived Rydberg excitation, the protocol can lead to blockade of strong coherent multiphoton field. This long lived Rydberg excitation is possible, since the blocked signal field induces an effective dephasing on the excitation, through its loss. The proposed device, could be alternatively used as an efficient optical single-photon detector.

Acknowledgements

First of all, I would like to thank my main supervisor Anders Søndberg Sørensen for his wonderful idea that led to the realization of this work, who together with my co-supervisior Oleksandr Kyriienko, introduced me to the quantum thinking through countless fruitful discussions.

Afterwards, I would like to thank all the members of the Theoretical Quantum Optics group in Niels Bohr Institute, for their knowledge sharing during the weekly group meeting and for making everyday life in the institure a pleasent experience.

Special thanks to the Sofoklis Achillopoulos foundation, for their support through the scholarship for post-graduate studies abroad, that made this work and my studies in Denmark possible.

Thanks to Vasil Rokaj for proofreading part of the thesis, to Georgios Magdis for his mentoring and to Mary Chatzieleftheriou for providing me with the lucky coin.

Finally I would like to thank my family for their support.

Contents

In memory of Konstantinos Mavrikakis,
who sails in distant worlds.

Chapter 1

Introduction

Since the original proposal of a photon as a quantum of light in 1905 by Einstein, and its discovery by Compton in 1923, the field of quantum optics has always been in the frontline of the modern quantum science and technology.

One of the most recently discovered properties of quantum optical systems, is that they can serve as suitable platforms for the implementation of quantum information processing (QIP). The initial idea of quantum information processing was suggested by Feynman in 1982, motivated by the inability of classical computers to give quantitative predictions about the complicated quantum systems. Ever since, the field of QIP attracted massive scientific attention, being actively developing in the 1990s both from theoretical and experimental point of view. This triggered the growth of quantum technologies, for instance leading to proposals and implementation of secure communications protocols, e.g. quantum key distribution. Finally, the ultimate goal of inventing a universal quantum computer can be foreseen in the nearest future.

The rapid evolution of the quantum technologies demanded, the development of efficient tools - quantum gates - to control and modify quantum signals. These devices commonly rely on the optical and microwave signals, serving similar role to controls in conventional electronics, and are largely insipired by the preceding classical electrical circuits. One of the devices from this family is an optical quantum transistor, being an analog of a classical field effect transistor. Similart to its electronic counterpart, it is a device where a small optical 'control' field is used to switch on and off the propagation of another optical 'signal' field via a nonlinear optical interaction. The fundamental limiting case of an optical quantum transistor is a single-photon transistor, where the presence or absence of a single photon in the gate field, controls the propagation of the 'signal' field. The single photon transistor was proposed by Chang et. al. [1] for an atom coupled to a nanowire in 2007 and several alternative schemes have been proposed and realised experimentally the past decade.

In this thesis we study a single photon transistor model, which consists of an ensemble of Rydberg atoms located inside a single-sided cavity, coupled to two driving fields. A 'signal' field incident on the ensemble can be reflected or lost, conditioned by the absence or presence of a 'control' field that is mapped to a collective Rydberg excitation, which leads to Rydberg blockade. An advantage the current proposal compared to previous models, is that driving fields are continuously turned on throughout the entire protocol, leading to the continuous wave version of the single photon transistor. This largely simplifies the protocol and possible experimental realization of the transistor. The scheme relies on the impedance matching condition for a signal photon, working in the presence of a probe, on the contrary of previous proposals. Namely, once we send the single-photon 'control' signal, it can be mapped to a very long lived Rydberg excitation exploiting the dephasing processes imposed by the pobing. This leads to the blockade and reflection of a coherent multiphoton field, where the number

of scattered probe photon defines the gain transistor. Noteworthly, the proposed device could be used also as an efficient optical single-photon detector. In this case the protocol allows for the detection of a presence or absence of the single control field by measuring the reflected signal field . The efficient multiphoton blockade is enabled by the long lifetime of the Rydberg excitation, potentially leadung to the detection with large signal-to-noise ratio.

The thesis outline is:

Chapter 1 (current chapter) presents the motivation and objectives of this thesis.

Chapter 2 introduces open quantum systems, a formalism to describe them, and an effective method to significantly simplify the treatment of these systems.

Chapter 3 describes the interaction of a Ξ-scheme atomic ensemble coupled to two fields and its scattering dynamics for different systems. The studied systems are the atomic ensemble being confined inside a single-sided cavity, 2) being confined inside a two-sided cavity; and 3) being in the free space, where corresponding analogy between the free space and cavity model is derived. Finally the single-sided cavity case is described for the case an atomic ensemble of Rydberg atoms.

Chapter 4 derives the necessary conditions for the 'control' field to be impedance matched to the Rydberg excitation, responsible for the blockade of the 'signal' field and stay there until it dephases.

Chapter 5 describes the effective dephasing introduced to the Rydberg excitation, responsible for Rydberg blockade induced by the decay of the blockaded 'signal' field.

Chapter 6 outlines the protocol of the proposed continuous single photon quantum transistor by making use of the results derived in the previous chapters.

Chapter 7 concludes on this work and provides an outlook for further research on this topic.

Chapter 2

Open Quantum Systems

The axioms of quantum mechanics describe the behaviour of closed quantum systems that do not interact with their surrounding environment. Unfortunately, ideal closed quantum systems do not exist in nature and the measurements (observations) we make, are limited to a small fraction of a much larger quantum system. The inconsistency of the basic formalism of quantum mechanics with the description of open quantum systems, lies in the fact that through their interaction with their environment, they become correlated with it. As a consequence of these quantum correlations, system and environment can no longer be considered 2 different systems but a whole entity, described by an entangled pure state, that is no longer seperable. The open system is described by a mixed state, and its evolution is not unitary.

2.1 Density Operator

A system in a mixed state, that is a probabilistic ensemble of pure states, cannot be described by a state ket as one in a pure state. In order to describe it, we need to introduce the notion of the density operator

$$\hat{\rho} = \sum_i p_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| . \tag{2.1}$$

The sum is over a statistical ensemble, where p_i is the probability of the system being in the i-th state of the enemble $|\Psi_i\rangle$, given that ket states are normalized, meaning $\langle\Psi_i|\Psi_i\rangle = 1$. Since p_i is a probability, it is clear that the following relations are satisfied

$$0 \leq p_i \leq 1, \qquad \sum_i p_i = 1, \qquad \sum p_i^2 \leq, 1 \tag{2.2}$$

$$tr\hat{\rho} = \sum_n \left\langle\Psi_n\right| \hat{\rho} \left|\Psi_n\right\rangle = \sum_n \sum_i p_i \left\langle\Psi_n|\Psi_i\right\rangle \left\langle\Psi_i|\Psi_n\right\rangle = \sum_i p_i \left\langle\Psi_i|\Psi_i\right\rangle = \sum_i p_i = 1. \tag{2.3}$$

The last equality shows that the trace of the density operator is always 1, since p_i is a probability. A special case of the density operator is when all p_i vanish, except for the j-th one, $p_i = \delta_{ij}$, where we obtain

$$\hat{\rho} = \left|\Psi_j\right\rangle \left\langle\Psi_j\right| , \tag{2.4}$$

this is the density operator for the pure state $|\Psi_j\rangle$. From which we get the following property

$$\hat{\rho}^2 = \left|\Psi\right\rangle \left\langle\Psi|\Psi\right\rangle \left\langle\Psi\right| = \left|\Psi\right\rangle \left\langle\Psi\right| = \hat{\rho}, \tag{2.5}$$

for a pure state, and

$$\hat{\rho}^2 = \sum_i \sum_j p_i p_j \left|\Psi_i\right\rangle \left\langle\Psi_i|\Psi_j\right\rangle \left\langle\Psi_j\right| \neq \hat{\rho}, \tag{2.6}$$

for a mixed state. The above inequality is violated only for $p_i = \delta_{ij}$ which corresponds to the density operator for a pure state.

The last important relation is about the expectation value of some operator $\hat{A}$ in a mixed state.

$$\langle \hat{A} \rangle = \sum_i p_i \langle \Psi_i | \hat{A} | \Psi_i \rangle = \sum_i p_i \langle \Psi_i | (\sum_s a_s |\alpha_s\rangle \langle \alpha_s|)(\sum_j |\Psi_j\rangle \langle \Psi_j|) |\Psi_i\rangle = tr(\hat{\rho}\hat{A}), \tag{2.7}$$

where $|\alpha_s\rangle$ and a_s the eigenstates and eigenvalues of $\hat{A}$ respectively and we used the completeness relation for $|\Psi_j\rangle$. By rearranging the terms we obtain the following expression

$$\langle \hat{A} \rangle = \sum_j \langle \Psi_j | (\sum_i p_i |\Psi_i\rangle \langle \Psi_i|)(\sum_s a_s |\alpha_s\rangle \langle \alpha_s|) |\Psi_j\rangle = tr(\hat{\rho}\hat{A}), \tag{2.8}$$

It is also important to show the time evolution of the density operator in the Schrödinger picture, which is equivalent the Schrödinger equation.

$$\hat{\rho}(t) = \sum_i p_i |\Psi_i(t)\rangle \langle \Psi_i(t)|, \tag{2.9}$$

$$\begin{aligned} \dot{\hat{\rho}}(t) &= \sum_i p_i ((\frac{d}{dt}|\Psi_i(t)\rangle) \langle \Psi_i(t)| + |\Psi_i(t)\rangle (\frac{d}{dt} \langle \Psi_i|(t))) \\ &= -\frac{i}{\hbar} \sum_i p_i ((\hat{\mathcal{H}}|\Psi_i(t)\rangle) \langle \Psi_i(t)| + |\Psi_i(t)\rangle (\langle \Psi_i|(t)\hat{\mathcal{H}})) \\ &= -\frac{i}{\hbar}(\hat{\mathcal{H}}\hat{\rho}(t) - \hat{\rho}(t)\hat{\mathcal{H}}) = -\frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{\rho}(t)]. \end{aligned} \tag{2.10}$$

2.2 Master equation

Now that we have introduced the necessary formalism we can study an open quantum system. We consider the case of a simple harmonic oscillator, coupled with its surounding environment, which is modeled as a reservoir of a large number of harmonic oscillators. The entire system is described by the Hamiltonian

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_s + \hat{\mathcal{H}}_r + \hat{\mathcal{H}}_{sr}, \tag{2.11}$$

where

$$\hat{\mathcal{H}}_s = \hbar\Omega \hat{a}^\dagger \hat{a}, \tag{2.12}$$

is the Hamiltonian of the small system, which we will call system for the rest of the chapter. Ω is the frequency of the system and $\hat{a}, \hat{a}^\dagger$ the creation and annihilation operators of the system with commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$.

$$\hat{\mathcal{H}}_r = \hbar \sum_i \omega_i \hat{b}_i^\dagger \hat{b}_i, \tag{2.13}$$

is the Hamiltonian of the reservoir, where ω_i is the frequency of the i-th mode of the reservoir and $\hat{b}_i, \hat{b}_i^\dagger$ the creation and annihilation operators of the i-th mode of the reservoir with commutation relation $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij}$.

$$\hat{\mathcal{H}}_{sr} = \hbar \sum_i (g_i \hat{a}^\dagger \hat{b}_i + g_i^* \hat{b}_i^\dagger \hat{a}), \tag{2.14}$$

is the interaction Hamiltonian between the reservoir and the system and g_i is the coupling constant between the i-th harmonic oscillator of the reservoir and the system.

The reservoir, which consists of a large number of degrees of freedom, is described by a time-independent density operator $\hat{\rho}_r$, in thermal equilibrium at the temperature T. A valid description for the systems modeled throughout this thesis,

$$\hat{\rho}_r(\hat{\mathcal{H}}_r) = \frac{e^{-\beta\hat{\mathcal{H}}_r}}{tr_r(e^{-\beta\hat{\mathcal{H}}_r})}. \tag{2.15}$$

We assume that the system and the reservoir start interacting at $t = t_0$ and that they did not exhibit any correlations at that moment. Then the initial state of the entire system is described by the tensor product of the density operators of the two subsystems, because the two subsystems are totally independent at $t = t_0$,

$$\hat{\rho}_{sr}(t_0) = \hat{\rho}_s(t_0) \otimes \hat{\rho}_r(\hat{\mathcal{H}}_r). \tag{2.16}$$

Since we are interested in the system's evolution and the obsevations we make are strictly on observables of the system, we will calculate the expectation value of a system operator $\hat{\mathcal{C}}$, i.e. it acts only on the states of the system,

$$\langle\hat{\mathcal{C}}(t)\rangle = tr_{sr}(\hat{\mathcal{C}}\hat{\rho}_{sr}(t)) = tr_s(\hat{\mathcal{C}}tr_r\hat{\rho}_{sr}(t)) \equiv tr_s(\hat{\mathcal{C}}\hat{\rho}_s(t)), \tag{2.17}$$

where tr_{sr} is the trace over both the system and the reservoir and $\hat{\mathcal{C}}$ is a system operator alone, for this reason it is not affected when we trace over the reservoir in the second equality. In the last equality of the above equation, we have defined the reduced density operator for the system $\hat{\rho}_s(t)$), as the trace over the reservoir of the density operator of the entire system.

It is easy to see that if we know $\hat{\rho}_s(t)$, we can determine the expectation value of any system operator at all times. This property makes it an extremely useful tool for studying open quantum systems and this is why, the rest of this section will be focused on the derivation of the master equation for the reduced denstity operator. In order to tackle this problem, we move to the interaction picture using the following unitary transformations of the reduced density operator $\hat{\rho}_{sr}$ and the interaction Hamiltonian (2.14)

$$\hat{\rho}_{sr,I} = e^{i(\hat{\mathcal{H}}_s+\hat{\mathcal{H}}_r)(t-t_0)/\hbar}\hat{\rho}_{sr}e^{-i(\hat{\mathcal{H}}_s+\hat{\mathcal{H}}_r)(t-t_0)/\hbar}, \tag{2.18}$$

$$\hat{\mathcal{H}}_{sr,I} = e^{i(\hat{\mathcal{H}}_s+\hat{\mathcal{H}}_r)(t-t_0)/\hbar}\hat{\mathcal{H}}_{sr}e^{-i(\hat{\mathcal{H}}_s+\hat{\mathcal{H}}_r)(t-t_0)/\hbar} = \hbar\sum_i g_i\hat{a}^\dagger e^{i\Omega(t-t_0)}\hat{b}_i e^{-i\omega_i(t-t_0)} + h.c., \tag{2.19}$$

where in the last equality we used the Baker-Hausdorf lemma,

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ..., \tag{2.20}$$

which can be seen by the Taylor expansion of the exponentials. For a more rigorous mathematical proof of the more general Baker-Campbell-Hausdorff theorem, result of which is the lemma above, the reader is referred to [9].

Using the Schrödinger equation in the density operator formalism (2.10), and the transformation we get the equation of motion for the reduced density operator in the interaction picture

$$\frac{d\hat{\rho}_{sr}}{dt} = -\frac{i}{\hbar}[\hat{\mathcal{H}},\hat{\rho}_{sr}], \tag{2.21}$$

using the above 2 equalities we have

$$\frac{d\hat{\rho}_{sr,I}}{dt} = -\frac{i}{\hbar}[\hat{\mathcal{H}}_{sr,I},\hat{\rho}_{sr,I}], \tag{2.22}$$

Generally, this equation of motion cannot be exactly solved, for system-resorvoir coupling. In order to solve it, we use the approximate iterative method of petrubation theory up to second order. We integrate over time equation (2.22) from t_0 to t, using the approximation $\hat{\rho}_{sr,I}(t) \approx \hat{\rho}_{sr,I}(t_0)$ in the commutator to obtain a first-order solution for $\hat{\rho}_{sr,I}(t)$

$$\hat{\rho}_{sr,I}(t) = \hat{\rho}_{sr,I}(t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt'[\hat{\mathcal{H}}_{sr,I}(t'-t_0), \hat{\rho}_{sr,I}(t_0)]. \tag{2.23}$$

Subsequently, we use the improved value of first order accuracy $\hat{\rho}_{sr,I}(t)$ in the commutator and integrate over time again to obtain the $\hat{\rho}_{sr,I}(t)$ accurate to second order.

$$\begin{aligned}\hat{\rho}_{sr,I}(t) = \hat{\rho}_{sr,I}(t_0) &- \frac{i}{\hbar}\int_{t_0}^{t} dt'[\hat{\mathcal{H}}_{sr,I}(t'-t_0), \hat{\rho}_{sr,I}(t_0)] \\ &- \frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''[\hat{\mathcal{H}}_{sr,I}(t'-t_0), [\hat{\mathcal{H}}_{sr,I}(t''-t_0), \hat{\rho}_{sr,I}(t_0)]].\end{aligned} \tag{2.24}$$

This is an expression for the density operator of the entire system in the interaction picture up to second order. In order to find the reduced density operator for the system in the interaction picture $\hat{\rho}_{s,I}(t)$, we perform the trace over the reservoir on $\hat{\rho}_{sr,I}(t)$

$$\hat{\rho}_{s,I}(t) \equiv tr_r(\hat{\rho}_{sr,I}(t)). \tag{2.25}$$

We now define a *coarsed grained* equation of motion for $\hat{\rho}_{s,I}(t)$

$$\dot{\hat{\rho}}_{s,I}(t) \simeq \frac{\hat{\rho}_{s,I}(t) - \hat{\rho}_{s,I}(t-\tau)}{\tau}, \tag{2.26}$$

which is justified by the fact, that the time interval $\tau = t - t_0$ is considered short compared to times yielding significant changes in the system variables.

Then by setting $t \to t+\tau$ and taking into account that the $\dot{\hat{\rho}}_s(t)$ does not vary significantly in the time τ, we can rewrite (2.26) as

$$\dot{\hat{\rho}}_{s,I}(t+\tau) \simeq \frac{\hat{\rho}_{s,I}(t+\tau) - \hat{\rho}_{s,I}(t-\tau)}{\tau} \simeq \dot{\hat{\rho}}_{s,I}(t). \tag{2.27}$$

This is an equation of motion for the reduced density operator in the interaction picture. Substituing the second order expression for the reduced density operator in the interaction picture $\hat{\rho}_{s,I}(t)$ from (2.25) and (2.27) we get

$$\begin{aligned}\dot{\hat{\rho}}_{s,I}(t) \simeq &- \frac{i}{\hbar\tau}\int_0^{\tau} d\tau' tr_r(\hat{\mathcal{H}}_{sr,I}(\tau'), \hat{\rho}_{sr,I}(t)) \\ &- \frac{1}{\hbar^2\tau}\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' tr_r(\hat{\mathcal{H}}_{sr,I}(\tau')\hat{\mathcal{H}}_{sr,I}(\tau'')\hat{\rho}_{sr,I}(t) - \hat{\mathcal{H}}_{sr,I}(\tau')\hat{\rho}_{sr,I}(t)\hat{\mathcal{H}}_{sr,I}(\tau'')) + h.c.,\end{aligned} \tag{2.28}$$

where we have used the fact that the double commutator of (2.24) can be written as

$$[\hat{\mathcal{H}}_{sr,I}(\tau'), [\hat{\mathcal{H}}_{sr,I}(\tau''), \hat{\rho}_{sr,I}(t)]] = \hat{\mathcal{H}}_{sr,I}(\tau')\hat{\mathcal{H}}_{sr,I}(\tau'')\hat{\rho}_{sr,I}(t) - \hat{\mathcal{H}}_{sr,I}(\tau')\hat{\rho}_{sr,I}(t)\hat{\mathcal{H}}_{sr,I}(\tau'') + h.c., \tag{2.29}$$

using commutation relations and the hermiticity of all the operators involved. Equation (2.28) is esentially the master equation for $\hat{\rho}_{s,I}(t)$, during the rest of the section we will analyze it. We can see that it has two time dependencies t and τ, but as we will show τ is associated only with

reservoir operators and this dependence disappears, if the reservoir is considered stationary, with infinitely short memory. From (2.19) we have

$$\hat{\mathcal{H}}_{sr,I}(\tau) = \hbar\hat{a}^\dagger \hat{F}(\tau) + \hbar\hat{a}\hat{F}^\dagger(\tau), \tag{2.30}$$

where

$$\hat{F}(\tau) = -i\sum_i g_i \hat{b}_i e^{i(\Omega-\omega_i)\tau}, \tag{2.31}$$

$F(\tau)$ is an operator that acts only in the Hilbert space of the reservoir and in the Heisenberg picture it's identied as a noise operator.

In the first part of the equation of motion (2.28) has terms of the following form

$$tr_r(\hat{a}^\dagger \hat{F}(\tau)\rho_{sr,I}(t)) = \hat{a}^\dagger \hat{\rho}_{s,I}(t) tr_r(\hat{F}(\tau)\hat{\rho}_r(\hat{\mathcal{H}}_r)). \tag{2.32}$$

The trace in the right hand side is the expectation value F_r of the reservoir operator $\hat{F}(\tau)$. This value vanishes if the density operator for the reservoir $\hat{\rho}_r$ is diagonal, as in of the thermal reservoir, defined in (2.15). Now we rewrite the second part of (2.28), using (2.29) and the cyclic properties of the trace

$$\begin{aligned}\dot{\hat{\rho}}_{s,I}(t) = -\frac{1}{\hbar^2\tau}\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' [&\hat{a}^\dagger\hat{a}\hat{\rho}_{s,I}(t)\left\langle \hat{F}(\tau')\hat{F}^\dagger(\tau'')\right\rangle_r - \hat{a}\hat{\rho}_{s,I}(t)\hat{a}^\dagger\left\langle \hat{F}(\tau'')\hat{F}^\dagger(\tau')\right\rangle_r \\ &+ \hat{a}\hat{a}^\dagger\hat{\rho}_{s,I}(t)\left\langle \hat{F}^\dagger(\tau')\hat{F}(\tau'')\right\rangle_r - \hat{a}^\dagger\hat{\rho}_{s,I}(t)\hat{a}\left\langle \hat{F}^\dagger(\tau'')\hat{F}(\tau')\right\rangle_r \\ &+ \hat{a}\hat{a}\hat{\rho}_{s,I}(t)\left\langle \hat{F}(\tau')\hat{F}^\dagger(\tau'')\right\rangle_r - \hat{a}\hat{\rho}_{s,I}(t)\hat{a}\left\langle \hat{F}^\dagger(\tau'')\hat{F}^\dagger(\tau')\right\rangle_r \\ &+ \hat{a}^\dagger\hat{a}^\dagger\hat{\rho}_{s,I}(t)\left\langle \hat{F}(\tau')\hat{F}(\tau'')\right\rangle_r - \hat{a}^\dagger\hat{\rho}_{s,I}(t)\hat{a}^\dagger\left\langle \hat{F}(\tau'')\hat{F}(\tau')\right\rangle_r] + h.c..\end{aligned} \tag{2.33}$$

By the use of (2.31), we can identify the reservoir average terms

$$\begin{aligned}\left\langle \hat{F}(\tau')\hat{F}^\dagger(\tau'')\right\rangle_r &= \sum_{i,j} g_i g_j^* \left\langle b_i b_j^\dagger\right\rangle_r e^{i\Omega(\tau'-\tau'')} e^{i(\omega_j\tau''-\omega_i\tau')} \\ &= \sum_i |g_i|^2 \left\langle \hat{b}_i\hat{b}_i^\dagger\right\rangle_r e^{i(\Omega-\omega_i)(\tau'-\tau'')},\end{aligned} \tag{2.34}$$

where for the second equality to stand, we considered the fact that the reservoir density matrix is diagonal.

These terms are the first order correlation functions of the reservoir and they only depend on the time difference $T = \tau' - \tau''$, meaning that the reservoir is stationary, as expected by the time independent, thermal equilibrium density operator $\hat{\rho}_r(\hat{\mathcal{H}}_r)$. This first order correlation function shows how fast the reservoir correlations decay away. We now perform the *Markoff approximation*, which assumes that the correlation time of the reservoir τ_c, which is the time for which the correlation function is not zero, is infinitely short compared to all times of interest for the system. This allows us to shift the limit of integration of the second integral in the following terms to infinity

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \left\langle F(\tau')F^\dagger(\tau'')\right\rangle_r = \int_0^\tau d\tau' \sum_i |g_i|^2 \left\langle \hat{b}_i\hat{b}_i^\dagger\right\rangle_r \int_0^{\tau'} dT e^{i(\Omega-\omega_i)T}, \tag{2.35}$$

$$\int_0^\tau d\tau' \int_0^{\infty} d\tau'' \left\langle F(\tau')F^\dagger(\tau'')\right\rangle_r = \int_0^\tau d\tau' \sum_i |g_i|^2 \left\langle \hat{b}_i\hat{b}_i^\dagger\right\rangle_r \int_0^{\infty} dT e^{i(\Omega-\omega_i)T}. \tag{2.36}$$

In order to treat these integrals we need to evaluate the sum over the modes and the can be found by finding the formulation of the density of electromagnetic states of the reservoir.
We will calculate it for the case of free space, this can be done by considering a three dimensional cubic cavity with edge L, the boundary conditions lead to modes with wavenumber $k_i = 2n/L, i = x, y, z; n_i = 1, 2, ..,$ so the number of modes in the volume element $dk_x dk_y dk_z$ is $dn = d^3kL^3/2\pi$. For large L the sumation over K can be written in integral form

$$\frac{1}{V}\sum_K \to \frac{1}{8\pi^3}\int d\mathbf{k}^3 = \frac{1}{8\pi^3}\int_0^\infty dk k^2 \int_0^\pi d\theta sin\theta \int_0^{2\pi} d\phi f(K). \tag{2.37}$$

Substituting $\omega = ck$, we substitute the sum over the modes in (2.35) with an integral over frequencies

$$\sum_i \to \frac{V}{8\pi^3 c^3}\int_0^\infty d\omega^3 \omega^2 \int_0^\pi d\theta sin\theta \int_0^{2\pi} d\phi. \tag{2.38}$$

Now considering the two polarization of the electric field carry out the three integrals taking into account the two possible field polarizations. However, if f(K) is independent ofthe field polarization we multiply by 2 and if f(K) doesnt depend on ϕ and θ we obtain 4π for the angular integrations. These simplifications give the correspondence

$$\begin{aligned}&\int_0^\tau d\tau' \frac{2V}{8\pi^3 c^3}\int_0^\infty d\omega\omega^2 \int_0^\pi d\theta sin\theta \int_0^{2\pi} d\phi |g(\omega)|^2 \left\langle \hat{b}(\omega)\hat{b}^\dagger(\omega)\right\rangle_r \int_0^\infty dT e^{i(\Omega-\omega)T} = \\ &= \frac{\tau V}{\pi^2 c^3}\int_0^\infty d\omega\omega^2 |g(\omega)|^2 \left\langle \hat{b}(\omega)\hat{b}^\dagger(\omega)\right\rangle_r \int_0^\infty dT e^{i(\Omega-\omega)T} = \frac{\tau V\Omega^2}{\pi^2 c^3}|g(\Omega)|^2 \left\langle \hat{b}(\Omega)\hat{b}^\dagger(\Omega)\right\rangle_r ,\end{aligned} \tag{2.39}$$

where in the last equality we have the following relation for the last integral

$$\lim_{t\to\infty}\int_{t_0}^t dt' e^{-i(\Omega-\omega)(t-t')} = \pi\delta(\Omega-\omega) - \mathcal{P}\left[\frac{i}{\Omega-\omega}\right], \tag{2.40}$$

which allows us to evaluate the product. The principal part $\mathcal{P}\left[\frac{i}{\Omega-\omega}\right]$ term leads to a frequency shift related to the Lamb shift, but we can neglect it.

We introduce now the decay rate

$$\gamma \equiv 2\pi D(\Omega)|g(\Omega)|^2, \tag{2.41}$$

where $D(\Omega)$ is the density of states between Ω and $\Omega + d\Omega$. For this specific case of free space and the coupling constant $g(\omega)$ being independent of the filed polarization and angular direction (θ,ϕ), its value is $D(\Omega) = \frac{V\Omega^2}{\pi^2 c^3}$. The value of $D(\Omega)$ is different for electric-dipole interactions, since in that case the coupling constant depends on the angle between the electric field polarization and the atomic quantization axis. The expectation value of the number operator $\overline{n} = \left\langle \hat{b}(\omega)\hat{b}^\dagger(\omega)\right\rangle_r$, of the reservoir, is given by the thermal distribution (2.15) and is a geometric series that leads to $\overline{n} = \frac{1}{1+e^{\hbar\beta\Omega}}$, that depends only on the temperature. Furthermore $\left\langle \hat{b}^\dagger(\omega)\hat{b}(\omega)\right\rangle_r = \overline{n} + 1$. By using (3.39) and (3.41) and the fact that terms like $\langle\hat{F}(\tau')\hat{F}(\tau'')\rangle$,$\langle\hat{F}^\dagger(\tau')\hat{F}^\dagger(\tau'')\rangle$ are neglected, since the density operator of the reservoir is diagonal, we get the final expression of the reduced density operator in the interaction picture.

$$\dot{\hat{\rho}}_{s,I}(t) = -\frac{\gamma}{2}(\overline{n}+1)(\hat{a}^\dagger\hat{a}\hat{\rho}_{s,I}(t) - \hat{a}\hat{\rho}_{s,I}(t)\hat{a}^\dagger) - \frac{\gamma}{2}\overline{n}(\hat{\rho}_{s,I}(t)\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{s,I}(t)\hat{a}) + h.c. \tag{2.42}$$

If we consider zero temperature for the reservoir the expectation value of the number operator is zero and the expression reduces to

$$\dot{\hat{\rho}}_{s,I}(t) = -\frac{\gamma}{2}(\{\hat{a}^\dagger\hat{a}, \hat{\rho}_{s,I}(t)\} - 2\hat{a}\hat{\rho}_{s,I}(t)\hat{a}^\dagger) \tag{2.43}$$

It is also possible to move back to the Schrödinger picture, by using the inverse unitary transformation of the one used in (2.18). Using it on equation 2.42, we get the equation of motion for the reduced density operator.

$$\begin{aligned}\dot{\hat{\rho}}_s(t) &= \frac{d}{dt}\left(e^{-i(\hat{\mathcal{H}}_s+\hat{\mathcal{H}}_r)}\hat{\rho}_{s,I}e^{i(\hat{\mathcal{H}}_s+\hat{\mathcal{H}}_r)}\right) \\ &= -\frac{i}{\hbar}[\hat{\mathcal{H}}_s,\hat{\rho}_s(t)] - \frac{\gamma}{2}(\overline{n}+1)(\hat{a}^\dagger\hat{a}\hat{\rho}_{s,I}(t) - \hat{a}\hat{\rho}_{s,I}(t)\hat{a}^\dagger) - \frac{\gamma}{2}\overline{n}(\hat{\rho}_{s,I}(t)\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{s,I}(t)\hat{a}) + h.c. \\ &= -\frac{i}{\hbar}[\hat{\mathcal{H}}_s,\hat{\rho}_s(t)] + \mathcal{L}[\rho_s]\end{aligned} \tag{2.44}$$

where in the equality we have introduced the Linbland superoperator $\mathcal{L}[\rho_s]$ acting on the reduced density matrix, which describes the non unitary evolution of the system to its coupling to the reservoir, that leads to irreversible dissipation. $\mathcal{L}[\rho_s]$ can be written as

$$\mathcal{L}[\rho_s] = -\frac{1}{2}\sum_i(\hat{L}_i^\dagger\hat{L}_i\hat{\rho}_s + \hat{\rho}_s\hat{L}_i^\dagger\hat{L}_i) + \sum_i L_i\rho_s L_i^\dagger \tag{2.45}$$

where the Linbland operators $\hat{L}_i$ of the system, are operators that express the decay processes. In the case of the damped hamonic oscillator, described above we have

$$\hat{L}_1 = \sqrt{\gamma_e(\overline{n}+1)}\hat{a} \tag{2.46}$$

$$\hat{L}_2 = \sqrt{\gamma_e\overline{n}}\hat{a} \tag{2.47}$$

We will describe the Linbland form and the physical meaning behind it in the following section, where we discuss Monte Carlo wavefunctions.
The above result is completely analogous to the case in which our system is not a harmonic oscillator, but a two level atom coupled to the same reservoir.
In this case the system Hamiltonian is

$$\hat{\mathcal{H}}'_s = \hbar\Omega\hat{\sigma}_+\hat{\sigma}_- \tag{2.48}$$

where $\hat{\sigma}_+ = |e\rangle\langle g|$ the transition operator and $\hat{\sigma}_- = |g\rangle\langle e|$ the lowering operator. The interaction Hamiltonian in the interaction picture corresponding to (2.30) is

$$\hat{\mathcal{H}}'_{sr,I} = \hbar\left(\hat{\sigma}_+\hat{F}(\tau) + \hat{\sigma}_+\hat{F}(\tau)\right) \tag{2.49}$$

By replacing $\hat{a}$ with $\hat{\sigma}_-$ and $\hat{a}^\dagger$ with $\hat{\sigma}_+$, all the above results stand, apart from the density of states $D(\Omega)$ and as a consequence the decay rate γ. This happens due to the dependence of the atom-reservoir's coupling constant on the angle between the direction of the atom's electric dipole moment and the polarization of the electric fields of the reservoir.

and using the explicit form of the coupling constant we get the equation corresponding to (2.39) as

$$\begin{aligned}&\int_0^\tau d\tau' \frac{V}{8\pi^3c^3}\int_0^\infty d\omega\omega^2\int_0^\pi d\theta sin\theta\int_0^{2\pi}d\phi\frac{\omega}{2\hbar\epsilon_0 V}|\langle g|\hat{d}|e\rangle|^2 sin^2\theta\,\langle\hat{b}_i\hat{b}_i^\dagger\rangle_r\int_0^\infty dT e^{i(\Omega-\omega)T} = \\ &= \frac{\tau}{3\pi^2c^3}\int_0^\infty d\omega\omega^2\frac{\omega}{2\hbar\epsilon_0}|\langle g|\hat{d}|e\rangle|^2\int_0^\infty dT e^{i(\Omega-\omega)T} = \frac{\tau\Omega^3}{6\pi^2\epsilon_0\hbar c^3}|\langle g|\hat{d}|e\rangle|^2\langle\hat{b}(\Omega)\hat{b}^\dagger(\Omega)\rangle_r\end{aligned} \tag{2.50}$$

From the equation above we can see that unlike the harmonic oscillator, the decay rate of a two level atom, coupled to the free space, modeled as a thermal reservoir under the Markov approximation is

$$\gamma' = \frac{\Omega^3}{\pi c^3} |\langle g|\hat{d}|e\rangle|^2 \tag{2.51}$$

and the equation of motion

$$\dot{\hat{\rho}}'_{s,I}(t) = -\frac{\gamma'}{2}(\overline{n}+1)(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}'_{s,I}(t) - \hat{\sigma}_-\hat{\rho}'_{s,I}(t)\hat{\sigma}_+) - \frac{\gamma'}{2}\overline{n}(\hat{\rho}_{s,I}(t)\hat{\sigma}_-\hat{\sigma}_+ - \hat{\sigma}_+\hat{\rho}_{s,I}(t)\hat{\sigma}_-) + h.c. \tag{2.52}$$

2.3 Monte Carlo Wave Function

In this section we will discuss the non unitary dynamics of dissipative open systems and how to describe them in the so called quantum jump formalism.
In the previous section we showed that the Schrödinger evolution of a small system coupled to a reservoir can be described in terms of a master equation of the general form 2.44, where the Liouvillian superoperator $\mathcal{L}[\hat{\rho}_s]$ describes the non-Hermitian evolution of the system due to its coupling to the reservoir, and is responsible for irreversible dissipation. We can rewrite 2.44 in the following form

$$\dot{\hat{\rho}}_s = -\frac{i}{\hbar}(\hat{\mathcal{H}}_{NH}\hat{\rho}_s - \hat{\rho}_s\hat{\mathcal{H}}_{NH}) + \mathcal{L}_{jump}[\hat{\rho}_s] \tag{2.53}$$

where we have introduced the non-Hermitian effective Hamiltonian

$$\mathcal{H}_{NH} = \hat{\mathcal{H}}_s - \frac{i\hbar}{2}\sum_i \hat{L}_i^\dagger \hat{L}_i \tag{2.54}$$

and the "jump" term, which is the last term of the Lindblad superoperator (2.66)

$$\mathcal{L}_{jump}[\hat{\rho}_s] = \sum_i \hat{L}_i \hat{\rho} \hat{L}_i^\dagger \tag{2.55}$$

For the case of the damped harmonic oscillator, discussed in this chapter the effective Hamiltonian becomes

$$\hat{\mathcal{H}}_{NH} = \hbar\Omega\hat{a}^\dagger\hat{a} - i\hbar\gamma(\overline{n} + \frac{1}{2})\hat{a}^\dagger\hat{a} \tag{2.56}$$

and the jump superoperator

$$\mathcal{L}_{jump}[\hat{\rho}_s] = \gamma(2\overline{n}+1)\hat{a}\hat{\rho}_s\hat{a}^\dagger \tag{2.57}$$

The evolution of the system density operator can, therefore, be thought of as resulting from two contributions: a Schrödinger-like part governed by the effective non-Hermitian Hamiltonian $\hat{H}_{NH}$, and a quantum jump part resulting from $\mathcal{L}_{jump}[\hat{\rho}_s]$.
The Monte Carlo wave functions method of solution of the master equation initially considers the evolution of pure states of the system, and carries out a statistical average over such systems in the end. But in contrast to the situation for closed systems, where this is straightforwardly achieved, this approach is not so simple for the open dissipative systems that we are interested in. The evolution of a pure state in this case, cannot be described by a Schrödinger evolution. Rather, it is intrinsically stochastic, and results from the combination of a nonhermitian, but Schrödinger-like evolution and random quantum jumps.

2.4 Heisenberg-Langevin equations

To gain some more insight into the system-reservoir interaction, we need to treat the previous problem in the Heisenberg picture, as well.
Using the Hamiltonian (2.11) we can find the Heisenberg equations of motion for the annihilation operators $\hat{a}(t)$ and $\hat{b}_j(t)$, as following

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{a}(t)] = -i\Omega\hat{a}(t) - i\sum_i g_i\hat{b}_i(t) \tag{2.58}$$

$$\dot{\hat{b}}_j(t) = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{b}_j(t)] = -i\omega_j\hat{b}_j(t) - ig_j^*\hat{a}(t) \tag{2.59}$$

If we now formally integrate the above Heisenberg equation of $\hat{b}_j$, we get

$$\hat{b}_j(t) = \hat{b}_j(t_0) - ig_j^*\int_{t_0}^{t} dt'\hat{a}(t')e^{-i\omega_j(t-t')} \equiv \hat{b}_{free}(t) + \hat{b}_{radiated}(t) \tag{2.60}$$

The term $\hat{b}_{free}(t)$ is the homogeneous solution of (2.41), describes the free evolution of $\hat{b}_j$ in the absence of any interaction with the system, where the second term $\hat{b}_{radiated}(t)$ gives the modification of this free evolution due to the coupling with the system and $\hat{a}(t)$ is the source for $\hat{b}_j(t)$.
Now using (2.60) in (2.58) we find

$$\dot{\hat{a}}(t) = -i\Omega\hat{a} - i\sum_i g_i\hat{b}_i(t_0)e^{-i\omega_i(t-t_0)} - \sum_i |g_i|^2\int_{t_0}^{t} dt'\hat{a}(t')e^{-i\omega_i(t-t')} \tag{2.61}$$

Here, the first term is the free evolution of the system in the absence of the reservoir, the first summation gives fluctuations of the reservoir and the second gives the radiation reaction.
We now move to an interaction picture, introducing the slowly varying operator

$$\hat{A}(t) = \hat{a}(t)e^{i\Omega t} \tag{2.62}$$

where $[\hat{A}(t), \hat{A}^\dagger(t)] = 1$ stands.
This will seperate separate the rapid free evolution of $\hat{a}(t)$ at the frequency Ω from the fast evolution due to the large bandwidth of the bath and (2.61) will become

$$\dot{\hat{A}}(t) = -\sum_j |g_j|^2\int_{t_0}^{t} dt'\hat{A}(t')e^{-i(\omega_j-\Omega)(t-t')} + \hat{F}_A(t) \tag{2.63}$$

where operator $\hat{F}_A(t)$ is the noise operator, which varies rapidly in time due to the presence of all the reservoir frequencies.

$$\hat{F}_A(t) = -i\sum_j g_j b_j(t_0)e^{i(\Omega-\omega_j)(t-t_0)} \tag{2.64}$$

As we mentioned before if the reservoir is described by a density operator diagonal in energy representation, which is in our case, then $\langle\hat{F}_A(t)\rangle_r = 0$. If we replace the sum of (2.63) by an integral and perform the Markoff approximation i.e. taking the limit of integration to infinity, by claiming that $\hat{A}(t)$ varies little over the inverse reservoir bandwidth, we find

$$\dot{\hat{A}}(t) = -\frac{\gamma}{2}\hat{A}(t) + \hat{F}_A(t) \tag{2.65}$$

Eventhough noise operator $\hat{F}_A(t)$ fluctuates rapidly and averages to zero, we need to keep it in the above equation in order to preserve the commutation relation of $\hat{A}(t)$, since otherwise $[\hat{A}(t), \hat{A}^\dagger(t)] \to 0$ as time grows large compared to γ.

We will now show that it is possible to transform the Lindblad master equation of the density operator, to equations of motion for operators of the systems in the Heisenberg picture. We can rewrite (2.44) as

$$\dot{\hat{\rho}}_s(t) = \mathcal{L}'[\hat{\rho}_s] \tag{2.66}$$

where we he have defined the Lindblad superoperator $\mathcal{L}'$ as

$$\mathcal{L}'[\hat{\rho}_s] = -\frac{i}{\hbar}[\hat{\mathcal{H}}_s, \rho_s] - \frac{1}{2}\sum_i (\hat{L}_i^\dagger \hat{L}_i \hat{\rho}_s + \hat{\rho}_s \hat{L}_i^\dagger \hat{L}_i) + \sum_i L_i \rho_s L_i^\dagger \tag{2.67}$$

We can also define the adjoint Liouvillian superoperator $\mathcal{L}'^\dagger$ by

$$tr_s(\hat{M}\mathcal{L}'[\hat{N}]) = tr_s(\mathcal{L}'^\dagger[\hat{M}]\hat{N}) \tag{2.68}$$

where $\hat{M}$, $\hat{N}$ are arbitraty system operators. Using the cyclic property of the trace we get the explicit form of $\mathcal{L}'^\dagger$

$$\mathcal{L}'^\dagger[\hat{M}] = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{M}] - \frac{1}{2}\sum_i (\hat{L}_i^\dagger \hat{L}_i \hat{M} + \hat{M}\hat{L}_i^\dagger \hat{L}_i) + \sum_i \hat{L}_i^\dagger \hat{M} \hat{L}_i \tag{2.69}$$

Now formally integrating (2.66) we obtain

$$\hat{\rho}_s(t) = e^{\mathcal{L}'}[\hat{\rho}_s(0)] \tag{2.70}$$

In the Schrödinger picture the expectation value of a system operator is given by (2.7) and using (2.68) we have

$$\langle \hat{a} \rangle = tr_s(\hat{\rho}_s(t)\hat{a}(0)) = tr_s(e^{\mathcal{L}'}[\hat{\rho}_s(0)]\hat{a}(0)) = tr_s(\hat{\rho}_s(0) e^{\mathcal{L}'^\dagger}[\hat{a}(0)]) \tag{2.71}$$

and since in the Heisenberg picture the expectation value of $\hat{a}$ is $\langle \hat{a} \rangle = tr_s(\hat{\rho}_s(0)\hat{a}(t))$ we have

$$\hat{a}(t) = e^{\mathcal{L}'^\dagger}[\hat{a}(0)] \tag{2.72}$$

We conclude to the Lindland equation for an arbitrary system operator $\hat{a}(t)$

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{a}] - \frac{1}{2}\sum_i (\hat{L}_i^\dagger \hat{L}_i \hat{a} + \hat{a}\hat{L}_i^\dagger \hat{L}_i) + \sum_i \hat{L}_i^\dagger \hat{a} \hat{L}_i + \hat{F}_a \tag{2.73}$$

where we have introduced the noise operator $\hat{F}_a$, that is necessary in order to preserve the commutation relation, as we've shown earlier in this section.

2.5 Effective Operator Formalism

So far in this chapter we have introduced open quantum systems and the formalism to describe them. We have seen that they involve both unitary and dissipative dynamics, which can lead to very complex evolution. In the present section we will introduce the effective operator formalism, an effective theory that reduces the complexity of the open system, through adiabatic elimination of it's rapidly evolving part, introduced by Reiter et al.[3]

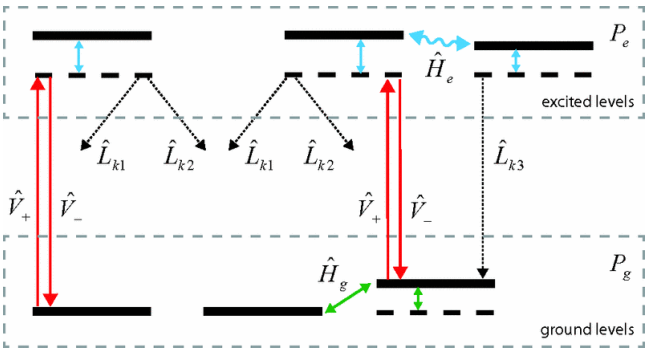

Figure 2.1: Ground and excited subspaces and couplings.

2.5.1 Ground and Excited Subspaces

We assume the open system to consist of two distinct subspaces, one for the ground states and one for the decaying excited states. The couplings of these two subspaces are assumed to be perturbative. Furthermore, we consider the dynamics of the system are Markovian, as we did in the previous sections, such that the time evolution of the density operator of the system ρ, can be described by the master equation (2.44). We note that we have dropped the subscript s from the densisty operator of the system, since it will be the only density operator we will be dealing with from now on.

Using the projection-operator method of [10] to divide the Hilbert space into two subspaces, one for the ground states and one for the excited states, represented by the projection operators $\hat{P}_g$ and $\hat{P}_e$, with $\hat{P}_g + \hat{P}_e = \hat{I}$ and $\hat{P}_g\hat{P}_e = 0$. The Hilbert space is now divided into two parts, one for the rapidly decaying (excited) states, and one for the comparably stable (ground) states.
By the use of the projection operators defined above, we divide the Hamiltonian into four parts:

$$\hat{H} = \hat{H}_g + \hat{H}_e + \hat{V}_+ + \hat{V}_- \tag{2.74}$$

where $\hat{H}_g = \hat{P}_g\hat{H}\hat{P}_g$ and $\hat{H}_e = \hat{P}_e\hat{H}\hat{P}_e$ are the Hamiltonians describing the ground and excited subspace respectively.
The perturbative excitations $\hat{V}_+ = \hat{P}_g\hat{H}\hat{P}_e$ and de-excitations $\hat{V}_- = \hat{P}_g\hat{H}\hat{P}_e$ couple the two subspaces.
We have assumed the ground states to be stable and the excited states to be decaying to the ground states, as seen in the example of figure 2.1. The Lindblad operators can then be written as $\hat{L}_k = \hat{P}_e\hat{L}_k\hat{P}_g$. We can rewrite now the master equation (2.44)

$$\dot{\hat{\rho}} = -((\hat{H}_{NH} + \hat{H}_g + \hat{V})\hat{\rho} - \hat{\rho}(\hat{H}^\dagger_{NH} + \hat{H}_g + \hat{V})) + \sum_k \hat{L}_k\hat{\rho}\hat{L}^\dagger_k \tag{2.75}$$

where the $\hat{H}_{NH}$ is the non-Hermitian Hamiltonian of the quantum jump formalism, as in (2.54), but involving only the excited Hamiltonian

$$\hat{\mathcal{H}}_{NH} = \hat{\mathcal{H}}_e - \frac{i\hbar}{2}\hat{L}^\dagger_i\hat{L}_i \tag{2.76}$$

Since we have assumed the couplings of the ground and excited subspaces $\hat{V}_\pm$ to be sufficiently weak to be described as perturbations of the evolution governed by an unperturbed Hamiltonian $\hat{H}_0 = \hat{H}_g + \hat{H}_{NH}$, we perform perturbation theory of the density operator. Moving to the interaction picture, using the unitary transformation

$$\hat{U}(t) = e^{-i(\hat{H}_{NH}+\hat{H}_g)t} \tag{2.77}$$

The operators in the interaction picture transfromed as

$$\hat{\rho}_I(t) = \hat{U}^{-1}(t)\hat{\rho}\left(\hat{U}^{-1}(t)\right)^\dagger \tag{2.78}$$

$$\hat{V}_I = \hat{U}^{-1}(t)(\hat{H}_0 + \hat{V})\hat{U}(t) + i\frac{d\hat{U}^{-1}}{dt}\hat{U}(t) = \hat{U}^{-1}(t)\hat{V}\hat{U}(t) \tag{2.79}$$

$$\hat{L}_{k,I}(t) = \hat{U}^{-1}\hat{L}_k\hat{U}(t) \tag{2.80}$$

and the reduced master equation transforms to

$$\dot{\hat{\rho}}_I(t) = -i\left(\hat{V}_I(t)\hat{\rho}_I(t) - \hat{\rho}_I(t)\hat{V}_I^\dagger(t)\right) + \sum_k \hat{L}_{k,I}(t)\hat{\rho}_I(t)\hat{L}_{k,I}^\dagger(t) \tag{2.81}$$

Now we perform a perturbative expansion of the density operator in a small parameter ϵ

$$\hat{\rho}_I(t) = \frac{1}{N}(\hat{\rho}_I^{(0)}(t) + \epsilon\hat{\rho}_I^{(1)}(t) + \epsilon^2\hat{\rho}_I^{(2)}(t) + ...) \tag{2.82}$$

and obtain a recursive formulation of the reduced master equation in powers of ϵ ,

$$\dot{\hat{\rho}}_I^{(n)}(t) = -i\left(\hat{V}_I(t)\hat{\rho}_I^{(n-1)}(t) - \hat{\rho}_I^{(n-1)}(t)\hat{V}_I^\dagger(t)\right) + \sum_k \hat{L}_{k,I}(t)\hat{\rho}_I^{(n)}(t)\hat{L}_{k,I}^\dagger(t) \tag{2.83}$$

We evaluate the first three orders

$$\dot{\hat{\rho}}_I^{(0)}(t) = \sum_k \hat{L}_{k,I}(t)\hat{\rho}_I^{(0)}(t)\hat{L}_{k,I}^\dagger(t) \tag{2.84}$$

$$\dot{\hat{\rho}}_I^{(1)}(t) = -i\left(\hat{V}_I(t)\hat{\rho}_I^{(0)}(t) - \hat{\rho}_I^{(0)}(t)\hat{V}_I^\dagger(t)\right) + \sum_k \hat{L}_{k,I}(t)\hat{\rho}_I^{(1)}(t)\hat{L}_{k,I}^\dagger(t) \tag{2.85}$$

$$\dot{\hat{\rho}}_I^{(2)}(t) = -i\left(\hat{V}_I(t)\hat{\rho}_I^{(1)}(t) - \hat{\rho}_I^{(1)}(t)\hat{V}_I^\dagger(t)\right) + \sum_k \hat{L}_{k,I}(t)\hat{\rho}_I^{(2)}(t)\hat{L}_{k,I}^\dagger(t) \tag{2.86}$$

If we now consider all particles initially in the ground stay, decay processes can be neglected for orders $n \leq 1$ so we obtain the following expressions,

$$\dot{\hat{\rho}}_I^{(0)}(t) = 0 \tag{2.87}$$

$$\dot{\hat{\rho}}_I^{(1)}(t) = -i\left(\hat{V}_I(t)\hat{\rho}_I^{(0)}(t) - \hat{\rho}_I^{(0)}(t)\hat{V}_I^\dagger(t)\right) \tag{2.88}$$

$$\dot{\hat{\rho}}_I^{(2)}(t) = -i\left(\hat{V}_I(t)\hat{\rho}_I^{(1)}(t) - \hat{\rho}_I^{(1)}(t)\hat{V}_I^\dagger(t)\right) + \sum_k \hat{L}_{k,I}(t)\hat{\rho}_I^{(2)}(t)\hat{L}_{k,I}^\dagger(t) \tag{2.89}$$

We can now seperate the evolution of ground and excited states, by using the projection operators on density operator. Then the evolution of the ground states is

$$\hat{P}_g\dot{\hat{\rho}}_I^{(0)}(t)\hat{P}_g = \hat{P}_g\dot{\hat{\rho}}_I^{(1)}(t)\hat{P}_g = 0 \tag{2.90}$$

$$\hat{P}_g\dot{\hat{\rho}}_I^{(2)}(t)\hat{P}_g = -i\hat{P}_g\left(\hat{V}_I(t)\hat{\rho}_I^{(1)}(t) - \hat{\rho}_I^{(1)}(t)\hat{V}_I^\dagger(t)\right)\hat{P}_g + \sum \hat{L}_{k,I}(t)\hat{P}_e\hat{\rho}^{(2)}(t)\hat{P}_e\hat{L}_{k,I}^\dagger(t) \tag{2.91}$$

The ground states are connected by unitary and dissipative processes of second order to the excited states, since there is only decay from the excited to the ground. Now for the dynamics of the excited states we find

$$\hat{P}_e\dot{\hat{\rho}}_I^{(0)}(t)\hat{P}_e = \hat{P}_e\dot{\hat{\rho}}_I^{(1)}(t)\hat{P}_e = 0 \tag{2.92}$$

$$\hat{P}_e\dot{\hat{\rho}}_I^{(2)}(t)\hat{P}_e = -i\hat{P}_e\left(\hat{V}_I(t)\hat{\rho}_I^{(1)}(t) - \hat{\rho}_I^{(1)}(t)\hat{V}_I^\dagger(t)\right)\hat{P}_e \tag{2.93}$$

The evolution of the excited unitary dynamics is only due to the interaction Hamiltonian $\hat{V}_I(t)$. The second order terms connect the states either in the ground or in the excited subspace and the interaction between the subspaces is given but the first order terms $\hat{P}_g\hat{\rho}_I^{(1)}(t)\hat{P}_e$ and $\hat{P}_e\hat{\rho}_I^{(1)}(t)\hat{P}_g$.

2.5.2 Adiabatic Elimination of the Excited States

We will now proceed by performing adiabatic elimination of the excited states, in order to reduce the copmlexity of the dynamics, by restricting it to the ground subspace. We do so by considering

$$\hat{P}_e \dot{\hat{\rho}}_I^{(2)}(t) \hat{P}_e \approx 0 \tag{2.94}$$

We now formally integrate (2.93) to obtain an expression for $\hat{P}_e \hat{\rho}_I^{(2)}(t) \hat{P}_e$ and also integrate (2.88) to get an expression for $\hat{\rho}_I^{(1)}(t)$. Now substituting these expressions to (2.91) we find the following equation of motion, that describes the evolution of the open system, after the exclusion of the excited states under adiabatic elimination.

$$\begin{aligned}\hat{P}_g \dot{\hat{\rho}}_I^{(2)}(t) \hat{P}_g = &-\hat{P}_g \hat{V}_I(t) \left(\int_0^t dt' \hat{V}_I(t') \hat{\rho}_I^{(0)}(t') \right) \hat{P}_g - \hat{P}_g \left(\int_0^t dt' \hat{\rho}_I^{(0)}(t') \hat{V}_I^\dagger(t') \right) \hat{V}_I^\dagger(t) \hat{P}_g \\ &+ \hat{P}_g \sum_k \hat{L}_{k,I}(t) \hat{P}_e \int_0^t dt' \int_0^{t'} dt'' \left(\hat{V}_I(t') \hat{\rho}_I^{(0)}(t'') \hat{V}_I^\dagger(t'') + \hat{V}_I(t'') \hat{\rho}_I^{(0)}(t'') \hat{V}_I^\dagger(t') \right) \hat{P}_e \hat{L}_{k,I}^\dagger(t) \hat{P}_g\end{aligned} \tag{2.95}$$

Where we have neglected terms sandwiched between perturbations $\hat{P}_g \hat{V}_I$ and $\hat{V}_I \hat{P}_g$, since $\hat{\rho}_I^{(0)}$ lives in the ground subspace, these terms do not contribute to the ground-state evolution.
As we can see the equation of motion of the ground state contains two Hamiltonian and two Lindblad terms, for which we need to evaluate the integrals

$$I_1 \equiv \hat{P}_g \hat{V}_I(t) \int_0^t dt' \hat{V}_I(t') \hat{\rho}_I^{(0)}(t') \hat{P}_g \tag{2.96}$$

$$I_2 \equiv \hat{P}_e \int_0^t dt' \int_0^{t'} dt'' \hat{V}_I(t') \hat{\rho}_I^{(0)}(t'') \hat{V}_I^\dagger(t'') \hat{P}_e \tag{2.97}$$

Now we assume the direct interactions within the ground subspace to be perturbative. Hence, the ground- state evolution is negligibly small compared to the one for the excited states so that we have $\hat{U}(t)\hat{P}_g \approx \hat{P}_g$. Then the I_1 can be written as

$$I_1 \approx \hat{V}_- \hat{O}(t) \left(\int_0^t dt' \hat{O}^{-1}(t') \right) \hat{V}_+ \hat{\rho}_I^{(0)} \tag{2.98}$$

By carrying out the integral we obtain

$$I_1 \approx \hat{V}_- e^{-i\hat{H}_{NH}t} \left[(i\hat{H}_{NH})^{-1} e^{i\hat{H}_{NH}t'} \right]_0^t \hat{V}_+ \hat{\rho}_I^{(0)} \approx \hat{V}_- (i\hat{H}_{NH})^{-1} \hat{V}_+ \hat{\rho}_I^{(0)} \tag{2.99}$$

where in the last step we used an approximation similar to the rotating wave approximation to $1 - e^{-i\hat{\mathcal{H}}_{NH}t} \approx 1$, which is by justified by the assumption that the dynamics of the ground states are slow compared to the time scale set by $\hat{\mathcal{H}}_{NH}$.
To evaluate the last two Lindblad terms in master equation (2.95) we carry out the double integral I_2. To do so, we use $\hat{\rho}_I^{(0)}(t'') \approx \hat{\rho}_I^{(0)}(t)$, since the ground states evolve slowly and to second order in $\hat{V}_I$. We neglect the higher order terms. Thus, we can separate the integral and write

$$I_2 \approx \frac{1}{2} \left(\int_0^t dt' \hat{O}^{-1}(t') \right) \hat{V}_+ \hat{\rho}_I^{(0)}(t) \hat{V}_- \int_0^t dt' (\hat{O}^{-1})^\dagger(t') \approx \frac{1}{2} (i\hat{H}_{NH})^{-1} \hat{V}_+ \hat{\rho}_I^{(0)} \hat{V}_- (-i\hat{H}_{NH}^\dagger)^{-1} \tag{2.100}$$

in the ground-state subspace, these terms do not contribute to the ground-state evolution and can therefore be neglected. We can then obtain the effective unitary and dissipative dynamics of the ground states,

$$\hat{P}_g\dot{\hat{\rho}}\hat{P}_g = -i\left(\hat{H}_{eff} - \frac{i}{2}\sum_k(\hat{L}^k_{eff})^\dagger\hat{L}^k_{eff}\right)\hat{\rho}^{(0)} + h.c. + \sum_k \hat{L}^k_{eff}\hat{\rho}^{(0)}(\hat{L}^k_{eff})^\dagger \tag{2.101}$$

with an effective Hamiltonian and effective Lindblad operators as defined above. To reach this form we have used the equality

$$\sum_k(\hat{L}^k_{eff})^\dagger\hat{L}^k_{eff} = \hat{V}_-(\hat{H}^{-1}_{NH})^\dagger\left(\sum_k\hat{L}^\dagger_k\hat{L}_k\right)\hat{H}^{-1}_{NH}\hat{V}_+ = -i\hat{V}_-\left(\hat{H}^{-1}_{NH} - (\hat{H}^{-1}_{NH})^\dagger\right)\hat{V}_+ \tag{2.102}$$

where in the last term we have defined

$$\hat{\mathcal{H}}_{eff} = -\frac{1}{2}(\hat{\mathcal{H}}^{-1}_{NH} + (\hat{\mathcal{H}}^{-1}_{NH})^\dagger)\hat{V}_+ + \hat{\mathcal{H}}_g \tag{2.103}$$

and

$$\hat{L}^k_{eff} = \hat{L}_k\hat{\mathcal{H}}^{-1}_{NH}\hat{V}_+. \tag{2.104}$$

Chapter 3

Interface Between Light and 3-Level Atomic Ensembles

In the current chapter we analyse the interaction of light with an ensemble of atoms. The interaction of light with multiatom ensembles has been a basic building block for quantum information processing and it will be the basic platform for the modeling of the continuous single photon transistor. After introducing the necessary formalism, we will thoroughly describe the scattering dynamics for the cases of 3-level atomic ensembles in a single-sided cavity, a two-sided cavity and in free space. Furthermore, we will derive a mapping relation between free space and single cavity that will allow us to relate the results between the two systems. In the last section of the chapter we will describe the scattering processes of a ensemble of Rydberg atoms in a single-sided cavity.

3.1 Collective Operators

Many body quantum systems are systems consisting of a large number of particles, which lead to a large number of degrees of freedom and thus to high complexity. In order to describe them, one needs to introduce symmetric collective states, a description which becomes valid by considering the particles of the system identical and indistiguishable, in the sense of the second quantization formalism. For the case of our interest, which consists of an atomic ensemble interacting with light beams, these symmetric collective states will have the form of collective excitations of the atomic ensemble and will be described by collective atomic operators[11].
An integral part of the systems, which we will be describing throughout this thesis, is a three level atomic ensemble. That is an ensemble constituted by a large number of atoms, where the atomic energy structure, consists of three energy levels, one ground $|g\rangle$ and two excited $|e\rangle, |r\rangle$ states, in a Ξ-scheme, as seen in Fig.3.1*b*.
We will begin by describing a single atom of the atomic ensemble and for now we will consider the state $|r\rangle$ as inaccessible. Then the description of the i-th atom of the ensemble reduces to that of a two level atom with states $|g\rangle$ and $|e\rangle$, which can be described by the angular momentum operators, as for the case of states $|e\rangle$ and $|g\rangle$,

$$\hat{j}^{ge}_{z,i} = \frac{\hbar}{2}(|g_i\rangle\langle g_i| - |e_i\rangle\langle e_m|) = \frac{\hbar}{2}(\hat{\sigma}^i_{gg} - \hat{\sigma}^i_{ee}), \tag{3.1}$$

$$\hat{j}^{ge}_{x,i} = \frac{\hbar}{2}(|e_i\rangle\langle g_i| + |g_i\rangle\langle e_i|) = \frac{\hbar}{2}(\hat{\sigma}^i_{eg} - \hat{\sigma}^i_{ge}), \tag{3.2}$$

$$\hat{j}^{ge}_{y,i} = \frac{i\hbar}{2}(|e_i\rangle\langle g_i| - |g_i\rangle\langle e_i|) = \frac{i\hbar}{2}(\hat{\sigma}^i_{eg} - \hat{\sigma}^i_{ge}), \tag{3.3}$$

$$\hat{j}^{ge}_{+,i} = \hat{j}^{ge}_{x,i} + i\hat{j}^{ge}_{y,i} = \hbar|g_i\rangle\langle e_i| = \hbar\hat{\sigma}^i_{ge}, \tag{3.4}$$

$$\hat{j}^{ge}_{-,i} = \hat{j}^{ge}_{x,i} - i\hat{j}^{ge}_{y,i} = \hbar \left|e_i\right\rangle \left\langle g_i\right| = \hbar\hat{\sigma}^{i}_{ge}, \tag{3.5}$$

where the quantization axis is chosen to be z, and $\hat{j}^{ge}_{+(-),i}$ is the operator which raises (lowers) $\hat{j}^{ge}_{z,i}$ by $\hbar$. We have also introduced operator $\hat{\sigma}^{i}_{\mu\mu} = \left|\mu_i\right\rangle \left\langle \mu_i\right|$ which is the projection operator on the state $\left|\mu\right\rangle$ for the i-th atom and internal state operator $\hat{\sigma}^{i}_{\mu\nu} = \left|\mu_i\right\rangle \left\langle \nu_i\right|$ between states $\left|\mu\right\rangle$ and $\left|\nu\right\rangle$ for the i-th atom.The standard angular momentum commutation relation is fullfilled

$$[\hat{j}^{ge}_{x,i}, j^{ge}_{y,i}] = i\hbar\hat{j}^{ge}_{z,i}. \tag{3.6}$$

Since we are interested in collective variables, we look at the total angular momentum operators $J^{ge}_{l} = \sum_i \hat{j}^{ge}_{l,i}$, where $l = x, y, z$, which also fulfills the angular momentum commutation relation $[\hat{J}^{ge}_{x}, \hat{J}^{ge}_{y}] = i\hbar\hat{J}^{ge}_{z}$ as is obvious from (3.6). The collective state with all atoms in the ground state is denoted $\left|g^N\right\rangle = \left|J = N\hbar/2, M = N\hbar/2\right\rangle$, and has total angular momentum number $J^{ge} = N\hbar/2$ and eigenvalue of J^{ge}_{z} equal to $N\hbar/2$, where n is the number of atoms. For a sufficiently large number of atoms and a weakly perturbed system, i.e. only a few atoms out of the ground state, we can approximate the $\hat{J}^{ge}_{z}$ operator by its expectation value $\hat{J}^{ge}_{z} \approx \langle J^{ge}_{z}\rangle$. Since it is mathematically inapropriate to replace an operator with a number, a rigorous formulation can be done by using the Holstein-Primakoff transformation. Since we assumed that the $\langle J^{ge}_{z}\rangle \approx N\hbar/2 \geq 0$ we can introduce the collective canonical position and momentum operators associated with the transition between the ground state $\left|g\right\rangle$ and the excited $\left|e\right\rangle$

$$\hat{X}^{ge} = \frac{\hat{J}^{ge}_{x}}{\sqrt{\langle \hat{J}^{ge}_{z}\rangle}}, \tag{3.7}$$

$$\hat{P}^{ge} = \frac{\hat{J}^{ge}_{y}}{\sqrt{\langle \hat{J}^{ge}_{z}\rangle}}, \tag{3.8}$$

which satisfy the canonical commutation relation

$$[\hat{X}^{ge}, \hat{P}^{ge}] = \frac{1}{\langle \hat{J}^{ge}_{z}\rangle}[\hat{J}^{ge}_{x}, \hat{J}^{ge}_{y}] = i\hbar \tag{3.9}$$

Now we can define the symmetric collective annihilation operator for the state $\left|e\right\rangle$.

$$\hat{P} = \frac{\hat{X}^{ge} + i\hat{P}^{ge}}{\sqrt{2\hbar}} = \frac{\sum_m \hat{j}^{ge}_{+,m}}{\sqrt{2\hbar \left\langle \hat{J}^{ge}_{z}\right\rangle}} = \frac{\hbar\sum_m \left|g_m\right\rangle \left\langle e_m\right|}{\sqrt{2 \left\langle \hat{J}^{ge}_{z}\right\rangle}} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left|g_i\right\rangle \left\langle e_i\right| \tag{3.10}$$

which satisfy the commutation relation $[\hat{P}, \hat{P}^{\dagger}] = 1$. $\hat{P}$ stands for polarization and is a symmetric collective operator, not to be confused with $\hat{P}^{ge}$, which is the canonical momentum operator defined in (3.17).
If we apply the symmetric collective creation operator to the collective ground state $\left|g^N\right\rangle$, where all atoms are in state $\left|g\right\rangle$, we create a symmetric superposition of one atom being in the excited state $\left|e\right\rangle$.

$$\hat{P}^{\dagger} \left|g^N\right\rangle = \hat{P}^{\dagger} \left|g, g, g, ..., g\right\rangle = \frac{1}{\sqrt{N}} \sum_{i=1} \left|g, g, ..., g, e^i, g, ..., g, g\right\rangle \tag{3.11}$$

If we now take into consideration the second excited state $\left|r\right\rangle$. We can follow the same process as above, while this time considering the two states $\left|g\right\rangle$ and $\left|r\right\rangle$. We can define the angular momentum operators for the two states $\left|r\right\rangle$ and $\left|g\right\rangle$

$$\hat{j}^{gr}_{z,i} = \frac{\hbar}{2}(\left|g_i\right\rangle \left\langle g_i\right| - \left|r_i\right\rangle \left\langle r_m\right|) = \frac{\hbar}{2}(\hat{\sigma}^{i}_{gg} - \hat{\sigma}^{i}_{rr}), \tag{3.12}$$

$$\hat{j}^{ge}_{x,r} = \frac{\hbar}{2}(|e_i\rangle\langle g_i| + |g_i\rangle\langle e_i|) = \frac{\hbar}{2}(\hat{\sigma}^i_{eg} - \hat{\sigma}^i_{ge}), \tag{3.13}$$

$$\hat{j}^{ge}_{y,i} = \frac{i\hbar}{2}(|e_i\rangle\langle g_i| - |g_i\rangle\langle e_i|) = \frac{i\hbar}{2}(\hat{\sigma}^i_{eg} - \hat{\sigma}^i_{ge}), \tag{3.14}$$

$$\hat{j}^{re}_{+,i} = \hat{j}^{re}_{x,i} + i\hat{j}^{re}_{y,i} = \hbar\,|g_i\rangle\langle r_i| = \hbar\hat{\sigma}^i_{re}, \tag{3.15}$$

$$\hat{X}^{gr} = \frac{\hat{J}^{gr}_x}{\sqrt{\langle \hat{J}^{gr}_z\rangle}}, \tag{3.16}$$

$$\hat{P}^{gr} = \frac{\hat{J}^{gr}_y}{\sqrt{\langle \hat{J}^{gr}_z\rangle}}. \tag{3.17}$$

Now having defined the above canonical position and momentum associated with the transition between states $|g\rangle$ and $|g\rangle$, we can define the symmetric collective operator for state $|r\rangle$

$$\hat{S} = \frac{X^{gr} + iP^{gr}}{\sqrt{2\hbar}} = \frac{\sum_i j^{gr}_{+,i}}{\sqrt{2\hbar\,\langle J^{gr}_z\rangle}} = \frac{\hbar\sum_i |g_i\rangle\langle r_i|}{\sqrt{2\,\langle J^{ge}_z\rangle}} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}|g_i\rangle\langle r_i|\,, \tag{3.18}$$

where we have used the fact that

$$\langle \hat{J}^{rg}_z\rangle \approx \langle \hat{J}^{eg}_z\rangle \approx \hbar N/2. \tag{3.19}$$

The symmetric collective operator $\hat{S}$ stands for spin wave, since $|r\rangle$, will be later considered a metastable state. Acting with the symmetric collective creation operator $\hat{S}$ on the collective ground state $|g^N\rangle$, we end up in a symmetric superposition of one atom being in state $|r\rangle$

$$\hat{S}^\dagger|g^N\rangle = \hat{P}^\dagger|g,g,g,...,g\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}|g,g,...,g,r^i,g,...,g,g\rangle \tag{3.20}$$

. The symmetric collective operators $\hat{S}$,$\hat{P}$ will play an important role for the description of the scattering process, of the interface between atomic ensembles and light beams.

3.2 Atomic Ensemble in a Cavity

In the current section we will describe a Ξ-scheme 3-level atomic ensemble, confinded in a cavity, which is interacting with 2 monochromatic fields, one quantum sinlge photon cavity field and one strong classical driving field propagating in the same direction as the cavity field.[4] The cavity field is coupled to the lower transition between states $|e\rangle$ and $|g\rangle$ and the classical one drives the higher transition, between $|r\rangle$ and $|e\rangle$, as seen in figure 3.1b. In figure 3.1a we can see a schematic representation of the atomic cloud inside the cavity, interacting with the cavity field denoted by the field operator $\hat{\mathcal{E}}$ and the classical field denoted by the Rabi frequency Ω.

The electric field vector operator for the cavity field is given by

$$\hat{\mathcal{E}}(z) = \boldsymbol{\epsilon_1}\left(\frac{\hbar\omega_1}{\epsilon_0 V}\right)^{1/2}(\hat{a} + \hat{a}^\dagger)sin(\omega_1 z/c) \tag{3.21}$$

where ω_1 is the frequency of the cavity field and $\hat{a}^\dagger, \hat{a}$ the creation and annihilation operators for the cavity field and where $\boldsymbol{\epsilon_1}$ are the unit vector in the direction of the polarization of the cavity field, ϵ_0 the perimittivity of free space, V the quantization volume of the field and c the

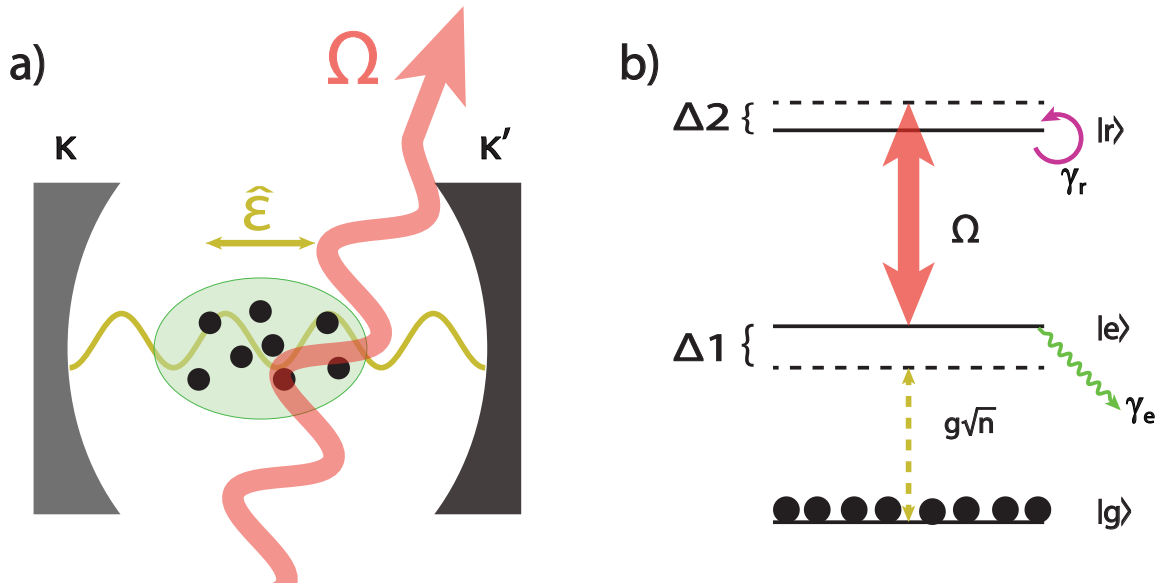

Figure 3.1: a)Schematic representation of a cloud of atoms in a cavity, interacting with the single mode field of the cavity $\hat{\mathcal{E}}$ coupled to transition $|g\rangle \leftrightarrow |e\rangle$ and a strong classical driving field Ω, which drives the transition $|e\rangle \leftrightarrow |r\rangle$. The fields are assumed to co-propage, but they have been drawn to cross for clarity. b)Energy level diagram of the system. All the atoms are initially in the ground state, γ_e is the decay of the excited state $|e\rangle$, γ_r is the dephasing of the metastable state $|r\rangle$. Matching colour code has been used between the figures.

speed of light.
The co-propagating single mode classical plane-wave field with frequency ω_2 is described by the electric field vector

$$E_2(z,t) = \boldsymbol{\epsilon_2}\mathcal{E}_2(t)cos(\omega_2(t-z/c)) = \boldsymbol{\epsilon_2}\frac{\mathcal{E}_2(t)}{2}(e^{i\omega_2(t-z/c)} + e^{-i\omega_2(t-z/c)}) \tag{3.22}$$

where $\boldsymbol{\epsilon_2}$ are the unit vector in the direction of the polarization of the classical field and $\mathcal{E}(t)$ its amplitude.
The Hamiltonian of the system is

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V} \tag{3.23}$$

where $\hat{\mathcal{H}}_0$ is the unperturbed Hamiltonian and $\hat{V}$ is the interaction Hamiltonia between the atomic ensemble and the fields.

$$\hat{\mathcal{H}}_0 = \hbar\omega_1\hat{a}^\dagger\hat{a} + \sum_i^N(\hbar\omega_g\hat{\sigma}^i_{gg} + \hbar\omega_e\hat{\sigma}^i_{ee} + \hbar\omega_r\hat{\sigma}^i_{rr}) \tag{3.24}$$

where as defined in the previous section $\hat{\sigma}^i_{\mu\mu} = |\mu_i\rangle\langle\mu_i|$ is the projection operator on the state $|\mu\rangle$ for the i-th atom and $\hbar\omega_g, \hbar\omega_e, \hbar\omega_r$ the energies of the states $|g\rangle, |e\rangle, |r\rangle$ respectively.
Now using the fact that the Hilbert space of the atomic cloud consists of all combinations of the atoms being in either of the three states, we can define the identity operator for the system $\hat{I} = \sum_i^N(\hat{\sigma}^i_{gg} + \hat{\sigma}^i_{ee} + \hat{\sigma}^i_{rr})$, so that we can rewrite the unperturbed Hamiltonian as

$$\hat{\mathcal{H}}_0 = \hbar\omega_1\hat{a}^\dagger\hat{a} + \sum_i^N(\hbar\omega_g\hat{\sigma}^i_{gg} + \hbar\omega_e\hat{\sigma}^i_{ee} + \hbar\omega_r\hat{\sigma}^i_{rr}) - \sum_i^N\omega_g(\hat{\sigma}^i_{gg} + \hat{\sigma}^i_{ee} + \hat{\sigma}^i_{rr}) + \omega_g\hat{I} \tag{3.25}$$

where $\omega_g\hat{I}$ is just a constant phase and it can be neglected. The unperturbed Hamiltonian can now be written as

$$\hat{\mathcal{H}}_0 = \hbar\omega_1\hat{a}^\dagger\hat{a} + \sum_i^N(\hbar\omega_{eg}\hat{\sigma}^i_{ee} + \hbar\omega_{rg}\hat{\sigma}^i_{rr}) \tag{3.26}$$

which is equivalent to setting ω_g as the zero energy. Above we used $\omega_{eg} = \omega_e - \omega_g$ and $\omega_{rg} = \omega_r - \omega_g$.

We now turn to the interaction Hamiltonian $\hat{V}$ and using the dipole approximation we have

$$\hat{V} = -\hbar \sum_i^N \hat{d}_i(\hat{\mathcal{E}}_1(z_i) + E_2(z_i, t)) = \hat{V}_1 + \hat{V}_2 \tag{3.27}$$

where $\hat{V}_1$ the Hamiltonian representing the interaction between the cavity field and the ensemble and $\hat{V}_2$ the one representing the interaction of the classical field with the ensemble. We have also denoted $\hat{d}_i$ the transition dipole vector operator of the i-th atom, which is a 3×3 matrix, where the transition between $|g\rangle \leftrightarrow |r\rangle$ is dipole forbidden i.e. ${}_i\langle g|\hat{\mathbf{d}}_i|r\rangle_i = {}_i\langle r|\hat{\mathbf{d}}_i|g\rangle_i = 0$.

We now consider the following assumptions, each field is coupled to one transition and in particular the cavity filed $\hat{\mathcal{E}}_1(z_i)$ is only coupled to the $|g\rangle \leftrightarrow |e\rangle$ transition and the classical $E_2(z_i, t)$ to the $|e\rangle \leftrightarrow |r\rangle$ i.e. ${}_i\langle e|\hat{\mathbf{d}}_i\boldsymbol{\epsilon_1}|r\rangle_i = {}_i\langle e|\hat{\mathbf{d}}_i\boldsymbol{\epsilon_2}|g\rangle_i = 0$.
Under these assumptions, the interactions Hamiltonians can be written as

$$\hat{V}_1 = -\hbar \sum_{i=1}^N (\hat{a} g_i \hat{\sigma}^i_{eg} sin(\omega_1 z_i/c) + \hat{a} g_i^* \hat{\sigma}^i_{ge} sin(\omega_1 z_i/c)) + h.c. \tag{3.28}$$

where $g_i = \langle e_i| (\boldsymbol{d}_i\boldsymbol{\epsilon_2}) |g_i\rangle \sqrt{\frac{\omega_1}{\hbar\epsilon_0 V}}$ is the coupling constant between the i-th atom and the cavity field, which for the sake of simplicity, will be considered real and equal for all atoms i.e. $g_i = g_i^* = g$.
The interaction Hamiltonian of the classical field with the ensemble is

$$\hat{V}_2 = -\hbar \sum_{j=1}^N (\Omega(t)\hat{\sigma}^j_{er} e^{-i\omega_2(t - z_j/c)} + \Omega^*(t)\hat{\sigma}^j_{re} e^{-i\omega_2(t - z_j/c)}) + h.c. \tag{3.29}$$

where $\Omega(t) = \langle e_i| (\boldsymbol{d}_i\boldsymbol{\epsilon_2}) |r_i\rangle \frac{\mathcal{E}_2(t)}{2\hbar}$ is the Rabi frequency of the classical field, defined as half of the traditional Rabi frequency, in order to avoid carrying factor of 2 around and a sign, so that e.g. a π pulse takes time $\pi/(2\Omega)$. We will consider the amplitude of the classical field $\mathcal{E}_2$ to be constant in time, which leads to time independent Rabi frequency Ω.
Now in order to remarkably simplify the interacting Hamiltonians $\hat{V}_1$ and $\hat{V}_2$, by the standard technique of standard rotating wave approximation technique, we move to a rotating frame of reference, through the following unitary transformation

$$\hat{U} = e^{-i\sum_{j=1}^N (\omega_1 \hat{a}^\dagger \hat{a} + \omega_1 \hat{\sigma}^j_{ee} + (\omega_1 + \omega_2)\hat{\sigma}^j_{rr})t} \tag{3.30}$$

This will transform our Hamiltonian $\hat{\mathcal{H}}$ to

$$\hat{\tilde{\mathcal{H}}} = \hat{U}^{-1}\hat{\mathcal{H}}\hat{U} - i\hbar\hat{U}^{-1}\frac{d\hat{U}}{dt} \tag{3.31}$$

We will look at each part of the Hamiltonian independently, first the transformed unperturbed Hamiltonian $\hat{\tilde{\mathcal{H}}}_0$ will be

$$\hat{\tilde{\mathcal{H}}}_0 = \hat{\mathcal{H}}_0 - i\hbar\hat{U}^{-1}\frac{d\hat{U}}{dt} = \hbar \sum_{i=1}^N ((\omega_{eg} - \omega_1)\hat{\sigma}^i_{ee} + \hbar(\omega_{rg} - \omega_1 - \omega_2))\hat{\sigma}^i_{rr} \tag{3.32}$$

where we have used the Baker-Hausdorf lemma (2.20). Now by defining the collective projection operators

$$\sigma_{\mu\mu} = \sum_{i}^{N} \hat{\sigma}^{i}_{\mu\mu} \tag{3.33}$$

and the detunings $\Delta_1 = \omega_{eg} - \omega_1$, $\Delta_2 = \omega_2 - \omega_{re}$, $\delta = \Delta_1 - \Delta_2$ we can rewrite the unperturbed Hamiltonian in the rotating frame as

$$\hat{\tilde{\mathcal{H}}}_0 = \hbar\Delta_1\hat{\sigma}_{ee} + \hbar(\Delta_1 - \Delta_2)\hat{\sigma}_{rr} = \hbar\Delta_1\hat{\sigma}_{ee} + \hbar\delta\hat{\sigma}_{rr} \tag{3.34}$$

Now we will transform the interaction Hamiltonian $\hat{V}_1$

$$\begin{aligned}\hat{\tilde{V}}_1 &= \hat{U}^{-1}\hat{V}_1\hat{U} - i\hbar\hat{U}^{-1}\frac{d\hat{U}}{dt} \\ &= -\hbar\sum_{j=1}^{N}(g\hat{a}\hat{\sigma}^{j}_{eg}sin(\omega_1 z_j/c) + \hat{a}^{\dagger}g\sigma^{j}_{eg}sin(\omega_1 z_j/c)e^{i2\omega_1 t} + g\hat{a}^{\dagger}\hat{\sigma}^{j}_{ge}sin(\omega_1 z_j/c) + \hat{a}g\sigma^{j}_{ge}sin(\omega_1 z_j/c)e^{-i2\omega_1 t})\end{aligned} \tag{3.35}$$

where we have used again the lemma (2.20). The second term and forth term oscillate rapidly compared to the first and the third, so they can be dropped under the rotating wave approximation. We also define the collective operator

$$\hat{\sigma}_{eg}(t) = \sum_{j=1}^{N}\hat{\sigma}^{j}_{eg}e^{-i\omega_1 t}sin(\omega_1 z_j/c) \tag{3.36}$$

and the slowly varying cavity field annihilation operator

$$\hat{\mathcal{E}}(t) = \hat{a}e^{i\omega_1 t} \tag{3.37}$$

Then we can write the transformed interaction Hamiltonian as

$$\hat{\tilde{V}}_1 = -\hbar(g\hat{\mathcal{E}}\hat{\sigma}_{eg} + g\hat{\mathcal{E}}^{\dagger}\hat{\sigma}_{ge}) \tag{3.38}$$

Finally we transform the interaction Hamiltonian $\hat{V}_2$

$$\begin{aligned}\hat{\tilde{V}}_2 &= \hat{U}^{-1}\hat{V}_2\hat{U} - i\hbar\hat{U}^{-1}\frac{d\hat{U}}{dt} \\ &= -\hbar\sum_{j=1}^{N}(\Omega\hat{\sigma}^{j}_{er}e^{-i\omega_2 z_j/c)} + \Omega\hat{\sigma}^{j}_{er}e^{i\omega_2 z_j/c}e^{-i2\omega_2 t} + \Omega^{*}\sigma^{j}_{re}e^{i\omega_2 z_j/c)} + \Omega^{*}\hat{\sigma}^{j}_{re}e^{i\omega_2 z_j/c}e^{-i2\omega_2 t}\end{aligned} \tag{3.39}$$

where again we made use of (2.20). The second term and the forth term oscillate rapidly compared to the first and the third, so they can again be dropped under the rotating wave approximation. Now we define the collective operator

$$\hat{\sigma}_{er} = \sum_{i}\hat{\sigma}^{i}_{er}e^{i\omega_2 z_i/c} \tag{3.40}$$

The transformed interaction Hamiltonian $\hat{\tilde{V}}_2$ can be written as

$$\hat{\tilde{V}}_2 = -\hbar(\Omega\hat{\sigma}_{re} + \Omega^{*}\hat{\sigma}_{er}) \tag{3.41}$$

So now we can write the effective Hamiltonian of the system in the rotating frame (3.31) after the rotating wave approximation as

$$\hat{\tilde{\mathcal{H}}} = \hat{\tilde{\mathcal{H}}}_0 + \hat{\tilde{V}}_1 + \hat{\tilde{V}}_2 = \hbar\Delta_1\hat{\sigma}_{ee} + \hbar\delta\hat{\sigma}_{rr} + \hbar(g\hat{\mathcal{E}}\hat{\sigma}_{eg} + g\hat{\mathcal{E}}^\dagger\hat{\sigma}_{ge} + \Omega\hat{\sigma}_{se} + \Omega^*\hat{\sigma}_{es}) \tag{3.42}$$

We will consider two sources of dissipation, the decay from the state $|e\rangle$ to the ground state $|g\rangle$ of the i-th atom

$$\hat{L}^i_{eg} = \sqrt{2\gamma_e}\sigma^i_{ge} \tag{3.43}$$

and the dephasing of the state $|r\rangle$ of the i-th atom

$$\hat{L}^i_{rr} = \sqrt{2\gamma_r}\sigma^i_{rr} \tag{3.44}$$

We can now derive the equations of motion for the collective atomic operators defined in (3.33), (3.36), (3.40) and their hermitian conjugates, by using the Lindblad equation for system operators (2.73), derived in the previous chapter

$$\dot{\hat{\sigma}}_{\mu\nu}(t) = \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{\sigma}_{\mu\nu}] + \sum_k (L_k^\dagger\hat{\sigma}_{\mu\nu}L_k - \frac{1}{2}(L_k^\dagger L_k\hat{\sigma}_{\mu\nu} + \hat{\sigma}_{\mu\nu}L_k^\dagger L_k)) + \hat{F}_{\mu\nu} \tag{3.45}$$

where $\sum_k \hat{L}_k = \sum_i^N \hat{L}^i_e g + \sum_i^N \hat{L}^i_r r$ since we have defined two Lindblad decay operators for our system. So we have the following 9 equations of motion of the collective atomic operators of our system.

$$\dot{\hat{\sigma}}_{gg} = \gamma_e\hat{\sigma}_{ee} - ig\hat{\mathcal{E}}\hat{\sigma}_{eg} + ig\hat{\mathcal{E}}^\dagger\hat{\sigma}_{ge} + \hat{F}_{gg} \tag{3.46}$$

$$\dot{\hat{\sigma}}_{rr} = -i\Omega^*\hat{\sigma}_{er} + i\Omega\hat{\sigma}_{re} + \hat{F}_{ss} \tag{3.47}$$

$$\dot{\hat{\sigma}}_{ee} = -\gamma_e\hat{\sigma}_{ee} + i\Omega^*\hat{\sigma}_{er} - i\Omega\hat{\sigma}_{re} + ig\hat{\mathcal{E}}\hat{\sigma}_{eg} - ig\hat{\mathcal{E}}^\dagger\hat{\sigma}_{ge} + \hat{F}_{ee} \tag{3.48}$$

$$\dot{\hat{\sigma}}_{ge} = -(\gamma_e + i\Delta_1)\hat{\sigma}_{ge} + i\Omega^*\hat{\sigma}_{gr} + ig\hat{\mathcal{E}}\hat{(}\sigma_{gg} - \hat{\sigma}_{ee}) + \hat{F}_{ge} \tag{3.49}$$

$$\dot{\hat{\sigma}}_{er} = -(\gamma_e + \gamma_r - i\Delta_2)\hat{\sigma}_{er} + i\Omega(\hat{\sigma}_{ee} - \hat{\sigma}_{rr}) - ig\hat{\mathcal{E}}^\dagger\hat{\sigma}_{gr} + \hat{F}_{er} \tag{3.50}$$

$$\dot{\hat{\sigma}}_{gr} = -(\gamma_r + i\delta)\hat{\sigma}_{gr} + i\Omega\hat{\sigma}_{ge} - ig\hat{\mathcal{E}}\hat{\sigma}_{er} + \hat{F}_{gr} \tag{3.51}$$

$$\dot{\hat{\sigma}}_{eg} = -(\gamma_e - i\Delta_1)\hat{\sigma}_{eg} - i\Omega\hat{\sigma}_{rg} - ig\hat{\mathcal{E}}^\dagger\hat{(}\sigma_{gg} - \hat{\sigma}_{ee}) + \hat{F}_{eg} \tag{3.52}$$

$$\dot{\hat{\sigma}}_{re} = -(\gamma_e + \gamma_r + i\Delta_2)\hat{\sigma}_{re} - i\Omega^*(\hat{\sigma}_{ee} - \hat{\sigma}_{rr}) + ig\hat{\mathcal{E}}\hat{\sigma}_{rg} + \hat{F}_{re} \tag{3.53}$$

$$\dot{\hat{\sigma}}_{rg} = -(\gamma_r - i\delta)\hat{\sigma}_{rg} - i\Omega^*\hat{\sigma}_{eg} + ig\hat{\mathcal{E}}^\dagger\hat{\sigma}_{re} + \hat{F}_{rg} \tag{3.54}$$

In the equations $(3.46) - (3.54)$ we introduced the Langevin noise operators $\hat{F}_{\mu\nu}$ for the atomic operators, which are neccesary in order to preserve the commutation relations, since we introduced decays.
For simplicity, for the rest of the thesis we will consider the Langevin noise operators negligible. In order to describe the scattering processes of the system we only need 2 of these 9 equations, plus the equation of motion of the cavity field and an equation known as the input output relation. We will acquire these two equations for two slightly different cases, first for a single-sided cavity and afterwards for a two-sided cavity.

Now we use the collective operators introduced in the previous section we have equations of motion for $\hat{S}, \hat{P}$

$$\hat{P} = -(\gamma_e + i\Delta_1)\hat{\sigma}_{ge} + i\Omega^*\hat{\sigma}_{gr} + ig\hat{\mathcal{E}} \tag{3.55}$$

$$\dot{\hat{S}} = -(\gamma_r + i\delta)\hat{S} + i\Omega\hat{P}. \tag{3.56}$$

As mentioned above we will use these equations of motion, but in order describe the scattering dynamics of the system we need the equations of motion for the cavity field, which are diferrent depending on the type of the cavity.

3.2.1 Single-Sided Cavity

We have described the equations of motion for the atomic operators of our system, but in order to be able to fully describe the scattering dynamics of the system we need the equations of motion for the cavity field. In this section we do so for the case of a one sided cavity.
To find the equation of motion for the cavity field, we need to define an extra Hamiltonian, similar to (2.18), which will describe the coupling of the single-sided cavity to the environment, which following the process used in the previous chapter, will be modeled by a reservoir of a large number of harmonic oscillators.
The entire system will be described by

$$\hat{\mathcal{H}}_{tot} = \hat{\mathcal{H}} + \hat{\mathcal{H}}_r + \hat{\mathcal{H}}_{cr} \tag{3.57}$$

where $\hat{\mathcal{H}}$ is defined in (3.23) as the Hamiltonian of the system.
$\hat{\mathcal{H}}_r$ is the Hamiltonian of the reservoir, descrbing the field external to the cavity, defined as in

$$\hat{\mathcal{H}}_r = \hbar \sum_i \omega_i \hat{b}_i^\dagger \hat{b}_i \tag{3.58}$$

$\hat{\mathcal{H}}_{cr}$ is the interaction Hamiltonian between reservoir and the cavity

$$\hat{\mathcal{H}}_{cr} = \hbar \sum_i (g_i \hat{\mathcal{E}}^\dagger(t) \hat{b}_i(t) + g_i^* \hat{b}_i^\dagger(t) \hat{\mathcal{E}}(t)) \tag{3.59}$$

where $\hat{b}$ is the annihilation operator of. The Heisenberg equation of motion of the reservoir's mode $\hat{b}(\omega)$ with frequency ω is

$$\dot{\hat{b}}(\omega, t) = \frac{i}{\hbar}[\hat{\mathcal{H}}_{tot}, \hat{\mathcal{E}}] = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}} + \hat{\mathcal{H}}_r + \hat{\mathcal{H}}_{cr}, \hat{\mathcal{E}}(t)] = -i\omega \hat{b}(\omega, t) + ig(\omega)\hat{\mathcal{E}}(t) \tag{3.60}$$

Now we can integrate formaly the above equation and using as initial condition the time $t_0 < t$, where t_0 the input time and get the expression for the operator $\hat{b}(\omega)$

$$\hat{b}(\omega, t) = e^{-i\omega(t-t_0)} \hat{b}(\omega, t_0) + ig(\omega) \int_t^{t_1} dt' e^{-i\omega(t-t')} \hat{a}(t') \tag{3.61}$$

or with the final condition at time $t_1 > t$, where t_1 the output time and get

$$\hat{b}(\omega, t) = e^{-i\omega(t-t_1)} \hat{b}(\omega, t_1) - ig(\omega) \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\mathcal{E}}(t') \tag{3.62}$$

$$\dot{\hat{\mathcal{E}}}(t) = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}} + \hat{\mathcal{H}}_r + \hat{\mathcal{H}}_{cr}, \hat{\mathcal{E}}(t)] = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}}, \hat{\mathcal{E}}(t)] - \int_{-\infty}^{\infty} d\omega g(\omega) \hat{b}(\omega, t) \tag{3.63}$$

Substituing (3.17)

$$\dot{\hat{\mathcal{E}}}(t) = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}}, \hat{\mathcal{E}}(t)] - \int_{-\infty}^{\infty} d\omega g(\omega) e^{-i\omega(t-t_0)} \hat{b}(\omega, t_0) - \int_{-i\infty}^{\infty} d\omega g^2(\omega) \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\mathcal{E}}(t') \tag{3.64}$$

We define input field

$$\hat{\mathcal{E}}_{in}(t) = -i \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} d\omega e^{-\omega(t-t_0)} \hat{b}(\omega, t_0) \tag{3.65}$$

with $[\hat{\mathcal{E}}_{in}(t), \hat{\mathcal{E}}_{in}^\dagger(t')] = \delta(t - t')$
Now as in Langevin, we do Markov Approximation

$$\int_{-\infty}^{\infty} d\omega g^2(\omega) \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\mathcal{E}}(t') \approx g^2(\omega) \int_{-\infty}^{\infty} d\omega \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\mathcal{E}}(t') = \frac{\gamma}{2} \hat{\mathcal{E}}(t) \tag{3.66}$$

Substitiuing in (3.17)

$$\dot{\hat{\mathcal{E}}}(t) = \frac{i}{\hbar}[\hat{\hat{\mathcal{H}}}, \hat{\mathcal{E}}] - \kappa\hat{\mathcal{E}}(t) + \sqrt{2\kappa}\hat{\mathcal{E}}_{in} \tag{3.67}$$

where $\kappa = \gamma/2$
Now following the same process using (3.18)

$$\dot{\hat{\mathcal{E}}}(t) = \frac{i}{\hbar}[\hat{\hat{\mathcal{H}}}, \hat{\mathcal{E}}] + \kappa\hat{\mathcal{E}}(t) + \sqrt{2\kappa}\hat{\mathcal{E}}_{out} \tag{3.68}$$

where

$$\hat{\mathcal{E}}_{out}(t) = i\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} d\omega e^{-i\omega(t-t_1)}\hat{b}(\omega, t_1) \tag{3.69}$$

We note that $\hat{\mathcal{E}}_{out}$ has been defined with a minus sign, in respect to $\hat{\mathcal{E}}_{in}$. This has been done, due to the fact that we consider $\hat{\mathcal{E}}_{out}$ to be the reflected part of $\hat{\mathcal{E}}_{in}$, and so it should acquire a π phase shift upon reflection and $e^{i\pi} = -1$.
From equation (3.67),(3.68) we get the input output relation

$$\hat{\mathcal{E}}_{out}(t) + \hat{\mathcal{E}}_{in}(t) = \sqrt{2\kappa}\hat{\mathcal{E}}(t) \tag{3.70}$$

From (3.67), we have the equation of motion for the cavity field

$$\dot{\hat{\mathcal{E}}}(t) = -\kappa\hat{\mathcal{E}}(t) + ig\sqrt{N}\hat{P}(t) + \sqrt{2\kappa}\hat{\mathcal{E}}_{in}(t) \tag{3.71}$$

We rewrite the equation of motion for the polarization and the spin operators

$$\dot{\hat{P}}(t) = -(\gamma_e + i\Delta)\hat{P}(t) + ig\sqrt{N}\hat{\mathcal{E}}(t) + i\Omega\hat{S}(t) \tag{3.72}$$

$$\dot{\hat{S}}(t) = -(\gamma_r + i\delta)\hat{S}(t) + i\Omega^*\hat{P}(t) \tag{3.73}$$

This system of four equations will be enough to describe the scattering dynamics of the system. To bring this equation to an easier form, we perform the Fourier transform, which is the decomposition of any function into a sum of sinusoidal basis functions. Each of these basis functions is a complex exponential of a different frequency. The Fourier Transform therefore gives us a unique way of viewing any function - as the sum of simple sinusoids.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega f(\omega)e^{-i\omega t} \tag{3.74}$$

where

$$f(\omega) = \int_{-\infty}^{\infty} dt f(t)e^{i\omega t} \tag{3.75}$$

It is important to stress that since negative frequency ω, doesn't have physical sense, we consider ω with respect to the cavity frequency ω_1. We show how this transforms the above equations from the time domain to the frequency domain, starting with

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\hat{\mathcal{E}}_{out}(\omega)e^{-i\omega t} + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\hat{\mathcal{E}}_{in}(\omega)e^{-i\omega t} = \frac{2\kappa}{2\pi}\int_{-\infty}^{\infty} d\omega\hat{\mathcal{E}}(\omega)e^{-i\omega t} \tag{3.76}$$

We gather the terms to one side

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\left(\hat{\mathcal{E}}_{out}(\omega) + \hat{\mathcal{E}}_{in}(\omega) - 2\kappa\hat{\mathcal{E}}(\omega)\right)e^{-i\omega t} = 0 \tag{3.77}$$

and it is easy to see, for this to be true we need

$$\hat{\mathcal{E}}_{out}(\omega) + \hat{\mathcal{E}}_{in}(\omega) = 2\kappa\hat{\mathcal{E}}(\omega) \tag{3.78}$$

This equation is equivalent to equation (3.70), but all functions are in the frequency domain. Following the same process we bring the other three equations to the frequency domain

$$-i\omega\hat{\mathcal{E}}(\omega) = -\kappa\hat{\mathcal{E}}(\omega) + ig\sqrt{N}\hat{P}(\omega) + \sqrt{2\kappa}\hat{\mathcal{E}}_{in}(\omega) \tag{3.79}$$

$$-i\omega\hat{P}(\omega) = -(\gamma_e + i\Delta)\hat{P}(\omega) + ig\sqrt{N}\hat{\mathcal{E}}(\omega) + i\Omega\hat{S}(\omega) \tag{3.80}$$

$$-i\omega\hat{S}(\omega) = -(\gamma_r + i\delta)\hat{S}(\omega) + i\Omega^*\hat{P}(\omega) \tag{3.81}$$

We can now solve this system of three equations. We solve (4.41) for $\hat{P}(\omega)$ and then substitute in (4.40) to get

$$\frac{(\gamma_e + i(\Delta-\omega))(\gamma_r + i(\delta-\omega))}{i\Omega^*}\hat{S}(\omega) = ig\sqrt{N}\hat{\mathcal{E}}(\omega) + i\Omega\hat{S}(\omega) \tag{3.82}$$

$$\hat{S}(\omega) = -\frac{\sqrt{2\kappa}\hat{\mathcal{E}}_{in}(\omega)g\sqrt{N}\Omega^*}{g^2N(\gamma_r + i(\delta-\omega)) + (\kappa - i\omega)((-i\gamma_e + \Delta - \omega)(i\gamma_r - (\delta-\omega) + \Omega^2)} \tag{3.83}$$

$$\hat{P}(\omega) = \frac{\sqrt{2\kappa}\hat{\mathcal{E}}_{in}(\omega)g\sqrt{N}(i\gamma_r - (\delta-\omega))}{g^2N(\gamma_r + i(\delta-\omega)) + (\kappa - i\omega)((-i\gamma_e + \Delta - \omega)(i\gamma_r - (\delta-\omega)) + \Omega^2)} \tag{3.84}$$

$$\hat{\mathcal{E}}(\omega) = \frac{\sqrt{2\kappa}\hat{\mathcal{E}}_{in}(\omega)((\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2)}{g^2N(\gamma_r + i(\delta-\omega)) + (\kappa - i\omega)((-i\gamma_e + \Delta - \omega)(i\gamma_r - (\delta-\omega)) + \Omega^2)} \tag{3.85}$$

Since we use single-sided cavity the reflection coefficeint will be given by the

$$R(\omega) = \frac{\hat{\mathcal{E}}_{out}(\omega)}{\hat{\mathcal{E}}_{in}(\omega)} = -1 + \sqrt{2\kappa}\frac{\hat{\mathcal{E}}(\omega)}{\hat{\mathcal{E}}_{in}(\omega)} \tag{3.86}$$

In this point we will have to comment on the informal notation used in the last equation, which we will be using throughout this thesis. Mathematically of course it doesn't make sense to divide an operator by an operator, but in our case all the operators of our system are proportional to the operator of the input field $\hat{\mathcal{E}}_{in}(\omega)$, e.g. $\hat{\mathcal{E}}_{out}(\omega) = R(\omega)\hat{\mathcal{E}}_{in}(\omega)$. This mathematically "inappropriate" division allows us to define and work with these proposortionality factors, which are very useful for our calculations. Overall, this informal division, represents the proportionality factor between the operator in the numerator and the input field operator $\hat{\mathcal{E}}_{in}(\omega)$ which will always be in the denominator.
Using (3.85) and (4.41) we find

$$R(\omega) = -1 + \frac{2\kappa((\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2)}{g^2N(\gamma_r + i(\delta-\omega)) + (\kappa - i\omega)((-i\gamma_e + \Delta - \omega)(i\gamma_r - (\delta-\omega)) + \Omega^2)} \tag{3.87}$$

which can be brought in the following form

$$R(\omega) = -1 + 2\kappa\left(\kappa - i\omega + \frac{g^2N}{\gamma_e + i(\Delta-\omega) + \frac{|\Omega|^2}{\gamma_r + i(\delta-\omega)}}\right)^{-1} \tag{3.88}$$

A very important relation, that we will be using in order to describe the scattering processes of the system, is the conservation of probability stated as following

$$\begin{aligned}\int_0^\infty dt\hat{\mathcal{E}}_{in}(t)\int_0^\infty dt'\hat{\mathcal{E}}^\dagger_{in}(t') = \int_0^\infty dt\hat{\mathcal{E}}_{out}(t)\int_0^\infty dt'\hat{\mathcal{E}}^\dagger_{out}(t') + (\sqrt{2\gamma_e}\int_0^\infty dt\hat{P}(t))(\sqrt{2\gamma_e}\int_0^\infty dt'\hat{P}^\dagger(t')) \\ + (\sqrt{2\gamma_r}\int_0^\infty dt\hat{S}(t))(\sqrt{2\gamma_r}\int_0^\infty dt'\hat{S}^\dagger(t'))\end{aligned} \tag{3.89}$$

This equation basically states that the total probability of entering the system, over the entire time domain, is equal to the total probability of exiting the system plus the total probability that of being lost to the environment through any of the two dissipative processes defined for our system.
It is easy to move to the frequenct domain, by Fourier transforming the above equation, This way we view the equation in terms of intensity, i.e. rates at which energy enters and exits our system plus the dissipative loss.

$$|\hat{\mathcal{E}}_{in}(\omega)|^2 = |\hat{\mathcal{E}}_{out}(\omega)|^2 + 2\gamma_e|\hat{P}(\omega)|^2 + 2\gamma_r|\hat{S}(\omega)|^2 \tag{3.90}$$

Now we turn to the proportionality factors, using the trick defined above.

$$1 = |R(\omega)|^2 + 2\gamma_e\frac{|\hat{P}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} + 2\gamma_r\frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} \tag{3.91}$$

For convinience we define the proportionality factors

$$\tilde{P}(\omega) = \frac{\hat{P}(\omega)}{\hat{\mathcal{E}}_{in}(\omega)} \tag{3.92}$$

and

$$\tilde{S}(\omega) = \frac{\hat{S}(\omega)}{\hat{\mathcal{E}}_{in}(\omega)} \tag{3.93}$$

Now in order to gain better understanding of how this system behaves, we first turn off the laser field that couples to the higher transition i.e. $\Omega = 0$.
From (3.83) it is clear that in this case, the last term of $\tilde{S}(\omega)$ is zero. Our system that is initially in the ground state cannot couple to the $|r\rangle$ state, so the other two rates are given by the following equations.

$$2\gamma_e|\tilde{P}(\omega)|^2 = 2\gamma_e\frac{|\hat{P}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} = \frac{4g^2N\gamma_e\kappa}{g^4N^2 + 2g^2N(\Delta\kappa - (\gamma+\kappa)\omega) + (\gamma^2 + (\Delta-\omega)^2)(\kappa^2+\omega^2)} \tag{3.94}$$

$$\begin{aligned} |R(\omega)|^2 =& \frac{g^4N^2 - 2g^2N(\gamma\kappa + (-\Delta+\omega)\omega) + (\gamma^2 + (\Delta-\omega)^2)(\kappa^2+\omega^2)}{g^4N^2 + 2g^2N(\gamma\kappa + (\Delta-\omega)\omega) + (\gamma^2 + (\Delta-\omega)^2)(\kappa^2+\omega^2)} \\ =& 1 - \frac{4g^2N\gamma_e\kappa}{g^4N^2 + 2g^2N(\gamma\kappa + (\Delta-\omega)\omega) + (\gamma^2 + (\Delta-\omega)^2)(\kappa^2+\omega^2)} \end{aligned} \tag{3.95}$$

We now plot these rates as functions of the dimensionless frequency ω/κ with respect to the lower transition frequency, for zero detuning. We see that on resonance the photon interacts strongly with the lower atomic transition and as a result, the energy decays from the excited state. Moving away from resonance the photon nteracts less with the atoms, while far from then it doesn't see the atoms and is just reflected by the cavity.

If we now turn the laser coupled to the higher transition on, i.e. $\Omega \neq 0$. The equations are

$$|R(\omega)|^2 = 1 - \frac{4g^2N(\gamma_e\kappa\gamma_r^2 + \omega(-\Delta+\omega)(\delta-\omega)^2 + 2\gamma_r\kappa\Omega^2)}{A1} \tag{3.96}$$

$$2\gamma_e|\tilde{P}(\omega)|^2 = \frac{4g^2N(\gamma_e\kappa\gamma_r^2 + \omega(-\Delta+\omega)(\delta-\omega)^2)}{A1} \tag{3.97}$$

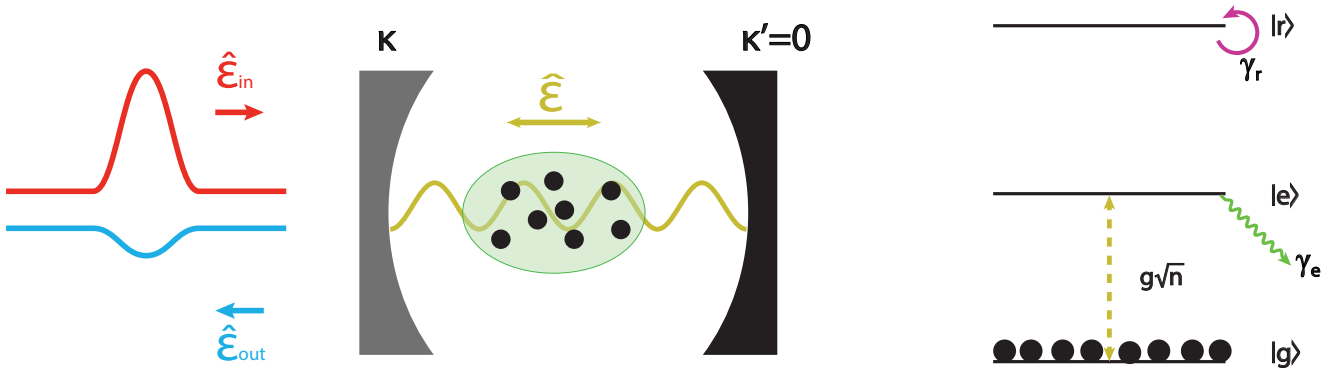

Figure 3.2: Schematic representation of atoms in the cavity. The cavity field is resonant to the $|g\rangle \leftrightarrow |e\rangle$ transition. Very low reflection is observed as it is evident from the following graph, most of the input energy is lost through decay of the excited state.

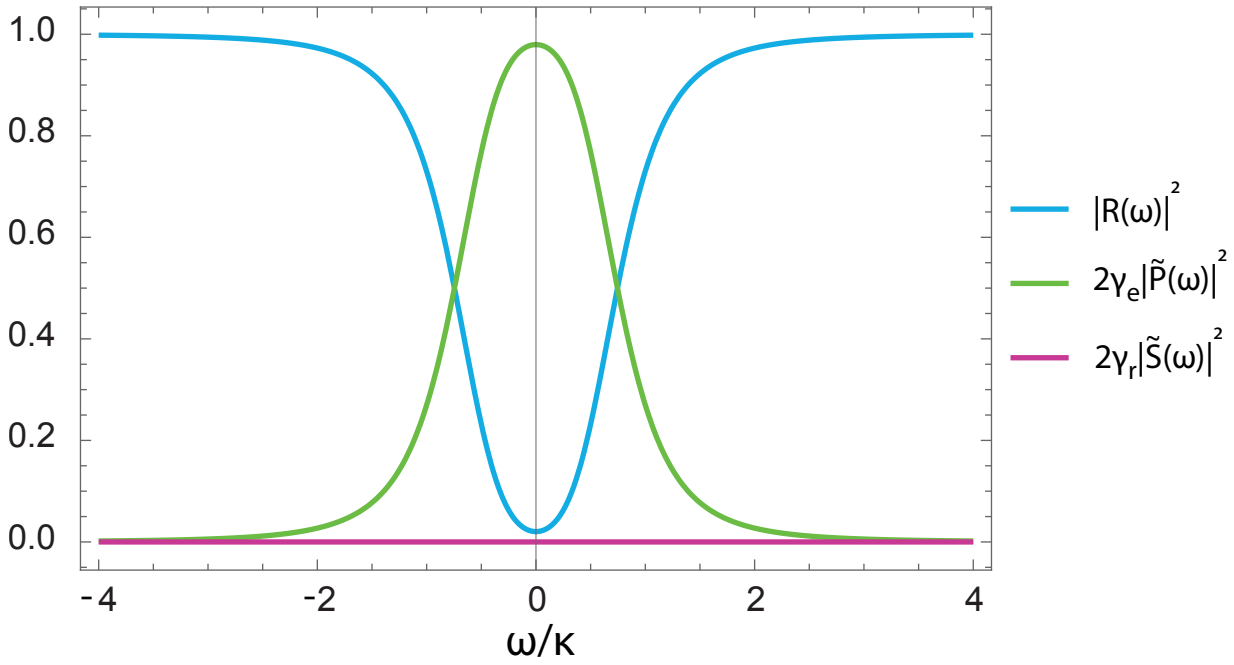

Figure 3.3: Diagram of the reflection rate $|R(\omega)|^2$, the loss rate through decay of the excited state $|e\rangle$, $2\gamma_e|\tilde{P}(\omega)|^2$ and the loss rate through dephasing of the state $|r\rangle$ as a function of the dimensionless frequency of the sugnal, for the case of the driving field being turned off $\Omega = 0$. The values used g=0.1, n=40,γ_e=0.3,κ=1,$\Delta = \delta = 0$.

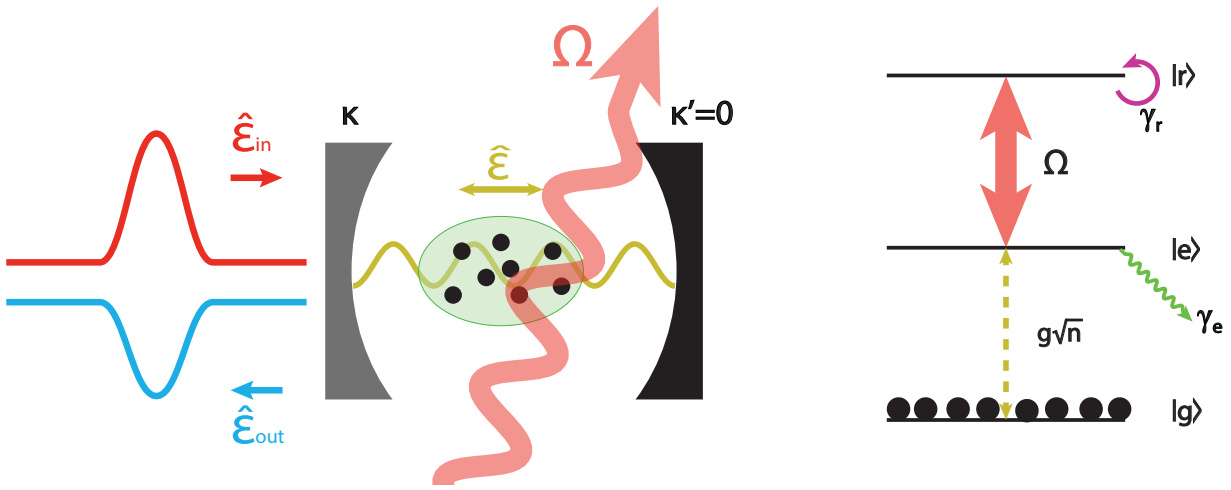

Figure 3.4: Schematic representation of atoms in the cavity. The cavity field is resonant to the $|g\rangle \leftrightarrow |e\rangle$ transition, as is the driving field Ω to the $|e\rangle \leftrightarrow |r\rangle$ transition. Strong reflection is observed, through the destructive of the 2 transitions a phenomenon called Electromagnetically Induced Transparency.

$$2\gamma_r|\tilde{S}(\omega)|^2 = \frac{8g^2N\gamma_r\kappa\Omega^2}{A1} \tag{3.98}$$

where

$$\begin{aligned} A1 =& g^4N^2(\gamma_r^2+(\delta-\omega)^2)+2g^2N((\gamma_r^2+(\delta-\omega)^2)(\gamma_e\kappa+(\Delta-\omega)\omega)+(\gamma_r\kappa+\omega(-\delta+\omega))\Omega^2)+ \\ & (\kappa^2+\omega^2)((\gamma_r^2+(\delta-\omega)^2)(\gamma_e^2+(\Delta-\omega)^2)+2(\gamma_e\gamma_r+(\delta-\omega)(-\Delta+\omega))\Omega^2+\Omega^4) \end{aligned} \tag{3.99}$$

It is obvious equation (4.50) stands.

We now plot the graph as before with zero detunings and zero dephasing of the $|r\rangle$ state. We observe that now there is perfect reflection on resonance and the field doesn't interact with the atoms. This is effect is called Electromagnetically Induced Transparency and is due to the destructive interference berween lower and higher transition, that that leads to the inaccesibility of the excited state $|e\rangle$,

3.2.2 2-sided Cavity

If we now consider a a 2-sided cavity, we have coupling with the environment through the left and the right imperfect mirror. The field external to the cavity, is divided into two reservoirs one next to the right mirror and one next to left mirror of the cavity.

$$\hat{\mathcal{H}}_{r2} = \hbar\sum_i \omega_{L,i}\hat{b}^\dagger_{L,i}\hat{b}_{L,i} + \hbar\sum_i \omega_{R,i}\hat{b}^\dagger_{R,i}\hat{b}_{R,i} \tag{3.100}$$

The interactions of the cavity field, with the external fields in each side is given by the interaction Hamiltonian

$$\hat{\mathcal{H}}_{cr2} = \hbar\sum_i (g_{i,L}\hat{\mathcal{E}}^\dagger(t)\hat{b}_{i,L}(t) + g_{i,R}\hat{\mathcal{E}}^\dagger(t)\hat{b}_{i,R}(t) + g^*_{i,L}\hat{b}^\dagger_{i,L}(t)\hat{\mathcal{E}}(t) + g^*_{i,R}\hat{b}^\dagger_{i,R}(t)\hat{\mathcal{E}}(t)) \tag{3.101}$$

$$\dot{\hat{b}}_L(\omega,t) = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}} + \hat{\mathcal{H}}_r + \hat{\mathcal{H}}_{cr}, \hat{\mathcal{E}}(t)] = -i\omega\hat{b}_L(\omega,t) + ig_L(\omega)\hat{\mathcal{E}}(t) \tag{3.102}$$

$$\dot{\hat{b}}_R(\omega,t) = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}} + \hat{\mathcal{H}}_r + \hat{\mathcal{H}}_{cr}, \hat{\mathcal{E}}(t)] = -i\omega\hat{b}_R(\omega,t) + ig_R(\omega)\hat{\mathcal{E}}(t) \tag{3.103}$$

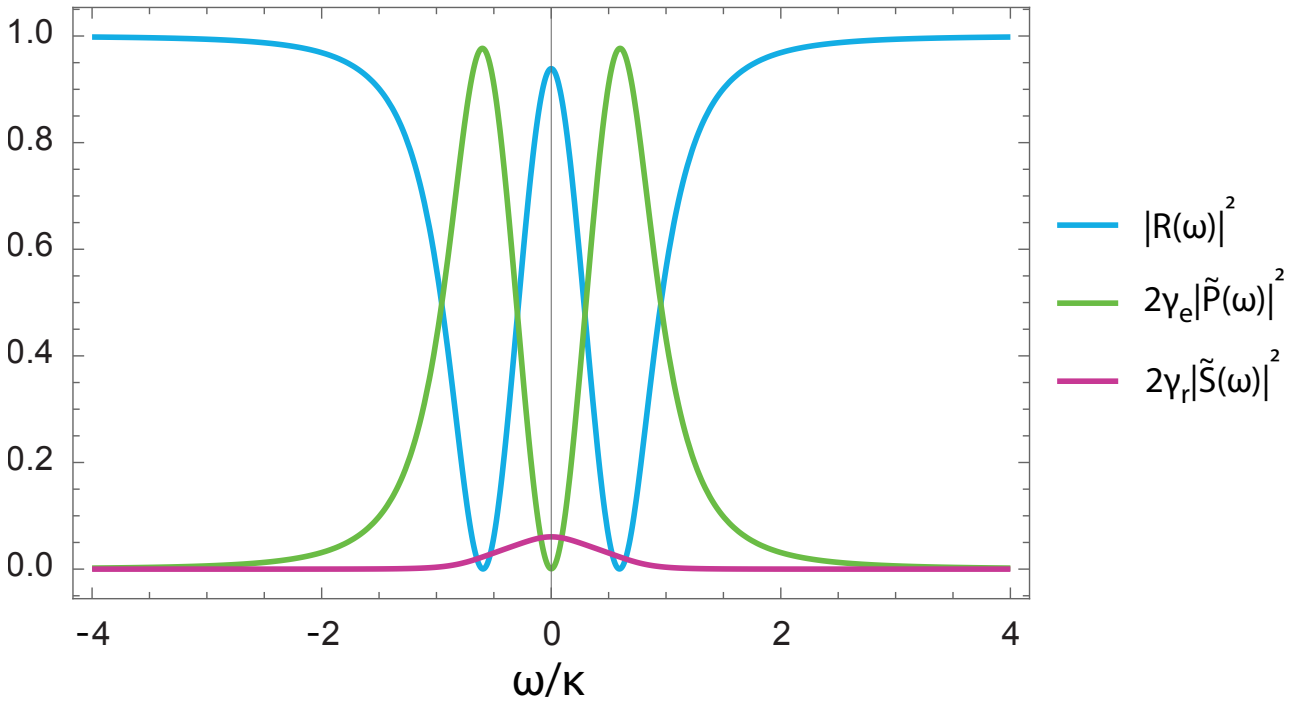

Figure 3.5: Diagram of the reflection rate $|R(\omega)|^2$, the loss rate through decay of the excited state $|e\rangle$, $2\gamma_e|\tilde{P}(\omega)|^2$ and the loss rate through dephasing of the state $|r\rangle$ as a function of the dimensionless frequency of the cavity, The driving field is on, with strength $\Omega = 0.5$. The rest of the values used are g=0.1, n=40,γ_e=0.3,κ=1,$\Delta = \delta = 0$.

Integrating formally with the initial condition at time $t_0 < t$, the input,

$$\hat{b}_L(\omega,t) = e^{-i\omega(t-t_0)}\hat{b}_L(\omega,t_0) + ig_L(\omega)\int_t^{t_1} dt' e^{-i\omega(t-t')}\hat{\mathcal{E}}(t') \tag{3.104}$$

or with the initial condition at time $t_1 > t$, the output,

$$\hat{b}_L(\omega,t) = e^{-i\omega(t-t_1)}\hat{b}_L(\omega,t_1) - ig_L(\omega)\int_t^{t_1} dt' e^{-i\omega(t-t')}\hat{\mathcal{E}}(t') \tag{3.105}$$

$$\hat{b}_R(\omega,t) = e^{-i\omega(t-t_1)}\hat{b}_R(\omega,t_1) - ig_R\omega)\int_t^{t_1} dt' e^{-i\omega(t-t')}\hat{\mathcal{E}}(t') \tag{3.106}$$

Substituing (3.17)

$$\begin{aligned}\dot{\hat{\mathcal{E}}}(t) = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}},\hat{\mathcal{E}}(t)] &- \int_{-\infty}^{\infty} d\omega g(\omega) e^{-i\omega(t-t_0)}\hat{b}_R(\omega,t_0) - i\int_{-\infty}^{\infty} d\omega g_R^2(\omega)\int_{t_0}^{t} dt' e^{-i\omega(t-t')}\hat{\mathcal{E}}(t') \\ &- \int_{-\infty}^{\infty} d\omega g(\omega) e^{-i\omega(t-t_0)}\hat{b}_L(\omega,t_0) - i\int_{-\infty}^{\infty} d\omega g_L^2(\omega)\int_{t_0}^{t} dt' e^{-i\omega(t-t')}\hat{\mathcal{E}}(t')\end{aligned} \tag{3.107}$$

We define input field

$$\hat{\mathcal{E}}_{in,R}(t) = -i\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} d\omega e^{-\omega(t-t_0)}\hat{b}_R(\omega,t_0) = 0 \tag{3.108}$$

$$\hat{\mathcal{E}}_{in,L}(t) = -i\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} d\omega e^{-\omega(t-t_0)}\hat{b}_L(\omega,t_0) \tag{3.109}$$

with $[\hat{\mathcal{E}}(t), \hat{\mathcal{E}}^\dagger(t')] = \delta(t-t')$
Now as in Langevin, we do Markov Approximation

$$\int_{-\infty}^{\infty} d\omega g_L^2(\omega) \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\mathcal{E}}(t') + \int_{-\infty}^{\infty} d\omega g_R^2(\omega) \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\mathcal{E}}(t') \approx$$
$$g_L^2(\omega) \int_{-\infty}^{\infty} d\omega \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\mathcal{E}}(t') + g_R^2(\omega) \int_{-\infty}^{\infty} d\omega \int_{t_0}^{t} dt' e^{-i\omega(t-t')} \hat{\mathcal{E}}(t') = \frac{\gamma_L + \gamma_R}{2} \hat{\mathcal{E}}(t) \tag{3.110}$$

Substitiuing in (3.17)

$$\dot{\hat{\mathcal{E}}}(t) = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}}, \hat{\mathcal{E}}] - (\kappa_R + \kappa_L)\hat{\mathcal{E}}(t) + \sqrt{2\kappa_L}\hat{\mathcal{E}}_{in} \tag{3.111}$$

where $\kappa_L = \gamma_L/2$ and $\kappa_R = \gamma_R/2$
Now following the same process using (3.18)

$$\dot{\hat{\mathcal{E}}}(t) = \frac{i}{\hbar}[\hat{\tilde{\mathcal{H}}}, \hat{\mathcal{E}}] + (\kappa_R + \kappa_L)\hat{\mathcal{E}}(t) - \sqrt{2\kappa_L}\hat{\mathcal{E}}_{out,L} + \sqrt{2\kappa_R}\hat{\mathcal{E}}_{out,R} \tag{3.112}$$

where

$$\hat{\mathcal{E}}_{out,R}(t) = -i\frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} d\omega e^{-\omega(t-t_1)} \hat{b}_R(\omega, t_1) \tag{3.113}$$

$$\hat{\mathcal{E}}_{out,L}(t) = i\frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} d\omega e^{-\omega(t-t_1)} \hat{b}_L(\omega, t_1) \tag{3.114}$$

From equation (3.111), (3.112) we get the input output relations for each side, similar to the (3.70)

$$\hat{\mathcal{E}}_{out,L(t)} + \hat{\mathcal{E}}_{in,L}(t) = \sqrt{2\kappa_L}\hat{\mathcal{E}}(t) \tag{3.115}$$

$$\hat{\mathcal{E}}_{out,R}(t) = \sqrt{2\kappa_R}\hat{\mathcal{E}}(t) \tag{3.116}$$

From equation (3.111) we get the equation of motion for the two sided cavity field.

$$\dot{\hat{\mathcal{E}}}(t) = -(\kappa_R + \kappa_L)\hat{\mathcal{E}}(t) + ig\sqrt{N}\hat{P}(t) + \sqrt{2\kappa_L}\hat{\mathcal{E}}_{in}(t) \tag{3.117}$$

These three equations together with (3.72), (3.73) is a system of 5 equations that describe the scattering dynamics of the 3-level atomic ensemle in a 2 sided cavity. As for the single sided case we can move to the frequency domain by Fourier transforming the 5 equations and have the following 5 equations.

$$\hat{\mathcal{E}}_{out,L(\omega)} + \hat{\mathcal{E}}_{in,L}(\omega) = \sqrt{2\kappa_L}\hat{\mathcal{E}}(\omega) \tag{3.118}$$

$$\hat{\mathcal{E}}_{out,R}(\omega) = \sqrt{2\kappa_R}\hat{\mathcal{E}}(\omega) \tag{3.119}$$

$$\dot{\hat{\mathcal{E}}}(\omega) = -(\kappa_R + \kappa_L)\hat{\mathcal{E}}(\omega) + ig\sqrt{N}\hat{P}(\omega) + \sqrt{2\kappa_L}\hat{\mathcal{E}}_{in}(\omega) \tag{3.120}$$

$$-i\omega\hat{P}(\omega) = -(\gamma_e + i\Delta)\hat{P}(\omega) + ig\sqrt{N}\hat{\mathcal{E}}(\omega) + i\Omega\hat{S}(\omega) \tag{3.121}$$

$$-i\omega\hat{S}(\omega) = -(\gamma_r + i\delta)\hat{S}(\omega) + i\Omega^*\hat{P}(\omega) \tag{3.122}$$

Solving this system we get the last 3 equations

$$\hat{S}(\omega) = -\frac{\sqrt{2\kappa_L}\hat{\mathcal{E}}_{in}(\omega)g\sqrt{N}\Omega^*}{(g^2N(\gamma_r + i(\delta-\omega)) + (\kappa_R + \kappa_L - i\omega)((-i\gamma_e + \Delta - \omega)(i\gamma_r + \omega) + \Omega^2)} \tag{3.123}$$

$$\hat{P}(\omega) = \frac{\sqrt{2\kappa_L}\hat{\mathcal{E}}_{in}(\omega)g\sqrt{N}(i\gamma_r - (\delta-\omega))}{(g^2N(\gamma_r + i(\delta-\omega)) + (\kappa_R + \kappa_L - i\omega)((-i\gamma_e + \Delta - \omega)(i\gamma_r + \omega) + \Omega^2)} \tag{3.124}$$

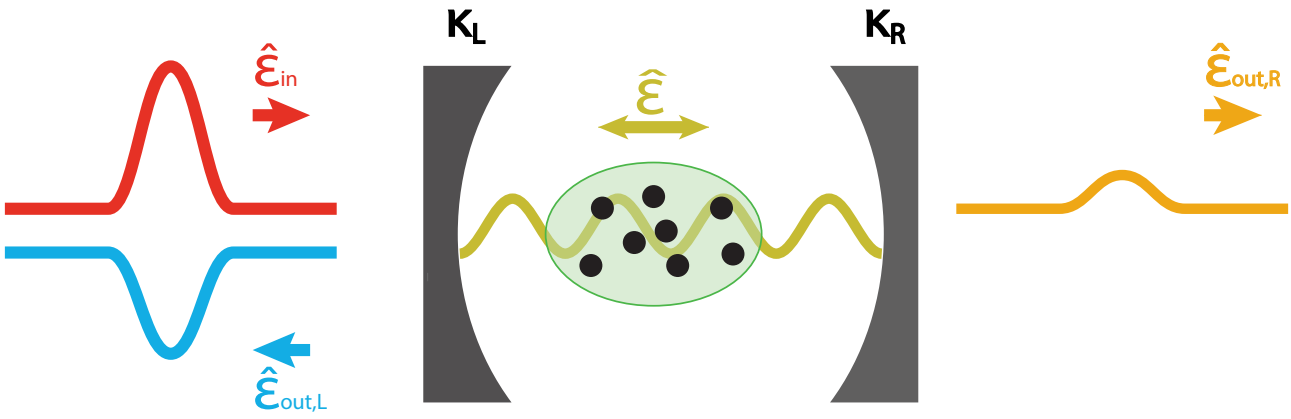

Figure 3.6: Schematic representation of atoms in the two sided cavity. The cavity field is resonant to the $|g\rangle \leftrightarrow |e\rangle$ transition, as the driving field Ω turned off. Strong reflection is observed, due to resonant interaction with the two level system.

$$\hat{\mathcal{E}}(\omega) = \frac{\sqrt{2\kappa_L}\hat{\mathcal{E}}_{in}(\omega)((\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2)}{(\gamma_r + i(\delta-\omega))((\kappa - i\omega)(\gamma_e + i(\Delta-\omega)) + |\Omega|^2(\kappa_R + \kappa_L - i\omega) + g^2N)} \tag{3.125}$$

Probability conservation equation (3.90) for the case of two sided cavity is

$$|\hat{\mathcal{E}}_{in,L}(\omega)|^2 = |\hat{\mathcal{E}}_{out,L}(\omega)|^2 + |\hat{\mathcal{E}}_{out,R}(\omega)|^2 + 2\gamma_e|\hat{P}(\omega)|^2 + 2\gamma_r|\hat{S}(\omega)|^2 \tag{3.126}$$

and as in the one sided case, we work with the proportionality factors

$$1 = |R(\omega)|^2 + |T(\omega)|^2 + 2\gamma_e|\tilde{P}(\omega)|^2 + 2\gamma_r\tilde{S}(\omega)|^2 \tag{3.127}$$

where we have defined the proportionality factors,

$$R(\omega) = \frac{\hat{\mathcal{E}}_{out,L}}{\hat{\mathcal{E}}_{in,L}} = -1 + \frac{2\kappa_L((\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2)}{(\gamma_r + i(\delta-\omega))((\kappa_R + \kappa_L - i\omega)(\gamma_e + i(\Delta-\omega)) + |\Omega|^2(\kappa - i\omega) + g^2N)} \tag{3.128}$$

and

$$T(\omega) = \frac{\hat{\mathcal{E}}_{out,R}}{\hat{\mathcal{E}}_{in,L}} = -\frac{2\sqrt{\kappa_L\kappa_R}((\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2)}{(\gamma_r + i(\delta-\omega))((\kappa_R + \kappa_L - i\omega)(\gamma_e + i(\Delta-\omega)) + |\Omega|^2(\kappa - i\omega) + g^2N)} \tag{3.129}$$

The equations are

$$|R(\omega)|^2 = 1 - \frac{4g^2N\kappa_L(\gamma_r\Omega^2) + \gamma_e(\gamma_r^2 + (\delta-\omega)^2) + 2\sqrt{\kappa_L\kappa_R}((\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2)}{A2} \tag{3.130}$$

$$|T(\omega)|^2 = \frac{4\kappa_R\kappa_L\gamma_e^2((\gamma_r^2 - (\delta-\omega)^2) + \gamma_r^2(\Delta-\omega)^2 + 2\gamma_e\gamma_r\Omega^2 + ((\delta-\omega)(-\Delta+\omega) + \Omega^2)^2)}{A2} \tag{3.131}$$

$$2\gamma_e|\tilde{P}(\omega)|^2 = \frac{4g^2N\gamma_e\kappa_L(\gamma_r^2 + (\delta-\omega)^2)}{A2} \tag{3.132}$$

$$2\gamma_r|\tilde{S}(\omega)|^2 = \frac{4g^2N\gamma_r\kappa_L\Omega^2}{A2} \tag{3.133}$$

where

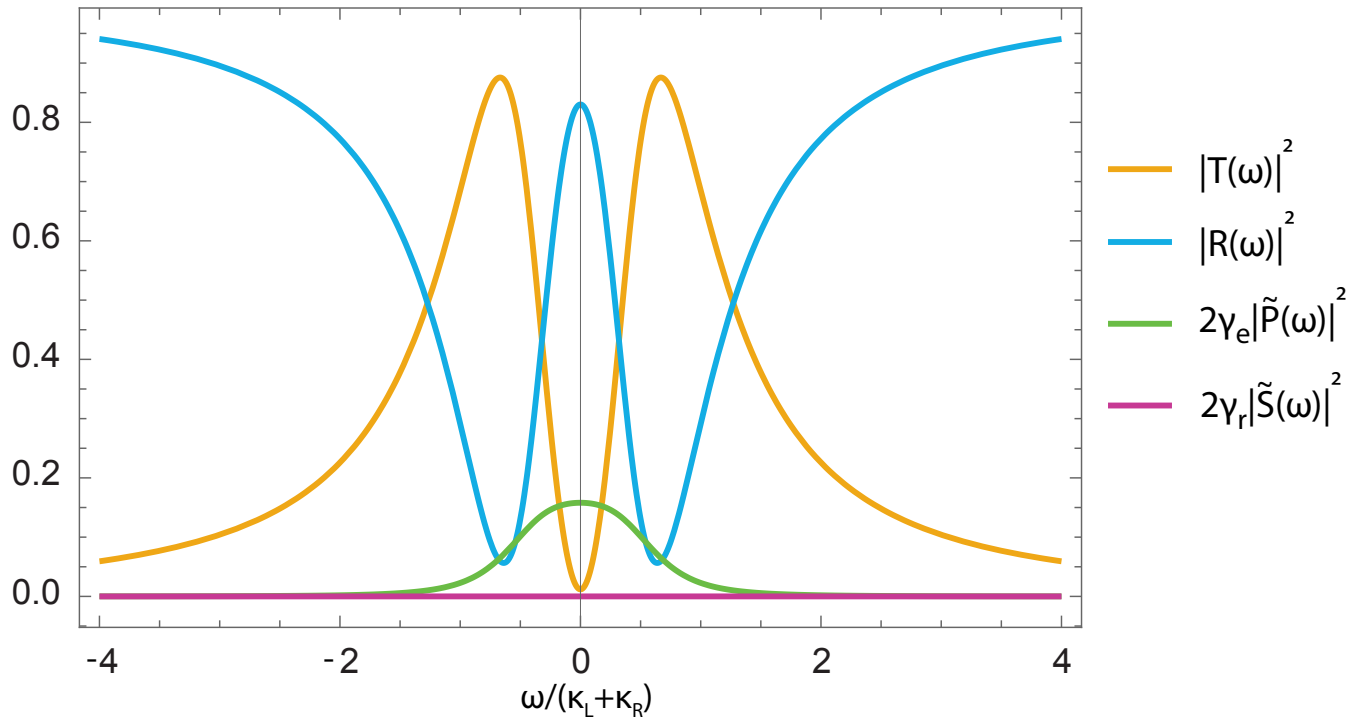

Figure 3.7: Diagram of the transmission and reflection rates $|T(\omega)|^2$, $|R(\omega)|^2$, the loss rate through decay of the excited state $|e\rangle$, $2\gamma_e|\tilde{P}(\omega)|^2$ and the loss rate through dephasing of the state $|r\rangle$ as a function of the dimensionless frequency of the two sided cavity. The driving field is off, $\Omega = 0$. The rest of the values used are g=0.1, n=40,γ_e=0.05,κ_L=0.6,κ_R=0.4,$\Delta = \delta = 0$.

$$\begin{aligned} A2 =& g^4N^2(\gamma_r^2 + (\delta-\omega)^2) + 2g^2N((\gamma_r^2 + (\delta-\omega)^2)(\gamma_e(\kappa_L+\kappa_R) + (\Delta-\omega)\omega) + (\gamma_r(\kappa_L+\kappa_R)+ \\ & \omega(-\delta+\omega))\Omega^2) + ((\kappa_L+\kappa_R)^2 + \omega^2)((\gamma_r^2 + (\delta-\omega)^2)(\gamma_e^2 + (\Delta-\omega)^2)+ \\ & 2(\gamma_e\gamma_r + (\delta-\omega)(-\Delta+\omega))\Omega^2 + \Omega^4) \end{aligned} \tag{3.134}$$

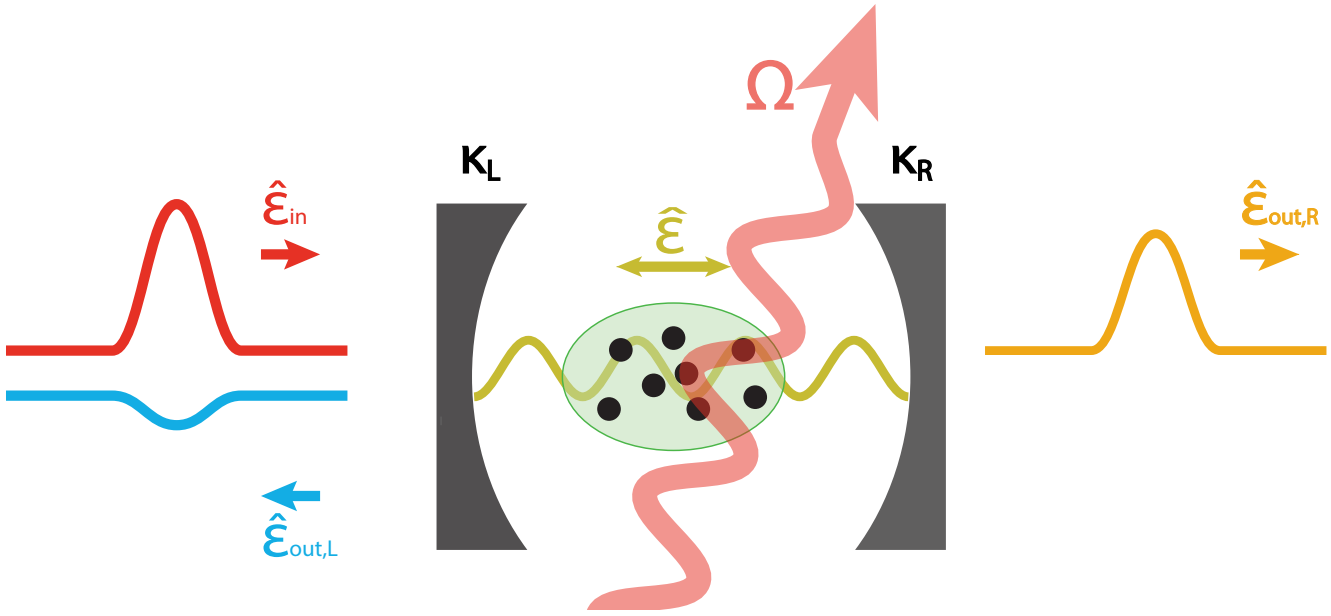

Figure 3.8: Schematic representation of atoms in the 2 sided cavity. The cavity field is resonant to the $|g\rangle \leftrightarrow |e\rangle$ transition, as is the driving field Ω to the $|e\rangle \leftrightarrow |r\rangle$ transition. Strong transmission is observed, through the destructive of the 2 transitions a phenomenon called Electromagnetically Induced Transparency.

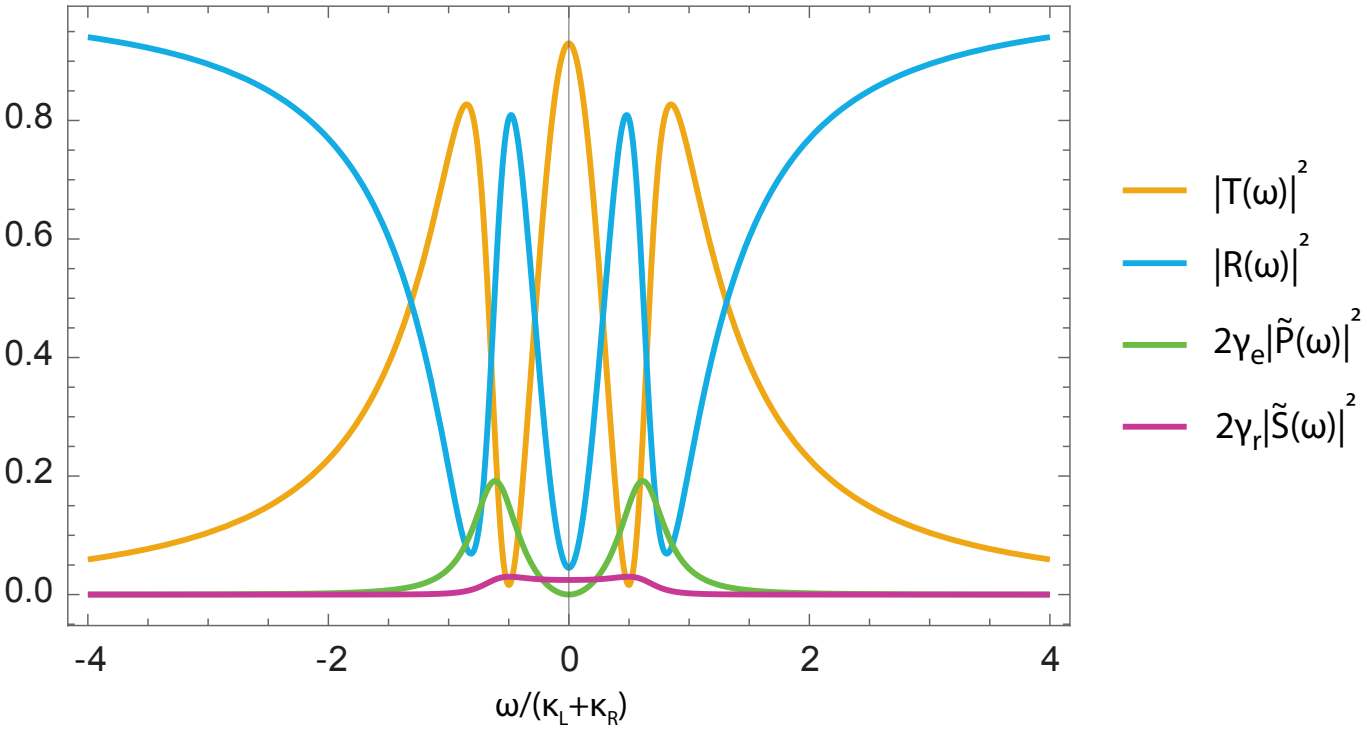

Figure 3.9: Diagram of the transmission and reflection rates $|T(\omega)|^2$, $|R(\omega)|^2$, the loss rate through decay of the excited state $|e\rangle$, $2\gamma_e|\tilde{P}(\omega)|^2$ and the loss rate through dephasing of the state $|r\rangle$ as a function of the dimensionless frequency of the two sided cavity. The driving field is off, $\Omega = 0$. The rest of the values used are g=0.1, n=40,γ_e=0.05,κ_L=0.6,κ_R=0.4,$\Delta = \delta = 0$.

3.3 Atomic Ensemble in Free Space

After having described the cases of the atomic ensemble in a one-sided and a two-sided cavity, we turn to the description of an atomic cloud in the free space. We will consider the linear dispertion in first order, the L/N is a normalization constant so that $\int_0^L n(z) = N$
The electric field vector operator for the cavity field is given by

$$\hat{E}_1(z) = \boldsymbol{\epsilon_1}\left(\frac{\hbar\omega_1}{4\pi c\epsilon_0 A}\right)^{1/2}(\int d\omega \hat{a}_\omega e^{i\omega z/c} + \int d\omega \hat{a}^\dagger_\omega e^{-i\omega z/c}) \tag{3.135}$$

where ω_1 is the frequency of the cavity field and $\hat{a}^\dagger_\omega, \hat{a}_\omega$ the creation and annihilation operators for the cavity field and where $\boldsymbol{\epsilon_1}$ are the unit vector in the direction of the polarization of the cavity field, ϵ_0 the perimittivity of free space, V the quantization volume of the field and c the speed of light.

$$[\hat{a}_\omega, \hat{a}^\dagger_{\omega'}] = \delta(\omega - \omega') \tag{3.136}$$

The co-propagating single mode classical plane-wave field with frequency ω_2 is the same as in the cavity case and is described by the electric field vector

$$E_2(z,t) = \boldsymbol{\epsilon_2}\mathcal{E}_2(t)cos(\omega_2(t - z/c)) = \boldsymbol{\epsilon_2}\frac{\mathcal{E}_2(t)}{2}(e^{i\omega_2(t-z/c)} + e^{-i\omega_2(t-z/c)}) \tag{3.137}$$

where $\boldsymbol{\epsilon_2}$ are the unit vector in the direction of the polarization of the classical field and $\mathcal{E}(t)$ its amplitude.
The Hamiltonian of the system is

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V} \tag{3.138}$$

where $\hat{\mathcal{H}}_0$ is the unperturbed Hamiltonian and $\hat{V}$ is the interaction Hamiltonia between the atomic ensemble and the fields.

$$\hat{\mathcal{H}}_0 = \hbar\int d\omega\omega\hat{a}^\dagger_\omega\hat{a}_\omega + \sum_i^N(\hbar\omega_g\hat{\sigma}^i_{gg} + \hbar\omega_e\hat{\sigma}^i_{ee} + \hbar\omega_r\hat{\sigma}^i_{rr}) \tag{3.139}$$

and the interaction Hamiltonian in defined the same way as in the cavity case. In a similar manner as before we get to the three equations of motion for our system.

$$(\partial_t + c\partial_z)\hat{\mathcal{E}}(z,t) = ig\sqrt{N}\hat{P}(z,t)\frac{n(z)L}{N} \tag{3.140}$$

$$\dot{\hat{P}}(t) = -(\gamma_e + i\Delta)\hat{P}(t) + ig\sqrt{N}\hat{\mathcal{E}}(t) + i\Omega\hat{S}(t) \tag{3.141}$$

$$\dot{\hat{S}}(t) = -(\gamma_r + i\delta)\hat{S}(t) + i\Omega^*\hat{P}(t) \tag{3.142}$$

Now we are going into a co-moving frame with new coordinates $t' = t - z/c$, $z' = z$, then we transform the partial derivatives to the coordinates

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial z} = \frac{\partial}{\partial t'} \tag{3.143}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial z} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial z} = -\frac{1}{c}\frac{\partial}{\partial t'} + \frac{\partial}{\partial z'} \tag{3.144}$$

and we choose the density to be constant in space $n(z) = \frac{N}{L}$.

$$\partial_{z'}\hat{\mathcal{E}}(z,t) = i\frac{g\sqrt{N}}{c}\hat{P}(z,t) \tag{3.145}$$

$$\partial_{t'}\hat{P}(t) = -(\gamma_e + i\Delta)\hat{P}(t) + ig\sqrt{N}\hat{\mathcal{E}}(t) + i\Omega\hat{S}(t) \tag{3.146}$$

$$\partial_{t'}\hat{S}(t) = -(\gamma_r + i\delta)\hat{S}(t) + i\Omega^*\hat{P}(t) \tag{3.147}$$

We will now use the Fourier transform as before

$$\partial_{z'}\hat{\mathcal{E}}(z,\omega) = i\frac{g\sqrt{N}}{c}\hat{P}(z,\omega) \tag{3.148}$$

$$-i\omega\hat{P}(\omega) = -(\gamma_e + i\Delta)\hat{P}(\omega) + ig\sqrt{N}\hat{\mathcal{E}}(t) + i\Omega\hat{S}(\omega) \tag{3.149}$$

$$-i\omega\hat{S}(\omega) = -(\gamma_r + i\delta)\hat{S}(\omega) + i\Omega^*\hat{P}(\omega) \tag{3.150}$$

We can now solve the system of the three equations as following

$$\hat{S}(\omega, z') = \frac{i\Omega^*}{\gamma_r + i(\delta - \omega)}\hat{P} \tag{3.151}$$

$$ig\sqrt{N}\hat{\mathcal{E}} = \frac{(\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2}{\gamma_r + i(\delta-\omega)}\hat{P} \tag{3.152}$$

$$\partial_{z'}\hat{\mathcal{E}}(z,\omega) = -\frac{\frac{g^2N}{c}(\gamma_r + i(\delta-\omega))}{(\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2}\hat{\mathcal{E}}(z,\omega) \tag{3.153}$$

If we now integrate from 0 to L, where is the length of the ensemble we have

$$\frac{\hat{\mathcal{E}}(L,\omega)}{\hat{\mathcal{E}}(0,\omega)} = exp(\frac{\gamma d(\gamma_r + i(\delta-\omega))}{(\gamma_r + i(\delta-\omega))(\gamma_e + i(\Delta-\omega)) + |\Omega|^2}) \tag{3.154}$$

where we have defined the optical depth $d = \frac{g^2NL}{\gamma c}$. From (3.145) we have

$$\hat{P}(z',\omega) = -i\frac{c}{g\sqrt{N}}\partial_{z'}\hat{\mathcal{E}}(z',\omega) \tag{3.155}$$

and then we can find a similar relation for $\hat{S}(\omega, z')$

$$\hat{S}(\omega, z') = -i\frac{c}{g\sqrt{N}}\frac{i\Omega^*}{\gamma_r + i(\delta-\omega)}\partial_{z'}\hat{\mathcal{E}}(z',\omega). \tag{3.156}$$

The probability conservation relation, similar to (3.90) in the free space case is

$$|\hat{\mathcal{E}}(0,\omega)|^2 = |\hat{\mathcal{E}}(L,\omega)|^2 + 2\gamma_e\int_0^L dz'|\hat{P}(\omega,z')|^2 + 2\gamma_r\int_0^L dz'|\hat{S}(\omega,z')|^2, \tag{3.157}$$

which basically states that the rate at which energy enters the atomic cloud, is equal to the rate of energy exiting it, plus the lost energy through the two sources of dissipation defined above.

$$1 = \frac{|\hat{\mathcal{E}}(L,\omega)|^2}{|\hat{\mathcal{E}}(0,\omega)|^2} + 2\gamma_e\frac{\int_0^L dz'|\hat{P}(\omega,z')|^2}{|\hat{\mathcal{E}}(0,\omega)|^2} + 2\gamma_r\frac{\int_0^L dz'|\hat{S}(\omega,z')|^2}{|\hat{\mathcal{E}}(0,\omega)|^2}, \tag{3.158}$$

3.4 Mapping between Cavity and Free Space Models

In the previous section we studied the interaction of a 3-level atomic ensemble with 2 fields in free space. We found that the behaviour of the system is very similar to the one of an atomic ensemble in a cavity. In this section we will derive a relation between the two different systems. We start from the single sided cavity Hamiltonian (3.42) in the dipole and rotating wave approximation. We consider being in the bad cavity limit, i.e. $\kappa \gg g\sqrt{N}$, so we can treat the coupling of the cavity field with the atoms perturbatively. This way we can, following the effective operator formalism developed in chapter 2, adiabatically eliminate the cavity filed operator $\hat{\mathcal{E}}$. We consider the excited subspace to contain only the cavity field so the non Hermitian Hamiltonian describing it, will be

$$\hat{\mathcal{H}}_{NH} = -i\hbar 2\kappa \hat{\mathcal{E}}^\dagger \hat{\mathcal{E}} \tag{3.159}$$

since we work in the rotating frame of the field, the $\hat{\mathcal{H}}_{NH}$ contains only the decay of the cavity field. No following the formalism (2.103) we define a new effective Hamiltonian as

$$\hat{\mathcal{H}}_{eff} = i\hat{\mathcal{E}} g\sqrt{N}\hat{\sigma}_{eg}(\frac{\hat{\mathcal{E}}^\dagger\hat{\mathcal{E}}}{\kappa})\hat{\mathcal{E}}^\dagger g\sqrt{N}\hat{\sigma}_{ge} = i\frac{g^2N}{\kappa}\hat{\sigma}_{ee}. \tag{3.160}$$

Then we can find the new equations of motion in this effective formalism,

$$\hat{P}(\omega) = -(\gamma_e + i\Delta)\hat{P}(\omega) - \frac{g^2N}{\kappa}\hat{P}(\omega) + i\Omega^*\hat{S}(\omega) + i\sqrt{\frac{2g^2N}{\kappa}}\hat{\mathcal{E}}_{in}(\omega), \tag{3.161}$$

$$\hat{S}(\omega) = -(\gamma_r + i\delta)\hat{S}(\omega) + i\Omega\hat{P}(\omega), \tag{3.162}$$

where the equation of $\hat{S}$ remained unchanged since it was not coupled to the cavity field, which we adiabatically eliminated.
Subsequently using (2.104) to find the effective decay of the cavity, we can write the new input output cavity relation as

$$\hat{\mathcal{E}}_{out}(\omega) = \hat{\mathcal{E}}_{in}(\omega) + \hat{L}^{cav}_{eff} = \hat{\mathcal{E}}_{in}(\omega) + i\sqrt{2\frac{g^2N}{\kappa}}\hat{P}(\omega) \tag{3.163}$$

By solving this system of the three equations we find the relation for the reflection coefficient

$$R(\omega) = 1 - \frac{i2\frac{g^2N}{2\kappa}(\gamma_r + i(\delta-\omega))}{(\gamma_e + i(\Delta + \frac{g^2N}{2\kappa} - \omega)(\gamma_r + i(\delta-\omega)) + |\Omega|^2)}. \tag{3.164}$$

If we now compare this result with the equation for the transmition coefficient in free space (3.154), we see that by expanding (3.154) for small optical depth d, to first order, we get a similar result to (3.164). Up to a factor in the denominator of (3.164), called Purcell factor, we can have perfect correspondance between free space model and single-sided cavity model, under the correspondance relation

$$C = d\frac{c}{L\kappa} \tag{3.165}$$

between the ensemble's cooperativity $C = \frac{g^2N}{2\kappa\gamma}$ in the cavity model and the ensemble's optical depth $d = \frac{g^2N}{2\kappa\gamma}$ in the free space model. For very small cooperativity $C = \frac{g^2N}{2\kappa\gamma} \ll 1$, the Purcell factor can be neglected $\gamma_e \approx \gamma_e(1-C) \approx \gamma_e$ $C = \frac{g^2N}{2\kappa\gamma} \ll 1$ and we can have perfect correspondance between the two models.
This relation is very useful, since it allows us to directly relate our results for the single cavity model to the free space model.

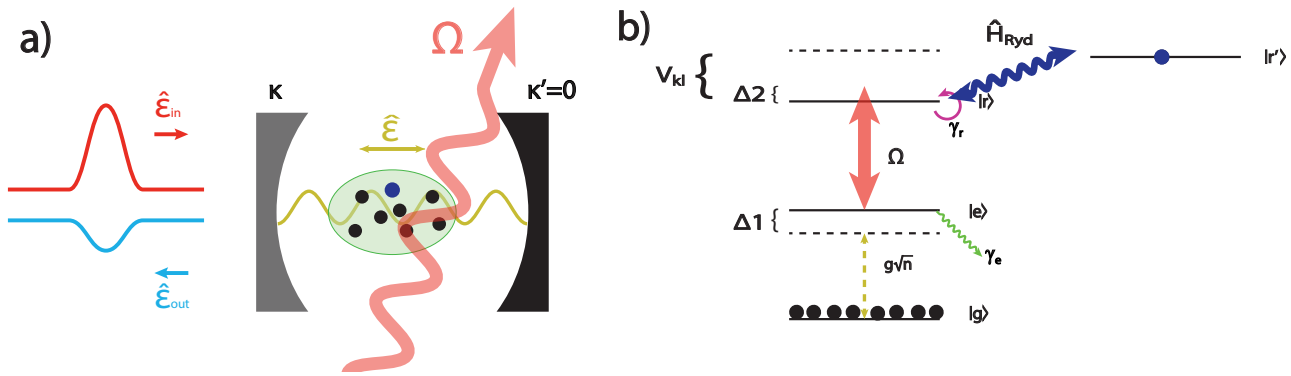

Figure 3.10: a)Schematic representation of atoms in the single sided cavity. The cavity field is coupled to the $|g\rangle \leftrightarrow |e\rangle$ transition, as is the driving field Ω to the $|e\rangle \leftrightarrow |r\rangle$ transition. Low reflection is observed, through the Rydberg interaction the energy of r state is shifted by V_{kl} and no longer accesible from e through Ω.

3.5 Rydberg Atoms

If we now consider the atoms consisting the ensemble, to be Rydberg atoms, i.e. atoms with large dipole moments when in Rydberg state. This leads to an extra term in the Hamiltonian, describing strong dipole-dipole interactions between atoms in states $|r\rangle$,$|r'\rangle$,

$$\hat{\mathcal{H}}_{Ryd} = \hbar \sum_l \sum_{k \neq l} \mathcal{V}_{kl} \, |r_l\rangle \langle r_l| \otimes |r'_k\rangle \langle r'_k| \tag{3.166}$$

For now we will consider the k-th atom to be excited in state $|r'_k\rangle$ then, we are left with

$$\hat{\mathcal{H}}_{Ryd,k} = \hbar \sum_l \mathcal{V}_{kl} \, |r_l\rangle \langle r_l| \tag{3.167}$$

which contributes only to the equation of motion of the spin operator $\dot{\hat{S}}(t)$ as

$$\begin{aligned} \dot{\hat{S}}_{Ryd,k}(t) =& \frac{i}{\hbar}[\hat{\mathcal{H}}_{Ryd,k}, \hat{S}(t)] = -i\frac{1}{N} \sum_i |g_i\rangle \langle r_i| \sum_l \mathcal{V}_{kl} \, |r_l\rangle \langle r_l| = \\ & - i \sum_l \frac{1}{N} \sum_i \mathcal{V}_{kl} \, |g_i\rangle \, \delta_{li} \langle r_l| = -i\frac{1}{N} \sum_l \mathcal{V}_{kl} \, |g_l\rangle \langle r_l| \end{aligned} \tag{3.168}$$

From equations (3.23),(3.24) we get the input output relation

$$\hat{\mathcal{E}}_{in} + \hat{\mathcal{E}}_{out} = \sqrt{2\kappa}\hat{\mathcal{E}} \tag{3.169}$$

$$\dot{\hat{\mathcal{E}}}(t) = -\kappa\hat{\mathcal{E}}(t) + ig\sqrt{N}\hat{P}(t) + \sqrt{2\kappa}\hat{\mathcal{E}}_{in}(t) \tag{3.170}$$

$$\dot{\hat{P}}(t) = -(\gamma_e + i\Delta)\hat{P}(t) + ig\sqrt{N}\hat{\mathcal{E}}(t) + i\Omega\hat{S}(t) \tag{3.171}$$

$$\dot{\hat{S}}'(t) = \dot{\hat{S}}(t) + \dot{\hat{S}}_{Ryd,k}(t) = -(\gamma_r + i\delta)\hat{S}(t) - i\sum_l \mathcal{V}_{kl} \, |r_l\rangle \langle g_l| + i\Omega^*\hat{P}(t) \tag{3.172}$$

We see that the equations of motion of the system operators do not change apart from the spin operator, that gets this new energy shift from the Rydberg interaction term. In order to deal with the last equation we have to write the spin operator in the decomposed form.

$$\dot{\hat{S}}'(t) = \sum_l \dot{\hat{\sigma}}^l_{gr}(t) = -(\gamma_r + i\delta) \sum_l \hat{\sigma}^l_{gr}(t) - i\sum_l \mathcal{V}_{kl} \, |r_l\rangle \langle g_l| + i\Omega^* \sum_l \hat{\sigma}^l_{ge}(t) \tag{3.173}$$

where we can neglect the sum and have

$$\dot{\hat{\sigma}}^l_{gr}(t) = -(\gamma_r + i\delta)\hat{\sigma}^l_{gr}(t) - i\mathcal{V}_{kl}\hat{\sigma}^l_{gr}(t) + i\Omega^*\hat{\sigma}^l_{ge}(t) \tag{3.174}$$

Then writting the other two equations in the same form, we end up with a system of 3 equations

$$\dot{\hat{\sigma}}^l_{ge}(t) = -(\gamma_e + i\Delta)\hat{\sigma}^l_{ge}(t) + ig\hat{\mathcal{E}}(t) + i\Omega\hat{\sigma}^l_{gr}(t) \tag{3.175}$$

$$\dot{\hat{\mathcal{E}}}(t) = -\kappa\hat{\mathcal{E}}(t) + ig\sum_l \hat{\sigma}^l_{ge}(t) + \sqrt{2\kappa}\hat{\mathcal{E}}_{in}(t) \tag{3.176}$$

We treat the system as we did in the previous sections of the chapter. We move to the frequency domain through Fourier transform and solve it. We get the equations

$$\hat{\mathcal{E}}(\omega) = \frac{\sqrt{2\kappa}\hat{\mathcal{E}}_{in}}{(\kappa - i\omega) + \sum_l \left(\frac{g^2(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega))}{(\gamma_e + i(\Delta - \omega))(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega)) + |\Omega|^2}\right)} \tag{3.177}$$

$$\hat{\sigma}^l_{ge} = \frac{i\sqrt{2\kappa}gn(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega))\hat{\mathcal{E}}_{in}}{((\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega))(\gamma_e + i(\Delta - \omega)) + |\Omega|^2)\left((\kappa - i\omega) + \sum_l \left(\frac{g^2(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega))}{(\gamma_e + i(\Delta - \omega))(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega)) + |\Omega|^2}\right)\right)} \tag{3.178}$$

$$\hat{\sigma}^l_{gr} = \frac{-\sqrt{2\kappa}gn\Omega^*\hat{\mathcal{E}}_{in}}{((\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega))(\gamma_e + i(\Delta - \omega)) + |\Omega|^2)\left((\kappa - i\omega) + \sum_l \left(\frac{g^2(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega))}{(\gamma_e + i(\Delta - \omega))(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega)) + |\Omega|^2}\right)\right)} \tag{3.179}$$

and the equations for $\hat{S}(\omega)$ can be find $\hat{P}(\omega)$ by their definition. It becomes evident that we need to evaluate the sum in the denominator, in order to understand the scatering dynamics of the system.

$$\sum_l \left(\frac{g^2(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega))}{(\gamma_e + i(\Delta - \omega))(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega)) + |\Omega|^2}\right) = \sum_l \left(\frac{g^2}{(\gamma_e + i(\Delta - \omega)) + \frac{|\Omega|^2}{(\gamma_r + i(\delta + \mathcal{V}_{kl} - \omega))}}\right) \tag{3.180}$$

We can convert the sum over all atoms to an integral over the volume of the ensemble, $\sum_l \to \int \rho_a(r)dV$ and write the explicit value of $\mathcal{V}_{kl} = -\frac{C_6}{r^6}$. Since the cloud is assumed homogeneous we have $\rho_a(r) = \frac{n}{V_a}$ and

$$\begin{aligned} &\frac{1}{V_a}\int_0^{R_a} drr^2 \int_0^{\pi} d\theta sin\theta \int_0^{2\pi} d\phi \left(\frac{g^2 N}{(\gamma_e + i(\Delta - \omega)) + \frac{|\Omega|^2}{(\gamma_r + i(\delta - \frac{C_6}{r^6} - \omega))}}\right) = \\ &\frac{1}{V_a}4\pi\int_0^{R_a} drr^2 \left(\frac{g^2 N}{(\gamma_e + i(\Delta - \omega)) + \frac{|\Omega|^2}{(\gamma_r + i(\delta - \frac{C_6}{r^6} - \omega))}}\right) \end{aligned} \tag{3.181}$$

where R_a is the radious of the atomic sphere and V its volume $V_a = \frac{4}{3}\pi R_a^3$. We can set R_a to be infinite, if we consider it larger than the radious of the Rydberg blockade R_{Ryd}. Since the Rydberg interaction term $V_{kl} \propto r^{-6}$ reduces very fast, in sixth order with distance between the two excited atoms and will be negligible for values higher than R_{Ryd}.
Now this integral can be difficult to evaluate, even after the asumption $R_a \to \infty$ valid for $R_a > R_{Ryd}$, so we choose to treat it for the resonant case of $\delta = \Delta = \omega = 0$, where also assume

the dephasing of the $|r\rangle$ state is to be zero i.e. $\gamma_r = 0$.

$$\begin{aligned}\frac{4\pi}{V_a}\int_0^\infty drr^2\left(\frac{g^2N}{\gamma_e+\frac{|\Omega|^2}{-i\frac{C_6}{r^6}}}\right) =& 4\pi\frac{g^2N}{V_a\gamma_e}\int_0^\infty drr^2\left(\frac{1}{1+i\frac{|\Omega|^2}{C_6\gamma_e}r^6}\right) = \\ & 4\pi\frac{g^2N}{V_a\gamma_e}\int_0^\infty drr^2\left(\frac{1-i\frac{|\Omega|^2}{C_6\gamma_e}r^6}{1+\frac{|\Omega|^4}{(C_6\gamma_e)^2}r^{12}}\right) = \\ & 4\pi\frac{g^2N}{V_a\gamma_e}\left(\int_0^\infty drr^2\frac{1}{1+\frac{|\Omega|^4}{(C_6\gamma_e)^2}r^{12}} - i\int_0^\infty drr^2\frac{\frac{|\Omega|^2}{C_6\gamma_e}r^6}{1+\frac{|\Omega|^4}{(C_6\gamma_e)^2}r^{12}}\right) = \\ & \left(\frac{\sqrt{2}g^2n\pi^2\sqrt{C_6\gamma_e}}{3V_a\gamma_e|\Omega|} - i\frac{\sqrt{2}g^2n\pi^2\sqrt{C_6\gamma_e}}{3V_a\gamma_e|\Omega|}\right)\end{aligned} \tag{3.182}$$

So we have the value of $\hat{\mathcal{E}}(\omega=0)$

$$\hat{\mathcal{E}}(\omega=0) = \frac{\sqrt{2\kappa}\hat{\mathcal{E}}_{in}}{\kappa+\frac{\sqrt{2}g^2n\pi^2\sqrt{C_6\gamma_e}}{3V_a\gamma_e|\Omega|}(1-i)} \tag{3.183}$$

since γ_r is zero, there is no loss from the $|r\rangle$, so $\hat{\sigma}^l_{gr}$ doesn't contribute to the scattering dynamics of the system. But γ_e is not zero, so we evaluate the equation $\hat{\sigma}^l_{ge}$, since it plays a role in the scattering of the incoming photons.

$$\hat{\sigma}^l_{ge}(\omega=0) = -\frac{\sqrt{2\kappa}gn\mathcal{V}_{kl}\hat{\mathcal{E}}_{in}}{(i\gamma_e\mathcal{V}_{kl}+|\Omega|^2)\left(\kappa+\frac{\sqrt{2}g^2n\pi^2\sqrt{C_6\gamma_e}}{3V_a|\Omega|}(1-i)\right)} \tag{3.184}$$

we now write in the polarization form and transforming the sum over atoms to a volume integral as before and we set $R_a \to \infty$ valid for $R_a > R_{Ryd}$, since $\mathcal{V}_{kl}(r > R_{ryd}) \approx 0$.

$$\hat{P}(\omega) = \frac{1}{\sqrt{n}}\sum_l \hat{\sigma}^l_{ge}(\omega) \to \frac{4\pi}{\sqrt{n}}\int_0^{R_a} dr\rho_a(r)r^2\hat{\sigma}^l_{ge}(\omega) \tag{3.185}$$

Considering as before the radius of the atomic cloud R_a to be larger than the Rydberg blockade radius R_{Ryd} and the density of atoms to be constant $\rho_a(r) = \frac{n}{V_a}$, we have

$$\hat{P}(\omega) \approx \frac{4\pi\sqrt{n}}{V_a}\int_0^\infty drr^2\hat{\sigma}^l_{ge}(\omega) \tag{3.186}$$

which leads to an expression containing the integral (3.182)

$$\hat{P}(\omega=0) = i4\pi\sqrt{2\kappa}\hat{\mathcal{E}}_{in}\frac{g\sqrt{N}}{V_a\gamma_e}\left(\kappa+\frac{\sqrt{2}g^2n\pi^2\sqrt{C_6\gamma_e}}{3V_a\gamma_e|\Omega|}(1-i)\right)^{-1}\int_0^\infty drr^2\frac{1}{1+i\frac{|\Omega|^2}{\gamma_e\mathcal{V}_{kl}}} \tag{3.187}$$

Using the explicit form of Rydberg interaction $\mathcal{V}_{kl} = -\frac{C_6}{r^6}$ we evaluate the integral as above and have

$$\hat{P}(\omega=0) = i\sqrt{2\kappa}\hat{\mathcal{E}}_{in}g\sqrt{N}\left(\kappa+\frac{\sqrt{2}g^2n\pi^2\sqrt{C_6\gamma_e}}{3\gamma_e|\Omega|}(1-i)\right)^{-1}\frac{\sqrt{2}\pi^2\sqrt{C_6\gamma_e}}{3\gamma_e|\Omega|}(1-i) \tag{3.188}$$

and bringing it in a nicer form,

$$\hat{P}(\omega=0)=\frac{i\sqrt{2\kappa}g\sqrt{N}\hat{\mathcal{E}}_{in}}{g^2N+\frac{3\kappa\gamma_e|\Omega|}{\sqrt{2}g^2n\pi^2\sqrt{C_6\gamma_e}(1-i)}} \tag{3.189}$$

The effect of the Rydberg interaction is an energy shift, that depends on the distance between the excited atoms, in order to gain some better intuition of this interaction we can will look at it in terms of the single atom cooperaticity $C_1=\frac{g^2}{\kappa\gamma}$ and the Rydberg radius. Dividing over $\hat{\mathcal{E}}_{in}$, we have the proportionality factors:

$$R(\omega)=2\kappa\left(\kappa-i\omega+\sum_l\frac{g^2N}{\gamma_e+i(\Delta+\omega)+\frac{|\Omega|^2}{\gamma_r+i(\delta+\mathcal{V}_{kl}+\omega)}}\right)^{-1}-1 \tag{3.190}$$

while for the resonant case, we dealt with above, it becomes

$$R(\omega)=2\left(1+\sum_l\frac{\frac{g^2}{\kappa\gamma_e}}{1-i\frac{|\Omega|^2}{\mathcal{V}_{kl}\gamma_e}}\right)^{-1}-1=2\left(1+\sum_l\frac{C_1}{1-i\frac{|\Omega|^2}{\mathcal{V}_{kl}\gamma_e}}\right)^{-1}-1 \tag{3.191}$$

where C_1 the cooperaticity of a single atom. If the Rydebrg interaction was considered constant in space, then our result, would have been just a phase shift, resulting to an AC Stark shift, similar to having a detuning δ

$$R(\omega)=2\left(1+\frac{C}{1-i\frac{|\Omega|^2}{\mathcal{V}\gamma_e}}\right)^{-1}-1 \tag{3.192}$$

While now, considering the Rydberg term having the r^{-6} behaviour, from the previous results.

$$R(\omega)=\frac{2}{1+nC\left(\frac{\sqrt{2}\pi^2\sqrt{C_6\gamma_e}}{3V_a|\Omega|}(1-i)\right)}-1=\frac{2}{1+C_b(1-i)}-1 \tag{3.193}$$

where $C_b=nC\frac{\sqrt{2}\pi^2\sqrt{C_6\gamma_e}}{3V_a|\Omega|}$, is the effective blockade cooperativit and it is basically the cooperativy of the atoms contributing to the interaction i.e. the atoms inside the Rydberg sphere, with volume $V_{ryd}=\frac{4}{3}\pi R_{ryd}^3$. The Rydberg radius is defined as

$$R_{Ryd}=\sqrt[6]{\frac{\Omega_{eff}}{C_6}}=\sqrt[6]{\frac{\Omega^2}{\gamma_eC_6}} \tag{3.194}$$

It is easy to see now that

$$C_b=nC\frac{\sqrt{2}\pi^2\sqrt{C_6\gamma_e}}{3V_a|\Omega|}=nC\sqrt{2}\pi\frac{V_{Ryd}}{V_a} \tag{3.195}$$

We can see that for no Rydberg interaction $C_6=0$ we have full reflection with no sign flip $R(\omega)=1$, while for very large Rydberg interaction strength $C_6\to\infty$, we have full reflection with a π phase flip, $R(\omega)=1$. This property has been used for the proposal of C-phase gate by Das et al. [5]. In our case we do not care about the phase, only about the reflection rate $|R(\omega)|^2$, which goes to unity in both cases, regardless of the field strength Ω.
The rate $\tilde{P}(\omega)$ in terms of the effective blocade cooperativity is

$$\tilde{P}(\omega)=i\frac{\sqrt{\frac{2}{\gamma_e}}\sqrt{C_b}}{1+C_b} \tag{3.196}$$

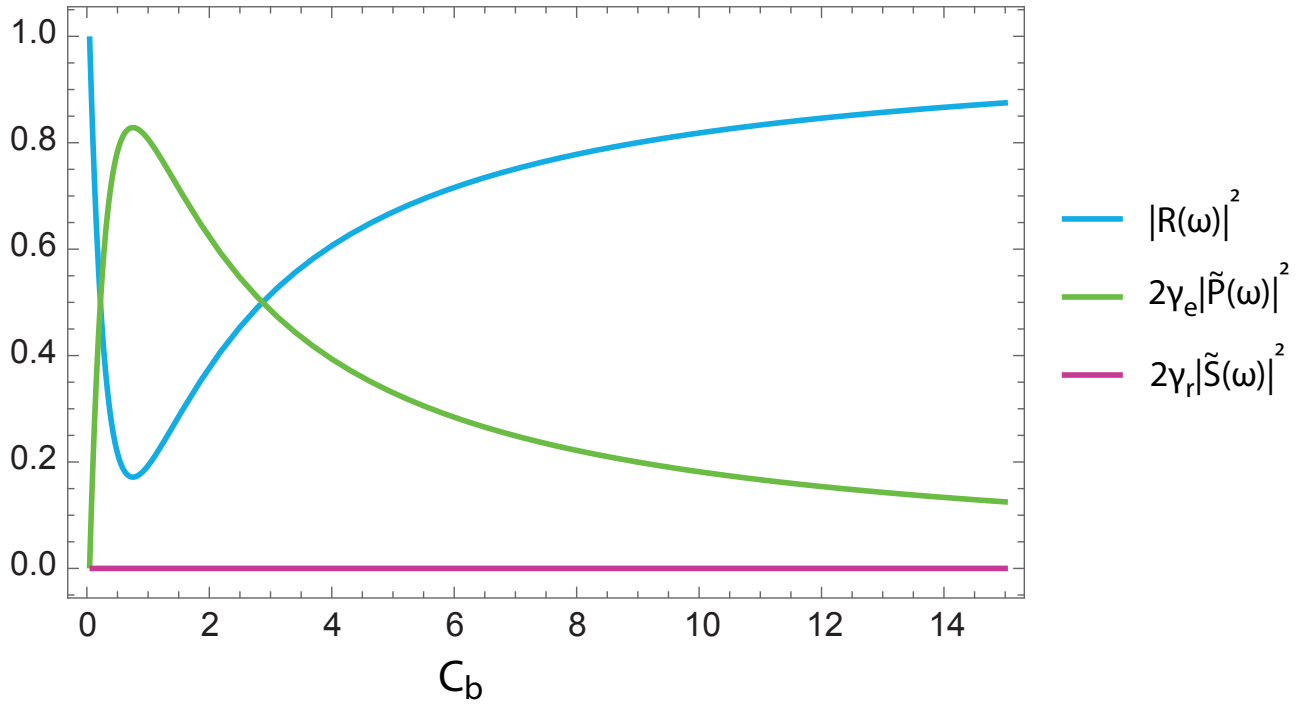

Figure 3.11: Diagram of the reflection rate $|R(\omega)|^2$, the loss rate through decay of the excited state $|e\rangle$, $2\gamma_e|\tilde{P}(\omega)|^2$ and the loss rate through dephasing of the state $|r\rangle$ as a function of the blockade cooperativity. We can see that we can at the maximum we can achieve is almost 83% losses of the incoming field, due to decay. The values used are γ_e=0,$\Delta = \delta = 0$.

Which is 0 on resonance as expected, for $C_b = 0$, due to EIT, as in the case viewed in the first section of this chapter. For $C_b \to \infty$, the rate goes to 0, again due to very strong blockade. The two rates as function of the blockade cooperativity are C_b.

$$|R(C_b)|^2 = 1 - \frac{4C_b}{1 + 2C_b(1 + C_b)} \tag{3.197}$$

and

$$2\gamma_e|P(C_b)|^2 = \frac{\sqrt{4C_b}}{1 + 2C_b(1 + C_b)} \tag{3.198}$$

We find the first derivative of the reflactance

$$\frac{d|R(C_b)|^2}{dC_b} = \frac{4(1 - C_b^2)}{1 + 2C_b(1 + C_b)} \tag{3.199}$$

and by setting it to 0, we can see that there is a minimum point

$$C_{b,min} = \sqrt{\frac{1}{2}} \tag{3.200}$$

that leads us to the minimum value of reflectance

$$|R(C_b)|^2_{min} = 1 - \frac{4}{2(1 + \sqrt{2})} = 3 - 2\sqrt{2} \tag{3.201}$$

We plot the two rates as functions of the blockade cooperativity C_b in figuree 3.11.

We can see that there is no value of C_b for which the reflectance $|R(\omega)|$ goes to zero, so that we have perfect Rydberg blockade. In order to investigate if there are conditions under which

we can achieve perfect blockade, so that all energy will be lost through spontaneous emission and there will be no reflectance, we will look a case out of resonance. We choose to solve the integral (3.182) for $\Delta = -\gamma_e$ and $\omega = \delta = 0$. This will lead to the following integral

$$\begin{aligned}
\frac{4\pi}{V_a}\int_0^\infty drr^2\left(\frac{g^2N}{(\gamma_e+i\Delta)+i\frac{|\Omega|^2}{\frac{C_6}{r^6}}}\right) &= 4\pi\frac{g^2N}{V_a\gamma_e}\int_0^\infty drr^2\left(\frac{1}{1-i+i\frac{|\Omega|^2}{\gamma_e C6}r^6}\right) \\
&= 4\pi\frac{g^2N}{V_a\gamma_e}\int_0^\infty drr^2\left(\frac{1-i(-1+\frac{|\Omega|^2}{\gamma_e C_6}r^6)}{1+(-1+\frac{|\Omega|^2}{\gamma_e C_6}r^6)^2}\right) \\
&= 4\pi\frac{g^2N}{V_a\gamma_e}\int_0^\infty drr^2\left(\frac{1}{1+(-1+\frac{|\Omega|^2}{\gamma_e C_6}r^6)^2}\right) - i\int_0^\infty drr^2\left(\frac{(-1+\frac{|\Omega|^2}{\gamma_e C_6}r^6)}{1+(-1+\frac{|\Omega|^2}{\gamma_e C_6}r^6)^2}\right) \\
&= 4\pi\frac{g^2N}{V_a\gamma_e}\frac{\sqrt{\gamma_e C_6}}{|\Omega|}\left(\pi\frac{\sqrt{1+\sqrt{2}}}{12}+i\pi\frac{\sqrt{-1+\sqrt{2}}}{12}\right)
\end{aligned} \tag{3.202}$$

We can see that we can define the effective cooperativity in this case as

$$C_{eff} = nC\frac{\sqrt{\gamma_e C_6}}{V_a|\Omega|}\left(4\pi^2\frac{\sqrt{1+\sqrt{2}}}{12}+i4\pi^2\frac{\sqrt{-1+\sqrt{2}}}{12}\right) \tag{3.203}$$

The reflection coefficient for this case becomes

$$R(C_b') = \frac{2}{1+C_{eff}} - 1 = \frac{2}{1+C_b'\left(4\pi^2\frac{\sqrt{1+\sqrt{2}}}{12}+i4\pi^2\frac{\sqrt{-1+\sqrt{2}}}{12}\right)} - 1 \tag{3.204}$$

where $C_b' = NC\frac{\sqrt{\gamma_e C_6}}{V_a|\Omega|}$. We can now plot the reflectance and we see that in this case we can get much closer to perfect blockade for a certain value of C_b'. As we can see in figure 3.12, there is a minimum of the reflectance for the value $C_b' = \frac{3}{\pi^2 2^{3/4}}$, we where $|R|^2 = \frac{4}{2\sqrt{2+\sqrt{2}}} - \approx 0.04$. This result means that for this value of C_b, if there is a Rydberg excitation somewhere in the ensemble, the probability of reflectance on resonance is a bit less than 0.4%, where the loss probability is over 99.6%.

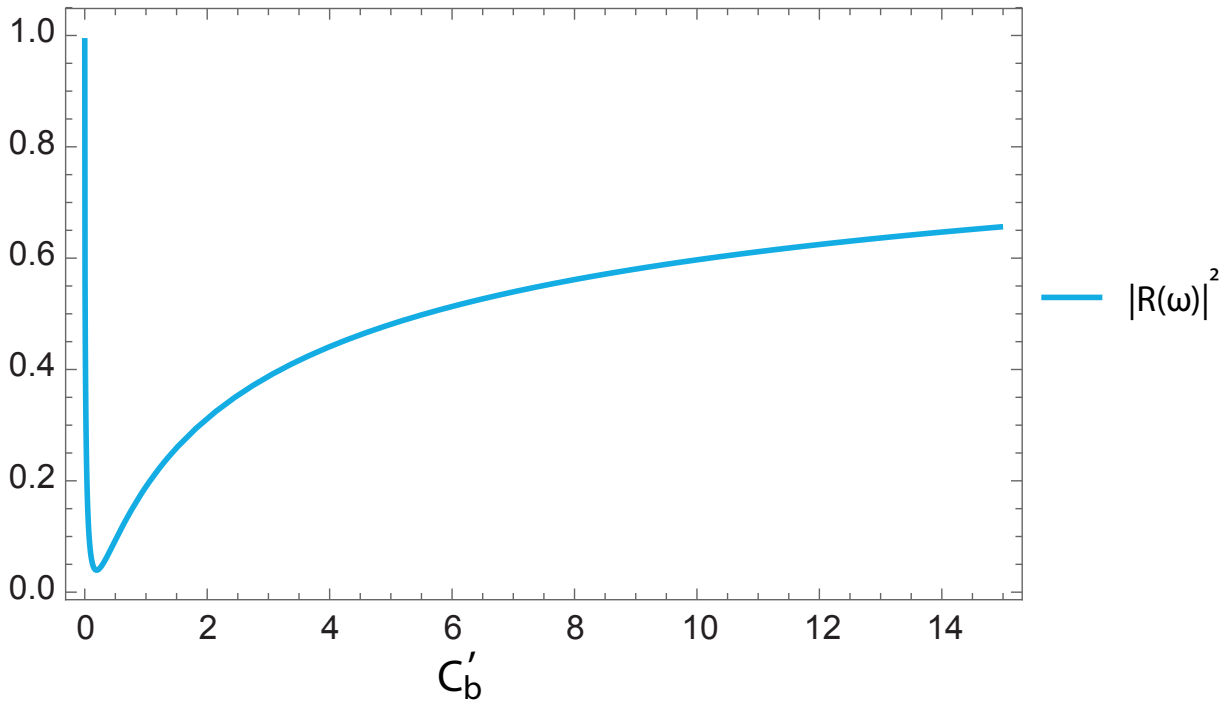

Figure 3.12: Diagram of the reflection rate $|R(\omega)|^2$, as a function of the blockade cooperativity, where for the value $C_b' = \frac{3}{\pi^2 2^{3/4}}$, we achive almost zero reflectance $|R|^2 = \frac{4}{2\sqrt{2+\sqrt{2}}} - \approx 0.04$. The condition used is $\Delta = -\gamma_e, \delta = 0$.

Chapter 4

Impedance Matching

In the previous chapter we showed that it is possible to condition the signal field, based on the absence or presence of a Rydberg excitation. In the current chapter we will investigate the conditions for our control field to excite an atom to the Rydberg state. In particular, we will look for the required conditions, so that a photon we send into the system, coupled to the second branch transition $|g\rangle \leftrightarrow |e'\rangle \leftrightarrow |r'\rangle$, will end up as an excitation in $|r'\rangle$ and stay there untill it dephases. For this purpose we use the following equation

$$|\hat{\mathcal{E}}_{in}(\omega)|^2 = |\hat{\mathcal{E}}_{out}(\omega)|^2 + 2\gamma_e|\hat{P}(\omega)|^2 + 2\gamma_r|\hat{S}(\omega)|^2 \tag{4.1}$$

This equation simply describes the conservation of energy in the system plus the coupling with its envirnment, taking into account all the effects of dissipation introduced. In particular, it shows that the rate of enegy of a certain frequency, entering the system will be equal to the rate exiting the system plus the energy lost by decay of the excited state and dephasing of the Rydberg state.
The fact that this equation describe rates is evidable by the units of $2\gamma_e|\hat{P}|^2$, which can better understood in the more familiar time domaim, by Fourier transforming this equation.
If we divide equation (4.1) by $|\hat{\mathcal{E}}_{in}(\omega)|^2$ we get

$$1 = \frac{|\hat{\mathcal{E}}_{out}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} + 2\gamma_e\frac{|\hat{P}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} + 2\gamma_r\frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} \tag{4.2}$$

We recognize the first ratio of the right hand side as the reflectance $|\hat{R}(\omega)|^2$.
Our goal is to optimize the conditions, so that all the incoming energy will exit the system through the dephasing of $|r\rangle$, described by the last term in the above equation.

$$2\gamma_r\frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} = 1 \tag{4.3}$$

4.0.1 Single Sided Cavity

For the single cavity case

$$2\gamma_r\frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} = 2\gamma_r\left(\frac{g\sqrt{N}\Omega}{(\gamma_r + i\delta - i\omega)(\gamma_e + i\Delta - i\omega) + \Omega^2}\right)^2\frac{|\mathcal{E}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} = 1 \tag{4.4}$$

and using the input output relation, we get

$$2\gamma_r\frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} = 2\gamma_r\left(\frac{g\sqrt{N}\Omega}{(\gamma_r + i\delta - i\omega)(\gamma_e + i\Delta - i\omega) + \Omega^2}\right)^2\frac{1}{2\kappa}(-1 + |\hat{R}(\omega)|)^2 = 1 \tag{4.5}$$

$$2\gamma_r \left(\frac{g\sqrt{N}\Omega}{(\gamma_r + i\delta - i\omega)(\gamma_e + i\Delta - i\omega) + \Omega^2} \right)^2 \frac{1}{2\kappa} (-1 + 2\kappa \left(\kappa - i\omega + \frac{g^2 N}{\gamma_e + i(\Delta + \omega) + \frac{|\Omega|^2}{\gamma_r + i(\delta + \omega)}} \right)^{-1} - 1)^2 = 1 \tag{4.6}$$

Solving this equation, we get the result for γ_r

$$\gamma_r = \frac{(g^2 n\kappa\Omega^2 - \gamma_e(\kappa^2 + \omega^2)\Omega^2 \pm \frac{1}{2}\sqrt{A})}{(g^4 n^2 + 2g^2 n(\gamma_e\kappa - \omega(\Delta + \omega)) + (\kappa^2 + \omega^2)(\gamma_e^2 + (\Delta + \omega)^2))} \tag{4.7}$$

$$\begin{aligned} A = {} & 4(g^2 n\kappa - \gamma_e(\kappa^2 + \ \omega^2))^2\Omega^4 - 4(g^4 n^2 + 2g^2 n(\gamma_e\kappa - \omega(\Delta + \ \omega)) + \\ & (\kappa^2 + \omega^2)(\gamma_e^2 + (\Delta + \omega)^2))(g^4 n^2(\delta + \omega)^2 + 2g^2 n(\delta + \omega)(\gamma_e\kappa(\delta \ + \omega) + \\ & \omega(-(\delta + \omega)(\Delta + \ \omega) + \Omega^2)) + (\kappa^2 + \omega^2)\ (\gamma^2(\delta + \omega)^2 + (-(\delta + \omega)(\ \Delta + \omega) + \Omega^2)^2)) \end{aligned} \tag{4.8}$$

This expression minimizes for $\omega = -\delta$

$$\gamma_r = -\frac{((\delta^2 + \kappa^2)\Omega^4}{(-g^2 n\ \kappa + \gamma(\delta^2 + \kappa^2))\Omega^2 \pm \sqrt{B})} \tag{4.9}$$

where,

$$B = -(g^4 n^2\delta^2 - 2g^2 n(\delta^2 - \delta\Delta - 2\gamma\kappa)(\delta^2 + \kappa^2) + (\delta \ - \Delta)^2(\delta^2 + \kappa^2)^2)\Omega^4 \tag{4.10}$$

B cannot be possitive, since γ_r has to be a real value. So we find the condtions for B=0, that is

$$\Delta = \delta - \frac{g^2 N\delta}{\delta^2 + \kappa^2} \pm \frac{2\sqrt{-g^2 N\gamma\kappa(\delta^2 + \kappa^2)^3}}{(\delta^2 + \kappa^2)^2} \tag{4.11}$$

For Δ to be real, it is obvious that we have to set $\gamma = 0$

$$\Delta = \delta - \frac{g^2 N\delta}{\delta^2 + \kappa^2} \tag{4.12}$$

And from this equation, another condition becomes evitable, that for non zero detunings we need the conditions $\Delta < \delta < 0$ or $\Delta > \delta > 0$, since the square of the coupling constant $g^2 N$ has to be real and positive.
So we have the condition for impedance matching

$$\gamma_r = \frac{(\delta^2 + \kappa^2)\Omega^2}{g^2 n\ \kappa} \tag{4.13}$$

in addition with requirement (4.12), $\gamma_e = 0$ and $\Delta < \delta < 0$ or $\Delta > \delta > 0$.
An easy way to test the validity of this result, is by considering the case of zero detunings $\delta = \Delta = 0$. For this case the condition (4.12) disappears and the only condition is $\gamma = 0$. Then, the dephasing becomes

$$\gamma_r = \frac{\kappa\Omega^2}{g^2 n} \tag{4.14}$$

We can easily check this result by using it to calculate the reflection amplitude $R(\omega) = 1$, that leads to $\frac{|\hat{\mathcal{E}}_{out}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} = 0$ and since $\gamma_e = 0$, from (4.2) we can see that the impedance matching condition is fulfilled.
Although (4.12) is the condition for perfect impedance mathing, we would like to have the dephasing being proportional to the number of atoms, since this condition will lead to long lived Rydberg excitations. So we try to find an approxiamte solution of impedance matching,

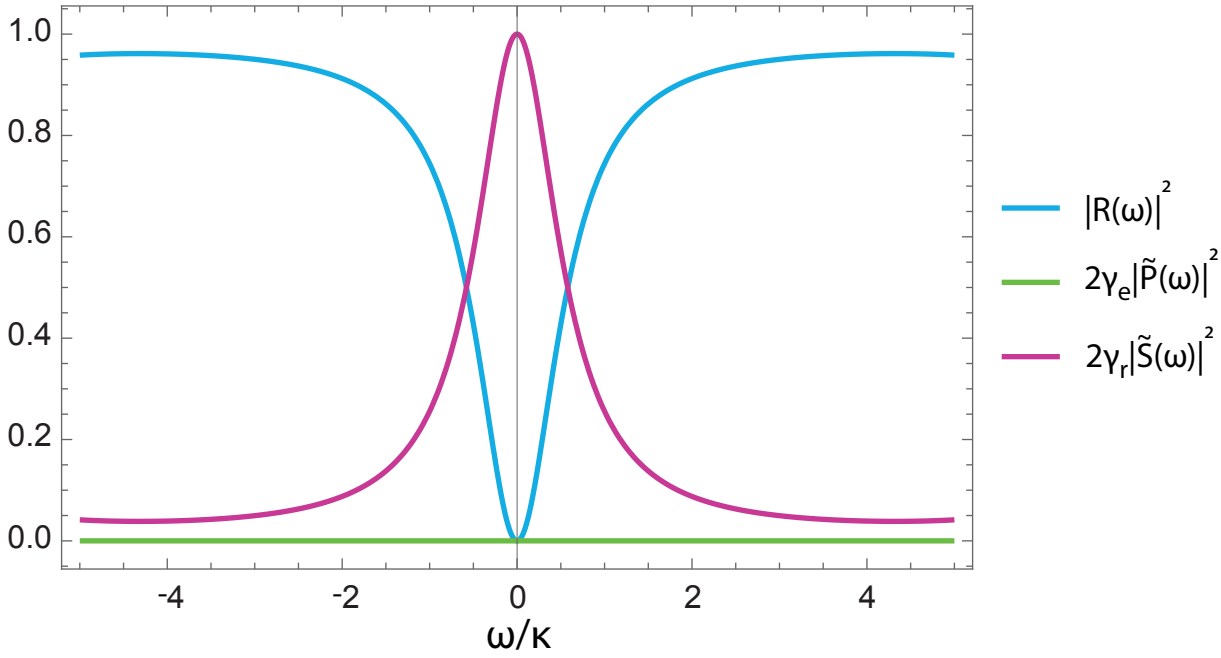

Figure 4.1: Diagram of the reflection rate $|R(\omega)|^2$, the loss rate through decay of the excited state $|e\rangle$, $2\gamma_e|\tilde{P}(\omega)|^2$ and the loss rate through dephasing of the state $|r\rangle$ as a function of the dimensionless frequency of the cavity. We can see that on resonance are impedance matched, with strength $\Omega = 0.5$. The rest of the values used are g=0.1, n=40,γ_e=0,κ=1,$\Delta = \delta = 0$.

where dephasing will be proportional to the number of atoms n, the reason for the advantage of this choice will become clear in the last chapter. But we can argue that the motivation behind it, comes from the fact that if we adiabatically eliminate the excited state $|e\rangle$, we find an effective decay of the state $|r\rangle$ equal to $\frac{g^2 N\Omega^2}{\kappa\Delta^2}$, this motivates us to look for an impedance matching at this value of γ_r.
We choose

$$\gamma_r = \frac{g^2 N\Omega^2}{\kappa\Delta^2} \tag{4.15}$$

and we try to find optimal conditions for $2\gamma_r \frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2}$ to approach unity. We eliminate decay from $\gamma = 0$

$$2\gamma_r \frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} = \frac{4g^4 n^2\Omega^4}{D} \tag{4.16}$$

$$\begin{aligned} D =& \Delta^2(g^4n^2((\delta+\omega)^2 + \frac{g^4n^2\Omega^4}{\Delta^4\kappa^2}) + 2g^2n\left(\omega(\delta+\omega)\Omega^2 + \frac{g^2n\Omega^4}{\Delta^2} - \omega(\Delta+\omega)((\delta+\omega)^2 + \frac{g^4n^2\Omega^4}{\Delta^4\kappa^2})\right) + \\ & (\kappa^2+\omega^2)(-2(\delta+\omega)(\Delta+\omega)\Omega^2 + \Omega^4 + (\Delta+\omega)^2((\delta+\omega)^2 + \frac{g^4n^2\Omega^4}{\Delta^4\kappa^2}))) \end{aligned} \tag{4.17}$$

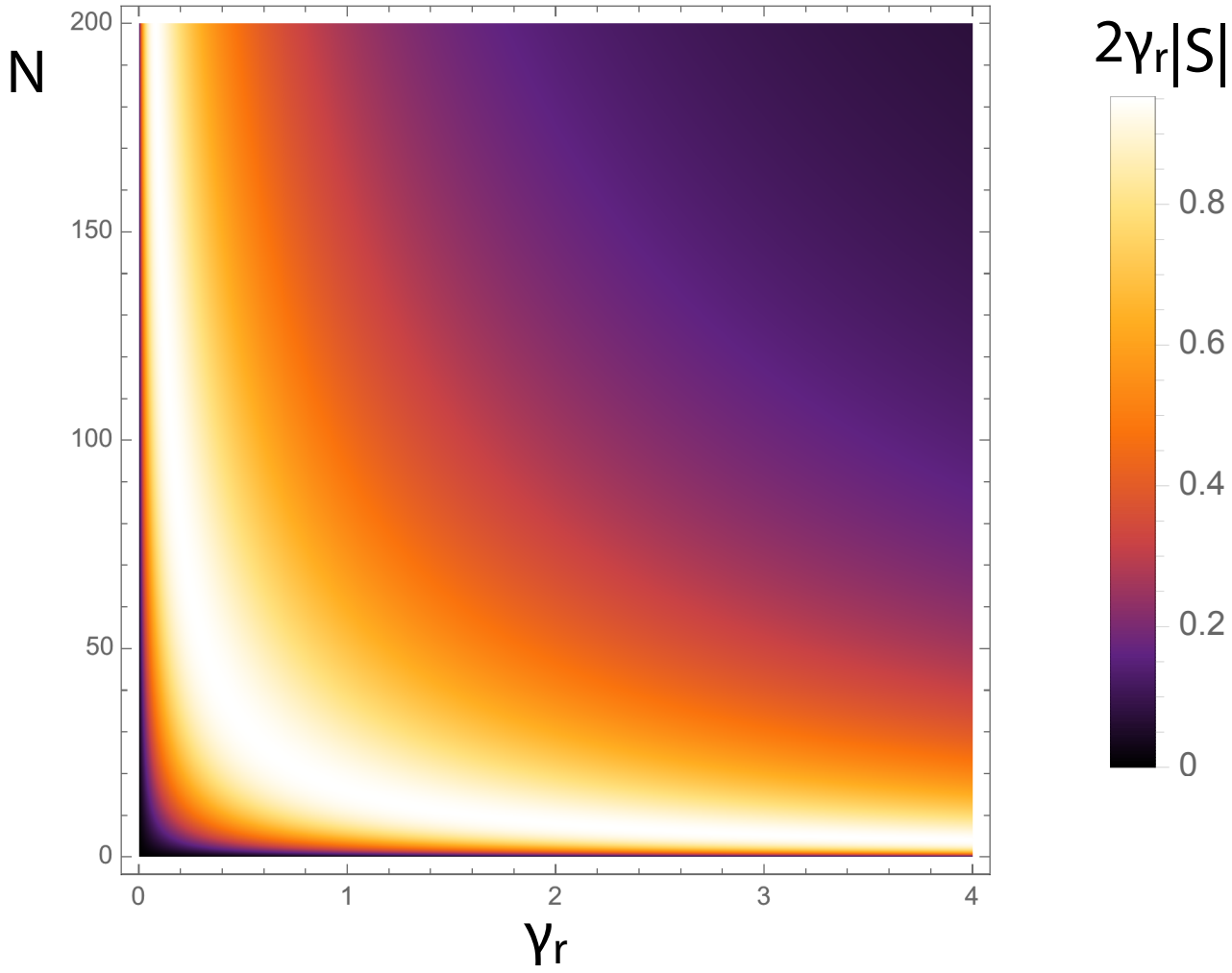

Figure 4.2: Diagram of $2\gamma_r|\tilde{S}(\omega)|^2$, the loss rate through dephasing of the Rydberg state $|r\rangle$, as a function of the number of atoms on the and the dephasing rate. We can see that γ_r is inversely proportional to the number of atoms on the collective excitation. The values used above are the driving field being $\Omega = 4$, g=1, γ_e=0, κ=1, $\Delta = \delta = \omega = 0$.

We look at the inverse of $2\gamma_r \frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2}$.

$$\frac{\hat{\mathcal{E}}_{in}(\omega)|^2}{2\gamma_r|\hat{S}(\omega)|^2} = \frac{D}{4g^4N^2\Omega^4} = \frac{1}{4}(\frac{\Delta^2(\delta+\omega)^2}{\Omega^4} + \frac{g^4N^2}{\Delta^2\kappa^2} + \frac{2\omega(\delta+\omega)\Delta^2}{g^2N\Omega^2} + 2 - \frac{2\omega\Delta^2(\Delta+\omega)(\delta+\omega)^2}{g^2N\Omega^4}$$
$$- \frac{2g^2N\omega(\Delta+\omega)}{\Delta^2\kappa^2} - \frac{2(\kappa^2+\omega^2)\Delta^2(\delta+\omega)(\Delta+\omega)}{g^4N^2\Omega^2} + \frac{(\kappa^2+\omega^2)\Delta^2}{g^4N^2}$$
$$+ \frac{\Delta^2(\kappa^2+\omega^2)(\Delta+\omega)^2(\delta+\omega)^2}{g^4N^2\Omega^4} + \frac{(\kappa^2+\omega^2)(\Delta+\omega)^2}{\Delta^2\kappa^2} \tag{4.18}$$

We make the following approximation, we consider large detuning $\Delta \gg \omega$ compared to the frequency, so we can write

$$(\Delta+\omega)^2 \approx \Delta^2 \tag{4.19}$$

$$\frac{\hat{\mathcal{E}}_{in}(\omega)|^2}{2\gamma_r|\hat{S}(\omega)|^2} \approx \frac{1}{4}(\frac{\Delta^2(\delta+\omega)^2}{\Omega^4} + \frac{g^4N^2}{\Delta^2\kappa^2} + 2 + \frac{2\omega(\delta+\omega)\Delta^2(\Omega^2-\Delta(\delta+\omega))}{g^2N\Omega^4}$$
$$- \frac{2g^2N\omega}{\Delta\kappa^2} - \frac{(\kappa^2+\omega^2)\Delta^2(2\Delta(\delta+\omega)-\Omega^2)}{g^4N^2\Omega^2} + \frac{(\kappa^2+\omega^2)\Delta^2(\Delta+\omega)^2(\delta+\omega)^2}{g^4N^2\Omega^4} + \frac{(\kappa^2+\omega^2)\Delta^2}{\kappa^2\Delta^2} \tag{4.20}$$

Now by choosing the following relation

$$\omega + \delta = \frac{\Omega^2}{\Delta} \tag{4.21}$$

we end up with

$$\frac{\hat{\mathcal{E}}_{in}(\omega)|^2}{2\gamma_r|\hat{S}(\omega)|^2} \approx \frac{1}{4}(1 + \frac{g^4N^2}{\Delta^2\kappa^2} + 2 - \frac{2g^2N\omega}{\Delta\kappa^2} + 1 + \frac{\omega^2}{\kappa^2}) \tag{4.22}$$

In the spirit of the approximations used above, considering $\Delta \gg 0$ we can neglect the second and the forth term and by considering $\omega \ll \kappa$ the last term can be set to zero and equation (4.21) approaces unity. So the inverse of $\frac{\hat{\mathcal{E}}_{in}(\omega)|^2}{2\gamma_r|\hat{S}(\omega)|^2}$ also approaches unity, under the same approximations

$$2\gamma_r \frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} \approx 1 \tag{4.23}$$

This result can be again verified by looking at the reflected ammplitude, which under these approximaitons give

$$R(\omega) \approx \frac{\kappa}{\kappa - i\omega} - 1 \tag{4.24}$$

$$|R(\omega)|^2 \approx 1 - \frac{8\kappa^2}{4\kappa^2+\omega^2} + \frac{4\kappa^2}{4\kappa^2+\omega^2} \approx 0 \tag{4.25}$$

We have basically shown that, if we have zero decay of the excited state $\gamma = 0$, large decay of the cavity , large detuning Δ, strong driving field's strength Ω, and choose the frequency of the probe as $\omega = \frac{\Omega^2}{\Delta} - \delta$ we can approximate perfect impedance matching with the desired value of dephasing $\gamma_r = \frac{g^2N\Omega^2}{\kappa\Delta^2}$.
We will now relax the condition $\gamma_e = 0$ and under the same approximations we will look for approximate impedance matching condtions for non zero decay rate γ_e.
We choose again

$$\gamma_r = \frac{g^2N\Omega^2}{\kappa\Delta^2} \tag{4.26}$$

and we try to find optimal conditions for $2\gamma_r \frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2}$ to approach unity.

$$2\gamma_r \frac{|\hat{S}(\omega)|^2}{|\hat{\mathcal{E}}_{in}(\omega)|^2} = \frac{4g^4 n^2 \Omega^4}{D} \tag{4.27}$$

$$\begin{aligned} D1 =& \Delta^2 (g^4 n^2 ((\delta+\omega)^2 + \frac{g^4 n^2 \Omega^4}{\Delta^4 \kappa^2}) + 2g^2 n \left(\omega(\delta+\omega)\Omega^2 + \frac{g^2 n \Omega^4}{\Delta^2} + (\gamma\kappa - \omega(\Delta+\omega))((\delta+\omega)^2 + \frac{g^4 n^2 \Omega^4}{\Delta^4 \kappa^2}) \right) \\ &+ (\kappa^2+\omega^2) \left(-2(\delta+\omega)(\Delta+\omega)\Omega^2 + \Omega^4 + \frac{2g^2 N \gamma \Omega^4}{\Delta^2 \kappa} + (\gamma^2 + (\Delta+\omega)^2)((\delta+\omega)^2 + \frac{g^4 n^2 \Omega^4}{\Delta^4 \kappa^2}) \right)) \end{aligned} \tag{4.28}$$

Now as before we look at the inverse of $2\gamma_r |\tilde{S}(\omega)|^2$

$$\begin{aligned} \frac{1}{2\gamma_r |\tilde{S}(\omega)|^2} = \frac{D1}{4g^4 N^2 \Omega^4} = \frac{1}{4}(& \frac{\Delta^2(\delta+\omega)^2}{\Omega^4} + \frac{g^4 N^2}{\Delta^2 \kappa^2} + \frac{2\omega(\delta+\omega)\Delta^2}{g^2 N \Omega^2} + 2 + \frac{2\gamma_e \kappa \Delta^2 (\delta+\omega)^2}{g^2 N \Omega^4} + \\ & \frac{2g^2 N \gamma}{\Delta^2 \kappa} - \frac{2\omega \Delta^2 (\Delta+\omega)(\delta+\omega)^2}{g^2 N \Omega^4} - \frac{2g^2 N \omega (\Delta+\omega)}{\Delta^2 \kappa^2} - \frac{2(\kappa^2+\omega^2)\Delta^2(\delta+\omega)(\Delta+\omega)}{g^4 N^2 \Omega^2} + \\ & \frac{(\kappa^2+\omega^2)\Delta^2}{g^4 N^2} + \frac{(\kappa^2+\omega^2)\gamma_e}{g^2 N \kappa} + \frac{\Delta^2(\kappa^2+\omega^2)(\gamma_e)^2(\delta+\omega)^2}{g^4 N^2 \Omega^4} + \frac{(\kappa^2+\omega^2)\gamma_e^2}{\Delta^2 \kappa^2} + \\ & \frac{\Delta^2(\kappa^2+\omega^2)(\Delta+\omega)^2(\delta+\omega)^2}{g^4 N^2 \Omega^4} + \frac{(\kappa^2+\omega^2)(\Delta+\omega)^2}{\Delta^2 \kappa^2} \end{aligned} \tag{4.29}$$

By using as before $\Delta \gg \omega$ and $\omega + \delta = \frac{\Omega^2}{\Delta}$ the equation reduces to

$$\begin{aligned} \frac{\hat{\mathcal{E}}_{in}(\omega)|^2}{2\gamma_r |\hat{S}(\omega)|^2} \approx \frac{1}{4}(& 1 + \frac{g^4 N^2}{\Delta^2 \kappa^2} + 2 - \frac{2g^2 N \omega}{\Delta \kappa^2} + 1 + \frac{\omega^2}{\kappa^2} + \frac{2g^2 N \gamma}{\Delta^2 \kappa} + \frac{(\kappa^2+\omega^2)\gamma_e^2}{\Delta^2 \kappa^2} + \\ & \frac{2g^2 N \gamma_e}{\Delta^2 \kappa} + \frac{(\kappa^2+\omega^2)\gamma_e^2}{g^4 N^2} + \frac{(\kappa^2+\omega^2)\gamma_e}{g^2 N \kappa} + \frac{2\gamma_e \kappa}{g^2 N} \end{aligned} \tag{4.30}$$

Now by considering $\Delta \gg g\sqrt{N}, \Delta \gg \gamma_e$ we can neglect the five terms, which have Δ and Δ^2 in the denominator. Furthermore by considering $\omega \ll \kappa$ we can neglect the fifth term and also use the followin relation $\kappa^2 + \omega^2 \approx \kappa^2$. We have

$$\frac{\hat{\mathcal{E}}_{in}(\omega)|^2}{2\gamma_r |\hat{S}(\omega)|^2} \approx \frac{1}{4}(4 + \frac{3\kappa^2 \gamma_e^2}{g^4 N^2} + \frac{\kappa^2 \gamma_e}{g^2 N \kappa}) \tag{4.31}$$

and now, by writting the above equation in terms of the collective cooperativity $C = \frac{g^2 N}{\kappa \gamma_e}$

$$\frac{\hat{\mathcal{E}}_{in}(\omega)|^2}{2\gamma_r |\hat{S}(\omega)|^2} \approx 1 + \frac{3}{4C} + (\frac{1}{2C})^2 \tag{4.32}$$

So turning now to the

$$\frac{2\gamma_r |\hat{S}(\omega)|^2}{\hat{\mathcal{E}}_{in}(\omega)|^2} \approx \frac{1}{1 + \frac{3}{4C} + (\frac{1}{2C})^2} \tag{4.33}$$

for large cooperativity we can approach unity. We first neglect the second order term by considering $C \gg 0$ which leads to $\frac{1}{C^2} \to 0$

$$\frac{2\gamma_r |\hat{S}(\omega)|^2}{\hat{\mathcal{E}}_{in}(\omega)|^2} \approx \frac{1}{1 + \frac{3}{4C}} \to 1 \tag{4.34}$$

We found the conditions, under which we can be achieve impedance matching for the value $\gamma_r = \frac{g^2 N \Omega^2}{\kappa \Delta^2}$, which will be responsible for long lived Rydeberg excitations, as we will see in the last chapter.

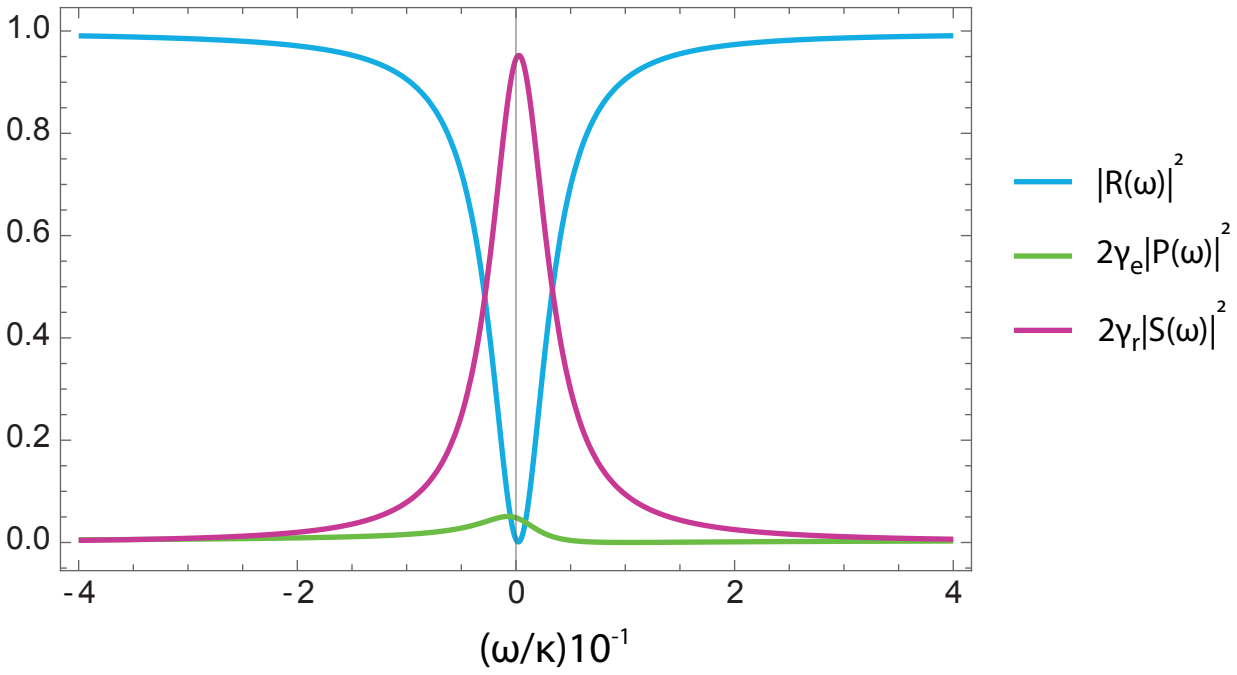

Figure 4.3: Diagram of the reflection rate $|R(\omega)|^2$, the loss rate through decay of the excited state $|e\rangle$, $2\gamma_e|\tilde{P}(\omega)|^2$ and the loss rate through dephasing of the state $|r\rangle$ as a function of the dimensionless frequency of the sugnal, for the case of the driving field being $\Omega = 5$. The values used g=1, n=40,γ_e=0.2,κ=10, γ_r = 0.16,$\Delta = 25, \delta = 1$.

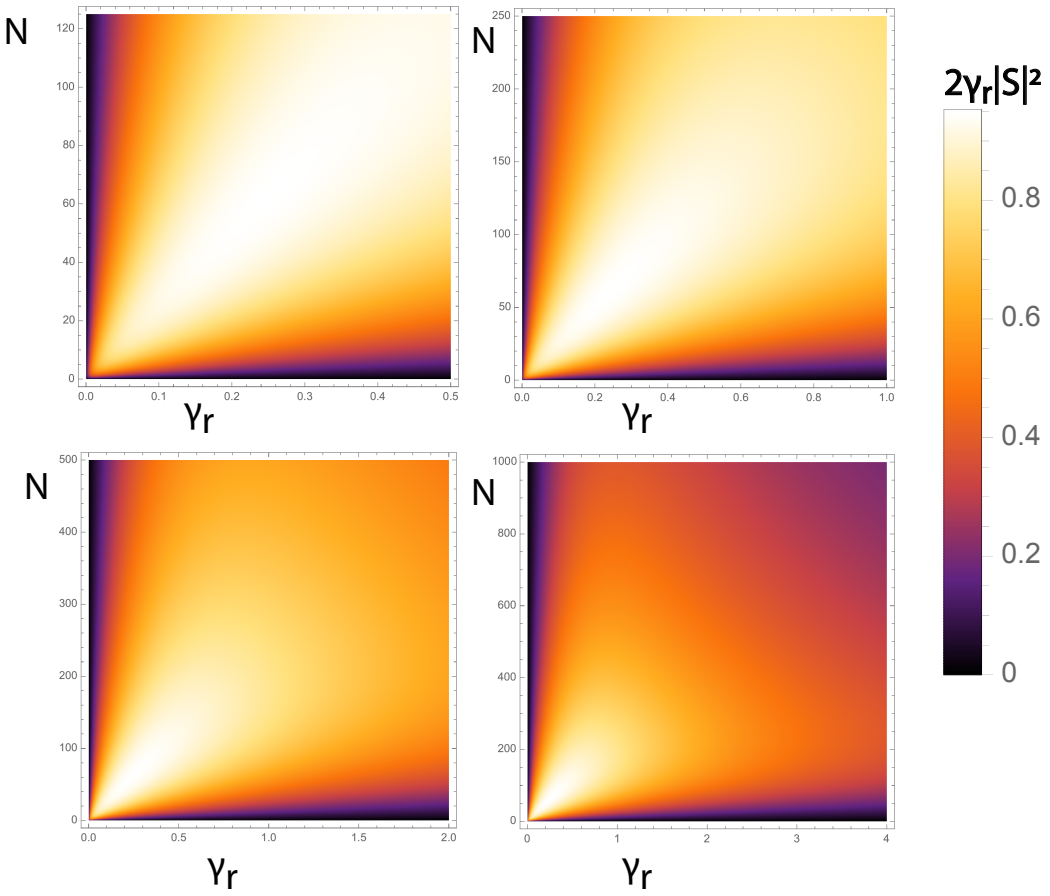

Figure 4.4: Diagram of the dephasing rate $2\gamma_r|\tilde{S}(\omega)|^2$, the loss rate through dephasing of the Rydberg state $|r\rangle$, as a function of the number of atoms on the and the dephasing rate. We can see that the system for linear of beviour of γ_r and n for small values. As we grow larger this behaviour fades, as a result of the n being very large and the approximation of $\Delta \gg g^2N$ breaks down. The values used above are the $\Omega = 5$, g=1,γ_e=0.2,κ=10,$\Delta = 25, \delta = 1$.

Chapter 5

Rydberg Induced Dephasing

Our goal in this section is to describe how the incoming signal field induces a dephasing on an atom being in the $|r'\rangle$ state. To do this we will use the effective operator formalism derived in chapter 2, dividing the system into an excited and a ground subspace and treating the coupling between them pertubatevily. This will allow us to adiabatically eliminate the exctited subspace, which will contain the incoming signal field and leads the an effective dephasing. The Hamiltonian of the entire system is

$$\begin{aligned}\hat{\mathcal{H}} =&\hbar(\Delta\hat{\sigma}_{ee} + \delta\hat{\sigma}_{rr} + \Delta'\sigma_{e'e'} + \delta'\hat{\sigma}_{r'r'} - \Omega_1\hat{\sigma}_{eg} - \Omega_1^*\hat{\sigma}_{ge} - \Omega_2\hat{\sigma}_{re} - \Omega_2^*\hat{\sigma}_{er} - \Omega\hat{\sigma}_{r'e'} - \Omega^*\hat{\sigma}_{e'r'}) \\ &- \hbar(g\hat{\mathcal{E}}\hat{\sigma}_{e'g} + g^*\hat{\mathcal{E}}^\dagger\hat{\sigma}_{ge'}) + \hbar\sum_{l=1}^{N}\sum_{\substack{k=1\\l\neq k}}^{N}\mathcal{V}_{kl}\,|r_l\rangle\langle r_l|\otimes|r'_k\rangle\langle r'_k|\end{aligned} \tag{5.1}$$

In order to describe our system we first need to define its Hilbert space, that is the vector

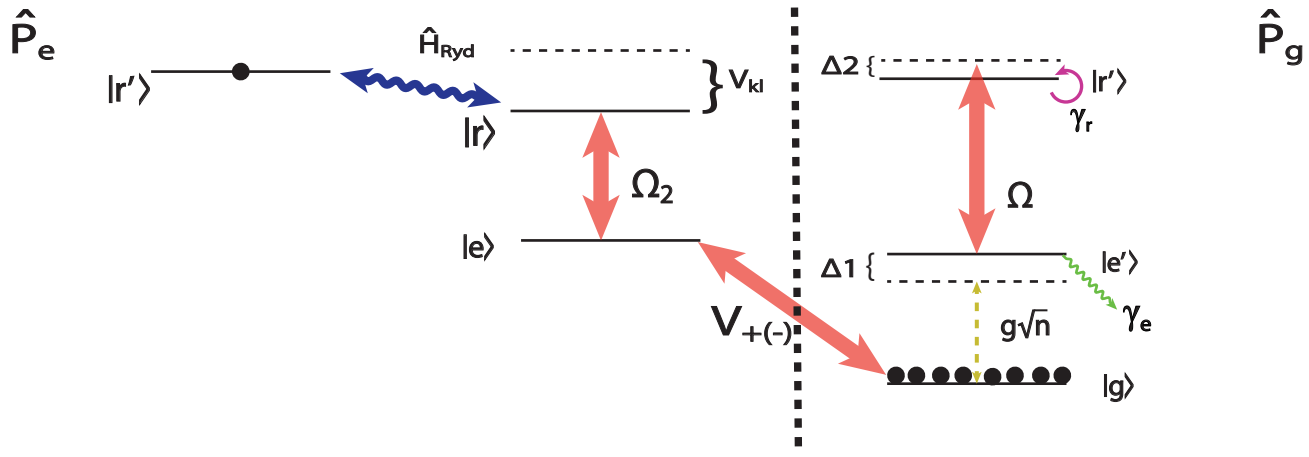

Figure 5.1: Energy diagram of two subspaces excited and groud. Rydberg interaction is part of the excited subspace. γ_r is the induced dephasing, by the elimination of the excited subspace

space of all the possible states of our system. We introduce the general state of our system $|\Psi\rangle$,

$$\begin{aligned}|\Psi\rangle =& c_g\,|g^N\rangle + \sum_{l=1}^{N} c_{el}\,|g^{N-1}, e_l\rangle + \sum_{l=1}^{N} c_{rl}\,|g^{N-1}, r_l\rangle + \sum_{k=1}^{N} c_{e'k}\,|g^{N-1}, e'_k\rangle \\ &+ \sum_{k=1}^{N} c_{r'k}\,|g^{N-1}, r'_k\rangle + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} c_{ee'kl}\,|g^{(N-2)}, e_l, e'_k\rangle + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} c_{er'kl}\,|g^{N-2}, e_l, r'_k\rangle \\ &+ \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} c_{e'rkl}\,|g^{N-2}, r_l, e'_k\rangle + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} c_{rr'kl}\,|g^{N-2}, r_l, r'_k\rangle\,,\end{aligned} \tag{5.2}$$

where $c_{\mu\nu mn}$ is the coefficient of m-th atom being in state $|\mu\rangle$ and n-th atom being in state $|\nu\rangle$ and $c_{\mu\nu mn}$ and $c_{\mu m}$ the probability of the m-th atom being in state $|\mu\rangle$. Any state of our system can be written as $|\Psi\rangle$ as long the coefficients are normalized i.e. the sum of their amplitudes should be equal to one.
From this generalized state $|\Psi\rangle$ of the system, we can indentify the nine terms that consist the basis of the dimensional Hilbert space of our system with dimensions $4N^2+4N+1$. So we can define the identity operator of the Hilbert space of our system $\hat{I}$ as following

$$\begin{aligned}\hat{I} =& |g^N\rangle\langle g^N| + \sum_{l=1}^{N} |g^{N-1}, e_l\rangle\langle g^{N-1}, e_l| + \sum_{l=1}^{N} |g^{N-1}, r_l\rangle\langle g^{N-1}, r_l| + \sum_{k=1}^{N} |g^{N-1}, e'_k\rangle\langle g^{N-1}, e'_k| \\ &+ \sum_{k=1}^{N} |g^{N-1}, r'_k\rangle\langle g^{N-1}, r'_k| + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{(N-2)}, e_l, e'_k\rangle\langle g^{(N-2)}, e_l, e'_k| + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, e_l, r'_k\rangle\langle g^{N-2}, e_l, r'_k| \\ &+ \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, r_l, e'_k\rangle\langle g^{N-2}, r_l, e'_k| + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, r_l, r'_k\rangle\langle g^{N-2}, r_l, r'_k|\,.\end{aligned} \tag{5.3}$$

Now following the effective operator formalsim introduced in chapter 2, we use the projection-operator method of Feshbach [17] to divide the Hilbert space into two subspaces, one for the ground states and one for the excited states, represented by the projection operators $\hat{P}_g$ and $\hat{P}_e$, with $\hat{P}_g + \hat{P}_e = \hat{I}$ and $\hat{P}_g\hat{P}_e = 0$.

$$\hat{P}_g = |g^N\rangle\langle g^N| + \sum_{k=1}^{N} |g^{N-1}, e'_k\rangle\langle g^{N-1}, e'_k| + \sum_{k=1}^{N} |g^{N-1}, r'_k\rangle\langle g^{N-1}, r'_k| \tag{5.4}$$

and

$$\begin{aligned}\hat{P}_e =& \sum_{l=1}^{N} |g^{N-1}, e_l\rangle\langle g^{N-1}, e_l| + \sum_{l=1}^{N} |g^{N-1}, r_l\rangle\langle g^{N-1}, r_l| \\ &+ \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, e_l, e'_k\rangle\langle g^{N-2}, e_l, e'_k| + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, e_l, r'_k\rangle\langle g^{N-2}, e_l, r'_k| \\ &+ \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, r_l, e'_k\rangle\langle g^{N-2}, r_l, e'_k| + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, r_l, r'_k\rangle\langle g^{N-2}, r_l, r'_k|\end{aligned} \tag{5.5}$$

The third and forth terms, do not contribute to our perturbative dynamics, since they belong to the excited subspace and they do not couple to the ground subspace, subsequently they can be neglect from our current analysis. Accordingly, the redefined excited space consists of

$$\begin{aligned}\hat{P}_e = &\sum_{l=1}^{N} |g^{N-1}, e_l\rangle \langle g^{N-1}, e_l| + \sum_{l=1}^{N} |g^{N-1}, r_l\rangle \langle g^{N-1}, r_l| \\ &+ \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, e_l, r'_k\rangle \langle g^{N-2}, e_l, r'_k| + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, r_l, r'_k\rangle \langle g^{N-2}, r_l, r'_k|\end{aligned} \tag{5.6}$$

The excited Hamiltonian is found as following

$$\begin{aligned}\hat{\mathcal{H}}_e &= \hat{P}_e \hat{H} \hat{P}_e \\ &= \hbar(\Delta \hat{P}_e \hat{\sigma}_{ee} \hat{P}_e + \delta \hat{P}_e \hat{\sigma}_{rr} \hat{P}_e + \delta' \hat{P}_e \hat{\sigma}_{r'r'} \hat{P}_e + \Omega_2 \hat{P}_e \hat{\sigma}_{re} \hat{P}_e + \Omega_2^* \hat{P}_e \hat{\sigma}_{er} \hat{P}_e) \\ &+ \hbar \hat{P}_e \sum_{k}^{N}\sum_{l\neq k}^{N} \mathcal{V}_{kl} \, |r\rangle_{l\,l}\langle r| \otimes |r'\rangle_{k\,k}\langle r'| \hat{P}_e\end{aligned} \tag{5.7}$$

In order to simplify further the system, we divide the excited space, into to subspaces, one for the absence of an excitation $|r'\rangle$ and one in the presence, denoted by the projection operators $\hat{P}_{e1}$, $\hat{P}_{e2}$, respectively.

$$\hat{P}_{e1} = \sum_{l=1}^{N} |g^{N-1}, e_l\rangle \langle g^{N-1}, e_l| + \sum_{l=1}^{N} |g^{N-1}, r_l\rangle \langle g^{N-1}, r_l| \tag{5.8}$$

$$\hat{P}_{e2} = \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, e_l, r'_k\rangle \langle g^{N-2}, e_l, r'_k| + \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, r_l, r'_k\rangle \langle g^{N-2}, r_l, r'_k| \tag{5.9}$$

$$\begin{aligned}\hat{H}_{e1} = \hat{P}_{e1}\hat{\mathcal{H}}\hat{P}_{e1} = \hbar\Delta \sum_{l=1}^{N} |g^{N-1}, e_l\rangle \langle g^{N-1}, e_l| + \hbar\delta \sum_{l=1}^{N} |g^{N-1}, r_l\rangle \langle g^{N-1}, r_l| \\ - \hbar\Omega_2 \sum_{l=1}^{N} |g^{N-1}, r_l\rangle \langle g^{N-1}, e_l| - \hbar\Omega_2^* \sum_{l=1}^{N} |g^{N-1}, e_l\rangle \langle g^{N-1}, r_l|\end{aligned} \tag{5.10}$$

and if we use the collective operators notation, we have

$$\hat{H}_{e1} = \hbar\Delta\hat{\sigma}_{ee} + \hbar\delta\hat{\sigma}_{rr} - \hbar\Omega_2\sigma_{re} - \hbar\Omega_2^*\sigma_{er} \tag{5.11}$$

Now the Hamiltonian in the second subspace, where the excitation $|r'\rangle$ is present, is

$$\begin{aligned}\hat{H}_{e2} = \hat{P}_{e2}\hat{\mathcal{H}}\hat{P}_{e2} = \\ = \hbar(\Delta+\delta')\sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, e_l, r'_k\rangle \langle g^{N-2}, e_l, r'_k| + \hbar\sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N}(\delta+\delta'+\mathcal{V}_{kl}) |g^{N-2}, r_l, r'_k\rangle \langle g^{N-2}, r_l, r'_k| \\ - \hbar\Omega_2 \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, r_l, r'_k\rangle \langle g^{N-2}, e_l, r'_k| - \hbar\Omega_2^* \sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} |g^{N-2}, e_l, r'_k\rangle \langle g^{N-2}, r_l, r'_k|\,,\end{aligned} \tag{5.12}$$

where in the detuning δ' can be neglected, since it doesn't play any role in the perturbative dynamics between the ground and the excited subspace. It will be also chosen to be zero $\delta' = 0$ for the rest of the thesis.

In order to show how these excited Hamiltonians were calculated we will show the explicit calculation for the last term, which contains the Rydberg interaction, the rest of the terms can be found in the same spirit.
We will begin by first acting on the Rydberg Hamiltonian with the projection operator for the second excited subspace, from the right side as following

$$\hat{\mathcal{H}}_{ryd,1} = \hat{\mathcal{H}}_{ryd}\hat{P}_{e2} = \left(\hbar \sum_{l=1}^{N} \sum_{\substack{k=1 \\ l\neq k}}^{N} \mathcal{V}_{kl} \left|r\right\rangle_{l\,l}\!\left\langle r\right| \otimes \left|r'\right\rangle_{k\,k}\!\left\langle r'\right| \right) \left(\sum_{l'=1}^{N} \sum_{\substack{k'=1 \\ l'\neq k'}}^{N} \left|g^{N-2}, r_{l'}, r'_{k'}\right\rangle \left\langle g^{N-2}, r_{l'}, r'_{k'}\right| \right) \tag{5.13}$$

Only the last term of the second excited subspace projection operator survives, since it is the only term with excited atoms both in state $|r\rangle$ and $|r'\rangle$. The other term contains projection operators to states orthogonal to the states that the Rydberg Hamiltonian project to. Using the orthogonality condtions $\langle r_k|r_{k'}\rangle = \delta_{k,k'}$ and $\langle r'_k|r'_{k'}\rangle = \delta_{k,k'}$ we get

$$\begin{aligned}
\hat{\mathcal{H}}_{Ryd,1} &= \hbar \sum_{l=1}^{N} \left|r_l\right\rangle \left\langle r_l\right| \left(\sum_{l'=1}^{N} \sum_{\substack{k=1 \\ l\neq k}}^{N} \sum_{\substack{k'=1 \\ l'\neq k'}}^{N} \mathcal{V}_{kl} \left|r'_k\right\rangle \left\langle r'_k|g^{n-2}, r_{l'}, r'_{k'}\right\rangle \left\langle g^{n-2}, r_{l'}, r'_{k'}\right| \right) \\
&= \hbar \sum_{l=1}^{N} \left|r_l\right\rangle \left\langle r_l\right| \left(\sum_{l'=1}^{N} \sum_{\substack{k=1 \\ l\neq k}}^{N} \sum_{\substack{k'=1 \\ l'\neq k'}}^{N} \mathcal{V}_{kl} \left|r'_k\right\rangle \delta_{k'k} \left|g^{n-2}, r_{l'}, r'_k\right\rangle \left\langle g^{n-2}, r_{l'}, r'_{k'}\right| \right) \\
&= \hbar \sum_{l=1}^{N} \left(\sum_{l'=1}^{N} \sum_{\substack{k=1 \\ k\neq l \\ k\neq l'}}^{N} \mathcal{V}_{kl} \left|r_l\right\rangle \left\langle r_l|g^{n-2}, r_{l'}, r'_k\right\rangle \left\langle g^{n-2}, r_{l'}, r'_k\right| \right)
\end{aligned} \tag{5.14}$$

$$\hat{\mathcal{H}}_{Ryd,1} = \hbar \sum_{l=1}^{N} \sum_{\substack{k=1 \\ l\neq k}}^{N} \mathcal{V}_{kl} \left|g^{N-2}, r_l, r'_k\right\rangle \left\langle g^{N-2}, r_l, r'_k\right| \tag{5.15}$$

In a very similar way, using the orthogonality relations, it is easy to show that

$$\hat{P}_{e2}\hat{\mathcal{H}}_{Ryd}\hat{P}_{e2} = \hat{P}_{e2}\hat{\mathcal{H}}_{Ryd,1} = \hbar \sum_{l=1}^{N} \left(\sum_{l'=1}^{N} \sum_{\substack{k=1 \\ k\neq l \\ k\neq l'}}^{N} \mathcal{V}_{kl} \left|r_l\right\rangle \left\langle r_l|g^{n-2}, r_{l'}, r'_k\right\rangle \left\langle g^{n-2}, r_{l'}, r'_k\right| \right). \tag{5.16}$$

By a similar way we find all the terms of the excited Hamiltonians defined above.

It is evident that

$$\hat{\mathcal{H}}_e = \hat{\mathcal{H}}_{e1} + \hat{\mathcal{H}}_{e2}, \tag{5.17}$$

since we have not considered $|e'\rangle$ part of our system, and state $|g\rangle$ doesn't couple to $|r'\rangle$, we have that the two excited subspaces are independent.

Taking into account that the only source of dissipation in both the excited subsystems is the decay γ_{eg} of the sate $|e\rangle$, we can define the non Hemitian Hamiltonian.

$$\hat{\mathcal{H}}_{NH} = \hat{\mathcal{H}}_e - i\frac{\gamma_e}{2}(|g^{N-2}, e_l, r'_k\rangle \langle g^{N-2}, e_l, r'_k| + |g^{N-1}, e_l\rangle \langle g^{N-1}, e_l|) \tag{5.18}$$

Accordingly, we can find the inverse non Hermitian Hamiltonians for the two excited subspaces

$$\hat{\mathcal{H}}^{-1}_{NH1} = \frac{1}{\hbar((\Delta - i\gamma_e)\delta - |\Omega_2|^2)}((\Delta - i\gamma_e)\hat{\sigma}_{rr} + \delta\hat{\sigma}_{ee} + \Omega_2\hat{\sigma}_{re} + \Omega_2^*\hat{\sigma}_{er}), \tag{5.19}$$

$$\begin{aligned}\hat{\mathcal{H}}^{-1}_{NH2} = &\sum_{l=1}^{N}\sum_{\substack{k=1\\ l\neq k}}^{N} \frac{1}{\hbar((\delta + \mathcal{V}_{kl})(\Delta - i\gamma_e) - |\Omega_2|^2)}((\delta + \mathcal{V}_{kl})\,|g^{N-2}, e_l, r'_k\rangle \langle g^{N-2}, e_l, r'_k| \\ &+ (\Delta - i\gamma_e)\,|g^{N-2}, r_l, r'_k\rangle \langle g^{N-2}, r_l, r'_k| + \Omega_2\,|g^{N-2}, r_l, r'_k\rangle \langle g^{N-2}, e_l, r'_k| \\ &+ \Omega_2^*\,|g^{N-2}, e_l, r'_k\rangle \langle g^{N-2}, r_l, r'_k|)\end{aligned} \tag{5.20}$$

We can now derive the effective decay in the first excited subspace, introduced by the decay of the l-th from the excited state $|e\rangle$.

$$\begin{aligned}\hat{L}^{ge,l}_{eff,1} =&\sqrt{2\gamma_e}\hat{\sigma}^l_{ge}\hat{\mathcal{H}}^{-1}_{NH1}\hat{V}_+ = \frac{\sqrt{2\gamma_e}\,|g_l\rangle \langle e_l|}{\hbar((\Delta - i\gamma_e)\delta - |\Omega_2|^2)}((\Delta - i\gamma_e)\hat{\sigma}_{rr} + \delta\hat{\sigma}_{ee} + \Omega_2\hat{\sigma}_{re} + \Omega_2\hat{\sigma}_{er})(\hbar\Omega_1\hat{\sigma}_{eg}) \\ &= \frac{\sqrt{2\gamma_e}\delta\Omega_1}{(\Delta - i\gamma_e)\delta - |\Omega_2|^2}\,|g_l\rangle \langle g_l|\end{aligned} \tag{5.21}$$

Furthermore we can find the effective decay in the second excited subspace, introduced by the decay of the l-th from the excited state $|e\rangle$, given that the k-th atom is in the $|r'\rangle$ state.

$$\begin{aligned}\hat{L}^{ge,lk}_{eff,2} &= \sqrt{2\gamma_e}\hat{\sigma}^l_{ge}\hat{\mathcal{H}}^{-1}_{NH2}\hat{V}_+ \\ &= \sum_{l''=1}^{N} \frac{\sqrt{2\gamma_e}\,|g_l\rangle \langle e_l|}{\hbar((\delta + \mathcal{V}_{kl''})(\Delta - i\gamma_e) - |\Omega_2|^2)}((\delta + \mathcal{V}_{kl''})\,|g^{N-2}, e_{l''}, r'_k\rangle \langle g^{N-2}, e_{l''}, r'_k| \\ &+ (\Delta - i\gamma_e)\,|g^{N-2}, r_{l''}, r'_k\rangle \langle g^{N-2}, r_{l''}, r'_k| + \Omega_2\,|g^{N-2}, r_{l''}, r'_k\rangle \langle g^{N-2}, e_{l''}, r'_k| \\ &+ \Omega_2^*\,|g^{N-2}, e''_l, r'_k\rangle \langle g^{N-2}, r_{l''}, r'_k|)(\hbar\Omega_1 \sum_{l'=1}^{N} |e_{l'}\rangle \langle g_{l'}|)\end{aligned} \tag{5.22}$$

After using commutation relations, using the property of Kronecker delta, as before we find

$$\begin{aligned}\hat{L}^{ge,lk}_{eff,2} = &\sum_{l''=1}^{N} \frac{\sqrt{2\gamma_e}\,|g_l\rangle \langle e_l|}{((\delta + \mathcal{V}_{kl''})(\Delta - i\gamma_e) - \Omega_2^2)}((\delta + \mathcal{V}_{kl''})\Omega_1\,|g^{N-2}, e_{l''}, r'_k\rangle \langle g^{N-1}, r'_k| \\ &- \Omega_2\Omega_1\,|g^{N-2}, r_{l''}, r'_k\rangle \langle g^{N-1}, r'_k|),\end{aligned} \tag{5.23}$$

and using again the commutation relations, we get to the final form

$$\hat{L}^{ge,lk}_{eff,2} = \frac{\sqrt{2\gamma_e}(\delta + \mathcal{V}_{kl})\Omega_1}{((\delta + \mathcal{V}_{kl})(\Delta - i\gamma_e) - \Omega_2^2)}\,|g^{N-1}, r'_k\rangle \langle g^{N-1}, r'_k|\,. \tag{5.24}$$

Now we can find the effective dephasing rate of the ground state $\frac{1}{N}\sum_{k=1}^{N}|g^{N-1}, r'_k\rangle$, induced

by the decay of the l-th atom, given that the k-th atom in the $|r'\rangle$ state.

$$\begin{aligned}
\gamma_1^{lk} &= \left| (\frac{1}{\sqrt{N}} \sum_{k''=1}^{N} \langle g^{N-1}, r'_{k''}|) \hat{L}_{eff}^{ge,lk} (\frac{1}{\sqrt{N}} \sum_{k'=1}^{N} |g^{N-1}, r'_{k'}\rangle) \right|^2 \\
&= \left| (\frac{1}{\sqrt{N}} \sum_{k''=1}^{N} \langle g^{N-1}, r'_{k''}|) (\hat{L}_{eff,1}^{ge,lk} + \hat{L}_{eff,2}^{ge,lk}) (\frac{1}{\sqrt{N}} \sum_{k'=1}^{N} |g^{N-1}, r'_{k'}\rangle) \right|^2 \\
&= \left| (\frac{1}{N} \sum_{k''=1}^{N} \sum_{k'=1}^{N} \langle g^{N-1}, r'_{k''}|) (\frac{\sqrt{2\gamma_e}(\delta + \delta' + \mathcal{V}_{kl})\Omega_1 |g^{N-1}, r'_k\rangle \langle g^{N-1}, r'_k|}{\hbar((\delta + \delta' + \mathcal{V}_{kl})(\delta' + \Delta - i\gamma_e) - \Omega_2^2)}) (|g^{N-1}, r'_{k'}\rangle) \right|^2 \\
&= \left| \frac{1}{N} \sum_{k''=1}^{N} \sum_{k'=1}^{N} \delta_{k,k'} \delta_{k,k''} \frac{\sqrt{2\gamma_e}(\delta + \delta' + \mathcal{V}_{kl})\Omega_1}{\hbar((\delta + \delta' + \mathcal{V}_{kl})(\delta' + \Delta - i\gamma_e) - \Omega_2^2)} \right|^2
\end{aligned} \tag{5.25}$$

where in the last line we used the commutation relations of the braket notation, that lead to

$$\gamma_1^{lk} = \left| \frac{1}{N} \frac{\sqrt{2\gamma_e}(\delta + \delta' + \mathcal{V}_{kl})\Omega_1}{((\delta + \delta' + \mathcal{V}_{kl})(\delta' + \Delta - i\gamma_e) - \Omega_2^2)} \right|^2 \tag{5.26}$$

If we now consider being in the ground state $|g^N\rangle$, then the effective dephasing rate is

$$\gamma_2^l = |\langle g^N| \hat{L}_{eff,1}^{ge} |g^N\rangle|^2 = \left| \frac{\sqrt{2\gamma_e}\delta\Omega_1}{(\Delta - i\gamma_e)\delta - \Omega_2^2} \right|^2 \tag{5.27}$$

The effective Hamiltonian is

$$\begin{aligned}
\hat{\mathcal{H}}_{eff} &= -\frac{1}{2}\hat{V}_-(\hat{\mathcal{H}}_{NH}^{-1} + \hat{\mathcal{H}}_{NH}^{-1\dagger})\hat{V}_+ + \hat{\mathcal{H}}_g \\
&= \frac{\hbar\Omega_1^2\delta}{(\Delta - i\gamma_e)\delta - |\Omega|^2} |g^N\rangle \langle g^N| + \frac{\hbar\Omega_1^2(\delta + \delta' + \mathcal{V}_{kl})}{(\delta + \delta' + \mathcal{V}_{kl})(\delta' + \Delta - i\gamma_e) - |\Omega_2|^2} |g^{N-1}, r'_k\rangle \langle g^{N-1}, r'_k| + \hat{\mathcal{H}}_g
\end{aligned} \tag{5.28}$$

Now let us consider the superposition of the the two ground states $|\Psi\rangle = c_1 \frac{1}{N} \sum_{k=1}^{N} |g^{N-1}, r'_k\rangle + c_2 |g^N\rangle = c_1 |R\rangle + c_2 |g^N\rangle$, for simplicity. It's equation of motion in the effective operator formalism, is given by equation 2.101,

$$\begin{aligned}
\dot{\hat{\sigma}}_{\Psi\Psi} &= \frac{i}{\hbar}[\hat{\mathcal{H}}_{eff}, \hat{\sigma}_{\Psi\Psi}] - \frac{1}{2}(\hat{L}_{eff}^{\dagger eg}\hat{L}_{eff}^{eg}\hat{\sigma}_{\Psi\Psi} + \hat{\sigma}_{\Psi\Psi}\hat{L}_{eff}^{\dagger eg}\hat{L}_{eff,i}^{eg}) + \sum \hat{L}_{eff}^{\dagger eg}\hat{\sigma}_{\Psi\Psi}\hat{L}_{eff}^{eg} \\
&= \frac{i}{\hbar}[\hat{\mathcal{H}}_{eff}, \hat{\sigma}_{\Psi\Psi}] \\
&- \frac{1}{2}(\gamma_1 |g^N\rangle \langle g^N| + \gamma_2 |R\rangle \langle R|)(|c_1|^2 |g^N\rangle \langle g^N| + |c_2|^2 |R\rangle \langle R| + c_1c_2^* |g^N\rangle \langle R| + c_1^*c_2 |R\rangle \langle g^N|) \\
&- \frac{1}{2}(|c_1|^2 |g^N\rangle \langle g^N| + |c_2|^2 |R\rangle \langle R| + c_1c_2^* |g^N\rangle \langle R| + c_1^*c_2 |R\rangle \langle g^N|)(\gamma_1 |g^N\rangle \langle g^N| + \gamma_2 |R\rangle \langle R|) \\
&+ (\sqrt{\gamma_1} |g^N\rangle \langle g^N| + \sqrt{\gamma_2} |R\rangle \langle R|)(|c_1|^2 |g^N\rangle \langle g^N| + |c_2|^2 |R\rangle \langle R| + c_1c_2^* |g^N\rangle \langle R| + c_1^*c_2 |R\rangle \langle g^N|)\times \\
&\times (\sqrt{\gamma_1} |g^N\rangle \langle g^N| + \sqrt{\gamma_2} |R\rangle \langle R|)
\end{aligned} \tag{5.29}$$

which leads to

$$\begin{aligned}
\dot{\hat{\sigma}}_{\Psi\Psi} &= \frac{i}{\hbar}[\hat{\mathcal{H}}_{eff}, \hat{\sigma}_{\Psi\Psi}] - \frac{1}{2}(\gamma_1 + \gamma_2 - 2\sqrt{\gamma_1}\sqrt{\gamma_2})c_1c_2^* |g^N\rangle \langle R| - \frac{1}{2}(\gamma_1 + \gamma_2 - 2\sqrt{\gamma_1}\sqrt{\gamma_2})c_2c_1^* |R\rangle \langle g^N| \\
&= \frac{i}{\hbar}[\hat{\mathcal{H}}_{eff}, \hat{\sigma}_{\Psi\Psi}] - (\sqrt{\gamma_2} - \sqrt{\gamma_1})^2(c_1c_2^* |g^N\rangle \langle R| + c_2c_1^* |R\rangle \langle g^N|)
\end{aligned} \tag{5.30}$$

we see that the diagonal terms of the superposition are decaying and that To get a better understanding of the dynamics we look at the operator

$$\hat{\sigma}_{g\Psi} = |g^N\rangle \langle R| \tag{5.31}$$

$$\begin{aligned}
\dot{\hat{\sigma}}_{g\Psi} &= \frac{i}{\hbar}[\hat{\mathcal{H}}_{eff}, \hat{\sigma}_{g\Psi}] - \frac{1}{2}\sum_{\substack{k=1\\l\neq k}}^{N}((\hat{L}_{eff}^{\dagger eg,lk}\hat{L}_{eff}^{eg,lk}\hat{\sigma}_{g\Psi} + \hat{\sigma}_{g\Psi}\hat{L}_{eff}^{\dagger eg,lk}\hat{L}_{eff,i}^{eg,lk}) + \hat{L}_{eff}^{\dagger eg,lk}\hat{\sigma}_{\Psi\Psi}\hat{L}_{eff}^{eg,lk}) \\
&= \frac{i}{\hbar}[\hat{\mathcal{H}}_{eff}, \hat{\sigma}_{\Psi\Psi}] + \sum_{\substack{k=1\\l\neq k}}^{N}(-\gamma_2^l - \gamma_1^{lk} + \sqrt{\gamma_2^l}\sqrt{\gamma_1^{lk}})(c_1^* |g^N\rangle \langle R|) \\
&= \frac{i}{\hbar}[\hat{\mathcal{H}}_{eff}, \hat{\sigma}_{\Psi\Psi}] - \sum_{l}^{N}\sum_{\substack{k=1\\l\neq k}}^{N}\left|\sqrt{\gamma_1^{lk}} - \sqrt{\gamma_2^l}\right|^2
\end{aligned} \tag{5.32}$$

from the above equation of motion, we can see that the effective dephasing introduced to the collective excitation $|r'\rangle$ by the decay if l-th atom is

$$\gamma_{eff}^{lk} = \left|\sqrt{\gamma_1^{lk}} - \sqrt{\gamma_2}^l\right|^2 = \left|\frac{1}{N}\frac{\sqrt{2\gamma_e}(\delta + \mathcal{V}_{kl})\Omega_1}{(\delta + \mathcal{V}_{kl})(\Delta - i\gamma_e) - |\Omega_2|^2} - \frac{\sqrt{2}\gamma_e\delta\Omega_1}{(\Delta - i\gamma_e)\delta - |\Omega_2|^2}\right|^2 \tag{5.33}$$

We have arrived at the result for the dephasing induced by the decay of $|e\rangle$ state, under the adiabatic elmination of the excited of the excited subspace.

Chapter 6

Continuous Wave Single Photon Transistor

In this chapter, we will use our results from the previous three chapters in order to achieve the complete description of our model for a continuous wave single photon transistor.

Our model consists of a 3-level atomic ensemble, inside a single-sided cavity. All atoms are initially in the ground state $|g\rangle$ and a coherent signal field with Rabi frequency Ω_1 couples the gound state to the excited state $|e\rangle$. Two driving fields, characterized by Rabi frequencies Ω_2 and Ω, are shined on the ensemble. Field Ω_2 drives the $|e\rangle \leftrightarrow |r\rangle$ transition and Ω drives the $|e'\rangle \leftrightarrow |r'\rangle$, both fields are assumed to be on resonance with the transition frequency they are associated with. Since there is no field to couple the ground state $|g\rangle$ to the excited state $|e'\rangle$ no atom occupy state $|e'\rangle$ and the transition $|e'\rangle \leftrightarrow |r'\rangle$, is inactive during this step of the protocol. The diagram of the energy level and the transitions can be seen in figure 6.1b.
The Hamiltonian of the system under the rotating wave approximation and dipole approximation is

$$\hat{\mathcal{H}} = \hbar\Delta\hat{\sigma}_{ee} - \hbar(\Omega_1\hat{\sigma}_{eg} + \Omega_2\hat{\sigma}_{re} + \Omega\hat{\sigma}_{e'r'}) + h.c. + \hbar\sum_{l}^{N}\sum_{\substack{k=1\\l\neq k}}^{N}\mathcal{V}_{kl}\,|r_l\rangle\langle r_l| \otimes |r'_k\rangle\langle r'_k|\,, \tag{6.1}$$

where the Rydberg interaction term is $\mathcal{V}_{kl} = \frac{C_6}{r^6}$, r is the distance between the k-th and the l-th atom and C_6 is the Rydberg strength, which is a charasteristic property of the atoms. Following the notation previously used, $\hat{\sigma}_{\mu\nu} = \sum_{i=1}^{N}\sigma^i_{\mu\nu}$ is the collective operator associated with the transition between states $|\mu\rangle$ and $|\nu\rangle$. Furthermore we consider two sources of dissipation in our system, these are spontaneous emission from the excited states $|e'\rangle$,$|e\rangle$ described by the following Lindblad operators

$$\hat{L}^{eg} = \sqrt{2\gamma_e}\,|g\rangle\langle e|\,, \tag{6.2}$$

$$\hat{L}^{e'g} = \sqrt{2\gamma'_e}\,|g\rangle\langle e'|\,, \tag{6.3}$$

where $2\gamma_e$,$2\gamma'_e$ the spontaneous decay rate from state $|e\rangle$ and state $|e'\rangle$ respectively.
We have chosen the value of the detuning to be equal to minus the decay rate i.e. $\Delta = -\gamma_e$. Furthermore the probe Ω_1 field is taken on resonance with the cavity i.e. $\omega = 0$. Another condition is

$$\Omega_2 = \frac{\pi^2 2^{3/4}\sqrt{\gamma_e C_6}}{3nC} \tag{6.4}$$

Under this condition our signal field Ω_1 is fully reflected due to scattering on the ensemble, a schematic representation of the scattering process can be seen in figure 6.1b. The reason for the perfect reflection, can be seen by looking at equation (3.96), where we have solved the scattering dynamics for basically the same system. Eventhough equation (3.96) was derived

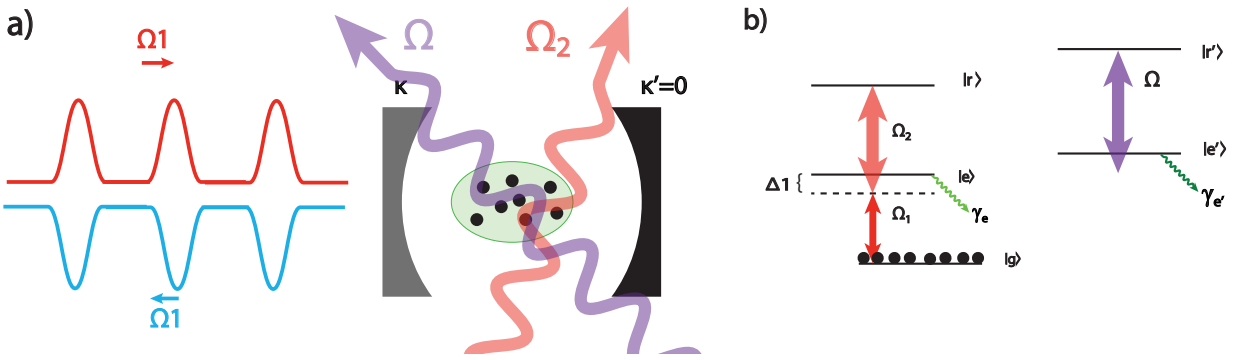

Figure 6.1: a) Schematic representation of our system in the initial case. The signal field Ω_1 couples to the transition $|e\rangle \leftrightarrow |g\rangle$ and being on resonance is totally reflected under EIT condition. Ω_2 couples the transition $|e\rangle \leftrightarrow |g\rangle$, driving field Ω is inactive in the absence of atoms in state $|e'\rangle$. b) Energy diagram of the system. The right branch transition doesn't couple to the atoms at this step.

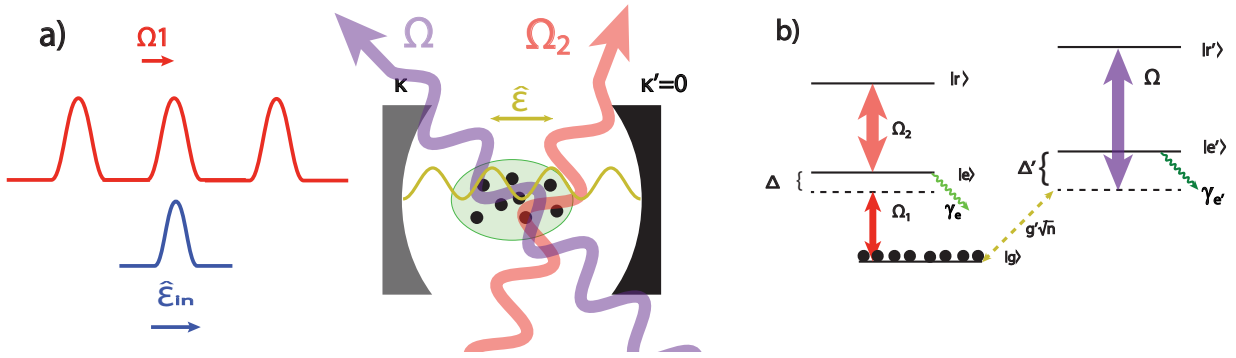

Figure 6.2: a) Schematic representation of the first part of the protocol a single photon $\hat{\mathcal{E}}_{in}$ is sent to the ensemble and couples to the transition $|g\rangle \leftrightarrow |e\rangle$ and subsequently to the transition $|e'\rangle \leftrightarrow |r'\rangle$ through driving field Ω. b) Energy diagram of the system.

for the case of single photon incoming field, the result can be used for coherent input. On resonance we are in Electromagnetically Induced Transparency and since we consider $|r\rangle$ a long lived metastable Rydberg state, with practically zero decay, for the protocols timescales we have perfect reflection, i.e. $\gamma_r = 0$ in (3.96).

Now we move to the second part of the protocol, where we block the signal field, using a weak control field, which is the basic function of any transistor. To achieve that, we send a single photon to the ensemble that couples to the transition $|g\rangle \leftrightarrow |e'\rangle$, this can be seen schematically in figure 6.2. This single photon should be far detuned $\Delta' \gg g'\sqrt{N}, \Delta' \gg \gamma'_e$, it's frequency with respect to the cavity frequency should be equal to the AC Stark shift induced by Ω i.e. $\omega' = \frac{\Omega^2}{\Delta'}$. We proved in chapter 4 that if we are in the strong coupling regime i.e. $g'^2 N \gg \kappa\gamma'_e$ we will approximate perfect impedance matching, meaning that our photon will be mapped to a symmetric collective state $\hat{S}'^{\dagger}|g\rangle = \sum_i^N |g, g, ..., g, r', g, .., g, g\rangle$ and stay there untill it dephases, as seen in figure 6.3.
Subsequently we reach the last part of the protocol that is now having a collective excitation in $|r'\rangle$, the signal field is now with 99.6% probabilty lost through decay of the excited state $|e\rangle$, as we show in chapter 4, which is the reason for choosing (6.4) for Ω_2 in the begining of the chapter. So far, we have achieved a continuous wave single photon transistor, where the presence of the control field blocks the signal, with the driving field Ω turned on during the entire protocol.

Now we move to the last part of the descrption, where we will study the effect of the induced dephasing. As we've shown in chapter 5, the decay of the state $|e\rangle$ as a result of the absoption

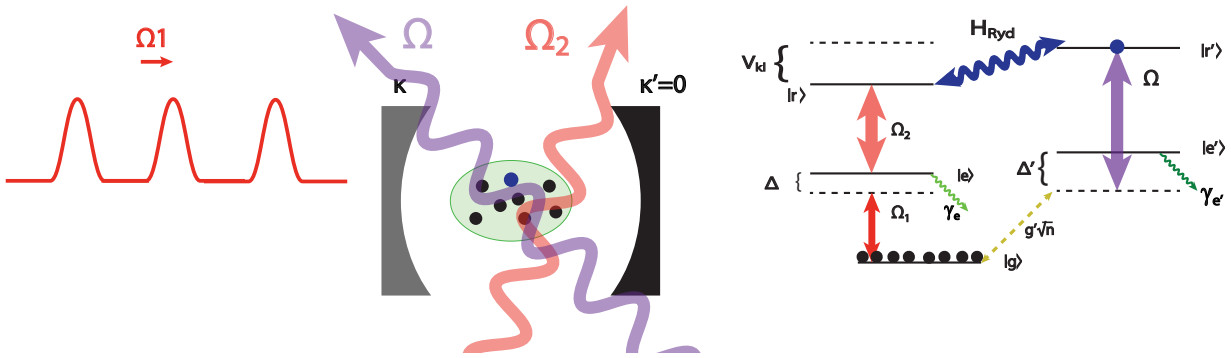

Figure 6.3: a) Schematic representation of the subsequent evolution of the second part of the protocol the single photon $\hat{\mathcal{E}}_{in}$ has been mapped to the excitation $|r'\rangle$ and stays there, since empedance matching conditions are fullfiled. The Rydberg intection between $|r\rangle$,$|r'\rangle$ shifts energy level $|r\rangle$ by $\mathcal{V}_{kl}$ and Ω_2 no longer couples with the transition $|e\rangle \leftrightarrow |r\rangle$, resulting in loss of the input field Ω_1. b) Energy diagram of the system.

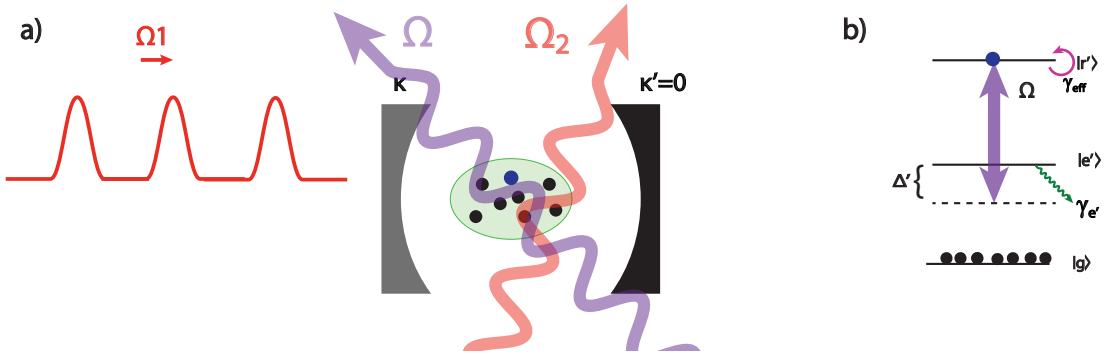

Figure 6.4: a) Schematic representation of the subsequent evolution of the second part of the protocol the single photon $\hat{\mathcal{E}}_{in}$ has been mapped to the excitation $|r'\rangle$ and stays there, since empedance matching conditions are fullfiled. With blue colour we represent the Rydberg excited $|r\rangle$, but it is a loose illustration since the excitation is a symmetric one. The Rydberg interaction between $|r\rangle$ and $|r'\rangle$, shifts energy level by V_{kl} and Ω_2 doesn't couple to the transition $|e\rangle \leftrightarrow |r'\rangle$. b) Energy diagram of the system, where we see that we have adiabatically eliminated the left branch and the signal field, by acquiring an induced dephasing

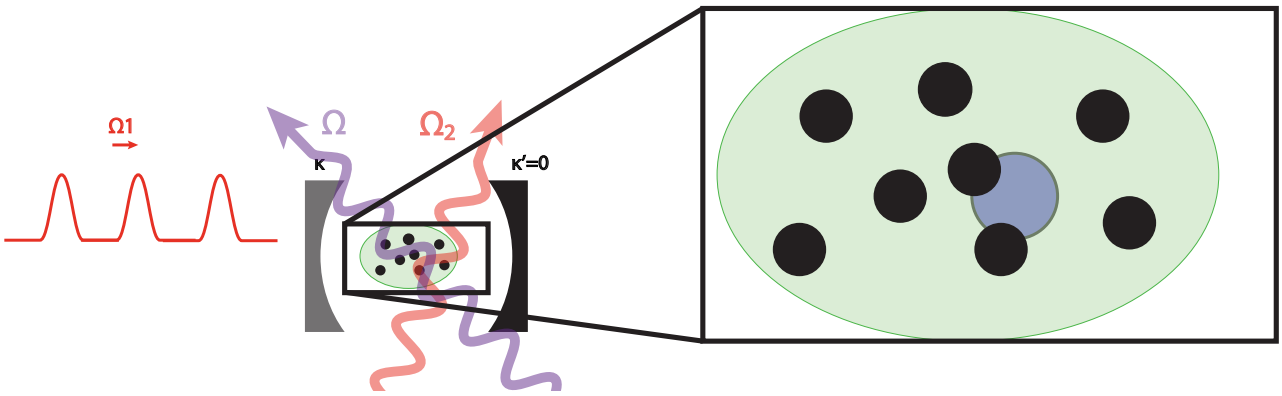

Figure 6.5: The l-th atom driven by the incoming singal field Ω_1 decays from the excited state $|e\rangle$, a process will lead to localization of the symmetric excitation $|r'_k\rangle$ to a sphere around the de-excited atom l-th. This enhaces the lifetime of the excitation $|r'_k\rangle$.

of the signal field, induces a dephasing on the collective excitation in the state $|r'\rangle$. Using the effective operator formalism in the previous chapter we introduced an effective Hamiltonian to describe our system, by adiabatically eliminating the probe field and the states $|e\rangle$, $|r\rangle$, through the introducuction of an effective dephasing. In order to be consistent with our previous analysis we need to use the values for the Rydberg blockade to be 99.6% efficient, so we choose the two photons detunings to be zero $\delta = \delta' = 0$ and the detuning Δ to have the same value as the negative value of the decay for the excited state $|e\rangle$ i.e. $\Delta = -\gamma_e$. Under these conditions the effective decay is

$$\gamma_{eff} = \sum_k \sum_l \left| \frac{1}{N} \frac{\sqrt{2\gamma_e}\mathcal{V}_{kl}\Omega_1}{\mathcal{V}_{kl}(-\gamma_e - i\gamma_e) - |\Omega_2|^2} \right|^2 \tag{6.5}$$

Now we will consider all states in the superposition of the Ryderg state to have equal contribution i.e. $\sum_k \frac{1}{N}\mathcal{V}_{kl} = \mathcal{V}_l$ and now moving from sum over atoms to a space integral as we did in (3.185), we find

$$\gamma_{eff} = \frac{4\pi}{\gamma_e^2 V_a N^2} \sum_k \int_0^{R_a} dr r^2 \left| \frac{\sqrt{2\gamma_e}\Omega_1}{-1 - i - \frac{|\Omega_2|^2}{\gamma_e \mathcal{V}_{kl}}} \right|^2 \tag{6.6}$$

and by considering the radius of the atomic ensemble larger than the one of the Rydberg blockade and set $R_a \to \infty$ we can evaluate this integral, which is similar to (3.202), which we solved in chapter 3. The result is proportional to R^3_{Ryd} and we have chosen the optimal value of Ω_2 in (6.4).

We now have that the value of the induced effective dephasing is equal to $\gamma_{eff} = \frac{g^2 N \Omega^2}{\kappa \Delta'^2}$, that is the condition for impedance matching, we derived in chapter 4. Furthermore, since above we chose $\delta' = 0$, the detuning in the in order to have impedance matching, we need to set the control field at frequency $\omega' = \Omega^2/\Delta'$ and Δ' should be large, i.e.$\Delta' \gg g'\sqrt{N}, \Delta' \gg \gamma'_e$.

At this step of the protocol we are at the collective state $\hat{S}^\dagger |g^N\rangle$ where the excitation $|r'\rangle$ is shared between all the atoms of the ensemble. The incoming control signal with 96% probability will end up in the excited state $|e\rangle$ and will decay. Once this first decay happens our collective excitation will collapse to an excitation consisting of the atoms inside the sphere, with center the l-th atom that decayed and radius equal to the Rydberg radius $R_{Ryd} = \sqrt[6]{\frac{\Omega^2}{\gamma_e C_6}}$. This is illustrated in figure 6.5, the decay of l-th atom from the excited state $|e\rangle$ induces a dephasing that acts on the collective superposition $\hat{S}^\dagger |g^N\rangle$ and transfoms it to the state $\hat{L}^{eg}_{eff}\hat{S}^\dagger |g^N\rangle$, that is localized around the decayed l-th atom. Because we have chosen the conditions, so that the excitation $|r'\rangle$ would be impedance matched, its dephasing rate would be equal the effective rate of decay of $|r'\rangle$ under adiabatic elimination of the state $|e'\rangle$, which is $\frac{g^2 N \Omega}{\kappa \Delta'^2}$, so proportional to

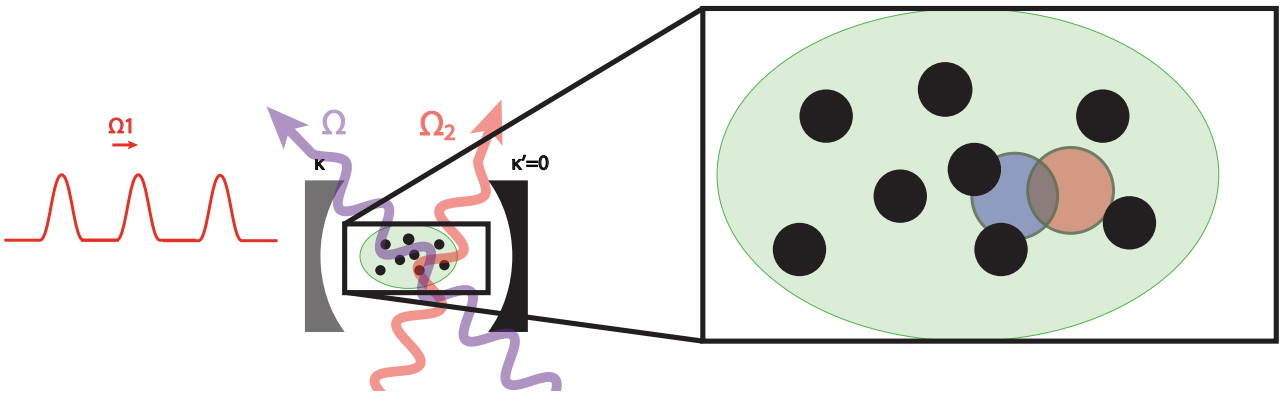

Figure 6.6: A second decay of a l'-th atom will result in a new localization of the excitation of $|r'\rangle$ on the intersection of the two spheres with radius R_{Ryd} and centers the l-th and l'-th atoms.

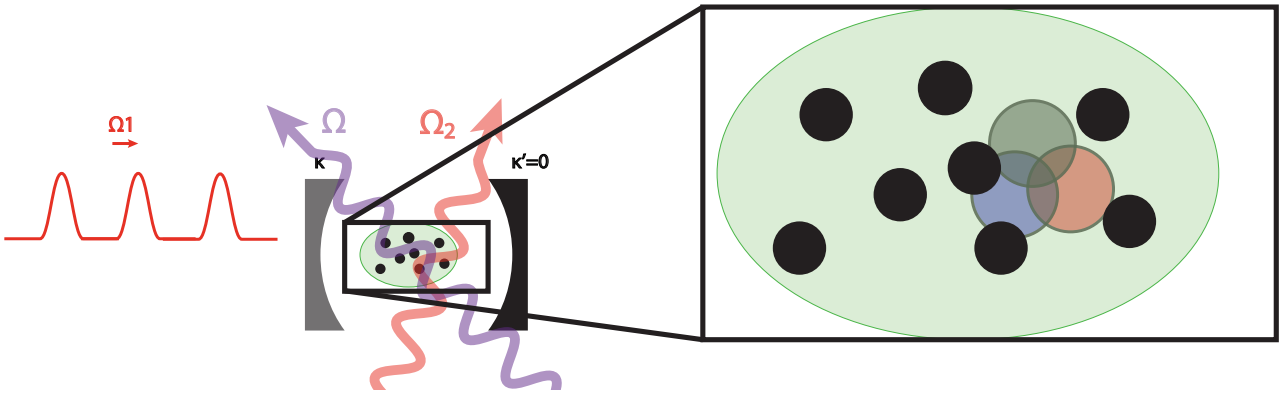

Figure 6.7: Schematic representation of the decay of the l''-th atom, which will lead to even more localized on the intersection between the three spheres, this continuation process can lead to very long lifetimes of the excitation $|r'_k\rangle$.

number of atoms participating in the excitation. This means that the lifetime of the excitation of $|r'\rangle$, will be multiplied by a factor of R_a/R_{Ryd}, the ratio of the radius of the atomic cloud over the Rydberg radius, which can be chosen to be large.

Subsequently the continues to scatter on the ensemble, which with 96% will lead to decay from the excited state of the l'-th atom. The second decay of the l'-th atom will result in a new localization of the excitation of $|r'\rangle$, by transformation to the state $\hat{L}^{eg}_{eff'}\hat{L}^{eg}_{eff}\hat{S}^{\dagger}\,|g^N\rangle$ which will be localized on the intersection of the two spheres with radius R_{Ryd} and centers the l-th and l'-th atoms, leading to a longer lifetime. This process can continue for a long time, under the same protocol. A representation of the next step of the protocol, can be seen for a third decay of the l" atom in figure 6.7.

This process of very long lifetime of the excitation can be very useful for another possible use of our model, that of a photon detector, where our signal field serves as the probe, for revealing the presence of a single photon, the process of long live can result in posible detection of strong coherent fields.

Chapter 7

Conclusion and Outlook

We have demonstrated a way to realize a continuous wave single photon transistor consisting of an ensemble of Rydberg atoms, located inside a single-sided cavity, coupled to two driving fields.

The proposed protocol consists of the use of a control field that is mapped to a collective Rydberg excitation, which leads to Rydberg blockade. Conditioned by the absence or presence of the 'control' field that is mapped to a collective Rydberg excitation, a signal field is reflected or lost respectively. In the absence of the control photon the signal field would be perfectly reflected on resonance through electromagnetically induced transparency(EIT). Under the impedance matching condition, we have proven the continuity of the scheme. This result leads to simplification of the protocol, since driving fields will be turned on during the entire protocol and could possible lead to experimetal simplification.

At this point it is constructive to mention that we have derived a mapping between the single-sided cavity model and free space, in the bad cavity limit. This strongly indicates that the protocol could be realized also for an ensemble of Rydberg atoms in the free space, a model which might be simpler to realize experimentally, but it needs deeper study.

Furthermore we have shown that the signal field once Rydberg blockaded by a collective excitation, would decay with almost unity probabiblity for certain condtions, through decay of the excited state. This decay would induce a dephasing on the Rydberg excitation that would lead to it's partially localization. This process will rise the lifetime of the Rydberg excitation and it is important to mention, that it could be nicely modeled by a Monte Carlo simulation.

Apart from the apparent observation that this process simplifies significantly our model, since a single excitation can block a large number of photons until it decays, this result might be useful for the realization of a high efficiency single photon detector for strong coherent fields. This alternative use of our model as a single photon detector, can be realized by detecting the reflectance of the singal field, which leads to determination of the presence or absence of the single photon control field.

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