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An Ion Crystal Quantum Repeater

Theory concerning the implementation of a DLCZ quantum repeater in a particular experimental realisation of ion Coulomb crystals

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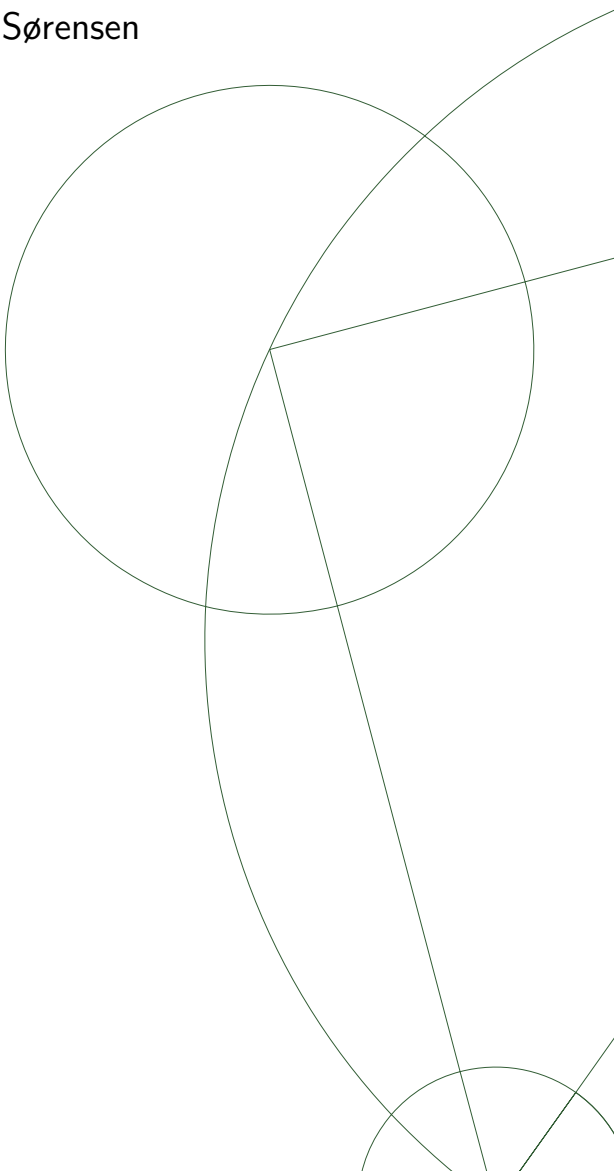

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Contents

Chapter 1

Introduction

A fundamental goal of the technology of quantum information science is reliable quantum communication. To send and to recieve quantum states over reasonably long distances are immense challenges given the rapid decoherence of quantum systems through any kind of transmission line. A possible way forward is to use so-called *quantum repeaters* which – instead of transmitting the actual quantum state – establish entanglement between the sender and the receiver. When entanglement is established it is then possible to do quantum communication with the transmission of only classical information.

In 2001 an important proposal for such a quantum repeater was published by Duan, Lukin, Cirac, and Zoller [Duan *et al.*]. This DLCZ-repeater employs atomic ensembles and single photons to swap entanglement between atomic ensembles. Several conditions must be met by the atomic ensembles for a successful operation. Most important is the light-matter quantum interface but there is also the fact that each ensemble must work as a quantum memory. We shall in this work focus on the requirements on the light-matter quantum interface.

The idea in this master's thesis is to use *Ion Coulomb Crystals* as the atomic ensembles for the DLCZ repeater. At Aarhus University in Denmark an experiment has created large crystals of calcium ions trapped in an electric potential. Here several thousand of trapped ions form regular three dimensional structures [Herskind *et al.*]. The crytals are formed in an optical cavity and strong coupling to the cavity light field can be realised. Other proporties make these crystals interesting for the quantum repeater scheme including their long life time and expected long coherence time. A potential useful feature is the fact that it is possible to trap different species in the same crystal and sympathetically cool the inner ions that do not themselves interact with the cooling light, See Figure 1.1.

With respect to the implementation of the repeater scheme the main challenge in this system is the fact that a standing wave cavity is employed

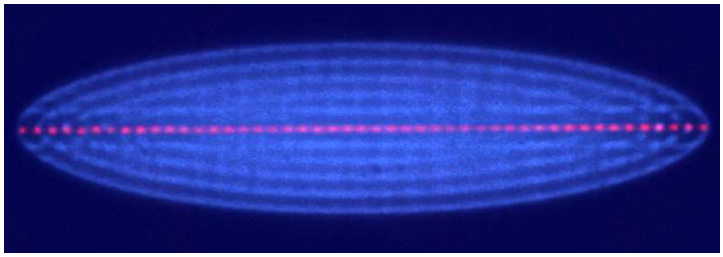

Figure 1.1: This picture is from the experiment in Aarhus: Several shells of lasercooled ions (blue) are sympathetically cooling a string of ions (red) of another species. Fluorescence imaging - false colors. Image available at: *http://www.phys.au.dk/iontrapgroup/*

to get stong coupling. The central questions we seek to resolve in this thesis are, consequently, the following: How does a standing wave in a cavity affect the light-atoms quantum interface? And if the effect is adverse to the quantum repeater scheme is there something simple to be done such that these ion crystals can still be used for the quantum repeater?

The thesis is structured as follows:

Chapter 1 motivates and explains what a quantum repeater is, in particular how the DLCZ quantum repeater works and what it requires for its physical realisation.

In Chapter 2 the ensemble model is derived that describes how the interaction between light and ions ideally works. The model is that we can consider the ions as non-interacting identical particles except for their position with respect to the standing wave cavity field.

Chapter 3 gives a quick overview of the particular experiment with ion Coulomb crystals. Then, we motivate the use of these crystals for the quantum repeater. Finally, a calculation and a nummerical simulation of the quantum interface between one such crystal and a cavity field is presented that highlights the problem of the standing wave in a cavity with respect to necessary operation of the quantum repeater.

The *write-in* process of the quantum repeater is treated in Chapter 4. The interaction with light is considered as a perturbation in the limit of large detuning. A solution to the problem of the standing wave is introduced: We consider moving the crystal during the interaction with light.

Lastly, in Chapter 5 the *read-out* process is described. The interaction is no longer just a perturbation since in the read-out we cannot work with a large detuning. Again, the effects of moving the crystal during the interaction is investigated to overcome the problem of the standing wave.

Chapter 2

Quantum Repeaters

2.1 Motivation

In the decade following the succes of the (first) quantum revolution Albert Einstein, together with Podolsky and Rosen, voiced his concerns that quantum mechanics was not a complete theory [Einstein *et al.*]. They, in a sense, proved that quantum mechanics was not complete, because if it were, then there would exist instantaneous interaction or *spooky action at a distance* between particles that had interacted in the past but are now separated. Erwin Scrödinger said that quantum mechanics was complete and to prove it he proposed a rather sinister experiment involving a cat that was both dead and alive [E. Schrödinger]. The experiment was and is fantastically impossible to perform and he knew it, and in essence he offered very little to prove his point. What he did do was to offer the interpretation which today is the preferred interpretation: The spooky action at a distance is an aspect of a fundamental quantum mechanical property of nature called *entanglement*.

Some thirty years later in the sixties John Bell came up with a theory that made it possible to concieve of experiments to test whether the quantum description was wrong [J. S. Bell]. Bell's work went even further such that it became possible to test an alternative explanation based on the arguments in [Einstein *et al.*]. Based on these ideas Alain Aspect and colleagues in the eighties did a series of very clever experiments which in a very convincing manner showed that the quantum mechanical model was not wrong while the alternative *was* wrong [Aspect *et al.*]. The conclusion everybody drew, including Bell who was rather unhappy about it, was that the concept of entanglement is a very good one.

The last twenty years or so a lot of effort in the scientific community has been put into the development of a technology that utilises the unique properties of entanglement. Optimistically, this is called the second quantum revolution.

2.1.1 Superposition and entanglement

Certain phenomena of quantum mechanics are very different from anything possible in classical mechanics and far removed form daily experience in particular. These phenomena are at the heart of Quantum Information Science (QIS) and therefore necessary for the motivation of the quantum repeater. One such phenomenon is *the superposition principle*. If a prepared system can be represented as being in either state of a set of eigenstates $\{|0\rangle, |1\rangle, ...\}$, then it is possible to prepare a system that can be represented as being in a superposition of these eigenstates:

$$|\psi\rangle_A = \sum_i \alpha_i |i\rangle_A \quad , \; i = 1, 2, ...$$

This fundamental characteristic is not part of the classical description of nature. A phenomenon related to the superosition priciple is *entanglement*. The interaction of two systems each in a quantum superpostion gives rise to the composite system:

$$|\psi\rangle_{AB} = \sum_{ij} \gamma_{ij} |i\rangle_A \otimes |j\rangle_B \; .$$

This state is in general an entangled state - unless the composite state is *separable*, upon which we can write:

$$|\psi\rangle_{AB} = \sum_i \alpha_i |i\rangle_A \otimes \sum_j \beta_j |j\rangle_B \; .$$

This description can be extended to many interacting systems, altough *multipartite* entanglement is not completely straight forward. Entangled systems are not possible in a classical description since superpositions are not part of a classical description.

The simplest quantum system is the two-level system or *the qubit*:

$$|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A \; .$$

To be precise: The quantum system is described by a two-dimensional Hilbert space $\mathcal{H}_A$ with a basis $\mathcal{B}_A = \{|0\rangle, |1\rangle\}$. The simplest example of entanglement is, then, two identical entangled qubits e.g.:

$$|\psi\rangle_{AB} = a_1|00\rangle_{AB} + a_2|11\rangle_{AB} \quad , \; a_i \neq 0 \; ,$$

where we have supressed the explicit sign for the tensor product.

2.1.2 Quantum information science

QIS seeks to exploit the peculiar characteristics of quantum systems introduced above to improve classically based information processing. While a

generalisation to three-level systems *qutrits*, or even more levels, is possible – so far QIS is very much the about the application of entangled qubits.

Information science is, classically, the manipulation of bits. In the technological realm a bit is a certain state of a physical system and the information is the knowledge about that system. Quantum information, then, is the knowledge of quantum systems - qubits. Bits are *either* $|0\rangle$ *or* $|1\rangle$. This means that if we have N bits the information stored is one particular permutation out of 2^N possible permutations. In contrast: Each *qubit* is in a superposition of $|0\rangle$ *and* $|1\rangle$. Consequently, N entangled qubits are in a superpostition of *all* 2^N possible permutations and, therefore, the information in the N qubits is very much larger than in the classical bits. Figure 2.1 illustrates this point.

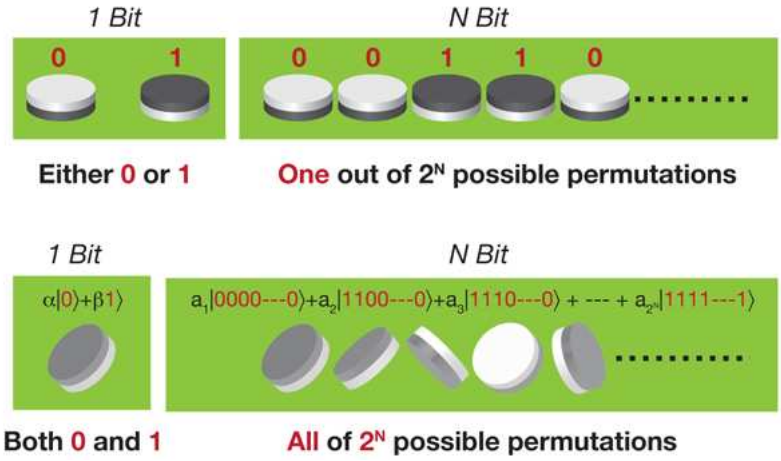

Figure 2.1: An illustration of the large state space accesible from N qubits. The illustration is taken from [Sjöqvist].

This exponential growth of the state space is one of the key motivators in the science and technology of quantum information science. One of the hopes is to model the evolution of quantum systems which likewise have very large state spaces intractable with classical computers [Feynman].

Another application is to use the 'quantum parallelism' of the entangled qubits to perform computations that are otherwise too large for classical computers to finish the computation within reasonable time limits. So far the best example is Shor's algorithm which speeds up the process of factoring large integers exponentially [Shor].

2.1.3 Quantum communication

A different motivation for quantum information science is the application of entanglement to communication schemes. Two applications should be

mentioned here: First, the actual communication of quantum states. A goal of modern research is to build quantum networks - that is, a distribution of quantum states between many nodes to do more complicated computations [Kimble]. Second, the use of certain peculiar characteristics of entanglement to perform completely secure transfer of information.

The Bell states. For the following example of quantum cryptography and for later use we shall need the so-called *Bell states*. These are states composed of two entangled qubits - and one of the states looks like this:

$$|\Phi^+\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}} \ . \tag{2.1}$$

This state is *maximally* entangled which means that measurements of the two qubits is completely correlated. We can do a measurement on the A qubit in the basis $\mathcal{B}_A = \{|0\rangle, |1\rangle\}$ measuring the observable

$$M_A = \lambda_0 |0\rangle_{AA}\langle 0| + \lambda_1 |1\rangle_{AA}\langle 1| \ ,$$

where λ_i is a real eigenvalue measurement outcome that occur with probability $\frac{1}{2}$. After the measurement the states are projected to

$$\lambda_0 : \ P_0|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle_{AB} \qquad \lambda_1 : \ P_1|\Phi^+\rangle = \frac{1}{\sqrt{2}}|11\rangle_{AB} \ ,$$

where $P_i = |i\rangle_{AA}\langle i|$ are the projection operators onto the eigenspace corresponding to λ_i. This means that if e.g. λ_0 was obtained in a measurement of qubit A we not only know the state of A but we also know, with complete certainty, the state of qubit B.

This is actually also possible in a classical description of systems. Imagine, we pick at random from a large ensemble of paper slips, half of which is labelled '00' and the other half '11', then we would obtain the same result as repeated trials of the quantum measurement above. This example was borrowed from [Brask master thesis]. What is *not* possible, classically, is the following: In quantum mechanics superpositions can be a good description of a system, so we can choose the basis for the measurement $\mathcal{B}'_A = \{|+\rangle, |-\rangle\}$, where

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \ .$$

Rewriting $|\Phi^+\rangle_{AB}$ in this basis we see that

$$|\Phi^+\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}} = \frac{|++\rangle_{AB} + |--\rangle_{AB}}{\sqrt{2}} \ .$$

So, measurements in this new basis are also completely correlated. The correlations are independent of the chosen basis and can give statistics not

possible in the classical description [J. S. Bell]. The other three Bell States are (surpressing the AB subscript):

$$|\Phi^-\rangle = (|00\rangle - |11\rangle)/\sqrt{2} \tag{2.2}$$

$$|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \tag{2.3}$$

$$|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}\ . \tag{2.4}$$

Quantum cryptography [Ekert, A]. Let A and B share n entangled Bell state qubits e.g. of the form

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\ .$$

A measures these n qubits in either the $\{|0\rangle, |1\rangle\}$ basis or the $\{|+\rangle, |-\rangle\}$ basis – which basis is chosen randomly. B does the same. A and B publish their choice of basis - if they are not the same they discard the measurement. But the rest of the measurements – around $n/2$ bits – make up a key that only A and B know. An attempt at eavesdropping would destroy the correlations in a way that A and B can detect by publishing a small part of the common key: If this part of the key is not the same for both A and B then somebody has listened (or the transmission line is broken).

Note that it is impossible for an eavesdropper to listen in (this would collapse the wave-function and destroy the entanglement) and then copy the quantum state (or in the opposite order) – this is called the *no-cloning theorem of quantum mechanics*, see below.

Real world quantum communication

Entanglement in nature is actually commonplace: An atom that decays is entangled with the photon it emits, two atoms colliding are entangled, etc. The challenge is to get the desired entangled state and to keep the entanglement intact during evolution of the systems. To prepare the entangled state the systems need to interact, but to keep the entanglement intact the systems must not interact again or with the environment, since any new interaction would give another entangled state and thus the desired entangled state is lost. These two conflicting properties will play a major part in the following.

Single photons as qubits. First, let us try to solve the problem of keeping the quantum state and therefore entanglement intact during evolution. Photons interact rarely with the environment and even more rarely with each other. They are, as is also evident from the revolution in fibre optics communication, very efficient carriers of information. In classical information the bit is encoded as sequence of pulses each consisting of many photons. The idea in quantum cmmunication is to encode the quantum information of a qubit in a single photon.

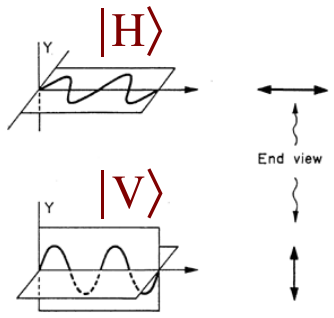

Figure 2.2: Using linear polarisation as a basis for encoding a qubit in a single photon. Image available at: *www.opticsforkids.org*

Encoding qubits in polarisation. The simplest example is to make a basis for the qubit in the vertical and horizontal polarisation of the electromagnetic field: $\mathcal{B} = \{|V\rangle, |H\rangle\}$, see Figure 2.2. To obtain a quantum superposition state all that is needed in principle is a half-waveplate at an angle θ with respect to the direction of the polarisation of the electro-magnetic field of the single photon. Classically rotating the polarisation of the electromagnetic field would correspond to putting the single photon into a superposition of the two orthogonal polarisations. After the passage through the waveplate the photon will be in the state:

$$|H\rangle \rightarrow \cos(2\theta)|H\rangle + \sin(2\theta)|V\rangle$$
$$|V\rangle \rightarrow \cos(2\theta)|H\rangle - \sin(2\theta)|V\rangle \ .$$

In QIS the transformation arising from $\theta = 22.5$ degrees is called a *Hadamard transformation.*

Spontaneous parametric down-conversion. Certain crystals have the remarkable non-linear characteristic that they can convert one incident photon into two photons. These photons travel along different axes through the crystal and for the type II crystals their polarisation is orthogonal, see Figure 2.3.

Furthermore, one can pump the laser such that at most one pair of photons is created in the crystal from each pulse. Now it is only a matter of aligning the crystal correctly, and then collecting the photons where the two different polarisation cones overlap, see Figure 2.4. Then one has succesive copies of Bell states made from pairs of photons, each pair in the polarization-entangled singlet state:

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B\big) \ .$$

Now, we have the entangled pairs in the form of photons that are relatively

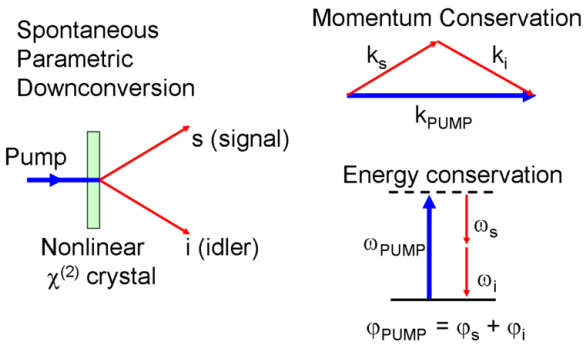

Figure 2.3: Illustration of the process of spontaneous parametric down-conversion giving the constraints on the wavenumbers and the frequencies of the photon pair created from the more energetic pump-photon. Image address: *http://en.wikipedia.org/wiki/Parametric(underscore)down-conversion*

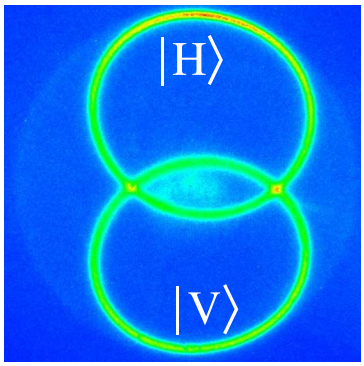

Figure 2.4: Emission pattern generated in the nonlinear optical process of spontaneous parametric down-conversion. Polarisation-entangled photon pairs are produced in the regions where the two rings cross. Here, obviously, many more than one pair of down-converted photons were created to make this image. The image image is available here: *http://www.rug.nl/natuurkunde/nieuws/colloquia/msc/colloquia/20060302lloyds*

robust against the decoherence form the environment. But they are not robust enough.

Noise in the transmission of photon qubits

Attenuation in fibres. So far the best bet for a realistic quantum communication is using optical fibres. The advantages are plenty e.g.: Small-cored fibers would support single-mode only transmission essential for a quantum channel, the fibre can be bent and it can be put almost anywhere – indeed much of the communcation infra-structure is already in place. The problem here, though, is foremost the loss of photons due to the fact that the photon *does* interact with the environment. The losses have been greatly reduced with the rise of the photo-communications technology but it cannot be eliminated, see Figure 2.5.

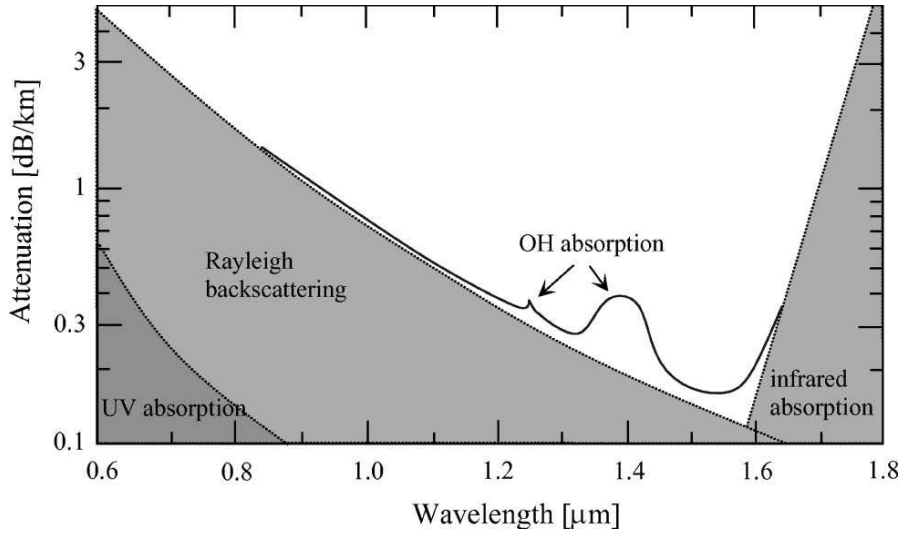

Figure 2.5: The so-called second and third telecommunications windows at 1.3 and 1.5 μm where the attenuation is minimal. For historical reasons the *first* telecommunications window is around 0.8-0.9 μm. From [Gisin *et al.*].

For example, at 1310-nm wavelength the attenuation is 0.35dB per km. This means that after approximately 10km half the photons would be lost. This is not a deal-breaker in classical communication, one just encodes the bit in many photons. Though, in quantum communication a single photon carries all the information of the qubit. This is potentially not a problem either: One looses photons, but this would just have as a consequence that the qubit rate would be slower. Actually, it is a problem since the rate drops exponentially with distance – on the other hand with a quantum repeater the rate only drops polynomially in the distance. The real problem is in the

detection of the photons.

Dark counts in single photon detectors. A single photon detector (SPD) would be indespensible to any of these implementations using photons as qubits. So far, any SPD has dark counts – meaning that it registers a photon even if there isn't any [1]. A Typical number is a dark count rate of 25 kHz for a 77K device [Gisin *et al.*]. Also the quantum efficiency of the SPD is still rather modest, up to maybe 50% in the telecommunication windows if you are lucky. Again, this low efficiency would just translate into a lower qubit rate. The big (bad) deal is that usually when one employs the SPD it is only open for a very short time exactly as to minimise the probability of a dark count. But in the case of the lost photons, we need to have the SPD open, since we do not know which photons are lost and which are transmitted. The result is that not only are the photons lost but any realistic SPD will introduce – via dark counts – false information for which there is no possible error correction.

Loss of determined polarisation. We will just briefly mention that there is another problem when using fibers for transmitting qubits related to the case when the qubits are encoded in the polarisation of the photon. Despite much progress in the fabrication of fibers they are not perfectly homogenous structures. There is always some level of birefringence in the fiber material which means that one different polarisation state would travel with different velocity than the orthogonal polarisation state. It would then destroy the qubit in its superposition of the two different polariations.

Free space transmission. In principle sending the photons through the air avoids the polarisation problem. But is very susceptible to changes in the weather, actually! Also one would need a free line of site between the transmission stations.

The facts in the couple of paragraphs above are some years old (it is from 2002) [Gisin *et al.*]. There have been improvements but the fundamental problems persist. The most spectacular display of improvements, though, has been the implementation of quantum cryptography in connection with the municipal elections in Geneva, 2007, done with a network of optical fibres. The record in free space transmission is now the 144 km transmission of a quantum key which makes a global distribution via satellites look realistic [Schmitt-Manderbach *et al.*].

The no-cloning problem/resource [Wootters *et al.*]. The most problematic of the problems listed above is the combination of the loss of photons and the dark counts in SPDs. A classical signal in which each single bit is made of many photons could just be amplified: You measure some of the photons and make plenty of copies and send them on. But what if we tried

[1] So does our eyes – or more precisely the individual rods that make up the retina. Therefore, the nerve from the eye to the brain has a threshold of a minimum of 4-5 rods that must fire before the nerve fires.

to copy the quantum state? Certainly it is not possible to measure the quantum state and re-send a reconstruction. A quantum copy machine should definitely be able to copy the basis states of the qubit subspace. That would look something like this:

$$|0, c, f\rangle \longrightarrow |0, 0, f_0\rangle \text{ and } |1, c, f\rangle \longrightarrow |1, 1, f_1\rangle \ ,$$

where the first entrance in the wave-function is the state that need to be copied, the second entrance is the system that is to carry the copy, and the third entrance is some final state of the quantum copy machine. Though, by the linearity of quantum mechanics, when we try to copy a quantum state - here a qubit - we get:

$$\begin{aligned}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|c, f\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0, 0, f_0\rangle + |1, 1, f_1\rangle) \\ \neq (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)|f_i\rangle \ .\end{aligned}$$

This final state of the quantum copy machine does *not* contain a copy of the initial quantum state to be copied. [Wootters *et al.*].

This is a problem for transmission, but as we saw it is a resource for cryptography. It is excactly the no-cloning theorem that prevents an eavesdropper from taking copies of the transmission and therefore betrays the eavesdropping.

2.2 The concept of the quantum repeater

The basic idea of the quantum repeater is to not send the quantum state through a noisy quantum channel. Instead, let sender and reciever share an entangled state and 'teleport' the quantum state between them using classical communication. A conceptual sketch is presented in Figure 2.6.

2.2.1 Quantum teleportation

The idea of quantum teleportation originates in [Bennett *et al.* (1993)]: Two separate systems A and B share a Bell state composed of two qubits say, $|\Phi^+\rangle$. System A wishes to quantum teleport the state $|\psi\rangle = \gamma_0|0\rangle + \gamma_1|1\rangle$ of a third qubit onto the qubit of B:

$$|\Psi_{system}\rangle = |\psi\rangle_Q|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(\gamma_0|0\rangle + \gamma_1|1\rangle)_Q(|00\rangle + |11\rangle)_{AB} \ .$$

This would be a way of representing that physical system. We can rewrite this expression by using the definition of the Bell states equations (2.1)-(2.4):

$$\begin{aligned}|\Psi_{system}\rangle = \frac{1}{2}\big(|\Phi^+\rangle_{QB} + |\Phi^-\rangle_{QB}\big)\gamma_0|0\rangle_B + \frac{1}{2}\big(|\Phi^+\rangle_{QB} - |\Phi^-\rangle_{QB}\big)\gamma_1|1\rangle_B \\ + \frac{1}{2}\big(|\Psi^+\rangle_{QB} + |\Psi^-\rangle_{QB}\big)\gamma_0|1\rangle_B + \frac{1}{2}\big(|\Psi^+\rangle_{QB} - |\Psi^-\rangle_{QB}\big)\gamma_1|0\rangle_B \ ,\end{aligned}$$

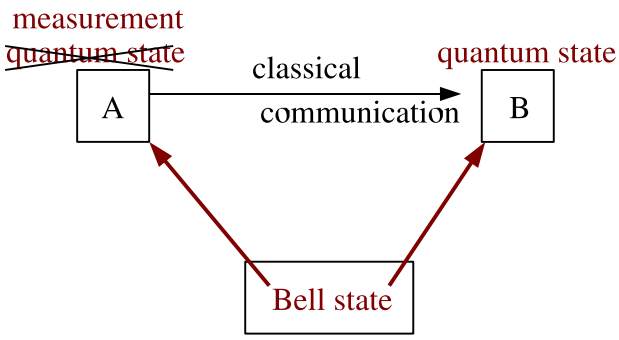

Figure 2.6: Illustration of the quantum repeater idea: A has a quantum state that A wishes to send to B. A and B share an entangled state. A does a measurement that collapses the quantum state to an eigenstate, but through classical communication the original quantum state is now at B's place.

and further manipulations result in:

$$|\Psi_{system}\rangle = \frac{1}{2}|\Phi^+\rangle_{QA}(\gamma_0|0\rangle + \gamma_1|1\rangle)_B + \frac{1}{2}|\Phi^-\rangle_{QA}(\gamma_0|0\rangle - \gamma_1|1\rangle)_B$$
$$+ \frac{1}{2}|\Psi^+\rangle_{QA}(\gamma_1|0\rangle + \gamma_0|1\rangle)_B + \frac{1}{2}|\Psi^-\rangle_{QA}(-\gamma_1|0\rangle + \gamma_0|1\rangle)_B \ .$$

Now A does a Bell state measurement, that is, determine which Bell state describes AQ. Then A transmits this information classically with two bits to B. There are only four, equally probable, outcomes. Then B knows exactly which quantum state is in the B-qubit. Finally, B can recover the original quantum state by suitable unitary rotations of the qubit:

$$\gamma_0|0\rangle - \gamma_1|1\rangle = \hat{\sigma}_3\gamma_0\big(|0\rangle + \gamma_1|1\rangle\big)$$
$$\gamma_1|0\rangle + \gamma_0|1\rangle = \hat{\sigma}_1\gamma_0\big(|0\rangle + \gamma_1|1\rangle\big)$$
$$\gamma_0|1\rangle - \gamma_1|0\rangle = \hat{\sigma}_2\gamma_0\big(|0\rangle + \gamma_1|1\rangle\big) \ .$$

Here we have introduced the Pauli spin matrices:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

This is the famous quantum teleportation protocol. It is very strange – one has the feeling 'that nothing really happened', system A just made a certain measurement. It is a good example of just how important the measurement process is in quantum mechanics.

Real world Bell state measurements. Up until now we have discussed using photons for qubits. Actually, the quantum teleportation protocal has

only been deterministically implemented in atomic systems. Using only linear optics[2] it is simply not possible to do a Bell state measurement on two photon qubits that distinguishes between all four of the Bell states eqn's (2.1)-(2.4) [Lütkenhaus *et al.*]. In [Furusawa *at al.*] they did light-to-light teleportation but relied on post selection to prove the successful teleportation. On the other hand, a few trapped ions have been the perfect vehicle for deterministic atom-to-atom quantum teleportation with very 'spooky' results [Riebe *et al.*] and [Barrett *et al.*]. A last successful protocol should be mentioned: [Sherson *et al.*]. Here the protocol is one of light-to-atoms teleportation. The conclusion is to use some form of atomic medium to do the Bell state analysis.

Entanglement swapping/connection

Let A and B share an entangled state - then they can transmit quantum information. The problem is that the quantum channel that distributes the entagled state is still noisy. The reach is thus limited as we saw earlier – maybe it is twice as large if the Bell-state machine is sitting in the middle of line between A and B. Again, we face the problem of loss of photons and loss of entanglement due to the non-determinstic transmission of the polarisation of the photons, both loss processes scale exponentially with distance. Likewise we are not able to amplify the quantum signal. But, there exists an engineous way to further distribute entanglement, called 'entanglement swapping' [Żukowski] or 'entanglement connection'. This allows us to distribute the entangled state in principle over as long a distance as is needed.

The idea is to entangle particles that do not even have a common past in the form of an interaction. The process is illustrated in Figure 2.7. A shares an entangled state with C while B and D share another entangled state. We next implement the quantum teleportation protocol on e.g. the CD and B systems. Then, by quantum teleportation the C qubit is transfered to B. Now, A and B share an entangled Bell state.

It is important to realize that the qubits at A and C must have been locally in contact. Entanglement creation requires an interaction - in the repeaters that we consider this interaction happens on a beam splitter. Then, after the entanglement creation each of the qubits can be transmitted a 'safe' distance, let's call the distance $l_0/2$. Also, the qubits at B and D must have interacted, and in the schemes we shall consider they are at the same location, let's call that a node. This process could be repeated with as many nodes as required to reach the desired transmission distance between A and B, as illustrated in Figure 2.8 for a distance $L = 4l_0$, where $l_0/2$

[2]Linear optics is basically the experimental practices which are best understood and therefore it is very important that a technology is based on operations that can be done with linear optics components such as beam splitters, photonic fibres etc.

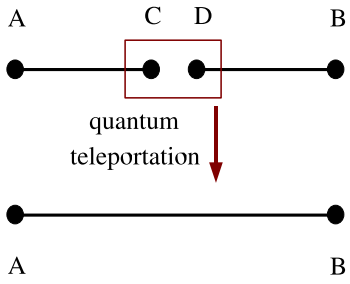

Figure 2.7: The entanglement connection - solid lines represent entanglement: A shares an entangled state with C while B and D share another entangled state. A Bell measurement is done on the two qubits of C and D with projects the qubits of A and B into an entangled state.

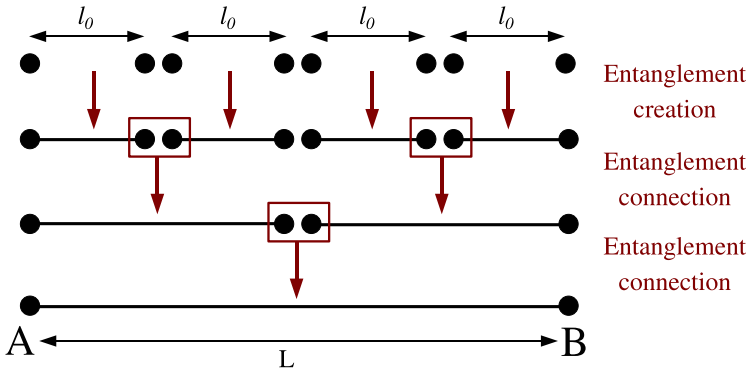

Figure 2.8: The quantum repeaters scheme that could work over a distance of $L = 4l_0$, where $l_0/2$ is a distance over which the loss in the transmission is small such that the of the quantum communication process is acceptable.

is a distance over which the loss in the transmission is small such that the fidelity of the quantum communication process is acceptable.

Fidelity. We shall return to this concept in some detail later cf. chapter 4. It is defined as $\mathcal{F} = \langle\psi|\rho|\psi\rangle$, where ρ is the density matrix of the evolved system and ψ, in our case, is the initial wave-function. Basically, it is a measure of how closely the transmitted qubit resembles the qubit before transmission.

2.2.2 Noise in the repeater scheme

In theory, multiple entanglement connections can overcome the limitations of the loss in the transmission of photon qubits. But, the problem of the noise problems still persist - only, it is no longer a problem that the photon is lost or changes it polarisation. Instead, for each entanglement creation and every connection the chance of error is present. This means that the entanglement procedure might fail and/or the entanglement connection might fail, and fidelity of the communication would drop to $\frac{1}{2}$ which is just complete lack of information. Also, the operations might succeed partially reducing the fidelity only slightly for each operation, but each transmission consists of many operations and even small reductions in fidelity would ruin the communication.

The different repeater schemes are basically different ways to have the noise from all the operations scale less quickly than the exponential loss due to attentation in photonic fibers.

Entanglement purification

[Briegel *et al.*] proposed to solve the problem of the noise in the entanglement creation and connection by employing a purification protocol. This means that after each connection any loss in fidelity would be restored to a better fidelity. This builds upon the fact that with a collection of low fidelity states it is possible to distill a one state of better fidelity - in the limit of infinite states the resulting distilled fidelity is 1 [Bennett *et al.* (1996)]. Already with a modest collection of states with moderate loss of fidelity it is possible to obtain a very good purified state with high fidelity. They further showed that the amount of operations and time the implementation required scaled slower than the exponential noise in transmission fibers.

This protocol requires, then, multiple entangled pairs spanning each l_0 distance. This is not impossible, but still very involved experimentally.

2.3 The DLCZ quantum repeater

There is another way to deal with noise in the quantum repeater scheme. This is the DLCZ repeater [Duan *et al.*], so-called from the authors' last

names. There are two new ideas in this article: First, there is the implementation of a probabilistic quantum repeater scheme, which promises to do away with the need to do a (so far) unrealistic purification step. Second, they gave a very precise recipe how to make the actual nodes work. They proposed to use atomic ensembles and then write, store, and read a single photon from these atoms. Third, the whole scheme can be implemented using only linear optics and, finally, [Duan *et al.*] showed that the time needed to establish entanglement grew polynomial with distance.

This article caused quite a stir since, to quote [Duan *et al.*], "The scheme involves laser manipulation of atomic ensembles, beam splitters, and single-photon detectors with moderate efficiencies, and is therefore compatible with current experimental technology." Everything was already in place in the laboratories around the world, the only thing missing is in principle easy – a reliable quantum memory – but here, approximately 10 years later, this is still the bottleneck.

2.3.1 Probabilistic implementation

The first fundamental breakthrough was the idea of using a probabilistic protocol. We have illustrated the idea in Figure 2.9. To entangle the qubits

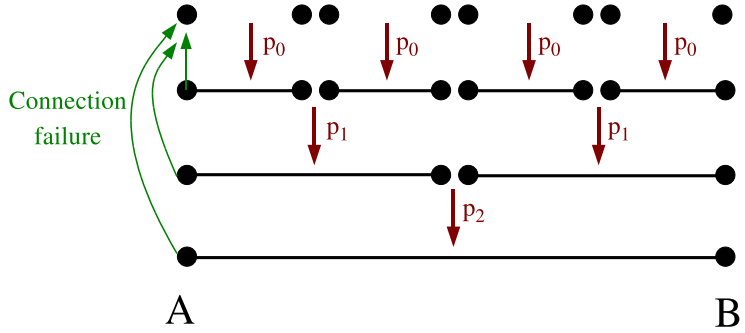

Figure 2.9: The probabilistic quantum repeaters scheme. The workings of the scheme is detailed in the text.

at A and B divide - as before - the transmission distance up into smaller distances that support quantum trasnmission with good fidelity. Then do the entanglement connection step for pairs of qubits - this happens with the probability p_0 for each pair. This time, do *not* do any purification to obtain high F after the connection. Instead, detect if the connection was not 'perfect' and start all over in the segment that did not succeed in entanglement connecting. The other segments that did succeed in the connection just wait – after some time all the nodes are entangled with a neighbour and the connection can proceed on the next level. Now, there

is a probability p_1 that the connection succeeds for each pair on this new level, but if it fails then one must restart all the way from the top on the whole segment involved. When entanglement connection is reached in all the nodes on this level one can proceed until finally the qubits at A and B are entangled.

So, how could this be realized and is this at all better than the exponential loss in optical fibers?

2.3.2 Using atomic ensembles

The second breakthrough was the interaction of light with the atomic ensemble. In [Duan *et al.*] they showed how it is possible to store a photon deterministically (conditioned on a click in a detector) and how this photon could be deterministically read out again. Additionally, this scheme has a build in entanglement purification effect. In the following we shall disregard noise in the description. Later we will see that noise introduces delays but not false information in establishing the quantum communication channel.

First, take the generic Λ structure of the energy levels of an idealized atom: There is the ground state $|g\rangle$, only one excited state $|e\rangle$, and then a different ground state $|s\rangle$ only accessible through the excited level. This system supports Raman transition from the level $|g\rangle$ to level $|s\rangle$. Next, take an esemble of these atoms, say N_a large, and have them interact with a classical field described by the Rabi frequency Ω, see Figure 2.10.

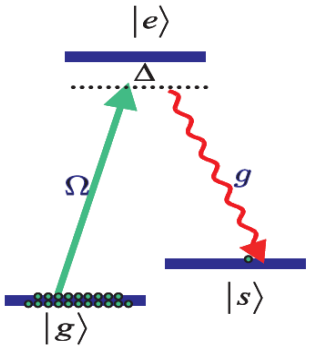

Figure 2.10: A generic Λ structure of the energy levels of an idealized atom. Ω is the Rabi frequency, g the coupling to the quantum field, and Δ the absolute detuning (to be explained in detail in chapter 3). Illustration taken from [Duan *et al.*].

Using an ensemble of atoms has two important consequences: One is that the interaction with light is $\sqrt{N_a}$ larger than with a single atom which makes it easier to achieve strong coupling. The other, very important consequence is the following: Retrieving the photon stored in the ensemble has

to happen completely deterministic: We must know in which ensemble, if any, the photon was stored in. Present technology limits the detection of the single photon to a photon emitted into the cavity mode and coupled out through the cavity. Any photon lost in the read-out because of emission into the many other modes would destroy the repeater scheme. The ensemble, if it is e.g. elongated in the direction of the longitudinal cavity mode, would collectively enhance the emission into the wished-for cavity mode propagating along the long axis of the cavity. A more detailed treatment of this aspect of the physical process is given in the reveiw [Sangouard *et. al*]

In the initial interaction to store the photon many atoms involved in the interaction and the different scattering amplitudes interfere making this enhancement less pronounced. This is different from the retrieval interaction where there is only one atom interacting with the light field. Fortunately, in the intial interaction lost photons are just lost and we can try again until we get the right mode and get a click in the detector. After the absorption of the single classical photon and the detection of the quantum photon the ensemble is left in a so-called 'symmetric Dicke state' generated by the operator:

$$\hat{S} = \frac{1}{\sqrt{N_a}} \sum_i |g\rangle_{ii}\langle s| \ .$$

The sum is over the atoms in the ensemble.

The write-in/entanglement process

To generate the entangleent between ensembles we take two elongated ensembles, call them L and R, and illuminate them with synchronous classical pusles. The pulses are sufficiently weak such that the probability of excitation, p_c, is low. After the interaction – *given that the emitted quantum photon is emitted into the forward scattered Stoke's mode which is coinciding with the longitudinal cavity mode* – each individual ensemble is left in the (un-normalised) state:

$$|\phi\rangle = |g\rangle_a|0\rangle_p + \sqrt{p_c}\hat{S}^\dagger\hat{a}^\dagger|g\rangle_a|0\rangle_p + \mathcal{O}(p_c) \ ,$$

where the subscripts a and p refers to the atoms and the single photon quantum field respectively.

We make sure that p_c is so small that chances of multiple excitations in each ensemble is negligible, in fact even the occurence of a simultaneous excitations in both ensembles can be disregarded. But once in a while there will be an exciation in either ensemble, we just do not know which. The quantum photon from this interaction would first be filtered from the classical photons and then mixed with vacuum from the other ensemble on a 50:50 beamsplitter (BS) and then afterwards the fields from the beamsplitter is collected in single photon detectors. The process is illustrated in Figure 2.11.

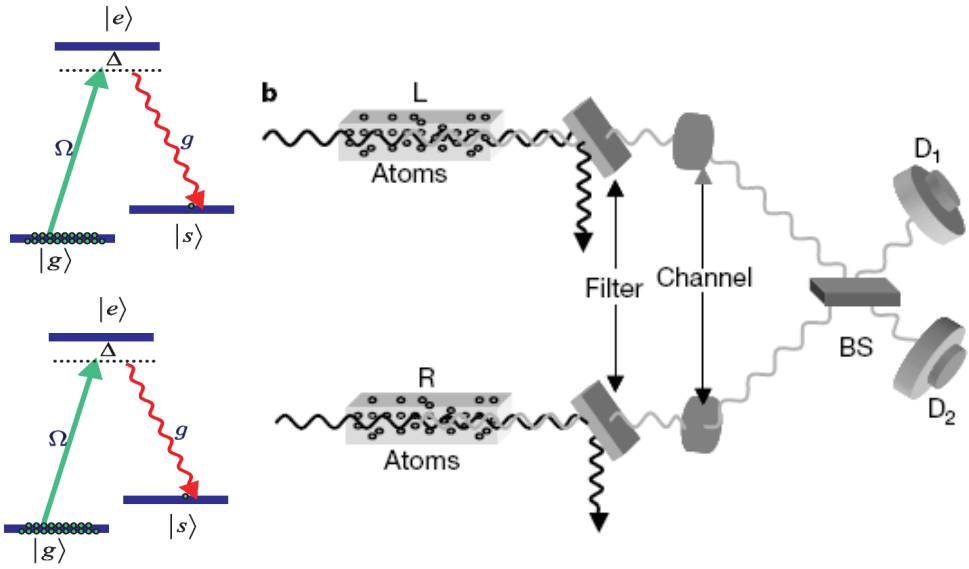

Figure 2.11: The quantum repeaters scheme to generate entanglement between two atomic ensembles. Adapted from [Duan *et al.*].

Now, given a click in one of the detectors with very high probability we have realized the state:

$$|\Psi\rangle_\phi = \frac{1}{\sqrt{2}}(\hat{S}_L^\dagger \pm \mathrm{e}^{i\phi}\hat{S}_R^\dagger)|g\rangle_{a,L}|g\rangle_{a,R} \ . \tag{2.5}$$

Here $\mathrm{e}^{i\phi}$ is the phase difference of the two paths and the sign is given beamsplitter transformation and depends on which detector clicks. This state is excactly what we are looking for - apart from a sign that we can compensate for since we know which detector clicked, and apart from a phase that can be cancelled - this is the Bell state from equation (2.1).

Storage

It is important to be clear about what quantum state it is that needs to be stored. It is not the qubit subject to the whole quantum communication scheme, it is rather the quantum coherence that ensures the effective retrieval of the excitation into a specific mode. We shall return to this matter later.

The goal is to establish entanglement between many ensembles, thus there will be some waiting time. For each pairs of ensembles the transmission of the synchronous pulses through the atoms must be done $1/p_c$ times. If each pulse cycle takes t_Δ then, on average, waiting for a click takes $T_0 = t_\Delta/p_c$. This could already be in the tenths of seconds range since we need to optically pump the ensembles after each trial, and p_c really needs to be very small or the whole repeater is prone to error.

Furthermore, on subsequent steps in the repeater scheme the waiting time increases since failure to entanglement connect forces a re-trial on a long part of the communication route. But it is substantial and requires a medium suitable for storage of a quantum state, a so-called quantum memory. The hope is that something like an atomic ensemble would work as quantum memory, and indeed impressive progress has been made [Julsgaard *et al.*]. Still, the succes reported herein is only miliseconds worth of quantum memory and clearly here is a major bottleneck that need to find a solution before quantum repeaters are possible.

The read-out/entanglement connection process

The final conceptual step in the quantum repeater is the entanglement connection. In the DLCZ protocol this is implemented via the read-out step. The stored photon in one of the ensembles is retrieved by shining a powerful field resonant with the quantum transition $|s\rangle \rightarrow |e\rangle$. Hereby the Raman transition is reversed with the emission of a photon resonant with the $|g\rangle \rightarrow |e\rangle$ transition. This photon it is extremely important to detect - fortunately, here the collective enhancement from the ensemble comes into play: Given that the photon read in into the ensemble came from the longitudinal cavity mode then the emitted photon in the reverse process will also be emitted into the same mode. This is further conditioned on the fact that *the ensemble of atoms is the same as during the write-in as during the read-out*. We shall return to this point later in some detail.

The process is illustrated in Figure 2.12. Here there is a doubling of the previous figures: Initially, the entanglement is established between the ensembles L and l_1 and between the ensembles R and l_2. Then the two read-

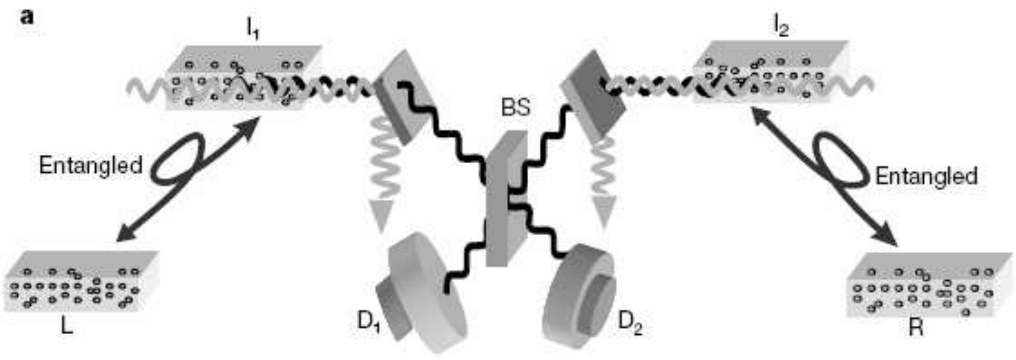

Figure 2.12: Entanglement connection between ensembles L and R via entanglement between L and l_1 and between R and l_2. From [Duan *et al.*].

out control pusles is shone onto the l_1 and l_2 ensembles respectively. Again,

the pulses should be simultaneous or rather the mixing of the two read-out beams should be simultaneoulsy as they impinge on the beam splitter.

Before this operation the four ensemble can be described as being in the state (with suitable unitary operations on the states from equation (2.5)):

$$\begin{aligned}|\Psi\rangle_{\text{system}} &= \frac{1}{2}\big(|g\rangle_L|s\rangle_{l_1} + |s\rangle_L|g\rangle_{l_1}\big)\big(|g\rangle_{l_2}|s\rangle_R + |s\rangle_{l_2}|g\rangle_R\big) \\ &= \frac{1}{2}\big(|gsgs\rangle + |gssg\rangle + |sggs\rangle + |sgsg\rangle\big)_{L,l_1,l_2,R}\,. \end{aligned} \tag{2.6}$$

Here the state $|g\rangle_L|s\rangle_{l_1}$ means $\hat{S}^\dagger_{l_1}|g,\cdots,g\rangle_L \otimes |g,\cdots,g\rangle_{l_1}$ etc.

Now, the state $|gssg\rangle_{L,l_1,l_2,R}$ corresponds to the detection of two photons at the read-out. Then, we know for sure that there is *not* a photon in either L or R. This counts as a futile attempt to entanglement connect and we have to start from the top in this segment of the transmission[3] Also, the state $|sggs\rangle_{L,l_1,l_2,R}$ means that there is no photon detection and then the knowledge is obtained that both the ensembles L and R contain a photon – again the connection fails and the process must be re-done. Conversely, the result of having the two states $|gsgs\rangle_{L,l_1,l_2,R}$ and $|sgsg\rangle_{L,l_1,l_2,R}$ means that one and only one photon is detected in the read-out. As before, we do not know from which ensemble it came. This is a successful entanglement connection since now after the read-out process the following system is achieved:

$$|\Psi\rangle_{\text{system}} = \frac{1}{\sqrt{2}}\big(|gs\rangle + |sg\rangle\big)_{L,R}\,. \tag{2.7}$$

The perfect Bell state! This entanglement could then be cascaded all the way through many pairs of entangled ensembles until A and B share a Bell state and are ready for quantum teleportation based quantum communication.

The forward scattered Stoke's mode. Now we can see why it is so important that the read-out photon is emitted into the longitudinal cavity mode and subsequently is detected with 'complete' certainty. Imagine that one of the stored photons was lost in the read-out into some mode that did not couple out though the cavity to be detected. That could potentially have the effect that the state $|gsgg\rangle_{L,l_1,l_2,R}$ is mistaken for the state $|gssg\rangle_{L,l_1,l_2,R}$. This in turn would mean that the we register the succes indicating single photon in the read-out process but the state of the ensembles L and R are

[3]Here is actually a serious problem: If there are two single photons and we mix them simultaneously on a 50:50 beam splitter then they will *both of them* go to the same detector [Gerry *et al.*], and SPDs have a hard time distinguishing two simultaneously incoming photons. We therefore need a photodetector with number resolution. These do not exist yet so a lot of proposals have been published on how to improve DLCZ specifically with regards this aspect of the scheme. We shall not write more about it here but a review is given in [Sangouard *et. al*]. The focus of the present work is on the physics of each particular atomic ensemble.

not an entangled state but $|gg\rangle_{LR}$. The quantum repeater would simply not work. One could later detect that there was no entanglement present in the communication channel and start over again from the very beginning, but this would make the time to establish the quantum channel forbiddingly long.

Built-in entanglement purification. Also, at this point we can see that the scheme has sort of a built-in entanglement purification. The primary source of noise is still the loss of photons in the fibers, but any loss of photons results in missing clicks in the detector. Therefore, given a click in the detector we have purified the entanglement. Unless off course a dark count is introduced. But we know exactly during which time interval to open the detector thus dark counts can be effectively minimized. Considering that the generation of the quantum photon via the Raman transition is very reliable, in effect we can disregard this problem.

Polynomial time of operations with increasing distance in the presence of noise

[Duan *et al.*] gives the formula for the communication time, the time to establish entanglement between A and B, as a function of distance:

$$T_{tot} \approx \frac{2(L/l_0)^2}{\eta_p p_a \Delta F_n \prod_{i=1}^{n} p_i} \ , \tag{2.8}$$

where η_p is the overall efficiency, ΔF_n is the overall fidelity imperfection, and p_i is the succes probability for the i'th entanglement connection. The expression in equation (2.8) is seen to grow only polynomially in L, the communication length.

Chapter 3

Light-atom interaction

3.1 The Hamiltonian of Quantum Optics

Our concern is the interaction of atoms with light. The interaction is a coupling of the electromagnetic field and the electrons bound to the atom - typically just the outermost and most weakly bound electron. The Hamiltonian that describes an electron in a central potential in the presence of an external field is [Gerry *et al.*] (p.75):

$$\hat{H}(\mathbf{r},t) = \frac{1}{2m}[\hat{\mathbf{P}} + e\mathbf{A}(\mathbf{r},t)]^2 - e\Phi(\mathbf{r},t) + V(r) \ . \tag{3.1}$$

Here the electron momentum is $\hat{\mathbf{P}}$ and the central potential $V(r)$ describe the isolated atom while the vector potential $\mathbf{A}$ and the scalar potential Φ describe the external field. Finally, e is the fundamental unit of charge, here taken to be a positive quantity, and m is the mass of the electron.

The above Hamiltonian is hard to use for calculations. Here follows the standard way of obtaining a (much) simpler Hamiltonian via clever *gauge transformations* of the fields in the interaction. The fields are

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= -\nabla\Phi(\mathbf{r},t) - \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} \ , \\ \mathbf{B}(\mathbf{r},t) &= \nabla \times \mathbf{A}(\mathbf{r},t) \ , \end{aligned} \tag{3.2}$$

and they are invariant under the gauge transformations

$$\begin{aligned} \Phi'(\mathbf{r},t) &= \Phi(\mathbf{r},t) - \frac{\partial\chi(\mathbf{r},t)}{\partial t} \ , \\ \mathbf{A}'(\mathbf{r},t) &= \mathbf{A}(\mathbf{r},t) + \nabla\chi(\mathbf{r},t) \ , \end{aligned} \tag{3.3}$$

which can be checked by pluggin them into (3.2). Note that χ is any scalar function which we are free to choose. We next make a unitary transformation

of the Schrödinger equation via the unitary operator $\hat{R} = \exp(-ie\chi(\mathbf{r},t)/\hbar)$. Then $\Psi'(\mathbf{r},t) = \hat{R}\Psi(\mathbf{r},t)$ and the transformed Schrödinger equation becomes

$$\hat{H}'\Psi'(\mathbf{r},t) = i\hbar\frac{\partial\Psi'(\mathbf{r},t)}{\partial t} \ , \tag{3.4}$$

where

$$\hat{H}' = \hat{R}\hat{H}\hat{R}^\dagger + i\hbar\frac{\partial\hat{R}}{\partial t}\hat{R}^\dagger \ . \tag{3.5}$$

With $\hat{\mathbf{P}} = -i\hbar\nabla$ the transformed Hamiltonian reads

$$\hat{H}'(\mathbf{r},t) = \frac{1}{2m}[\hat{\mathbf{P}} + e\mathbf{A}'(\mathbf{r},t)]^2 - e\Phi'(\mathbf{r},t) + V(r) \ . \tag{3.6}$$

The domain of interaction with light of optical frequencies or lower can be well described in the non-relativistic regime. Here the energy of a photon is much less than the kinetic energy of an electron moving with speeds above 10% of the speed of light. We can then choose the *Coulomb gauge* which is not relativistically invariant, but has the advantage that the radiation fields are completely described by the vector potential. In this gauge $\Phi = 0$ and the fields are transversal i.e. $\nabla\mathbf{A} = 0$ and the vector potential satisfies the homogenous wave equation. Then the Hamiltonian becomes

$$\hat{H}'(\mathbf{r},t) = \frac{1}{2m}[\hat{\mathbf{P}} + e(\mathbf{A}(\mathbf{r},t) + \nabla\chi)]^2 + e\frac{\partial\chi}{\partial t} + V(r) \ . \tag{3.7}$$

Since the vector potential satisfies the wave-equation the solutions take the form: $\mathbf{A} = \mathbf{A}_0 \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t) + c.c.$, with the length of the wave vector of the radiation given by $k = 2\pi/\lambda$. Furthermore, the fields that we shall consider are all of a wavelength much larger than the diameter of the atom. The atomic diameter is the range over which the single outer valens electron of the ionised alkali earth atom moves and this sets the length scale for when variations in the external field becomes important. This means that $\mathbf{k}\cdot\mathbf{r} \ll 1$ which allows for the *dipole approximation* to be made: $\mathbf{A}(\mathbf{r},t) \simeq \mathbf{A}(t)$, and we can take the vector potential to be spatially uniform across the atom.

The last piece of the puzzle is to choose the gauge function $\chi(\mathbf{r},t) = -\mathbf{A}(t)\cdot\mathbf{r}$. Then

$$\nabla\chi(\mathbf{r},t) = -\mathbf{A}(t) \quad \text{and} \quad \frac{\partial\chi}{\partial t}(\mathbf{r},t) = -\mathbf{r}\cdot\frac{\partial\mathbf{A}}{\partial t} = \mathbf{r}\cdot\mathbf{E(t)} \ .$$

Finally, the Hamiltonian then can be written as:

$$\hat{H}' = \frac{\hat{\mathbf{P}}}{2m} + V(r) + \mathbf{r}\cdot\mathbf{E}(t) = \hat{H}_0 - \hat{\mathbf{d}}\cdot\hat{\mathbf{E}}(t) \ , \tag{3.8}$$

where $\hat{H}_0$ is the Hamiltonian for the isolated atom for which we know the relevant eigenvalues and eigenfunctions of the time-independent Schrödinger

equation and where $-\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}$ models the interaction. The expression $\hat{\mathbf{d}} = -e\mathbf{r}$ is the dipole operator of the atom while $\hat{\mathbf{E}}$ is the operator of the EM-field.

This Hamiltonian can be used both for semiclassical description and for the interaction of an atom with a quantized field. The simple structure of the interaction part of the Hamiltonian makes it easy to see how the quantum system can be manipulated e.g. to transfer an electron from one eigenstate of H_0 to another eigenstate.

3.1.1 The Hamiltonian of the three level Λ structure

For the quantum repeater we need a three-level system as explained before. The basic interaction with light that is needed is a transfer of one atom from the ground state to a metastable state with the simultaneous emission of a photon distinguishable from the driving field photons. An important concept for our purposes is *adiabatic population transfer*. Essentially, it is the following: Start with an atom in a specified quantum state. Next, expose the atom to a controlled sequence of pulses of radiation that forces the atom into a specific quantum target state. The adiabatic part of this population transfer is the fact that the system stays in an eigenstate of the Hamiltonian of the atoms and the light. This makes the system reasonably robust against perturbations. Also, if things are done 'right' the passage is done via a 'dark state' that never populates the excited level thus minimizing the loss in the process due to spontaneous emission from the excited level.

Such a scheme can appropriately be implemented in a three-level approximation to the energy level structure of an atom in the lambda-configuration, as shown in Figure 3.1. The stationary energy level $|g\rangle$ is the stable ground state that is the initial internal state of the atom, the $|s\rangle$-level is the metastable target state and $|e\rangle$ is an unstable intermediate excited level. We need the population transfer to 'travel' via the excited level since the transition $|g\rangle \rightarrow |s\rangle$ is forbidden. It is the fact that the transition is forbidden which makes it possible that the atom will stay in the target state. On the other hand the excited level is prone to spontaneous emission.

To go further we expand $\hat{H}$ from equation (3.8) in the three basis states $|g\rangle$, $|e\rangle$ and $|s\rangle$ to obtain the matrix elements of the Hamiltonian. There are nine matrix elements, three describe the energy of the unperturbed atom: $\langle g|\hat{H}|g\rangle = \hbar\omega_g$, $\langle e|\hat{H}|e\rangle = \hbar\omega_e$, and $\langle s|\hat{H}|s\rangle = \hbar\omega_s$. Two elements are zero since these transitions are dipole 'forbidden': $\langle g|\hat{H}|s\rangle$ and its complex conjugate. The interaction of the classical field and the atom is assumed to drive the $|g\rangle \rightarrow |e\rangle$ transition and is described by $\langle e|\hat{H}|g\rangle = \hbar\Omega$, where Ω is the Rabi frequency which is an angular frequency (we will define an explicitly time varying Rabi frequency below). The interaction of the quantum field and the atom is assumed to drive the $|s\rangle \rightarrow |e\rangle$ transition and is described by $\langle e|\hat{H}|s\rangle = \hbar g$, where g is the magnitude of the coupling $\mathbf{E} \cdot \mathbf{d}_{eg}$.

The difference between classical and quantum field lies in the way that

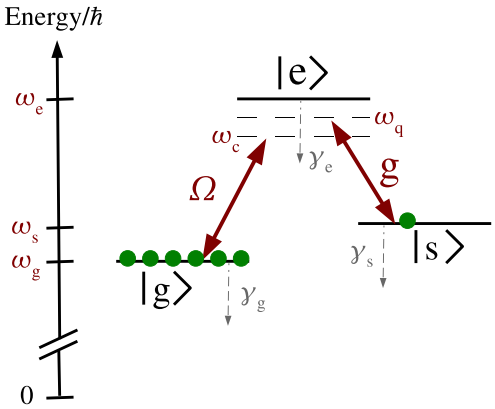

Figure 3.1: Λ-scheme of a three-level atom for the implementation of adiabatic population transfer. The decay from the ground state is represented just for completeness - usually we shall work only with systems where γ_g can be completely neglected

each quantum photon counts while the classical photons come from an inextinguishable source (a laser). The interacting fields are here drawn with a detuning from the resonance that would drive the actual transition. As we shall see it is by detuning the fields that we can model the process as if the atom is never in the excited state but nevertheless makes the 'forbidden' transition to $|s\rangle$. The γ's are the rates of spontaneous emission from the stationary levels and we have $\gamma_e \gg \gamma_g, \gamma_s$.

To differentiate the two fields such that they are active on different transitions and not the same - we have drawn the two meta-stable levels with different energies. Another way of differentiating is to use two meta-stable levels that have different polarisations channel up to the excited level. This has been the original idea of the experimenters [Mortensen] - we shall later comment on which strategy will be the most appropriate.

Most generally, the dynamics of the system is given by (3.8). If we ignore the decay - it can be dealt with when computing the equations of motion - and take the layout of the energy levels from Figure 3.1 we have:

$$\hat{H} = \hat{H}_{\text{field}} + \hat{H}_{\text{atom}} + \hat{H}_{\text{interaction}} \ ,$$

where 'field' refers to the quantum field. In second quantisation the field is described by operators and is proportional to a combination of the creation and annihilation operators: $\hat{E} \propto \hat{a} + \hat{a}^\dagger$ [Gerry *et al.*]. Furthermore, it is convenient to introduce the *atomic transition operators*: $\hat{\sigma}_{ij} = |i\rangle\langle j|$ which has the effect of taking an atom form the j'th energy level to the i'th. If the transition between the two stable levels is indeed forbidden the dipole operator on the transition interacting with the classical field has the following

structure:

$$\hat{d}_{\rm ge} \propto \hat{\sigma}_{ge} + \hat{\sigma}_{eg} \ , \tag{3.9}$$

while the same operator for the transition interacting with the quantum field looks like:

$$\hat{d}_{\rm se} \propto \hat{\sigma}_{se} + \hat{\sigma}_{es} \ . \tag{3.10}$$

Now the structure of the part of the interaction Hamiltonian describing the quantum field interaction becomes:

$$H^{(q)}_{\rm interaction} \propto (\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_{se} + \hat{\sigma}_{es}) \ , \tag{3.11}$$

where two terms represent processes that are unphysical: $\hat{a}\hat{\sigma}_{se}$ represents absorption of a quantum photon simultaneous with a transition from a higher energy state to a lower and $\hat{a}^\dagger\hat{\sigma}_{es}$ represents an emission of a quantum photon while the atom transitions to a higher energy state. We shall neglect these terms, this is called the *rotating wave approximation* (RWA) [Gerry *et al.*]. We do the RWA in the same fashion with respect to the interaction with the classical field:

$$H^{(c)}_{\rm interaction} \propto \cos[\omega_c(t - z/c)](\hat{\sigma}_{se} + \hat{\sigma}_{es}) \simeq \frac{1}{2}(\mathrm{e}^{-i\omega_c(t-z/c)}\hat{\sigma}_{eg} + \mathrm{H.c}) \ . \tag{3.12}$$

After these manipulations we are left with the following Hamiltonian for the three level atom interacting with the two fields:

$$\hat{H} = \sum_n^{modes} \hbar\omega_q \hat{a}^\dagger_{q,n}\hat{a}_{q,n} \tag{3.13}$$

$$+ \hbar\omega_g\hat{\sigma}_{gg} + \hbar\omega_s\hat{\sigma}_{ss} + \hbar\omega_e\hat{\sigma}_{ee} \tag{3.14}$$

$$- \left(\hat{\mathbf{E}}_q(\mathbf{r}) \cdot \mathbf{d}_{es}\mathrm{e}^{-i\omega_q t}\hat{\sigma}_{es} + \mathrm{H.c}\right) \tag{3.15}$$

$$- \left(\hbar\frac{\Omega(t)}{2}\mathrm{e}^{-i\omega_c t}\hat{\sigma}_{eg} + \mathrm{H.c}\right) \quad . \tag{3.16}$$

The term in (3.13) is the energy in the quantum field, while the terms in (3.14) describe the energy of the internal state of the atoms. With regards to the interaction (3.15) models the exchange of a quantum of excitation between the quantum field and the atom with the resulting transition of the atom. This interaction is dependent on the vectorial nature of light and the dipole moment of that particular quantum channel in the atom. For the relevant interaction the actual spatial distribution of the quantum field is also important, especially since the crystal is kept in an optical cavity, and we define for the spatial part (after RWA):

$$\hat{\mathbf{E}}_q(\mathbf{r}) \equiv \sqrt{\frac{2\hbar\omega_q}{\epsilon_0 V}} \sum_n \hat{a}_{q,n} u_n(\mathbf{r}_\perp)\sin(k_n z)\mathbf{e}_{q,n} \ , \tag{3.17}$$

where V is the cavity volume, z is the propagation axis of the light and where u_n is a mode function that describes the field in its extention transverse to the z-axis. Finally, $\mathbf{e}_{q,n}$ is the unit vector of the polarisation. Note that we assume a spatially oscillating part along the propagation or cavity axis – this is the standing wave due to the cavity, as we shall investigate later. Already here we shall do the following approximation:

$$u_n(\mathbf{r}_\perp) = u_0(\mathbf{r}_\perp) \propto \mathrm{e}^{i\omega_c z/c} \exp\left\{\frac{-2(x^2+y^2)}{w^2(z)}\right\} , \tag{3.18}$$

where $w(z)$ is the spotsize of the Gaussian mode of the cavity. That is, only the TEM_{00} mode of the cavity is excited by the quantum field, see e.g. [Milonni *et al.*].

Finally, (3.16) is the interaction of the classical-like field addressing the $|g\rangle \rightarrow |e\rangle$ transition. Here the field is so strong and comprising so many photons that we can model the interaction with the scalar values of the coupling. Furthermore, we assume the strong field limit where Rabi oscillations are introduced in the system by the driving field with the frequency:

$$\Omega(t) = \Omega(\mathbf{r}, t) = |\mathbf{d}_{ge}| \frac{E_0(t)}{\hbar} u(\mathbf{r}) . \tag{3.19}$$

Here $E_0(t)$ is the amplitude of the classical-like field that can be varied and $u(\mathbf{r})$ is the spatial mode function which is identical to the mode function for the quantum field. The Rabi model is explained in e.g. [Gerry *et al.*], both for the classical field as we model it and for a few photon field.

3.1.2 A Dark State of the interaction

First, we find a suitable representation of the Hamiltonian in equations (3.13)-(3.16) that also takes into account the quantum photon. We can use the basis of the uncoupled atomic states $\mathfrak{B} = \{|g,-\rangle, |e,-\rangle, |s,+\rangle\}$, where the $|+\rangle$ indicates a q-photon in the cavity and $|-\rangle$ the absence. If we neglecgt the energy in the quantum field and look only at the interaction, then we have for each atom

$$\hat{H} = -\frac{\hbar}{2} \begin{bmatrix} -2\omega_g & \Omega(t)\mathrm{e}^{-i\omega_c t} & 0 \\ \Omega^*(t)\mathrm{e}^{i\omega_c t} & -2\omega_e & 2g^*(t)\mathrm{e}^{i\omega_q t} \\ 0 & 2g\mathrm{e}^{-i\omega_q t} & -2\omega_s \end{bmatrix} .$$

To account for the energy of the photons in the system and to get rid of the oscillations we do the following: First, we again observe that the Schrödinger equation with a certain Hamiltonian

$$i\hbar \frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle , \tag{3.20}$$

can be written as

$$i\hbar\frac{\partial}{\partial t}|\psi'\rangle = \hat{H}'|\psi'\rangle \ , \tag{3.21}$$

where $\hat{U}$ is a unitary transformation such that $\hat{U}|\psi\rangle = |\psi'\rangle$ and

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^\dagger + i\hbar\frac{\partial\hat{U}}{\partial t}\hat{U}^\dagger \ . \tag{3.22}$$

If we use the following $\hat{U}$

$$\hat{U} = \begin{bmatrix} \mathrm{e}^{-i\omega_c t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-i\omega_q t} \end{bmatrix} \quad \text{with} \quad i\hbar\frac{\partial\hat{U}}{\partial t}\hat{U}^\dagger = \begin{bmatrix} \hbar\omega_c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \hbar\omega_q \end{bmatrix} \ ,$$

then we get that

$$\hat{H}' = -\frac{\hbar}{2}\begin{bmatrix} -2(\omega_g+\omega_c) & \Omega(t) & 0 \\ \Omega^*(t) & -2\omega_e & 2g^* \\ 0 & 2g & -2(\omega_s+\omega_q) \end{bmatrix} \ .$$

This is then the RWA Hamiltonian in a doubly rotating frame. Such a Hamiltonian nicely displays energy conservation and all the terms are constants or slowly varying amplitudes of the fields.

Next, we take as the reference energy $\hbar(\omega_g+\omega_c) = 0$. Then the Hamiltonian reads

$$\hat{H}' = -\frac{\hbar}{2}\begin{bmatrix} 0 & \Omega(t) & 0 \\ \Omega^*(t) & -2\Delta_q & 2g^* \\ 0 & 2g & -2(\Delta_c-\Delta_q) \end{bmatrix} \ , \tag{3.23}$$

where $\Delta_c = \omega_e - \omega_g - \omega_c$ is the detuning of the classical field from the atomic resonance and where $\Delta_q = \omega_e - \omega_s - \omega_q$ is the detuning of the quantum field from the other atomic resonance.

The aim is to find an eigenstate of the Hamiltonian which does not have a contribution from the excited state, a so-called *dark state*. By diagonalizing (3.23) it becomes apparent that this is possible if two-photon detuning is zero $\Delta_c - \Delta_q = 0$. This is called the two-photon resonance condition. Now the Hamiltonian reduces to

$$\hat{H}' = -\frac{\hbar}{2}\begin{bmatrix} 0 & \Omega(t) & 0 \\ \Omega(t)^* & -2\Delta & 2g^* \\ 0 & 2g & 0 \end{bmatrix} \ , \tag{3.24}$$

with $\Delta_q = \Delta_c = \Delta$. We can take the couplings to be real without the loss of too much generality and we proceed to find the eigenvalues:

$$\lambda_0 = 0 \tag{3.25}$$

$$\lambda_\pm = -\frac{\hbar}{2}\left(\Delta \pm \sqrt{\Delta^2 + 4g^2 + \Omega(t)^2}\right) \ , \tag{3.26}$$

corresponding to the eigenvectors that can be expressed as

$$|\psi_0\rangle = \Omega(t)|g,-\rangle - 2g|s,+\rangle \tag{3.27}$$

$$|\psi_+\rangle = 2g|g,-\rangle - \frac{4g^2+\Omega(t)^2}{\sqrt{\Delta^2+4g^2+\Omega(t)^2}-\Delta}|e,-\rangle + \Omega(t)|s,+\rangle \tag{3.28}$$

$$|\psi_-\rangle = 2g|g,-\rangle + \frac{4g^2+\Omega(t)^2}{\sqrt{\Delta^2+4g^2+\Omega(t)^2}+\Delta}|e,-\rangle + \Omega(t)|s,+\rangle \ . \tag{3.29}$$

This can be rewritten such that we have coefficients between zero and one, the so-called *mixing angles*:

$$\cos\alpha = \frac{2g}{\sqrt{4g^2+\Omega(t)^2}} \quad , \quad \tan\alpha = \frac{\Omega(t)}{2g} \quad \text{and} \quad \tan\beta = \sqrt{-\frac{\lambda_-}{\lambda_+}} \ . \tag{3.30}$$

These definitions are adapted from [Møller *et al.*]. This results in

$$|\psi_0\rangle = \cos\alpha|g,+\rangle - \sin\alpha|s,-\rangle \tag{3.31}$$

$$|\psi_+\rangle = \sin\alpha\sin\beta|g,+\rangle + \cos\beta|e,-\rangle - \cos\alpha\sin\beta|s,-\rangle \tag{3.32}$$

$$|\psi_-\rangle = \sin\alpha\cos\beta|g,+\rangle - \sin\beta|e,-\rangle + \cos\alpha\cos\beta|s,-\rangle \ . \tag{3.33}$$

These are the so-called dressed atomic states to differentiate them with the *bare* atomic states without the interaction with light. For a treatment of the dressed states in the two-level atom see [Gerry *et al.*] (Chap. 4). Now it is clear that one of these states, (3.31), is a 'dark state', i.e. it will not emit light since it does not contain the light emitting excited state. The result is that we can keep the system in this eigenstate all during the interaction without adding or subtracting energy and without having the atom in the excited state at any time. In principle at least - as can be seen from (3.31) - this would imply that the quantum field could be turned on before the interaction. This, however, is not relevant for the quantum repeater scheme since the quantum field here should consist strictly of the one photon coming from the atomic excitation.

STIRAP. A final remark concerns the STIRAP scheme (Stimulated Raman Adiabatic Passage) – a review of the subject is given in [Vitanov *et al.*]. In this process the field on the $|s\rangle \rightarrow |e\rangle$ is controlled by a time-dependent many-photon field just like the field on the $|g\rangle \rightarrow |e\rangle$ is. The dark state then looks like

$$|\psi_0\rangle = \Omega(t)|g,+\rangle - g_{classical}(t)|s,-\rangle \ . \tag{3.34}$$

From inspection we see that we should begin by applying the field on the transition $|s\rangle \rightarrow |e\rangle$ even if $|s\rangle$ is completely unpopulated. Then adiabatically turn the $g_{classical}$ field down while the classical field is turned on. Then the atom is transferred from $|g\rangle$ into $|s\rangle$ without ever populating the excited

state. Interestingly, the Aarhus group has already demonstrated that it is feasible to implement a STIRAP process in these ions, albeit they used pulses of many photons on the quantum channel, they only had one or a few ions, and they did not work inside a cavity, see [Sørensen *et al.*] and [Møller *et al.*]. Conditions for adiabaticity are also given in [Møller *et al.*], we require that the rate of change the wave-function is small compared to the energy separation between the dressed state eigenvalues:

$$\frac{d}{dt}|\psi_0\rangle \ll |\lambda_\pm/\hbar| \ . \tag{3.35}$$

3.1.3 Adiabatic elimination of the excited level

It is not given that we perform the STIRAP process on the dark state of the Hamiltonian, so we seek another solution to transfer the atom from $|g\rangle$ to $|s\rangle$ without losing the excitation to spontaneous emission from the excited level. Consider the eigenstate of the Hamiltonian

$$|\psi_-\rangle = 2g|g,-\rangle + \frac{4g^2+\Omega(t)^2}{\sqrt{\Delta^2+4g^2+\Omega(t)^2}+\Delta}|e,-\rangle + \Omega(t)|s,+\rangle \ . \tag{3.36}$$

If the detuning dominates the coupling strengths of the interactions g and $\Omega(t)$ then the contribution of the $|e,-\rangle$ basis state becomes negligible in the eigenstate of the Hamiltonian and we can populate the $|s\rangle$ level by just slowly turning on the classical field.

We should also mention that adiabatic population transfer has already been experimentally implemented with the interaction of a single photon with single atoms in a cavity, see [Kuhn *et al.* 1] for the theory and numerical simulations, and [Kuhn *et al.* 2] for the experiment.

3.2 An ensemble of atoms

We are going to explore the interaction of light with the ion Coulomb crystals mentioned in the introduction and described in more detail later. Instead of one atom interacting with the fields we have up to a couple of thousand of atoms. On the other hand the ions are far apart due to the Coulomb repulsion and a typical interparticle distance is measured in tenths of microns. This means that there will be no overlap of the wavefunctions like in a BEC (well, that and the fact that the ions are fermions). There will be no emission near-field inter-ionic perturbation like in superradiance. Finally, there will be no dipole-dipole interaction. In short we will treat the ions as if they were isolated particles. This is call this the *ensemble model*.

The particles might interact via phonons but even these we will neglect since the confinement is very weak compared to the solid state. The dominant motion of each ion are then the thermal vibrations of the ion as if it

was isolated. Each ion is basically caught in a static potential given by the neighbouring ions (and for the surface ions also given be the applied external potential). Here the ion will wiggle about the minimum energy point of the potential due to thermal motion.

3.2.1 The ensemble Hamiltonian

Given the considerations above we propose to write the Hamiltonian for an ensemble of ions as a simple sum over single particle Hamiltonians in the effective rotating frame:

$$\hat{H} = \hbar\Delta \sum_{j=1}^{N_A} \hat{\sigma}_{ee} - \hbar\Big(\sum_{j=1}^{N_A} \Omega^{(j)}(t)\hat{\sigma}_{eg}^{(j)} + \sum_{j=1}^{N_A} g^{(j)}(t)\hat{\sigma}_{es}^{(j)} \; + \; H.c. \Big) \; . \qquad (3.37)$$

Here we have used equations (3.13-3.16) and the manipulations leading to (3.24). Note that $\Omega^{(j)}(t)$ is dependent on the position of the $j'th$ ion and so is $g^{(j)}$.This Hamiltonian will be our model for the write-in interaction and by interchanging the operators $\hat{\sigma}_{es}^{(j)}$ and $\hat{\sigma}_{eg}^{(j)}$ the same Hamiltonian can be used for the read-out process. From here we will derive the equations of motion in later chapters. In this chapter we will give a simplified model also found in [Sangouard *et. al*]: The interaction is basically creating an atomic excitation together with an excitation of the quantum field which can decribed by a Hamiltonian with the following structure:

$$\hat{H} = \chi(\hat{a}^\dagger \hat{s}^\dagger + \hat{a}\hat{s}) \; , \qquad (3.38)$$

where $\hat{a}^\dagger$ is a single excitation of the quantum field, $\hat{s}^\dagger$ is an atomic excitation in *the ensemble of ions*, and χ is given by the power of the laser relevant for the interaction, the number of ions, and the transition strengths.

Since we are dealing with an ensemble of particles interacting with a classical-like pulse of light comprising many photons we should take into account the fact that we could have more than one excitation in the ensemble. Now, the Hamiltonian in (3.38) describes the creation or annihilation of pairs of bosonic excitations. It is also the Hamiltonian describing the process of non-degenerate parametric down-conversion [Hammerer *et al.*] in which an energetic photon is converted into two photons with less energy. In general these two photons will be entangled. Also, this Hamiltonian is the model of the process known as two-mode squeezing [Gerry *et al.*]. The generated photons from this process have been thoroughly studied, see e.g. [Sangouard *et. al*] and the references therein. The process of parametric down-conversion is well described by evolving the initial vacuum state of

the modes a and s as follows:

$$\begin{aligned} \mathrm{e}^{-iHt}|0\rangle_a|0\rangle_s &= \frac{1}{\cosh(\chi t)}\mathrm{e}^{-i\tanh(\chi t)\hat{a}^\dagger\hat{s}^\dagger}|0\rangle_a|0\rangle_s \\ &= \frac{1}{\cosh(\chi t)}\sum_{m=0}^{\infty}(-i)^m\tanh^m(\chi t)|m\rangle_a|m\rangle_s \ , \end{aligned} \tag{3.39}$$

using the simplified Hamiltonian in (3.38). If χt is a small number - and we do have control over the physical parameters - we can expand (3.39) to get:

$$\begin{aligned} \mathrm{e}^{-iHt}|0\rangle_a|0\rangle_s = &\left(1-\frac{1}{2}(\chi t)^2\right)|0\rangle_a|0\rangle_s - i\chi t|1\rangle_a|1\rangle_s \\ &- (\chi t)^2|2\rangle_a|2\rangle_s + \mathcal{O}\big((\chi t)^3\big) \ . \end{aligned} \tag{3.40}$$

This means that there is a probability $(\chi t)^2$ of emitting one photon and creating one atomic excitation. Furthermore, there is a probability $(\chi t)^4$ for emitting two photons and two atomic excitations and so on. This puts a natural limit on the efficiency of the process since we cannot just pump photons into the interaction. We need that $(\chi t)^2 \ll 1$ to have at most one excitation. We shall not go into more details about this particular challenge concerning the quantum repeater, but it should be kept in mind for applications.

3.2.2 Dicke-states in ensembles of atoms

Stokes and Anti-Stokes photons. We just remind the reader of the two processes relevant for the quantum repeater that must take place in each individual ensembles, see Figure 3.2. The definition of the Stokes photon

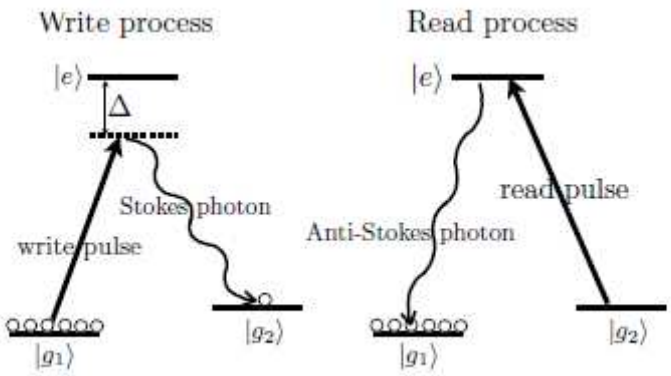

Figure 3.2: Simplified illustration of the write-in and the read-out. Taken from [Sangouard *et. al*] where $|g_1\rangle = |g\rangle$ in our text and $|g_2\rangle = |s\rangle$.

comes from the history of spectroscopy: If an atom absorbs a photon and afterwards emit a photon of less energy the less energetic photon is called a Stokes photon. Conversely, an Anti-Stokes photon is an emitted photon with more energy than the absorbed photon. If the two stable levels are degenerate the nomenclature can be carried over and the Stokes photon is the emission from a transition out of the ground state while the Anti-Stokes photon is the emission from the transition back into the ground state.

How should the single excitation in the ensemble of ions be described? Let us imagine that the light-matter interaction has resulted in one photon being absorbed from the classical-like field and one photon has been emitted into the quantum field. What is not known is in which of the ions the excitation is. This lack of information is enough to project the whole ensemble into a superposition of all of the possible realisations of a single excitation in the ensemble:

$$|\psi_{\text{write}}\rangle = \frac{1}{\sqrt{N_A}} \sum_{j=1}^{N_A} c_j \mathrm{e}^{i(\mathbf{k}_c - \mathbf{k}_q)\cdot \mathbf{r}_j} |g_1, ..., s_j, ..., g_{N_A}\rangle \ . \tag{3.41}$$

This is the state of the ensemble in the moment the interaction has taken place. Here c_j is a prefactor given by the position of the $j'th$ ion since the shape of the laser beam in the cavity is not homogeneous and most importantly, there is a standing wave along the cavity axis.

Dicke states: the state in (3.41) is a modified so-called Dicke state. A Dicke state is defined [Hume *et al.*] as the equal superposition of all basis states of N qubits having m excitations:

$$\left| D_N^{(m)} \right\rangle = \binom{N}{m}^{-\frac{1}{2}} \sum_k P_k \Big(|\downarrow^{\otimes(N-m)} \uparrow^{\otimes m}\rangle \Big) \ , \tag{3.42}$$

where the arrows designate the two possible states of the qubits and where the sum is over the Binom(N,m) permutations that P_k, the permutation operator, produces. These state were introduced by Dicke [Dicke] to study superradiance. The symmetric state with $m = 1$

$$\left| D_{N_A}^{(1)} \right\rangle = |\psi_{\text{W}}\rangle = \frac{1}{\sqrt{N_A}} \sum_{j=1}^{N_A} |g_1, ..., s_j, ..., g_{N_A}\rangle \ , \tag{3.43}$$

is called a W state but we shall also refer to it in the following as just a Dicke state.

The state in (3.43) is a quantum superposition state - it is *coherent* meaning that we can expect quantum interference phenomena in a sense that will be described below in 3.2.3. This quantum state should *not* be confused with the quantum state that is the object of the quantum communication, the qubit.

It is actually quite peculiar: Each ion is a quantum object but one would intuitively expect that one ion - and just one ion - picks up the excitation. But this seems not to be so: Theory and experiments have over the last ten years shown that the coherent superposition is a good description of what happens, e.g. in [Hald *et al.*] or [Simon *et al.*]. Also, the ion Coulomb crystals are very large objects, just look at Figure 3.3. Off course the fluores-

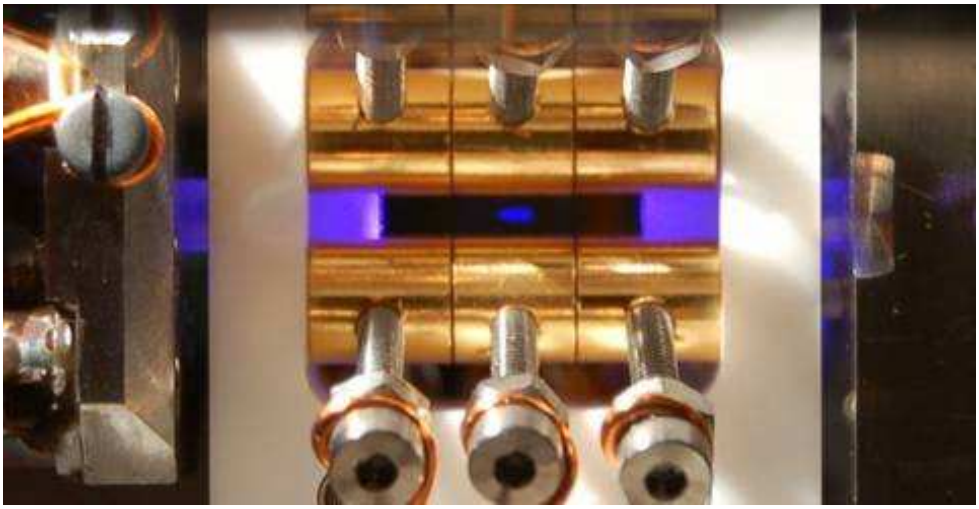

Figure 3.3: A macroscopic quantum object visible to the naked eye: An ion Coulomb crystal fluoresces in the visible spectrum. It is sitting between two mirrors 11mm apart that also scatter the light. This ordinary photograph is only very slightly enhanced to make it suitable for printing on paper - otherwise it is as found on *http://phys.au.dk/forskning/amo/cooling-and-trapping-of-ions/cqedwithioncoulombcrystals/*.

cence in this photograph constitutes a measurement such that the quantum state has collapsed and any quantum information lost.

A spin-wave quantum

This is also known as a *magnon*. The particular excitation we have described in the ensemble of three-level systems can be considered as a quantized spin wave. After the adiabatic elimination of the excited level and in the approximation that we never see two excitations the interaction can be described by the operators

$$\sum_j \hat{\sigma}_{sg}^{(j)} = \hat{J}_+ \ ,$$

and

$$\sum_j \hat{\sigma}_{gs}^{(j)} = \hat{J}_- \ ,$$

that obey the usual operator algebra for spin half angular momentum in quantum mechanics [Hammerer *et al.*].

3.2.3 Collective enhancement effects

The main reason to consider using atomic ensembles for the quantum interface is the enhancement of the interaction. The enhancement is different in the two interaction processes relevant for the quantum repeater.

Write-in

A simple collective enhancement effect is the fact that there are many ions that can interact with light. The probability amplitude for interaction is enhanced by a factor of $\sqrt{N_A}$, where N_A is the number of atoms/ions in the ensemble. This factor arises from the matrix element relevant for the interaction

$$\begin{aligned}\langle\psi_{\mathrm{W}}|\hat{J}_+|g,...,g\rangle &= \frac{1}{\sqrt{N_A}}\sum_{j=1}^{N_A}\langle g_1,...,s_j,...,g_{N_A}|\sum_{l}^{N_A}\hat{\sigma}_{sg}^{(l)}|g,...,g\rangle \\ &= \frac{1}{\sqrt{N_A}}\sum_{j,l=1}^{N_A}\delta_{lk} = \sqrt{N_A}\ .\end{aligned}$$

We have discussed above how we do *not* have information of which ion stores the excitation. A subtle point is that there is *which path* information [Scully and Zubairy] in the ensemble. So, *if* we made a fluorescence measurement *then* a single ion would light up as the one ion that had the excitation. Another way of saying this is that the target states are all orthogonal: $|g_1,...,s_i,...,g_{N_A}\rangle$ is orthogonal to $|g_1,...,s_j,...,g_{N_A}\rangle$ for $i \neq j$. The result is that there can be no quantum interference between the different paths to the end state [Sangouard *et. al*].

We have seen how the quantum repeater is initialised by a click in a detector - the detection of the Stokes photon. This also means that we have realised the phase matching condition $\mathbf{k}_c = \mathbf{k}_q$. After this detection we know that there is an excitation in the ensemble of ions. And we know that at the moment of interaction the ensemble was in a Dicke (W) state with one excitation, at least as long as there are not more than one excitations in the ensemble.

Read-out

First, there is the same $\sqrt{N_A}$ enhancement in the read-out as in the write-in. But, additionally, there can be quantum interference. This quantum interference is possible since in this case we do not have which path information

- we cannot make a measurement to determine which path the system took to get back into the ground state.

Following the description above we can write the successful read state after the absorption of a photon on the $|s\rangle \rightarrow |e\rangle$ transition and with the emission of a photon on the $|e\rangle \rightarrow |g\rangle$ transition:

$$|\psi_{\text{read}}\rangle = \frac{1}{\sqrt{N_A}} \sum_{j=1}^{N_A} c_j c'_j \mathrm{e}^{i(\mathbf{k}_c - \mathbf{k}_q)\cdot \mathbf{r}_j} \mathrm{e}^{i(\mathbf{k}_r - \mathbf{k}_{AS})\cdot \mathbf{r}'_j} |g_1, ..., g_{N_A}\rangle \ , \tag{3.44}$$

where the subscript r refers to the classical read photon while AS refers to the Anti-Stokes photon we want to produce from the interaction, and where $\mathbf{r}'_k$ is the position of the $k'th$ ion at the read interaction moment.

We now have two possibilities for a phase-matching condition that gives constructive interference: If the ions are at rest ($\mathbf{r}_j = \mathbf{r}_j$ for all j) between write and read, then

$$\mathbf{k}_q + \mathbf{k}_{AS} \stackrel{!}{=} \mathbf{k}_c + \mathbf{k}_r \ , \tag{3.45}$$

with the direction of the Anti-Stokes photon given by $\mathbf{k}_c + \mathbf{k}_r - \mathbf{k}_q$. If the ions are not at rest we shall need

$$\mathbf{k}_q + \mathbf{k}_{AS} \stackrel{!}{=} \mathbf{k}_c + \mathbf{k}_r \stackrel{!}{=} 0 \ . \tag{3.46}$$

These constitutes conditions - not only for getting the wished for photon in the Anti-Stokes mode - but also for having quantum interference between the paths getting from the excited ensemble to the ensemble in the ground state. The constructive interference leads to a large enhancement of the emission efficiency into the Anti-Stokes mode as demonstrated in [Simon *et al.*] from where we cite the graph on Figure 3.4. This figure comes from exactly the kind of interaction that we have considered, and they realised a small quantum memory of maybe $\sim$ 250ns of memory. The graph should be interpreted as the efficiency of getting a photon in the Anti-Stokes mode from the read-out procedure. The efficiency rises as a function of the optical depth - effectively the number of atoms along the propagation axis of the light.

This process should not be confused with *superradiance* [Dicke]. Here the emitters (the atoms) are so close that the near-field of the emitters overlap and they stimulate each other to emit into the same mode. The whole ensemble in superradiance works in a way not dissimilar to a laser medium as discussed in [Hilliard *et al.*].

3.3 Cavity physics

The experiment we consider is performed in a cavity to enhance the interaction of light with atoms by basically having the light pass the ensemble of

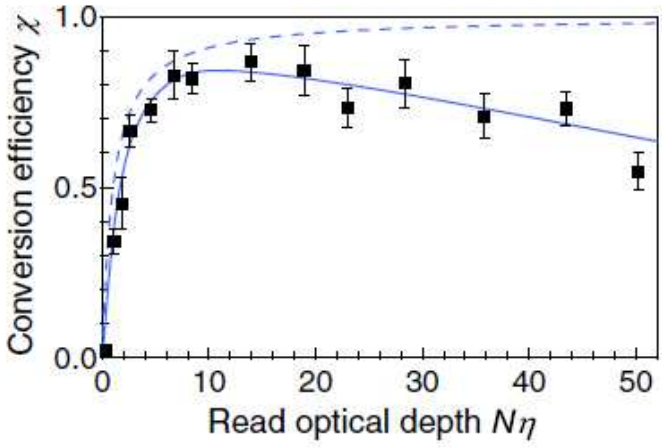

Figure 3.4: Conversion efficiency is plotted as a function of optical depth. The conversion is the successful read-out of a photon into the Anti-Stokes mode. A sharp rise in efficiency is observed growing to a very high efficiency as the optical depth increases. There is a drop off in efficiency as a function of optical depth because of dephasing from additional excited states basically due to the thermal notion of atoms in the larger and larger ensemble. Taken from [Simon *et al.*].

atoms many times. The cavity complicates the description (and the experiment) but it also helps in the collection of the all-important single photons needed for the quantum repeater. We will here only ever so lightly touch upon the subject of (quantum) physics in an cavity.

Most generally a cavity is characterised by the fact that the field inside it must vanish at the two mirrors. This implies that for the cavity to sustain a travelling field the wavelength of the field wave must be $2\lambda = n \cdot l_{cav}$, where n is an integer and l_{cav} is the length between the mirrors. The corresponding frequencies are called the cavity's resonance- or eigenfrequencies.

3.3.1 Classical cavity input-output relations

The aim is to find the modification of the equations of motion of the light field operators due to the cavity[1]. The actual modification can be found using classical arguments: Consider the empty cavity on Figure 3.5.

Cavity spectrum. First, we look for a relation between the fields $E^{\text{in}}(t), E_1^{\text{out}}(t)$, and $E_2^{\text{out}}(t)$ as a function of the frequency of the $E^{\text{in}}(t)$ field and the cavity

[1] In this subsection we follow [Herskind thesis] closely such that we can compare later calculations with the actual experiments

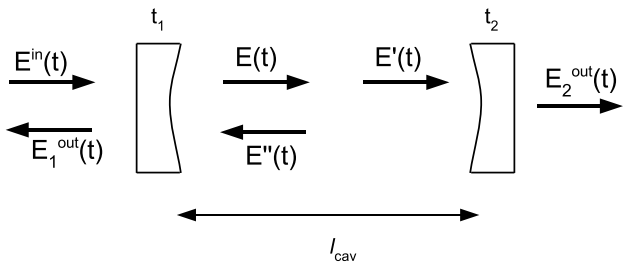

Figure 3.5: A simplified single mode cavity with two confocal mirrors with transmitivity t_1 and t_2.

parameters. Inside the cavity just after the first mirror we have

$$E(t) = t_1 E^{\text{in}}(t) + \alpha E''(t) r_1 \mathrm{e}^{i\pi} \ , \tag{3.47}$$

where α accounts for losses to absorption and scattering, t_1 the transmitivity, r_1 the reflectivity of the first mirror, and where the last factor is a phase due to reflection also in the first mirror. Let us denote with $\mathcal{L}$ the loss in the cavity *intensity per round trip*, then $\alpha^2 = 1 - \mathcal{L}/2$. We assume for each mirror

$$T_i + R_i + \frac{\mathcal{L}}{2} = 1 \ ,$$

where T_i is the transmittance and R_i reflectance, and where $t_i = \sqrt{T_i}$ and $r_i = \sqrt{R_i}$. Then for the field in the other direction inside the cavity we have

$$E''(t) = \alpha r_2 E(t-\tau) \mathrm{e}^{i\phi} \mathrm{e}^{i\pi} \ , \tag{3.48}$$

where $\tau = 2l_{cav}/c$ is the cavity round trip time and $\phi = (\omega - \omega_{cav})\tau = \Delta_{cav}\tau$ is the phase shift after one round trip. ω is the frequency of the field and ω_{cav} is the nearest cavity resonance. Plugging (3.48) into (3.47) gives:

$$E(t) = t_1 E^{\text{in}}(t) + \alpha^2 E(t-\tau) r_1 r_2 \mathrm{e}^{i\phi} \mathrm{e}^{i2\pi} \tag{3.49}$$

$$\simeq \sqrt{T_1} E^{\text{in}}(t) + \left(1 - \frac{\mathcal{L}}{2}\right) \sqrt{1 - T_1 - \frac{\mathcal{L}}{2}} \sqrt{1 - T_2 - \frac{\mathcal{L}}{2}} (1 + i\phi) E(t-\tau) \ . \tag{3.50}$$

Here we have expanded $\exp(i\phi) \simeq (1 + i\phi)$ for small detunings. If we expand the square roots and keep the parameters $T_1, T_2, \mathcal{L}$, and $\Delta_{cav}\tau$ up to first order we get:

$$E(t) \simeq \sqrt{T_1} E^{\text{in}}(t) + \left(1 - \mathcal{L} - \frac{T_1}{2} - \frac{T_2}{2} - i\Delta_{cav}\tau\right) E(t-\tau) \ . \tag{3.51}$$

Next, we consider the decay rate through the mirrors:

$$\kappa_i = \frac{1-r_i}{\tau} = \frac{1-\sqrt{1-T_i}}{\tau} \simeq \frac{T_i}{2\tau} \ ,$$

and define

$$\kappa_{\mathcal{L}} = \frac{\mathcal{L}}{2\tau} \ .$$

Then we get

$$\frac{E(t)-E(t-\tau)}{\tau} = -(\kappa_1+\kappa_2+\kappa_{\mathcal{L}}+i\Delta_{cav})E(t-\tau) + \frac{1}{\sqrt{\tau}}\sqrt{2\kappa_1}E^{\text{in}}(t) \ .$$

With $\kappa = \kappa_1+\kappa_2+\kappa_{\mathcal{L}}$ and in the limit where $\tau \to 0$ this becomes:

$$\dot{E} = -(\kappa+i\Delta_{cav})E(t) + \frac{1}{\sqrt{\tau}}\sqrt{2\kappa_1}E^{\text{in}}(t) \ , \tag{3.52}$$

where $\sqrt{\tau}$ is kept since it goes to 0 much slower than τ does. This is the equation of motion for the field inside the cavity. When dealing a with quantum field we shall use the same equation modified to be an operator equation. In the quantum case also noise will be important.

The output fields are

$$\begin{aligned} E_1^{\text{out}}(t) &= t_1\alpha E''(t) + r_1 E^{\text{in}}(t) \\ E_2^{\text{out}}(t) &= t_2\alpha E'(t) \ . \end{aligned}$$

In the steady state we get from (3.52)

$$E(t) = \frac{\sqrt{2\kappa_1/\tau}}{\kappa+i\Delta_{cav}}E_2^{\text{in}}(t) \ , \tag{3.53}$$

and we can calculate how much is transmitted through the cavity

$$\text{trans} = \left|\frac{E_2^{\text{out}}(t)}{E^{\text{in}}(t)}\right|^2 = \left|\frac{\alpha t_2\sqrt{2\kappa_1/\tau}}{\kappa+i\Delta_{cav}}\right|^2 = \frac{4\alpha^2 t_2^2\kappa_1/\tau}{\kappa^2+\Delta_{cav}^2} = \frac{4\kappa_1\kappa_2}{\kappa^2+\Delta_{cav}^2} \ , \tag{3.54}$$

since E' and E only differ by a phase and since $\alpha^2 \simeq 1$ for low losses. This expression is a Lorentzian in ω, the frequency of the light:

$$\text{trans} = \frac{4\kappa_1\kappa_2}{\kappa^2}\frac{1}{1+(\Delta_{cav}/\kappa)^2} \ , \tag{3.55}$$

with the full width at half maximum FWHM$= 2\kappa = 2\pi\cdot 6,3\cdot 10^6$Hz and with a peak transmission of $4\kappa_1\kappa_2/\kappa^2 \simeq 0.007$ for the realistic experimental parameters of $T_1 = 1500\cdot 10^{-6}$, $T_2 = 5\cdot 10^{-6}$, and $\mathcal{L} = 600\cdot 10^{-6}$.

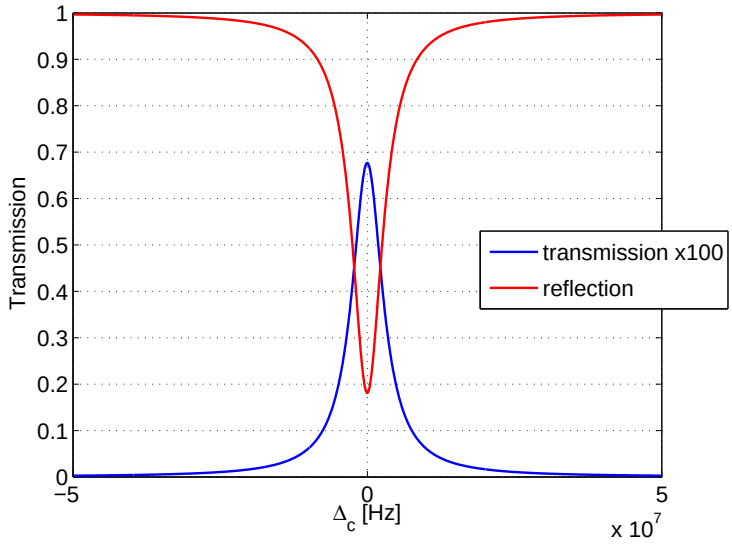

Figure 3.6: The theoretical model of cavity transmission and reflection as a function of the cavity detuning from the frequency of the light. The plot is generated using the experimental values for the parameters involved that are given in the text.

A similar calculation shows that what is reflected from the cavity is

$$\text{refl} = \left| \frac{E_1^{\text{out}}(t)}{E^{\text{in}}(t)} \right|^2 = \frac{(\kappa_{\mathcal{L}} - \kappa_1 + \kappa_2)^2 + \Delta_{cav}^2}{\kappa^2 + \Delta_{cav}^2} \ . \tag{3.56}$$

Figure 3.6 shows the reflection and 100×transmission as the detuning is varied but still small - the paramaters are as given above for a realistic experiment.

We shall need the concept of *finesse* later. The free spectral range is defined as $FSR = \tau^{-1}$ and it is the frequency spacing between the cavity resonances. Then, the finesse is defined as

$$\mathcal{F} \equiv \frac{FSR}{FWHM} = \frac{2\pi}{\mathcal{L} + T_1 + T_2} \ . \tag{3.57}$$

3.3.2 Cavity QED

Cavity quantum electrodynamics is the study of the coherent dynamics of the combined light-atom system in a cavity. The beginning of the field in the late 1940's is the description of spontaneous emission and how it is changed in a cavity as opposed to vacuum [E. M. Purcell]. This is an example of a *Perturbative interaction*. In non-pertubative interactions the interaction

exceeds the dissipation mechanisms and fundamental studies can be made of the coherence and decoherence of quantum systems.

Strong coupling. Consider Figure 3.7 with a schematic drawing of an ensemble cavity QED experiment. When the following condition

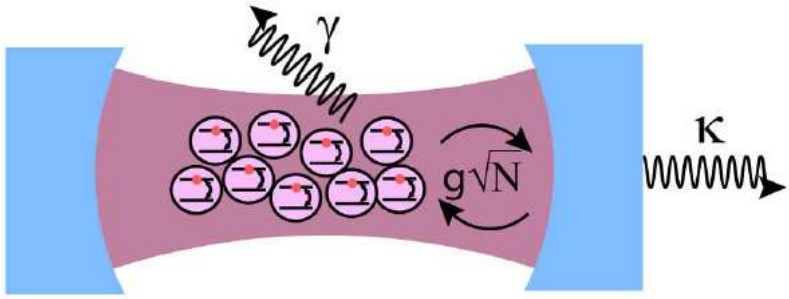

Figure 3.7: A generic ensemble cavity QED experiment where κ is the decay from the cavity, γ is the decay from the excited state of each of the identical N atoms, and where g describes the strength of the coupling in the two-level atoms. Taken from [Herskind thesis].

$$g\sqrt{N_A} > \kappa, \gamma$$

is met, so-called strong coupling is achieved. In this regime many tests of fundamental quantum mechanics have been carried out. One striking example is the highly coherent evolution of cavity-atom systems that allow for the observation of not only Rabi oscillations between the excited atom an the vacuum field - but also the collapse and subsequent revival of the coherent dynamics when a photon field is interacting with the already excited atom as it enters the cavity [Brune *et al.*] also cited in [Gerry *et al.*].

In the system we study the cavity is there to enhance the interaction between light and matter. In the bad cavity/free space model [Duan *et al.*] *optical depth* is defined as the factor of attenuation from the Lambert-Beer law $I/I_0 = e^{-\mathrm{od}}$:

$$\mathrm{od} \equiv \frac{d^2 \nu N_A}{2c\hbar\epsilon_0 A\gamma} \ .$$

The optical depth is the figure of merit of the efficiency of the interaction. Here d is the size of the atomic dipole moment, ν is the natural frequency of the light, and A is the cross section of the laser beam. High optical depth is most important for the read-out process where a high retrieval efficiency is a condition for the success of the repeater scheme.

For an ensemble in a cavity the efficiency depends on the *coorporativity* rather than on the optical depth, where the cooperativity is related to the

definition of strong coupling:

$$C = \frac{N_A g^2}{2\kappa\gamma} \ .$$

A Cooperativity above 2.5 characterises strong coupling and by inference we need to be in the strong coupling regime to see a good retrieval efficiency above 0.7.

In [Gorshkov *et al.*] they calculate the retrieval efficiency for a system like an ensemble in a cavity and find that the optimum efficiency possible is:

$$\eta = \frac{C}{1+C} \ . \tag{3.58}$$

This is in the limit where Ω is zero and κ is very large. We shall return to this expression as we find the efficiency of the read-out interaction for the ion crystals.

Chapter 4

Ion Coulomb crystals

We shall propose to use cooled and trapped ions for the quantum repeater ensembles to be used in the nodes to generate and connect entanglement. The special collective state of these ions is a so-called Coulomb crystal - a periodic structure involving maybe several thousand of ions[1].

These crystals might be a very serious contender in the race to build a functioning quantum repeater: The fact that the structures are made up of ions gives great spatial control over them, the fairly large number of ions gives strong coupling to light, and the lifetime is very, very long compared to any similar system. All the while these crystals are of so low density that atom-to-atom interactions are not the limiting factor for the coherence as it is in the solid state.

4.1 Coulomb crystals

4.1.1 General remarks - plasmas with a single sign of charge

Our primary reference for the following section is the article [Herskind *et al.*], while more details can be found in the Ph. D. thesis [Mortensen].

Historically, the point of departure was the addition of the Coulomb interaction between the electrons to the free electron model in metals. The free electron model works best as a description of the kinetic properties of the electrons e.g. the heat capacity of metals at higher temperatures. At low temperatures, however, the kinetic energy becomes comparable with the Coulomb interaction energy and the movement of the electrons should be dominated by the interatomic Coulomb forces and the electrons should form

[1]This is not the state of cooled ions usually used in applied quantum information. There, cooled ions are put in strings of ions with very few particles. Furthermore, these strings are cooled to such low temperatures that phonons are frozen out such that the interaction between the ions can happen deterministically via the interchange of vibrational quanta, see e.g [Riebe *et al.*] and [Barrett *et al.*] while important theoretical contributions have come from e.g. [I. Cirac and P. Zoller].

crystals. This of course requires that the electrons are still confined inside the structure of positively charged ions that make up the metal structure and serve as a neutralising background.

There is one further point: This crystallization can only occur if the density of the electrons is low enough. The quantum mechanical picture of this phase transition is the following: Only at very low densities will the overlap of the electronic wave functions be so small that the long-range Coulomb interaction will overcome the formation of energy bands with delocalized electrons. Therefore, the formation of this crystaline state, a so-called Wigner crystal [Wigner], requires that the density of electrons is lowered as compared to a normal metal where the density is exactly so high that the electrons become delocalized and form energy bands. Such crystals have subsequently been observed in two dimensions in a sheet of electrons on a liquid-He surface [Grimes and Adams], although 3-D crystals have not been observed.

A thermodynamic description of the single component plasma is the plasma coupling parameter Γ:

$$\Gamma = \frac{E_{Coul}}{E_{kin}} = \frac{Q^2}{4\pi\epsilon_0 a k_B T} \ . \tag{4.1}$$

Here Q is the charge and $a = 4/(3\pi n_0)^{1/3}$ is the Wigner-Seitz radius with n_0 as the zero temperature density of electrons. In normal metals $a \sim \Delta r$ at low temperature, where Δr is the kinetic motion of each particle in the lattice. By lowering the density, as discussed above, and adjusting the temperature one can achieve a liquid phase for $\Gamma \sim 2$ and at $\Gamma \sim 150$ or equivalently $a/\Delta r \sim 10$ the plasma becomes a solid.

Ion crystals

The parameter Γ and its values from above are relevant for other plasmas with a single charge, including singly positively charged ions which is our interest for the quantum repeater. The question is how to control the particles and the parameter Γ.

Photoionisation. First we need to ionise atoms. This can advantageously be done via resonance-enhanced photoionisation with the use of UV light resonant with a very energetic electronic transition. Then, the effect is such that the highly excited electron is so weakly bound that the absorbtion of a second UV photon ionises the atom. This process has several advantages. The most interesting for our application is the possible selectivity of ionisation of different atomic species or isotopes because of the different energy levels in different species/isotopes. Very precise ratios of e.g. two different isotopes is achievable.

Doppler laser cooling of particles with internal degrees of freedom. To have the Coulomb energy very much larger than the kinetic energy the ions

shoul be very cold. This can be achieved by interacting the atoms with two counter-propagating laser beams with a frequency, ω_L, close to an atomic transition, ω_a. Let us say that the lasers are red detuned $\omega_L < \omega_a$. For a certain velocity the atom is resonant with the laser light that propagates towards it because of the Doppler shift, while the light that propagates in the same direction is out of resonance. The atom is consequently slowed down in this direction because the atom will scatter photons that are counter-propagating. Six lasers would in effect create an optical molasses and cool the atom in all directions. A lower limit on the temperature achievable [Metcalf and van der Straten] is set by:

$$T_D = \frac{\hbar\Gamma_l}{2k_b} \ , \tag{4.2}$$

where Γ_l is the natural linewidth of the transition used for the Doppler cooling. There is one problem, though: Unlike for electron crystals in the neutralising background of positive ions in the metal we will need some mechanism that acts neutralising on the positive ions to make them form crystals. We will address this point below.

The linear Paul trap. Earnshaw's theorem says that it is not possible to confine a charged particle with static electric fields only. The linear Paul trap employs time varying electric fields to trap the ions. Four electrodes carry oscillating currents - two diagonally opposite electrodes carry the same voltage and a quadrupole field is created that has the frequency $\Omega_{\rm rf}$ and potential $U_{\rm rf}$. Finally, the trap is capped in the each end with a static field with the potential $U_{\rm ec}$. The advantage of the Paul trap is that it does not contain a magnetic field[2]. The geometry of such a trap is depicted in Figure 4.3.

In the trap each ion is to a good approximation moving in a harmonic pseudo potential. This is a time-averaged potential:

$$\Phi_{\rm ps}(r,z) = \frac{1}{2}M\left(\omega_r^2 r^2 + \omega_z^2 z^2\right) \ , \quad r^2 = x^2 + y^2 \ , \tag{4.3}$$

where M is the mass of the individual ion and where the axial trapping frequency and the secular frequency are:

$$\omega_z = \sqrt{\frac{-a}{2}}\Omega_{\rm rf} \ \text{ and } \ \omega_r = \frac{\sqrt{q^2/2+a}}{2} \ . \tag{4.4}$$

Also, these substitutions have been made:

$$a = -4\frac{\eta Q U_{\rm ec}}{z_0^2 M \Omega_{\rm rf}^2} \ \text{ and } \ q_x = -q_y = 2\frac{Q U_{\rm rf}}{M r_0^2 \Omega_{\rm rf}^2} \ , \tag{4.5}$$

[2]Usually in quantum computation with cold ions a Penning trap is used – here a static magnetic field superimposed on the electric fields trap the ions. The advantage of the Penning trap is that it does not heat up the ions as much as the Paul trap.

where η, z_0, and r_0 are geometric factors.

The point we are trying to make here is that the pseudo potential is independent of the mass of the ion in the axial direction, but in the radial direction the potential *is* dependent of the mass. The lighter ions are consequently trapped closer to the center than the heavier ions (if there are different ions in the trap, naturally). This becomes very interesting for our application later.

A cold confined positively charged ion plasma. The cooled ions coalsesce in the linear Paul trap and when the condition mentioned on the Γ from equation (4.1) is met crystals form. Were they infinite in size they would be body-centered cubic (bcc) crystals, but already modestly sized groups of ions show bcc structure. The particle density is then $n_0 \sim 10^8 \mathrm{cm}^{-3}$. The neutralizing background in this system is made up of the electric fields applied in the trap.

Sympathetic cooling in a multicomponent ion plasma crystal. The structures formed are further caracterized by being concentric shells of ions. If, then, two different species of ions are both trapped then the lighter ions would form a crystal near the center while the heavies ions would form concentric shells outside this inner crystal. Via the Coulomb interaction the outer heavier ions would collisionally cool the inner crystal. The advantage is that we could form and cool an ion crystal without the ions interact with the Doppler cooling lasers. This might be attractive for the quantum light-atom interface, since longer coherence times could be possible.

4.1.2 The Aarhus ion Coulomb crystals

The crystals we propose to analyse for the quantum repeater have already been realised in the laboratory – a nearly ten years effort led by M. Drewsen at the University of Aarhus.

The experimental set-up is kept in a fairly large vacuum chamber in which calcium is dispensed. Calcium is an alkaline earth metal with the atomic number 20 and has the electronic configuration $[\mathrm{Ar}]4s^2$. For a single species crystal two-photon dissociation $^{40}\mathrm{Ca}^+$ ions are created, the level scheme relevant for this operation is shown in Figure 4.1. After the ionisation the electronic structure is like the alkaline metal potassium and an easily accessible single outer electron is ready for the quantum interface.

The ions are cooled by Doppler cooling as illustrated on Figure 4.2. As the ions get caught in the cooling beam they loose kinetic energy and start to form the crystal in the linear Paul trap, see Figure 4.3.

There is a medium finesse ($\mathcal{F} \sim 3200$) cavity around the crystal in order to realize strong coupling to the quantum field, see Figure 4.4. The experimenters are interested in a quantum field of carefully prepared pulses. They further assume that the cavity is tuned to the resonance of the atomic transition targeted for the interaction. Lasers for the quantum interface

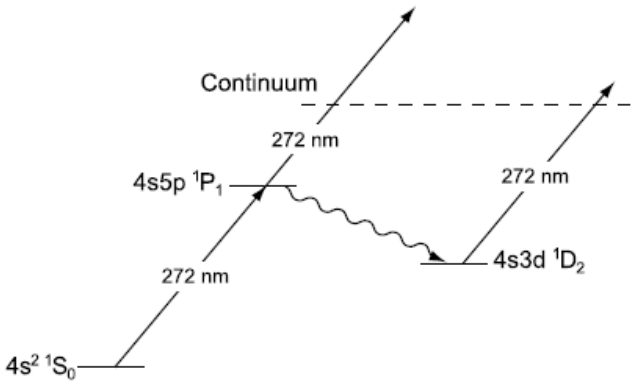

Figure 4.1: The level scheme for resonant two-photon ionisation of neutral ^{40}Ca . Taken from [Herskind *et al.*].

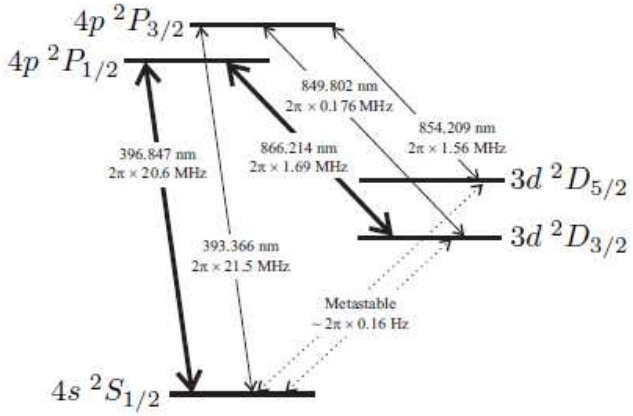

Figure 4.2: The level scheme for Doppler cooling of Ca^+ with wavelenths and partial decay rates for the dipole allowed transitions. The thick lines indicate the transitions most frequently used for the cooling scheme. Taken from [Mortensen].

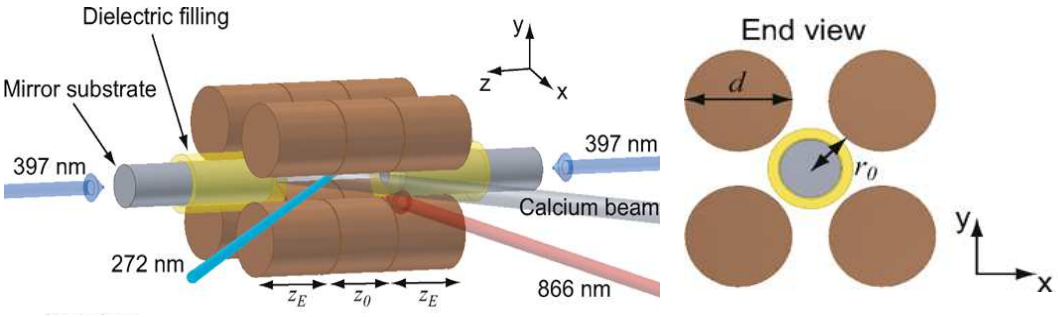

Figure 4.3: The geometry of the paul trap used in the experiment. Illustrated are the beams for ionisation (272 nm), cooling (397 nm), and driving (866 nm) . Taken from [Herskind *et al.*].

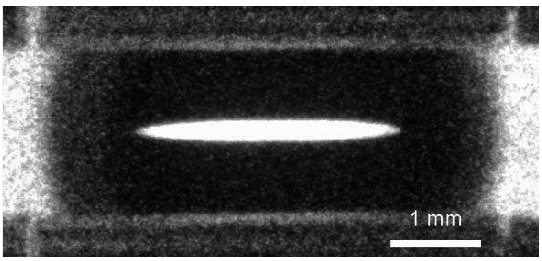

Figure 4.4: Scattering of laser light from the crystal and the surrounding mirrors of the cavity. For latter reference when we consider moving the crystal around it should be noted that the cavity can be considerably longer than on this picture. Taken from [Herskind *et al.*].

are coupled in via the cavity, both the quantum field and the classical-like driving/control field. The quantum interface could be realised in the calcium ions via a Λ level scheme using the magnetic substates of the $3D_{3/2}$ and the $4P_{1/2}$ states, as shown in Figure 4.5.

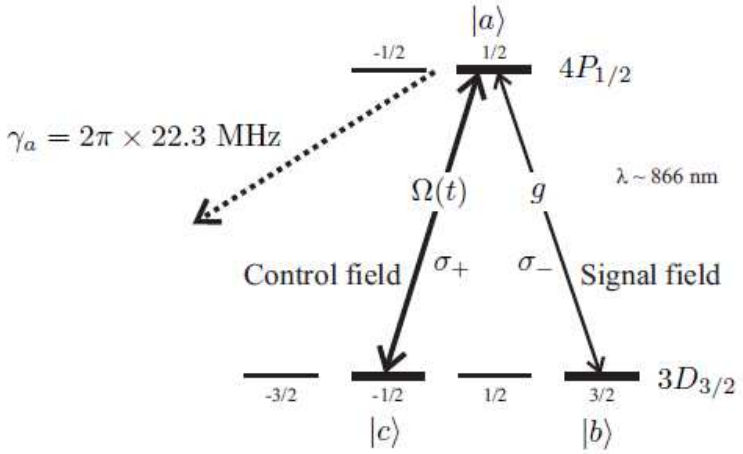

Figure 4.5: The three magnetic substates constituting the Λ-system for the quantum interface are indicated by their coupling to the classical control field (σ_+ polarised) and the quantum signal field (σ_- polarised). Taken from [Mortensen].

Additionally, two-species crystals are possible e.g. with two different isotopes of calcium: $^{44}Ca^+$ and $^{40}Ca^+$. The outer heavier ions are cooled by lasers and by collisions they sympathetically cool the inner crystal, without having the cooling lasers interacting with the inner crystal since this is a differerent isotope. This inner crystal could then constitute the medium for a quantum interface with light. Up until now the Aarhus group has made crystals with several tens of thousands of ions where the inner crystal contains maybe a tenth of the ions. A two-species crystal is shown in Figure 4.6. The top flourescent image shows the combined crystals while the middle image shows the outer crystal and the botoom image the inner crystal.

4.2 Advantages of an ion Coulomb crystal in an optical cavity

4.2.1 Strong coupling

A prerequisite of the light-atom interface in cavities to handle the deterministic interaction at the single-photon level is so-called strong coupling. Basically it says that a photon in the cavity should interact with the atoms

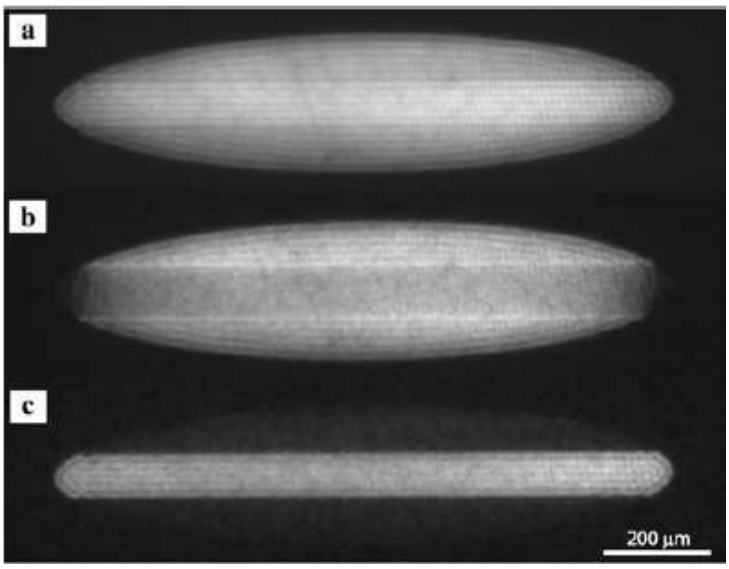

Figure 4.6: A two-species ion Coulomb crystal (a) with ~ 2000 $^{40}Ca^{+}$ ions in the inner crystal (c) and ~ 13000 $^{44}Ca^{+}$ in the outer crystal (b). Taken from [Herskind *et al.*].

before it escapes the cavity. That is, the rate of interaction should be larger than the loss from the cavity and larger than the spontaneous decay:

$$g > \kappa, \gamma \ . \tag{4.6}$$

In Aarhus the experimenters achieved a fantastic result in the winter of 2008-9: They realised strong coupling of the electro-magnetic field to an ion Coulomb crystal in an optical cavity [Herskind, Dantan *et al.*]. What they measured was the *cooperativity* which is proportinal to the coupling squared:

$$C = \frac{g_N^2}{2\kappa\gamma'} \ , \tag{4.7}$$

where g_N is the ensemble-enhanced coupling and where γ' is the spontaneous emmission rate 'dressed' by the cavity.

As can be seen from Figure 4.7 strong coupling can be achieved with as little as 600 ions. The condition for strong coupling is $C > 2.5$. The cooperativity grows linearly with a higher number of ions. The result is important because usually strong coupling is achieved with very high finesse cavities which again must be kept very small (we shall explain why later) which means that there is very little room for all the operations required to trap and cool the atomic medium and manipulate the interaction. This cavity is large enough for the relatively large crystal with access to both

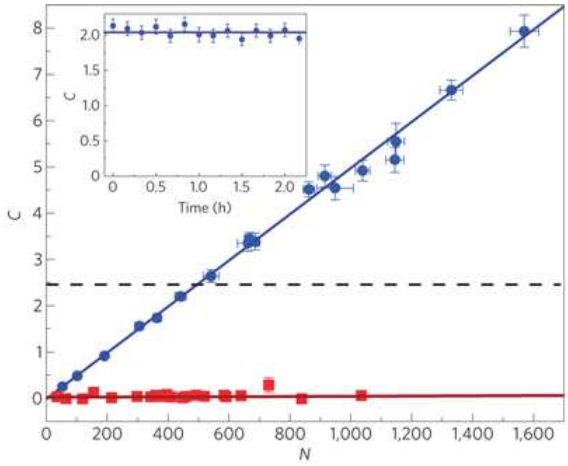

Figure 4.7: The large graph with the filled circles shows the cooperativity as a function of crystal size measured in number of ions. The squares are the cooperativity as the interaction is done with light that is that is not polarisation resonant. The solid lines are linear fits. The dashed horizontal line demarks the regime of strong coupling. The small insert graph depicts the lifetime of the same crystal plotted as cooperativity as a function of time. Taken from [Herskind, Dantan *et al.*].

cooling and interaction lasers. Also, high finesse cavities are hard to keep stable – in these experiments it is easier to keep the cavity adequately stable.

It should be pointed out that the cavity for this particular design had been made longer than the one depicted in Figure 4.4. The present cavity is 11.8 mm long.

4.2.2 Lifetime and coherence time

Also on Figure 4.7 is a smaller graph showing how the lifetime of the crystal is measured in hours. Or rather, how the cooperativity stays the same over very long times – which would indicate that the crystal is the same during the same time.

What is more important for a quantum memory is the coherence time. It is a measure of how well a quantum state after storage resembles the state before storage. In the experiment [Herskind, Dantan *et al.*] they measured the coherence time of up to 1.7ms. This is comparable to other schemes for quantum memories e.g. [Julsgaard *et al.*]. In the present experiment the coherence time was limited by the fluctuations of the magnetic field with which the coherence time was measured. It is to be expected that a considerably longer coherence time can be observed with a 'cleaner' set-up or maybe using a different measuring technique. In somewhat compara-

ble systems coherence times of many seconds have been observed, see e.g. [Berkeland *et al.*].

Finally, to compare with other quantum information interface proposals, an optical depth of 16 was measured.

4.3 Aspects of the quantum interface

We will return to the modelling of the atom-light interface in much greater detail in the following chapters, here we just will list a couple of points.

4.3.1 The wave-function in a standing wave cavity

In [Duan *et al.*] they specifically contemplate using a running wave cavity for the quantum repeater scheme. We will consider a realisation in a standing wave cavity, simply because this is what the experiment in Aarhus have implemented so far. Also, a standing wave cavity is much more simple to build and to make work in a reliable way, especially since they have to accommodate all the technical apparatus of the Paul trap.

As already mentioned, we will make the approximation that there will be only one transverse mode in the cavity, the lowest mode of cavity excitation is the TEM_{00}. This is reasonable and can be achieved in the experiment by good alignment of the cavity and by placement of the crystal exactly symmetrical on the cavity axis, and with most of the ions on or close to the axis. Also, the intensity on this mode is so evenly distributed in the area where the crystal is, that we take the intensity to be constant in the transverse plane. This is especially reasonable in the limit where the inner crystal, which is our quantum interface, is a string of ions lying along the axis of the cavity. Therefore, we only need to consider what happens along the axis of the cavity.

The bandwidth of the cavity is measured to be 4 MHz [Herskind *et al.*]. A single photon is a wavepacket comprised of many monochromatic waves with a certain bandwidth. If we take the same bandwidth of this wavepacket and of the cavity in principle the photon could pass into the cavity without the loss of any of its consituent frequencies and thus keep the same characteristics. Then it comes down to what kind of a photon we wish to store in the crystal. If the photon came from a pulsed femtosecond laser the bandwidth would be much broader ($\sim$ 10 terahertz). Such a pulse would be severely distorted upon entering the cavity.

Let us elaborate a little on the nature of our single photon. The time width of the pulse and the length in meters would be:

$$\Delta t = \frac{1}{\delta\nu} = 0.26\ \mu\mathrm{s} \quad \text{and} \quad \Delta s = c\Delta t \simeq 10\ \mathrm{m}\ , \tag{4.8}$$

where $\delta\nu$ is the bandwidth of the single photon. This is obviously much longer than the cavity and the pulse would interfere with itself and this would be the cause of standing waves in the cavity. The carrier wave length would still be 866nm since the cavity detuning is only on the order of some hundreds of megahertz. The standing wave will have the same intensity along every top in the pattern since the different components making up the photon wave packet does not extinguish each other because of the narrow bandwidth.

Putting it all together, we expect a monochromatic standing wave along the cavity axis with nodes every half wavelength and the intensity described by:

$$I(z) \propto A\sin^2(k_0 z) \quad \text{with} \quad k_0 = \frac{2\pi}{866\text{nm}} \quad . \tag{4.9}$$

Evolution of the system of the crystal and cavity-field

We will here consider process of the cavity field interactiong with the crystal: First, the excitation of the photon is written into the ions. Then, the time passes with the excitation kept in the ions and, last, the photon is read out again. Generally, we consider the read-in and write-out process as reversible and use unitary operators to model this - we have that

$$|\psi_{out}\rangle = \hat{U}_r(t_r)\hat{U}_s(\Delta t_s)\hat{U}_w(t=0)|\psi_{in}\rangle \ , \tag{4.10}$$

where the subscript 'in' refers to the wavefunction just when the excitation is read into the ion-ensemble as opposed to the subscript 'out' which is the wavefunction at the moment of read-out, that is, after the storage. Furthermore, with regards to the subscripts on the unitary evolution operators w means 'write', s 'storage' and r means 'read'.

For the wavefunction of the system of the crystal and the cavity photon before the interaction we propose the following:

$$|\psi_{in}\rangle = \hat{a}^\dagger_{k_0}|\underbrace{0,...,0}_{m}\rangle_{\text{atoms}}|vac.\rangle_{\text{field}} \ , \tag{4.11}$$

In the following we drop the sub- and subcripts for 'atoms' and 'field'.

Now, the write-in process we take to be 'very fast' and with perfect coupling and represent it by this operator expression:

$$\hat{U}_w(t=0) = \frac{1}{\mathcal{N}_1}\sum_m \sin(k_0 z_m(t=0))\hat{\sigma}^m_{10}\hat{a}_{k_0} \ . \tag{4.12}$$

Then

$$|\psi_w\rangle = \frac{1}{\mathcal{N}_1}\sum_m \sin(k_0 z_m(0))\hat{\sigma}^m_{10}\hat{a}_{k_0}\hat{a}^\dagger_{k_0}|0,...,0\rangle|vac.\rangle \tag{4.13}$$

$$= \frac{1}{\mathcal{N}_1}\sum_m \sin(k_0 z_m(0))|0,...,0,1_m,0,...,0\rangle|vac.\rangle \ . \tag{4.14}$$

$\mathcal{N}_1$ is a normalisation constant to be determined below (it is found numerically in the Matlab code that produces the values plotted in Figure 4.10), see appendix A.

Next, we let time pass. In this storage time Δt_s the ions might move around but, ideally, nothing else happens and we shall take $\hat{U}_s(t) = id.$ Later, it might become interesting to incorporate ion-ion collisions or other such non-unitary processes.

To model the fact that the ions have moved since the write-in we put it into the operator representing the read-out step. We shall take $\hat{U}_r = \hat{U}_w^\dagger$. For a time $t \neq 0$ this gives

$$\begin{aligned}|\psi_{out}\rangle &= \frac{1}{\mathcal{N}_2}\sum_{n,k}\sin(kz_n(t))\hat{\sigma}_{01}^n\hat{a}_k^\dagger \\ &\quad\times\left(\frac{1}{\mathcal{N}_1}\sum_m \sin(k_0 z_m(0))|0,...,0,1_m,0,...,0\rangle|vac.\rangle\right) \\ &= \frac{1}{\mathcal{N}}\sum_{m,k}\sin(kz_m(t))\sin(k_0 z_m(0))\hat{a}_k^\dagger|0,...,0\rangle|vac.\rangle\ , \end{aligned} \tag{4.15}$$

since $\hat{\sigma}_{01}^n|0,...,0,1_m,0,...,0\rangle = \delta_{nm}|0,...,0\rangle$. Finally, we can write that

$$|\psi_{out}\rangle = \sum_k c_k\hat{a}_k^\dagger|0,...,0\rangle|vac.\rangle \quad \text{where} \tag{4.16}$$

$$c_k = \frac{1}{\mathcal{N}}\sum_m \sin(kz_m(t))\sin(k_0 z_m(0))\ . \tag{4.17}$$

4.3.2 Movement of the ions during operations

The crux of the matter is this: We know the initial wave-function right as the ions have stored the excitation of the photon. Then we wish to store the excitation for a given period of time – the longer the better: The race is on to make a 100 millisecond quantum memory. Then later we wish to read out the excitation.

Now, the ions have moved in the time between write-in and read-out such that the wave-function has changed into the expression given by equation (4.16). We take it that $z_m(t) = z_m(0) + \Delta z_m(t) = z_m + \Delta z_m$, where Δz_m is how much the $m'th$ ion moved in the z-direction during the storage. Again, we need only worry about the one-dimensional case, at least for now.

Here we propose a simple model based on considerations from classical physics to give a realistic set of $\{\Delta z_m\}$. First off, we assume that we can take the movement of the ion to be a restricted kind of random walk: There is equal probability that the ion will go to the right as to the left but as it goes it will be stopped by the Coulomb potential of the neighbouring ion:

$$U_c(z_1) = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2}{|z_2 - z_1|}\ . \tag{4.18}$$

Z_i is the charge of the $i'th$ particle, where the ions under investigation each have the charge $-e$. Here we consider only point particles as the ions are very far apart compared to their size. Actually, the moving ion will feel all the potentials of all the ions in the z-direction. By the superposition principle we can just add these potentials - the two sums are the contribution to the potential from the left respectively to the right of the $m'th$ ion:

$$U_m(z_m) = \sum_{j=1}^{\text{left}} \frac{e^2}{4\pi\epsilon_0} \frac{1}{|z_j + z_m|} + \sum_{j=1}^{\text{right}} \frac{e^2}{4\pi\epsilon_0} \frac{1}{|z_j - z_m|} \ . \tag{4.19}$$

Next, we uncouple the motion of each ions from the motion of all the other ions. So, we assume that there are no collective effects of the motion in the form of phonons etc. Additionally, we assume that the distance between all pairs of ions initially is the same, d:

$$U_m(z_m) = \frac{e^2}{4\pi\epsilon_0} \left(\sum_{j=1}^{\text{left}} \frac{1}{|jd + z_m|} + \sum_{j=1}^{\text{right}} \frac{1}{|jd - z_m|} \right) \ . \tag{4.20}$$

U_m is well-defined continuous and differentiable on the set of open intervals $(-kd, kd)$ where k is an integer. To plot this potential we choose $d = 22\mu$m as it is a reasonable number based on the quoted density of $6.1 \times 10^8 \text{cm}^{-3}$ [Herskind *et al.*]. Furthermore, we assume that the $m'th$ ion is sitting in the middle of a 1-D crystal with 250 ions to the left and 250 to the right of it, see Figure 4.8. The infinite sum of n^{-1} does not converge but the partial

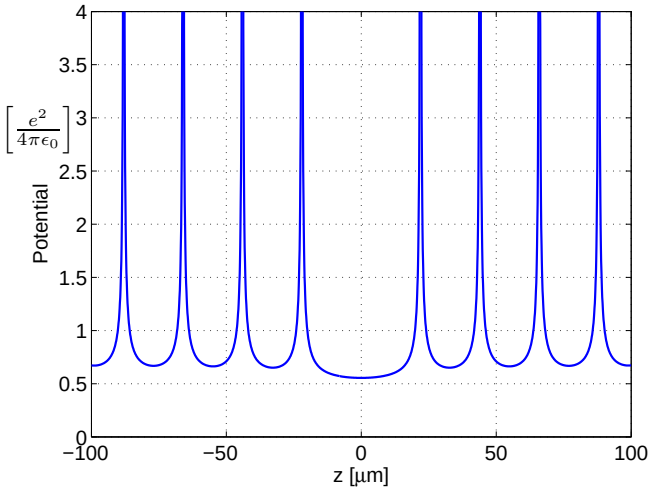

Figure 4.8: The potential seen by an ion in $z = 0$ and with other ions $j \cdot 22\mu m$ to either side.

sums do, of course, and they do so reasonably fast: An ion trapped ten ions from the edge of the crystal sees the same potential as the middle ions but the initial off-set is approximately 20% smaller.

As can be seen from the figure an ion in the crystal will be trapped in a potential with steep barriers. Here the ion will move back and forth with a movement given by the initial velocity, and if we do a Taylor expansion around zero and up to second order we see that this might be a good description given that the initial velocity is not too high, see Figure 4.9. The

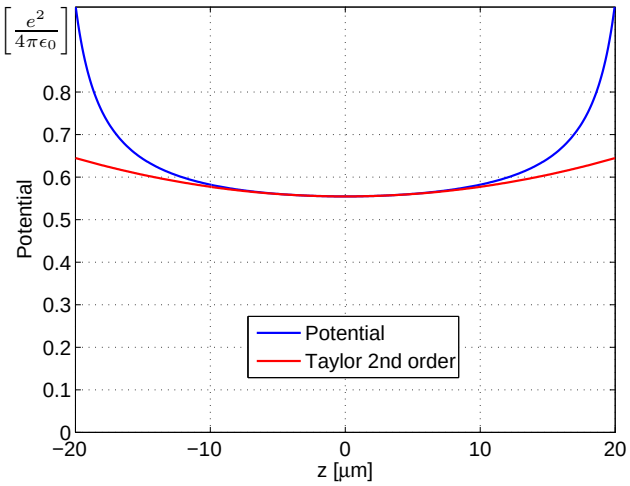

Figure 4.9: Close-up of the center of the potential from Figure 4.8 with the Taylor expansion of the potential up to second order superimposed.

Taylor expansion is, since $U_m(z_m)$ is an even function:

$$U_m(z_m) = U_m(0) + \frac{1}{2}U_m''(0)z^2 + \mathcal{O}(z^4) \ , \tag{4.21}$$

and the force on the ion is:

$$F_m(z_m) = -\frac{d}{dz}U_m(z) = -U''(0)z = -kz \ . \tag{4.22}$$

This is the equation of simple harmonic undamped oscillation, with the angular frequency of oscillation:

$$\omega = \sqrt{\frac{k}{m}} \ . \tag{4.23}$$

We are not completely uncomfortable by the fact that the oscillations are undamped – where should any friction come from? This on the other hand, via the fluctuation-dissipation theorem, entails that we have no dissipation

or diffusion in the system. This makes an analysis based on Brownian motion or other models for noise difficult. What follows is a simple minded alternative.

Now, we can find k by differentiating (4.20) twice in the set of open intervals $(-kd, kd)$ and then evaluating in $z = 0$:

$$U_m''(0) = \frac{e^2}{4\pi\epsilon_0}\frac{2}{d^3}\left(\sum_{j=1}^{\text{left}}\frac{1}{m^3} + \sum_{j=1}^{\text{right}}\frac{1}{m^3}\right) = \frac{e^2}{4\pi\epsilon_0}\frac{2}{d^3} \times 2 \cdot 1.2 \ , \tag{4.24}$$

where we again have considered the ion in the middle of the 501 ion 1-D crystal. Actually, the series $\sum_n^\infty 1/n^3$ does converge, namely to $\zeta(3)$, the Riemann zeta-function, which is equal to the partial sum of 250 terms by the first 5 significant digits. Moving on with the analysis of the simple harmonic oscillator we can write a time-dependent position function that will solve equation (4.23):

$$z_m(t) = A\sin(\omega t + \phi) \ \text{ and } \ \dot{z}_m(t) = \omega A\cos(\omega t + \phi) \ , \tag{4.25}$$

where A is the amplitude and where ϕ is a phase that we set to zero. Now, we can set $z(0) = 0$ and $\dot{z}(0)$ we assume we know as the mean speed of the Maxwell-Boltzmann velocity distribution:

$$\dot{z}_m(0) = \sqrt{\frac{8k_BT}{\pi m}} \ . \tag{4.26}$$

Here k_B, T and m are Boltzmann's constant, temperature, and the mass of the ion. A quick calculation at $T = 10\text{mK}$ gives $A \sim 1\mu m$ which gives some credence to the small amplitude approximation which was the basis for the simple harmonic oscillator model.

Next, we proceed to find the deviation of the ion's position away from zero:

$$\Delta z = \sqrt{\langle z^2\rangle - \langle z\rangle^2} = \sqrt{\langle z^2\rangle} = \sqrt{\frac{1}{2}A^2} = \frac{1}{\sqrt{2}}A \ , \tag{4.27}$$

where the averages are time-averages. We can now just plug in what we have found above:

$$\Delta z_m = \frac{1}{\sqrt{2}}\frac{\dot{z}_m(0)}{\omega} = \frac{\sqrt{8k_BT/\pi m}}{\sqrt{2}\sqrt{k/m}} = 2\sqrt{\frac{k_B\epsilon_0}{e^2\cdot 1.2}}\sqrt{Td^3} \ . \tag{4.28}$$

This takes a while to equilibrate - the model depends on the time average above.

At last, we can quantitatively describe the movement of the ion in the trap: We assume that the set $\{\Delta_m\}$ to be used in (4.16) is distributed as a normal distribution with mean value equal to zero and with the deviation given by (4.28). This, in principle, would model both the random aspect

of an ion's movement but also the fact that it is trapped in a potential. The expression (4.28) allows us to quickly determine the parameters for better write-in vs. read-out fidelity: Lower temperature lowers the speed of the ions while smaller distances between the ions traps the individual ion tighter. Finally, more ions in the crystal traps the ions even tighter, although this contribution is very small and not expected to grow significantly with more ions.

We observe, furthermore, that the distance between the ions can be obtained from the density given *a priori* from the experimental parameters [Herskind *et al.*]:

$$\rho = \frac{\epsilon_0 U_{rf}^2}{m r_0^4 \Omega_{rf}^2} \ , \tag{4.29}$$

where U_{rf} and Ω_{rf} are the voltage and trap frequency characterizing the Paul trap and where r_0 is the distance from the trap center to electrodes that carry the current making the electric potential.

4.3.3 Fidelity of storing a single photon

In the following we take a crystal for the quantum interface to consist of 500 ions on a string. This would be a string $500 \cdot 0.02\mu\text{m} = 10\text{mm}$ long which obviously exceeds the dimensions of the cavity - but it is of no consequence for the ideas presented in the following.

Fidelity is an overlap of two functions, and is defined by $\mathcal{F} \equiv |\langle\psi_{out}|\psi_{in}\rangle|^2$. This definition pertains to pure states as we have in this experiment. Using (4.16) and (4.11) we get

$$\mathcal{F} = \Big| \sum_k c_k^* \langle vac.|\langle 0,...,0|\hat{a}_k \hat{a}_{k_0}^\dagger |0,...,0\rangle|vac.\rangle \Big| \tag{4.30}$$

$$= |c_{k_0}| = \Big| \frac{1}{\mathcal{N}} \sum_m \sin(k_0 z_m(t)) \sin(k_0 z_m(0)) \Big| \ , \tag{4.31}$$

since $\langle vac.|\hat{a}_k \hat{a}_{k_0}^\dagger |vac.\rangle = \delta_{k,k_0}$. Plots of the fidelity as a function of density of ions are obtained for different values of the temperature in Figure 4.10. How to actually generate the plot is described in appendix A, where we have chosen densities and temperatures around the experimental values quoted in [Herskind *et al.*]. It is clear that the current experimental setup is inadequate. As expected the fidelity is low given the thermal motion of the ions. Although, it seems as if we only need to take the temperature down by a factor of a hundred or raise the density by a factor of ten to be in an interesting regime. The experimentalists in Aarhus already promises somewhat larger crystals but making the density higher by a factor of 10 or more is not realistic. Lowering the temperature by as much as a two orders of magnitude is properly not realistic either since the ions continously heat up in the trap.

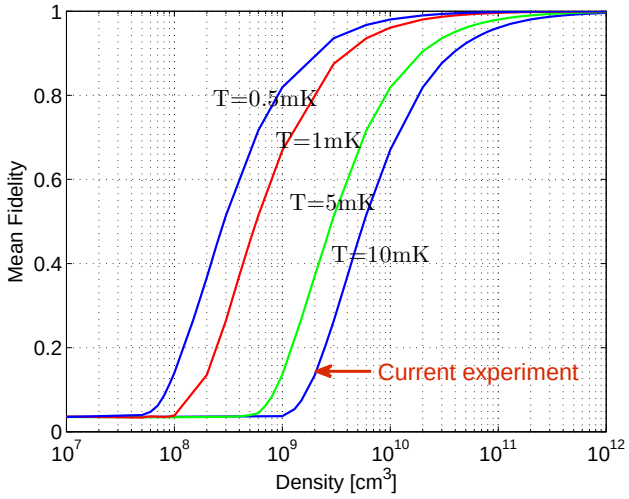

Figure 4.10: The Mean Fidelity of the write-in followed by the read-out process modeled by the sinusoidal wavefunction in standing wave cavity plotted as a function of the density of ions. Each point making up the curve is an average of 500 realisations. The different curves are for different temperatures and the current experiment is pointed out.

The Lamb-Dicke limit. The above analysis is another way of stating the so-called Lamb-Dicke limit: The ions are cooled so low that the individual ion motion is much smaller than the wavelength of light that is used to excite the desired transition. Formally:

$$k\sqrt{\langle x^2 \rangle} \ll 1 \ . \tag{4.32}$$

For quantum computation with a couple of ions and their coupling via the vibrational sidebands this limit must be met. In our system with thousands of ions that are shuffled together heating will allways be an issue. The rest of this thesis could be considered as an attempt to answer the question: Since we cannot get to the Lamb-Dicke limit, then what?

Chapter 5

Write-in

In the previous chapter we saw how we cannot expect the same photonic waveform that is written-in to be read-out. The goal, then, is construct a symmetric Dicke-state in the ions during the write-in independent of where each ion is in relation to the standing wave of the field during the interaction. This will give high efficiency in the write process of the quantum repeater. It means that we always couple to a symmetric state such that the protocol is insensitive to thermal motion.

The standing wave in the cavity

First, we let us state the problem in more detail. On Figure 5.1 we have drawn, once again, the one-dimensional potential as seen by an ion together with the second order Taylor expansion. Because of the cavity there will be a standing wave with two tops and two crests per wavelength. Consequently, there will be dramtic changes within, say, 200 nm of the electro.magnetic field. Given the analysis in the previous chapter the most probable amplitude of the simple harmonic oscillations will be on the order of 1μm with a period of around $5 \cdot 10^{-6}$s at $T \sim 10$mK. This should be compared to the interaction time that is on the order of 5μs [Kuhn *et al.* 1]. In summary, each individual ion *will* see dramatic changes in the field during the interaction, immediately after, and in the time between the write-in and the read-out.

We can illustrate the implication of this motion of each ion by 'drawing' the wavefunction of the write-in and read-out as in Figure 5.2. Here the amplitude squared of the contribution to the wavefunction is drawn as a function of the indices of the individual ions. If the ions are just left to thermally move in their potential the wavefunction cannot look like a Dicke State. The write-in wavefunction will have contributions that vary pseudo-randomly in size over the ions and a short moment later ($\sim 10^{-6}$s) the wavefunction has a new distribution of contributions from the ions.

Finally, in figure 5.3 is drawn the Dicke-state that is the result we look for. Here each ion contributes equally to the wavefunction describing the

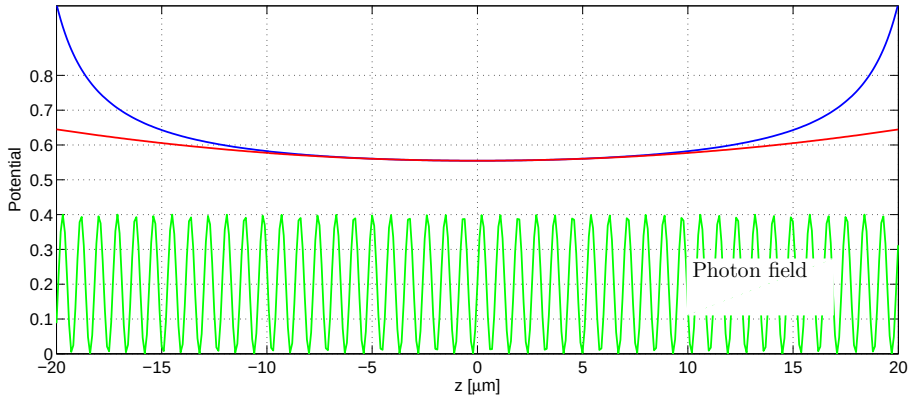

Figure 5.1: The blue curve is the one-dimensional Coulomb potentional as seen by a single ion with two neighbourions. The red curve is the sencond order Taylor expansion of the potential and the green is the standing wave of the light field. All curves are drawn as a function of distance and the potential is given in arbitrary units.

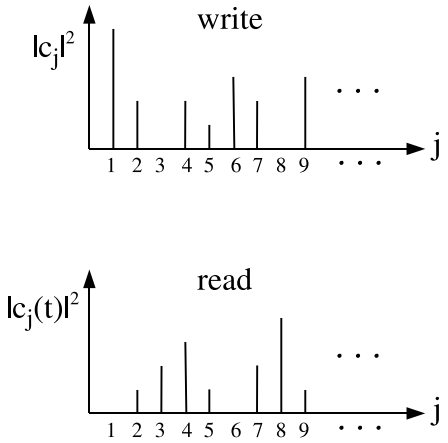

Figure 5.2: An illustration of the write-in and read-out wavefunctions. The magnitude of the $j'th$ ions coupling to the field is graphed as a funtion of j.

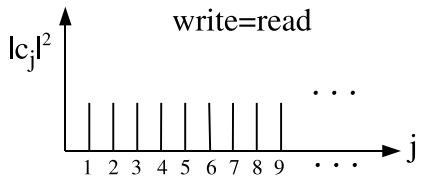

Figure 5.3: The magnitude of the $j'th$ ions coupling to the field is now a constant as a funtion of j for a symmetric Dicke state.

collective excitation. The main line of our argument is that this state is attainable by suitably moving the ions during the interaction with light. All that should be observed is that it is done fast enough such that the collective movement dominates the individual movement of each ion. The result should be that each ion will interact with the field in the same amount.

5.1 Equations of motion

We proceed by calculating the interaction. The beginning is the ensemble Hamiltonian derived earlier:

$$\hat{H} = \hbar \sum_{j=1}^{\text{ions}} \hat{\sigma}_{ee} - \hbar \Big(\sum_{j=1}^{\text{ions}} \Omega^{(j)}(t) \hat{\sigma}_{eg}^{(j)} + \sum_{j=1}^{\text{ions}} g^{(j)}(t) \hat{\sigma}_{es}^{(j)} \; + \; H.c. \Big) \; . \tag{5.1}$$

In the Heisenberg picture the equations of motion can be found via the Heisenberg equation of motion:

$$i\hbar \frac{d}{dt} \hat{L}(t) = \big[\hat{L}(t), \hat{H}(t) \big] + i\hbar \frac{\partial \hat{L}(t)}{\partial t} \; , \tag{5.2}$$

where $\hat{L}(t)$ is an operator which we shall take *not* to be *explicitly* time-dependent such that the partial derivative with respect to time is identically

zero. Calculating the seven pertinent operators gives:

$$\begin{aligned}
\frac{d}{dt}\hat{a} &= i\sum_j g^{(j)}\hat{\sigma}_{se}^{(j)} \\
\frac{d}{dt}\sum_j \hat{\sigma}_{ee}^{(j)} &= i\sum_j \left(\Omega^{(j)}\hat{\sigma}_{eg}^{(j)} - \Omega^{*(j)}\hat{\sigma}_{ge}^{(j)} + g^{(j)}\hat{a}\hat{\sigma}_{es}^{(j)} - g^{(j)}\hat{a}^\dagger\hat{\sigma}_{se}^{(j)}\right) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{gg}^{(j)} &= i\sum_j \left(\Omega^{*(j)}\hat{\sigma}_{ge}^{(j)} - \Omega^{(j)}\hat{\sigma}_{eg}^{(j)}\right) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{ss}^{(j)} &= i\sum_j \left(g^{(j)}\hat{a}^\dagger\hat{\sigma}_{se}^{(j)} - g^{(j)}\hat{a}\hat{\sigma}_{es}^{(j)}\right) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{ge}^{(j)} &= i\sum_j \left(\Omega^{(j)}\left(\hat{\sigma}_{gg}^{(j)} - \hat{\sigma}_{ee}^{(j)}\right) + g^{(j)}\hat{a}\hat{\sigma}_{gs}^{(j)} - i\Delta\hat{\sigma}_{ge}^{(j)}\right) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{gs}^{(j)} &= i\sum_j \left(g^{(j)}\hat{a}^\dagger\hat{\sigma}_{ge}^{(j)} - \Omega^{(j)}\hat{\sigma}_{es}^{(j)}\right) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{es}^{(j)} &= i\sum_j \left(i\Delta\hat{\sigma}_{es}^{(j)} - g^{(j)}\hat{a}^\dagger\hat{\sigma}_{ss}^{(j)} + \Omega^{*(j)}\hat{\sigma}_{gs}^{(j)}\right) .
\end{aligned}$$

Here we have summed over the atomic operators as we should do in the ensemble model of the interaction.

5.1.1 Noise

Noise in this system comes from two processes: First, there is the flux of photons in an out of the cavity with respect to the field. Second, there is decay with respect to the atomic operators. We can describe the noise in the field via the result from the classical analysis of the input-output relations of the cavity performed earlier. For the atomic operators we introduce the decay out of the respective energy level accompanied by the relevant Langevin operator [Scully and Zubairy] for the $j'th$ atom, $\hat{F}_{kl}^{(j)}$, while still

supressing j and hats on operators. Then the equations of motion become:

$$\begin{aligned}
\frac{d}{dt}\hat{a} &= -\big(\kappa + i\Delta_{cav}\big)\hat{a}(t) + i\sum_j g^{(j)}\hat{\sigma}_{se} + \sqrt{\frac{2\kappa}{\tau}}\hat{a}_{in}(t) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{ee}^{(j)} &= \sum_j \Big(-\gamma_e\hat{\sigma}_{ee}^{(j)} + i\Omega^{(j)}\hat{\sigma}_{eg}^{(j)} - i\Omega^{*(j)}\hat{\sigma}_{ge}^{(j)} + ig^{(j)}\hat{a}\hat{\sigma}_{es}^{(j)} - ig^{(j)}\hat{a}^\dagger\hat{\sigma}_{se}^{(j)} \\
&\qquad + \hat{F}_{ee}^{(j)}\Big) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{gg}^{(j)} &= \sum_j \Big(\gamma_{eg}\hat{\sigma}_{ee}^{(j)} + i\Omega^{*(j)}\hat{\sigma}_{ge}^{(j)} - i\Omega^{(j)}\hat{\sigma}_{eg}^{(j)} + \hat{F}_{gg}^{(j)}\Big) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{ss}^{(j)} &= \sum_j \Big(\gamma_{es}\hat{\sigma}_{ee}^{(j)} + ig^{(j)}\hat{a}^\dagger\hat{\sigma}_{se}^{(j)} - ig^{(j)}\hat{a}\hat{\sigma}_{es}^{(j)} + \hat{F}_{ss}^{(j)}\Big) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{ge}^{(j)} &= \sum_j \Big(-\gamma\hat{\sigma}_{ge}^{(j)} + i\Omega^{(j)}\big(\hat{\sigma}_{gg}^{(j)} - i\hat{\sigma}_{ee}^{(j)}\big) + ig^{(j)}\hat{a}\hat{\sigma}_{gs}^{(j)} - i\Delta\hat{\sigma}_{ge}^{(j)} + \hat{F}_{ge}^{(j)}\Big) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{gs}^{(j)} &= \sum_j \Big(-\gamma_s\hat{\sigma}_{gs}^{(j)} + ig^{(j)}\hat{a}^\dagger\hat{\sigma}_{ge}^{(j)} - i\Omega^{(j)}\hat{\sigma}_{es}^{(j)} + \hat{F}_{gs}^{(j)}\Big) \\
\frac{d}{dt}\sum_j \hat{\sigma}_{es}^{(j)} &= \sum_j \Big(-\gamma'\hat{\sigma}_{es}^{(j)} + i\Delta\hat{\sigma}_{es}^{(j)} - ig^{(j)}\hat{a}^\dagger\hat{\sigma}_{ss}^{(j)} + i\Omega^{*(j)}\hat{\sigma}_{gs}^{(j)} + \hat{F}_{es}^{(j)}\Big) \ .
\end{aligned}$$

In the following we elaborate on the description in [Gorshkov *et al.*]. The decays are described as follows: The radiative decay out of the excited state is $\gamma_e = \gamma_{es} + \gamma_{eg}$ which in turn is the sum of the decays from the excited state into the two stable states. $\gamma = \gamma_e/2 + \gamma_{deph}$ is the decay rate of the optical coherence, where, in principle, dephasing can take place, given perhaps by collisions between the atoms. Dephasing is also present in $\gamma' = \gamma_e/2 + \gamma'_{deph}$, while the decay γ_s is given entirely by dephasing.

It is given that for any Langevin operator $\langle\hat{F}(t)\rangle = 0$. Therefore we consider only products of operators. We work in a perturbation regime and will need only the products of two such Langevin operators. In all of our analysis hitherto we have presumed that all the atoms are in the ground state. This can conveniently be achieved with optical pumping. This means that we only need to worry about the noise from $\hat{F}_{ge}, \hat{F}_{ge}^\dagger, \hat{F}_{gs}$, and $\hat{F}_{gs}^\dagger$ as long as there is no decay from the ground state. We proceed by using the generalised Einstein relations [Hald *et al.*]

$$\langle\hat{F}_{\mu\nu}(t)\hat{F}_{\alpha\beta}(t')\rangle = \langle D(\hat{\sigma}_{\mu\nu}\hat{\sigma}_{\alpha\beta}) - D(\hat{\sigma}_{\mu\nu})\hat{\sigma}_{\alpha\beta} - \hat{\sigma}_{\mu\nu}D(\hat{\sigma}_{\alpha\beta})\rangle\delta(t-t') \ . \quad (5.3)$$

$D(\hat{\sigma}_{\mu\nu})$ is the deterministic part of the equation of motion for $\dot{\hat{\sigma}}_{\mu\nu}$, that is, without the Langevin operators. By treating each Langevin operator for each atom individually we have assumed that different atoms couple to independent vacuum modes [Hald *et al.*]. Now the correlation functions can

be computed for each j e.g.:

$$\begin{aligned}\langle \hat{F}_{ge}(t)\hat{F}^{\dagger}_{ge}(t')\rangle &= \langle \hat{F}_{ge}(t)\hat{F}_{eg}(t')\rangle \\ &= \langle D(\hat{\sigma}_{ge}\hat{\sigma}_{eg}) - D(\hat{\sigma}_{ge})\hat{\sigma}_{eg} - \hat{\sigma}_{ge}D(\hat{\sigma}_{eg})\rangle\delta(t-t') \\ &= \langle \gamma_{eq}\hat{\sigma}_{ee} + 2\gamma\hat{\sigma}_{gg}\rangle\delta(t-t') \simeq 2\gamma\delta(t-t') \ .\end{aligned}$$

Here $\hat{\sigma}_{ge}\hat{\sigma}_{eg} = \hat{\sigma}_{gg}$ and $\hat{\sigma}_{gg} - \hat{\sigma}_{ee} \simeq \hat{\sigma}_{gg} \simeq 1$, since we are treating each atom individually. Computing the sixteen combinations gives only two non-zero terms and they are both *antinormally* ordered. The other non-zero term is

$$\langle \hat{F}_{gs}(t)\hat{F}^{\dagger}_{gs}(t')\rangle \simeq 2\gamma_s\delta(t-t') \ .$$

Consequently, we can ignore e.g. $\hat{F}_{se}$ as it is done below. The importance of the fact that normally ordered correlations are zero is that the noise in the system is vacuum noise, that is, only loss out of the system that mixes in vacuum. This can also be seen from the equations of motion where we have not included decay from the ground state into the states $|e\rangle$ and $|s\rangle$. It is a reasonable assumption since the energy gap to the excited state is in the optical regime and since the transistion from $|e\rangle$ to $|s\rangle$ is 'dipole forbidden'.

5.2 Solving for the field operator

5.2.1 The relevant system of three coupled equation

The target for these calculations is to find an expression for the product $\hat{a}^{\dagger}\hat{a}$ as a funtion of the interaction with the atoms. From the equations of motion a system of three coupled equations with three unknowns $\hat{a}, \hat{\sigma}_{gs}, \hat{\sigma}_{se}$ can be found (using that $\hat{\sigma}_{se}$ is the adjoint to $\hat{\sigma}_{es}$):

$$\frac{d}{dt}\hat{a} = -\big(\kappa + i\Delta_{cav}\big)\hat{a}(t) + i\sum_j g^{(j)}\hat{\sigma}^{(j)}_{se} + \sqrt{\frac{2\kappa}{\tau}}\hat{a}_{in}(t) \tag{5.4}$$

$$\frac{d}{dt}\sum_j \hat{\sigma}^{(j)}_{se} = \sum_j \Big(-(\gamma' + i\Delta)\hat{\sigma}^{(j)}_{se} + ig^{(j)}\hat{a}\hat{\sigma}^{(j)}_{ss} + i\Omega^{(j)}\hat{\sigma}^{(j)}_{sg}\Big) \tag{5.5}$$

$$\frac{d}{dt}\sum_j \hat{\sigma}^{(j)}_{gs} = \sum_j \Big(-\gamma_s\hat{\sigma}^{(j)}_{gs} + ig^{(j)}\hat{a}^{\dagger}\hat{\sigma}^{(j)}_{ge} - i\Omega^{(j)}\hat{\sigma}^{(j)}_{es} + \hat{F}^{(j)}_{gs}\Big) \ . \tag{5.6}$$

Here, obviously, there is a fourth unknown: $\hat{\sigma}^{(j)}_{ge}$. If the interaction is sufficiently weak such that it can be regarded as a perturbation we can get rid of this fourth unknown as we shall show below.

5.2.2 The interaction as a perturbation

Compare $ig\hat{a}^{\dagger}\hat{\sigma}_{ge}$ to the other atomic operators figuring in equation (5.6), namely $i\Omega\hat{\sigma}_{es}$ and $\gamma_s\hat{\sigma}_{gs}$. The real decay is important and is kept, so the

important comparison is between the two imaginary terms. Here Ω can be chosen such that it is much larger than $g\hat{a}$. Aditionally, the action of the operator $\hat{\sigma}_{ge}$ essentially only 'happens once'. A large detuning will allow a large Ω while keeping the actual number of transitions very small. Consequently, we can ignore $ig\hat{a}^\dagger\hat{\sigma}_{ge}$.

In line with the comments above we can take $\hat{\sigma}_{ss} \simeq 0$. Also, we shall consider the input of the quantum field to be zero – this is before the interaction that produces the quantum field. Finally, then, we must solve the following system where we only need the summation over the atomic operators in the field equation:

$$\dot{\hat{a}} = -\big(\kappa + i\Delta_{cav}\big)\hat{a}(t) + i\sum_j g^{(j)}\hat{\sigma}_{se}^{(j)} \tag{5.7}$$

$$\dot{\hat{\sigma}}_{se}^{(j)} = -(\gamma' + i\Delta)\hat{\sigma}_{se}^{(j)} + i\Omega^{(j)}\hat{\sigma}_{sg}^{(j)} \tag{5.8}$$

$$\dot{\hat{\sigma}}_{gs}^{(j)} = -\gamma_s\hat{\sigma}_{gs}^{(j)} - i\Omega^{(j)}\hat{\sigma}_{es}^{(j)} + \hat{F}_{gs}^{(j)} \ . \tag{5.9}$$

In the above equation we cannot just consider a steady state and set the differentiations equal to zero, since this would in effect decouple light and atoms completely. Instead we shall do formal integrations of the equations of motion. What we will do is to ignore the equation of motion for $\hat{\sigma}_{sg}$ and eventually any hint of noise in the atomic operators dissapears. The physical interpretation is that the only noise is vacuum mixed in because of spontaneous emission from the excited level out of the system. With our approximations this in effect eliminates the excited level. More precisely we integrate (5.7) and (5.8):

$$\hat{a}(t) = \hat{a}(0)\mathrm{e}^{-(\kappa+i\Delta_{cav})t} + \int_0^t \mathrm{d}t'\mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')}i\sum_j g^{(j)}(t')\hat{\sigma}_{se}^{(j)}(t') \ ,$$

$$\hat{\sigma}_{se}^{(j)}(t) = \hat{\sigma}_{se}^{(j)}(0)\mathrm{e}^{(-\gamma'+i\Delta)t} + \int_0^t \mathrm{d}t'\mathrm{e}^{(-\gamma'+i\Delta)(t-t')}\big[ig^{(j)}\hat{a}(t') + \Omega^{(j)}\hat{\sigma}_{sg}^{(j)}(t')\big] \ .$$

Here it is understood that each operator is a the tensor product of operators e.g. $\hat{a} \to \hat{\mathbb{1}}_A \otimes \hat{a}_L$ and $\hat{\sigma}_A \to \hat{\sigma}_A \otimes \hat{\mathbb{1}}_L$, where subscripts refer to the Hilbert space describing the atom and the light respectively.

Invoking the interaction as a perturbation we take $\hat{a}(0) = 0$, $\hat{\sigma}_{se}(0) = 0$, and $\hat{\sigma}_{sg}(t) = \hat{\sigma}_{se}(0)$ =constant. Then we plug the result for the atomic operator into the field equation. In the process we take $g^2\hat{a} = 0$ in the perturbative limit and the result is

$$\hat{a}(t) = i\sum\int_0^t \mathrm{d}t'\int_0^{t'} \mathrm{d}t''\mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')}\mathrm{e}^{(-\gamma'+i\Delta)(t'-t'')}g^{(j)}(t')\Omega^{(j)}(t'')\hat{\sigma}_{sg}^{(j)}(0) \ . \tag{5.10}$$

To ease later computations we shall write this as

$$\hat{a}(t) = i\sum_j \theta_j(t)\hat{\sigma}_{sg}^{(j)}(0) \tag{5.11}$$

$$\theta_j = \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')} \mathrm{e}^{(-\gamma'+i\Delta)(t'-t'')} g^{(j)}(t')\Omega^{(j)}(t'') \ .$$

5.3 Write-in

The above description of the system and the interaction with light is quite general. The specific write-in process in the quantum repeater scheme is our next subject. As we saw earlier the quantum repeater is initialised when one of the detectors detects a photon from either of the two distinct ensembles interacting simultaneously with light. The successful write-in is therefore conditioned on a click in a detector accompanied by a specific configuration of the atomic system. Here we shall simplify the discription and look at only one ensemble; the successful write-in is then a click in a detector from light interacting with this single ensemble.

5.3.1 The Fidelity of a successful write-in

The question we wish to answer is this: Given a photon in the quantum field what is the Fidelity that the ensemble is in the desired one excitation Dicke State? This fidelity is then a number for the successful read-out of the excitation into the Anti-Stokes mode and thus into the a cavity mode that can be effectively coupled out and subsequently detected.

Fidelity. In quantum information the Fidelity is a measure of the overlap of two wavefunctions, that is, how close two vectors lie in a Hilbert space. It is a positive number and normalised such that $0 \leq \mathcal{F} \leq 1$. The definition of the fidelity for pure states follows from the theory of Hilbert spaces

The Dicke states are pure states so we can write the fidelity for our problem as

$$\mathcal{F} = \langle\psi_{\mathrm{W}}|\hat{\rho}_A^{\mathrm{out}}|\psi_{\mathrm{W}}\rangle \ , \tag{5.12}$$

where $|\psi_{\mathrm{W}}\rangle$ is defined in (3.43) and $\hat{\rho}_A$ is the density matrix of the atomic ensemble after the interaction. The whole system initially is described by

$$|\psi_{\mathrm{in}}\rangle = |0\rangle_A \otimes |0\rangle_L \ . \tag{5.13}$$

We take the out wavefunction to be a unitary evolution of the ground state wavefunction:

$$|\psi_{\mathrm{out}}\rangle = \hat{U}(t)|\psi_{\mathrm{in}}\rangle \ , \tag{5.14}$$

where $\hat{U}(t) \simeq \mathbb{1}$, the unit operator, since the probability of getting a single excitation in the ensemble is kept low. This justifies the approach taken that the interaction is just a perturbation. In a similar fashion the field operator is described by the unitary evolution

$$\hat{a}(t) = \hat{U}^{\dagger}(t)\hat{a}(0)\hat{U}(t) \ . \tag{5.15}$$

We shall only consider the contribution from the quantum field with a single excitation. This means that after taking the trace over the light field we get for the atomic ensemble the wave function:

$$\begin{aligned} |\psi^{\text{out}}(t)\rangle_A &= \frac{1}{\sqrt{p_1(t)}} \mathbb{1}_A \otimes |1\rangle\langle 1|_L \hat{U}(t)|0\rangle_A \otimes |0\rangle_L \\ &= \frac{1}{\sqrt{p_1(t)}} \hat{\mathbb{P}}\hat{U}(t)|0\rangle_A \otimes |0\rangle_L \ , \end{aligned} \tag{5.16}$$

where $|1(t)\rangle$ is the state of the quantum field with one excitation and $\hat{\mathbb{P}}$ is the projection operator $\mathbb{1}_A \otimes |1\rangle\langle 1|_L$. We take the out-coupling of the cavity to be lossless and the detection to be perfect and call the successful process a 'click' in a detector. Therefore the out wavefunction is normalised with $[p_1(t)]^{-1}$ where $p_1(t)$ is the probability of the click in the detector.

Now we can find the out wavefunction for the atomic ensemble $|\psi_A^{\text{out}}(t)\rangle$ in a form that connects to the calculations of the equations of motion we have done so far (here written a little more compactly):

$$\begin{aligned} \sqrt{p_1(t)}|\psi_A^{\text{out}}(t)\rangle &= {}_L\langle 1|\hat{U}(t)|\psi_{\text{in}}\rangle & (5.17)\\ &= {}_L\langle 0|\hat{a}(0)\hat{U}(t)|\psi_{\text{in}}\rangle & (5.18)\\ &= {}_L\langle 0|\hat{U}(t)\hat{U}^{\dagger}(t)\hat{a}(0)\hat{U}(t)|\psi_{\text{in}}\rangle & (5.19)\\ &= {}_L\langle 0|\hat{U}(t)\hat{a}(t)|\psi_{\text{in}}\rangle & (5.20)\\ &\simeq {}_L\langle 0|\hat{a}(t)|\psi_{\text{in}}\rangle \ , & (5.21) \end{aligned}$$

since $\hat{U}^{\dagger}(t)\hat{U}(t) = \hat{U}(t)\hat{U}^{\dagger}(t) = \mathbb{1}$ and $\hat{U}(t) \simeq \mathbb{1}$ in the perturbative limit. From the equation of motion for $\hat{a}(t)$, equation (5.12), we get

$$\begin{aligned} \sqrt{p_1(t)}|\psi_A^{\text{out}}(t)\rangle &= {}_L\langle 0|i\sum_j^{N_A} \theta_j \hat{\sigma}_{sg}^{(j)}|\psi_{\text{in}}\rangle & (5.22)\\ &= i\sum_j^{N_A} \theta_j(t)\hat{\sigma}_{sg}^{(j)}|g,...,g\rangle_A & (5.23)\\ &= i\sum_j^{N_A} \theta_j(t)|g_1,...,s_j,...,g_{N_A}\rangle_A \ , & (5.24) \end{aligned}$$

where we can ignore the light field since ${}_L\langle 0|0\rangle_L = 1$. From this expression we can derive the density matrix of the ensemble after all of the interaction.

It is a mixture of the pure states generated when a 'click' is recorded at time $t \in [0, t_c]$, where we treat time as a classical random variable. Then we integrate in time over the duration of the interaction from $t = 0$ to $t = t_c$:

$$\begin{aligned}\hat{\rho}_A^{\text{out}} &= \frac{1}{p_{\text{tot}}} \int_0^{t_c} dt\ p_1(t) |\psi_A^{\text{out}}(t)\rangle\langle\psi_A^{\text{out}}(t)| \\ &= \frac{1}{p_{\text{tot}}} \int_0^{t_c} dt {}_L\langle 0|\hat{a}(t)|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|\hat{a}^\dagger(t)|0\rangle_L \\ &= \frac{1}{p_{\text{tot}}} \int_0^{t_c} dt \sum_j^{N_A}\sum_k^{N_A} \theta_j^*(t)\theta_k(t) |g_1, ..., s_j, ..., g_{N_A}\rangle_{A\,A}\langle g_1, ..., s_k, ..., g_{N_A}| \ .\end{aligned}$$

Furthermore, p_{tot} is the integral over $p_1(t)$ that normalises $p_1(t)$:

$$\begin{aligned}p_{\text{tot}} &= \int_0^{t_c} dt\ p_1(t) \\ &= \int_0^{t_c} dt \langle\psi_{\text{in}}|\hat{a}^\dagger(t)|0\rangle_{L\,L}\langle 0|\hat{a}(t)|\psi_{\text{in}}\rangle \\ &= \int_0^{t_c} dt \sum_j^{N_A}\sum_k^{N_A} \theta_j^*(t)\theta_k(t)\, {}_A\langle g_1, ..., s_j, ..., g_{N_A}|g_1, ..., s_k, ..., g_{N_A}\rangle_A \\ &= \int_0^{t_c} dt \sum_j^{N_A}\sum_k^{N_A} \theta_j^*(t)\theta_k(t)\delta_{lk} \\ &= \sum_j^{N_A} \int_0^{t_c} dt |\theta_j(t)|^2 \ .\end{aligned}$$

Here we have used the orthogonality of $|g_1, ..., s_j, ..., g_{N_A}\rangle_A$ and $|g_1, ..., s_k, ..., g_{N_A}\rangle_A$, where $j \neq k$. Now we have all the information we need in order to find the Fidelity:

$$\begin{aligned}\mathcal{F} &= \langle\psi_{\text{W}}|\hat{\rho}_A^{\text{out}}|\psi_{\text{W}}\rangle \\ &= \frac{1}{N_A p_{\text{tot}}} \int_0^{t_c} dt \sum_j^{N_A}\sum_{j'}^{N_A}\sum_k^{N_A}\sum_{k'}^{N_A} \theta_j^*(t)\theta_k(t)\, {}_A\langle g_1, ..., s_{j'}, ..., g_{N_A}|g_1, ..., s_j, ..., g_{N_A}\rangle_A \\ &\quad \times {}_A\langle g_1, ..., s_k, ..., g_{N_A}|g_1, ..., s_{k'}, ..., g_{N_A}\rangle_A \\ &= \frac{1}{N_A p_{\text{tot}}} \int_0^{t_c} dt \sum_j^{N_A}\sum_{j'}^{N_A}\sum_k^{N_A}\sum_{k'}^{N_A} \theta_j^*(t)\theta_k(t)\delta_{jj'}\delta_{kk'} \\ &= \frac{\displaystyle\sum_j^{N_A}\sum_k^{N_A} \int_0^{t_c} dt\ \theta_j^*(t)\theta_k(t)}{\displaystyle N_A \sum_j^{N_A} \int_0^{t_c} dt |\theta_j(t)|^2} \ ,\end{aligned} \tag{5.25}$$

where $|\psi_{\mathrm{W}}\rangle$ is given by (3.43) and where the θ's are known from (5.12). It is apparent that if the position of the ions is unimportant, then the Fidelity reduces to unity. In principle this is the expression with which we can evaluate how effective the quantum interface is. If we consider the expression for θ_j,

$$\theta_j(t) = \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')} \mathrm{e}^{(-\gamma'+i\Delta)(t'-t'')} g^{(j)}(t')\Omega^{(j)}(t'') \ , \quad (5.26)$$

we see that the important parameters are the j-dependent couplings $g^{(j)}(t')$ and $\Omega^{(j)}(t'')$ where the j dependence is a function of the position of the j'th atom with respect to the nodes and tops of the electromagnetic field. We shall take both of the couplings to have the same j-dependence and in the direction along the optical axis in the cavity we have:

$$g^{(j)}(t) \propto \sin(k_q z^{(j)}(t)) \quad (5.27)$$

$$\Omega^{(j)}(t) \propto \sin(k_c z^{(j)}(t)) \quad (5.28)$$

$$\Rightarrow g^{(j)}(t) \propto \sin[k_q(z^{(j)}(t=0) + v^{(j)}(t)t + v(t)t)] \quad (5.29)$$

$$\Omega^{(j)}(t) \propto \sin[k_c(z^{(j)}(t=0) + v^{(j)}(t)t + v(t)t)] \ , \quad (5.30)$$

where $v^{(j)}(t)$ is the velocity of the individual ion with respect to the electromagnetic field and where $v(t)$ is an overall velocity identical for all the ions in the Coulomb crystal.

5.3.2 An analytical solution for a constant speed

One important case can be solved analytically in a straight forward manner. First we ignore the variation in the coupling due to displacement in the xy-plane. In equation (3.18) we have described change in the xy-plane by a slowly varying Gaussian envelope and the change in the z-direction, the standing wave at an optical frequency, dominates the spatial dependence.

From equations (5.27)-(5.30) we have the z-dependence and given $z^{(j)}(t = 0)$, the functional form of $v^{(j)}(t)$, and $v(t)$ the integral can be computed. Also, since all the j-dependence is accounted for in equations (5.27)-(5.30) this is all we will retain for the following computations. This means that we must compute

$$\theta_j(t) \propto \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')} \mathrm{e}^{(-\gamma'+i\Delta)(t'-t'')} \quad (5.31)$$
$$\times \sin[k_q(z_0^{(j)} + v^{(j)}(t')t' + v(t')t')] \sin[k_c(z_0^{(j)} + v^{(j)}(t'')t'' + v(t'')t'')] \ .$$

As in the the previous chapters we will consider the velocity of the individual ion as a constant given by a the most probable speed. In turn this velocity is a Gaussian distribution centered around the mean speed, but what is

important here is that it is constant for the j-th ion and the fact that it is a relatively narrow distribution such that most of the ions will move with speeds that are less than 5 $\frac{m}{s}$ at $T \sim 10$ mK. So, $v^{(j)}(t) = v^{(j)}(t=0) < 5\ \frac{m}{s}$. Finally, we consider the case where the overall velocity of the whole crystal is constant $v(t) = v$ and where the velocity of the individual ion is also constant but different for each ion. Now the problem is straight forward with

$$\theta_j(t) \propto \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')} \mathrm{e}^{(-\gamma'+i\Delta)(t'-t'')} \times \sin[k_q(z_j + v_j t')] \sin[k_c(z_j + v_j t'')] \ , \tag{5.32}$$

where $v_j = v + v^{(j)}$ and $z_j = z^{(j)}(t=0)$. Rewriting this expression using the Euler formula for the sine function gives

$$\theta_j(t) \propto \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')} \mathrm{e}^{(-\gamma'+i\Delta)(t'-t'')} \times \left\{ \mathrm{e}^{ik_q(z_j+v_j t')} - \mathrm{e}^{-ik_q(z_j+v_j t')} \right\} \left\{ \mathrm{e}^{ik_c(z_j+v_j t'')} - \mathrm{e}^{-ik_c(z_j+v_j t''c)} \right\} . \tag{5.33}$$

After some re-arranging,

$$\theta_j(t) \propto \int_0^t \mathrm{d}t' \mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')} \mathrm{e}^{(-\gamma'+i\Delta)t'} \left\{ \mathrm{e}^{ik_q(z_j+v_j t')} - \mathrm{e}^{-ik_q(z_j+v_j t')} \right\} \times \int_0^{t'} \mathrm{d}t'' \mathrm{e}^{-(-\gamma'+i\Delta)t''} \left\{ \mathrm{e}^{ik_c(z_j+v_j t'')} - \mathrm{e}^{-ik_c(z_j+v_j t'')} \right\} , \tag{5.34}$$

we can perform the integration over t'':

$$\theta_j(t) \propto \int_0^t \mathrm{d}t' \mathrm{e}^{-(\kappa+i\Delta_{cav})(t-t')} \mathrm{e}^{(-\gamma'+i\Delta)t'} \left\{ \mathrm{e}^{ik_q(z_j+v_j t')} - \mathrm{e}^{-ik_q(z_j+v_j t')} \right\} \times \mathrm{e}^{-(-\gamma'+i\Delta)t'} \left\{ \frac{\mathrm{e}^{ik_c(z_j+v_j t')} - 1}{-(-\gamma'+i\Delta) + ik_c v_j} - \frac{\mathrm{e}^{-ik_c(z_j+v_j t'')} - 1}{-(-\gamma'+i\Delta) - ik_c v_j} \right\} .$$

Then the integration over t' can be done. We drop eight terms that are proportional to $\mathrm{e}^{-\kappa t}$ since we are interested in times long enough such that the decay of the cavity has allowed for the detection of the photon, that is,

$\mathrm{e}^{-\kappa t} \simeq 0$ Thus, we are left with these eight terms:

$$\theta_j(t) \propto \frac{\mathrm{e}^{i(k_c+k_q)z_j}}{-\gamma' + i(\Delta + k_c v_j)} \frac{1}{\kappa + i\Delta_{cav} + i(k_q + k_c)v_j} \mathrm{e}^{i(k_q+k_c)v_j t} \tag{5.35}$$

$$- \frac{\mathrm{e}^{i(k_c+k_q)z_j}}{-\gamma' + i(\Delta + k_c v_j)} \frac{1}{\kappa + i\Delta_{cav} + i(-\Delta + k_q v_j)} \mathrm{e}^{i(-\Delta+k_q v_j)t} \tag{5.36}$$

$$- \frac{\mathrm{e}^{i(k_c-k_q)z_j}}{-\gamma' + i(-\Delta + k_c v_j)} \frac{1}{\kappa + i\Delta_{cav} + i(k_q - k_c)v_j} \mathrm{e}^{i(k_q-k_c)v_j t} \tag{5.37}$$

$$+ \frac{\mathrm{e}^{i(k_q-k_c)z_j}}{-\gamma' + i(\Delta - k_c v_j)} \frac{1}{\kappa + i\Delta_{cav} + i(-\Delta + k_q v_j)} \mathrm{e}^{i(-\Delta+k_q v_j)t} \tag{5.38}$$

$$- \frac{\mathrm{e}^{i(k_c-k_q)z_j}}{-\gamma' + i(\Delta + k_c v_j)} \frac{1}{\kappa + i\Delta_{cav} + i(k_c - k_q)v_j} \mathrm{e}^{i(k_c-k_q)v_j t} \tag{5.39}$$

$$+ \frac{\mathrm{e}^{i(k_c-k_q)z_j}}{-\gamma' + i(\Delta + k_c v_j)} \frac{1}{\kappa + i\Delta_{cav} - i(\Delta + k_q v_j)} \mathrm{e}^{-i(\Delta+k_q v_j)t} \tag{5.40}$$

$$+ \frac{\mathrm{e}^{-i(k_c+k_q)z_j}}{-\gamma' + i(\Delta - k_c v_j)} \frac{1}{\kappa + i\Delta_{cav} - i(k_c + k_q)v_j} \mathrm{e}^{-i(k_c+k_q)v_j t} \tag{5.41}$$

$$+ \frac{\mathrm{e}^{-i(k_c+k_q)z_j}}{-\gamma' + i(\Delta - k_c v_j)} \frac{1}{\kappa + i\Delta_{cav} - i(\Delta + k_q v_j)} \mathrm{e}^{-i(\Delta+k_q v_j)t} \,. \tag{5.42}$$

The interpretation of these terms is as follows: (5.35) describes the simultaneous absorption of a classical photon and a quantum photon while (5.41) is the simultaneous emission of the two different kinds of photons. (5.36) is the absorption of a quantum photon and the emission of a classical photon and (5.39) is the inverse process of the absorption of the classical photon and emission of the quantum photon. (5.36) and (5.37) are processess that are augmentet as the resonance is Doppler shiftet as the atom moves along the optical axis. Finally, (5.40) and (5.42) are processes that would be decay of the excitation out of the system.

Of all these terms we are interested only in the process in (5.39), which is exactly the write-in process of absorption of a classical photon and emission of a quantum photon. All the other processes we shall seek to minimise while trying to augment this particular term. One way to do this is first to observe that when we compute the Fidelity we eventually will need to make a further intergration over time, therefore any term with a time-oscilating complex phase will contribute very little. Then, the obvious thing to do is to take $k_c = k_q = k$ such that these important terms do not oscillate in time. The consequence is that the energies of the two stable states be degenerate which off course is a problem for the detection of the quantum photon since now it has the same energy as the classical photons, but it is a solvable problem – maybe through polarisation discrimination. We shall return to this question in a later section where we discuss the number of

classical photons given a successful interaction. Then we have

$$\theta_j(t) \propto \frac{\mathrm{e}^{i2z_j}}{-\gamma' + i(\Delta + kv_j)} \frac{1}{\kappa + i\Delta_{cav} + i2kv_j} \mathrm{e}^{i2kv_j t} \tag{5.43}$$

$$- \frac{\mathrm{e}^{i2kz_j}}{-\gamma' + i(\Delta + kv_j)} \frac{1}{\kappa + i\Delta_{cav} + i(-\Delta + kv_j)} \mathrm{e}^{i(-\Delta + kv_j)t} \tag{5.44}$$

$$- \frac{1}{-\gamma' + i(-\Delta + kv_j)} \frac{1}{\kappa + i\Delta_{cav}} \tag{5.45}$$

$$+ \frac{1}{-\gamma' + i(\Delta - kv_j)} \frac{1}{\kappa + i\Delta_{cav} + i(-\Delta + kv_j)} \mathrm{e}^{i(-\Delta + kv_j)t} \tag{5.46}$$

$$- \frac{1}{-\gamma' + i(\Delta + kv_j)} \frac{1}{\kappa + i\Delta_{cav}} \tag{5.47}$$

$$+ \frac{1}{-\gamma' + i(\Delta + kv_j)} \frac{1}{\kappa + i\Delta_{cav} - i(\Delta + kv_j)} \mathrm{e}^{-i(\Delta + kv_j)t} \tag{5.48}$$

$$+ \frac{\mathrm{e}^{-i2kz_j}}{-\gamma' + i(\Delta - kv_j)} \frac{1}{\kappa + i\Delta_{cav} - i2kv_j} \mathrm{e}^{-i2kv_j t} \tag{5.49}$$

$$+ \frac{\mathrm{e}^{-i2kz_j}}{-\gamma' + i(\Delta - kv_j)} \frac{1}{\kappa + i\Delta_{cav} - i(\Delta + kv_j)} \mathrm{e}^{-i(\Delta + kv_j)t} \ . \tag{5.50}$$

Next, we collect the terms with $\mathrm{e}^{i(-\Delta + kv_j)t}$ in (5.44) and (5.46) and put the prefactors on a common denominator:

$$\begin{aligned}
&\frac{1}{-\gamma' + i(\Delta - kv_j)} \frac{1}{\kappa + i\Delta_{cav} + i(-\Delta + kv_j)} \\
&- \frac{\mathrm{e}^{i2kz_j}}{-\gamma' + i(\Delta + kv_j)} \frac{1}{\kappa + i\Delta_{cav} + i(-\Delta + kv_j)} \\
&\propto \frac{-\gamma' + i(\Delta + kv_j) - \mathrm{e}^{i2kz_j}[-\gamma' + i(\Delta - kv_j)]}{[-\gamma' + i(\Delta - kv_j)][-\gamma' + i(\Delta + kv_j)]} \\
&= \frac{-\gamma' + i(\Delta + kv_j) - \mathrm{e}^{i2kz_j}[-\gamma' + i(\Delta - kv_j)]}{-\Delta^2 + (kv_j)^2 + (\gamma')^2 - 2i\gamma\Delta} \\
&\to \propto \frac{1}{\Delta} \ ,
\end{aligned}$$

for $\Delta^2 \gg (kv_j)^2, (\gamma')^2$. Since we can control Δ by detuning and pumping harder and v_j by moving the whole crystal during the interaction this requirement can be met. The same line of reasoning allows us to disregard the terms in (5.48) and (5.50). We can go on and apply the same arguments even to the term that we wish to stimulate and collect the terms of (5.45) and (5.47). The last two terms (5.43) and (5.49) cannot be collected as

easily, but we can always find a common denominator :

$$\begin{aligned}
&\frac{1}{-\gamma' + i(\Delta + kv_j)} \frac{1}{\kappa + i(\Delta_{cav} + 2kv_j)} \frac{1}{-\gamma' + i(\Delta - kv_j)} \frac{1}{\kappa + i(\Delta_{cav} - 2kv_j)} \\
&\approx \frac{1}{-\Delta^2\kappa^2 + 4\Delta^2(kv_j)^2 - \kappa^2(kv_j)^2 + 4(kv_j)^4} \\
&\to \frac{\kappa^2}{\Delta^2} \ .
\end{aligned}$$

if we neglect γ' and Δ_{cav} for a moment. And since the dominating term in the numerator is linear in Δ the terms in (5.43) and (5.49) would also be $\propto \Delta^{-1}$ if $\Delta^2 \gg (kv_j)^2, (\gamma')^2, \kappa^2$,

The point of this excercise is that the detuning is independent of the position of the ions and therefore the θ's are independent of j and the Fidelity can be reduced to unity for large detuning.

5.3.3 The estimated error

The above result is not too impressive, but it does give us the basic idea that with the right detuning relative to the the other time scales of the physics we can make the interaction independent of j, and thus obtain a Dicke state in the ions. There are natural limits as to how far a detuning is practical and we will return to this question in the next section.

Now we will look for an expression of the Fidelity such that we can estimate the deviation away from unity as a function of how fast we move the crystal compared to the other time scales of the problem. The idea is to consider the j-dependence of θ_j as a small perturbation to an other wise constant factor:

$$\theta_j(t) \propto \theta_0 + \epsilon f_j(t) + \mathcal{O}(\epsilon^2) \ , \tag{5.51}$$

where ϵ is a small positive number that is there for bookkeeping purposes. We have written "$\propto$" such that we do not have to include the constant terms that are equal for all θ's that would drop out when we calculate the Fidelity. Equation (5.51) above implies that

$$\begin{aligned}
\theta_j^*(t)\theta_l &\simeq (\theta_0^* + \epsilon f_j^*(t))(\theta_0 + \epsilon f_l(t)) &(5.52)\\
&= |\theta_0|^2 + \epsilon\theta_0^* f_l + \epsilon\theta_0 f_j^* + \epsilon^2 f_j^* f_l \ . &(5.53)
\end{aligned}$$

Now the Fidelity in (5.25) becomes (with $N = N_A$ and the without stating

the limits of the time integrations):

$$\mathcal{F} = \frac{N^2|\theta_0|^2 t + \epsilon N\theta_0^* \sum_j \int dt\ f_j(t) + \epsilon N\theta_0 \sum_j \int dt\ f_j(t) + \epsilon^2 \sum_{j,l} \int dt\ f_j^*(t) f_l(t)}{N^2|\theta_0|^2 t + \epsilon N\theta_0^* \sum_j \int dt\ f_j(t) + \epsilon N\theta_0 \sum_j \int dt\ f_j(t) + \epsilon^2 N \sum_j \int dt\ |f_j(t)|^2} \tag{5.54}$$

$$:= \frac{A + \epsilon B + \epsilon^2 C}{A + \epsilon B + \epsilon^2 D}\ , \tag{5.55}$$

where the integration over t has been done for the constant terms. The last definition (5.55) is for the ease of the following computation. Expand $\mathcal{F}$ to second order around $\epsilon = 0$:

$$\mathcal{F}(\epsilon) = \mathcal{F}(0) + \epsilon\mathcal{F}^{(1)}(0) + \frac{\epsilon^2}{2!}\mathcal{F}^{(2)}(0) + \mathcal{O}(\epsilon^3)\ . \tag{5.56}$$

Here the superscript in the parenthesis denotes the multiplicity of differentiation with respect to ϵ. We get

$$\mathcal{F}(0) = \frac{A}{A} = 1 \tag{5.57}$$

$$\mathcal{F}^{(1)}(0) = \frac{B - B}{A} = 0 \tag{5.58}$$

$$\mathcal{F}^{(2)}(0) = \frac{2(C - D)}{A}\ . \tag{5.59}$$

Then

$$\mathcal{F} = 1 - \frac{\epsilon^2}{2!}\frac{2(D - C)}{A} + \mathcal{O}(\epsilon^3) \tag{5.60}$$

$$= 1 - \frac{N\sum_j \int dt\ |f_j(t)|^2 - \sum_{j,l} \int dt\ f_j^*(t) f_l(t)}{N^2|\theta_0|^2 t}\ , \tag{5.61}$$

to leading order and where we have dropped the bookkeeping ϵ. From equations (5.43) to (5.50) we find

$$\theta_0 \propto -\frac{1}{-\gamma' + i(-\Delta + kv_j)}\frac{1}{\kappa + i\Delta_{cav}} - \frac{1}{-\gamma' + i(\Delta + kv_j)}\frac{1}{\kappa + i\Delta_{cav}}\ ,$$

while $f_j(t)$ is proportional to the other six terms of those eight equations. Now, considering the numerator of the fraction in (5.61) we are interested in terms that grow linearly with t since the denominator also grows linear in t. Thus when forming the products of different factors of f_j we need those terms that are not oscillating in time. Therefore, the calculation of $f_j^*(t) f_l(t)$ involves only the six expressions that are squares of terms. We

get:

$$
\begin{aligned}
&f_j^*(t)f_l(t) \propto \\
&\frac{\mathrm{e}^{-i2(z_j-z_k)}}{-\gamma'-i(\Delta+kv_j)}\frac{1}{\kappa-i(\Delta_{cav}+2kv_j)}\frac{1}{-\gamma'+i(\Delta+kv_l)}\frac{1}{\kappa+i(\Delta_{cav}+2kv_l)} \\
&+\frac{\mathrm{e}^{-i2k(z_j-z_l)}}{-\gamma'-i(\Delta+kv_j)}\frac{1}{\kappa-i(\Delta_{cav}-\Delta+kv_j)}\frac{1}{-\gamma'+i(\Delta+kv_l)}\frac{1}{\kappa+i(\Delta_{cav}-\Delta+kv_j)} \\
&+\frac{1}{-\gamma'-i(\Delta-kv_j)}\frac{1}{\kappa-i(\Delta_{cav}-\Delta+kv_j)}\frac{1}{-\gamma'+i(\Delta-kv_l)}\frac{1}{\kappa+i(\Delta_{cav}-\Delta+kv_l)} \\
&+\frac{1}{-\gamma'-i(\Delta+kv_j)}\frac{1}{\kappa-i(\Delta_{cav}-\Delta+kv_j)}\frac{1}{-\gamma'+i(\Delta+kv_l)}\frac{1}{\kappa+i(\Delta_{cav}-\Delta+kv_l)} \\
&+\frac{\mathrm{e}^{i2(z_j-z_k)}}{-\gamma'-i(\Delta-kv_j)}\frac{1}{\kappa-i(\Delta_{cav}-2kv_j)}\frac{1}{-\gamma'+i(\Delta-kv_l)}\frac{1}{\kappa+i(\Delta_{cav}-2kv_l)} \\
&+\frac{\mathrm{e}^{i2(z_j-z_k)}}{-\gamma'-i(\Delta-kv_j)}\frac{1}{\kappa-i(\Delta_{cav}-\Delta+kv_j)}\frac{1}{-\gamma'+i(\Delta-kv_l)}\frac{1}{\kappa+i(\Delta_{cav}-\Delta+kv_l)} \ .
\end{aligned}
$$

In the same manner as above we get

$$
\begin{aligned}
|f_j(t)|^2 \propto &\frac{1}{(\gamma')^2+(\Delta+kv_j)^2}\frac{1}{\kappa^2+(\Delta_{cav}+2kv_j)^2} && (5.62)\\
&+\frac{1}{(\gamma')^2+(\Delta+kv_j)^2}\frac{1}{\kappa^2+(\Delta_{cav}-\Delta+kv_j)^2} && (5.63)\\
&+\frac{1}{(\gamma')^2+(\Delta-kv_j)^2}\frac{1}{\kappa^2+(\Delta_{cav}-\Delta+kv_j)^2} && (5.64)\\
&+\frac{1}{(\gamma')^2+(\Delta+kv_j)^2}\frac{1}{\kappa^2+(\Delta_{cav}-\Delta+kv_j)^2} && (5.65)\\
&+\frac{1}{(\gamma')^2+(\Delta-kv_j)^2}\frac{1}{\kappa^2+(\Delta_{cav}-2kv_j)^2} && (5.66)\\
&+\frac{1}{(\gamma')^2+(\Delta-kv_j)^2}\frac{1}{\kappa^2+(\Delta_{cav}-\Delta+kv_j)^2} \ . && (5.67)
\end{aligned}
$$

To simplify these expressions we will make the additional approximation: Consider the overall motion of the ions that we control. If the constant speed is much larger than the mean speed of the individual ion due to thermal motion, $v \gg v^{(j)}$, then we can neglect the individual motion of each ion and set $v_j = v_l = v$. Also, when we sum over all the ions the phase factors will average to zero, e.g.:

$$
\left\langle \mathrm{e}^{-i2k(z_j-z_l)} \right\rangle_z \approx 0 \ . \qquad (5.68)
$$

Then we shall only need two terms from $f_j^*(t)f_l(t)$ that now reads:

$$\frac{1}{(\gamma')^2+(\Delta-kv)^2}\frac{1}{\kappa^2+(\Delta_{cav}-\Delta+kv)^2}\ ,$$
$$\frac{1}{(\gamma')^2+(\Delta+kv)^2}\frac{1}{\kappa^2+(\Delta_{cav}-\Delta+kv)^2}\ .$$

These two equations match (5.64) and (5.65) and they cancel. The last part of the Fidelity we need to calculate is

$$|\theta_0|^2 \propto \frac{4\Delta^2}{\kappa^2+\Delta_{cav}^2}\frac{1}{((\gamma')^2-\Delta^2+(kv)^2)^2+4(\gamma' kv)^2}\ . \tag{5.69}$$

This means that we can perform the integration and summations in the expression for the Fidelity. If we consider the correction to the Fidelity of unity it is apparent that the factors of t and N^2 drop out and we are left with:

$$\frac{N\sum_j \int dt\ |f_j(t)|^2-\sum_{j,l}\int dt\ f_j^*(t)f_l(t)}{N^2|\theta_0|^2 t} \tag{5.70}$$
$$\begin{aligned}
&= \frac{\kappa^2+\Delta_{cav}^2}{4\Delta^2}\big((\gamma')^2-\Delta^2+(kv)^2)^2+4(\gamma' kv)^2\big)\\
&\times\Bigg\{\frac{1}{(\gamma')^2+(\Delta+kv)^2}\frac{1}{\kappa^2+(\Delta_{cav}+2kv)^2}\\
&+\frac{1}{(\gamma')^2+(\Delta+kv)^2}\frac{1}{\kappa^2+(\Delta_{cav}-\Delta+kv)^2}\\
&+\frac{1}{(\gamma')^2+(\Delta-kv)^2}\frac{1}{\kappa^2+(\Delta_{cav}-2kv)^2}\\
&+\frac{1}{(\gamma')^2+(\Delta-kv)^2}\frac{1}{\kappa^2+(\Delta_{cav}-\Delta+kv)^2}\Bigg\}\ .
\end{aligned} \tag{5.71}$$

From the analysis of the Fidelity in the previous section we know that the interesting limit is high detuning. Then we can ignore the two terms in the bracket in (5.71) that are proportional to Δ^{-4}. Also, $\Delta^2 \gg (\gamma')^2, (kv)^2$ and we get in leading order

$$\begin{aligned}
\mathcal{F} = 1-\frac{\kappa^2+\Delta_{cav}^2}{4}\Delta^2\Bigg\{&\frac{1}{(\gamma')^2+(\Delta+kv)^2}\frac{1}{\kappa^2+(\Delta_{cav}+2kv)^2}\\
&+\frac{1}{(\gamma')^2+(\Delta-kv)^2}\frac{1}{\kappa^2+(\Delta_{cav}-2kv)^2}\Bigg\}\ .
\end{aligned} \tag{5.72}$$

And, finally, in the limit where aditionally $\Delta \gg kv$ and $kv \gg \Delta_{cav}$ and $(kv)^2 \gg \kappa^2$ we get

$$\mathcal{F} \simeq 1 - \frac{\kappa^2 + \Delta_{cav}^2}{8(kv)^2} \ . \tag{5.73}$$

This is the wished for expression for the Fidelity as a function of the physical parameters and then as a function of how fast we move the crystal during the interaction. For some physical parameters [Herskind, Dantan *et al.*] mentions $\kappa = 2\pi \times 2.15$MHz and the relevant atomic transition lies at 866nm. So, if we ignore the cavity detuning and chose e.g. $v = 20\frac{m}{s}$ we get $\mathcal{F} \approx 99.9\%$. Of course (5.73) does not take into account the important factor of the spontaneous decay but then we can use the expression in (5.72).

A different perspective is to consider the fact that by moving the crystal we are Doppler shifting the ions into resonance with one direction of the field. While moving further away form resonance with the field in the other direction. This means that the standing wave looks like two travelling waves, which is what it is. So, in frequency space only one frequency is picked out instead of two. In space it is like an average over the field tops.

5.3.4 The Number of classical photons

As mentioned above, we can also detune too far. The result would be that we would have to pump many more photons into the cavity to make sure that the interaction had the right chance of happening. This in turn would mean that when the quantum photon is in the cavity it would be there with the classical photons. And the more classical photons present the harder it would be to detect the quantum photon, especially if the two stable levels of the ions are degenerate. According to private correspondance the experimentors in Aarhus can detect a quantum photon even if the flux of classical photons is up to 10^4. Here we will do a quick calculation of the ratio of the flux of quantum photons to the flux of classical photons. Consider the simplified model on Figure 5.4.

Next, define an operator for the quantum field:

$$\hat{a}_{\text{out}} := \frac{1}{\sqrt{\kappa}} \sum_j \frac{\Omega_j g_j}{\Delta} \hat{\sigma}_{sg}^{(j)} \ , \tag{5.74}$$

where κ is the decay through the semi-transparent mirror. This operator is basically the expression in (5.12) but we just consider the dominating terms after the integrations. Then for the classical field we consider a pulse of a certain pulse length T. This pulse length is very long compared to the dynamics of the interaction and the read-out. So we consider the classical field as in the continous wave picture. For the quantum field we have the

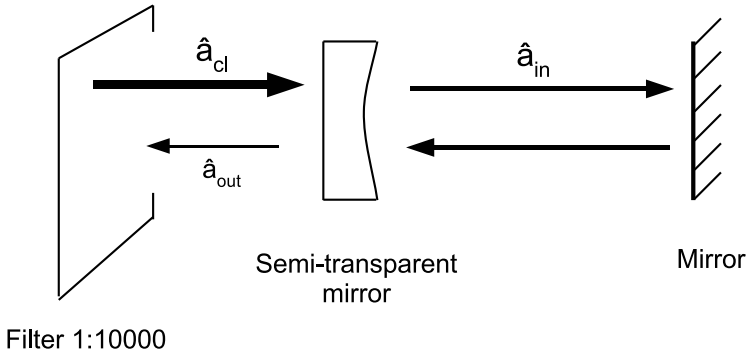

Figure 5.4: A simple model of the output of the cavity with regards to filtering the quantum photon from the classical photons.

number of photons out $\hat{N}_{\rm out} = \hat{a}^\dagger_{\rm out}\hat{a}_{\rm out}T$ and

$$\hat{N}_{\rm out} = \left(\frac{1}{\sqrt{\kappa}}\sum_j \frac{\Omega_j g_j}{\Delta}\hat{\sigma}^{(j)}_{sg}\right)^\dagger \frac{1}{\sqrt{\kappa}}\sum_j \frac{\Omega_j g_j}{\Delta}\hat{\sigma}^{(j)}_{sg} \tag{5.75}$$

$$\simeq \frac{1}{\kappa}\frac{|\Omega|^2|g|^2}{\Delta^2}N_A T \ , \tag{5.76}$$

where we have taken for granted that our scheme of moving the crystal during the interaction produces those nice Dicke states such that $\Omega_j = \Omega$ and $g_j = g$ independent of the position and individual movement of each ion. Then we can also quantise the classical field, and we have that $|\Omega| \simeq g\hat{a}_{cl}$ and

$$\hat{a}_{\rm in} = \frac{1}{\sqrt{\kappa}}\hat{a}_{\rm cl} \ . \tag{5.77}$$

The number of photons inside the cavity is $\hat{N}_{\rm phot} = \hat{a}^\dagger_{\rm in}\hat{a}_{\rm in}T$. This means that

$$\hat{N}_{\rm out} \simeq \frac{1}{\kappa^2}\frac{|g|^4}{\Delta^2}N_A\hat{N}_{\rm phot} \ . \tag{5.78}$$

Next, we use the definition of Cooperativity, $C = N_A|g|^2/2\kappa\gamma$ and we get

$$\hat{N}_{\rm out} \simeq \left(\frac{2C\gamma}{\Delta}\right)^2 \frac{\hat{N}_{\rm phot}}{N_A} \ , \tag{5.79}$$

$$\hat{N}_{\rm phot} \simeq \left(\frac{\Delta}{2C\gamma}\right)^2 \hat{N}_{\rm out}N_A \ . \tag{5.80}$$

With a realistic $C = 2.5$, $N_A = 600$, $\gamma = 1.2 \cdot 10^6$, $\Delta = 25 \times \kappa$, then we get $\hat{N}_{\rm phot} \simeq 650$, which means that we need around a thousand photons to get a quantum photon and that this ratio makes detection of the quantum photon possible.

Chapter 6

Read-out

In the previous chapter we saw that moving the crystal with just the right speed might be a good way to improve the quantum interface between light and matter during the write-in of the quantum reapeter operation. Here we consider the read-out with the intent of using the same trick. A special concern in the read-out is that it is crucial that we get the stored excitation out of the ensemble and into the detector. At the same time it is not possible just to make the field stronger and stronger since then we cannot detect the quantum photon. This means that a large detuning is not an option as it was in the write-in.

A second important difference is the fact that in the write-in we took the time-evolution of the system to be $\hat{U}(t) \simeq \mathbb{1}$, since the probability of exciting the ensemble was kept very small. For the reasons stated above we need the probability of read-out as high as possible. Thus, the evolutionen cannot be set equal to the unitary operator and the interaction is not a perturbation.

6.1 Model

We suppose that that by moving the crystal we have achieved the symmetric Dicke state

$$|\psi_{\text{in}}\rangle = |S_0\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |g_0, ..., s_j, ..., g_{N-1}\rangle \ . \tag{6.1}$$

We have here chosen to label the N atoms $0...N-1$. The reason will become apparent later when we introduce $\hat{S}_l$ as the discrete Fourier transform of the atomic operators $\hat{\sigma}_{sg}$. Then, we will investigate if indeed we get the read-out quantum field into the Anti-Stokes mode.

In the read-out we apply a classical field with tintohe frequency ω_c on the $|s\rangle \rightarrow |e\rangle$ transition and wait for the decay $|e\rangle \rightarrow |g\rangle$ to produce the quantum field with the frequency ω_q, see Figure 6.1. This means that the model is

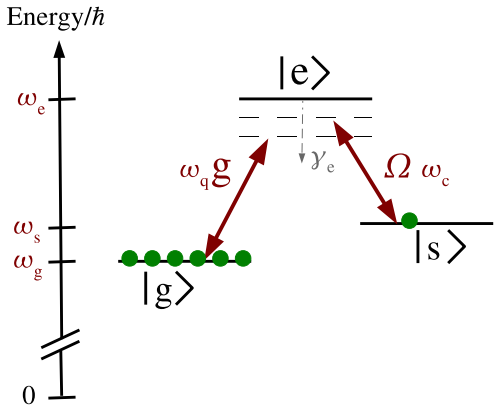

Figure 6.1: The Relevant energy levels of the lambda systems for the read-out process. Ω and ω_c are the Rabi frequency and the angular frequency of the calssical control field while g and ω_c are the coupling and the angular frequency of the quantum read-out field that we wish to come out into the Anti-Stokes mode and become detected.

exactly the same as for the read-out except for the interchange of the two fields. The Hamiltonian is thus after the rotating wave approximation and using a suitable rotating frame:

$$\hat{H} = \hbar \sum_{j=0}^{N-1} \left(\Delta \hat{\sigma}_{ee}^{(j)} - \left(g^{(j)} \hat{a}^\dagger \hat{\sigma}_{ge}^{(j)} + H.c. \right) - \left(\frac{\Omega^{(j)}}{2} \hat{\sigma}_{es}^{(j)} + H.c. \right) \right) , \tag{6.2}$$

where we consider a finite detuning and still under the assumption that the two-photon detuning is set to zero. An important point is that we will work with little or maybe no detuning.

6.2 Equations of motion

Using the same procedure as for the write-in we find the three coupled equations of motion that describe the evolution of the light field during the read-out:

$$\dot{\hat{a}} = -(\kappa + i\Delta_{\text{cav}})\hat{a} + i \sum_{j=0}^{N-1} g^{(j)} \hat{\sigma}_{ge}^{(j)} \tag{6.3}$$

$$\dot{\sigma}_{ge}^{(j)} = -(\gamma_e + i\Delta)\hat{\sigma}_{ge}^{(j)} + i g^{(j)} \hat{a} + i \frac{\Omega^{(j)}}{2} \hat{\sigma}_{gs}^{(j)} \tag{6.4}$$

$$\dot{\hat{\sigma}}_{gs}^{(j)} = i \frac{\Omega^{(j)}}{2} \hat{\sigma}_{ge}^{(j)} \ . \tag{6.5}$$

Here we have taken $\hat{a}^{\text{in}} = 0$. Also, the term $i\hat{a}\hat{\sigma}_{es}^{(j)}$ which would appear in equation (6.5) is zero, since we only have one excitation in the $|s\rangle$ level. The explanation is that we cannot have the transition from $|s\rangle$ to $|e\rangle$ simultaneous with the creation of photon on the $|e\rangle$ to $|g\rangle$ transition. Finally, $\hat{\sigma}_{gg}^{(j)} - \hat{\sigma}_{ee}^{(j)} \simeq \mathbb{1}$, as before.

We can find two equations that couple the light field with the set of $\hat{\sigma}_{gs}^{(j)}$ operators by elimination of the $\hat{\sigma}_{ge}^{(j)}$ operator by formal integration:

$$\hat{\sigma}_{ge}^{(j)}(t) = \mathrm{e}^{-(\gamma_e + i\Delta)t}\hat{\sigma}_{ge}^{(j)}(0) + \int_0^t dt' \mathrm{e}^{-(\gamma_e + i\Delta)(t-t')}\left\{ i\hat{a}g^{(j)} + i\frac{\Omega^{(j)}}{2}\hat{\sigma}_{gs}^{(j)} \right\} . \tag{6.6}$$

We take $\hat{\sigma}_{ge}^{(j)}(0) = 0$ and the j dependence of the coupling strengths to be the time dependent factors. This means that we introduce the same sinesoidal wavefunction to describe the j-dependence as we did for the write-in and we take the operators outside the intergral:

$$\hat{\sigma}_{ge}^{(j)}(t) = \left\{ ig\hat{a} + i\frac{\Omega}{2}\hat{\sigma}_{gs} \right\} \int_0^t dt' \mathrm{e}^{-(\gamma_e + i\Delta)(t-t')} \sin\left[k(x_j + v_j t')\right] \tag{6.7}$$

where $k = k_c = k_q$ and v_j is constant and where g and Ω are the part of the couplings that are not dependent on the postion of the j'th ion. Performing the integral gives

$$\hat{\sigma}_{ge}^{(j)}(t) = \left\{ ig\hat{a} + i\frac{\Omega}{2}\hat{\sigma}_{gs} \right\} \left(\frac{\mathrm{e}^{ikx_j}\mathrm{e}^{ikv_j t}}{(\gamma_e + i\Delta) + ikv_j} - \frac{\mathrm{e}^{-ikx_j}\mathrm{e}^{-ikv_j t}}{(\gamma_e + i\Delta) - ikv_j} \right) , \tag{6.8}$$

where we have neglected terms proportional to $\mathrm{e}^{-\gamma_e t}$ since we are interested in times long such that the system most probably has decayed. This expression is substituted into (6.3) and we get

$$\begin{aligned} \dot{\hat{a}} = -(\kappa + i\Delta_{\text{cav}})\hat{a} + \hat{a}\frac{g^2}{4}\sum_{j=0}^{N-1}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kv_j t} - 1}{\gamma_e + i(\Delta + kv_j)} - \frac{1 - \mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kv_j t}}{\gamma_e + i(\Delta - kv_j)} \right) \\ + \quad \frac{g\Omega}{8}\sum_{j=0}^{N-1}\hat{\sigma}_{gs}^{(j)}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kv_j t} - 1}{\gamma_e + i(\Delta + kv_j)} - \frac{1 - \mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kv_j t}}{\gamma_e + i(\Delta - kv_j)} \right) . \end{aligned} \tag{6.9}$$

The same substitution is done in (6.5) and we have

$$\dot{\hat{a}} = A\hat{a} + \sum_{j=0}^{N-1} B_j \hat{\sigma}_{gs}^{(j)} \tag{6.10}$$

$$\dot{\hat{\sigma}}_{gs}^{(j)} = B_j\hat{a} + C_j\hat{\sigma}_{gs}^{(j)} , \tag{6.11}$$

with

$$A = -(\kappa + i\Delta_{\text{cav}}) + \frac{g^2}{4}\sum_{j=0}^{N-1}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kv_jt}-1}{\gamma_e + i(\Delta + kv_j)} - \frac{1-\mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kv_jt}}{\gamma_e + i(\Delta - kv_j)}\right) \tag{6.12}$$

$$B_j = \frac{g\Omega}{8}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kv_jt}-1}{\gamma_e + i(\Delta + kv_j)} - \frac{1-\mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kv_jt}}{\gamma_e + i(\Delta - kv_j)}\right) \tag{6.13}$$

$$C_j = \frac{\Omega^2}{16}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kv_jt}-1}{\gamma_e + i(\Delta + kv_j)} - \frac{1-\mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kv_jt}}{\gamma_e + i(\Delta - kv_j)}\right) \tag{6.14}$$

We can represent the system of $N+1$ coupled equations in matrix form:

$$\dot{\mathbf{x}}(t) = \mathbf{M}'(t)\mathbf{x}(t)\ , \tag{6.15}$$

where

$$\mathbf{x}^T(t) = \left(\hat{a}, \hat{\sigma}_{gs}^{(0)}, \hat{\sigma}_{gs}^{(1)}, \ldots, \hat{\sigma}_{gs}^{(N-1)}\right)\ , \tag{6.16}$$

The $(N+1)\times(N+1)$ matrix in the basis

$$\mathfrak{B}' = \left\{\mathbf{v}_1\hat{a}, \mathbf{v}_2\hat{\sigma}_{gs}^{(0)}, \mathbf{v}_3\hat{\sigma}_{gs}^{(1)}, \ldots, \mathbf{v}_{N+1}\hat{\sigma}_{gs}^{(N-1)}\right\}\ ,\ \mathbf{v}_i^T = (0, ..., 1_i, ..., 0)\ ,$$

is then

$$\mathbf{M}'(t) = \begin{pmatrix} A & B_0 & B_1 & B_2 & \cdots & B_{N-1} \\ B_0 & C_0 & 0 & 0 & \cdots & 0 \\ B_1 & 0 & C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ B_{N-1} & 0 & 0 & \cdots & \cdots & C_{N-1} \end{pmatrix}\ . \tag{6.17}$$

6.3 Solving for the field operator

Equation (6.15) does not have a simple solution given that the matrixelements of (6.17) all depend on time in different ways. One way to proceed is to consider each matrix element and find the constant part of it. For example:

$$C_j = \bar{C}_j + C_j(t) \tag{6.18}$$

$$= -\frac{\Omega^2}{16}\left(\frac{1}{\gamma_e + i(\Delta + kv_j)} - \frac{1}{\gamma_e + i(\Delta - kv_j)}\right) + \frac{\Omega^2}{16}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kv_jt}}{\gamma_e + i(\Delta + kv_j)} + \frac{\mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kv_jt}}{\gamma_e + i(\Delta - kv_j)}\right) \tag{6.19}$$

Next, consider the velocity of the j'th ion $v_j = v + \delta v_j$. We will work in the situation where the constant speed imposed on the crystal is larger than the speed of the individual ion due to thermal vibrations $v > \delta v_j$. Then expand the following term around $\delta v_j = 0$:

$$\frac{1}{\gamma_e + i(\Delta + k(v + \delta v_j))} \simeq \frac{1}{\gamma_e + i(\Delta + kv)} - \frac{k\delta v_j}{[\gamma_e + i(\Delta + kv)]^2} \tag{6.20}$$

This implies that there is a constant term of C_j which is independent of j:

$$\begin{aligned} C_j &= \bar{C} + C_j(t) \\ &= -\frac{\Omega^2}{16}\left(\frac{1}{\gamma_e + i(\Delta + kv)} + \frac{1}{\gamma_e + i(\Delta - kv)}\right) \\ &\quad + \frac{\Omega^2}{16}\left(\frac{k\delta v_j}{[\gamma_e + i(\Delta + kv)]^2} - \frac{k\delta v_j}{[\gamma_e + i(\Delta - kv)]^2}\right) \\ &\quad + \frac{\Omega^2}{16}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kvt}}{\gamma_e + i(\Delta + kv)} + \frac{\mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kvt}}{\gamma_e + i(\Delta - kv)}\right). \end{aligned} \tag{6.21}$$

The same can be done for

$$\begin{aligned} A_j &= \bar{A} + A(t) \\ &= -(\kappa + i\Delta_{\mathrm{cav}}) - \frac{g^2 N}{4}\left(\frac{1}{\gamma_e + i(\Delta + kv)} + \frac{1}{\gamma_e + i(\Delta - kv)}\right) \\ &\quad + \frac{g^2}{4}\sum_{j=0}^{N-1}\left(\frac{k\delta v_j}{[\gamma_e + i(\Delta + kv)]^2} - \frac{k\delta v_j}{[\gamma_e + i(\Delta - kv)]^2}\right) \\ &\quad + \frac{g^2}{4}\sum_{j=0}^{N-1}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kvt}}{\gamma_e + i(\Delta + kv)} + \frac{\mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kvt}}{\gamma_e + i(\Delta - kv)}\right), \end{aligned} \tag{6.22}$$

where j dependence implies time-dependence, and

$$\begin{aligned} B_j &= \bar{B} + B_j(t) \\ &= -\frac{g\Omega}{8}\left(\frac{1}{\gamma_e + i(\Delta + kv)} + \frac{1}{\gamma_e + i(\Delta - kv)}\right) \\ &\quad + \frac{g\Omega}{8}\left(\frac{k\delta v_j}{[\gamma_e + i(\Delta + kv)]^2} - \frac{k\delta v_j}{[\gamma_e + i(\Delta - kv)]^2}\right) \\ &\quad + \frac{g\Omega}{8}\left(\frac{\mathrm{e}^{i2kx_j}\mathrm{e}^{i2kvt}}{\gamma_e + i(\Delta + kv)} + \frac{\mathrm{e}^{-i2kx_j}\mathrm{e}^{-i2kvt}}{\gamma_e + i(\Delta - kv)}\right). \end{aligned} \tag{6.23}$$

Now, it is apparent that the time-dependent terms all fast oscillations in time and since we later will be interested in expectation values and averages we take the time-dependent terms as perturbations of the constant terms. In this perturbation we also include the constant terms that are j-dependent

since those terms are small. Consequently, we write $\mathbf{M}'(t) = \mathbf{M}'_0 + \mathbf{M}'_1(t)$, where:

$$\mathbf{M}'_0 = \begin{pmatrix} \bar{A} & \bar{B} & \bar{B} & \bar{B} & \cdots & \bar{B} \\ \bar{B} & \bar{C} & 0 & 0 & \cdots & 0 \\ \bar{B} & 0 & \bar{C} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \bar{B} & 0 & 0 & \cdots & \cdots & \bar{C} \end{pmatrix} , \tag{6.24}$$

and

$$\mathbf{M}'_1(t) = \begin{pmatrix} A(t) & B_0(t) & B_1(t) & B_2(t) & \cdots & B_{N-1}(t) \\ B_0(t) & C_0(t) & 0 & 0 & \cdots & 0 \\ B_1(t) & 0 & C_1(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ B_{N-1}(t) & 0 & 0 & \cdots & \cdots & C_{N-1}(t) \end{pmatrix} . \tag{6.25}$$

6.3.1 Change of basis

Further progress towards a solution can be made by changing the basis such that $\mathbf{M}'_0$ is transformed into an almost diagonal matrix. This is done through a discrete Fourier transform of the atomic operators. Define

$$\hat{S}_l \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \mathrm{e}^{i\frac{2\pi}{N}jl} \hat{\sigma}_{gs}^{(j)} , \tag{6.26}$$

with the inverse

$$\hat{\sigma}_{gs}^{(j)} \equiv \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \mathrm{e}^{-i\frac{2\pi}{N}jl} \hat{S}_l . \tag{6.27}$$

We substitute this into (6.10):

$$\dot{\hat{a}} = A\hat{a} + \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \mathrm{e}^{-i\frac{2\pi}{N}jl} B_j \hat{S}_l . \tag{6.28}$$

With $B_j = \bar{B} + B_j(t)$ we get

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}\mathrm{e}^{-i\frac{2\pi}{N}jl}B_j\hat{S}_l &= \frac{\bar{B}}{\sqrt{N}}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}\mathrm{e}^{-i\frac{2\pi}{N}jl}\hat{S}_l \\ &\quad + \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}\mathrm{e}^{-i\frac{2\pi}{N}jl}B_j(t)\hat{S}_l \\ &= \sqrt{N}\bar{B}\hat{S}_0 + \sum_{l=0}^{N-1}B_l^{\hat{a}}(t)\hat{S}_l \ , \end{aligned} \tag{6.29}$$

where

$$B_l^{\hat{a}}(t) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\mathrm{e}^{-i\frac{2\pi}{N}jl}B_j(t) \ . \tag{6.30}$$

Differentiation of (6.26) is straight forward and we plug in (6.11):

$$\dot{\hat{S}}_l = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}jl}\dot{\hat{\sigma}}_{gs}^{(j)} \tag{6.31}$$

$$= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}jl}\Big(B_j\hat{a} + C_j\hat{\sigma}_{gs}^{(j)}\Big) \ . \tag{6.32}$$

Now,

$$\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}jl}B_j = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}jl}\Big(\bar{B} + B_j(t)\Big) \tag{6.33}$$

$$= \sqrt{N}\bar{B}\delta_{l,0} + \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}jl}B_j(t) \tag{6.34}$$

$$= \sqrt{N}\bar{B}\delta_{l,0} + B_l(t) \ , \tag{6.35}$$

and

$$\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}jl}C_j\hat{\sigma}_{gs}^{(j)} = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}jl}\Big(\bar{C} + C_j(t)\Big)\hat{\sigma}_{gs}^{(j)} \tag{6.36}$$

$$\begin{aligned} &= \frac{\bar{C}}{N}\sum_{j=0}^{N-1}\sum_{l'=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}j(l-l')}\hat{S}_{l'} \\ &\quad + \frac{1}{N}\sum_{j=0}^{N-1}\sum_{l'=0}^{N-1}\mathrm{e}^{i\frac{2\pi}{N}j(l-l')}C_j(t)\hat{S}_{l'} \\ &= \bar{C}\hat{S}_l + \sum_{l'=0}^{N-1}C_{l,l'}(t)\hat{S}_{l'} \ . \end{aligned} \tag{6.37}$$

In the above equation

$$C_{l,l'}(t) = \sum_{j=0}^{N-1} \mathrm{e}^{i\frac{2\pi}{N}j(l-l')} C_j(t) \ . \tag{6.38}$$

The $N+1$ coupled equations now read

$$\dot{\hat{a}} = A\hat{a} + \sqrt{N}\bar{B}\hat{S}_0 + \sum_{l=0}^{N-1} B_l^{\hat{a}}(t)\hat{S}_l \tag{6.39}$$

$$\dot{\hat{S}}_l = \sqrt{N}\bar{B}\delta_{l,0}\hat{a} + B_l(t)\hat{a} + \bar{C}\hat{S}_l + \sum_{l'=0}^{N-1} C_{l,l'}(t)\hat{S}_{l'} \ . \tag{6.40}$$

They can be represented in the following transformed matrix form:

$$\dot{\mathbf{y}}(t) = \mathbf{M}(t)\mathbf{y}(t) \ , \tag{6.41}$$

where

$$\mathbf{y}^T(t) = \left(\hat{a}, \hat{S}_0, \hat{S}_1, \ldots, \hat{S}_{N-1}\right) \ , \tag{6.42}$$

The $(N+1)\times(N+1)$ matrix in the basis

$$\mathfrak{B} = \left\{\mathbf{v}_1\hat{a}, \mathbf{v}_2\hat{S}_0, \mathbf{v}_3\hat{S}_1, \ldots, \mathbf{v}_{N+1}\hat{S}_{N-1}\right\} \ , \tag{6.43}$$

becomes $\mathbf{M}(t) = \mathbf{M}_0 + \mathbf{M}_1(t)$, where:

$$\mathbf{M}_0 = \begin{pmatrix} \bar{A} & \sqrt{N}\bar{B} & 0 & 0 & \cdots & 0 \\ \sqrt{N}\bar{B} & \bar{C} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \bar{C} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \bar{C} \end{pmatrix} \ , \tag{6.44}$$

and

$$\mathbf{M}_1(t) = \begin{pmatrix} A(t) & B_0^{\hat{a}}(t) & B_1^{\hat{a}}(t) & B_2^{\hat{a}}(t) & \cdots & B_{N-1}^{\hat{a}}(t) \\ B_0(t) & C_{0,0}(t) & C_{0,1}(t) & C_{0,2}(t) & \cdots & C_{0,N-1}(t) \\ B_1(t) & C_{1,0}(t) & C_{1,1}(t) & C_{1,2}(t) & \cdots & C_{1,N-1}(t) \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ B_{N-1}(t) & C_{N-1,0}(t) & C_{N-1,1}(t) & \cdots & \cdots & C_{N-1,N-1}(t) \end{pmatrix} \ . \tag{6.45}$$

6.3.2 Perturbation theory

We can consider $\mathbf{M}_1(t)$ to be a perturbation of the constant matrix $\mathbf{M}_0$. Progress can now be made using perturbation theory to find an aproximate solution to (6.41). We include up to second order in the small terms of the power series of $\mathbf{y}(t)$ and write

$$\mathbf{y}(t) = \mathbf{y}_0(t) + \lambda\mathbf{y}_1(t) + \lambda^2\mathbf{y}_2(t) \tag{6.46}$$

$$\dot{\mathbf{y}} = \dot{\mathbf{y}}_0 + \lambda\dot{\mathbf{y}}_1 + \lambda^2\dot{\mathbf{y}}_2 \tag{6.47}$$

$$\mathbf{M}(t) = \mathbf{M}_0 + \lambda\mathbf{M}_1(t) \ , \tag{6.48}$$

where λ is there to keep track of small terms and to what order we need them. Later we shall put $\lambda = 1$. We take $\mathbf{y}_0(t=0) = \mathbf{y}^{\text{in}}$ and therefore $\mathbf{y}_1(0) = \mathbf{y}_2(0) = 0$. Now we get

$$\dot{\mathbf{y}} = \mathbf{M}(t)\mathbf{y}(t) \tag{6.49}$$

$$\begin{aligned}\dot{\mathbf{y}}_0 + \lambda\dot{\mathbf{y}}_1 + \lambda^2\dot{\mathbf{y}}_2 &= \big[\mathbf{M}_0 + \lambda\mathbf{M}_1(t)\big]\big[\mathbf{y}_0(t) + \lambda\mathbf{y}_1(t) + \lambda^2\mathbf{y}_2(t)\big] \qquad (6.50)\\ &= \mathbf{M}_0\mathbf{y}_0(t) + \lambda\big(\mathbf{M}_0\mathbf{y}_1(t) + \mathbf{M}_1(t)\mathbf{y}_0(t)\big)\\ &\quad + \lambda^2\big(\mathbf{M}_0\mathbf{y}_2(t) + \mathbf{M}_1(t)\mathbf{y}_1(t)\big) + \mathcal{O}(\lambda^3) \qquad (6.51)\end{aligned}$$

Equating powers of λ and solving the ensuing equations gives

$$\lambda^0 :$$

$$\dot{\mathbf{y}}_0 = \mathbf{M}_0\mathbf{y}_0(t) \tag{6.52}$$

$$\Rightarrow\ \mathbf{y}_0(t) = \mathrm{e}^{\mathbf{M}_0 t}\mathbf{y}_0(0) \ , \tag{6.53}$$

$$\lambda^1 :$$

$$\dot{\mathbf{y}}_1 = \mathbf{M}_0\mathbf{y}_1(t) + \mathbf{M}_1(t)\mathbf{y}_0(t) \tag{6.54}$$

$$\Rightarrow\ \mathbf{y}_1(t) = \mathrm{e}^{\mathbf{M}_0 t}\mathbf{y}_1(0) + \int_0^t dt'\ \mathrm{e}^{\mathbf{M}_0(t-t')}\mathbf{M}_1(t')\mathrm{e}^{\mathbf{M}_0 t}\mathbf{y}_0(0) \tag{6.55}$$

$$= \int_0^t dt'\ \mathrm{e}^{\mathbf{M}_0(t-t')}\mathbf{M}_1(t')\mathrm{e}^{\mathbf{M}_0 t'}\mathbf{y}_0(0) \ , \tag{6.56}$$

$$\lambda^2 :$$

$$\dot{\mathbf{y}}_2 = \mathbf{M}_0\mathbf{y}_2(t) + \mathbf{M}_1(t)\mathbf{y}_1(t) \tag{6.57}$$

$$\begin{aligned}\Rightarrow\ \mathbf{y}_2(t) &= \mathrm{e}^{\mathbf{M}_0 t}\mathbf{y}_2(0)\\ &\quad + \int_0^t dt' \int_0^{t'} dt''\ \mathrm{e}^{\mathbf{M}_0(t-t')}\mathbf{M}_1(t')\mathrm{e}^{\mathbf{M}_0(t'-t'')}\mathbf{M}_1(t'')\mathrm{e}^{\mathbf{M}_0 t''}\mathbf{y}_0(0)\\ &= \int_0^t dt' \int_0^{t'} dt''\ \mathrm{e}^{\mathbf{M}_0(t-t')}\mathbf{M}_1(t')\mathrm{e}^{\mathbf{M}_0(t'-t'')}\mathbf{M}_1(t'')\mathrm{e}^{\mathbf{M}_0 t''}\mathbf{y}_0(0) \ . \qquad (6.58)\end{aligned}$$

With these terms we are ready to construct approximations to $\mathbf{y}(t)$ and thereby find an approximation for the time evolved field operator.

6.4 Read-out

The above analysis means that we can find an approximation for the field operator: $\hat{a}(t) \simeq \hat{a}_0(t) + \hat{a}_1(t) + \hat{a}_2(t)$. First we find $\hat{a}_0(t)$ from (6.53).

Diagonalising $\mathbf{M}_0$

Since we have that $\mathbf{y}_0^T(0) = (0, \hat{S}_0, ...)$ we can restrict our attention to the 2×2 submatrix of $\mathbf{M}_0$:

$$\mathbf{T} = \begin{pmatrix} \bar{A} & \sqrt{N}\bar{B} \\ \sqrt{N}\bar{B} & \bar{C} \end{pmatrix} , \tag{6.59}$$

which we diagonalise by a matrix $\mathbf{P}$ such that $\mathbf{T} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{T}\mathbf{P}$:

$$\mathbf{P} = \begin{pmatrix} \frac{-2\sqrt{N}\bar{B}}{-\bar{C}+\bar{A}-\sqrt{D}} & \frac{-2\sqrt{N}\bar{B}}{-\bar{C}+\bar{A}+\sqrt{D}} \\ 1 & 1 \end{pmatrix} , \tag{6.60}$$

where the constant

$$D = (\bar{C} - \bar{A})^2 + 4\bar{N}B^2 . \tag{6.61}$$

Further, we have

$$\mathbf{P}^{-1} = \frac{1}{2\sqrt{D}} \begin{pmatrix} \frac{-(-\bar{C}+\bar{A}-\sqrt{D})(-\bar{C}+\bar{A}+\sqrt{D})}{2\bar{B}} & \frac{-2\sqrt{N}\bar{B}}{-\bar{C}+\bar{A}-\sqrt{D}} \\ \frac{(-\bar{C}+\bar{A}-\sqrt{D})(-\bar{C}+\bar{A}+\sqrt{D})}{2\bar{B}} & \frac{-2\sqrt{N}\bar{B}}{-\bar{C}+\bar{A}+\sqrt{D}} \end{pmatrix} , \tag{6.62}$$

and

$$\mathbf{D} = \frac{1}{2} \begin{pmatrix} \bar{C} + \bar{A} + \sqrt{D} & 0 \\ 0 & \bar{C} + \bar{A} - \sqrt{D} \end{pmatrix} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} . \tag{6.63}$$

Now, since $\mathbf{D}$ is diagonal we can write

$$\mathrm{e}^{\mathbf{T}t} = \mathrm{e}^{\mathbf{P}\mathbf{D}\mathbf{P}^{-1}t} \tag{6.64}$$

$$= \mathbf{P}\mathbf{P}^{-1} + \mathbf{P}\mathbf{D}\mathbf{P}^{-1}t + \frac{1}{2!}\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P}\mathbf{D}\mathbf{P}^{-1}t^2 + \cdots \tag{6.65}$$

$$= \mathbf{P}\mathrm{e}^{\mathbf{D}t}\mathbf{P}^{-1} . \tag{6.66}$$

Then, we have

$$\begin{pmatrix} \hat{a}_0(t) \\ \hat{S}_0(t) \end{pmatrix} = \mathbf{P} \begin{pmatrix} \mathrm{e}^{\lambda_0 t} & 0 \\ 0 & \mathrm{e}^{\lambda_1 t} \end{pmatrix} \mathbf{P}^{-1} \begin{pmatrix} \hat{a}^{\mathrm{in}} = 0 \\ \hat{S}_0(0) \end{pmatrix} . \tag{6.67}$$

We can read off the expression for the field operator:

$$\hat{a}_0(t) = \left(P_{11}\mathrm{e}^{\lambda_0 t}P_{12}^{-1} + P_{12}\mathrm{e}^{\lambda_1 t}P_{22}^{-1} \right) \hat{S}_0(0) = f(t)\hat{S}_0(0) . \tag{6.68}$$

This gives

$$f(t) = \frac{\sqrt{N}\bar{B}}{\sqrt{D}} \mathrm{e}^{\frac{1}{2}(\bar{C}+\bar{A})t} \left(\mathrm{e}^{\frac{1}{2}\sqrt{D}t} - \mathrm{e}^{-\frac{1}{2}\sqrt{D}t} \right) . \tag{6.69}$$

6.4.1 Efficiency of the read-out interaction

We define the efficiency of the read-out interaction to zero'th order as

$$\eta = \int_0^\infty dt\ 2\kappa \hat{a}_0^\dagger(t)\hat{a}_0(t) = \int_0^\infty dt\ 2\kappa |f(t)|^2 \quad , \tag{6.70}$$

where 2κ ensures the correct normalisation since we defined the field from the classical equation of motion earlier. Now,

$$|f(t)|^2 = \frac{N|\bar{B}|^2}{|\sqrt{D}|^2} \mathrm{e}^{\mathrm{Re}(\bar{C}+\bar{A})t} \left| \mathrm{e}^{\frac{1}{2}\sqrt{D}t} - \mathrm{e}^{-\frac{1}{2}\sqrt{D}t} \right|^2 \ , \tag{6.71}$$

Consider $D = (\bar{C}-\bar{A})^2 + 4\bar{N}B^2$ together with equations (6.61) with (6.21) and (6.22). In the limit of week classical drive field, then $\bar{A}$ is the dominating term such that $|\sqrt{D}|^2 \simeq |\bar{A}|^2$. Furthermore, we take κ to be the dominating term of $\bar{A}$ such that the term proportional with $\mathrm{e}^{-\frac{1}{2}\sqrt{D}t}$ quickly decays and we ignore it. Then

$$|f(t)|^2 \approx \frac{N|\bar{B}|^2}{|\sqrt{D}|^2} \mathrm{e}^{\mathrm{Re}(\bar{C}+\bar{A}+\sqrt{D})t} \ . \tag{6.72}$$

We solve the integral and consider the decaying exponential to be zero at large t:

$$\eta = \frac{2\kappa N|\bar{B}|^2}{|\sqrt{D}|^2} \frac{-1}{\mathrm{Re}(\bar{C}+\bar{A}+\sqrt{D})} \ . \tag{6.73}$$

Then, we expand the square root in the leading term $(\bar{C}-\bar{A})$:

$$\sqrt{D} \simeq (\bar{C}-\bar{A}) + \frac{2N\bar{B}^2}{\bar{C}-\bar{A}} \ . \tag{6.74}$$

A further simplification gives

$$\bar{C}+\bar{A}+\sqrt{D} \simeq 2\bar{C} - \frac{2N\bar{B}^2}{\bar{A}} \ . \tag{6.75}$$

This means that the efficiency becomes:

$$\eta = \frac{-\kappa N|\bar{B}|^2}{|\bar{A}|^2\mathrm{Re}(\bar{C}) - N\mathrm{Re}(\bar{A}^*\bar{B}^2)} \ . \tag{6.76}$$

We have that

$$|\bar{B}|^2 = \frac{g^2\Omega^2}{16} \frac{\gamma_e^2+\Delta^2}{(\gamma_e^2+(\Delta+kv)^2)(\gamma_e^2+(\Delta-kv)^2)} \ , \tag{6.77}$$

while

$$\bar{B}^2 = \frac{g^2\Omega^2}{16}\left(\frac{-\gamma_e(\gamma_e^2+\Delta^2+(kv)^2)+i\Delta(\gamma_e^2+\Delta^2-(kv)^2)}{(\gamma_e^2+(\Delta+kv)^2)(\gamma_e^2+(\Delta-kv)^2)}\right)^2 . \tag{6.78}$$

Then we introduce the Cooperativity C again and compute

$$\bar{A}^* = -\kappa\left(1+C\gamma_e^2\frac{\gamma_e^2+\Delta^2+(kv)^2}{(\gamma_e^2+(\Delta+kv)^2)(\gamma_e^2+(\Delta-kv)^2)}\right) \tag{6.79}$$

$$+\, i\left(\Delta_{cav}-C\kappa\gamma_e\Delta\frac{\gamma_e^2+\Delta^2-(kv)^2}{(\gamma_e^2+(\Delta+kv)^2)(\gamma_e^2+(\Delta-kv)^2)}\right) \tag{6.80}$$

$$(\mathrm{Re}\bar{A})^2 = \kappa^2\left(1+C\gamma_e^2\frac{\gamma_e^2+\Delta^2+(kv)^2}{(\gamma_e^2+(\Delta+kv)^2)(\gamma_e^2+(\Delta-kv)^2)}\right)^2 \tag{6.81}$$

$$(\mathrm{Im}\bar{A})^2 = \left(\Delta_{cav}-C\kappa\gamma_e\Delta\frac{\gamma_e^2+\Delta^2-(kv)^2}{(\gamma_e^2+(\Delta+kv)^2)(\gamma_e^2+(\Delta-kv)^2)}\right)^2 \tag{6.82}$$

$$\mathrm{Re}\bar{C} = -\frac{\Omega^2}{16}2\gamma_e\frac{\gamma_e^2+\Delta^2+(kv)^2}{(\gamma_e^2+(\Delta+kv)^2)(\gamma_e^2+(\Delta-kv)^2)} \tag{6.83}$$

Now we have all the terms needed to compute the efficiency and after some algebra we get

$$\eta = \frac{C\kappa^2\frac{\gamma_e^2+\Delta^2}{\gamma_e^2+\Delta^2+(kv)^2}}{\kappa^2+\Delta_{cav}^2+C\kappa^2\frac{\gamma_e^2+\Delta^2}{\gamma_e^2+\Delta^2+(kv)^2}} . \tag{6.84}$$

This is the formula for the efficciency that describes the read-out to zero'th order in the field operator. One should worry about η being a positive quantity and normalised to $\eta \in [0,1]$. Inspection reveals that the efficiency is correctly normalised.

Next, imagine that kv is very large, then

$$\eta \approx \frac{C\kappa^2}{\kappa^2+\Delta_{cav}^2}\frac{\gamma_e^2+\Delta^2}{(kv)^2} , \tag{6.85}$$

which shows that the effeciency deteriorates with movement of the crystal. To some extent this can be countered by a larger coorperativity.

To compare with the expression computed in [Gorshkov *et al.*] we consider the limit $kv \to 0$. This is obviously a bit strange since this whole derivation has been done assuming $kv \neq 0$, but let us do it none the less to get

$$\eta = \frac{C\kappa^2}{\kappa^2+C\kappa^2+\Delta_{cav}^2} , \tag{6.86}$$

which in the limit of zero cavity detuning becomes

$$\eta = \frac{C}{1+C} , \tag{6.87}$$

exactly what was found in [Gorshkov *et al.*].

Let us continue the interpretation of (6.84) by considering our suggestion to move the crystal during the interaction with light. It is clear that the derivation was done under the assumption that we moved the crystal with a speed much larger than the speed of the thermal motion of each individual ion. But now we see that the speed of the crystal actually diminishes the efficiency, especially in the limit in (6.85). The good news for the experiment is that we can compensate this loss in effeciency due to the imposed motion by making the cooperativity larger.

In another way (6.84) makes good physical sense. If we divide by

$$\kappa^2 + \Delta_{cav}^2 \ , \tag{6.88}$$

and multiply by γ_e, then we get

$$\eta = \frac{C\gamma_e \frac{\kappa^2}{\kappa^2+\Delta_{cav}^2} \frac{\gamma_e^2+\Delta^2}{\gamma_e^2+\Delta^2+(kv)^2}}{\gamma_e + C\gamma_e \frac{\kappa^2}{\kappa^2+\Delta_{cav}^2} \frac{\gamma_e^2+\Delta^2}{\gamma_e^2+\Delta^2+(kv)^2}} \ . \tag{6.89}$$

This can be interpreted as the rate of decay into the cavity mode divided by the sum of all the possible decay rates. Here this sum is just two terms: The rate into the cavity mode and the scattering rate out of the cavity:

$$\eta = \frac{\Gamma_{cav}}{\Gamma_s + \Gamma_{cav}} \ . \tag{6.90}$$

Compared to the write-in it is clear that the balance for the experimentalists is more difficult to achieve for good efficiency for the read-out. But as we have indicated it is possible. There is also the fact that the cavity detuning should be kept minimal and preferably zero. Usually in these kinds of experiments one scans the cavity during the interaction such that at some point one reaches the optimal conditions. In our scheme this is probably not possible if the cavity detuning must be zero and a feeb-back mechanism should be employed to lock the cavity at the optimal condition.

Chapter 7

Conclusion and outlook

In this thesis we have considered the implementation of a DLCZ quantum repeater in a particular experimental realisation of ion Coulomb crystals. The circumstances of this experiment is such that the crystals are formed inside an optical standing wave cavity.

The central question we sought to resolve was the following: How does a standing wave in a cavity affect the light-atoms quantum interface? And immediately followed the question: If the effect is adverse to the quantum repeater scheme is there something simple to be done such that these ion crystals can still be used for the quantum repeater?

7.1 Conclusion

In answer to the first question we made a model and a numerical simulation of the interaction between the standing wave cavity field and the ion crystal. It became the measure of success to achieve a symmetric Dicke state in the ions during the interaction, since then the quantum photon would scatter into a cavity mode and be detected. This was not possible given the thermal motion of the ions in the standing wave cavity field as can be seen from the main result from Chapter 4, which is reproduced here in Figure 7.1. We conclude that an attempt to use the ion crystal for a single photon quantum-matter interface would be too susceptible to losses from scattering off the ions and out of the cavity undetected.

In answer to the second question we undertook a more detailed modelling of the quantum repeater write-in interaction. The ensuing calculation in the perturbative regime gave a general expression for the Fidelity of realising a

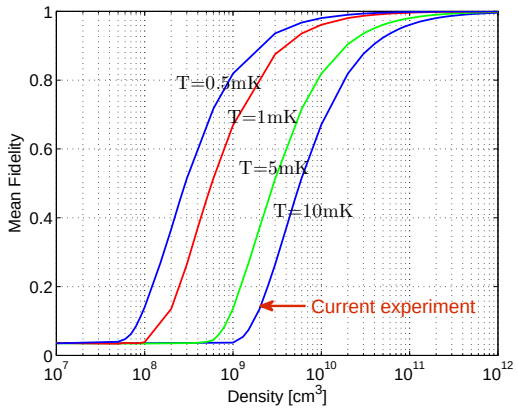

Figure 7.1: The Mean Fidelity of the write-in followed by the read-out process. The Fidelity is a measure of how close to a symmetric Dicke state ions are during the interaction.

symmetric Dicke state in the ions during the interaction:

$$\mathcal{F} = \langle\psi_{\rm W}|\hat{\rho}_A^{\rm out}|\psi_{\rm W}\rangle = \frac{\sum_j^{N_A}\sum_k^{N_A}\int_0^{t_c} dt\ \theta_j^*(t)\theta_k(t)}{N_A\sum_j^{N_A}\int_0^{t_c} dt|\theta_j(t)|^2}\ , \tag{7.1}$$

$$\theta_j(t) = \int_0^t {\rm d}t' \int_0^{t'} {\rm d}t'' {\rm e}^{-(\kappa+i\Delta_{cav})(t-t')}{\rm e}^{(-\gamma'+i\Delta)(t'-t'')}g^{(j)}(t')\Omega^{(j)}(t'')\ . \tag{7.2}$$

Then, we presented the idea to move the crystal during the interaction and to move it with a speed much larger than the thermal motion of the individaul ions. An important case was the case of a constant speed of movement, which could be solved analytically. We found that given a large detuning we could indeed realise a Dicke state in the ensemble of ions. Furthermore, we presented a calculation of the error to the Fidelity of unity as a function of the other physical parameters in the system given the movement of the whole crystal. The Fidelity with the correction to leading order was:

$$\mathcal{F} = 1 - \frac{\kappa^2 + \Delta_{cav}^2}{8(kv)^2}\ , \tag{7.3}$$

for a large detuning. This result suggests that moving the crystal during the interaction with light in effect averages the large variation in the standing waveform in the cavity. As a consequence we can achieve a Dicke state in the ions and detect the quantum photon since it will be scattered into a cavity mode.

The promising result when moving the crystal was conditioned on a large detuning but, as we argued, there are draw-backs to a very large detuning, notably the problem of detecting the quantum photon when meny classical photons are present. It is therefore important to know if the proposed scheme is at all realistic given the particular experimental setting. By calculating the number of classical photons necessary to trigger the quantum interaction:

$$\hat{N}_{\text{phot}} \simeq \left(\frac{\Delta}{2C\gamma}\right)^2 \hat{N}_{\text{out}} N_A \ , \tag{7.4}$$

we concluded that we could indeed initiate the interaction with few enough classical photons such that the quantum photon could still be detected.

With regards to the read-out process we discussed how it was not possible to be in a far detuned regime and that the interaction could no longer be considered as a simple perturbation. We found a way to solve for the operators as an expansion. We did perturbation calculation in this quantity and found the field to zero'th order by diagonalising the problem. The end result is an efficiency of the read-out interaction to zero'th order in the field:

$$\eta = \frac{C\kappa^2 \frac{\gamma_e^2+\Delta^2}{\gamma_e^2+\Delta^2+(kv)^2}}{\kappa^2 + \Delta_{cav}^2 + C\kappa^2 \frac{\gamma_e^2+\Delta^2}{\gamma_e^2+\Delta^2+(kv)^2}} \ . \tag{7.5}$$

This expression is derived on the premises that we move the crystal with a large speed compared to the thermal motion of the ions. In the relevant limits the efficiency never the less reduces to already established results that did not consider the speed of the crystal or the individual ions. A second important point is that moving the crystal actually lowers the efficiency of the read-out, but this could in priciple be corrected by a larger cooperativity.

7.2 Outlook

In this presentation we have considered in theory an idea that might overcome the standing wave problem. We found a regime where we could solve the equations analytically. This required a constant speed of movement of the crystal. Now it is time for feed-back from the experimentalists – is it at all possible to move the crystal as we suggest? Otherwise it could be interesting to make numerical simulations of not-constant speeds that *are* possible in the laboratory. Another immediate thing to do is to find the corrections to the efficiency of the read-out interaction in (7.5), and to see how they depend on the speed of movement.

The work presented in this thesis has focused solely on the DLCZ quantum repeater protocol as envisioned in the original proposal. Recently, work at the Niels Bohr institute and at Harvard University have suggested another way of implementing the quantum repeater node in an atomic ensemble

[Brask *et al.*]. Instead of using two ensembles in the entanglement process they use only one ensemble but with two possible excitations. The entanglement is done via fluorescence detection instead of reading out the photon and using photocounter detection. The efficiency of fluoresence detection is much higher than any single photon counting which could greatly improve the DLCZ repeater. Due to the fact that the same ensemble holds two excitations there are both lower and upper bounds on the number of atoms in the ensemble for good operation. What is interesting is that the number of ions in an ion crystal in the Aarhus experiment could fall within this interval.

Acknowledgements

The work in this thesis has benefitted from the input from a number of people:

Chapter 5 was done in collaboration with Post. Doc. Dirk Witthaut at the Niels Bohr institute, now at the Max-Planck-Institute in Göttingen. He also offered advice on many different questions scattered throughout the thesis.

We had some sobering input from the principal investigator of the Aarhus experiment Michael Drewsen that put a merciful end to some of our speculations.

Two Ph.D. students working on the experiment at Aarhus University have been helpful. Magnus Albert sent a preprint of some as of yet unpublished work and showed us around the laboratory and Peter Herskind provided a thesis [Herskind thesis] documenting much of the work done to achieve strong coupling.

Ph.D. student at the Niels Bohr institute Jonatan Brask provided a preprint of [Brask *et al.*]. Also, it has been helpful to read his master thesis [Brask master thesis] for writing the chapter on quantum repeaters of the present thesis.

Also, I would like to thank everybody at Quantop at the Niels Bohr Institute, especially Jörg Müller and Eugene Polzik. Their support has been very, very encouraging.

Finally, I want to thank Anders Sørensen at the Niels Bohr Institute who supervised the project. It was his suggestion to move the crystal during the interaction to solve the problem of the standing wave.

Summary / Resumé

Summary in English

This master's thesis considers a quantum repeater-type interaction in a particular experimental realisation of ion Coulomb crystals. To distribute quantum states over large distances is very difficult due to the decoherence of each quantum state during transmission. A series of quantum repeaters work by distributing entanglement between two distant locations and then transfering a quantum state with the transmission of only classical information. The DLCZ quantum repeater does this using atomic ensembles and only linear optics – ingredients which are already available experimentally.

A candidate for such an atomic ensemble is an Ion Coulomb crystal. At Aarhus University an experiment has realised such crystals with several thousand ions trapped in an electric trap where they form long-lived periodic structures. The crystal is in an optical cavity to realise strong coupling to light. The cavity is a standing wave cavity and we show in a simulation that the standing wave is detrimental to the quantum repeater interaction.

We present a detailed calculation of the quantum repeater so-called *write-in* interaction in a standing wave cavity. A possible solution to the problem of the standing wave is presented in the form of moving the crystal during the interaction with light. The result is an expression for the Fidelity of obtaining an interaction photon in the cavity mode which is a measure of success. The special case of a constant speed of the crystal allows for an analytical solution, and at sufficiently high detuning the Fidelity goes to unity. A correction to this Fidelity of unity is found using perturbation theory. The result is promising: In the limit of high detuning the correction can be made small by moving the crystal sufficiently fast.

Finally, we calculate the so-called *read-out* interaction. Again, fast movement of the crystal is considered. A complication is the fact that high detuning is not possible in the read-out. The result is given as an efficiency of the read-out that is shown to be in correspondence with already established results in other limiting cases. The efficiency is to some extent degraded by fast movement of the crystal but it can be recovered by having a high cooperativity.

Resumé på dansk

Dette speciale undersøger mulighederne for at anvende eksperimentelt realiserede ionkrystaller som del af en *kvanterepeater*. Det er en stor udfordring at sende kvantetilstande over lange afstande, idet en kvantetilstand dekohærer yderst let. En serie af kvanterepeatere ville kunne distribuere *entanglement* over store afstande, for derefter at overføre en kvantetilstand udelukkende ved klassisk informationsoverførsel. Den såkaldte DLCZ-kvanterepeater fungere ved brug af atomare ensembler og lineære optiske komponenter – ingredienser, der allerede er fast inventar i laboratorier.

En kandidat til det atomare ensemble er ioner organiseret i Coulombkrystaller. På Aarhus Universitet har man realiseret sådanne krystaller ved at fange ioner i et elektrisk felt, hvori de samler sig i periodiske strukturer. Krystallen sidder i en optisk kavitet for at opnå et regime, hvor krystallen kobler særlig effektivt til lysfeltet. Kaviteten understøtter stående bølger og disse stående bølge er ødelæggende for virkningsgraden af kvanterepeateren. Dette viser vi i en numerisk simulation.

Dernæst præsenterer vi en detaljeret udregning af kvanterepeaterens såkaldte *write-in*, hvor en eksitation overføres fra feltet til krystallen. En mulig løsning til problemet med den stående bølge er at flytte krystallen, mens den interagerer med feltet. Vores resultat er en formel for *Fidelity*, som er et mål for succes. I det specialtilfælde hvor hastigheden af krystallen er konstant, kan vi finde et analysisk udtryk for Fidelity. Hvis det klassiske felt, der driver interaktionen, er langt fra resonans, så går Fidelity mod 1. Vi beskriver korrektionen til dette et-tal og finder, at korrektionen kan minimeres ved at bevæge krystallen tilstrækkelig hurtigt.

Til sidst udregnes kvanterepeaterens *read-out*. Igen betragter vi tilfældet, hvor vi bevæger krystallen. Denne interaktion kompliceres ved, at vi ikke kan gå langt fra resonans, når vi driver systemet. Resultatet er en effiktivitet, som er en korrektion til allerede etablerede udtryk, og i de relevante grænser ses de at stemme overens. Effektiviteten degraderes ved at flytte krystallen hurtigt, men der kan kompenseres herfor ved en stor *cooperativity*.

Appendix A

Code for Matlab

fun_fid the function called in the following plot rutine

```
function mean_F=fun_fid(r,T)
k=1.38e-23;
m=40*1.7e-27;
e0=8.85e-12;
e1=1.602e-19;
a=1.202;
l=866e-9; % wavelength of quantum light
c=3e+8; % speed of light in vacuum
k_0=2*pi/l; % wavenumber of standing wave in cavity
N=500; % number of ions
z=(0:1:N-1)*22e-6; % position of the ions 22 micrometers apart
s=(sin(k_0*z));%.^2; % size of the standing wave at each ion-site
n_in=s*s'; % normalisation
psi_in=s./sqrt(n_in); % normalised wavefunction
M=2*sqrt(k*e0/(e1^2*a));
d=(1./r).^(1/3)/100;
% implement a 'for loop' such that we get many realisations of Fidelity:
for n=1:500
    si=M*sqrt(T*d^3);  % std deviation
    dz =random('Normal', 0, si, 1, N); %Rand walk is norm.distr, \mu=0
    z_p=z+dz; % new z-coordinate after random walk during storage time.
    p=(sin(k_0*z_p));%.^2; % the new waveform
    n_out=p*p'; % normalisation
    psi_out=p./sqrt(n_out); % normalised wavefunction
    F(n)=abs(psi_out*psi_in'); % Fidelity...
end
mean_F=mean(F); % taken from the distribution of n realisations
```

fun_test to plot Figure 4.10

```
clear figure
clear Fid
r=[0.0001 0.001 0.01 0.013 0.015 0.02 0.025 0.03 0.04 0.05 0.06 0.1
0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.3 1.6 2 2.5 3 4 5 8 10]*10^11;
for i=1:length(r)
    Fid(i)=fun_fid(r(i),10e-3);
end
semilogx(r,Fid);
hold on

clear Fid
r=[0.0001 0.001 0.003 0.004 0.005 0.006 0.007 0.008 0.009 0.01 0.015
0.02 0.025 0.03 0.06 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.3 1.6 2
2.5 3 4 5 8 10]*10^11;
for i=1:length(r)
    Fid(i)=fun_fid(r(i),5e-3);
end
semilogx(r,Fid, 'color','g');
clear Fid
r=[0.0001 0.0005 0.0006 0.0007 0.0008 0.0009 0.001 0.002 0.003 0.004
0.005 0.006 0.01 0.03 0.06 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.3
1.6 2 2.5 3 4 5 8 10]*10^11;
for i=1:length(r)
    Fid(i)=fun_fid(r(i),10e-4);
end
semilogx(r,Fid ,'color','r');
clear Fid
r=[0.0001 0.0005 0.0006 0.0007 0.0008 0.0009 0.001 0.0015 0.002 0.0025
0.003 0.006 0.01 0.03 0.06 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.3
1.6 2 2.5 3 4 5 8 10]*10^11;
for i=1:length(r)
    Fid(i)=fun_fid(r(i),5e-4);
end
semilogx(r,Fid ,'color','b');
title('Mean Fidelity as a function of density');
xlabel('Density/cm^3');
ylabel('Mean Fidelity');
```

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