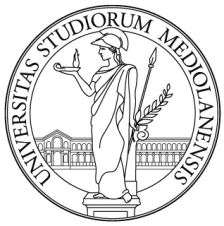


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FACOLTÀ DI SCIENZE MATEMATICHE,
FISICHE E NATURALI

CORSO DI LAUREA MAGISTRALE IN FISICA

# A SINGLE PHOTON TRANSISTOR REALIZED BY A THREE-LEVEL LADDER ATOM IN A CAVITY

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# Preface

This master's thesis presents the main results of the project concerning the proposal of a new kind of single photon transistor on which I have worked in the Theoretical Quantum Optics Group of the Niels Bohr Institute. The work has been part of a period as exchange student in the context of the Erasmus program that I have passed at the University of Copenhagen. The work has been carried out during the period February to December 2012.

The main objective of the project was the theoretical study of the efficiency of a new scheme for a single photon transistor which is meant to extend the possibility of experimental realization of that one proposed in Ref. [10].

The thesis is intended for physics master students readers or with an equivalent background. The main concepts of quantum optics and open systems theory are introduced in the first two chapters but the reader is assumed to have a background in quantum mechanics.

My theoretical knowledge of quantum optics, which at the beginning of the work was basic, has been increased constantly and significantly during all the period of the work. I also have learned to use the program *Matlab* which has been necessary to perform numerical simulations which often have proved to be a valid support to the analytical work.

A sincere gratitude goes to my supervisor Prof. Anders S. Sørensen, whose belongs all the main ideas of the project. He has provided patient help whenever I needed, explaining me new theoretical techniques, correcting my errors and showing me how to interpret the results. Among the other members of the group special thanks goes to Florentin Reiter who provided me useful help with the superconducting circuits literature. Furthermore I would like to thank Prof. Matteo Paris who offered to be my supervisor in Milan and made possible my Erasmus candidature and this work.

## English summary

In this thesis we propose and study a new scheme for the realization of a single photon transistor, i.e. a device in which a single control photon determines the transmission or reflection of a stronger photonic field. In 2007 a scheme for such a transistor has been proposed in Ref. [10] using the strong coupling between plasmonics modes on a one-dimensional nanowire and a nearby three-level atom with a $\Lambda$-configuration of the levels. The difficulty in realizing experimentally such a system has lead us to develop a new scheme, based on a three-level atom of $\Xi$ kind inserted in a cavity. Our scheme has the great advantage to be suited for an implementation with superconducting circuits which has been proved to realize cavity QED system with a coupling constant up to $10^4$ times larger than in traditional microwaves cavities.

The system used consists in a two-side cavity with the two walls coupled to two one-dimensional waveguides at a loss rate of $\kappa$. The cavity mode is resonant with the transition between the excited state $|e\rangle$ and the upper ground state $|g\rangle$ with coupling constant $g$. The transition between $|g\rangle$ and the lower ground state $|f\rangle$ is far off-resonance but can be driven by a classical external field. The two upper levels can decay spontaneously with rates respectively $\gamma_2$ and $\gamma_1$. The strong coupling regime is assumed, i.e $g \gg \kappa \gg \gamma_{1,2}$.

The protocol consists of two steps. In the first step the atom is prepared in the state $|a_i\rangle = 1/\sqrt{2}(|g\rangle + |f\rangle)$ and a pulse of time width $\sigma_T \gg 1/\kappa$ consisting of zero or one photon is sent. In the case of one photon the phase of the state $|f\rangle$ is flipped. After a time $T > \sigma_T$ a classical $\pi/2$ pulse is performed. We prove that this ideally realizes the map $|0\rangle|a_i\rangle \rightarrow |0\rangle|g\rangle, |1\rangle|a_i\rangle \rightarrow |1\rangle|f\rangle$. During the second step a coherent state is sent and we prove that for intensities lower than $g^2/4\kappa$ if the atom is in $|g\rangle$ a large part of the intensity is reflected while is it is in $|f\rangle$ there is complete transmission. In this way we have shown that all the conditions for the working of the transistor are satisfied. In the thesis we have not only limited our attention to the ideal limit, but we have also taken into account the imperfections and quantified their contributions to the probability of error.

The dynamics of the first step is studied in detail with a Schrödinger approach and the coefficients of reflection and transmission for a single photon are determined. We also consider the situation in which the lower transition is still off-resonance but cannot be neglected. Combining these result with a detailed analysis of the protocol

a compact expression, containing the different sources of errors, for the probability of fail is derived. For optimized values of $\kappa$ and $\sigma_T$ it goes as $(\gamma_1\gamma_2/g^2)^{2/5}$, thus it works remarkably well if $g \gg \gamma_{1,2}$. The dynamics of the multiphoton case is determined by solving numerically the master equation. In this way we obtain the curves of transmission and reflection as function of the input intensity. We find that the gain of the transistor for $T' = 1/\gamma_1$ is $g^2/4\kappa\gamma_1$, which with the optimized values of $\kappa$ and $\sigma_T$ scales as the inverse of the error probability of the first step.

## Riassunto in italiano

In questa tesi proponiamo e studiamo un nuovo schema per la realizzazione di un transistor a fotone singolo, vale a dire un dispositivo in cui un singolo fotone determina la trasmissione o riflessione di un campo più intenso. Nel 2007 uno schema per un tale transistor è stato proposto nella Ref. [10] usando il forte accoppiamento tra i modi plasmonici su un nanoconduttore monodimensionale e un atomo a tre livelli posto nelle vicinanze con configurazione $\Lambda$ dei livelli. La difficoltà nel realizzare sperimentalmente un tale schema ci ha condotto a proporne uno differente, basato su una atomo a tre livelli con configurazione $\Xi$ posto in una cavità. Il nostro schema ha inoltre il grande vantaggio di poter essere realizzato utilizzando circuiti superconduttori che è stato dimostrato possono realizzare sistemi di cavity QED con una costante di accoppiamento fino a $10^4$ volte più grande che non nelle tradizionali cavità a microonde.

Il sistema usato consiste in una cavità a due pareti, ciascuna accoppiata con una guida d'onda monodimensionale e un rate di perdita $\kappa$. Il modo della cavitÃ  è risonante con lo stato eccitato $|e\rangle$ e con il piú alto dei due stati fondamentali $|g\rangle$ con una costante di accoppiamento $g$. La transizione tra $|g\rangle$ e lo stato fondamentale inferiore $|f\rangle$ è lontana dalla risonanza ma puó essere stimolata da un campo classico esterno. I due livelli superiori possono decadere spontaneamente con rate rispettivamente $\gamma_2$ e $\gamma_1$. Assumiamo un regime di accoppiamento forte, cioè $g \gg \kappa \gg \gamma_{1,2}$.

Il protocollo consiste di due fasi. Nella prima l'atomo è preparato nello stato $|a_i\rangle = 1/\sqrt{2}(|g\rangle + |f\rangle)$ e un impulso di larghezza temporale $\sigma_T \gg 1/\kappa$ consistente in uno o nessun fotone è inviato. Nel caso di un fotone la fase dello stato $|g\rangle$ cambia di segno. dopo un tempo $T > \sigma_T$ un impulso classico di area $\pi/2$ viene mandato. Nella tesi proviamo che queste operazioni realizzano idealmente la mappa $|0\rangle|a_i\rangle \rightarrow |0\rangle|g\rangle, |1\rangle|a_i\rangle \rightarrow |1\rangle|f\rangle$. Durante la seconda fase un campo in uno stato coerente viene inviato e abbiamo dimostrato che se l'intensità è minore di $g^2/4\kappa$ l'atomo in $|g\rangle$ riflette gran parte dell'intensità mentre l'atomo in $|f\rangle$ realizza una trasmissione completa. In questo modo abbiamo dimostrato che tutte le condizioni necessarie per il funzionamento del transistor sono soddisfatte. La nostra analisi non si è limitata al limite ideale, ma le varie imperfezioni sono state prese in considerazione e il loro contributo alla probabilità di errore è stato quantificato.

La dinamica della prima fase è studiata in grande dettaglio con un approccio

alla Schrödinger e i coefficienti di riflessione e trasmissione vengono determinati. In oltre consideriamo anche il caso in cui la transizione inferiore é fuori risonanza ma non al punto di essere trascurata completamente. Combinando questi risultati con un'analisi dettagliata del protocollo ricaviamo un'espressione compatta per la probabilità di fallimento che contiene le diverse fonti di errore. Con i valori ottimizzati di $\kappa$ e $\sigma_T$ troviamo che questa probabilitá va come $(\gamma_1\gamma_2/g^2)^{2/5}$. La dinamica del caso multifotonico è determinata risolvendo numericamente la master equation. In questo modo otteniamo le curve di riflessione e trasmissione in funzione dell'intensità entrante. Troviamo che il guadagno del transistor, per $T' = 1/\gamma_1$, vale $g^2/4\kappa\gamma_1$ che con i valori ottimizzati di $\kappa$ e $\sigma_T$ scala come l'inverso della probabilità di errore del primo step.

La struttura della tesi è la seguente:

- **Capitolo 2**: Presenta i concetti fondamentali dell'ottica quantistica come la quantizzazione del campo elettromagnetico e il modello di Jaynes-Cumming per la descrizione dell'interazione di un atomo con un singolo modo del campo.

- **Capitolo 3**: Introduce la teoria dei sistemi aperti da un punto di vista operatoriale, cioè per mezzo delle equazioni di Heisenberg-Langevin, e da un punto di vista dello stato del sistema, attraverso la teoria della master equation. Inoltre la teoria del "quantum jump" è esposta.

- **Capitolo 4**: Descrive il modello di transistor fotonico proposto da Chang in *et al.* nel 2007 e la nostra proposta. Una sezione è dedicata ai sistemi superconduttivi.

- **Capitolo 5**: Presenta la derivazione analitica dei risultati. La dinamica del primo step del protocollo è risolta nel caso generale e nel caso di impulso gaussiano e rettangolare. La dinamica del secondo step viene trattata nell'approssimazione di stato stazionario per un campo entrante coerente.

- **Capitolo 6**: Presenta la derivazione della formula per l'errore del primo step. Vengono trovati i valori ottimali dei parametri $\kappa$ e $\sigma_T$ e viene fornita un'espressione della probabilità di errore e del guadagno del transistor nel caso ottimale.

# Contents

# Chapter 1

# Introduction

Quantum optics is a field of physics in which quantum mechanics is applied to the theory of light and of its interaction with matter [1, 2, 3, 4, 5]. It is at present one of livest area of research in physics, and the interest in it has increased in the last two decades with the developing of quantum computation and quantum information theory [6]. Despite the fact that the model systems for physical implementation of quantum information processing devices ranging from almost all areas of physics has been proposed and everyday studied, those one which have reached the status of "products on the market", i.e. communication devices using quantum cryptography protocols, rely on quantum optics systems such as optical fibers and photodetectors.

Central in quantum optics is the concept of *photon*, which has been introduced by Einstein in 1905 to explain the photoelectric effect, even if a complete and modern combination of wave-like and particle-like aspects of light is due to Dirac in 1927, one year after the born of quantum mechanics. In Dirac's theory each mode of the electromagnetic field is considered as a quantum harmonic oscillator and the number of photons of the field correspond to the number of excitations of the oscillators. One consequences of the quantization of the electromagnetic field is the rise of vacuum fluctuations, necessary to explain the spontaneous emission of atoms. Most of phenomena of lights are explainable by the Maxwell classical theory [7] and others by a semi-classical approach in which atoms are quantized but light is treated classically, but some phenomena like quantum beats in $\Lambda$-atom require a full quantum theory.

To make photons interact with others photons is one of the most important challenges of quantum optics. Photons rarely interact and though in some media is possible to obtain non-linear effects by exploiting the fact that their refraction index is a function of the intensity of the field, these effects are very small. Optical non-linearities can be obtained at the level of single photons, by making them interact with single atoms. Recently the interaction of photons propagating on one-dimensional waveguide with two-level emitters has been studied by some authors [8, 9] and the transmitted properties of the emitter has been exploited to project a single-photon transistor by the authors of Ref. [10] . The working of this proposal requires the use of a three-level atom in a $\Lambda$ configuration, i.e. an atom with an excited state and two ground states with different energies. Up to date, the absence of such an atom strongly coupled to the waveguides has, however, prevented the experimental realization of the scheme.

Strong coupling between atoms and light can be achieved placing the atoms in cavities with high quality factor. The subfield of quantum optics which deal with this subject is called *cavity QED*, and its importance cannot be underestimated. The use of cavities in which just a discrete set of modes are allowed permits the practical realization of the Jaynes-Cumming model for the interaction between atoms and light. This model is simple but extremely powerful and the dynamics generated by it can be solved exactly in a number of situation. Our work consists mainly in a new scheme for the photonic transistor relying on the properties of a three-level $\Xi$-atom. The main goal of this thesis is to validate it theoretically. The language and the formalism used in the work is that one of cavity QED, even if the ideal practical implementation of the scheme consists in the use of an artificial superconducting atoms. Indeed it has been shown that superconducting circuits can reproduce cavity QED systems with the great advantage of reaching a stronger coupling between (artificial) atoms and field than in microwave and optical cavities[11]. The field of superconducting circuits has risen in the last fifteen years and is a very promising alternative to the use of optical system for the realization of quantum information processes, since quantum effects are obtained at a mesoscopic level.

The thesis outline is:

- **Chapter 2**: Presents the fundamentals concepts of quantum optics, in par-

ticular the quantization of the electromagnetic field and the Jaynes-Cumming model for the description of the interaction of an atom with a single mode of the field.

- **Chapter 3**: Introduces the theory of open systems from both an operatorial and a density matrix approaches, i.e Heisenberg-Langevin equations and master equation theory. Furthermore the "quantum jump" theory is explained. The knowledge of open systems tools is essential to introduce in the Hamiltonian of our system the decay of atomic levels. Also some approximations used in this chapter, such as the Markov approximation, will prove useful to solve for the dynamics of our system.

- **Chapter 4**: Describes the scheme for the transistor proposed by Chang *et al.* nel 2007 and our proposal. A section of this chapter is dedicated to a brief review of superconducting systems.

- **Chapter 5**: Presents the analytical derivation of the results, i.e. the dynamics of the system atom-cavity-waveguides is solved for the general case of an incoming single photon, then the result is specialized to the case of a gaussian and a rectangular pulse. The transmission and reflection coefficients of the cavity for a single photon are derived. In the end the multi-photon case is treated in the situation of a coherent input field.

- **Chapter 6**: Presents the derivation of the error formula for our scheme and gives the optimized parameters of the scheme and the gain of the transistor, summarizes the results with the state-of-art available values of our scheme parameters and provides a brief description of some application of the photonic transistor.

# Chapter 2

# Interaction between light and atoms

## 2.1 The quantization of the electromagnetic field

[1]The quantization of light is fundamental in the complete explanation of several facts and experiments. Among them we find the spontaneous emission, the Lamb shift, the gyromagnetic moment of the electron and, for what concerns closely our work, the dynamics of an atom in a resonator.

Following [2] we briefly review the quantization of a single-mode cavity field. Let's consider a perfect cavity of volume $V$, i.e. a cavity without losses. A monochromatic mode polarized in the $\hat{x}$ direction, satisfying Maxwell equations and boundary condition, has form

$$\mathbf{E}(z,t) = \hat{x}q(t)\sqrt{\frac{2\omega^2}{\epsilon_0 V}}\sin kz, \tag{2.1}$$

where $q(t), \Omega$ and $k$ are the amplitude, the frequency and the wave number of the field. Of course, to satisfy the boundary condition, $\omega = cm\pi/L$, with $m$ a certain integer. From the Maxwell equations without sources is immediate to obtain the magnetic field

$$\mathbf{B}(z,t) = \hat{y}\frac{\dot{q}(t)}{c^2 k}\sqrt{\frac{2\omega^2}{\epsilon_0 V}}\cos kz. \tag{2.2}$$

[1]The literature for this chapter is contained in [1, 2, 3]. A more detailed treatment of the extension to the multi-mode field of the concepts exposed is contained in [4]

The electromagnetic energy density associated with the field is known to be

$$\mathcal{U} = \frac{1}{2}\left[\epsilon_0 E^2 + B^2/\mu_0\right] \tag{2.3}$$

so that the Hamiltonian that describe the single-mode field is

$$\mathcal{H} = \frac{1}{2}\int dV\left[\epsilon_0 E^2 + B^2/\mu_0\right]. \tag{2.4}$$

If we insert eqs. (2.1) and (2.2) in (2.4) we find that the Hamiltonian is simply

$$\mathcal{H} = \frac{1}{2}\Big(p^2 + \omega^2 q^2\Big), \tag{2.5}$$

where we have put $\dot{q} = p$. We recognize eq. (2.5) to be the Hamiltonian an harmonic oscillator of unitary mass. The quantization of an harmonic oscillator is a well known procedure, reported on every book of quantum mechanics. We take $q$ and $p$ as canonical variables for the classical system associated and replace them with the operators $\hat{q}$ and $\hat{p}$ satisfying the commutation relation

$$\left[\hat{q}, \hat{p}\right] = i\hbar. \tag{2.6}$$

It is convenient and traditional to replace the canonical hermitian operators with two non-hermitian operators, defined as

$$\hat{a} = (2\hbar\omega)^{-1/2}(\omega\hat{q} + i\hat{p}) \tag{2.7}$$

$$\hat{a}^\dagger = (2\hbar\omega)^{-1/2}(\omega\hat{q} - i\hat{p}). \tag{2.8}$$

It is just algebra to show that the commutation relation between these operators is

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \tag{2.9}$$

and that the Hamiltonian is

$$H = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big). \tag{2.10}$$

Replacing the position operator in (2.1) in term of the operators $\hat{a}$ and $\hat{a}^\dagger$ give us the

expression for the electric field

$$\hat{\mathbf{E}}(z,t) = \hat{x}\sqrt{\frac{2\omega^2}{\epsilon_0 V}}(\hat{a}^\dagger + \hat{a})\sin kz. \tag{2.11}$$

The quantization of a multi-mode field is quite obvious. We replace (2.1) with

$$\mathbf{E}(z,t) = \sum_s \hat{\epsilon}_s q_s(t)\sqrt{\frac{2\omega_s^2}{\epsilon_0 V}}\sin k_s z, \tag{2.12}$$

where $\hat{\epsilon}_s$ is the polarization of mode $s$, $\omega_s = ck_s$, $k_s = s\pi/L$ with $s = 1, 2, 3..$ and $L$ is the length of the cavity in the $z$ direction. Now the Hamiltonian is the sum of the single-mode Hamiltonians,

$$H = \sum_s H_s = \sum_s \hbar\omega_s\left(\hat{a}_s^\dagger\hat{a}_s + \frac{1}{2}\right), \tag{2.13}$$

with commutation relations of the annihilation and creation operators

$$\left[\hat{a}_s, \hat{a}_{s'}^\dagger\right] = \delta_{s,s'}. \tag{2.14}$$

## 2.2 Fock states

[2] The eigenstates of the Hamiltonian (2.10) are called *Fock states* and are denoted as $|n\rangle$. They have eigenvalues $E_n$, i.e.

$$H|n\rangle = E_n|n\rangle, \tag{2.15}$$

with

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, ... \tag{2.16}$$

It is clear from eq. (2.16) that $n$ is the number of quanta of energy $\hbar\omega$ present in the single-mode field when it is in the $|n\rangle$ state. It is common to say, in an equivalent formulation, that $n$ is the number of photon of the field. The states $|n\rangle$ are an orthonormal basis of the Hilbert space of the Hamiltonian (2.10). This means that every state of the field can be written as a superposition of Fock states.

[2] From here we use the convention to put an hat on operators only when there can be ambiguity. We denote with $\mathcal{H}$ the classical Hamiltonian and with $H$ the quantized operator Hamiltonian.

The action of the operator $\hat{a}$ on the state $|n\rangle$ is

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{2.17}$$

that means to remove a photon from the state. For this reason $\hat{a}$ is called *annihilation* operator. On the opposite the operator $\hat{a}^\dagger$ creates a photon,

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{2.18}$$

and so it is called *creation* operator. The operator $\hat{a}^\dagger\hat{a} = \hat{n}$, that appears in the Hamiltonian, is called number operator for obvious reasons and its expectation value on a generic state is the average number of photons in the field.

The generalization to the multi-mode case consists simply in taking the tensor product of single-mode basis. The general state of the field is

$$|\psi\rangle = |\psi_1\rangle\,|\psi_2\rangle\,|\psi_2\rangle\,.. \tag{2.19}$$

where $|\psi_1\rangle$,$|\psi_2\rangle$,.. are single-mode states, i.e. superpositions of single mode Fock states.

## 2.3 Coherent states

If we take the expectation value of the electric field (2.11) on a Fock state we get zero, no matter how large is the value of $n$. Classically we deal with electric and magnetic field that are not null, so that the ordinary light cannot be composed by Fock states but must be a superposition of them. If we replace in (2.11) operators with continuous variables we find that the electric field has the classical form. To do that we have to find some states that are eigenstates of the creation and annihilations operators. It turns out that just the annihilation operator has eigenstates. These states are called *coherent* states and it is possible to show that, in the Fock states basis, they have the form

$$|\alpha\rangle = \exp\big(-\frac{1}{2}|\alpha|^2\big)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{2.20}$$

The expectation value of the operator $\hat{n}$ and so the average number of photons of the state is clearly $|\alpha|^2$. The distribution of the probability to measure $n$ photons is

poissonian, i.e.

$$P_n = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}. \tag{2.21}$$

The coherent states are "the most classical state of light" in fact, as already stated, the expectation value of the electric field is

$$\langle \alpha | \, E_x(z,t) \, | \alpha \rangle = 2|\alpha| \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \sin(\omega t - kz - \theta), \tag{2.22}$$

that corresponds to a classical field, see Fig. 2.1. Furthermore, while the uncertainty of the electric field on a Fock state is

$$\Delta E_x = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left( n + \frac{1}{2} \right)^{1/2}, \tag{2.23}$$

for a coherent state it is just $\sqrt{\hbar\omega/2\epsilon_0 V}$, being equal to that one of the vacuum. Also, the fractional uncertainty in the number of photon and the uncertainty in the phase become smaller as the number of photon increases.

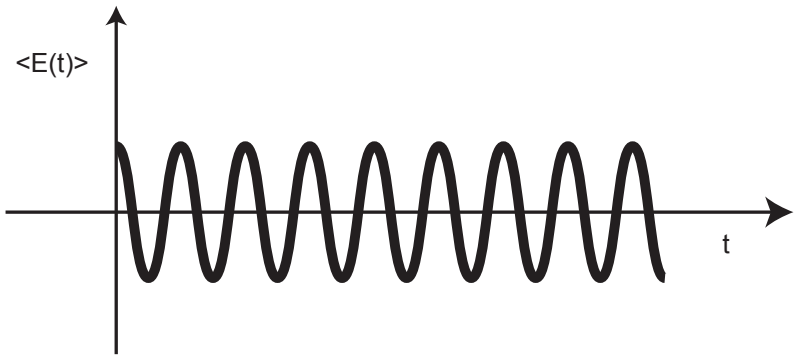


Figure 2.1: Coherent state expectation value of the electric field as function of time for a fixed point. The fluctuations of the field are the same as those of the vacuum state and are independent of the time.

The coherent states span all the Hilbert space but they are not orthonormal, indeed

$$| \langle \beta | \alpha \rangle |^2 = e^{-|\beta - \alpha|^2}, \tag{2.24}$$

and the completeness relation is

$$\int |\alpha\rangle \langle \alpha| \, \frac{d\alpha^2}{\pi} = 1 \tag{2.25}$$

## 2.4 The Jaynes-Cumming model

In this section we deal with the problem of the dynamics of an atom interacting with a single-mode field. The physical situation corresponding to this problem is that one of an atom placed in a cavity supporting only one mode. On the contrary, a free atom interacts with an infinite number of modes so that the dynamics it is not well described by assuming only a single-mode field.

We consider an atom of two levels $|g\rangle$ and $|e\rangle$, interacting with a field

$$\hat{\mathbf{E}}(z,t) = \hat{\mathbf{e}}\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}(\hat{a}^\dagger + \hat{a})\sin kz. \tag{2.26}$$

where $\hat{\mathbf{e}}$ is an arbitrary oriented polarization vector. The interaction Hamiltonian is

$$H^{(I)} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}, \tag{2.27}$$

where $\hat{\mathbf{d}} = -e\hat{\mathbf{r}}$ is the dipole moment of the atom. Eq. (2.27) can be rewritten as

$$H^{(I)} = -\hat{d}\lambda(\hat{a}^\dagger + \hat{a}), \tag{2.28}$$

where $\hat{d} = \hat{\mathbf{d}} \cdot \hat{\mathbf{e}}$ and

$$\lambda = -\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}. \tag{2.29}$$

It is convenient to introduce the atomic-transition operators

$$\sigma_+ = |e\rangle\langle g|, \quad \sigma_- = |g\rangle\langle e| = \sigma_+^\dagger \tag{2.30}$$

and the inversion operator

$$\sigma_3 = |e\rangle\langle e| - |g\rangle\langle g|. \tag{2.31}$$

Because the eigenstates of the atomic Hamiltonian have defined parity, just the off-diagonal elements of the dipole operator have nonzero values. Thus we can write

$$\hat{d} = d\sigma_- + d^*\sigma_+ = d(\sigma_- + \sigma_+), \tag{2.32}$$

where we set $\langle e|\,\hat{d}\,|g\rangle = d$ and assumed that $d$ is real. The interaction Hamiltonian assumes the form

$$H^{(I)} = \hbar g(\sigma_- + \sigma_+)(\hat{a}^\dagger + \hat{a}). \tag{2.33}$$

where $g = d\lambda/\hbar$.

If we define the zero of the energy halfway between the energy of the atomic ground state and of the excited state we have that the Hamiltonian of the atom is

$$H_0^{(a)} = \frac{1}{2}\hbar\omega_0\sigma_3, \tag{2.34}$$

where $\omega_0 = E_e - E_g$. If we drop the zero-point energy term the Hamiltonian of the free-field is

$$H_0^{(f)} = \hbar\omega\hat{a}^\dagger\hat{a}. \tag{2.35}$$

The interaction Hamiltonian consists in four products of operators. In the free-field case the operators $\hat{a}^\dagger$ and $\hat{a}$ evolve as

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}, \quad \hat{a}(t) = \hat{a}(0)e^{-i\omega t}, \tag{2.36}$$

and similarly

$$\sigma_+(t) = \sigma_+(0)e^{i\omega_0 t}, \quad \sigma_-(t) = \sigma_-(0)e^{-i\omega_0 t}. \tag{2.37}$$

Thus we can approximate the evolution of the four terms of the Hamltonian (2.33) as

$$\begin{aligned} &\sigma_+\hat{a} \sim e^{i(\omega_0-\omega)t} \\ &\sigma_-\hat{a}^\dagger \sim e^{-i(\omega_0-\omega)t} \\ &\sigma_+\hat{a}^\dagger \sim e^{i(\omega_0+\omega)t} \\ &\sigma_-\hat{a} \sim e^{-i(\omega_0+\omega)t} \end{aligned} \tag{2.38}$$

The last two terms vary much more rapidly than the first two and furthermore they do not conserve energy, indeed, for instance, the third one creates a photon and at the same time excites the atom. Thus they can be dropped, doing the so-called rotating wave approximation (RWA). The resulting Hamiltonian, called *Jaynes-Cumming* Hamiltonian, is

$$H = \frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\sigma_-\hat{a}^\dagger + \sigma_+\hat{a}). \tag{2.39}$$

## 2.5 The dressed states

There are several ways to solve for the dynamics generated by the Hamiltonian (2.39), including solving the time-dependent Schrödinger equation for a definite Fock state

and then extrapolating for the case of a general state. In this section we focus on the method consisting in finding the stationary states of the Hamiltonian. We noticed that (2.39) causes only transitions of the form

$$|n\rangle |e\rangle \leftrightarrow |g\rangle |n+1\rangle , \tag{2.40}$$

thus the Hilbert space is decomposed in a series of subspaces $\mathscr{H}^{(n)}$ of dimension two, in which is confined the dynamics. The product state $|n\rangle |e\rangle$ and $|g\rangle |n-1\rangle$ are referred as "bare" states of the Jaynes-Cumming model. Using the notation

$$\begin{aligned} |\psi_{1n}\rangle &= |n\rangle |e\rangle \\ |\psi_{2n}\rangle &= |g\rangle |n+1\rangle \end{aligned} \tag{2.41}$$

and defining $H_{ij}^{(n)} = \langle \psi_{in}| H |\psi_{jn}\rangle$, we find that the matrix element of the Jaynes-Cumming Hamiltonian are

$$\begin{aligned} H_{11}^{(n)} &= \hbar\big[n\omega + \frac{1}{2}\omega_0\big] \\ H_{22}^{(n)} &= \hbar\big[(n+1)\omega - \frac{1}{2}\omega_0\big] \\ H_{12}^{(n)} &= H_{21}^{(n)} = \hbar g\sqrt{n+1} \end{aligned} \tag{2.42}$$

or, equivalently, in matrix form

$$\mathbf{H}^{(n)} = \hbar \begin{bmatrix} n\omega - \frac{1}{2}\omega_0 & g\sqrt{n+1} \\ g\sqrt{n+1} & (n+1)\omega - \frac{1}{2}\omega_0 \end{bmatrix}. \tag{2.43}$$

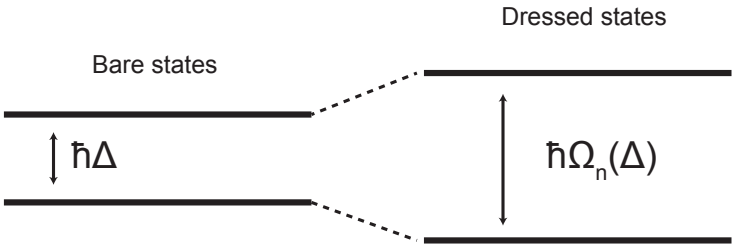


Figure 2.2: Splitting in the energies of the levels due to the interaction atom-field.

The eigenvalues of $H^{(n)}$ are

$$E_{\pm}(n) = \big(n + \frac{1}{2}\big)\hbar\omega \pm \frac{\hbar}{2}\Omega_n(\Delta), \tag{2.44}$$

where

$$\Omega_n(\Delta) = [\Delta^2 + 4g^2(n+1)]^{(1/2)}, \tag{2.45}$$

with $\Delta = \omega - \omega_0$, is the Rabi frequency, see Fig. 2.2. The eigenstates associated with these eigenvalues are

$$\begin{aligned} |n,+\rangle &= \cos(\Phi_n/2)\,|\psi_{1n}\rangle + \sin(\Phi_n/2)\,|\psi_{2n}\rangle \\ |n,-\rangle &= -\sin(\Phi_n/2)\,|\psi_{1n}\rangle + \cos(\Phi_n/2)\,|\psi_{2n}\rangle \end{aligned} \tag{2.46}$$

where

$$\Phi_n = \tan^{-1}\left(\frac{2g\sqrt{n+1}}{\Delta}\right). \tag{2.47}$$

The states $|n,\pm\rangle$ are referred ad "dressed" states of the Jaynes-Cumming model. To solve for the dynamics of a general state is straightforward, in fact every state can be written as a superposition of bare states $|\psi_{11}\rangle, |\psi_{12}\rangle, .., |\psi_{1n}\rangle, |\psi_{2n}\rangle, ..,$ which can all be expressed as combination of dressed states $|1,+\rangle, |1,-\rangle$,..,$|n,+\rangle$, $|n,-\rangle, ..$ Since the dressed states are eigenstates of the Hamiltonian the evolve in time simply as

$$|n,\pm\rangle\,(t) = \exp\big[-\frac{i}{\hbar}E_{\pm}(n)t\big]\,|n,\pm\rangle\,(0), \tag{2.48}$$

thus, once that we have written the general state as superposition of dressed states, we have only to multiply each term by the appropriate phase factor.

# Chapter 3

# Open systems

It is a common situation in physics that a small system, for instance an atom, is coupled to a large system, usually called environment or bath, for instance the continuum of modes of the electromagnetic field. Often one is just interested in dynamics of the small system and can ignore that one of the field. In general the approximations made to eliminate the environment dynamics, bring to an irreversible decay of the small system, as happens in the Weisskopf-Wigner theory of spontaneous emission.

Systems of this kind, coupled to a much larger system are called *open systems*. There are different approached to deal with them. The Heisenberg one leads to the introduction of *quantum noise operators*, while the Schrödinger one to the so-called *master equation*. The master equation can be solved with the method of MonteCarlo wave functions that permits to decompose the dynamics in two parts, a deterministic Schrödinger-like one and a probabilistic one.

In our work we are not going to consider any part of our composed system as an environment, nevertheless, in our calculations, we use some of the approximations of the open systems such as the Markov approximation. We also will write the Hamiltonian of our system as non-hermitian in the spirit of "quantum jump". For this reason we present here briefly the basic results of open system theory.[1]

[1] The literature for this chapter is contained in [2], [5] and [12].

## 3.1 Heisenberg-Langevin equations

Let's consider an harmonic oscillator (the system) coupled to a bath (or reservoir) of harmonic oscillators. The Hamiltonian describing the overall system is

$$H = H_S + H_B + H_{SB} \tag{3.1}$$

where

$$H_S = \hbar\Omega\hat{a}^\dagger\hat{a} \tag{3.2}$$

is the unperturbed Hamiltonian of the system,

$$H_B = \sum_j \hbar\omega_j\hat{b}_j^\dagger\hat{b}_j \tag{3.3}$$

is the unperturbed Hamiltonian of the bath, and

$$H_{SB} = \hbar\sum_j (g_j\hat{a}^\dagger\hat{b}_j + g_j^*\hat{a}\hat{b}_j^\dagger) \tag{3.4}$$

is the interaction Hamiltonian, for which when an excitation is created in the system one is destroyed in the bath and viceversa. From the Hamiltonian (3.1) is immediate to obtain the Heisenberg equation of motion for the system and bath operators:

$$\dot{\hat{a}}(t) = -i\Omega\hat{a}(t) - i\sum_j g_j\hat{b}_j(t), \tag{3.5}$$

$$\dot{\hat{b}}_j(t) = -i\omega_j\hat{b}_j(t) - ig_j^*\hat{a}(t). \tag{3.6}$$

We can integrate formally (3.6) to get

$$\hat{b}_j(t) = \hat{b}_j(t_0)e^{-i\omega_j(t-t_0)} - ig_j^*\int_{t_0}^{t} dt'\hat{a}(t')e^{-i\omega_j(t-t')}$$
$$\equiv \hat{b}_{free}(t) + \hat{b}_{radiated}, \tag{3.7}$$

where the first term is the solution of the homogeneous equation associated with (3.6) and represents the free evolution of $\hat{b}_j$ in the absence of interaction with the system, while the second term gives the modification to the free evolution given by

the coupling with the system. Inserting eq. (3.6) into eq. (3.5) we find

$$\dot{\hat{a}}(t) = -i\Omega\hat{a}(t) - i\sum_j g_j\hat{b}_j(t_0)e^{-i\omega_j(t-t_0)} - \sum_j |g_j|^2 \int_{t_0}^{t} dt'\hat{a}(t')e^{-i\omega_j(t-t')}. \quad (3.8)$$

Now we eliminate the first term introducing the operator

$$\hat{A}(t) = \hat{a}(t)e^{i\Omega t}, \quad \text{with } [\hat{A}(t), \hat{A}^\dagger(t)] = 1, \quad (3.9)$$

obtaining the equation

$$\dot{\hat{A}}(t) = -\sum_j |g_j|^2 \int_{t_0}^{t} dt'\hat{A}(t')e^{-i(\omega_j-\Omega)(t-t')} + \hat{F}(t), \quad (3.10)$$

where

$$\hat{F}(t) = -i\sum_j g_j\hat{b}_j(t_0)e^{-i(\omega_j-\Omega)(t-t_0)} \quad (3.11)$$

and it is called *noise operator*. The reason is that it varies rapidly in time due to the presence of all reservoir frequencies, giving origin to fluctuations. Also, if the density operator describing the bath is diagonal in the energy representation, and we always assume that, $\langle F(t)\rangle_b$ vanishes.

Now we replace the sum on $j$ in eq. (3.10) with an integral on the frequencies

$$\sum_j \to \int d\omega \mathcal{D}(\omega) \quad (3.12)$$

where $\mathcal{D}(\omega)$ is the density of state and we do the *Markov approximation*. This consists in assuming that $\hat{A}(t)$ varies little over a time of the order of the inverse of the bandwidth of the bath. This allows us to extend the time integration to infinity and so to get a $\pi\delta(t-t')$ when performing the integration over $\omega$. Thus we get

$$\dot{\hat{A}}(t) = -\frac{\gamma}{2}\hat{A}(t) + \hat{F}(t) \quad (3.13)$$

with $\gamma = 2\pi\mathcal{D}(\Omega)|g(\Omega)|^2$. If we take the expectation value of eq. (3.10) we get the simple equation

$$\langle\dot{\hat{A}}(t)\rangle = -\frac{\gamma}{2}\langle\hat{A}(t)\rangle \quad (3.14)$$

that can simply be integrated to get

$$\langle\hat{A}(t)\rangle = e^{-\gamma t/2}\langle\hat{A}(0)\rangle. \quad (3.15)$$

We can be tempted to extend this solution to the operator equation (3.13), neglecting the noise term, but this would be completely wrong since the commutation relation of $\hat{A}$ would not be preserved as it is easy to verify. The noise operator, which fluctuates rapidly and averages to zero, plays a role similar to that one of the Langevin force in the Brownian motion. For this reason equations of the form of (3.13) are called *quantum Langevin equations.*

## 3.2 Input-output relations for a cavity

An interesting application of the Heisenberg-Langevin formalism, explained in the previous section, is to find the input-output relations of a cavity coupled with an external field. We consider a single-side cavity. The Hamiltonian is the same as in (3.1), with the abstract sum on $j$ replaced by an integral on the frequencies $\omega$ of the external field. The operator $\hat{a}$ here represents clearly the single-mode intra-cavity field operator, while $\hat{b}$ is again the extra-cavity reservoir field operator. With this modification the interaction Hamiltonian is

$$H_{FC} = \hbar \int_{-\omega_b}^{\omega_b} d\omega\, \kappa(\omega)[\hat{b}(\omega)\hat{a}^\dagger + \hat{b}^\dagger(\omega)\hat{a}], \tag{3.16}$$

where $\kappa(\omega)$ is the coupling constant and $2\omega_b$ is the width of the field spectrum.

The Heisenberg equation of motion for the operators are

$$\dot{\hat{a}} = -i\Omega\hat{a} - i\int_{-\omega_b}^{\omega_b} d\omega\, \kappa(\omega)\hat{b}(\omega), \tag{3.17}$$

$$\dot{\hat{b}}(\omega) = -i\omega\hat{b}(\omega) - i\kappa(\omega)\hat{a}, \tag{3.18}$$

where the time dependence of the operator is sublet. Eq. (3.18) can be integrated in two ways, depending on whether we choose to solve it in terms of the initial conditions (the input) or in terms of the final conditions (the output). The two solutions are respectively

$$\hat{b}(\omega) = \hat{b}_0(\omega)e^{-i\omega(t-t_0)} - i\kappa(\omega)\int_{t_0}^{t} dt'\, \hat{a}(t')e^{-i\omega(t-t')}, \tag{3.19}$$

where $t > t_0$ and $\hat{b}_0$ is the value of $\hat{b}$ at time $t_0$, and

$$\hat{b}(\omega) = \hat{b}_1(\omega)e^{-i\omega(t-t_1)} + i\kappa(\omega)\int_{t}^{t_1} dt'\, \hat{a}(t')e^{-i\omega(t-t')}, \tag{3.20}$$

where $t_1 > t$ and $\hat{b}_1$ is the value of $\hat{b}$ at time $t_1$. Inserting the solution (3.19) in (3.17) we get

$$\dot{\hat{a}} = -i\Omega\hat{a} - i\int_{-\omega_b}^{\omega_b} d\omega\, \kappa(\omega)\hat{b}_0(\omega)e^{-i\omega(t-t_0)} - \int_{-\omega_b}^{\omega_b} d\omega\, \kappa(\omega)^2 \int_{t_0}^{t} dt'\, \hat{a}(t')e^{-i\omega(t-t')}. \tag{3.21}$$

We make the assumption that the coupling constant $\kappa$ can be considered constant over a band of frequencies around the cavity frequency $\Omega$, and, as above, we define $\gamma = 2\pi\kappa(\Omega)^2$. We also define the input field operator

$$\hat{a}_{IN}(t) = -\frac{i}{\sqrt{2\pi}}\int_{-\omega_b}^{\omega_b} d\omega\, \hat{b}_0(\omega)e^{-i\omega(t-t_0)}. \tag{3.22}$$

Doing the Markov approximation in the last term of eq. (3.21) and using (3.22) we can rewrite it as

$$\dot{\hat{a}} = -i\Omega\hat{a} - \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\hat{a}_{IN}. \tag{3.23}$$

This equation is of the Heisenberg-Langevin kind and the noise term is given by the input field. If we do the steps from eq. (3.21) using the solution in terms of the final conditions and with the definition of an output field operator

$$\hat{a}_{OUT}(t) = -\frac{i}{\sqrt{2\pi}}\int_{-\omega_b}^{\omega_b} d\omega\, \hat{b}_1(\omega)e^{-i\omega(t-t_1)}. \tag{3.24}$$

we get

$$\dot{\hat{a}} = -i\Omega\hat{a} + \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\hat{a}_{OUT}. \tag{3.25}$$

If we subtract eq. (3.25) from eq. (3.23) we get the important boundary condition

$$\hat{a}_{IN}(t) - \hat{a}_{OUT}(t) = \sqrt{\gamma}\hat{a}(t). \tag{3.26}$$

If we Fourier transform eq. (3.23) we get the equivalent equation in frequency space

$$\big[i(\Omega - \omega) + \frac{\gamma}{2}\big]\hat{a}(\omega) = \sqrt{\gamma}\hat{a}_{IN}, \tag{3.27}$$

and similarly Fourier transforming (3.25) we get

$$\big[i(\Omega - \omega) + \frac{\gamma}{2}\big]\hat{a}(\omega) = \sqrt{\gamma}\hat{a}_{OUT}. \tag{3.28}$$

Now combining eqs. (3.27), (3.28) and (3.26) we obtain the desired input-output

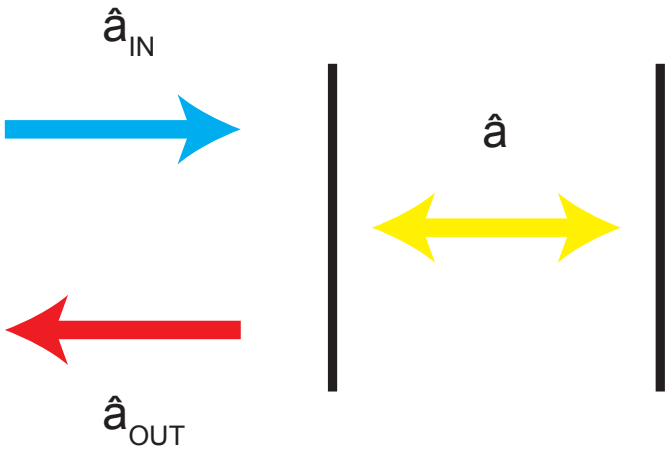


Figure 3.1: Representation of a one-side cavity, with the intra-cavity, the input and the output fields.

relation

$$\hat{a}_{OUT} = -\frac{\omega - \Omega - i\gamma/2}{\omega - \Omega + i\gamma/2}\hat{a}_{IN}, \tag{3.29}$$

which shows that an incoming field interacting with a cavity get a frequency-depending phase shift.

## 3.3 Quantum regression theorem

In this section we demonstrate an important theorem related to the expectations values of products of system operators, the *quantum regression theorem*. In section 3.1 we have shown the evolution of a system operator $\hat{A}_\mu$ is described by an Heisenberg-Langevin equation, which has form

$$\frac{d}{dt}\hat{A}_\mu(t) = \hat{D}_\mu(t) + \hat{F}_\mu(t) \tag{3.30}$$

where $\hat{D}_\mu(t)$ is the drift term and $\hat{F}_\mu(t)$ is the noise term which averages to zero. We can use this equation to calculate the expectation value of the product of two system operators. In fact multiplying on the right eq. (3.30) for $\hat{A}_\nu(t')$, with $t' < t$, and

taking the expectation value we get

$$\frac{d}{dt}\langle \hat{A}_\mu(t)\hat{A}_\nu(t')\rangle = \langle \hat{D}_\mu(t)\hat{A}_\nu(t')\rangle + \langle \hat{F}_\mu(t)\hat{A}_\nu(t')\rangle\,. \tag{3.31}$$

Since in the Markov approximation the system operator $\hat{A}_\nu(t')$ at cannot be influenced by the future noise, the second term on the r.h.s. factorizes in a product of two expectation values and so vanishes. We have obtained the quantum regression theorem:

$$\frac{d}{dt}\langle \hat{A}_\mu(t)\hat{A}_\nu(t')\rangle = \langle \hat{D}_\mu(t)\hat{A}_\nu(t')\rangle\,, \tag{3.32}$$

which simply states that the expectation value of the two-time correlation function satisfies the same equation of motion as the single operator expectation value. This theorem is very useful when one has to hand with several two-time correlation functions since it proves that it is sufficient to calculate the equations of motion for the operators.

## 3.4 Master equation

An alternative way to solve for the dynamics of a system coupled to reservoir is to solve the Schrödinger equation. Denoting as $\rho_{sb}$ the density matrix of the overall system, the Schrödinger equation is

$$\dot{\rho}_{sb} = -\frac{i}{\hbar}\big[H, \rho_{sb}\big]. \tag{3.33}$$

Since we are not interested in the dynamics of the bath, we can trace out the degrees of freedom of the bath, to get the *reduced density operator* of the system $\rho_s(t) = \mathrm{Tr}_b\rho_{sb}(t)$. Once that we know $\rho_s(t)$ for all times we can determine the expectation values of the system operators simply taking

$$\langle \hat{O}(t)\rangle = \mathrm{Tr}_s\{\hat{O}\rho_s(t)\}. \tag{3.34}$$

The equation of motion for $\rho_s(t)$ is called *master equation*.

It is convenient to move to the interaction picture, where the density operator is defined as

$$\tilde{\rho}_{sb} = e^{iH_0(t-t_0)}\rho_{sb}e^{-iH_0(t-t_0)}. \tag{3.35}$$

where $H_0 = H_S + H_B$ Differentiating (3.35) with respect to the time we get the equation of motion of the density operator in the interaction picture

$$\frac{\partial \tilde{\rho}_{sb}}{\partial t} = -\frac{i}{\hbar}\big[\tilde{H}_I(t-t_0), \tilde{\rho}_{sb}\big], \tag{3.36}$$

where

$$\tilde{H}_I(t-t_0) = e^{iH_0(t-t_0)} H_I e^{-iH_0(t-t_0)} \tag{3.37}$$

is the interaction Hamiltonian in the interaction picture. We find a second order solution of eq. (3.36) in perturbation theory using an iterative method. We start integrating eq. (3.36) from $t_0$ to $t$ to obtain a first order solution. We get

$$\tilde{\rho}_{sb}(t) = \tilde{\rho}_{sb}(t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt' \big[\tilde{H}_I(t'-t_0), \tilde{\rho}_{sb}(t')\big]. \tag{3.38}$$

Now we do another integration and we insert the first order expression of $\tilde{\rho}_{sb}(t)$ as approximation. We obtain

$$\begin{aligned}\tilde{\rho}_{sb}(t) = \tilde{\rho}_{sb}(t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt' \big[\tilde{H}_I(t'-t_0), \tilde{\rho}_{sb}(t_0)\big] + \\ - \frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \big[\tilde{H}_I(t'-t_0), \big[\tilde{H}_I(t''-t_0), \tilde{\rho}_{sb}(t_0)\big]\end{aligned} \tag{3.39}$$

where in the last commutator we have put $\tilde{\rho}_{sb}(t) \approx \tilde{\rho}_{sb}(t_0)$ to get a close expression. Now we can trace out the degree of freedom of the bath and denoting $\tilde{\rho}_s(t) = \mathrm{Tr}_b\{\tilde{\rho}_{sb}(t)\}$, we define a *coarse-grained equation of motion*

$$\dot{\tilde{\rho}}_s(t) = \frac{\tilde{\rho}_s(t) - \tilde{\rho}_s(t-\tau)}{\tau}, \tag{3.40}$$

where $\tau$ is a time large compared to the bath memory time $\tau_c$ but small compared to the time $\rho_s$ varies significantly. Using the explicit form of the last commutator of eq. (3.39)

$$\big[\tilde{H}'_I, \big[\tilde{H}''_I, \tilde{\rho}_{sb}\big] = \tilde{H}'_I \tilde{H}''_I \tilde{\rho}_{sb} - \tilde{H}'_I \tilde{\rho}_{sb} \tilde{H}''_I + \text{adj.}, \tag{3.41}$$

we have that the coarse-grained equation of motion deriving from eq. (3.39) is

$$\begin{aligned}\dot{\tilde{\rho}}_{sb}(t) = -\frac{i}{\hbar\tau}\int_0^{\tau} d\tau' \,\mathrm{Tr}_b\big[\tilde{H}_I(\tau'), \tilde{\rho}_{sb}(t)\big] + \\ - \frac{1}{\hbar^2\tau}\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \,\mathrm{Tr}_b\{\tilde{H}_I(\tau')\tilde{H}_I(\tau'')\tilde{\rho}_{sb}(t) - \tilde{H}_I(\tau')\tilde{\rho}_{sb}(t)\tilde{H}_I(\tau'')\} + \text{adj.}\end{aligned} \tag{3.42}$$

Using the explicit forms of the Hamiltonians (3.2)-(3.4) it can be shown easily that

$$\tilde{H}_I(\tau) = \hbar\hat{a}^\dagger\hat{F}(\tau) + \hbar\hat{a}^\dagger\hat{F}^\dagger(\tau), \tag{3.43}$$

where

$$\hat{F}(\tau) = -i\sum_j g_j\hat{b}_j e^{i(\Omega-\omega_j)\tau} \tag{3.44}$$

and is the analogous of the noise operator in the Heisenberg picture. When we trace over the bath the first term of eq. (3.42) is composed of two terms of the form

$$\mathrm{Tr}_b\{\hat{a}^\dagger\hat{F}(\tau)\tilde{\rho}_{sb}(t)\} = \hat{a}^\dagger\tilde{\rho}_s(t)\mathrm{Tr}_b\{\hat{F}(\tau)\tilde{\rho}_b(t)\}. \tag{3.45}$$

The second trace is the expectation value of the operator $F(\tau)$, which, assuming that the density operator of the bath is diagonal in the energy, is zero. Substituting eq. (3.43) in eq. (3.42) and using the cyclic property of the trace we get

$$\begin{aligned}\dot{\tilde{\rho}}_s(t) = -\frac{1}{\hbar^2\tau}\int_0^\tau d\tau'\int_0^{\tau'} d\tau''\big[&\hat{a}^\dagger\hat{a}\tilde{\rho}_s(t)\left\langle\hat{F}(\tau')\hat{F}^\dagger(\tau'')\right\rangle_b - \hat{a}\tilde{\rho}_s(t)\hat{a}^\dagger\left\langle\hat{F}(\tau'')\hat{F}^\dagger(\tau')\right\rangle_b + \\ &+\hat{a}\hat{a}^\dagger\tilde{\rho}_s(t)\left\langle\hat{F}^\dagger(\tau')\hat{F}(\tau'')\right\rangle_b - \hat{a}^\dagger\tilde{\rho}_s(t)\hat{a}\left\langle\hat{F}^\dagger(\tau'')\hat{F}(\tau')\right\rangle_b + \\ &+\hat{a}\hat{a}\tilde{\rho}_s(t)\left\langle\hat{F}^\dagger(\tau')\hat{F}^\dagger(\tau'')\right\rangle_b - \hat{a}\tilde{\rho}_s(t)\hat{a}\left\langle\hat{F}^\dagger(\tau'')\hat{F}^\dagger(\tau')\right\rangle_b + \\ &+\hat{a}^\dagger\hat{a}^\dagger\tilde{\rho}_s(t)\left\langle\hat{F}(\tau')\hat{F}(\tau'')\right\rangle_b - \hat{a}^\dagger\tilde{\rho}_s(t)\hat{a}^\dagger\left\langle\hat{F}(\tau'')\hat{F}(\tau')\right\rangle_b\big] + \text{adj.}\end{aligned} \tag{3.46}$$

Using eq. (3.44) we find that the bath average terms present in (3.46) are of the form

$$\begin{aligned}\left\langle\hat{F}(\tau')\hat{F}^\dagger(\tau'')\right\rangle_b = \sum_{i,j} g_i g_j^*\left\langle\hat{b}_i\hat{b}_j^\dagger\right\rangle_b e^{i\Omega(\tau'-\tau'')}e^{i(\omega_j\tau''-\omega_i\tau')} = \\ = \sum_i |g_i|^2\left\langle\hat{b}_i\hat{b}_i^\dagger\right\rangle_b e^{i(\Omega-\omega_i)(\tau'-\tau'')}\end{aligned} \tag{3.47}$$

where the second equality is valid provided that the bath operators are diagonal. This kind of averages are first order correlation function of the bath and they depend only on the time difference $T = \tau' - \tau''$.

Now, as we did using the Heisenberg formalism, we perform the Markov approximation, assuming that the correlation time $\tau_c$, that is the time for which the correlation function are non-null, is infinitely short compared to all the other times

of interest. This permits us to extend the time integral in

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \, \langle \hat{F}(\tau')\hat{F}^\dagger(\tau'')\rangle_b = \\ = \int_0^\tau d\tau' \sum_i |g_i|^2 \, \langle \hat{b}_i \hat{b}_i^\dagger \rangle_b \int_0^{\tau'} dT e^{i(\Omega-\omega_i)T} \quad (3.48)$$

to infinity. Thus, replacing the sum with an integral over frequencies we get

$$\int_0^\tau d\tau' \int_{-\omega_b}^{\omega_b} d\omega \, \mathcal{D}(\omega)|g(\omega)|^2 \, \langle \hat{b}(\omega)\hat{b}^\dagger(\omega)\rangle_b \int_0^\infty dT e^{i(\Omega-\omega)T} = \\ = \frac{\gamma\tau}{2} \, \langle \hat{b}(\Omega)\hat{b}^\dagger(\Omega)\rangle_b \, , \quad (3.49)$$

where, again, $\gamma = 2\pi\mathcal{D}(\Omega)|g(\Omega)|^2$. The average of the number operator $\hat{b}^\dagger\hat{b}$ is given by the thermal distribution an so it is a function of the temperature of the reservoir. Denoting by $\bar{n}$ and computing all terms in (3.46) by mean of (3.48), we get finally the master equation

$$\dot{\tilde{\rho}}_s(t) = -\frac{\gamma}{2}(\bar{n}+1)\big[\hat{a}^\dagger\hat{a}\tilde{\rho}_s(t) - \hat{a}\tilde{\rho}_s(t)\hat{a}^\dagger\big] - \frac{\gamma}{2}\bar{n}\big[\tilde{\rho}_s(t)\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\tilde{\rho}_s(t)\hat{a}\big] + \text{adj.} \quad (3.50)$$

which at zero temperature reduces to

$$\dot{\tilde{\rho}}_s(t) = -\frac{\gamma}{2}\big(\{\hat{a}^\dagger\hat{a}, \tilde{\rho}_s(t)\} - 2\hat{a}\tilde{\rho}_s(t)\hat{a}^\dagger\big). \quad (3.51)$$

It is easy to move back the Schrödinger picture, from eq. (3.50) we find

$$\dot{\rho}_s(t) = -\frac{i}{\hbar}\big[H_s, \rho_s(t)\big] - \frac{\gamma}{2}(\bar{n}+1)\big[\hat{a}^\dagger\hat{a}\rho_s(t) - \hat{a}\rho_s(t)\hat{a}^\dagger\big] - \frac{\gamma}{2}\bar{n}\big[\rho_s(t)\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\rho_s(t)\hat{a}\big] + \text{adj.} = \\ = -\frac{i}{\hbar}\big[H_s, \rho_s(t)\big] + \mathcal{L}[\rho_s], \quad (3.52)$$

where the term $\mathcal{L}[\rho_s]$ describe the interaction of the system with the bath. It can be shown that, to preserve the trace of the density operator, it has to be of the form

$$\mathcal{L}[\rho_s] = -\frac{1}{2}\sum_i \{\hat{C}_i^\dagger\hat{C}_i, \rho_s\} + \sum_i \hat{C}_i\rho_s\hat{C}_i^\dagger, \quad (3.53)$$

where $\hat{C}_i$ are system operators. In the case of the harmonic oscillator they are

$$\begin{aligned} \hat{C}_1 &= \sqrt{\gamma(\bar{n}+1)}\hat{a} \\ \hat{C}_2 &= \sqrt{\gamma\bar{n}}\hat{a}. \end{aligned} \quad (3.54)$$

In the case, for us more interesting, of a two-levels atom, the system operators are $\hat{\sigma}_-$, $\hat{\sigma}_+$ and $\hat{\sigma}_3$ with their well-known algebra. It can be easily proved that the operators in the master equation are the same as in (3.54) with $\hat{a}$ replaced by $\hat{\sigma}_-$.

It is an important fact that we can rewrite the master equation (3.52) as

$$\dot{\rho}_s(t) = -\frac{i}{\hbar}\big(H_{eff}\rho_s - \rho_s H_{eff}^\dagger\big) + \mathcal{L}_{jump}[\rho_s], \tag{3.55}$$

where

$$H_{eff} = H_s - \frac{i\hbar}{2}\sum_i \hat{C}_i^\dagger \hat{C}_i \tag{3.56}$$

is a non-hermitian effective Hamiltonian which provides a Schrödinger-like evolution, and

$$\mathcal{L}_{jump}[\rho_s] = \sum_i \hat{C}_i \rho_s \hat{C}_i^\dagger \tag{3.57}$$

is a "jump" term. In the case of the two-levels atom with decay from the upper level the two terms are

$$H_{eff} = \frac{\hbar\omega_0}{2}\sigma_3 - i\hbar\Gamma(\bar{n} + \frac{1}{2})\sigma_+\sigma_-, \tag{3.58}$$

and

$$\mathcal{L}_{jump}[\rho_s] = \Gamma(2\bar{n}+1)\sigma_-\rho_s\sigma_+. \tag{3.59}$$

It is not completely clear for now which is the role of the two terms in the time evolution of the density matrix and why the second term is called "jump". It will be much more clear considering the *Monte-Carlo wave functions method* in the next section.

## 3.5 Monte-Carlo wave functions method

As we have seen in the previous section the evolution of the density matrix of a system interacting with a reservoir cannot be reduced to a Schrödinger-like deterministic evolution. The Monte-Carlo wave function method is a stochastic method that results very powerful to solve the dynamics of open system. Let's take the system in a pure state $|\psi\rangle$ at time $t_0$, it will be clear how to generalize the method to a system starting in a mixed state. The master equation (3.55) for the system density operator is

$$|\dot{\psi}\rangle\langle\psi| + |\psi\rangle\langle\dot{\psi}| = -\frac{i}{\hbar}H_{eff}|\psi\rangle\langle\psi| + \frac{i}{\hbar}|\psi\rangle\langle\psi|H_{eff}^\dagger + \sum_i \hat{C}_i|\psi\rangle\langle\psi|\hat{C}_i^\dagger. \tag{3.60}$$

The first term on the right side derives from the non-hermitian Schrödinger-like evolution

$$i\hbar|\dot{\psi}\rangle = H_{eff}\,|\psi\rangle\,, \tag{3.61}$$

while the second term, which is clearly different, project the state from $|\psi\rangle$ to one of the states $|\psi\rangle_i = \hat{C}_i\,|\psi\rangle$, for this reason it is called "jump" term.

Decomposed the evolution in these two parts, we can solve the master equation numerically, evolving the state $|\psi\rangle$ under the action of $H_{eff}$ for small intervals $\delta t$. To the first order we have

$$|\psi(t+\delta t)\rangle = \left(1 - \frac{iH_{eff}\delta t}{\hbar}\right)|\psi(t)\rangle\,. \tag{3.62}$$

Because of the non-hermitian nature of $\hat{\mathcal{H}}_{eff}$ we have that $|\psi(t+\delta t)\rangle$ is not normalized. A quick calculation shows that the squared norm is 1-$\delta p$, where

$$\delta p = \delta t \sum_i \langle\psi(t)|\,\hat{C}_i^\dagger \hat{C}_i\,|\psi(t)\rangle \equiv \sum_i \delta p_i. \tag{3.63}$$

This means that if we want to preserve the norm during the evolution we have to take into account also the jump part. We are also brought to consider $\delta p_i$ as the probability for the jump $|\psi\rangle \to \hat{C}_i\,|\psi\rangle$ to occur. Now we have all the elements to implement a simple evolution scheme using Monte-Carlo method, i.e. random numbers. For each step of the evolution a number $0 < r \leq 1$ is generated, if $0 < r \leq \delta p_1$ the state is projected on $|\psi\rangle_1$, if $\delta p_1 < r \leq \delta p_1 + \delta p_2$ the state is projected on $|\psi\rangle_2$ and so on. If $\delta p_1 + ... + \delta p_n < r \leq 1$ the Schrödinger-like evolution of eq. (3.62) is performed. After each step the state is normalized. The evolution of the state obtained in this way from the initial time $t_0$ to the final time $t_f$ is called a *quantum trajectory*.

Since the process is stochastic, for the nature itself of the master equation, we obtain different trajectories for the same initial state, so that to have a reliable picture of the evolution we have to average many trajectories, i.e. to run the program many times. The generalization to a mixed initial state is straightforward, indeed, since the density matrix is linear, is enough to repeat the process with the different possible initial states, averaging the trajectories over the initial probabilities $P_i$ of the states. The great advantage of the Monte-Carlo wave functions method is that it deals just with state vectors. This means that if the number of state to consider is $N$ the memory required to use this method scales with $N$. On the contrary dealing

with the density matrix, which dimension is $N^2$, requires a memory space that scales with $N^2$!

In some situations we are not interested in all the states of the system, for instance we can sometimes neglect the levels of an atom to which an upper level decays, being just interested in the dynamics of the upper level. In these situations we can ignore the jump part of the evolution and use only the effective Hamiltonian to get the dynamics. In this spirit we will make use of non-hermitian Hamiltonians to describe our system in the chapters that follow.

# Chapter 4

# A photonic transistor

To make photons interacting with photons is one of the fundamental challenges of quantum optics. To have optical non-linearities, i.e. non-trivial interactions between photons, is in fact a requirement to realize quantum computation with photons or with atoms and photons [6]. A conceptually simple idea involving the problem of interaction between photons is that one of the photonic transistor. The electronic transistor is device in which two pairs of electric terminals are applied to a semiconductor material and a small current flowing between one of the pair changes the flowing between the other pair [13]. For analogy, it is called photonic transistor a device where a small optical field is used to control another optical field. Its fundamental limit is the single-photon transistor, where the control field consists in just one photon, so that is the presence or the absence of this control photon to permit or deny the propagation of the controlled field. The applications of this device are several, going from photo-detection to quantum computation processes.

## 4.1 Proposed $\Lambda$-atom model

A model of photonic transistor has been proposed in 2007 by *Chang et al.* [10]. Here strong nonlinear interactions are realized by exploiting the strong coupling between individual photon emitters and surface propagating plasmons confined to a conducting nanowire. Surface plasmons are propagating electromagnetic modes confined to the surface of a dielectric-conductor interface. The nanowire on which plasmons are confined can be thought as a one-dimensional guide and the modes can be indexed by

the wavevectors $k$. The advantage in using nanowires instead of optical fibres is that they exhibit good confinement also when the radius is much shorter than the optical wavelenght. The confinement results in a strong coupling between the plasmonic modes and any emitter with a dipole-allowed transition, for instance a two-levels emitter, placed in proximity of the wire. It has been shown [14], using just classical electrodynamics considerations, that for an emitter placed to an optimal distance from the nanowire the ratio between the spontaneous emission rate $\Gamma_{pl}$ in the plasmonics modes and the emission rate $\Gamma'$ into all the other possible channels can exceed $10^3$.

The Hamiltonian describing such a system, i.e. a single emitter strongly coupled to a continuum of one-dimensional modes, is

$$H = \hbar(\omega_{eg} - i\Gamma')\sigma_{ee} + \int dk\, \hbar c|k|\hat{a}_k^\dagger \hat{a}_k - \hbar g \int dk(\sigma_{eg}^+ \hat{a}_k + \sigma_{eg}^+ \hat{a}_k^\dagger), \tag{4.1}$$

where a linear dispersion relation ($\omega_k = ck$) for the plasmons has been assumed. The Hamiltonian is non-hermitian in the spirit of the quantum jump to account for the decay of the level $|e\rangle$ in modes different from the plasmonic ones, and the coupling constant is been assumed to be frequency-independent. Considering the scattering of the single photon one finds that the coefficient of reflection is

$$r(\delta_k) = -\frac{\Gamma_{pl}}{\Gamma_{pl} + \Gamma' - 2i\delta_k}, \tag{4.2}$$

where $\delta_k = ck - \omega_{eg}$ is the photon detuning. The transmission coefficient is given by $t(\delta_k) = 1 + r(\delta_k)$. On resonance $r \approx -(1 - 1/P)$ where $P = \Gamma_{pl}/\Gamma'$ is called *Purcell factor*, that for the value reported above makes the system acting as nearly perfect mirror. The situation is different for the scattering of a multiphoton state. In the case of a coherent input of amplitude $\xi_c$ on resonance the reflectance, i.e. the absolute square value of the reflection coefficient, are found to be

$$\mathcal{R} = |r|^2 = \left(1 + \frac{1}{P}\right)^{-2} \frac{1}{1 + 8(\Omega_c/\Gamma)^2}, \tag{4.3}$$

where $\Omega_c = \sqrt{2\pi} g \xi_c$ and $\Gamma = \Gamma_{pl} + \Gamma'$. It is clear from eq. (4.3) that the emitter exhibits saturation at high power since the reflectance decrease to zero.

The idea presented in [10] to realize the photonic transistor is represented in Fig. 4.1. It is based on the use of a three-level atom with levels configuration of the $\Lambda$ kind. Beside the two levels $|g\rangle$ and $|e\rangle$ is present a third level $|s\rangle$ which is

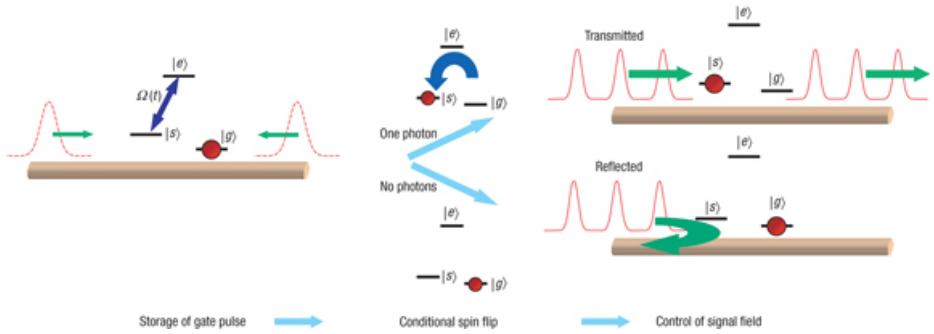


Figure 4.1: Representation of the scheme of the transistor proposed in ref. [10].

coupled to $|e\rangle$. The state $|s\rangle$ is assumed to be decoupled from the plasmon modes, but coupled to $|e\rangle$ through some classical control field. If this field is turned on an incoming photon can be captured and stored in $|s\rangle$. If the atom sits in $|s\rangle$ clearly an incoming photon does not see the atom and is completely transmitted. On the other way, if no photon is present the control field has no effect and the atom remains in $|g\rangle$. Thus the protocol proposed consists mainly in two steps: (1) the atom is started in $|g\rangle$ and the classical field is turned on for a time $T'$ during which it makes the operation $(\alpha|0\rangle + \beta|1\rangle)|g\rangle \rightarrow |0\rangle((\alpha|g\rangle + \beta|s\rangle)$, (2) a multiphoton state is sent for a time $T''$ to the atom and its propagation is conditioned by the state of the atom at the end of the first step. The efficiency of the first step is given by the probability of storing the photon. It has been shown that is $1 - 1/P$. During the second step the main limit is the characteristic time over which a spin flip can occur. These depends on the number ratio of the decay rates of $|e\rangle$ to $|g\rangle$ and $|s\rangle$. It is easy to see that the number of scattered photon is $n \sim \Gamma_{e\rightarrow g}/\Gamma_{e\rightarrow s}$. This is the "gain" of the transistor, i.e. the difference between the number of photons controlled and the number of photon in the control input.

## 4.2 Superconducting qubits

An obstacle in the realization of the transistor scheme exposed in the previous section is the absence of three-levels $\Lambda$ atoms coupled strongly with one-dimensional waveguides. Experimentally the most ideal coupling of (artificial) atoms to waveguides is achieved in superconducting system. Several systems have been proposed to realized qubit, i.e. ideal two-levels system, using microscopic degrees of freedom [6]. They range from nuclear and electron spins to atoms in cavities to ions trapped in mag-

netic fields. The realization of qubits using superconducting circuits is something radically different, indeed here qubits are constructed from collective electrodynamics modes of macroscopic electrical elements, rather than microscopic degrees of freedom [11, 15, 16].

In a classical circuits currents, voltage etc. in each branch have a definite value and can be measured simultaneously, conversely in quantum circuits these quantities obey the superposition principle and are described by operators that do not necessarily commute. A consequence of the quantum behavior is that we can have current flowing at the same time in two opposite directions, exactly like quantum particles. The superconductivity of the elements is a necessary condition for that, indeed to have a quantum behavior the first requirement is the absence of dissipation. The cooling of metals to temperature of the order of decimals of 1 K is the standard process to achieve superconductivity. The simplest quantum circuit is the LC circuit, i.e. a circuit with a capacitor and an inductor. As is known from circuits theory an LC circuit behaves like an harmonic oscillator in the sense the equation of motion for the current flowing in the circuit is exactly the second-order differential equation of the harmonic oscillator. From the equation of motion is straightforward to obtain a Lagrangian and then an Hamiltonian of the system, that results to be

$$\mathcal{H} = \frac{1}{2C} p_{\Phi_L}^2 + \frac{1}{2L} \Phi_L^2, \tag{4.4}$$

where

$$\Phi_i = \int_{-\infty}^{t} dt' \, V_i(t') \tag{4.5}$$

and $p_{\Phi_j}$ is the canonical conjugate momentum. This Hamiltonian can be quantized in the same way as the electromagnetic field has been quantized in section 2.1., i.e. with the replacing

$$\begin{aligned} &\Phi_L \to \hat{\Phi}_L \\ &p_{\Phi_L} \to \hat{p}_{\Phi_L} \\ &\left[\hat{\Phi}_L, \hat{p}_{\Phi_L}\right] = i\hbar. \end{aligned} \tag{4.6}$$

The LC resonator is not the best choice to realize a superconducting qubit, for the obvious reason that an harmonic oscillator has an uniform separation between the energy eigenvalues.

What is required to have a good approximation of a two-level system is an *an-*

*harmonicity* in the levels configuration, which is achieved by introducing a non-linear element in the circuit. There is only one element which is both non-linear and non-dissipative at arbitrary low temperature: the *superconducting tunnel junction*, also called *Josephson junction*. This element consists of a sandwich of two superconducting thin films separated by an insulating layer that is thin enough to allow tunnelling of discrete charges through the barrier (Fig. 4.2a). The phase difference $\delta$ between the two sides is the important variable characterizing the junction. The relations between this phase and the current and voltage across the junction are called Josephson relations. The first relation is

$$I = I_c \sin \delta, \tag{4.7}$$

where $I_c$ is the *critical current* of the junction, i.e. the maximal current that can flows through the junction, whose value depends on its material, geometry and dimension. The second relation is

$$V = \frac{\Phi_0}{2\pi} \frac{d\delta}{dt}, \tag{4.8}$$

where $\Phi_0 = h/2e$, is the dual of the Cooper pair charge, the *flux quantum*. Combining (4.5) and (4.8) we get

$$\delta = 2\pi \frac{\Phi}{\Phi_0}, \tag{4.9}$$

which inserted in eq. (4.7) gives

$$I = I_c \sin 2\pi \frac{\Phi}{\Phi_0}. \tag{4.10}$$

A relation of the kind $I = I(\Phi)$ corresponds in general to an inductor. We see that in this case the inductance is non-linear and is given by

$$L_J(\delta) = \left( \frac{\partial I}{\partial \Phi} \right)^{-1} = \frac{L_{J0}}{\cos \delta}, \tag{4.11}$$

where $L_{J0} = \Phi_0/2\pi I_c$. Thus the Josephson junction behave in a circuit as a non-linear inductor. Furthermore it brings an intrinsic capacity $C_j$, so that can be represented as a circuit with a non-linear inductor in parallel with a capacitor, like in Fig. 4.2c. From the expression of the inductance can be demonstrated that the energy associated with a Josephson induction is

$$E_J(\Phi_{ext}) = -E_J \cos \pi \frac{\Phi_{ext}}{\Phi_0}, \tag{4.12}$$

where $\Phi_{ext}$ is the flux externally imposed and $E_J = \Phi_0 I_c/2\pi$ is a constant called

*Josephson energy.*

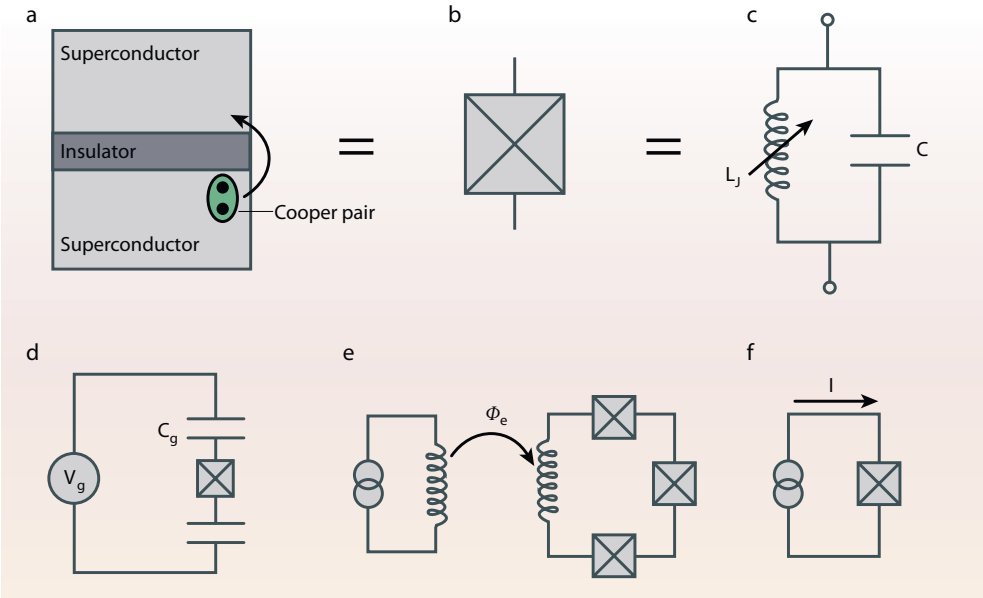


Figure 4.2: from Ref. [11]: **a**) basic scheme of a Josephson junction; **b**) symbol in circuit diagrams of a Josephson junction; **c**) equivalent circuit of a Josephson junction (the arrow represents the non-linearity of the inductance); **d**) charge qubit circuit; **e**) flux qubit circuit; **f**) phase qubit circuit;

Until now three main kinds of superconducting qubits have been used. The classification is based on the variables by which they are controlled and excited. We describe them briefly, without entering the technical details and omitting completely the important part of the read-out procedure.

The simplest of them is the *charge qubit*, or *Cooper pair box qubit*. Its circuit implementation consists of a Josephson junction, a capacitor $C_g$ and a voltage source $U$ in series, see Fig. 4.2d, so that one of the two superconducting pieces of the junction, being wired with one armature of the capacitor is isolated. The Hamiltonian of the circuit can be proved to be

$$H_{CPB} = E_C(N - N_g)^2 - E_J \cos\theta, \tag{4.13}$$

where $N$ is the operator number of Cooper pairs on the island and $\theta = \delta \bmod 2\pi$ is the canonical conjugate operator. $E_C = (2e)^2/2(C_J + C_C)$ is the charging energy of the island and $N_g = C_g U/2e$ is the number of pairs due to the polarization of the capacitor $C_g$ and, contrary to $N$, assumes a continuum of values. $N_g$ and $E_J$ are a parameter that can be controlled externally, the former by acting on the voltage $U$,

and the latter by applying a field through the junction or by using a split junction and applying a field through the loop. . The qubit states correspond to adjacent Cooper pair number states $|N = 0\rangle$ and $|N = 1\rangle$. For $N_g = 1/2$ the two levels are degenerate in energy. Near this point it can be shown that the Hamiltonian assumes the simple form

$$H_{CPB} = -E_z(\sigma_z + X_{control}\sigma_x), \tag{4.14}$$

where $E_z$ in the limit $E_J \ll E_C$, which is the normal regime for the Cooper pair box, reduces to $E_J/2$ and $X_{control} = 2(1/2 - N_g)E_C/E_J$. We see from this expression for the Hamiltonian how the qubit can be manipulated controlling the voltage $U$.

The second kind of qubit is the *flux qubit*, it consists in a superconducting loop (with inductance $L$) interrupted by one ore three Josephson junctions, see Fig. 4.2e. A current through an external inductor generates a magnetic flux which induces clockwise or anticlockwise supercurrents in the loop. Here the conjugate variables are the flux through the loop $\phi$ and the charge on the capacitance $q$ and the Hamiltonian is

$$H_F = \frac{q^2}{2C_J} + \frac{\phi^2}{2L} - E_J \cos\left[\frac{2e}{\hbar}(\phi - \Phi_{ext})\right]. \tag{4.15}$$

The potential in this case is no more a reversed cosine but has a parabolic shape with a cosine corrugation. The first two energy eigenstates are the symmetric and the antisymmetric combination of the two wavefunctions localized in the two wells. The external flux $\Phi_{ext}$ plays here the role of the charge $N_g$ in the Cooper pair box qubit.

The third type of qubit is the *phase qubit*, which consists in a single Josephson junction in series with a current source, see Fig. 4.2f. The current alters the phase at the two side of the junction according to eq. (4.7). Here the conjugate variables are the phase $\delta$ and the charge $2e$ on the capacitance. The Hamiltonian is

$$H_P = E_{CJ}p^2 - I\phi_0\delta - I_c\phi_0 \cos\delta, \tag{4.16}$$

where $E_{CJ} = (2e)^2/C_J$ and $I$ is the current in the current in the junction. The potential has the shape of a tilted washboard.

The reason for which we focused on the superconducting circuits is that it has been proved they can be used to reproduce cavity QED systems, realizing the so-called *circuit QED* [17, 18]. The *strong coupling* limit of cavity QED, i.e. the

condition $g \gg \kappa, \gamma$, where $\kappa$ and $\gamma$ are the decay rates of the cavity and the atom, has been reached using superconducting circuits for the first time in 2004. The structure used in the experiment consists in transmission-line resonator coupled to a Cooper pair box qubit, represented in Fig. 4.3. The transmission-line resonator is realized by mean of central superconducting nanowire running through two coplanar planes. Two gaps in the wire, placed an integer number of wave-lengths apart, are the 'mirrors' used to form a 'cavity'. The piece of wire placed between the gaps has an inductance $L$ and a capacitance $C$ in order to behave like an harmonic oscillator. The predominant source of dissipation are is the loss of the photons from the resonator through these ports at rate $\kappa$, which can be modified engineering the size and shape of the gaps, while the internal loss of the resonator are negligible. The qubit, an isolated Josephson junction is placed between the wire ant the ground planes at an antinode of the voltage standing wave on the line so that it couples with the electric field. The dipole atom of this artificial atom is very large, and so it is the electric field amplitude, due to the small confinement. This results in a coupling that is orders of magnitude larger than that one of real atoms and optical and microwaves photons. Another advantage of the circuit QED is that the position of the artificial atom is defined inside the cavity with great precision, so that all the problems relating to the trapping and cooling of the atom are avoided.

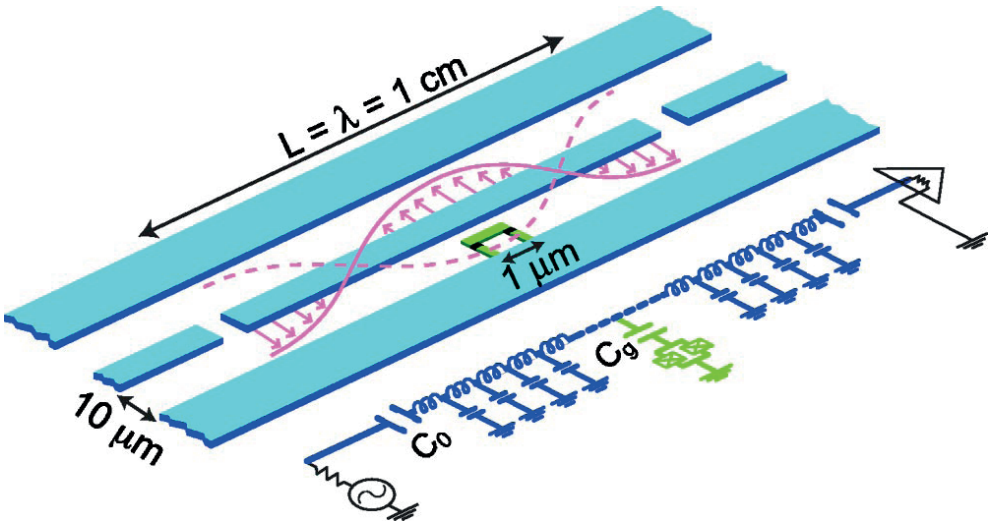


Figure 4.3: Schematic layout and equivalent lumped-circuit representation of the implementation of a cavity using superconducting circuits, from Ref. [17]. In blue is represented the 1-D transmission-line resonator, while in green the Cooper pair box qubit.

## 4.3 New Ξ-atom-in-cavity model

As we have seen in the previous section the developing of superconducting circuits technologies permits the achievement of strong coupling regime of CQED in an unprecedented way. The transmission properties of a cavity in that regime are the natural candidate for the realization of a photonic transistor. The use of superconducting qubit imposes a strict limitation on the model: the levels of the atom have to be in a ladder configuration.

The use of three-level atoms in cavity has already been proposed to realize quantum computation. Duan and Kimble in 2004 [19, 20] proposed a scheme to realized a phase-flip gate based on a ladder three-level atom in one side cavity using photons polarization to encode the quantum information. In their scheme the transition between the two upper levels $|e\rangle$ and $|g\rangle$ is resonant with the cavity frequency, while the transition between $|g\rangle$ and the lowest level $|f\rangle$ is strongly detuned. For an incoming photon pulse with time width $\sigma_T \gg 1/\kappa$ centered around $\omega_{ge}$, the properties of the reflected pulse depend on the state of the atom. If the atom sits in $|f\rangle$ the atom does not play any role and the pulse acquires a $\pi$ phase. If the atom is in $|g\rangle$ the reflection is modified by the coupling, and because of the detuning of the dressed frequency from that of the incoming pulse, the photon pulse acquires no phase change. On this property is based the scheme for quantum computation proposed. The dynamics of the a system has been studied computationally by the authors and empirical formulae for the error probability and the fidelity of the gate have been found.

Our model for the photonic transistor is based on essentially the same system used in [19], though we are mainly interested in transmission properties instead of phase shift. We choose to discuss a two-sides cavity, even if also a procedure with a one-side cavity and some linear optics element can be realized. As summarized above and represented in Fig. 4.3 we take a Ξ-system, assuming that it is sufficiently anharmonic such that the cavity is resonant with the upper transition between state $|g\rangle$ and $|e\rangle$, but off-resonant from the lower transition between $|f\rangle$ and $|g\rangle$. We therefore ignore the cavity coupling on the lower transition and assume that the cavity only interacts with the upper transition with a coupling constant $g$. With this arrangement we can be in a regime where there is a strong cavity enhancement of the decay of the upper level $|e\rangle$ such that it decays rapidly compared to the two lower levels $|g\rangle$ and $|f\rangle$, provided that the coupling constant is much larger than the decay rate of the lower transition $\gamma_1$ and that the cavity decay rate $\kappa$ is chosen

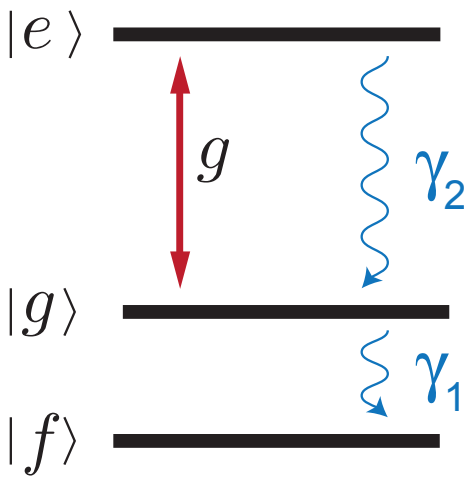


Figure 4.4: Assumed atomic level structure. The cavity field resonantly couples the upper states $|g\rangle$ and $|e\rangle$ of a Ξ-system with a coupling constant $g$, while it is far off resonance from the lower transition between $|f\rangle$ and $|g\rangle$. The levels $|g\rangle$ and $|e\rangle$ decay to the level below with decay rates of $\gamma_1$ and $\gamma_2$ respectively.

accordingly, where $\kappa$ is the total decay rate of the cavity mode into the input and output waveguides, and $\gamma_1$ the decay rates from $|g\rangle$ to $|f\rangle$. We also consider in our system a decay rate of $|e\rangle$ to $|g\rangle$ in modes different from the cavity one, while we assume completely negligible the losses of the cavity in channels different from the waveguides and the decay from $|e\rangle$ to $|f\rangle$. Fundamental for our transistor realization is the possibility to drive the transition $|g\rangle \leftrightarrow |f\rangle$ classically. In essence this combined system of the Ξ-atom and the cavity thus have two metastable levels which mimic the Λ-type atom in the original transistor proposal [10].

The Hamiltonian describing the system is

$$\begin{aligned} H &= H_0^A + H_0^C + H_0^W + H^{AC} + H^{CW} = \\ &= \hbar(\omega_e - i\gamma_2/2)\sigma_{ee} + \hbar(\omega_g - i\gamma_1/2)\sigma_{gg} + \hbar c \int dk\, k\big[\hat{b}_k^\dagger \hat{b}_k + \hat{c}_k^\dagger \hat{c}_k\big] + \hbar\omega_c \hat{a}^\dagger \hat{a} + \\ &\quad + i\hbar g\big[\hat{a}^\dagger \hat{\sigma}_{eg}^- - \hat{a}\hat{\sigma}_{eg}^+\big] + i\hbar c\sqrt{\kappa/2\pi} \int dk\big[\hat{b}_k \hat{a}^\dagger - \hat{b}_k^\dagger \hat{a} + \hat{c}_k \hat{a}^\dagger - \hat{c}_k^\dagger \hat{a}\big], \end{aligned} \quad (4.17)$$

where the first line contains the non-interacting Hamiltonians of atom, cavity and external fields, which are represented by operators $\hat{b}$ and $\hat{c}$ for the the field modes on the left and right side of the cavity. The second line contains the Hamiltonians of the interaction atom-cavity and cavity-external fields. A linear dispersion relation has been assumed for the external modes, and $\kappa$ has been assumed to be frequency-

independent, which is reasonable for pulses not too spread in momentum.

We explain for now our protocol in the ideal case, i.e. where $\gamma_1 = \gamma_2 = 0$, leaving a detailed analysis of the errors in a successive chapter. Similarly to the original transistor proposal our protocol consists of two steps, the first one is the interaction of the system with the gate photon, while the second consists of sending a multi-photon signal field. A key ingredient in our scheme is the difference in the scattering dynamics for photons scattering off the cavity depending on the state of the atom. If the atoms is in state $|f\rangle$ the atoms do not interact with the cavity field and the incoming resonant photon is transmitted through the cavity. If on the other hand the photon is in state $|g\rangle$ the atom and the cavity form dressed states at frequencies $\omega_c \pm g$. If the atoms is in $|g\rangle$ an incoming photon will effectively be off resonance and will be reflected. The state of the atom thus controls whether the cavity is transmitting or not and acts as a switch. We can thus realize the single photon transistor if we can flip the state of the atom conditioned on the presence of a photon in the first gate pulse. To do this we use the procedure of Ref. [19] and start with the atom initialized in the state $|\Psi_0\rangle_a = 1/\sqrt{2}(|g\rangle + |f\rangle)$ using the classical field on the transition between $|g\rangle$ and $|f\rangle$. Then for a time $T$ we send in the gate pulse containing a single photon or not. This gate pulse is sent in symmetrically as shown in Fig. 4.5. By taking the incoming photon to be a superposition of a pulse coming from the left and from the right we cannot determine the atomic state from whether the photon has been transmitted or reflected. There is, however, a phase difference between the reflection and transmission such that the whole evolution is

$$\begin{aligned} |0\rangle |\Psi_0\rangle_a &\to \frac{1}{\sqrt{2}} |0\rangle (|g\rangle + |f\rangle) \\ |1\rangle |\Psi_0\rangle_a &\to \frac{1}{\sqrt{2}} |1\rangle (|g\rangle - |f\rangle). \end{aligned} \tag{4.18}$$

Since the phase of the superposition now depends on the absence or presence of a single photon we can map this into the population with a $\pi/2$-pulse:

$$\begin{aligned} |g\rangle &\xrightarrow{\pi/2} \frac{1}{\sqrt{2}}(|g\rangle - |f\rangle) \\ |f\rangle &\xrightarrow{\pi/2} \frac{1}{\sqrt{2}}(|g\rangle + |f\rangle). \end{aligned} \tag{4.19}$$

The overall transformation in the first step is then given by

$$\begin{aligned} |0\rangle |\Psi_0\rangle_a &\rightarrow |0\rangle |g\rangle \\ |1\rangle |\Psi_0\rangle_a &\rightarrow |1\rangle |f\rangle , \end{aligned} \tag{4.20}$$

which exactly realizes a conditional flip of the atomic state.

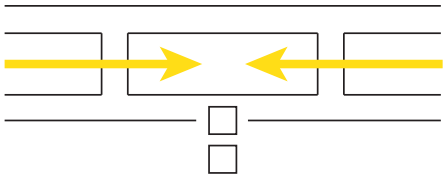

Figure 4.5: First step. A control field is sent in symmetrically from the left and right. This makes it indistinguishable whether the photon was reflected or transmitted, but still a phase difference is imparted on the atom (see text). This phase difference can be used to flip the atom depending on the presence or absence of a photon in the field.

The second step consists in sending in signal photons to the system for a time $T'$. In this step the beam of photons is only incident from the left as shown in Fig. 4.6. Since the transmission or reflection of these photons is determined by atomic state, the presence or absence of a photon in the first step controls the transmission in the second step.

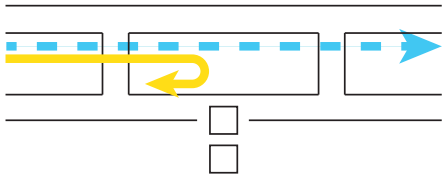

Figure 4.6: Second step. A strong field is sent in from the left. The field is reflected if the atom is in state $|g\rangle$ (full arrow), and transmitted if the atom is in state $|f\rangle$ (dashed arrow).

# Chapter 5

# Analytical solutions

In this chapter we search for an analytical solution of the dynamics generated by the Hamiltonian (4.17) in some cases of interest for the working of our proposal of transistor. First of all we solve for the general case of a single incoming photon, from which we derive the coefficients of transmission and reflection and the probability to loose the photon during the interaction with cavity. Then we specialize the general solution to the cases of a pulse with a gaussian or rectangular shape. Lastly we find a solution for the case of a coherent input state in the steady state condition.

## 5.1 Incoming single photon

In this section we solve the Schrödinger-like part of the dynamics generated by the Hamiltonian (4.17). Our system, considered as the ensemble of all its elements, atom, cavity and waveguides, is an open system, indeed the coupling of the atom with electromagnetic modes different from that one of the cavity is a source of quantum noise. To take it into account, in the spirit of the "quantum jump" explained in sec. 3.5 we have inserted in the Hamiltonian the non-hermitian terms $-i\gamma_2 |e\rangle \langle e|$ and $-i\gamma_1 |g\rangle \langle g|$. However to solve the Schrödinger equation with the effective Hamiltonian is not enough to describe completely the dynamics, it is indeed required to consider the "jump" part of the evolution. Thus it seems that we are prevented to give a deterministic solution. Analyzing the first step of the protocol we notice that at end of it an externally driven $\pi/2$ pulse mixes the states $|g\rangle$ and $|f\rangle$. This means that, if the probabilities obtained with Schödinger equation for the atom to stay in $|g\rangle$

or in $|f\rangle$ before the $\pi/2$ pulse are $P_g$ and $P_f$ (where $P_g + P_f < 1$ because of the non-hermitianity of the Hamiltonian), after the pulse the probabilities are

$$P'_g = P_g + \frac{1}{2}(1 - P_g - P_f) = \frac{1}{2}(1 + P_g - P_f) \tag{5.1}$$

and

$$P'_f = P_f + \frac{1}{2}(1 - P_f - P_g) = \frac{1}{2}(1 + P_f - P_g). \tag{5.2}$$

This fact is extremely important since it means that the probabilities $P_g$ and $P_f$ are sufficient and we can neglect completely the stochastic part of the evolution.

The Schrödinger equation is [21]

$$\frac{\partial}{\partial t}|\psi\rangle = -\frac{i}{\hbar} H_{eff}|\psi\rangle\,, \tag{5.3}$$

where $|\psi\rangle$ is a generic state of the system, which belongs to an Hilbert space that is the tensor product of four spaces: one for the atom, one for the cavity and two for the waveguides. The basis of these spaces are $\{|f\rangle, |g\rangle, |e\rangle\}$, $\{|0\rangle, |1\rangle,..,|n\rangle,..\}$,$\{|0_L\rangle, |k_L\rangle\}$ and $\{|0_R\rangle, |k_R\rangle\}$, where $k_L$ and $k_R$ are continuum indexes. Since we are interested in the case of one excitation we can restrict our attention to a subspace of the Hilbert space and define an appropriate basis. We take the internal states

$$\begin{aligned}
|g\rangle|1\rangle|0\rangle|0\rangle &\equiv |A\rangle \\
|e\rangle|0\rangle|0\rangle|0\rangle &\equiv |B\rangle \\
|g\rangle|0\rangle|0\rangle|0\rangle &\equiv |C\rangle \\
|f\rangle|1\rangle|0\rangle|0\rangle &\equiv |D\rangle \\
|f\rangle|0\rangle|0\rangle|0\rangle &\equiv |E\rangle
\end{aligned} \tag{5.4}$$

plus the four continuum infinity of states given by $\hat{b}_k^\dagger|C\rangle$, $\hat{c}_k^\dagger|C\rangle$, $\hat{b}_k^\dagger|E\rangle$ and $\hat{c}_k^\dagger|E\rangle$. We can express the generic state $|\Psi\rangle$ using this states:

$$\begin{aligned}
|\Psi\rangle = c_a|A\rangle + c_b|B\rangle + c_c|C\rangle + c_d|D\rangle + c_e|E\rangle + \\
\int dk\big[d_k\hat{b}_k^\dagger|C\rangle + d'_k\hat{b}_k^\dagger|E\rangle + e_k\hat{c}_k^\dagger|C\rangle + e'_k\hat{c}_k^\dagger|E\rangle\big].
\end{aligned} \tag{5.5}$$

Using (5.5) we find that the r.h.s. member of the Schrödinger equation (5.3) is

$$\begin{aligned}
&- i(\omega_g - i\gamma_1/2)c_a\,|A\rangle - i\omega_c c_a\,|A\rangle - g c_a\,|B\rangle - c\sqrt{\kappa/2\pi}c_a \int dk\,[\hat{b}_k^\dagger\,|C\rangle + \hat{c}_k^\dagger\,|C\rangle\,] + \\
&- i(\omega_e - i\gamma_2/2)c_b\,|B\rangle + g c_b\,|A\rangle - i(\omega_g - i\gamma_1/2)c_c\,|C\rangle - i\omega_c c_d\,|D\rangle + \\
&- c\sqrt{\kappa/2\pi}c_d \int dk\,[\hat{b}_k^\dagger\,|E\rangle + \hat{c}_k^\dagger\,|E\rangle\,] - i(\omega_g - i\gamma_1/2)\int dk\,d_k\hat{b}_k^\dagger\,|C\rangle + \\
&- ic\int dk\,k\,d_k\hat{b}_k^\dagger\,|C\rangle + c\sqrt{\kappa/2\pi}\int dk\,d_k\,|A\rangle - ic\int dk\,k\,d'_k\hat{b}_k^\dagger\,|E\rangle + \\
&+ c\sqrt{\kappa/2\pi}\int dk\,d'_k\,|D\rangle - i(\omega_g - i\gamma_1/2)\int dk\,e_k\hat{c}_k^\dagger\,|C\rangle - ic\int dk\,k\,e_k\hat{c}_k^\dagger\,|C\rangle + \\
&+ c\sqrt{\kappa/2\pi}\int dk\,e_k\,|A\rangle - ic\int dk\,k\,e'_k\hat{c}_k^\dagger\,|E\rangle + c\sqrt{\kappa/2\pi}\int dk\,e'_k\,|D\rangle\,. \qquad (5.6)
\end{aligned}$$

By multiplying the two members on the left for each one of the states contained in (5.5) we get the following decoupled systems for the coefficients

$$\begin{aligned}
\dot{c}_a &= -i(\omega_g + \omega_c - i\gamma_1/2)c_a + g c_b + c\sqrt{\kappa/2\pi}\int dk\,d_k + c\sqrt{\kappa/2\pi}\int dk\,e_k \\
\dot{c}_b &= -i(\omega_e - i\gamma_2/2)c_b - g c_a \\
\dot{d}_k &= -\sqrt{\kappa/2\pi}c_a - i(\omega_g + ck - i\gamma_1/2)d_k \\
\dot{e}_k &= -\sqrt{\kappa/2\pi}c_a - i(\omega_g + ck - i\gamma_1/2)e_k
\end{aligned} \qquad (5.7)$$

and

$$\begin{aligned}
\dot{c}_d &= -i\omega_c c_d + c\sqrt{\kappa/2\pi}\int dk\,d'_k + c\sqrt{\kappa/2\pi}\int dk\,e'_k \\
\dot{d}'_k &= -ick\,d'_k - \sqrt{\kappa/2\pi}c_d \\
\dot{e}'_k &= -ick\,e'_k - \sqrt{\kappa/2\pi}c_d
\end{aligned} \qquad (5.8)$$

The first system describe the dynamics when the atom is started in $|g\rangle$, while the second one when it is started in $|f\rangle$. The decoupling of the equations in the two groups follows the fact the in the Hamiltonian there is no term for the transition between those two atomic states. The transition can happen, as we have seen, just by a decay process and so by a "jump". Clearly the linearity of the Schrödinger equation makes obvious how to find a solution in the case of an atom in a superposition of states.

We assume the resonant condition, i.e. $\omega_{eg} = \omega_e - \omega_g = \omega_c$ and we move to a kind of interaction picture replacing the amplitudes of system (5.7) $c_{a,b}(t), d_k(t), e_k(t)$ with the amplitudes $C_{a,b}e^{-i(\omega_e - i\gamma_1)t}$, etc. In this way we get the following equations

for the new amplitudes:

$$\begin{aligned}
\dot{C}_a &= g\,C_b + c\sqrt{\kappa/2\pi}\int dk\,D_k + c\sqrt{\kappa/2\pi}\int dk\,E_k \\
\dot{C}_b &= -(\gamma_2-\gamma_1)/2\,C_b - g\,C_a \\
\dot{D}_k &= -\sqrt{\kappa/2\pi}c_a - i(ck-\omega_c)D_k \\
\dot{E}_k &= -\sqrt{\kappa/2\pi}c_a - i(ck-\omega_c)E_k.
\end{aligned} \tag{5.9}$$

The initial conditions are $C_a(t_0) = C_b(t_0) = C_c(t_0) = 0$, and for convenience we choose $t_0 = 0$. We formally integrate the third equation of (5.9) and we find

$$D_k(t) = e^{-i(ck-\omega_c)t}D_k(0) - \sqrt{\kappa/2\pi}\int_0^t dt'\,e^{-i(ck-\omega_c)(t-t')}C_a(t') \tag{5.10}$$

and similarly for the fourth equation. The equation for $C_a$ can as well be formally integrated:

$$C_a(t) = g\int_0^t dt'\,C_b(t') + c\sqrt{\kappa/2\pi}\int dk\,\int_0^t dt'\,(D_k(t') + E_k(t')) \tag{5.11}$$

If we insert (5.10) in (5.11) we get

$$\begin{aligned}
C_a(t) = g\int_0^t dt'\,C_b(t') + \sqrt{\kappa}\int_0^t dt'\,\big(D^{IN}(t') + E^{IN}(t')\big) - \\
- \frac{\kappa c}{\pi}\int dk\,\int_0^t dt'\,\int_0^{t'} dt''\,e^{-i(ck-\omega_c)(t'-t'')}C_a(t''),
\end{aligned} \tag{5.12}$$

where we have defined

$$D^{IN}(t) = \frac{c}{\sqrt{2\pi}}\int dk\,e^{-i(ck-\omega_c)t}D_k(0) \tag{5.13}$$

the photonic incoming field amplitude at the left entrance of the cavity, and similarly $E^{IN}(t)$. We treat the last integral in eq. (5.12) with the Markov approximation, discussed in Chapter 3. We get

$$\begin{aligned}
C_a(t) = g\int_0^t dt'\,C_b(t') + \sqrt{\kappa}\int_0^t dt'\,\big(D^{IN}(t') + E^{IN}(t')\big) - \\
- \kappa\int_0^t dt'\,C_a(t'),
\end{aligned} \tag{5.14}$$

which means that now $C_a$ is couple just with $C_b$, since both $D^{IN}(t)$ and $E^{IN}(t)$ are known from the initial conditions. Taking the time derivative of (5.14) and the second

equation in (5.9) we get a system of two first order coupled differential equations for $C_a$ and $C_b$:

$$\begin{aligned}\dot{C}_a &= -\kappa\, C_a(t) + g\, C_b(t) + \sqrt{\kappa}\big(D^{IN}(t) + E^{IN}(t)\big)\\ \dot{C}_b &= -(\gamma_2 - \gamma_1)/2\, C_b(t) - g\, C_a(t)\end{aligned} \tag{5.15}$$

The system is in the form

$$\dot{C}_i = \sum_j a_{ij} x_j + b_i, \tag{5.16}$$

and its solution it is well known from the theory of differential equations. First of all we have to find the eigenvalues of the matrix

$$a_{ij} = \begin{pmatrix} -\kappa & +g \\ -g & -\Gamma/2 \end{pmatrix} \tag{5.17}$$

with $\Gamma = (\gamma_2 - \gamma_1)/2$, that result to be $\lambda_{1,2} = -\big(\kappa + \Gamma \pm \sqrt{(\kappa - \Gamma)^2 - 4g^2}\big)/4$. Since we are in the strong coupling regime, i.e. $g \gg \kappa, \gamma_{1,2}$, they can be approximated as $\lambda_{1,2} = -(\kappa+\Gamma)/2 \mp ig = -\beta \mp ig$, with $\beta = (\kappa+\Gamma)/2$. With the same approximation the matrix of the eigenvectors is

$$S = \begin{pmatrix} i + \frac{\beta'}{g} & -i + \frac{\beta'}{g} \\ 1 & 1 \end{pmatrix} \tag{5.18}$$

where $\beta' = (\kappa - \Gamma)/2$.

The matrix whose columns are solutions of the associate homogeneous system is

$$X = \begin{pmatrix} (i + \frac{\beta'}{g})e^{\lambda_1 t} & (-i + \frac{\beta'}{g})e^{\lambda_2 t} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \tag{5.19}$$

and its inverse is

$$X^{-1} = \frac{1}{2}\begin{pmatrix} -ie^{-\lambda_1 t} & (1 + i\frac{\beta'}{g})e^{-\lambda_1 t} \\ +ie^{-\lambda_2 t} & (1 - i\frac{\beta'}{g})e^{-\lambda_2 t} \end{pmatrix} \tag{5.20}$$

We know that the solution of system (5.16) is $x_i(t) = \sum_j c_j X_{ij} + \sum_{j,k} X_{ij} \int dt X^{-1}_{jk} b_k$. In our case the first term on the r.h.s. is given by

$$c_1 \begin{pmatrix} i + \frac{\beta'}{g} \\ 1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} -i + \frac{\beta'}{g} \\ 1 \end{pmatrix} e^{\lambda_2 t}, \tag{5.21}$$

while the second term is equal to

$$\frac{1}{2}\begin{pmatrix}(i+\frac{\beta'}{g})e^{\lambda_1 t} & (-i+\frac{\beta'}{g})e^{\lambda_2 t}\\ e^{\lambda_1 t} & e^{\lambda_2 t}\end{pmatrix}\int dt\begin{pmatrix}-ie^{-\lambda_1 t} & (1+i\frac{\beta'}{g})e^{-\lambda_1 t}\\ +ie^{-\lambda_2 t} & (1-i\frac{\beta'}{g})e^{-\lambda_2 t}\end{pmatrix}\begin{pmatrix}\sqrt{\kappa}\big(D^{IN}(t)+E^{IN}(t)\big)\\ 0\end{pmatrix}=$$
$$\frac{1}{2}\begin{pmatrix}(i+\frac{\beta'}{g})e^{\lambda_1 t} & (-i+\frac{\beta'}{g})e^{\lambda_2 t}\\ e^{\lambda_1 t} & e^{\lambda_2 t}\end{pmatrix}\int dt\begin{pmatrix}-ie^{-\lambda_1 t}\\ ie^{-\lambda_2 t}\end{pmatrix}\sqrt{\kappa}\big(D^{IN}(t)+E^{IN}(t)\big). \quad (5.22)$$

Thus the generic solution of system the system that we are solving is

$$\begin{aligned}C_a(t) = c_1\Big(i+\frac{\beta'}{g}\Big)e^{\lambda_1 t} + c_2\Big(-i+\frac{\beta'}{g}\Big)e^{\lambda_2 t} + \\ + \frac{1}{2}\Big[\Big(1-i\frac{\beta'}{g}\Big)e^{\lambda_1 t}\int dt\, e^{-\lambda_1 t}\sqrt{\kappa}\big(D^{IN}(t)+E^{IN}(t)\big) + \\ + \Big(1+i\frac{\beta'}{g}\Big)e^{\lambda_2 t}\int dt\, e^{-\lambda_2 t}\sqrt{\kappa}\big(D^{IN}(t)+E^{IN}(t)\big)\Big] \quad (5.23)\end{aligned}$$

and

$$\begin{aligned}C_b(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \\ + \frac{i}{2}\Big[-e^{\lambda_1 t}\int dt\, e^{-\lambda_1 t}\sqrt{\kappa}\big(D^{IN}(t)+E^{IN}(t)\big) + \\ + e^{\lambda_2 t}\int dt\, e^{-\lambda_2 t}\sqrt{\kappa}\big(D^{IN}(t)+E^{IN}(t)\big)\Big]. \quad (5.24)\end{aligned}$$

For convenience we call the two function defined by an indefinite integral in (5.23) and (5.24) $F_1(t)$ and $F_2(t)$. To determine the coefficients $c_1$ and $c_2$ we use the initial conditions $C_a(0) = C_b(0) = 0$:

$$\begin{cases}c_1(i+\frac{\beta'}{g}) + c_2(-i+\frac{\beta'}{g}) + \frac{1}{2}\big(F_1(0)(1-i\frac{\beta'}{g}) + F_2(0)(1+i\frac{\beta'}{g})\big) = 0\\ c_1 + c_2 + \frac{i}{2}\big(-F_1(0)+F_2(0)\big) = 0\end{cases}$$

with solution

$$\begin{cases}c_1 = \frac{i}{2}F_1(0)\\ c_2 = -\frac{i}{2}F_2(0)\end{cases}$$

Thus, replacing this solution in (5.23) (5.24), we get the solution for $C_a$ and $C_b$ given by

$$C_a(t) = \frac{1}{2}\Big(1-i\frac{\beta'}{g}\Big)e^{\lambda_1 t}\big(F_1(t)-F_1(0)\big) + \frac{1}{2}\Big(1+i\frac{\beta'}{g}\Big)e^{\lambda_2 t}\big(F_2(t)-F_2(0)\big) \quad (5.25)$$

and

$$C_b(t) = -\frac{i}{2}e^{\lambda_1 t}\big(F_1(t) - F_1(0)\big) + \frac{i}{2}e^{\lambda_2 t}\big(F_2(t) - F_2(0)\big). \tag{5.26}$$

By the fundamental theorem of calculus we know that $\int_0^t dt' f(t') = F(t) - F(0)$, where $F$ is the primitive of $f$, so that $F(t) = \int_0^t dt'\, f(t') + F(0)$. Applying this formula and inserting the explicit values of $\lambda_1$ and $\lambda_2$ we find

$$\begin{aligned} C_a(t) = &\frac{\big(1 - i\frac{\beta'}{g}\big)\sqrt{\kappa}}{2}e^{-(\beta+ig)t}\int_0^t dt'\, e^{+(\beta+ig)t'}\big(D^{IN}(t') + E^{IN}(t')\big) + \\ &+ \frac{\big(1 + i\frac{\beta'}{g}\big)\sqrt{\kappa}}{2}e^{-(\beta-ig)t}\int_0^t dt'\, e^{+(\beta-ig)t'}\big(D^{IN}(t') + E^{IN}(t')\big) = \\ &= \sqrt{\kappa}\,\Re\left[\left(1 - i\frac{\beta'}{g}\right)e^{-(\beta+ig)t}\int_0^t dt'\, e^{+(\beta+ig)t'}\big(D^{IN}(t') + E^{IN}(t')\big)\right] \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} C_b(t) = &-\frac{i\sqrt{\kappa}}{2}e^{-(\beta+ig)t}\int_0^t dt'\, e^{+(\beta+ig)t'}\big(D^{IN}(t') + E^{IN}(t')\big) + \\ &+ \frac{i\sqrt{\kappa}}{2}e^{-(\beta-ig)t}\int_0^t dt'\, e^{+(\beta-ig)t'}\big(D^{IN}(t') + E^{IN}(t')\big) = \\ &= \sqrt{\kappa}\,\Im\left[e^{-(\beta+ig)t}\int_0^t dt'\, e^{+(\beta+ig)t'}\big(D^{IN}(t') + E^{IN}(t')\big)\right]. \end{aligned} \tag{5.28}$$

We can now solve in a similar manner the other system, that one for the atom initially in $|f\rangle$. The initial condition is $c_d(0) = 0$. The equation for $d'_k$ can be formally integrated as

$$d'_k(t) = e^{-i(ck-\omega_c)t}d'_k(0) - \sqrt{\kappa/2\pi}\int_0^t dt'\, c_d(t') \tag{5.29}$$

and similarly for the equation for $e'_k$. The equation for $c_d$ can be formally integrated as

$$c_d(t) = c\sqrt{\kappa/2\pi}\int dk \int_0^t dt'\,(d'_k(t') + e'_k(t')). \tag{5.30}$$

Inserting eq. (5.29) in eq. (5.30) we get

$$\begin{aligned} c_d(t) = \sqrt{\kappa}\int_0^t dt'\,\big(D'^{IN}(t') + E'^{IN}(t')\big) - \\ - \frac{\kappa}{\pi}c\int dk \int_0^t dt' \int_0^{t'} dt''\, e^{-i(ck-\omega_c)(t'-t'')}c_d(t'') \end{aligned} \tag{5.31}$$

where we have defined

$$D'^{IN}(t) = \frac{c}{\sqrt{2\pi}}\int dk e^{-i(ck-\omega_c)t}d'_k(t_0) \tag{5.32}$$

the photonic field amplitude at the left entrance of the cavity, and similarly for $E'^{IN}(t)$. Using the Markov approximation in the last term we get

$$c_d(t) = \sqrt{\kappa} \int_0^t dt' \left( D'^{IN}(t') + E'^{IN}(t') \right) - \kappa \int_0^t dt' c_d(t') \tag{5.33}$$

Taking the derivative we respect to $t$ we find

$$\dot{c}_d(t) = -\kappa c_d(t) + \sqrt{\kappa} \big( D'^{IN}(t') + E'^{IN}(t') \big) \tag{5.34}$$

which has an immediate solution

$$c_d(t) = \sqrt{\kappa} e^{-\kappa t} \int_0^t dt' \, e^{\kappa t'} \big( D'^{IN}(t') + E'^{IN}(t') \big). \tag{5.35}$$

## 5.2 Transmission and reflection coefficients

After having found the solution for the amplitudes $c_a, c_b$ and $c_d$, which represent the amplitudes of the states with some internal excitation, given the input field, we focus here on the solution for the output field, again splitting the two cases of the atom initialized in $|g\rangle$ or in $|f\rangle$.

Equation (5.10) can be formally integrated also in a different way by using the final condition on the field, i.e.

$$D_k(t) = e^{-i(ck-\omega_c)(t-T)} D_k(T) + \sqrt{\kappa/2\pi} \int_t^T dt' \, e^{-i(ck-\omega_c)(t-t')} C_a(t'). \tag{5.36}$$

We integrate in $k$ eqs. (5.10) and (5.36) using the Markov approximation for the terms with $C_a$, with the definitions

$$D^{IN}(t) = \frac{c}{\sqrt{2\pi}} \int dk \, e^{-i(ck-\omega_c)t} D_k(0) \tag{5.13}$$

and

$$D^{OUT}(t) = \frac{c}{\sqrt{2\pi}} \int dk \, e^{-i(ck-\omega_c)(t-T)} D_k(T), \tag{5.37}$$

and we subtract the second from the first one, getting the equation

$$D^{OUT}(t) - D^{IN}(t) = -\sqrt{\kappa} C_a(t), \tag{5.38}$$

which is an important boundary condition, relating the left incoming field, the left

outgoing field and the intra-cavity fields.

Let's go back to the system of differential equation (5.15). If we Fourier transform the amplitudes using $\delta(k) = ck - \omega_c$, and we denote $D_k(0)$ as $D^{IN}(k)$ and similarly for the other fields, we get

$$\begin{aligned} -i\delta(k)\, C_a(k) &= g\, C_b(k) + \sqrt{\kappa} D^{IN}(k) + \sqrt{\kappa} E^{IN}(k) - \kappa C_a(k) \\ -i\delta(k)\, C_b(k) &= -\Gamma\, C_b(k) - g\, C_a(k). \end{aligned} \tag{5.39}$$

From the second equation we can express $C_b$ in term of $C_a$:

$$C_b(k) = -\frac{ig}{\delta(k) + i\Gamma} C_a(k) \tag{5.40}$$

which inserted in the first one gives us

$$\frac{-i\delta(k)^2 + (\Gamma + \kappa)\delta(k) + i\kappa\Gamma + ig^2}{\delta(k) + i\Gamma} C_a(k) = \sqrt{\kappa}\big(D^{IN}(k) + E^{IN}(k)\big). \tag{5.41}$$

Using eq. (5.38) to eliminate $C_a$ in eq. (5.41) we get finally the involved but exact equation which relates the the left output field with the incoming fields

$$\begin{aligned} D^{OUT}(k) = \frac{\delta(k)^2 + i\Gamma\delta(k) - g^2}{\delta(k)^2 + i(\Gamma + \kappa)\delta(k) - \Gamma\kappa - g^2} D^{IN}(k) + \\ + \frac{-i\kappa\delta(k) + \kappa\Gamma}{\delta(k)^2 + i(\Gamma + \kappa)\delta(k) - \Gamma\kappa - g^2} E^{IN}(k), \end{aligned} \tag{5.42}$$

Since the system is symmetric is immediate to obtain the analogous relation for the right outgoing field

$$\begin{aligned} E^{OUT}(k) = \frac{\delta(k)^2 + i\Gamma\delta(k) - g^2}{\delta(k)^2 + i(\Gamma + \kappa)\delta(k) - \Gamma\kappa - g^2} E^{IN}(k) + \\ + \frac{-i\kappa\delta(k) + \kappa\Gamma}{\delta(k)^2 + i(\Gamma + \kappa)\delta(k) - \Gamma\kappa - g^2} D^{IN}(k), \end{aligned} \tag{5.43}$$

From the last equations it is immediate to get reflection and transmission coefficients. Indeed if we put, for instance, $E_R^{IN}(k)$ equal to zero we get that the former is the coefficient of $D^{OUT}(k)$ in (5.42), while the latter is the coefficient of $D^{OUT}(k)$ in (**??**), i.e.

$$r(k) = \frac{\delta(k)^2 + i\Gamma\delta(k) - g^2}{\delta(k)^2 + i(\Gamma + \kappa)\delta(k) - \Gamma\kappa - g^2} \tag{5.44}$$

and

$$t(k) = \frac{-i\kappa\delta(k) + \kappa\Gamma}{\delta(k)^2 + i(\Gamma + \kappa)\delta(k) - \Gamma\kappa - g^2}. \tag{5.45}$$

We consider now the simplest case in which the atom is started in $|f\rangle$. Replacing the amplitudes present in system (5.8) with $C_D e^{-i\omega_c t}$, etc we have

$$\dot{D}'_k = -i(ck - \omega_c)\, D'_k - \sqrt{\kappa/2\pi} C_d, \tag{5.46}$$

which as we have seen can be integrated in two ways, with the initial condition as

$$D'_k(t) = e^{-i(ck-\omega_c)t} D'_k(0) - \sqrt{\kappa/2\pi} \int_0^t dt'\, e^{-i(ck-\omega_c)(t-t')} C_d(t'), \tag{5.47}$$

or with the final condition as

$$D'_k(t) = e^{-i(ck-\omega_c)(t-T)} D'_k(T) + \sqrt{\kappa/2\pi} \int_t^T dt'\, e^{-i(ck-\omega_c)(t-t')} C_d(t'). \tag{5.48}$$

If we integrate in $k$ eqs. (5.47) and (5.48) using the Markov approximation for the terms with $C_d$, with the definitions

$$D'^{IN}(t) = \frac{c}{\sqrt{2\pi}} \int dk\, e^{-i(ck-\omega_c)t} D'_k(0) \tag{5.49}$$

and

$$D'^{OUT}(t) = \frac{c}{\sqrt{2\pi}} \int dk\, e^{-i(ck-\omega_c)(t-T)} D'_k(T), \tag{5.50}$$

and we subtract the second from the first one, we get the equation

$$D'^{OUT}(t) - D'^{IN}(t) = -\sqrt{\kappa} C_d(t), \tag{5.51}$$

which is the same boundary condition as eq. (5.38).

Fourier transforming eq. (5.34) we get

$$(-i\delta(k) + \kappa) C_d(k) = +\sqrt{\kappa}\big(D'^{IN}(k) + E'^{IN}(k)\big). \tag{5.52}$$

Combining eqs. (5.52) and (5.51) we find the relation

$$D'^{OUT}(k) = \frac{-i\delta(k) D'^{IN}(k) - \kappa E'^{IN}(k)}{\kappa - i\delta(k)} \tag{5.53}$$

and for symmetry

$$E'^{OUT}(k) = \frac{-i\delta(k) E'^{IN}(k) - \kappa D'^{IN}(k)}{\kappa - i\delta(k)}, \tag{5.54}$$

from which we derive the transmission and reflection coefficients

$$t'(k) = -\frac{i\kappa}{\delta(k) + i\kappa} \tag{5.55}$$

and

$$r'(k) = \frac{\delta(k)}{\delta(k) + i\kappa}. \tag{5.56}$$

Summarizing we have that the field amplitudes in the initial Schrödinger picture of (5.5) at time $T$ are given by the relations

$$\begin{aligned} d_k(T) &= \big(r(k)d_k(0) + t(k)e_k(0)\big)e^{-i(\omega_e - i\gamma_1/2)T} \\ e_k(T) &= \big(t(k)d_k(0) + r(k)e_k(0)\big)e^{-i(\omega_e - i\gamma_1/2)T} \\ d'_k(T) &= \big(r'(k)d'_k(0) + t'(k)e'_k(0)\big)e^{-i\omega_c t} \\ e'_k(T) &= \big(t'(k)d'_k(0) + r'(k)e'_k(0)\big)e^{-i\omega_c t}. \end{aligned} \tag{5.57}$$

Keeping in mind that we are in a regime in which $g \gg \kappa \gg \sigma, \gamma_1, \gamma_2$, where $\sigma$ is the width in momentum space of the pulse, we can expand the coefficients (5.45) and (5.44), we get that

$$t(k) \approx \frac{\kappa(i\delta(k) + \Gamma)}{g^2} \approx 0 \tag{5.58}$$

and

$$r(k) \approx 1 - \frac{\kappa(i\delta(k) + \Gamma)}{g^2} \approx 1, \tag{5.59}$$

as stated above. Similarly, we see that using just the condition $\kappa \gg \sigma$ we have $t'(k) \approx -1$ and $r'(k) \approx 0$, again as expected.

## 5.3 Gaussian pulse

In this section we use the general solution for the dynamics of an input consisting in a single photon pulse, derived in the previous sections, with the assumption that the pulse shape is gaussian . The gaussian case has been studied numerically in Refs. [19, 20], and we expect to obtain the same results using our analytical calculations. We take as input field, in the case the atom is $|g\rangle$ at $T = 0$,

$$D^{IN}(t) = \frac{1}{\sqrt[4]{2\pi\sigma_T^2}} e^{-(t-T/2)^2/4\sigma_T^2}. \tag{5.60}$$

The peak of the pulse reach the cavity at $t = T/2$, where $T$ is the time duration that we consider. With reference to our protocol we can identify $T$ with the duration of the first step.

We are firstly interested in the form of the amplitudes $C_a$ and $C_b$. Inserting (5.60) in (5.11) and (5.28), the argument of the real part in (5.27) is

$$\frac{1}{\sqrt[4]{2\pi\sigma_T^2}} e^{-(\beta+ig)t} \int_0^t dt'\, e^{(\beta+ig)t'} e^{-\delta(t')^2/4\sigma_T^2}, \tag{5.61}$$

where we have put $\delta(t) = t - T/2$. The integral can be easily solved by completing the square in the exponent, i.e.

$$\begin{aligned} &\frac{1}{\sqrt[4]{2\pi\sigma_T^2}} e^{-(\beta+ig)t} \int_0^t dt'\, e^{-[\delta(t')^2 - 4(\beta+ig)\delta(t')\sigma_T^2 + 4(\beta+ig)^2\sigma_T^4]/4\sigma_T^2} \times \\ &\qquad \times e^{(\beta+ig))^2\sigma_T^2 + (\beta+ig)T/2} = \\ &= \frac{\sqrt[4]{8\pi\sigma_T^2}}{2} e^{(\beta+ig)^2\sigma_T^2} e^{-(\beta+ig)\delta(t)} \left(1 + erf\big[\frac{\delta(t) - 2(\beta+ig)\sigma_T^2}{2\sigma_T}\big]\right), \end{aligned} \tag{5.62}$$

where $erf(x)$ is the error function, defined in the following way

$$erf(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z dy\, e^{-t^2}. \tag{5.63}$$

Now we can write the explicit expression, though rather involved, of the amplitudes $C_a$ and $C_b$:

$$\begin{aligned} C_a(t) = \frac{\sqrt[4]{8\pi\kappa^2\sigma_T^2}}{2} \Re\Bigg[ &\left(1 - i\frac{\beta'}{g}\right) e^{(\beta+ig)^2\sigma_T^2} e^{-(\beta+ig)\delta(t)} \times \\ &\times \left(1 + erf\big[\frac{\delta(t) - 2(\beta+ig)\sigma_T^2}{2\sigma_T}\big]\right)\Bigg] \end{aligned} \tag{5.64}$$

and

$$\begin{aligned} C_b(t) = \frac{\sqrt[4]{8\pi\kappa^2\sigma_T^2}}{2} \Im\Bigg[ &e^{(\beta+ig)^2\sigma_T^2} e^{-(\beta+ig)\delta(t)} \times \\ &\times \left(1 + erf\big[\frac{\delta(t) - 2(\beta+ig)\sigma_T^2}{2\sigma_T}\big]\right)\Bigg]. \end{aligned} \tag{5.65}$$

We have plotted in Figs. 5.1 and 5.2 the evolution in time of $c_a$ and $c_b$, which are given by $e^{-i(\omega_e - i\gamma_1/2)t} C_a(t)$ and $e^{-i(\omega_e - i\gamma_1/2)t} C_b(t)$, for values of the parameters in according to the regime $g \gg \kappa \gg 1/\sigma, \gamma_1, \gamma_2$, confronting the analytical expressions (5.64) and (5.65) with purely numerical solutions, i.e. obtained by the direct numerical

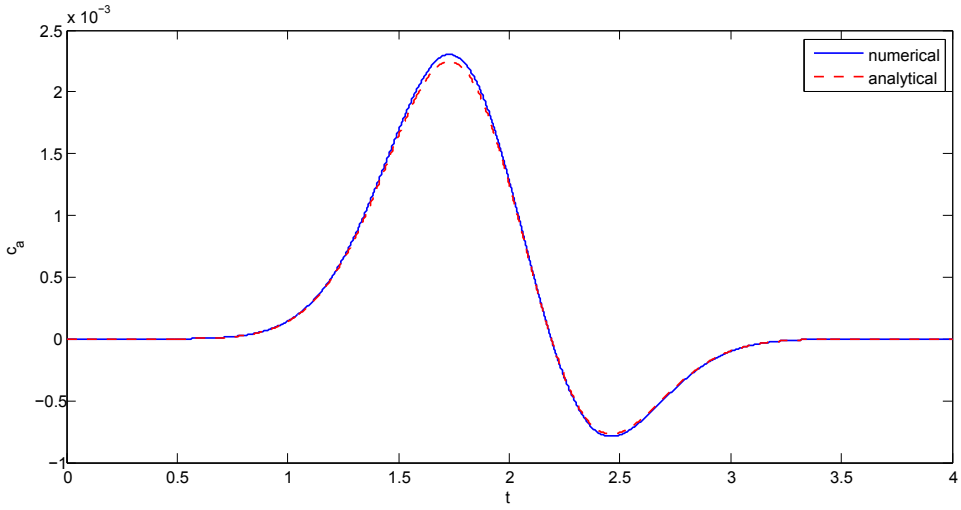


Figure 5.1: $c_a$ for a gaussian single-photon incoming pulse as function of time (with $g = 90, \kappa = 30, \gamma_1 = 0.01, \gamma_2 = 3, T = 4, \sigma_T = T/16$). Confrontation of the numerical (dashed line) and the analytical (solid line) results.

integration of the Schrödinger equation. We notice immediately that the agreement is very good.

In eq. (5.64) the absolute value of the argument of the *erf* function is large because $g \gg \sigma_T$ so that we can use the asymptotic approximation (to the first order) $erf(x) \approx -e^{-x^2}/\sqrt{\pi}x$. We find that

$$\begin{aligned} C_a(t) \approx -\frac{\sqrt[4]{8\pi\kappa^2\sigma_T^2}}{2}\Re\Bigg[\left(1 - i\frac{\beta'}{g}\right)e^{(\beta+ig)^2\sigma_T^2}e^{-(\beta+ig)\delta(t)}\times \\ \times \frac{e^{-\delta(t)^2/4\sigma_T^2}e^{(\beta+ig)\delta(t)}e^{-(\beta+ig)^2\sigma_T^2}}{\sqrt{\pi}(\delta(t) - 2(\beta+ig)\sigma_T^2)/2\sigma_T}\Bigg] = \\ = -\frac{\sigma_T}{g}\sqrt[4]{\frac{8\kappa^2\sigma_T^2}{\pi}}\Re\left[\frac{g - i\beta'}{\delta(t) - 2(\beta+ig)\sigma_T^2}e^{-\delta(t))^2/4\sigma_T^2}\right] = \\ = -\sigma_T\sqrt[4]{\frac{8\kappa^2\sigma_T^2}{\pi}}\frac{\delta(t) - 2\Gamma\sigma_T^2}{(\delta(t) - 2\beta\sigma_T^2)^2 + 4g^2\sigma_T^4}e^{-\delta(t)^2/4\sigma_T^2}. \end{aligned} \quad (5.66)$$

The second term in the denominator of the fraction is much larger than the first one, so that we can neglect it and approximate the expression as

$$C_a(t) \approx -\sqrt[4]{\frac{8\kappa^2\sigma_T^2}{\pi}}\frac{\delta(t) - 2\Gamma\sigma_T^2}{4g^2\sigma_T^3}e^{-\delta(t)^2/4\sigma_T^2}. \quad (5.67)$$

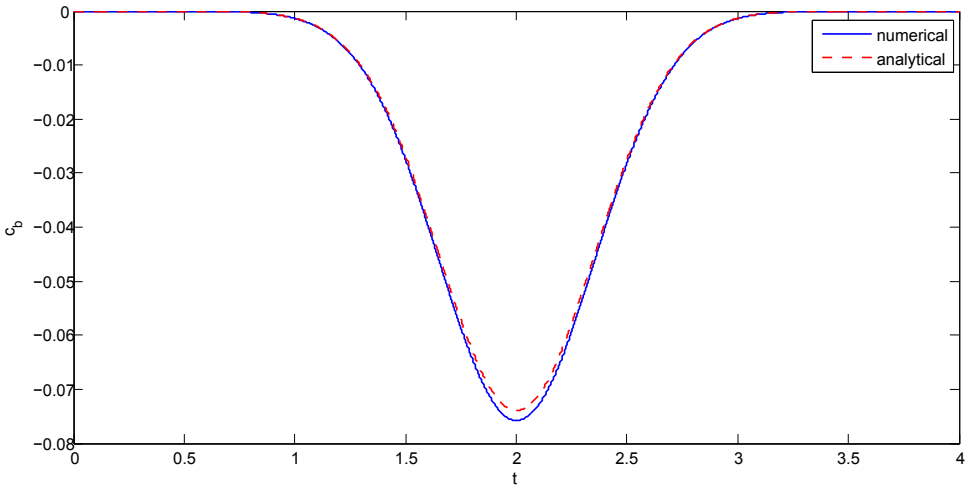


Figure 5.2: $c_b$ for a gaussian single-photon incoming pulse as function of time (with $g = 90, \kappa = 30, \gamma_1 = 0.01, \gamma_2 = 3, T = 4, \sigma_T = T/16$). Confrontation of the numerical (dashed line) and the analytical (solid line) results.

With the same asymptotic expansion we can show that

$$C_b(t) \approx -\sqrt[4]{\frac{8\kappa^2\sigma_T^2}{\pi}}\frac{1}{2g\sigma_T}e^{-\delta(t)^2/4\sigma_T^2}. \tag{5.68}$$

Similarly we deal with the case in which the atom is started in $|f\rangle$. Inserting the gaussian pulse (5.60) in the expression for $c_d$ (5.35), we get

$$\begin{aligned} c_d(t) = \frac{\sqrt{\kappa}}{\sqrt[4]{2\pi\sigma_T^2}}e^{-\kappa t}\int_0^t dt'\, e^{\kappa t'}e^{-\delta(t')^2/4\sigma_T^2} = \\ = \frac{\sqrt[4]{8\pi\kappa^2\sigma_T^2}}{2}e^{\sigma_T^2\kappa^2}e^{-\kappa\delta(t)}\left(1 + erf\big[\frac{\delta(t) - 2\kappa\sigma_T^2}{2\sigma_T}\big]\right). \end{aligned} \tag{5.69}$$

Using the asymptotic expansion of the $erf$ function we find

$$\begin{aligned} c_d(t) = -\sigma_T\sqrt[4]{\frac{8\kappa^2\sigma_T^2}{\pi}}\frac{e^{-\delta(t)^2/4\sigma_T^2}}{\delta(t) - 2\kappa\sigma_T^2} \approx \\ \approx \sqrt[4]{\frac{8\kappa^2\sigma_T^2}{\pi}}\frac{4\sigma_T^4\kappa^2 - 2\sigma_T^2\kappa\delta(t) + \delta(t)^2}{8\sigma_T^5\kappa^3}e^{-\delta(t)^2/4\sigma_T^2}, \end{aligned} \tag{5.70}$$

where in the last step we have expanded the denominator around $\delta(t) = 0$ to the second order.

## 5.4 Rectangular pulse

In this section we study the case of an incoming rectangular pulse, in a manner completely analogous to what we have done in the previous section for the gaussian pulse. The expression of $D^{IN}$ for a pulse of width $\sigma_T$ centered in $T/2$ is

$$D^{IN}(t) = \frac{1}{\sqrt{\sigma_T}}\Theta(t - T/2 + \sigma_T/2)\Theta(T/2 + \sigma_T/2 - t) = \\ = \frac{1}{\sqrt{\sigma_T}}\mathrm{rect}\left(\frac{\delta(t)}{\sigma_T}\right). \quad (5.71)$$

Inserting this expression in the formula for $C_a$ (5.27) we get

$$C_a(t) = \sqrt{\frac{\kappa}{\sigma_T}}\Re\left[\left(1 - i\frac{\beta'}{g}\right)e^{-(\beta+ig)t}\int_0^t dt'\, e^{+(\beta+ig)t'}\Theta(\delta(t') + \sigma_T/2)\Theta(\sigma_T/2 - \delta(t'))\right], \quad (5.72)$$

which is 0 for $t < T/2 - \sigma_T/2$ while for $T/2 - \sigma_T/2 < t < T/2 + \sigma_T/2$ is equal to

$$C_a(t) = \sqrt{\frac{\kappa}{\sigma_T}}\Re\left[\left(1 - i\frac{\beta'}{g}\right)e^{-(\beta+ig)t}\frac{e^{+(\beta+ig)t} - e^{+(\beta+ig)(T/2-\sigma_T/2)}}{\beta + ig}\right] = \\ = \sqrt{\frac{\kappa}{\sigma_T}}\Re\left[\left(1 - i\frac{\beta'}{g}\right)\frac{1 - e^{-(\beta+ig)(\delta(t)+\sigma_T/2)}}{\beta + ig}\right] = \\ = \sqrt{\frac{\kappa}{\sigma_T}}\Re\left[\frac{(g - i\beta')(\beta - ig)}{g(g^2 + \beta^2)}\left(1 - e^{-(\beta+ig)(\delta(t)+\sigma_T/2)}\right)\right] \approx \\ \approx -\sqrt{\frac{\kappa}{\sigma_T}}\frac{e^{-\beta(\delta(t)+\sigma_T/2)}\sin g(\delta(t) + \sigma_T/2)}{g}, \quad (5.73)$$

where in the last step the approximation sign is due to the fact the we have kept the term of order $g^{-1}$ and neglect the others. For for $t > T/2 + \sigma_T/2$ the integral is

$$C_a(t) = \sqrt{\frac{\kappa}{\sigma_T}}\Re\left[\left(1 - i\frac{\beta'}{g}\right)e^{-(\beta+ig)\delta(t)}\frac{e^{+(\beta+ig)\sigma_T/2} - e^{-(\beta+ig)\sigma_T/2}}{\beta + ig}\right] \approx \\ \approx \sqrt{\frac{\kappa}{\sigma_T}}\frac{1}{g}e^{-\beta\delta(t)}\left(e^{-\beta\sigma_T/2}\sin g(\delta(t) + \sigma_T/2) - e^{+\beta\sigma_T/2}\sin g(\delta(t) - \sigma_T/2)\right). \quad (5.74)$$

We can proceed in the same way for $c_b$, inserting (5.71) in (5.28) we get

$$C_b(t) = \sqrt{\frac{\kappa}{\sigma_T}}\Im\left[e^{-(\beta+ig)t}\int_0^t dt'\, e^{+(\beta+ig)t'}\Theta(\delta(t') + \sigma_T/2)\Theta(\sigma_T/2 - \delta(t'))\right], \quad (5.75)$$

which is 0 for $t < T/2 - \sigma_T/2$ while for $T/2 - \sigma_T/2 < t < T/2 + \sigma_T/2$ is equal to

$$C_b(t) \approx -\sqrt{\frac{\kappa}{\sigma_T}}\frac{\left(1 - e^{-\beta(\delta(t)+\sigma_T/2)}\cos g(\delta(t) + \sigma_T/2)\right)}{g}, \quad (5.76)$$

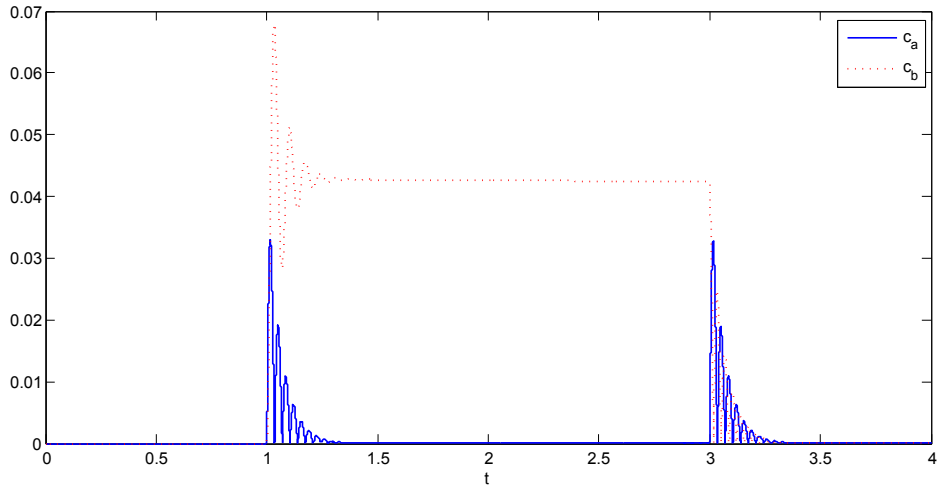


Figure 5.3: $c_a$ (dashed line) and $c_b$ (solid line) for a rectangular single-photon incoming pulse as function of time (with $g = 90, \kappa = 30, \gamma_1 = 0.01, \gamma_2 = 3, T = 4, \sigma_T = T/2$).

while for for $t > T/2 + \sigma_T/2$ is equal to

$$C_b(t) \approx \sqrt{\frac{\kappa}{\sigma_T}} \frac{1}{g} e^{-\beta\delta(t)} \left( e^{-\beta\sigma_T/2} \cos g(\delta(t) + \sigma_T) - e^{+\beta\sigma_T/2} \cos g(\delta(t) - \sigma_T) \right). \quad (5.77)$$

We have plotted in Fig. 5.3 the evolution of the amplitudes $c_a$ and $c_b$ for a choice of the parameters

In the case the atom is started in $|f\rangle$, inserting (5.71) in (5.35) we get

$$c_d(t) = \sqrt{\frac{\kappa}{\sigma_T}} e^{-\kappa t} \int_0^t dt' \, e^{+\kappa t'} \Theta(\delta(t') + \sigma_T/2) \Theta(\sigma_T/2 - \delta(t')), \quad (5.78)$$

which is 0 for $t < T/2 - \sigma_T/2$ while for $T/2 - \sigma_T/2 < t < T/2 + \sigma_T/2$ is equal to

$$c_d(t) = \frac{1}{\sqrt{\kappa\sigma_T}} \left( 1 - e^{-\kappa(\delta(t) + \sigma_T/2)} \right), \quad (5.79)$$

while for for $t > T/2 + \sigma_T/2$ is equal to

$$c_d(t) = \frac{1}{\sqrt{\kappa\sigma_T}} e^{-\kappa\delta(t)} \left( e^{+\sigma_T/2} - e^{-\sigma_T/2} \right). \quad (5.80)$$

## 5.5 Decoherence in the system

Up to this point we have considered the decay of the atomic levels in modes different from that one of the cavity, i.e. of $|e\rangle$ at a rate $\gamma_2$ and of $|g\rangle$ at a rate $\gamma_1$, but we have completed neglected the problem of decoherence, that is the decaying of the phase of a superposition of states or equivalently of the non-diagonal elements of the density matrix of the system [2].

Referring for simplicity to a two-level system with the upper level decaying at a rate $\gamma$ we have seen in sec. 3.4 that the dynamics of the system in presence of decay can be described by a master equation in the Lindblad form

$$\dot{\rho}_s(t) = -\frac{i}{\hbar}[H_s, \rho_s(t)] - \frac{\gamma}{2}\{\sigma_{eg}^+\sigma_{eg}^-, \rho_s(t)\} + \gamma\sigma_{eg}^-\rho_s(t)\sigma_{eg}^+, \tag{5.81}$$

where $H_s$ is the Hamiltonian of the system. If the non-diagonal element are decaying exponentially at a rate $\gamma/2 + \gamma_d$ is easy to show that this can be included in the master equation by introducing the Lindblad operators $\sqrt{\gamma_d}\sigma_{ee}$ and $\sqrt{\gamma_d}\sigma_{gg}$. In fact with this insertion (5.81) becomes

$$\begin{aligned}\dot{\rho}_s(t) = -\frac{i}{\hbar}[H_s, \rho_s(t)] - \frac{\gamma+\gamma_d}{2}\{\sigma_{ee}, \rho_s(t)\} - \frac{\gamma_d}{2}\{\sigma_{gg}, \rho_s(t)\} + \\ + \gamma\sigma_{eg}^-\rho_s(t)\sigma_{eg}^+ + \gamma_d\sigma_{ee}\rho_s(t)\sigma_{ee} + \gamma_d\sigma_{gg}\rho_s(t)\sigma_{gg},\end{aligned} \tag{5.82}$$

from which is immediate to obtain the equations of motion for the elements of the density matrix:

$$\begin{aligned}\dot{\rho}_{ee}(t) &= -\frac{i}{\hbar}[H_s, \rho_{ee}(t)] - \gamma\rho_{ee}(t) \\ \dot{\rho}_{gg}(t) &= -\frac{i}{\hbar}[H_s, \rho_{gg}(t)] + \gamma\rho_{ee}(t) \\ \dot{\rho}_{eg}(t) &= -\frac{i}{\hbar}[H_s, \rho_{eg}(t)] - \big(\frac{\gamma}{2} + \gamma_d\big)\rho_{eg}(t),\end{aligned} \tag{5.83}$$

as requested. According to sec. 3.5 we can transfer this to the quantum jump picture where the effective Hamiltonian is

$$H_{eff} = H_s - \frac{i\hbar(\gamma+\gamma_d)}{2}\sigma_{ee} - \frac{i\hbar\gamma_d}{2}\sigma_{gg}. \tag{5.84}$$

In our three-level system we have to introduce three decoherence rates, one for each of the three non-diagonal element of the density matrix. So we introduce in the

master equation the three Lindblad operators $\sqrt{\gamma_e}\sigma_{ee}$, $\sqrt{\gamma_g}\sigma_{gg}$, $\sqrt{\gamma_f}\sigma_{ff}$, in addition to $\sqrt{\gamma_2}\sigma_{eg}^{-}$ and $\sqrt{\gamma_1}\sigma_{gf}^{-}$. We get that the equations for the six independent elements of the density matrix are

$$\begin{aligned}
\dot{\rho}_{ee}(t) &= -\frac{i}{\hbar}[H_s, \rho_{ee}(t)] - \gamma_2\rho_{ee}(t) \\
\dot{\rho}_{gg}(t) &= -\frac{i}{\hbar}[H_s, \rho_{gg}(t)] - \gamma_1\rho_{gg}(t) + \gamma_2\rho_{ee}(t) \\
\dot{\rho}_{ff}(t) &= -\frac{i}{\hbar}[H_s, \rho_{ff}(t)] + \gamma_1\rho_{gg}(t) \\
\dot{\rho}_{eg}(t) &= -\frac{i}{\hbar}[H_s, \rho_{eg}(t)] - \left((\gamma_1+\gamma_2)/2 + \gamma_e + \gamma_g\right)\rho_{eg}(t) \\
\dot{\rho}_{gf}(t) &= -\frac{i}{\hbar}[H_s, \rho_{gf}(t)] - \left(\gamma_1/2 + \gamma_g + \gamma_f\right)\rho_{gf}(t) \\
\dot{\rho}_{ef}(t) &= -\frac{i}{\hbar}[H_s, \rho_{ef}(t)] - \left(\gamma_2/2 + \gamma_e + \gamma_f\right)\rho_{ef}(t),
\end{aligned} \tag{5.85}$$

which confronted with the expected equations for the non-diagonal elements give the system

$$\begin{aligned}
\gamma_{eg}^d &= \gamma_e + \gamma_g \\
\gamma_{gf}^d &= \gamma_g + \gamma_f \\
\gamma_{ef}^d &= \gamma_e + \gamma_f
\end{aligned} \tag{5.86}$$

with solution

$$\begin{aligned}
\gamma_e &= (\gamma_{eg}^d + \gamma_{ef}^d - \gamma_{gf}^d)/2 \\
\gamma_g &= (\gamma_{eg}^d + \gamma_{gf}^d - \gamma_{ef}^d)/2 \\
\gamma_f &= (\gamma_{ef}^d + \gamma_{gf}^d - \gamma_{eg}^d)/2
\end{aligned} \tag{5.87}$$

In the quantum jump picture this corresponds to the introduction of the terms $i\hbar\gamma_i/2\,\sigma_{ii}$ in the effective Hamiltonian. In the case of the two upper levels this consists in a simple replacing of $\gamma_2$ with $\gamma_2' = \gamma_2 + \gamma_e$ and of $\gamma_1$ with $\gamma_1' = \gamma_1 + \gamma_g$. The term $\gamma_f\sigma_{ff}$ introduces a new kind of decay in the system which afflicts the equations relative to the state $|f\rangle$. The system of equations (5.8) becomes

$$\begin{aligned}
\dot{c}_d &= -i(\omega_c - i\gamma_f/2)c_d + c\sqrt{\kappa/2\pi}\int dk\, d_k' + c\sqrt{\kappa/2\pi}\int dk\, e_k' \\
\dot{d}_k' &= -i(ck - i\gamma_f/2)\, d_k' - \sqrt{\kappa/2\pi}c_d \\
\dot{e}_k' &= -i(ck - i\gamma_f/2)\, e_k' - \sqrt{\kappa/2\pi}c_d.
\end{aligned} \tag{5.88}$$

We can move to an interaction picture replacing the three amplitudes with $C_d e^{-i(\omega_c - i\gamma_f/2)t}$

in order to have the equations

$$\begin{aligned}\dot{C}_d &= c\sqrt{\kappa/2\pi}\int dk\, D'_k + c\sqrt{\kappa/2\pi}\int dk\, E'_k \\ \dot{D}'_k &= -i\delta(k)\, D'_k - \sqrt{\kappa/2\pi}C_d \\ \dot{E}'_k &= -i\delta(k)\, E'_k - \sqrt{\kappa/2\pi}C_d.\end{aligned} \tag{5.89}$$

Thus we have to modify the last two eqs. (5.57) which become

$$\begin{aligned}d'_k(T) &= \big(r'(k)d'_k(0) + r'(k)e'_k(0)\big)e^{-i(\omega_c - i\gamma_f/2)t} \\ e'_k(T) &= \big(t'(k)d'_k(0) + r'(k)e'_k(0)\big)e^{-i(\omega_c - i\gamma_f/2)t}.\end{aligned} \tag{5.90}$$

The quantum jump part of the evolution reduces, for our purposes, to find the probability that the atom decoheres to $|e\rangle$, in fact, as we have seen the second $\pi/2$ pulse mixes the probabilities to be in $|g\rangle$ or in $|f\rangle$. The probability to have a jump to the state $|e\rangle$ is equal to $\delta t\, \gamma_e\, \rho_{ee}$. It is important to note that in the gaussian case the probability to be in the excited state $P_e(t) = \rho_{ee} = |c_b(t)|^2$, replacing the decay rates according to what said above and using eq. (5.68), is

$$P_e(t) = \sqrt{\frac{2}{\pi}}\frac{\kappa}{g^2\sigma_T}e^{-\gamma'_1 t}e^{-\delta(t)^2/2\sigma_T^2}, \tag{5.91}$$

so that time integral of the probability to decohere to the excited state is

$$P_e^{TOT} = \frac{2\kappa\gamma_e}{g^2}e^{-\gamma'_1 T/2}e^{\gamma'^2_1\sigma_T^2/2}. \tag{5.92}$$

Thus expression gives just an upper limit to the probability of being effectively in the excited state at time $T$, in fact after a jump the excited state can still decay in the cavity mode or in other modes, leaving the atom in one of the two lower states. Since this limit is small, and the probability to decay is high, we can neglect completely the possibility that the atom is in $|e\rangle$ at time $T$.

## 5.6 Coupling between the levels $|g\rangle$ and $|f\rangle$

In our description of the system in sec. 4.3 we have stated that the states $|g\rangle$ and $|f\rangle$ are completely decoupled so that in the quantum jump Hamiltonian no term coupling these states is present. The reason for the decoupling is the fact the atom is assumed

to have an anharmonicity such that the transition frequency $\omega_{gf}$ is far detuned from the cavity frequency $\omega_c = \omega_{eg}$.

A more general theory would include a second Jaynes-Cumming-like term in the Hamiltonian to provide a coupling between the state $|g, 0\rangle$ and the state $|f, 1\rangle$, where the first labels refer to the atomic state and the second to the number of photons. The energies of these states are respectively $\omega_g$ and $\omega_f + \omega_c - i\kappa/2$ where, keeping the quantum jump approach, the imaginary part account for the decaying of the states to the lower state $|f, 0\rangle$. The detuning of the transition is given by the difference of the energies and is $\tilde{\Delta} = \omega_{gf} - \omega_c + i\kappa/2 = \Delta + i\kappa/2$ where we have called $\Delta$ the real part. The inclusion of such a coupling complicates considerably the dynamics, because the state $|g\rangle$ can now decay emitting a cavity photon and so we have two-photon states, increasing the difficulty of solving the equations and troubling the definitions of transmission and reflection coefficient.

In the case of a far detuned but dipole allowed transition as seems to be our case we can use an effective Jaynes-Cummings Hamiltonian for far off-resonant interactions [3]. In fact, it can be shown that the Jaynes-Cummings Hamiltonian for the interactions atom-field in case of large detuning reduces to

$$H^{eff} = \frac{\hbar g'^2}{\tilde{\Delta}}\left[\sigma_{gf}^{+}\sigma_{gf}^{-} + \hat{a}^{\dagger}\hat{a}(\sigma_{gg} - \sigma_{ff})\right] = \\ = \hbar(\tilde{\chi} - i\tilde{\kappa}/2)\left[\sigma_{gg} + \hat{a}^{\dagger}\hat{a}(\sigma_{gg} - \sigma_{ff})\right], \quad (5.93)$$

where $g'$ is the coupling constant of the transition $|g\rangle \leftrightarrow |f\rangle$, and we have defined

$$\tilde{\chi} = \frac{g'^2\Delta}{\Delta^2 + \kappa^2/4} \quad (5.94)$$

and

$$\tilde{\kappa} = \frac{g'^2\kappa}{\Delta^2 + \kappa^2/4} \quad (5.95)$$

Of course the condition necessary to make the use of the effective Hamiltonian valid is that $\Delta \gg g'$. Adding this term to the Hamiltonian of the system we have the

equations for the amplitudes, obtained by the Schrödinger equation, change becoming

$$\begin{aligned}
\dot{c}_a &= -i(\omega_g + 2\tilde{\chi} + \omega_c - i\gamma_1'/2 - i\tilde{\kappa})c_a + gc_b + c\sqrt{\kappa/2\pi}\int dk\, d_k + c\sqrt{\kappa/2\pi}\int dk\, e_k \\
\dot{c}_b &= -i(\omega_e - i\gamma_2'/2)c_b - gc_a \\
\dot{d}_k &= -\sqrt{\kappa/2\pi}c_a - i(\omega_g + \tilde{\chi} + ck - i\gamma_1'/2 - i\tilde{\kappa}/2)d_k \\
\dot{e}_k &= -\sqrt{\kappa/2\pi}c_a - i(\omega_g + \tilde{\chi} + ck - i\gamma_1'/2 - i\tilde{\kappa}/2)e_k \\
\dot{c}_d &= -i(\omega_c - \tilde{\chi} - i\gamma_f/2 + i\tilde{\kappa}/2)c_d + c\sqrt{\kappa/2\pi}\int dk\, d_k' + c\sqrt{\kappa/2\pi}\int dk\, e_k' \\
\dot{d}_k' &= -i(ck - i\gamma_f/2)\, d_k' - \sqrt{\kappa/2\pi}c_d \\
\dot{e}_k' &= -i(ck - i\gamma_f/2)\, e_k' - \sqrt{\kappa/2\pi}c_d.
\end{aligned} \tag{5.96}$$

We see that the equations are still decoupled in two groups, one referring to the atom in $|g\rangle$, the other to the atom in $|f\rangle$. The difference with the previous situation is that there seems to be some shift in the energy of the bare atomic state $|g\rangle$ and a further decay contribute. We replace the resonant condition between the upper transition and the cavity mode with the condition $\omega_c = \omega_e - \omega_g - \tilde{\chi}$.

We can proceed in the same way we have done previously, and solve the first system of equations in a frame moving with complex frequency $\omega_e - i(\gamma_1' + \tilde{\kappa})/2$, to get

$$\begin{aligned}
\dot{C}_a &= (-i\tilde{\chi} - \tilde{\kappa}/2)\, C_a + g\, C_b + c\sqrt{\kappa/2\pi}\int dk\, D_k + c\sqrt{\kappa/2\pi}\int dk\, E_k \\
\dot{C}_b &= -(\Gamma' + \tilde{\kappa}/2)\, c_b - g\, C_a \\
\dot{D}_k &= -\sqrt{\kappa/2\pi}C_a - i\delta(k)\, D_k \\
\dot{E}_k &= -\sqrt{\kappa/2\pi}C_a - i\delta(k)\, E_k
\end{aligned} \tag{5.97}$$

Formally integrating the equation for $D_k$ we get

$$D_k(t) = e^{-i\delta(k)t}D_k(0) - \sqrt{\kappa/2\pi}\int_0^t dt'\, e^{-i\delta(k)(t-t')}C_a(t') \tag{5.98}$$

and similarly for $E_k$. The equation for $C_a$ can as well be formally integrated:

$$\begin{aligned}
C_a(t) = e^{-i(\tilde{\chi}-\tilde{\kappa}/2)t}C_a(0) + g\int_0^t dt'\, e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)(t-t')}C_b(t') + \\
+ c\sqrt{\kappa/2\pi}\int dk\int_0^t dt'\, e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)(t-t')}(D_k(t') + E_k(t'))
\end{aligned} \tag{5.99}$$

If we insert (5.98) in (5.99) we get

$$C_a(t) = e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)t}C_a(0) + g\int_0^t dt'\, e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)(t-t')}C_b(t') + \\ + \sqrt{\kappa}\int_0^t dt'\, e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)(t-t')}\big(E_L^{IN}(t') + E_R^{IN}(t')\big) - \\ - \frac{\kappa c}{\pi}\int dk \int_0^t dt'\, e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)(t-t')}\int_0^{t'} dt''\, e^{-i\delta(k)(t-t')}c_a(t''), \quad (5.100)$$

where we have defined

$$D^{IN}(t) = \frac{c}{\sqrt{2\pi}}\int dk\, e^{-i\delta(k)t}D_k(0) \quad (5.101)$$

and similarly for $E^{IN}(t)$. We treat the last integral in eq. (5.100) with the Markov approximation. We get

$$C_a(t) = e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)t}C_a(0) + g\int_0^t dt'\, e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)(t-t')}C_b(t') + \\ + \sqrt{\kappa}\int_0^t dt'\, e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)(t-t')}\big(D^{IN}(t') + E^{IN}(t')\big) - \\ - \kappa\int_0^t dt'\, e^{-i(\tilde{\chi}-i\tilde{\kappa}/2)(t-t')}C_a(t'), \quad (5.102)$$

Multiplying both sides of eq. (5.102) for $e^{+i(\tilde{\chi}-i\tilde{\kappa}/2)t}$ and taking the time derivative of it and the second equation in (5.97) we get a system of first order coupled differential equations for $C_a$ and $C_b$ analogous to (5.15):

$$\begin{aligned}\dot{C}_a &= -(i\tilde{\chi} + \kappa + \tilde{\kappa}/2)\, C_a(t) + g\, C_b(t) + \sqrt{\kappa}(D^{IN}(t) + E^{IN}(t)) \\ \dot{C}_b &= -(\Gamma' + \tilde{\kappa}/2)\, C_b(t) - g\, C_a(t)\end{aligned} \quad (5.103)$$

Fourier transforming the amplitudes, so that the time derivative results in the multiplication for $-i\delta(k)$, we get

$$\begin{aligned}(-i\delta(k) - i\tilde{\chi} + \kappa + \tilde{\kappa}/2)\, C_a(k) &= +g\, C_b(k) + \sqrt{\kappa}D^{IN}(k) + \sqrt{\kappa}E^{IN}(k) \\ (-i\delta(k) + \Gamma' + \tilde{\kappa}/2)\, C_b(k) &= -g\, C_a(k)\end{aligned} \quad (5.104)$$

From the second equation we can express $C_b$ in term of $C_a$:

$$C_b(k) = -\frac{ig}{\delta(k) + i(\Gamma' + \tilde{\kappa}/2)}C_a(k) \quad (5.105)$$

which inserted in the first one gives us

$$\frac{\delta(k)^2 - \tilde{\chi}(\delta(k) + i(\Gamma' + \tilde{\kappa}/2)) + i\delta(k)(\kappa + \Gamma' + \tilde{\kappa}) - (\kappa + \tilde{\kappa}/2)(\Gamma' + \tilde{\kappa}/2) - g^2}{\delta(k) + i(\Gamma' + \tilde{\kappa}/2)} C_a(k) = i\sqrt{\kappa}\big(D^{IN}(k) + E^{IN}(k)\big). \quad (5.106)$$

Using the relation

$$D^{OUT}(t) - D^{IN}(t) = -\sqrt{\kappa}\, C_a(t), \quad (5.107)$$

and operating in the same way we did above we find the reflection and transmission coefficients

$$\tilde{r}(k) = \frac{\delta(k)^2 - \tilde{\chi}(\delta(k) + i(\Gamma' + \tilde{\kappa}/2)) + i\delta(k)(\Gamma' + \tilde{\kappa}) - \tilde{\kappa}/2(\Gamma' + \tilde{\kappa}/2) - g^2}{\delta(k)^2 - \tilde{\chi}(\delta(k) + i(\Gamma' + \tilde{\kappa}/2)) + i\delta(k)(\kappa + \Gamma' + \tilde{\kappa}) - (\kappa + \tilde{\kappa}/2)(\Gamma' + \tilde{\kappa}/2) - g^2} \quad (5.108)$$

and

$$\tilde{t}(k) = \frac{-i\kappa\delta(k) + \kappa(\Gamma' + \tilde{\kappa}/2)}{\delta(k)^2 - \tilde{\chi}(\delta(k) + i(\Gamma' + \tilde{\kappa}/2)) + i\delta(k)(\kappa + \Gamma' + \tilde{\kappa}) - (\kappa + \tilde{\kappa}/2)(\Gamma' + \tilde{\kappa}/2) - g^2} \quad (5.109)$$

which generalize (5.44) and (5.45) .

In a similar manner working as in the previous section with the last three equations in (5.96) we get the equation

$$\dot{C}_d(t) = -i(\tilde{\chi} - i\tilde{\kappa}/2)\, C_d(t) + \sqrt{\kappa}(D'^{IN}(t) + E'^{IN}(t)) \quad (5.110)$$

with

$$D'^{IN}(t) = \frac{c}{\sqrt{2\pi}} \int dk\, e^{-i\delta(k)t} D'_k(0). \quad (5.111)$$

Fourier transforming and using the boundary condition we find the transmission and reflection coefficients for the atom in the lowest state

$$\tilde{t}'(k) = -\frac{i\kappa}{\delta(k) - \tilde{\chi} + i(\kappa + \tilde{\kappa}/2)} \quad (5.112)$$

and

$$\tilde{r}'(k) = \frac{\delta(k) - \tilde{\chi} + i\tilde{\kappa}}{\delta(k) - \tilde{\chi} + i(\kappa + \tilde{\kappa}/2)} \quad (5.113)$$

which generalize (5.55) and (5.56).

To sum up if we consider decoherence and a weak coupling of the two lower

states we have that eqs. (5.57) are generalized by

$$\begin{aligned} d_k(T) &= \big(\tilde{r}(k)d_k(0) + \tilde{t}(k)e_k(0)\big)e^{-i(\omega_e - i(\gamma_1'+\tilde{\kappa})/2)T} \\ e_k(T) &= \big(\tilde{t}(k)d_k(0) + \tilde{r}(k)e_k(0)\big)e^{-i(\omega_e - i(\gamma_1'+\tilde{\kappa})/2)T} \\ d_k'(T) &= \big(\tilde{r}'(k)d_k'(0) + \tilde{t}'(k)e_k'(0)\big)e^{-i(\omega_c - i\gamma_f'/2)T} \\ e_k'(T) &= \big(\tilde{t}'(k)d_k'(0) + \tilde{r}'(k)e_k'(0)\big)e^{-i(\omega_c - i\gamma_f'/2)T}. \end{aligned} \tag{5.114}$$

## 5.7 Coherent incoming field in steady state approximation

In this section we move from the single-photon dynamics to the multi-photons case. We are primarily interested in the situation in which the input field consists of a coherent state, that, as we have seen in sec. 2.3, is the "most classical" state of light, and the atom is started in $|g\rangle$,

Because of this fact we can assume a semi-classical description of the external fields and replace the term responsible for the interaction waveguides modes-cavity mode in the Hamiltonian (4.17) with the term

$$H^{CW} = i\sqrt{\kappa}E(\hat{a}^\dagger - \hat{a}), \tag{5.115}$$

where $E$ is the amplitude of the classical external incoming field, that we have supposed real and on resonance with the cavity frequency. We also have to account for the decay of the cavity through the two walls introducing two identical Lindblad operators $\hat{L}_\kappa = \sqrt{\kappa}\hat{a}$. The master equation for the system is

$$\dot{\rho} = -\frac{i}{\hbar}\big[H, \rho\big] - \sum_{i=\gamma_1,\gamma_2,\kappa} \frac{1}{2}\big\{\rho, \hat{L}_i^\dagger \hat{L}_i\big\} + \hat{L}_i \rho \hat{L}_i^\dagger, \tag{5.116}$$

where in this case $H$ is the hermitian part of the Hamiltonian, and $\hat{L}_i$ are the Lindblad operators which account for the three decays.

Since we are interested in the transmission probability and we want to assume a steady state approximation, we consider a time $T' \ll 1/\gamma_1$, otherwise the steady state solution would be trivially the atom decayed in $|f\rangle$. Thus, in the master equation we can assume $\gamma_1 = 0$ and eliminate the corresponding Lindblad operator. With this approximation the Hilbert space reduces to the tensor product between the two-

dimensional atoic space spanned by $|g\rangle$ and $|e\rangle$ and the Fock space of the cavity mode excitations. We get that eq. (5.116) is equivalent to the system of equations

$$\begin{aligned}\dot{\rho}_{ng,ng} &= +\gamma_2\rho_{ne,ne} - 2\kappa n\rho_{ng,ng} + 2\kappa(n+1)\rho_{n+1g,n+1g} + \\ &+ g\sqrt{n}D_{n-1e,ng} + \sqrt{\kappa}ED_{n-1g,ng} - \sqrt{\kappa}ED_{n+1g,ng} \\ \dot{\rho}_{ne,ne} &= -\gamma_2\rho_{ne,ne} - 2\kappa n\rho_{ng,ng} + 2\kappa(n+1)\rho_{n+1g,n+1g} + \\ &- g\sqrt{n}D_{n+1g,ne} + \sqrt{\kappa}ED_{n-1e,ne} - \sqrt{\kappa}ED_{n+1e,ne}\end{aligned} \tag{5.117}$$

where we have defined $D_{a,b} = \rho_{a,b} + \rho_{b,a}$, which satisfy the equations

$$\begin{aligned}\dot{D}_{ng,mg} &= +\gamma_2 D_{ng,mg} - \kappa(n+m)D_{ng,mg} + 2\kappa\sqrt{n+1}\sqrt{m+1}D_{n+1g,m+1g} + \\ &+ g(\sqrt{n}D_{n-1e,mg} + \sqrt{m}D_{ng,m-1e}) + \sqrt{\kappa}E(\sqrt{n}D_{n-1g,mg} + \sqrt{m}D_{ng,m-1g} - \\ &- \sqrt{n+1}D_{n+1g,mg} - \sqrt{m+1}D_{ng,m+1g}), \\ \dot{D}_{ne,me} &= -\gamma_2 D_{ne,me} - \kappa(n+m)D_{ne,me} + 2\kappa\sqrt{n+1}\sqrt{m+1}D_{n+1e,m+1e} + \\ &- g(\sqrt{n+1}D_{n+1g,me} + \sqrt{m+1}D_{ne,m+1g}) + \sqrt{\kappa}E(\sqrt{n}D_{n-1e,me} + \sqrt{m}D_{ne,m-1e} - \\ &- \sqrt{n+1}D_{n+1e,me} - \sqrt{m+1}D_{ne,m+1e}),\end{aligned} \tag{5.118}$$

and similarly for $D_{ng,mg}$. In principle this is an infinite dimensional linear system, but for a given field amplitude $E$ the Hilbert space can be truncated to a number of excitations $n'$. The number of states is clearly $2n'+1$ so that the independent elements of the density matrix, which is also the dimension of the linear problem, is $d = (2n'+1)(n'+1)$.

The system can be solved by a computer for reasonable values of $n'$. The transmission intensity is given by the formula

$$I_T = \kappa \langle \hat{a}^\dagger \hat{a} \rangle_{SS} = \kappa \mathrm{Tr}[\hat{a}^\dagger \hat{a} \rho_{SS}] = \kappa \sum_n n\, \rho_{n,n} \tag{5.119}$$

where $\rho_{n,n} = \rho_{ng,ng} + \rho_{ne,ne}$. Similarly the intensity lost because of the decay of the excited state in modes different from the cavity one is

$$I_L = \gamma_2 \langle \sigma_{ee} \rangle_{SS} = \gamma_2 \mathrm{Tr}[\sigma_{ee}\rho_{SS}] = \gamma_2 \sum_n \rho_{ne,ne}. \tag{5.120}$$

In Fig. 5.4 we report the transmission curve, where we see that the ratio between the transmitted intensity and the input intensity grows quickly and saturate to one

when the incoming field $E \sim g/\sqrt{\kappa}$. This behavior can be explained considering the average number of photon in the cavity. In fact for low intensities it replicates the single photon dynamics and we have a large reflection rate, while at higher intensities $\bar{n} > 1$ and most of the photons do not see the atom so that the behavior of the system becomes similar to that one of the empty cavity for which we have total transmission. Looking at the intensity reflected (Fig. 5.5) we see that it reach a maximum and then

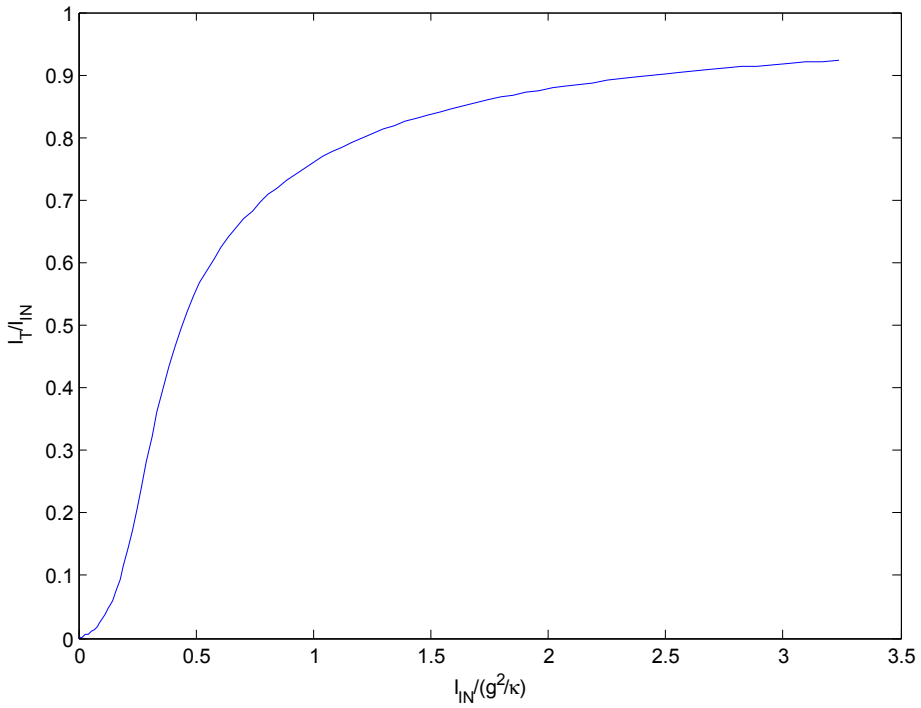


Figure 5.4: Ratio between the transmitted intensity and the input intensity as function of the input field in unity of $g/\sqrt{\kappa}$, with $g = 400$, $\kappa = 100$ and $\gamma_2 = 2$.

remains constant at $g^2/4\kappa$, which can be considered as the maximum intensity that can be reflected by the system. The intensity lost because of the decay of the upper level in modes different from the cavity one (Fig. 5.6) saturates to $\gamma_2/2$, because the population of the excited level for strong driving field is $1/2$. The value of the maximum reflected intensity can be derived changing approach in the description of the dynamics of the system. Up to this point we have used the Schrödinger picture, now for convenience we move to the Heisenberg picture. From Hamiltonian (4.17) we can derive the equation of motion of the operator $\hat{a}(t)$:

$$\frac{d}{dt}\hat{a}(t) = -\kappa\hat{a}(t) + g\sigma_{eg}^{-}(t) + \sqrt{\kappa}\hat{b}_{IN}, \tag{5.121}$$

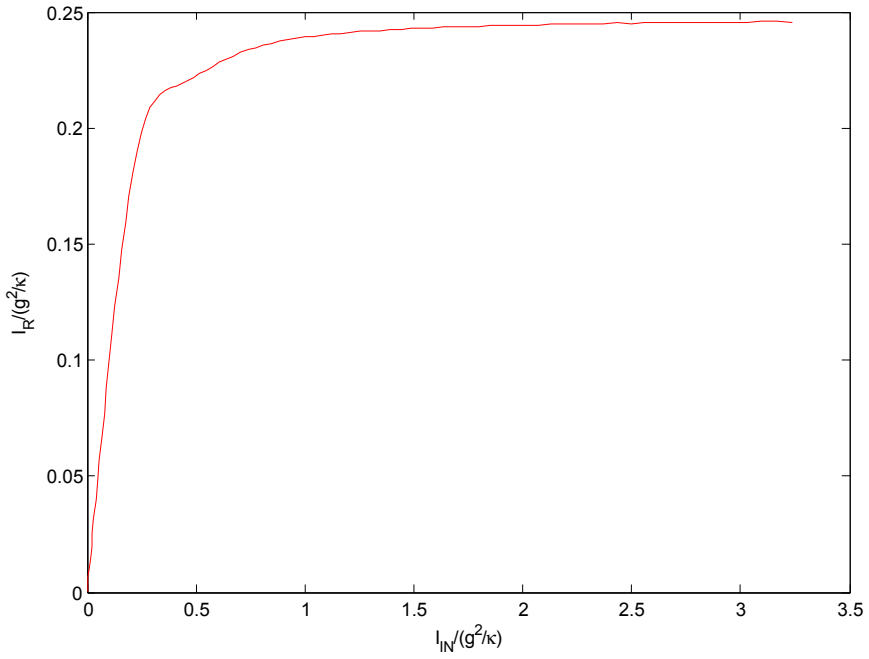


Figure 5.5: Reflected intensity as function of the input field in unity of $g/\sqrt{\kappa}$, with $g = 400$, $\kappa = 100$ and $\gamma_2 = 2$.

where we have performed the Markov approximation and defined, in manner analogous to what is done in sec. 3.2, the operator

$$\hat{b}_{IN}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \hat{b}_0(k) e^{-i(ck-\omega_c)(t-t_0)}. \tag{5.122}$$

Similarly we can derive the equations of motion of the operators $\hat{a}^\dagger, \sigma_{eg}^-, \sigma_{eg}^+$ and $\sigma_3$ which form a system of coupled nonlinear differential equations. For our purposes we do not need to deal with it.

We notice that since the input field is in a coherent state we can rewrite eq. (5.121) replacing the operator $\hat{b}_{IN}(t)$ with a classical field $\xi(t)$ which we assume real:

$$\frac{d}{dt}\hat{a}(t) = -\kappa\big(\hat{a}(t) - \frac{\xi(t)}{\sqrt{\kappa}}\big) + g\sigma_{eg}^-(t). \tag{5.123}$$

From this equation we are lead to define the operator

$$\hat{a}'(t) = \hat{a}(t) - \frac{\xi(t)}{\sqrt{\kappa}}, \tag{5.124}$$

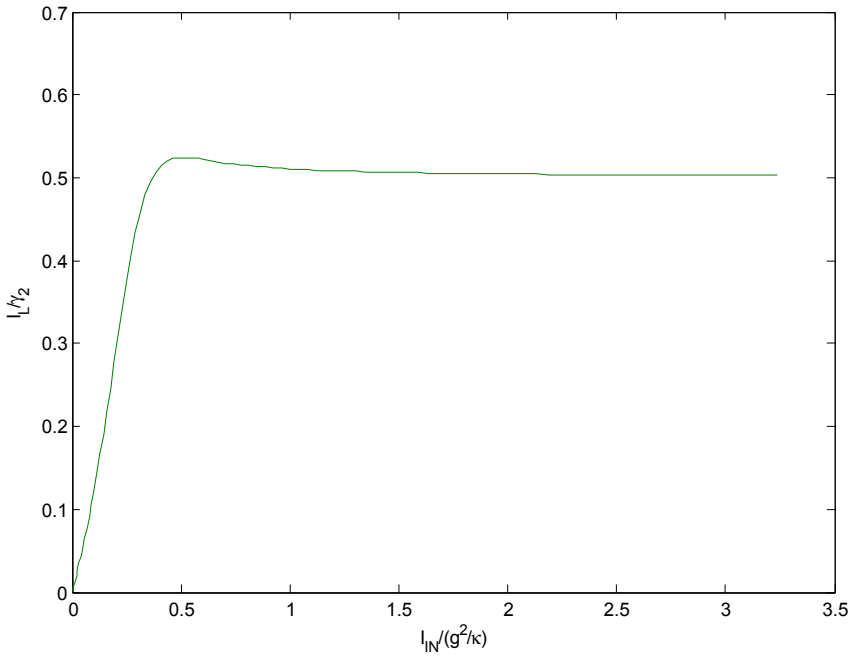


Figure 5.6: Lost intensity as function of the input field in unity of $g/\sqrt{\kappa}$, with $g = 400$, $\kappa = 100$ and $\gamma_2 = 2$.

which has equation of motion

$$\frac{d}{dt}\hat{a}'(t) = -\kappa\hat{a}'(t) + g\sigma_{eg}^{-}(t). \tag{5.125}$$

Doing this is equivalent to move to a frame in which the cavity has no input field. In fact, as we have already stated, the field in the cavity without atom would be $\xi(t)/\sqrt{\kappa}$, and subtracting it we get the field produced by the atom. The input field now acts as a classical drive directly on the atom. The reflected intensity is now given by the operator $\hat{a}'(t)$, i.e.

$$I_R = \kappa \left\langle \hat{a}'^{\dagger}\hat{a}' \right\rangle . \tag{5.126}$$

If we integrate formally eq. (5.125), with the initial condition $\hat{a}'(0) = 0$, we get

$$\hat{a}'(t) = g\int_{-\infty}^{t} dt' e^{-\kappa(t-t')}\sigma_{eg}^{-}(t'). \tag{5.127}$$

Inserting eq. (5.127) and the equivalent for $\hat{a}'^{\dagger}$ we get

$$I_R = \kappa g^2 \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' e^{-\kappa(2t-t'-t')} \left\langle \sigma_{eg}^{+}(t'')\sigma_{eg}^{-}(t') \right\rangle . \tag{5.128}$$

We can change the coordinates of integration introducing $\tau = t'' - t'$ and $T = t' + t''$. Paying attention to the limits of integration we can rewrite the last equation as

$$I_R = \frac{\kappa g^2}{2} \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{2t-|\tau|} dT e^{-\kappa(2t-T)} \left\langle \sigma_{eg}^{+}(\tau)\sigma_{eg}^{-}(0)\right\rangle , \tag{5.129}$$

where we have used the fact that in steady state $\langle \sigma_{eg}^{+}(t'')\sigma_{eg}^{-}(t')\rangle$ depends only on the time difference $t'' - t'$. Performing the integral in $T$ we get

$$I_R = \frac{g^2}{2} \int_{-\infty}^{+\infty} d\tau e^{-\kappa|\tau|} \left\langle \sigma_{eg}^{+}(\tau)\sigma_{eg}^{-}(0)\right\rangle . \tag{5.130}$$

Now $\langle \sigma_{eg}^{+}(\tau)\sigma_{eg}^{-}(0)\rangle$ has to be evaluated in steady state and integrated for a long time, compared to the classical Rabi frequency, over the time difference. In steady state and strong driving there is around $1/2$ probability that the atom is in the excited state and hence $\sigma_{eg}^{-}$ gives a factor $1/2$. At time $t = 0$ $\sigma_{eg}^{-}$ brings the atom to the ground state, but because of time evolution at later times there is a $1/2$ probability to find the atom in the ground state so that $\sigma_{eg}^{+}$ gives another factor $1/2$. Hence averaged over time we have $\langle \sigma_{eg}^{+}(\tau)\sigma_{eg}^{-}(0)\rangle = 1/4$. Inserting this in (5.130) and doing the last integral, we get $I_R = g^2/4\kappa$.

# Chapter 6

# Results and conclusions

## 6.1 Errors

The validity of the scheme for the photonic transistor we propose depends on the efficiency of the operations performed with the system. Crucial is the fact that at the end of the first step the presence of a photon is mapped on the atomic state $|f\rangle$, while the absence is mapped on $|g\rangle$. We have to analyze the evolution of the system during the first step to determine the fidelity of the map. We assume that the operations performed by a classical control, i.e. the preparation of the atom at $t = 0$ in the state $1/\sqrt{2}(|g\rangle + |f\rangle)$ and the $\pi/2$ pulse at $t = T$, have fidelity one. We also assume that the control pulse has a gaussian shape.

In detail what happens during the first step when the control photon is present is that at $t < 0$ a photon pulse consisting of a symmetric superposition of a pulse coming from the left and one coming from the right propagates toward the cavity while the atom is assumed to sit in the lower state $|f\rangle$:

$$|\psi\rangle^{(1)}_{t<0} = \int dk \left( l_k(t<0)\hat{b}^\dagger_k + r_k(t<0)\hat{c}^\dagger_k \right) |E\rangle \tag{6.1}$$

where $l_k(t) = r_k(t)$ and the state is normalized to one. At $t = 0$ the classical pulse

mixes the atomic states so that

$$|\psi\rangle^{(1)}_{t<0} \xrightarrow{\pi/2} |\psi\rangle^{(1)}_{t=0} = \frac{1}{\sqrt{2}} \int dk \left( l_k(0)\hat{b}^\dagger_k + r_k(0)\hat{c}^\dagger_k \right) \left[ \, |C\rangle + |E\rangle \, \right] =$$
$$= \int dk \left[ \left( d_k(0)\hat{b}^\dagger_k + e_k(0)\hat{c}^\dagger_k \right) |C\rangle + \left( d'_k(0)\hat{b}^\dagger_k + e'_k(0)\hat{c}^\dagger_k \right) |E\rangle \right] \quad (6.2)$$

where in the last step we have freely renamed the four amplitudes. Note that for now $d_k(0) = e_k(0) = d'_k(0) = e'_k(0)$ and the state is still normalized to one. Then the system consisting of the external fields, the cavity field and the atom evolves for a time $T$, after which the second classical pulse performs the transformation

$$\begin{aligned} |g\rangle &\xrightarrow{\pi/2} \frac{1}{\sqrt{2}}(|g\rangle - |f\rangle) \\ |f\rangle &\xrightarrow{\pi/2} \frac{1}{\sqrt{2}}(|g\rangle + |f\rangle). \end{aligned} \quad (6.3)$$

so that the final state is

$$|\psi\rangle^{(1)}_{FIN} = \frac{1}{\sqrt{2}} \int dk \Big[ d_k(T)\hat{b}^\dagger_k(|C\rangle - |E\rangle) + e_k(T)\hat{c}^\dagger_k(|C\rangle - |E\rangle) + c_a(T)(|A\rangle - |D\rangle +$$
$$+ \sqrt{2}c_b(T)\,|B\rangle + d'_k(T)\hat{b}^\dagger_k(|C\rangle + |E\rangle) + e'_k(T)\hat{c}^\dagger_k(|C\rangle + |E\rangle) + c_d(T)(|A\rangle + |D\rangle) \Big]. \quad (6.4)$$

As we have anticipated at the beginning of sec. 5.1, since the ideal first step maps the presence of a control photon into an atomic state $|f\rangle$, the error probability is $P'_g$ given by eq. (5.1):

$$P'_g = P_g + \frac{1}{2}(1 - P_g - P_f) = \frac{1}{2}(1 + P_g - P_f). \quad (5.1)$$

This form for the probability, as we have explained, depends on the fact that we have considered a non Hermitian evolution so that at time $T$ the sum of the probabilities for the atomic states is $p < 1$, but since the $\pi/2$ pulse at time $T$ mixes equally the two atomic states we can write the normalized probabilities as $P_i = P_i + (1-p)/2$. The probabilities $P_g$ and $P_f$ are readily obtained from the expression of the final state (6.4):

$$P_g = \frac{1}{2} \int dk \left[ |d_k(T) + d'_k(T)|^2 + |e_k(T) + e'_k(T)|^2 + |c_a(T) + c_d(T)|^2 \right] \quad (6.5)$$

and

$$P_f = \frac{1}{2}\int dk\Big[|d_k(T) - d'_k(T)|^2 + |e_k(T) - e'_k(T)|^2 + |c_d(T) - c_a(T)|^2\Big]. \qquad (6.6)$$

The last contributions in eqs. (6.5) and (6.6) originate from the fact at time $T$ the photon could be still inside the cavity as excitation of the intra-cavity mode. Also, and we have neglect this fact in the above discussion, the atom in principle could be in $|e\rangle$ at time $T$. Nevertheless if we look at eqs. (5.67), (5.68) and (5.70) we see that the amplitudes $c_a$, $c_b$ and $c_d$ are proportional to $e^{-\delta(t)^2/4\sigma_T^2}$ so that for a ratio $n = T/\sigma_T$ large enough the three contributions are completely negligible.

Now we have just to use the expressions (5.114) for the amplitudes $d_k(T)$, $e_k(T)$, $d'_k(T)$, $e'_k(T)$. The problem is to perform the integral on $k$, since the reflection and transmission coefficients are quite involved. We use the fact that we are in the strong coupling regime and that the pulse time width is much larger than the time spent in the cavity by the photon to approximate them. Since $d_k(0) = e_k(0)$ we need an expansion of $\tilde{t}(k) + \tilde{r}(k)$ which expanding the denominator to the first order and neglecting the terms odd in $\delta(k)$ which will be wash out by the integration with a gaussian with exponent $\delta(k)$ gives

$$\tilde{t}(k) + \tilde{r}(k) \approx 1 - \frac{2\kappa(\Gamma' + \tilde{\kappa}/2)}{g^2} - \frac{2\kappa^2\delta(k)^2}{g^4}, \qquad (6.7)$$

where we also have omitted all the other terms deriving from the first order expansion since they are much smaller than that one kept. Using similar arguments we can expand $\tilde{t}'(k) + \tilde{r}'(k)$ to get

$$\tilde{t}'(k) + \tilde{r}'(k) \approx -1 - \frac{2(i\tilde{\chi} + \tilde{\kappa})}{\kappa} + \frac{2\delta(k)^2}{\kappa^2}. \qquad (6.8)$$

Inserting (6.7) and (6.8) in eqs. (5.114) and using the fact that $e_k = d_k$ and $e'_k = e_k$ and the normalization of the initial amplitudes we find

$$P_g = \frac{1}{4}\left|e^{-(\gamma'_1+\tilde{\kappa})T/2}\left(1-\frac{2\kappa(\Gamma'+\tilde{\kappa}/2)}{g^2}-\frac{\kappa^2}{g^4\sigma_T^2}\right)+e^{-\gamma'_f T/2}\left(-1+\frac{2(i\tilde{\chi}+\tilde{\kappa})}{\kappa}+\frac{1}{\kappa^2\sigma_T^2}\right)\right|^2 \qquad (6.9)$$

and similarly

$$P_f = \frac{1}{4}\left|e^{-(\gamma'_1+\tilde{\kappa})T/2}\left(1-\frac{2\kappa(\Gamma'+\tilde{\kappa}/2)}{g^2}-\frac{\kappa^2}{g^4\sigma_T^2}\right)-e^{-\gamma'_f T/2}\left(-1+\frac{2(i\tilde{\chi}+\tilde{\kappa})}{\kappa}+\frac{1}{\kappa^2\sigma_T^2}\right)\right|^2 \qquad (6.10)$$

Inserting these expressions in (5.1) we find finally the probability of error in the case that the control photon is present:

$$P_{err}^{(1)} = \frac{1}{2}\left[1 - e^{-(\gamma_1+\tilde{\kappa}+\gamma_f)T/2}\left(1 - \frac{2\kappa(\Gamma' + \tilde{\kappa}/2)}{g^2} - \frac{\kappa^2}{g^4\sigma_T^2}\right)\left(1 - \frac{2\tilde{\kappa}}{\kappa} - \frac{1}{\kappa^2\sigma_T^2}\right)\right] \quad (6.11)$$

We have to consider now the case in which the control photon is not present. In this case the evolution is much simpler, at time $t = 0$ the system is prepared in the state

$$|\psi\rangle_{t=0}^{(0)} = \frac{1}{\sqrt{2}}\big(\,|C\rangle + |E\rangle\,\big). \quad (6.12)$$

The time evolution consists in this case simply in a decay of the amplitude of $|C\rangle$ so that at time $t = T$ before the $\pi/2$ pulse the state of the system is

$$|\psi\rangle_{t=T}^{(0)} = \frac{1}{\sqrt{2}}\big(e^{-(\gamma_1'+\tilde{\kappa})T/2}\,|C\rangle + e^{-\gamma_f T/2}\,|E\rangle\,\big), \quad (6.13)$$

and after the classical pulse is

$$|\psi\rangle_{FIN}^{(0)} = \frac{1}{2}\big(e^{-(\gamma_1'+\tilde{\kappa})T/2}(|C\rangle - |E\rangle) + e^{-\gamma_f T/2}(|C\rangle + |E\rangle)\big). \quad (6.14)$$

It is immediate to get from (6.14) the probabilities for the atomic states:

$$P_g = \frac{1}{4}\big|e^{-(\gamma_1'+\tilde{\kappa})T/2} + e^{-\gamma_f T/2}\big|^2 \quad (6.15)$$

and

$$P_f = \frac{1}{4}\big|e^{-(\gamma_1'+\tilde{\kappa})T/2} - e^{-\gamma_f T/2}\big|^2, \quad (6.16)$$

so that the probability of error for the case of no control photon is

$$P_{err}^{(0)} = \frac{1}{2}\big(1 - e^{-(\gamma_1'+\tilde{\kappa}+\gamma_f)T/2}\big) \quad (6.17)$$

Assuming that the transistor is used in a way such that the probabilities to have or not a control field are 1/2 the average probability of error during the first step is

$$\begin{aligned} P_{err} &= \frac{1}{2}\big(P_{err}^{(0)} + P_{err}^{(1)}\big) = \\ &= \frac{1}{4}\left[2 - e^{-(\gamma_1'+\tilde{\kappa}+\gamma_f)n\sigma_T/2}\left(\big(1 - \frac{2\kappa(\Gamma' + \tilde{\kappa}/2)}{g^2} - \frac{\kappa^2}{g^4\sigma_T^2}\big)\big(1 - \frac{2\tilde{\kappa}}{\kappa} - \frac{1}{\kappa^2\sigma_T^2}\big) + 1\right)\right] \end{aligned} \quad (6.18)$$

If we consider $(\gamma_1' + \tilde{\kappa} + \gamma_f)T$, $\frac{\kappa\gamma_2}{g^2}$, $\tilde{\kappa}/\kappa$ and $\frac{1}{\kappa^2\sigma_T^2}$ as small parameters, and we expand eq. (6.18) to the first order in them we find that

$$P_{err} \approx \frac{1}{4}\left((\gamma_1' + \tilde{\kappa} + \gamma_f)n\sigma_T + \frac{2\kappa(\Gamma' + \tilde{\kappa}/2)}{g^2} + \frac{2\tilde{\kappa}}{\kappa} + \frac{1}{\kappa^2\sigma_T^2}\right). \tag{6.19}$$

We note that for $\kappa \leq g$ we have that $\sigma_T \gg \kappa/g^2$ so that we can eliminate some terms from the last expression and reduce it to

$$P_{err} \approx \frac{1}{4}\left((\gamma_1' + \gamma_f + \tilde{\kappa})n\sigma_T + \frac{\kappa\gamma_2'}{g^2} + \frac{1}{\kappa^2\sigma_T^2}\right). \tag{6.20}$$

These five terms represent the sources of error that we have, the first, the second and the fourth terms are due to the decay and decoherence of the atom, the third one is due to the finite anharmonicity and the last one to the fact that the finite width of the pulse does not give a perfect phase shift. In general in an experiment of CQED the parameters that can be engineered quite easily are the coupling constant $g$, the cavity decay $\kappa$ and the pulse width $\sigma_T$.

It is interesting to find the optimal values of the last two parameters, i.e. the values that minimize the error probability in the simpler case in which the terms due to the finite anharmonicity and decoherence can be neglected. In this case

$$Err(\kappa, \sigma_T) = \frac{1}{4}\left(\gamma_1 n\sigma_T + \frac{\kappa\gamma_2}{g^2} + \frac{1}{\kappa^2\sigma_T^2}\right) \tag{6.21}$$

Let's calculate the zero of the partial derivatives of the error function (6.19):

$$\frac{\partial}{\partial\kappa}Err(\kappa, \sigma_T) = \frac{1}{4}\left(\frac{\gamma_2}{g^2} - \frac{2}{\kappa^3\sigma_T^2}\right) = 0 \tag{6.22}$$

and

$$\frac{\partial}{\partial\sigma_T}Err(\kappa, \sigma_T) = \frac{1}{4}\left(\gamma_1 n - \frac{2}{\kappa^2\sigma_T^3}\right) = 0 \tag{6.23}$$

The first implies $\kappa = \left(\frac{\gamma_2\sigma_T^2}{2g^2}\right)^{-1/3}$ which substituted in the second one gives the optimal values

$$\begin{aligned} \kappa &= 2^{1/5}(\gamma_1 n)^{2/5}\left(\frac{\gamma_2}{g^2}\right)^{-3/5} \\ \sigma_T &= 2^{1/5}(\gamma_1 n)^{-3/5}\left(\frac{\gamma_2}{g^2}\right)^{2/5} \end{aligned} \tag{6.24}$$

Since the product of them is proportional to $g^2/n\gamma_1\gamma_2$ they clearly satisfy the condi-

tion $\sigma_T \gg 1/\kappa$. To verify that this is a minimum for $Err$ we have to calculate the Hessian matrix of the function:

$$\mathbf{H}_{Err} = \begin{bmatrix} Err_{\kappa\kappa} & Err_{\kappa\sigma_T} \\ Err_{\sigma_T\kappa} & Err_{\sigma_T\sigma_T} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\frac{1}{\kappa^4\sigma_T^2} & \frac{1}{\kappa^3\sigma_T^3} \\ \frac{1}{\kappa^3\sigma_T^3} & \frac{3}{2}\frac{1}{\kappa^2\sigma_T^4} \end{bmatrix} \tag{6.25}$$

Since both the determinant of $\mathbf{H}_{Err}$, that is $\frac{5}{4}\frac{1}{\kappa^6\sigma_T^6}$, and $Err_{\kappa\kappa}$ are positive, values in (6.24) represent a minimum of $Err$. Inserting these values in the expression of $Err$ we find that the optimal error is

$$Err_{opt} = 2^{-9/5}\left(\frac{n\gamma_1\gamma_2}{g^2}\right)^{2/5}. \tag{6.26}$$

## 6.2 Gain of the transistor

Characterizing the second step of the protocol depends a little on the exact characteristics which is sought for the transistor. Above we have seen that the transmission is highly sensitive to the atomic state, but the transmission also depends on the intensity. Ideally the transistor should not transmit any signal photons if there was no photons in the gate pulse. From the results of section 5.7, we see that this requires an intensity $|\xi|^2 \ll g^2/\kappa$. If on the other hand we are mainly interested in having a large difference in the transmission depending on the state of the first gate pulse, we can work with a somewhat higher intensity $|\xi|^2 \sim g^2/4\kappa$. Here we will take the latter approach. The gain of the transistor is then given by the ratio between the difference in the number of signal photons and the number of photons in the gate pulse. The former is of the order of $T'g^2/4\kappa$, where $T'$ is the time duration of the second steps, while the latter is one. The time $T'$ is limited by the decay of state $|g\rangle$ to $|f\rangle$. If we want the gain to be the same in all events we should pick a short time $\gamma_1 T' \ll 1$ to ensure a small probability of decay. On the other hand if we are interested in a large average gain, we should let the second step last until $T' \sim 1/\gamma_1$. Inserting the optimal value of $\kappa$ and setting $T' = 1/\gamma_1$ we find

$$G = \left(\frac{\gamma_2^3 g^4}{2^9 n^2 \gamma_1^7}\right)^{1/5} \tag{6.27}$$

which can be very large.

## 6.3 Conclusions and outlook

In this way we have proved that our scheme can provide a valid model for the realization of a single photon transistor. For large anharmonicity and optimal values of $\kappa$ and $\sigma_T$, we can do a rough esteem of the error probability and of the gain assuming that the decay and decoherence rates are all similar to a unique parameter $\gamma$. With this approximation we find readily that the error probability of the first step. i.e. that the control photon provides a wrong information, scales as $s^{-1}$, where $s = (g/\gamma)^{4/5}$. On the other hand the gain of the transistor scales as $s$. Superconducting circuits state-of-the-art experiments, such as those ones realized in Zürich by the group lead of A. Wallraf [22, 23] have decoherence time ranging from 0.5 to few microseconds. The coupling rate $g/2\pi$ in the same experiments is around 0.4 GHz, $10^4$ times that one of real atoms in microwaves cavities, so that with these values we have that $s \approx 10$.

Problems with these values arise because the typical anharmonicity $\alpha/2\pi$ obtained with superconducting qubits is around 0.4 GHz, so that the condition $g/\alpha \ll 1$ is not satisfied [23]. Since this condition is required for the working of the transistor it is necessary to decrease $g$, reducing also in this way the efficiency and the gain. It is important to note that the condition that can be more relaxed is $g \gg \kappa$, in fact, despite the fact that we have assumed to be in the strong coupling regime, we have never used it. As already pointed out by Kimble in Ref. [19], the ratio between these two parameters has almost no effects on the phase shift of the photon, while is required that $g^2/\kappa \gg \gamma_2$ to have a low probability to loose the photon during the first step.

We conclude mentioning some possible applications of the single photon transistor. As pointed out in Ref. [10] the most important and natural application is high precision photo detection, where the large gain in the signal field enables the detection of the weak field of the single photon. In this situation, the difference between the transmitted intensities of the signal fields in the two case is large provided that that field is not too strong (compared to $g^2/\kappa$) and can be set to an optimal value with respect to the sensibility of the classical detector. Another application is the realization of Schrödinger cat states. These in quantum optics are states of light consisting in a superposition of states that are macroscopically distinguishable [3]. In our case if we send during the first step a superposition of zero and one photon we get that this pulse becomes entangled with a much stronger field.

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