FACULTY OF SCIENCE
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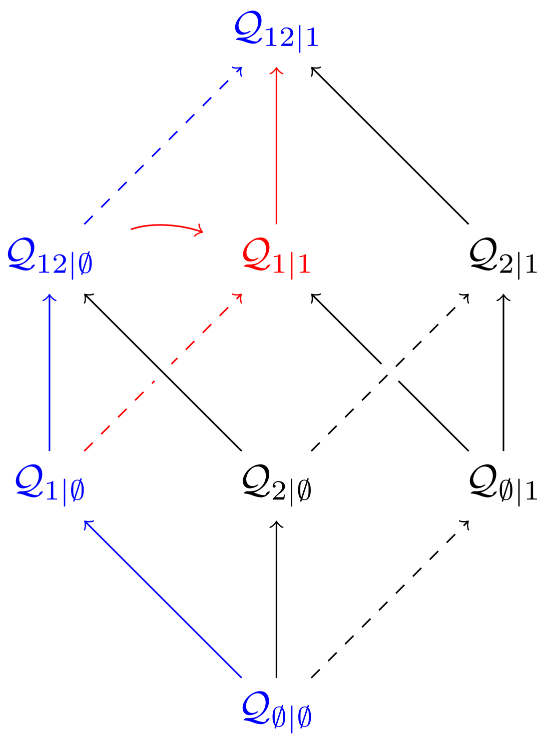
Fermionic Duality for Integrable Super Spin Chains



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Abstract

This thesis examines a fermionic duality present in the super spin chain interpretation of the $\mathfrak{su}(2|1)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory. By a direct Feynmann diagram calculation, the two-point correlation functions are determined in the planar limit at one-loop order for the $\mathfrak{so}(6)$ sector and the gauge-invariant operators with definite conformal dimension are shown to correspond to solutions of one-dimensional spin chains for the $\mathfrak{su}(2)$ sector. The spin chains are then solved by the Coordinate Bethe Ansatz for the $\mathfrak{su}(2)$ sector and by the Nested Bethe Ansatz for the $\mathfrak{su}(2|1)$ sector and are shown to include a fermionic duality transformation. The Algebraic Bethe Ansatz is reviewed and used to prove a closed-form formula that relates the norm of Bethe states to the Gaudin matrix. A defect version of the field theory is investigated where one-point correlation functions can be expressed in terms of the superdeterminant of the Gaudin matrix. The dual spin chain solutions are then shown to correspond to the different choices of grading of the Lie superalgebra of the theory. The transformation rules of the superdeterminant appearing in the subsector $\mathfrak{su}(2|1)$ are then examined under the fermionic duality transformation of a non-momentum-carrying node of the Dynkin diagram. Finally, an attempt to prove that the superdeterminant transforms covariantly under this duality is made with promising results.

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Contents

Chapter 1

Introduction

The standard model of particle physics unionizes the electromagnetic, weak, and strong forces of nature into a quantum field theory (QFT) that is currently our best model for describing physics on a small scale. General relativity (GR) on the other hand is the theory of gravity, the last of the fundamental forces, and is currently our best model describing physics of large scale. In the search for a theory that combines GR and QFT, string theory has been one of the most successful candidates so far. In particular, a ten-dimensional superstring theory is favored over its 26-dimensional bosonic counterpart since it includes both bosonic and fermionic excitations which are related by supersymmetry [1].

Supersymmetries have also been implemented in QFTs where in particular $\mathcal{N} = 4$ super Yang-Mills (SYM) theory first formulated by Brink, Scherk, and Schwarz in [2] has proved to possess a wide range of applications. The reason for its many applications is linked to the fact that it carries the maximal amount of supersymmetries for a four-dimensional theory without gravity [3]. This causes the theory to be conformally invariant even at the quantum level with its beta function conjectured to be zero at all loop orders [4]. Furthermore, the symmetries make the theory integrable in the planar limit where the number of colors is taken to infinity making it possible to calculate some physical quantities analytically [5]. The conformal symmetry greatly restricts the structure of two- and three-point correlation functions such that their space-time dependence is determined by the conformal data of the primary operators. The conformal invariance also implies that there are no inherent mass scales in the theory like the ones we find in quantum chromodynamics (QCD) and the reason for its popularity might therefore seem unjustified. However, QCD is asymptotically free and at high energies, it is approximately conformal making it possible to study high-energy gluon scattering by calculating amplitudes in $\mathcal{N} = 4$ SYM.

In 1998 Juan Maldacena conjectured that string theory and QFT were not two competing theories, but instead dual theories in the sense that all observables can be mapped from one theory to the other [6–8]. Specifically, his anti-de Sitter/conformal field theory (AdS/CFT) correspondence claims that type IIB superstring theory living in the product space[1] $AdS_5 \times S^5$ is holographically dual to $\mathcal{N} = 4$ SYM theory living on the boundary of the AdS_5 part of the theory which is locally isomorphic to four-dimensional Minkowski-space [6, 8]. The AdS/CFT correspondence is a strong/weak duality meaning that the strong coupling regime in $\mathcal{N} = 4$ SYM maps to the weak coupling regime in the superstring theory. The upside of this is that it enables us to perturbatively calculate quantities in one theory that cannot be calculated in the other if the conjecture does indeed hold. The downside is that proving Maldacena's conjecture might be difficult for the same reasons.

[1] Here S^5 is the five-dimensional sphere while AdS_5 is the five dimensional anti-de Sitter (AdS) space which is the maximally symmetric space with constant negative curvature.

In 2003 it was found that the observables of $\mathcal{N} = 4$ SYM theory correspond to solutions of one-dimensional super spin chains [9, 10]. One way to prove the AdS/CFT correspondence might therefore be by using these spin chains as intermediary systems for mapping observables between the two theories. The simplest spin chain, the $\mathfrak{su}(2)$ Heisenberg spin chain, was solved already in 1931 by Hans Bethe when he on a research stay in Rome came up with a clever ansatz now known as the Coordinate Bethe Ansatz (CBA) [11]. The Heisenberg spin chain was then already well-known by solid-state physicists to whom the spin chains were not abstract constructions in a QFT but posed as a real physical model describing metals. Since then the spin chains has been studied among others by Baxter and Faddeev who systematized the study of integrable models and introduced the Algebraic Bethe Ansatz (ABA) which has since been used by Beisert and Staudacher to generalize the Heisenberg spin chain to the $\mathcal{N} = 4$ superconformal algebra of $\mathfrak{psu}(2, 2|4)$ [12, 13].

Since its formulation, $\mathcal{N} = 4$ SYM has continued to reveal new features. For instance, defect versions of the field theory which are holographically dual to certain D-branes living in the string theory have revealed several integrable boundary states of the super spin chains related to the theory [14, 15]. The overlaps between Bethe states and these boundary states include information about the one-point functions in the defect theory and can be expressed through the superdeterminant of the Gaudin matrix which is related to the norm of the Bethe state. Apart from being interesting object in the QFT, the overlaps have also been studied in statistical mechanics where the overlaps with Neel states were first found [16] and have recently been useful in the study of quantum quenches [17]. So far these overlaps have been found in different gradings of the Lie superalgebra using both direct QFT calculations as well as symmetry arguments with some gradings being more easily accessible than others [18].

Recently, Beisert, Kazakov, Sakai, and Zarembo showed that there exists a fermionic duality transformation that can be used to transform the Bethe equations between the different gradings of the Lie superalgebra of the spin chain [19]. This enables us to transform the overlaps in the defect theory and arrive at expressions for other gradings.

In this thesis, we will show how the tools of integrability can be used to study how the overlaps present in the defect theory transform under these dualities.

1.1 Structure

This thesis is organized as follows: In chapter 2 we derive the $\mathcal{N} = 4$ SYM theory by dimensional reduction of its ten-dimensional counterpart first considered by Brink et al. We then review the symmetries of the theory and in particular show how the conformal symmetry constrains the form of the two-point functions of the operators with definite conformal dimensions. We then show that in the t' Hooft limit where the number of colors is taken to infinity the correlation functions of interest dramatically simplify. In this limit, we then do a direct Feynman diagram calculation at one-loop order to determine correlation functions in the $\mathfrak{so}(6)$ sector and show that finding the operators with definite conformal dimensions is equivalent to diagonalizing a Hamiltonian describing one-dimensional spin chains.

In chapter 3 we review the rather physical Coordinate Bethe Ansatz (CBA) which diagonalizes the Hamiltonian of the spin chains related to our theory. We start by solving the $\mathfrak{su}(2)$ sector where the dilatation operator corresponds to the $\text{XXX}_{1/2}$ Heisenberg spin chain known from condensed matter physics. We then go on and introduce the slightly more advanced Nested Bethe Ansatz (NBA) to derive the Bethe equations solving the super spin chain of the $\mathfrak{su}(2|1)$ sector which is of special interest in this thesis. In the process of solving the spin chain through the Bethe Ansätze, we will show that there is a manifest ambiguity as to which fields constitute the vacuum of our chain and which constitute the different levels of excitations. This is the first hint of the existence of the duality transformations we will investigate more thoroughly later on.

In chapter 4 we outline the Algebraic Bethe Ansatz (ABA) which, as its name hints, is slightly less intuitive than the CBA but despite that has the advantage of formally proving the integrability of the theory. We then use the tools developed in the ABA to derive a norm function for Bethe states expressed in terms of the determinant of the Gaudin matrix.

In chapter 5 we discuss a certain defect version of $\mathcal{N} = 4$ SYM which is holographically dual to a D5-D3 probe brane living in the $AdS_4 \times S^2$ part of the string theory. In this dCFT one-point correlation functions are no longer trivial but can instead be written in terms of overlaps between boundary states and Bethe states which in turn can be expressed by the superdeterminant of the Gaudin matrix decorated with some Baxter polynomials.

In chapter 6 we show how the bosonic and fermionic duality transformations of the Bethe equations found in chapter 3 are algebraic implications encompassed by a set of $\mathcal{QQ}$ relations. Furthermore, we show how the fermionic duality transformation can be used to transform between the different possible Bethe equations each of which corresponds to a choice of simple roots of the underlying superconformal algebra of the theory, $\mathfrak{psu}(2,2|4)$.

Finally, in chapter 7 we set out to prove the conjecture given by Kristjansen, Müller, and Zarembo in [20] where they claim that the overlap formulae found in the dCFT transform covariantly. Keeping with the previous part of the thesis, we restrict ourselves to the $\mathfrak{su}(2|1)$ sector since it is the simplest sector containing the fermionic duality of interest. Specifically, we show how the superdeterminant appearing in the overlap formula of interest transforms under the duality transformation on the non-momentum-carrying node from the ···○······⊗— to the —⊗——⊗··· representation of the algebra. Chapter 8 contains our conclusion and outlook.

Chapter 2

$\mathcal{N} = 4$ Super Yang-Mills Theory

In this chapter, we derive the $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory in four dimensions via dimensional reduction of the $\mathcal{N} = 1$ SYM theory in ten dimensions following Brink, Scherk, and Schwarz in [2]. We then go over the symmetries of the theory and their consequences and in particular, we show how the conformal symmetry greatly restricts the form of two-point correlations functions of operators with definite conformal dimensions. We then go on to show that in the planar limit, where the number of colors is taken to infinity, the correlation functions simplify and it becomes possible to determine the conformal dimension of operators in the $\mathfrak{so}(6)$ sector. This is then done by introducing a renormalization scheme and by explicit Feynman diagram calculations at one-loop order. Finally, we turn our focus to the $\mathfrak{su}(2)$ subsector of the theory where the good conformal operators correspond to solutions of the one-dimensional Heisenberg spin chain, which we will investigate in the next chapter.

2.1 Dimensional Reduction

In order for a theory to be supersymmetric, it must have the same amount of fermionic and bosonic degrees of freedom. specializing to the case where we only have a massless vector particle and an unrestricted massless Dirac spinor which have $D-2$ and $2^{D/2}$ degrees of freedom respectively there is no integer dimension D where these match. However, by restricting the spinor to satisfy Majorana and Weyl conditions the number of fermionic degrees of freedom reduces to $2^{D/2-2}$ which matches the number of bosonic degrees of freedom for $D = 10$. Starting with the 10 dimensional $\mathcal{N} = 1$ SYM theory, the goal is to derive a four-dimensional SYM theory by means of dimensional reduction. We chose all fields to transform in the adjoint representation of $U(N)$ where all fields are represented by $N \times N$ Hermitian matrices. The action of the ten-dimensional theory is then given by [2]

$$S_{10} = \int d^{10}x \, \mathrm{tr} \left(-\frac{1}{2} F^{MN} F_{MN} + i\bar{\Psi}\Gamma^M D_M \Psi \right), \tag{2.1}$$

where $M, N = 0, \ldots, 9$ are Lorentz indices, Ψ is the Grassmann-valued Majorana-Weyl fermion, Γ^M are the ten-dimensional gamma matrices satisfying the Clifford algebra $\{\Gamma^M, \Gamma^N\} = -2\eta^{MN}$, $\bar{\Psi} = \Psi^\dagger \Gamma^0$, and

$$F_{MN} = \frac{i}{g_{\mathrm{YM}}}[D_M, D_N] = \partial_M A_N - \partial_N A_M - ig_{\mathrm{YM}}[A_M, A_N], \quad D_M\Psi = \partial_M \Psi - ig_{\mathrm{YM}}[A_M, \Psi], \tag{2.2}$$

are the field strength and the gauge covariant derivative with A_M being the gauge field. The action is invariant under the non-abelian local gauge transformations

$$\Psi \to U\Psi U^{-1}, \qquad A_M \to UA_M U^{-1} + \frac{i}{g_{\mathrm{YM}}} U\partial_M U^{-1}, \quad F^{MN} \to UF^{MN}U^{-1}, \tag{2.3}$$

with $U(x) \in U(N)$ and the supersymmetric transformations

$$\delta A_M = i\bar{\alpha}\Gamma_M \Psi, \qquad \delta\Psi = \frac{1}{2}\Gamma_{MN}F^{MN}\alpha, \tag{2.4}$$

where $\Gamma_{MN} \equiv [\Gamma_M, \Gamma_N]$ and α is a Grassmann-valued spinor. The spinors satisfy the Majorana and Weyl conditions

$$\Psi = \mathcal{C}_{10}\bar{\Psi}^t, \qquad \Gamma_{11}\Psi = -\Psi, \tag{2.5}$$

where $\Gamma_{11} = i\Gamma_0 \dots \Gamma_9$ and $\mathcal{C}_{10}$ is the charge conjugation operator defined by $\mathcal{C}_{10}\Gamma_M\mathcal{C}_{10}^{-1} = -\Gamma_M^t$. To find a representation of the gamma matrices satisfying the Clifford algebra in ten dimensions, it is convenient to use the four-dimensional gamma matrices in the Weyl basis given by

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad \sigma^\mu = (\mathbb{1}, \sigma^i), \quad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i) \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.6}$$

for $\mu = 0, 1, 2, 3$, since we want our theory to reduce to four dimensions. The gamma matrices satisfy

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i, \qquad \gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger. \tag{2.7}$$

With the four-dimensional charge conjugation given by

$$\mathcal{C}_4 = \begin{bmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{bmatrix}, \tag{2.8}$$

the ten-dimensional gamma matrices can be represented by the 32×32 matrices

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}_8, \qquad \mu = 0, 1, 2, 3, \tag{2.9a}$$

$$\Gamma^4 = \gamma_5 \otimes \mathbb{1}_2 \otimes i\gamma^0, \tag{2.9b}$$

$$\Gamma^{4+a} = \gamma_5 \otimes \mathbb{1}_2 \otimes \gamma^a, \qquad a = 1, 2, 3, \tag{2.9c}$$

$$\Gamma^8 = \gamma_5 \otimes \sigma_1 \otimes i\gamma_5, \tag{2.9d}$$

$$\Gamma^9 = \gamma_5 \otimes \sigma_2 \otimes i\gamma_5. \tag{2.9e}$$

Then, by introducing

$$U = i\mathbb{1}_4 \otimes \begin{bmatrix} 0 & 0 & -\sigma_2 & 0 \\ 0 & -\sigma_2 & 0 & 0 \\ \sigma_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_0 \end{bmatrix}, \tag{2.10}$$

and transforming the gamma matrices and the charge conjugation operator according to

$$\Gamma^M \longrightarrow U\Gamma^M U^\dagger, \qquad \mathcal{C}_{10} \longrightarrow U^*\mathcal{C}_{10}U^\dagger, \tag{2.11}$$

the charge conjugation and the eleventh gamma matrix factorizes into

$$\mathcal{C}_{10} = \mathcal{C}_4 \otimes \begin{bmatrix} 0 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{bmatrix}, \qquad \Gamma_{11} = i\Gamma_0 \dots \Gamma_9 = \gamma_5 \otimes \begin{bmatrix} -\mathbb{1}_4 & 0 \\ 0 & \mathbb{1}_4 \end{bmatrix}. \tag{2.12}$$

This hints that the Majorana-Weyl spinor decompose into

$$\Psi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_8 \end{pmatrix}, \tag{2.13}$$

where χ_i are in turn to be constrained spinors in four dimensions. The Weyl condition, $\Gamma_{11}\Psi = -\Psi$, implies that

$$\chi_i = \gamma_5 \chi_i, \qquad \chi_j = -\gamma_5 \chi_j \qquad \text{for } i = 1, \ldots, 4, \quad j = 5, \ldots, 8, \tag{2.14}$$

which in turn implies that the first four spinors are left projections and the last four are right projections of another spinor

$$\chi_i = L\psi_i, \qquad \chi_j = R\psi_j, \tag{2.15}$$

since the projection operators satisfy $\gamma_5 L = L$ and $\gamma_5 R = -R$ and are given by

$$L = \frac{1}{2}(\mathbb{1}_4 + \gamma_5), \qquad R = \frac{1}{2}(\mathbb{1}_4 - \gamma_5). \tag{2.16}$$

From the Majorana condition, $\Psi = \Psi^C = \mathcal{C}_{10}\Gamma_0\Psi^*$, it follows that

$$L\psi_i = \mathcal{C}_4\gamma_0 R\psi^*_{i+4} = L\psi_{i+4}, \qquad R\psi_{i+4} = \mathcal{C}_4\gamma_0 L\psi^*_i = R\psi_i, \tag{2.17}$$

and consequently the Majorana-Weyl spinors decompose into four chiral and four anti-chiral 4 dimensional Majorana spinors

$$\Psi = \begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_1 \\ \vdots \\ R\psi_4 \end{pmatrix}, \tag{2.18}$$

since the spinors still satisfy $\psi_i = \mathcal{C}_4\bar{\psi}^t_i$. Now, the reduction ansatz is to freeze all dependencies of the last six coordinates $x^i, i = 5, \ldots, 9$ and decompose the Gauge field into

$$A^\mu = A^\mu, \qquad \mu = 0, \ldots 3, \tag{2.19a}$$

$$A^i = \phi^i, \qquad i = 4, \ldots 9, \tag{2.19b}$$

where ϕ_i are to be considered six real scalar fields. The kinetic term in the action involving the field strength then decomposes to

$$F^{MN}F_{MN} = F^{\mu\nu}F_{\mu\nu} + 2F^{\mu i}F_{\mu i} + F^{ij}F_{ij}, \tag{2.20}$$

with

$$F^{\mu i}F_{\mu i} = \left(\partial^\mu\phi^i - \partial^i A^\mu - ig_{\text{YM}}[A^\mu, \phi^i]\right)\left(\partial_\mu\phi_i - \partial_i A_\mu - ig_{\text{YM}}[A_\mu, \phi_i]\right) = D^\mu\phi^i D_\mu\phi_i, \tag{2.21}$$

and

$$F^{ij}F_{ij} = \left(\partial^i\phi^j - \partial^j\phi^i - ig_{\text{YM}}[\phi^i, \phi^j]\right)\left(\partial_i\phi_j - \partial_j\phi_i - ig_{\text{YM}}[\phi_i, \phi_j]\right) = -g^2_{\text{YM}}[\phi^i, \phi^j][\phi_i, \phi_j], \tag{2.22}$$

where the ∂_i terms vanish due to the reduction ansatz. Likewise, the fermionic kinetic term reduces to

$$i\bar{\Psi}\Gamma^M D_M \Psi = i\bar{\Psi}\Gamma^\mu D_\mu \Psi + g_{\mathrm{YM}}\bar{\Psi}\Gamma^i[\phi_i, \Psi]. \tag{2.23}$$

Consequently, the ten-dimensional supersymmetric $\mathcal{N} = 1$ SYM theory reduces to the four-dimensional $\mathcal{N} = 4$ SYM with the action

$$\begin{aligned} S_{\mathcal{N}=4} = 2\int d^4x \, \mathrm{tr}\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi_i D^\mu\phi_i \\ &+\frac{i}{2}\bar{\psi}\Gamma^\mu D_\mu\psi + \frac{g_{\mathrm{YM}}}{2}\bar{\psi}\Gamma^i[\phi_i, \psi] + \frac{g_{\mathrm{YM}}^2}{4}[\phi_i, \phi_j][\phi_i, \phi_j]\Big], \end{aligned} \tag{2.24}$$

where Γ^μ are still the ten-dimensional gamma matrices and $F_{\mu\nu}$ and D_μ are now given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{\mathrm{YM}}[A_\mu, A_\nu], \tag{2.25a}$$

$$D_\mu\phi_i = \partial_\mu\phi_i - ig_{\mathrm{YM}}[A_\mu, \phi_i], \quad D_\mu\psi = \partial_\mu\psi - ig_{\mathrm{YM}}[A_\mu, \psi], \tag{2.25b}$$

with $\mu, \nu = 0, 1, 2, 3$ and $i = 1, \ldots, 6$. The supersymmetries (2.4) become

$$\delta A_\mu = i\bar{\alpha}\Gamma_\mu\Psi, \quad \delta\phi_i = i\bar{\alpha}\Gamma_i\Psi, \quad \delta\Psi = \frac{1}{2}\Gamma_{\mu\nu}F^{\mu\nu}\alpha + D_\mu\phi_i\Gamma^{\mu i}\alpha - \frac{ig_{\mathrm{YM}}}{2}[\phi_i, \phi_j]\Gamma^{ij}\alpha. \tag{2.26}$$

It is often useful to expand the fields in the N^2 generators of $U(N)$

$$\phi^i = \phi^i_a T_a, \qquad F^{\mu\nu} = F^{\mu\nu}_a T_a, \tag{2.27}$$

where the generators T_a are Hermitian and satisfy

$$T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk}, \qquad [T_a, T_b] = if_{abc}T^c, \qquad \mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}. \tag{2.28}$$

2.2 Conformal Symmetry

The action (2.24) exhibits a lot of symmetry which it inherits from its 10 dimensional ancestor and importantly it is invariant under conformal transformations which preserves angles between any two line segments[1]. These transformations belong to the special subset of coordinate transformations that scale the metric according to

$$g_{\mu\nu} \to g'_{\mu\nu}(x') = \Omega^2(x)g_{\mu\nu}(x). \tag{2.29}$$

Comparing this with a general coordinate transformation of the metric

$$g_{\mu\nu} \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x), \tag{2.30}$$

under the infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \epsilon\xi^\mu(x), \tag{2.31}$$

and by expanding $\Omega^2(x) = 1 + \epsilon\omega(x) + \mathcal{O}(\epsilon^2)$, we get that to first order in ϵ

$$\omega(x)g_{\mu\nu} = -\partial_\mu\xi_\nu - \partial_\nu\xi_\mu. \tag{2.32}$$

[1]The conformal transformations are known form the Mercator projection which projects the surface of the Earth onto a cylinder such that angles are conserved while areas far away from the equator are distorted.

By tracing (2.32), we arrive at the differential equation

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d} g_{\mu\nu} \partial_\sigma \xi^\sigma(x), \tag{2.33}$$

and by contracting this with ∂^μ we obtain

$$d\partial^2 \xi_\nu = (2-d)\partial_\nu \partial^\mu \xi_\mu, \tag{2.34}$$

which for $d = 2$ is simply the Laplace equation. Thus, in two dimensions there are an infinite amount of conformal parameters. Contracting again with ∂^ν, we see that $\partial^2 \partial_\mu \xi^\mu = 0$ and thus ξ^μ can depend at most quadratically in x. The solutions for (2.34) for $d \neq 2$ are

$$\xi^\mu(x) = \alpha^\mu + \omega^\mu_\nu x^\nu + \sigma x^\mu + \beta_\nu (g^{\mu\nu} x^2 - 2x^\mu x^\nu), \tag{2.35}$$

where α^μ, ω^μ_ν, and σ are the generators of infinitesimal translations, Lorentz transformations and dilatations respectively, the latter simply being scaling of the metric. The last two terms proportional to β constitute the special conformal transformations given by an inversion followed by an infinitesimal translation followed by an inversion again,

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + \alpha^\mu \to \frac{\left(\frac{x^\mu}{x^2} + \alpha^\mu\right)}{\left(\frac{x^\nu}{x^2} + \alpha^\nu\right)\left(\frac{x_\nu}{x^2} + \alpha_\nu\right)} = \alpha_\nu (g^{\mu\nu} x^2 - 2x^\mu x^\nu) + \mathcal{O}(\alpha^2). \tag{2.36}$$

To determine a differential operator representation of the conformal generators, we consider a scalar field $\phi(x)$ that is invariant under conformal transformations (2.31). We then have that

$$\phi'(x') = \phi(x' - \epsilon\xi) = \phi(x') - \epsilon \xi^\mu \partial_\mu \phi(x') + \mathcal{O}(\epsilon^2). \tag{2.37}$$

Comparing this with the infinitesimal transformation generated by a conformal generator G_a,

$$\phi'(x) = \phi(x) - i\delta_a G_a \phi(x), \tag{2.38}$$

we identify the d generators of translation P^μ and the $d(d-1)$ generators of Lorentz transformations $J_{\mu\nu}$, which constitute the Poincaré algebra by

$$P_\mu = -i\partial_\mu, \qquad J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \tag{2.39}$$

while the dilatation generator D and the d special conformal generators K^μ are given by[2]

$$D = -ix^\mu \partial_\mu, \qquad K^\mu = i(g^{\mu\nu} x^2 - 2x^\mu x^\nu)\partial_\nu. \tag{2.40}$$

Counting the number of conformal generators, one gets $\frac{1}{2}(d+2)(d+1)$ which is the same amount of generators of $SO(d,2)$ and in fact the conformal algebra of d dimensional spacetime is isomorphic to the Lorentz algebra of $d+2$ dimensional spacetime. Working out their commutation relations, we obtain

$$[D, P_\mu] = iP_\mu, \tag{2.41a}$$
$$[D, K_\mu] = -iK_\mu, \tag{2.41b}$$
$$[K_\mu, P_\nu] = -2i\left(J_{\mu\nu} - \eta_{\mu\nu} D\right), \tag{2.41c}$$
$$[P_\mu, J_{\rho\sigma}] = i\left(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho\right), \tag{2.41d}$$
$$[K_\mu, J_{\nu\sigma}] = i(\eta_{\mu\nu} K_\sigma - \eta_{\mu\sigma} K_\nu), \tag{2.41e}$$
$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\left(\eta_{\mu\rho} J_{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma), \tag{2.41f}$$

[2] Not to be confused with generators of Lorentz boost which are often denoted $K_m = J_{0m}$

while the rest vanish. It turns out that the conformal symmetry of the action (2.24) greatly restricts the form of correlation functions. All one-point functions vanish due to scale invariance while the spacetime dependence of two-point functions are fixed by the conformal dimension of the operators. Fields with definite conformal dimension Δ_i transform under scale transformations according to

$$x \to \lambda x, \qquad \phi_i(x) \to \lambda^{-\Delta_i}\phi_i(\lambda x). \tag{2.42}$$

With $\lambda = 1+\sigma$, the infinitesimal form of the transformation is

$$\phi_i(x) \to (1 - i\sigma D)\phi_i(\lambda x) = (1-\sigma\Delta_i)\phi_i(\lambda x), \tag{2.43}$$

and the dilatation operator thus measures the conformal dimension of a field. The conformal dimension is classically equal to the mass dimension of the field, but it will receive quantum corrections. When quantizing the theory, the action of the Poincaré generators is replaced with the commutator such that $D\phi_i \to [D,\phi_i]$ and instead of considering the classical fields we now consider composite operators $\mathcal{O}_\Delta$ consisting of multiple fields. In order to find out how the operators of definite conformal dimension transform, we consider the action of the generators on the fields at origin where they transform in a representation of the rotation group $SO(d)$ given by

$$[J_{\mu\nu}, \mathcal{O}^a_\Delta(0)] = (\Sigma_{\mu\nu})^a_b \mathcal{O}^b_\Delta(0), \quad [K_\mu, \mathcal{O}^a_\Delta(0)] = 0, \quad [D, \mathcal{O}^a_\Delta(0)] = -i\Delta^a_b \mathcal{O}^b_\Delta(0), \tag{2.44}$$

with $\Sigma_{\mu\nu}$ being a matrix satisfying the Lorentz algebra and a, b indices for the $SO(d)$ representation which we will henceforth suppress. The action of the translation generator is still given by

$$[P_\mu, \mathcal{O}_\Delta(x)] = -i\partial_\mu \mathcal{O}_\Delta(x). \tag{2.45}$$

We can then find the action at any spacetime by translating with $e^{iP^\mu x_\mu}$

$$\begin{aligned}[D, \mathcal{O}_\Delta(x)] &= [D, e^{-iP_\mu x^\mu}\mathcal{O}_\Delta(0)e^{iP_\mu x^\mu}] = e^{-iP_\mu x^\mu}[e^{iP_\mu x^\mu} D e^{-iP_\mu x^\mu}, \mathcal{O}_\Delta(0)]e^{iP_\mu x^\mu} \\ &= e^{-iP_\mu x^\mu}[D - ix^\mu P_\mu, \mathcal{O}_\Delta(0)]e^{iP_\mu x^\mu} = -i\left(\Delta + x^\mu\partial_\mu\right)\mathcal{O}_\Delta(x),\end{aligned} \tag{2.46}$$

where we used the Baker-Hausdorff formula $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots$. Similarly, we get for the other generators

$$[J_{\mu\nu}, \mathcal{O}_\Delta(x)] = -i\left(\Sigma_{\mu\nu} - x_\mu\partial_\nu + x_\nu\partial_\mu\right)\mathcal{O}_\Delta(x), \tag{2.47a}$$

$$[K_\mu, \mathcal{O}_\Delta(x)] = -i\left(2x_\mu\Delta + 2x^\nu\Sigma_{\nu\mu} + 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right)\mathcal{O}_\Delta(x). \tag{2.47b}$$

We will only be interested in spinless states and will thus take $\Sigma_{\mu\nu}$ to be zero. The prototypical two-point correlation function is given by

$$f_{12}(x_1,x_2) = \langle 0|\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)|0\rangle \equiv \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle\,. \tag{2.48}$$

Then, by using the commutation relation (2.45), we get that

$$\begin{aligned}\left(\frac{\partial}{\partial x_1^\mu} + \frac{\partial}{\partial x_2^\mu}\right) f_{12}(x_1,x_2) &= \langle \frac{\partial}{\partial x_1^\mu}\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle + \langle \mathcal{O}_{\Delta_1}(x_1)\frac{\partial}{\partial x_2^\mu}\mathcal{O}_{\Delta_2}(x_2)\rangle \\ &= \langle [P_\mu, \mathcal{O}_{\Delta_1}(x_1)]\mathcal{O}_{\Delta_2}(x_2)\rangle + \langle \mathcal{O}_{\Delta_1}(x_1)[P_\mu, \mathcal{O}_{\Delta_2}(x_2)]\rangle = 0,\end{aligned} \tag{2.49}$$

where we used that the vacuum is annihilated by all symmetry generators $G\,|0\rangle = 0$. The correlation functions can therefore not depend on $x_1 + x_2$ but only on the distance $z = x_1 - x_2$, which makes sense since the theory is translation invariant. Without loss of generality, we set $x_2 = 0$ and using commutation relation (2.46) we get the differential equation

$$\left(\Delta_1 + \Delta_2 + z^\mu \frac{\partial}{\partial z^\mu}\right) f_{12}(z) = 0, \tag{2.50}$$

which is solved by

$$f_{12}(z) = C_{12} z^{-\Delta_1 - \Delta_2}. \tag{2.51}$$

Finally, by considering the commutation relation (2.47b), we get the equation

$$\left(2\Delta_1 z_\mu + 2z_\mu z^\nu \frac{\partial}{\partial z^\nu} - z^2 \frac{\partial}{\partial z^\mu}\right) z^{-\Delta_1 - \Delta_2} = 0, \tag{2.52}$$

which is solved only if $\Delta_1 = \Delta_2$. Consequently, the two-point functions between operators with definite conformal dimension are restricted to be on the form

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{C_{ij}}{|x - y|^{2\Delta_i}}, \tag{2.53}$$

with $C_{ij} = 0$ if $\Delta_i \neq \Delta_j$. We will use this to determine the gauge-invariant operators of definite conformal dimension at one loop order.

2.3 $\mathcal{N} = 4$ Superconformal Group

In addition to the conformal symmetry, the action has a global $SU(6) \cong SU(4)$ symmetry where the 6 scalars transform in the six-dimensional representation, the 4 chiral fermions in the fundamental representation, and the 4 anti-chiral fermions in the antifundamental representation. This is known as bosonic R-symmetry which constitutes the largest group that commutes with all other symmetry transformations. Finally, the supersymmetry (2.26) gives rise to 16 supercharges $Q_{\alpha a}$ and $\tilde{Q}^a_{\dot{\alpha}}$ where $a = 1, 2, 3, 4$ and α and $\dot{\alpha}$ are spinor indices belonging to the $SU(2)_L$ and $SU(2)_R$ part of the Lorentz group $SO(1,3) \simeq SU(2)_L \times SU(2)_R$. The commutators are

$$\{Q_{\dot{\alpha} a}, \tilde{Q}^b_{\alpha}\} = \delta^b_a P_{\alpha\dot{\alpha}}, \quad \{Q_{\dot{\alpha} a}, Q_{\dot{\beta} b}\} = \{\tilde{Q}^a_\alpha, \tilde{Q}^b_\beta\} = 0. \tag{2.54}$$

Combining the supersymmetries with the conformal ones we introduce an additional 16 charges $S^a_\alpha, \tilde{S}_{a\dot{\alpha}}$. Together, the conformal symmetries, the supersymmetries and the R-symmetries form the $\mathcal{N} = 4$ superconformal group of $PSU(2,2|4)$. The algebra generating a Lie supergroup is know as a Lie superalgebra and an introduction to this as well as the notion of Dynkin diagrams can be found in appendix A. Through the remainder of this chapter we will only be interested in the $\mathfrak{so}(6)$ subsector of the theory consisting of only the six scalars ϕ_i.

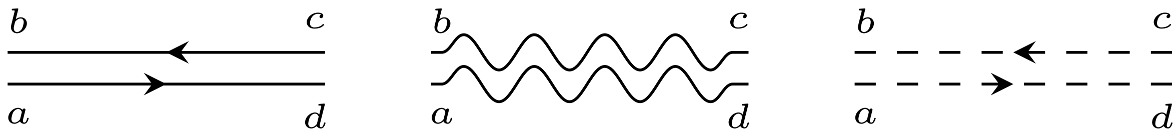
2.4 Correlation Functions

In Feynman's path integral formalism the correlation functions are given by

$$\langle 0|\mathcal{T}\phi(x_1)\dots\phi(x_n)|0\rangle \equiv \langle \phi(x_1)\dots\phi(x_n)\rangle = N \int \mathcal{D}\phi \ \phi(x_1)\dots\phi(x_n) e^{iS}, \tag{2.55}$$

where $\mathcal{T}$ is the time-ordering operator, N a normalization constant such that $\langle 0|0\rangle = 1$ and $\mathcal{D}\phi$ denotes that the integral is to be taken over all field configurations. By introducing a generating functional

$$Z[J] = N \int \mathcal{D}\phi \ \exp\left(i \int d^d x \mathcal{L} + J(x)\phi(x)\right), \tag{2.56}$$



(a) Scalar propagator **(b)** Gluon propagator **(c)** Fermion propagator

Figure 2.1: Propagators carrying two indices represented with two lines. Breaking with tradition, we use dashed lines for the fermion propagator and a full line for the scalar propagator.

correlation functions can be obtained by taking the functional derivatives

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle = (-i)^n \frac{\delta Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\Big|_{J=0}, \tag{2.57}$$

where we have introduced a source term J for the field ϕ. For a free theory this can be solved explicitly while for interacting theories we use perturbation theory and expand the interacting part of the Lagrangian in the coupling constant which produces the Feynman rules. In our case the non-interacting part is given by

$$\mathcal{L}_{\text{free}} = 2\,\mathrm{tr}\left[\frac{1}{2}A_\mu\left(\Box\eta^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu + \frac{1}{2}\phi_i\Box\phi_i + \frac{i}{2}\bar{\psi}\Gamma^\mu\partial_\mu\psi\right], \tag{2.58}$$

from which we can read of the scalar propagator given by the two-point Green's function of the Laplacian[3]

$$\langle\phi^i_{ab}(x)\phi^j_{cd}(y)\rangle = \frac{1}{2}\delta_{ij}\delta_{ad}\delta_{bc}\int\frac{d^{4-2\varepsilon}p}{(2\pi)^{4-2\varepsilon}}\frac{e^{ip(x-y)}}{p^2} = \delta_{ij}\delta_{ad}\delta_{bc}\frac{\Gamma(1-\varepsilon)}{8\pi^{2-\varepsilon}(x-y)^{2(1-\varepsilon)}}, \tag{2.59}$$

where we have introduced a dimensional regularization with $d = 4-2\varepsilon$ to keep track of the divergences we are about to encounter. Likewise, the gluon propagator in Feynman–'t Hooft gauge becomes

$$\langle A^\mu_{ab}(x)A^\nu_{cd}(y)\rangle = \delta_{\mu\nu}\delta_{ad}\delta_{bc}\frac{\Gamma(1-\varepsilon)}{8\pi^{2-\varepsilon}(x-y)^{2(1-\varepsilon)}} \equiv \delta_{\mu\nu}\delta_{ad}\delta_{bc}K_\varepsilon(x,y), \tag{2.60}$$

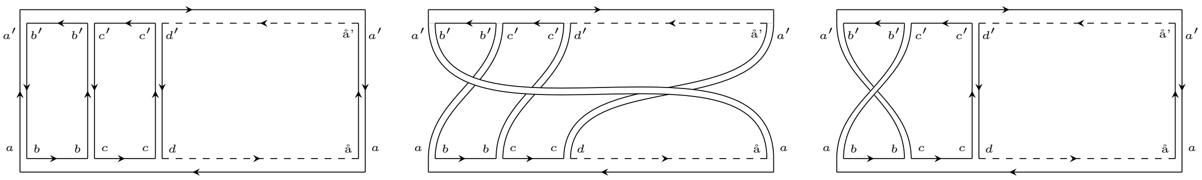
where we introduced $K_\varepsilon(x,y)$ to lighten notation. The Kronecker deltas $\delta_{ad}\delta_{bc}$ suggest that we represent the propagators by a double line like seen in Fig. 2.1 in order to keep track of the color indices. The even n-point functions are then given by all the possible combinations of the two-point propagator

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle = \sum_{\text{combinations}}\langle\phi(x_1)\phi(x_2)\rangle\ldots\langle\phi(x_{n-1})\phi(x_n)\rangle, \tag{2.61}$$

while the odd n-point functions vanish. If we now consider the two-point correlation functions of a gauge-invariant composite operator in the $\mathfrak{so}(6)$ sector which consists of traces of the six scalar fields ϕ_i such as

$$\mathcal{O}_I = \mathrm{tr}\left[\phi^{i_1}\phi^{i_2}\ldots\phi^{i_L}\right] = \phi^{i_1}_{ab}\phi^{i_2}_{bc}\ldots\phi^{i_L}_{\mathring{a}a}, \tag{2.62}$$

[3]If the gauge group is $SU(N)$ instead of $U(N)$ the completeness relation (2.28) reads $T^a_{ij}T^a_{kl} = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}$ and we would have to replace $\delta_{ad}\delta_{bc}$ with $\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ac}\delta_{bd}$, which in the planar limit, $N\to\infty$ would be the same.



(a) **(b)** **(c)**

Figure 2.2: The planar Feynman diagrams (a,b) carry two extra factors of N compared to a non-planar diagram (c) which is consequently suppressed in the 't Hooft Limit where $N \to \infty$.

where $I = \{ i_1, i_2, \ldots, i_L \}$ is an ordered set specifying the operator. We need to consider every possible contraction to determine the two-point function. We then have that

$$\begin{aligned}\langle \mathcal{O}_I(x)\bar{\mathcal{O}}_I(y)\rangle &= \langle \phi^{i_1}_{ab}(x)\phi^{i_2}_{bc}(x)\ldots\phi^{i_L}_{na}(x)\phi^{i_L}_{a'n'}(y)\phi^{i_2}_{c'b'}(y)\ldots\phi^{i_1}_{b'a'}(y)\rangle \\ &= K_\varepsilon(x,y)^L\left(\delta_{aa'}\delta_{aa'}\delta_{bb'}\delta_{bb'}\delta_{cc'}\delta_{cc'}\ldots + \delta_{i_1,i_2}\delta_{aa'}\delta_{ab'}\delta_{bc'}\delta_{ba'}\delta_{cb'}\delta_{cc'}\ldots + \ldots\right).\end{aligned} \tag{2.63}$$

We see that there are $L!$ possible ways to contract the fields, but not every contraction is of the same type. The first diagram (Fig. 2.2a) is a so called planar diagram corresponding to the first term of (2.63) and by summing over repeated indices we get

$$\ldots\delta_{aa'}\delta_{aa'}\delta_{bb'}\delta_{bb'}\delta_{cc'}\delta_{cc'}\ldots = \ldots N^3\ldots, \tag{2.64}$$

which should be compared to the non-planar diagram (Fig. 2.2c) corresponding to the second term

$$\ldots\delta_{aa'}\delta_{ab'}\delta_{bc'}\delta_{ba'}\delta_{cb'}\delta_{cc'}\ldots = \ldots N\ldots, \tag{2.65}$$

which is suppressed by two orders of N. Fig. 2.2b represents another planar diagram which is non-zero if the cyclic permutation $(i_1 \to i_2, i_2 \to i_3, \ldots)$ leaves $\mathcal{O}_I$ invariant. Consequently, in the planar or 't Hooft limit, where $g_{\rm YM} \to 0, N \to \infty$ while the effective coupling $g^2 \equiv \frac{g^2_{YM}N}{16\pi^2}$ is kept fixed, only planar Feynman diagrams contribute to the correlation functions. The two-point function of the composite operator then becomes

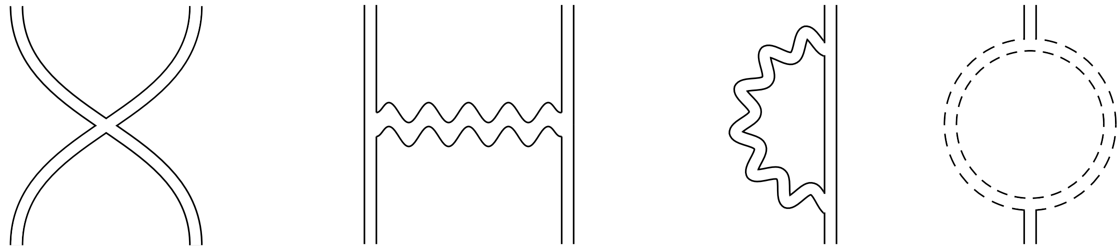
$$\langle \mathcal{O}^{\rm bare}_I(x)\bar{\mathcal{O}}^{\rm bare}_I(y)\rangle = c_I N^L K_\varepsilon(x,y)^L, \tag{2.66}$$

where c_I is the number of cyclic permutations that leave $\mathcal{O}_I$ invariant. Comparing this with (2.53) we see that the conformal dimension in the free theory is simply L. Here we have introduced the superscript to distinguish the bare operators from the renormalized operators that we introduce in the next section where we turn our attention to the interacting part of the theory. The correlation functions then become divergent and we therefore need to renormalize our theory.

2.5 Renormalization Scheme

Because the action (2.24) does not have any conformal anomalies, i.e. it stays conformal after quantizing the fields, the two-point functions still have the form of (2.53) in perturbation theory but the conformal dimension receives quantum corrections and we thus expand it in powers of g

$$\Delta_I = \sum_{n=0}^{\infty} g^{2n}\Delta^{(n)}_I, \tag{2.67}$$



(a) Four-point vertex **(b)** Gluon exchange **(c)** Self-energy diagrams

Figure 2.3: The four Feynman diagrams contributing to the two-point functions at one-loop order in the $\mathfrak{so}(6)$ sector. The four point vertex (a) contributes to operator mixing while the others do not.

were we have $\Delta^{(0)} = L$. The goal is then to calculate the anomalous dimension $\Delta - \Delta^{(0)}$, which we will do to one-loop order. The two-point functions of the composite operators (2.62) are likewise expanded in powers of g

$$\langle \mathcal{O}_I^{\text{bare}}(x)\bar{\mathcal{O}}_J^{\text{bare}}(y)\rangle_\varepsilon = \sqrt{c_I c_J} N^{\Delta^{(0)}} K_\varepsilon(x,y)^{\Delta^{(0)}} \left(\delta_{IJ} + g^2 M_{IJ}(\varepsilon)[\mu(x-y)]^{2\varepsilon}\right) + \mathcal{O}(g^4), \quad (2.68)$$

where δ_{IJ} is one if I is any cyclic permutation of J. Here, μ is a renormalization scale with dimension of mass that has been introduced in order to keep g dimensionless and avoid weird logarithms of dimension full terms. As we will show in the next section M_{IJ} is a matrix with a simple pole at $\varepsilon = 0$ and we can therefore write it as

$$M_{IJ}(\varepsilon) = -\frac{1}{\varepsilon} D_{IJ} + M_{IJ}^{\text{fin}} + \mathcal{O}(\varepsilon). \quad (2.69)$$

Instead of adding counter terms to the Lagrangian, the correlation functions can become finite by rescaling the operators

$$\mathcal{O}_I^{\text{ren}} = \mathcal{Z}_{IJ}(g,\varepsilon)\frac{\mathcal{O}_J^{\text{bare}}}{\sqrt{c_J}}, \qquad \mathcal{Z}_{IJ}(g,\varepsilon) = \delta_{IJ} + \frac{g^2}{2\varepsilon} D_{IJ} - \frac{g^2}{2} M_{IJ}^{\text{fin}}. \quad (2.70)$$

With this choice, the renormalized two-point correlation function becomes

$$\begin{aligned}\langle \mathcal{O}_I^{\text{ren}}(x)\bar{\mathcal{O}}_J^{\text{ren}}(y)\rangle_{\varepsilon=0} &= \lim_{\varepsilon\to 0} \mathcal{Z}_{IA}(g,\varepsilon)\mathcal{Z}_{JB}(g,\varepsilon)\frac{\langle \mathcal{O}_A^{\text{bare}}(x)\bar{\mathcal{O}}_B^{\text{bare}}(y)\rangle_\varepsilon}{\sqrt{c_A c_B}} \\ &= N^{\Delta^{(0)}} K_\varepsilon(x,y)^{\Delta^{(0)}} \left(\delta_{IJ} - g^2 D_{IJ}\log(\mu^2(x-y)^2) + \mathcal{O}(g^4)\right).\end{aligned} \quad (2.71)$$

Evidently, by determining and diagonalizing D_{IJ} we will find the good conformal operators that satisfy (2.53) at one-loop order.

2.6 The Feynman Diagrams at One-loop Order

In order to determine D_{IJ}, we will do an explicit Feynman diagram calculation, and we will therefore need to look at the interacting part of the Lagrangian given by

$$\begin{aligned}\mathcal{L}_{\text{int}} =& 2g_{\text{YM}}\,\text{tr}\left[\frac{1}{2}\bar{\psi}\Gamma^i[\phi_i,\psi] + iA_\mu A_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + i[A_\mu,\phi_i]\partial^\mu\phi_i + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu,\psi]\right] \\ &+ \frac{g_{\text{YM}}^2}{2}\,\text{tr}\left[[\phi_i,\phi_j][\phi_i,\phi_j] + [A_\mu,A_\nu][A^\mu,A^\nu] + 2[A_\mu,\phi_i][A^\mu,\phi_i]\right].\end{aligned} \quad (2.72)$$

We can then find the correlation functions at a given order by expanding the interacting part in powers of g_{YM}

$$\begin{aligned}\langle F(\phi)\rangle &= \int \mathcal{D}\phi F(\phi)\exp\left(i\int d^dx\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} + J\phi\right) \\ &= \left\langle F(\phi)\left(1 + i\int d^dx\mathcal{L}_{\text{int}} - \frac{1}{2}\int d^dx d^dy \mathcal{L}_{\text{int}}(x)\mathcal{L}_{\text{int}}(y) + \dots\right)\right\rangle_0 ,\end{aligned} \tag{2.73}$$

where the zero-subscript denotes the correlation function is found in the free theory. This results in the Feynman diagram expansion. The only three Feynman diagrams that contribute to the correction of the two-point functions in the $\mathfrak{so}(6)$ sector at one loop order are given in Fig. 2.3. Starting with the four-point vertex (Fig. 2.3a) corresponding to the $\text{tr}([\phi^i,\phi^j][\phi^i,\phi^j])$ part of (2.72), the contribution is given by

$$\langle[\phi_i\phi_j]_{ab}(x)[\phi_{j'}\phi_{i'}]_{b'a'}(y)\rangle^{(a)} = \frac{ig_{\text{YM}}^2}{2}\int d^{4-2\varepsilon}z\,\langle[\phi_i\phi_j]_{ab}(x)\,(\text{tr}[\phi_k,\phi_l][\phi_k,\phi_l](z))\,[\phi_{j'}\phi_{i'}]_{b'a'}(y)\rangle_0 . \tag{2.74}$$

There are four planar ways to contract $\phi_k\phi_l\phi_k\phi_l$ and $\phi_k\phi_k\phi_l\phi_l$ with the in-going and out-going fields, and we thus get

$$\begin{aligned}\langle[\phi_i\phi_j]_{ab}(x)\,(\text{tr}[\phi_k,\phi_l][\phi_k,\phi_l](z))\,[\phi_{j'}\phi_{i'}]_{b'a'}(y)\rangle_0 &= K_\varepsilon(x,z)^2K_\varepsilon(z,y)^2 \\ &\times 2N^2\delta_{aa'}\delta_{bb'}(4\delta_{ij'}\delta_{ji'} - 2\delta_{ii'}\delta_{jj'} - 2\delta_{ij}\delta_{i'j'}),\end{aligned} \tag{2.75}$$

where the first term contracts in-going with outgoing scalars in order, the second in reverse order, while the last term contracts in-going with in-going and outgoing with out-going. Substituting K into the integral we get

$$\begin{aligned}\int d^{4-2\varepsilon}zK_\varepsilon(x,z)^2K_\varepsilon(z,y)^2 &= \left(\frac{\Gamma(1-\varepsilon)}{8\pi^{2-\varepsilon}}\right)^4\int d^{4-2\varepsilon}z\frac{1}{[(x-z)(z-y)]^{4(1-\varepsilon)}} \\ &= \left(\frac{\Gamma(1-\varepsilon)}{8\pi^{2-\varepsilon}}\right)^4\pi^{2-\varepsilon}\frac{\Gamma(2-3\varepsilon)\Gamma(\varepsilon)^2}{\Gamma(2-2\varepsilon)^2\Gamma(2\varepsilon)}\frac{1}{(x-y)^{4-6\varepsilon}} \\ &= K_\varepsilon(x,y)^2\frac{1}{2^6\pi^{2-\varepsilon}}\frac{\Gamma(1-\varepsilon)^2\Gamma(2-3\varepsilon)\Gamma(\varepsilon)^2}{\Gamma(2-2\varepsilon)^2\Gamma(2\varepsilon)}(x-y)^{2\varepsilon},\end{aligned} \tag{2.76}$$

where we used that

$$\int d^{4-2\varepsilon}u\frac{1}{u^{2\alpha}(u+y-x)^{2\alpha}} = \pi^{2-\varepsilon}G(\alpha)\frac{1}{(x-y)^{2(2\alpha+\varepsilon-2)}}, \tag{2.77}$$

with

$$G(\alpha) = \frac{\Gamma(2\alpha+\varepsilon-2)\Gamma(2-\varepsilon-\alpha)^2}{\Gamma(\alpha)^2\Gamma(4-2\alpha-2\varepsilon)}. \tag{2.78}$$

Expanding the gamma functions in powers of ε, we obtain

$$\frac{\Gamma(1-\varepsilon)^2\Gamma(2-3\varepsilon)\Gamma(\varepsilon)^2}{\Gamma(2-2\varepsilon)^2\Gamma(2\varepsilon)} = 2\left(\frac{1}{\varepsilon} + 1 + \gamma_E\right) + \mathcal{O}(\varepsilon), \tag{2.79}$$

and

$$\pi^\varepsilon = 1 + \varepsilon\log\pi + \mathcal{O}(\varepsilon^2), \qquad (x-y)^{2\varepsilon} = 1 + \varepsilon\log(x-y)^2 + \mathcal{O}(\varepsilon^2). \tag{2.80}$$

Keeping the ε dependent terms up to zeroth order, we get

$$\left(\frac{1}{\varepsilon}+1+\gamma_E\right)(1+\varepsilon\log\pi)\left(1+\varepsilon\log(x-y)^2\right)=\frac{1}{\varepsilon}+1+\gamma_E+\log\pi(x-y)^2+\mathcal{O}(\varepsilon). \quad (2.81)$$

Thus, we find that the four-point vertex is given by

$$\begin{aligned}\langle[\phi_i\phi_j]_{ab}(x)[\phi_{j'}\phi_{i'}]_{b'a'}(y)\rangle^{(a)} &= g^2NK_\varepsilon^2(x,y)\delta_{aa'}\delta_{bb'}(2\delta_{ij'}\delta_{ji'}-\delta_{ii'}\delta_{jj'}-\delta_{ij}\delta_{i'j'}) \\ &\times\left(\frac{1}{\varepsilon}+1+\gamma_E+\log\pi(x-y)^2+\mathcal{O}(\varepsilon)\right).\end{aligned} \quad (2.82)$$

Similarly, it can be shown that the gluon exchange diagram (2.3b) is given by [21]

$$\begin{aligned}\langle[\phi_i\phi_j]_{ab}(x)[\phi_{j'}\phi_{i'}]_{b'a'}(y)\rangle^{(b)} &= g^2NK_\varepsilon(x,y)^2\delta_{ii'}\delta_{jj'}\delta_{aa'}\delta_{bb'} \\ &\times\left(\frac{1}{\varepsilon}+3+\gamma_E+\log(\pi(x-y)^2)+\mathcal{O}(\varepsilon)\right),\end{aligned} \quad (2.83)$$

and the self-energy (2.3c) has the contribution

$$\begin{aligned}\langle[\phi_i]_{ab}(x)[\phi_j]_{b'a'}(y)\rangle^{(c)} &= K_\varepsilon(x,y)\delta_{ij}\delta_{aa'}\delta_{bb'} \\ &\times\left[1-2g^2\left(\frac{1}{\varepsilon}+2+\gamma_E+\log(\pi(x-y)^2)+\mathcal{O}(\varepsilon)\right)+\mathcal{O}(g^4)\right].\end{aligned} \quad (2.84)$$

To determine the correlation functions of the composite operators at one-loop order we can use one of the contributions (2.82-2.84) to make a contraction. The four-point vertex and the gluon exchange diagram can only contract neighboring fields in order to form a planar diagram. To keep track of the Kronecker deltas used to match field types we introduce a permutation operator

$$\mathbb{P}_{n,n+1}\ldots\delta_{i_n,j_m}\delta_{i_{n+1},j_{m+1}}\ldots=\ldots\delta_{i_{n+1},j_m}\delta_{i_n,j_{m+1}}\ldots \quad (2.85)$$

that permutes neighboring fields in $\mathcal{O}_I$ and a trace operator

$$\mathbb{K}_{n,n+1}\delta_{i_n,j_m}\delta_{i_{n+1},j_{m+1}}=\delta_{i_n,j_{n+1}}\delta_{i_m,j_{m+1}}, \quad (2.86)$$

that contracts neighboring fields. Then the non-trivial part of the diagram, where (2.3a) is inserted at site n and $n+1$, can be written as

$$2\delta_{i_nj_{n+1}}\delta_{j_ni_{n+1}}-\delta_{i_ni_{n+1}}\delta_{j_nj_{n+1}}-\delta_{i_nj_n}\delta_{i_{n+1}j_{n+1}}=(2\mathbb{P}_{n,n+1}-\mathbb{K}_{n,n+1}-1)\delta_{i_nj_n}\delta_{i_{n+1}j_{n+1}}. \quad (2.87)$$

Combining all contributions to the two-point correlation function, we get

$$\begin{aligned}\langle\mathcal{O}_I^{\text{bare}}(x)\mathcal{O}_J^{\text{bare}}(y)\rangle_\varepsilon &= \sqrt{c_Ic_J}N^LK_\varepsilon(x,y)^L \\ &\times\left[\delta_{IJ}-g^2\left(\frac{1}{\varepsilon}+1+\gamma_E+\log(\pi(x-y)^2)\right)D_{IJ}+\mathcal{O}(\varepsilon,g^4)\right],\end{aligned} \quad (2.88)$$

where

$$D_{IJ}=\frac{1}{\sqrt{c_Ic_J}}\sum_{n=1}^{L}(2-2\mathbb{P}_{n,n+1}+\mathbb{K}_{n,n+1})(\delta_{i_1,j_1}\delta_{i_2,j_2}\ldots\delta_{i_L,j_L}+\text{cyclic permutations}), \quad (2.89)$$

is the dilatation matrix defined in (2.69). Here, the L cyclic permutations is due to the L cyclic permutations of J and the sum over n corresponds to the L different places the three corrections can be put into the planar diagrams. Furthermore, we identify sites n with $n+L$, such that the operators $\mathbb{P}_{L,L+1}=\mathbb{P}_{L,1}$ and $\mathbb{K}_{L,L+1}=\mathbb{K}_{L,1}$ connect the fields in the ends. We have thus found the dilatation matrix in the $\mathfrak{so}(6)$ sector of the theory, and we need only to diagonalize this to find the good conformal operators and their conformal dimension.

2.7 The Subsectors

The spectral problem is significantly simplified if we specialize to the $\mathfrak{su}(2)$ subsector of the $\mathfrak{so}(6)$ sector consisting of two complex scalar fields X and Y given by

$$X = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4. \tag{2.90}$$

Using (2.59), their propagators become

$$\langle \bar{X}_{ab}(x) X_{b'a'}(y)\rangle = \langle \bar{Y}_{ab}(x) Y_{b'a'}(y)\rangle = 2\delta_{aa'}\delta_{bb'}K_\varepsilon(x,y), \tag{2.91}$$

while the rest of the propagators vanish. Considering single trace composite operators of X and Y, most of the derivation of the one-loop dilatation operator follows in the same manner as before. The composite operators still take the form of (2.62) but with ϕ_i being either X or Y. However, now the $\mathbb{K}_{n,n+1}$ part of (2.89), which contracts neighboring fields within one operator, will be proportional to $1 + i^2$ and vanish, leaving the $\mathfrak{su}(2)$ dilatation operator

$$D_{IJ} = \frac{2}{\sqrt{c_I c_J}} \sum_{n=1}^{L} (1 - \mathbb{P}_{n,n+1})(\delta_{i_1,j_1}\delta_{i_2,j_2}\ldots\delta_{i_L,j_L} + \text{cyclic permutations}). \tag{2.92}$$

The full planar $\mathfrak{psu}(2,2|4)$ dilatation operator to one loop order was found in [19], but throughout the rest of the thesis we will primarily work with the $\mathfrak{su}(2)$ subsector as well as the $\mathfrak{su}(1|1)$ and $\mathfrak{su}(2|1)$ sectors consisting of one scalar and one or two fermions. The dilatation operator has the same form as (2.92) but with a graded permutation operator that picks up a minus when exchanging two fermions. In the next chapter, we will show how the problem of diagonalizing each of these sectors corresponds to finding eigenstates of one-dimensional spin chains.

Chapter 3

Spin Chains and the Coordinate Bethe Ansätze

In this chapter, we map the composite operators of certain subsectors in $\mathcal{N} = 4$ SYM to configurations of one-dimensional spin chains. As we will show, the spin chain eigenstates have a very physical interpretation where magnons parameterized by a set of momenta are traveling down the spin chains and scatter with each other. With this interpretation, the Coordinate Bethe Ansatz (CBA) reduces the spectral problem of diagonalizing the dilatation operator to the problem of solving a set of coupled algebraic equations. The $\mathfrak{su}(2|1)$ and $\mathfrak{su}(1|2)$ sectors are somewhat more involved but can be solved by the nested Bethe ansatz with a similar physical interpretation only with two types of excitations where one is nested and is considered as an excitation on a "short" spin chain. As an extra treat, the CBA showcases how different physical interpretations of the spin chains lead to different but dual solutions which will be further investigated in chapter 6.

3.1 The Heisenberg Spin Chain and the $\mathfrak{su}(2)$ Bethe Ansatz

As we found in the previous chapter, the operators in the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM consist of only two scalar fields $X = \phi_1 + i\phi_2$ and $Y = \phi_3 + i\phi_4$. We now consider the gauge-invariant composite single trace operators of length L (2.62) and map them to elements of the Hilbert space $\mathcal{H} = \mathbb{C}^{2\otimes L}$ by opening up the trace, $\mathrm{tr}\,[XYY \ldots XYX] \to |XYY \ldots XYX\rangle$ modulo cyclicity, such that the Dilatation operator (2.92) defines an operator $H : \mathcal{H} \to \mathcal{H}$ defined by

$$H = \sum_{x=1}^{L} \mathbb{1}_2^{\otimes L} - \mathbb{P}_{x,x+1}, \qquad \mathbb{P}_{x,x+1} = \frac{1}{2}\sum_{i=0}^{3} \ldots \sigma_x^i \otimes \sigma_{x+1}^i \ldots, \tag{3.1}$$

which we will refer to as the Hamiltonian and likewise we will refer to the eigenvalues as energies instead of the conformal dimension. Here $\mathbb{P}_{x,x+1}$ permutes the fields at position x and $x+1$ which we in the second equation have written as a 4×4 matrix with respect to the basis $\{XX, XY, YX, YY\}$ and decomposed into the three Pauli matrices σ^i along with the identity matrix $\mathbb{1}_2 = \sigma^0$. This is exactly the Hamiltonian of the ferromagnetic $\mathrm{XXX}_{1/2}$ Heisenberg spin chain known from solid-state physics, the one-dimensional lattice consisting of L particles with spin 1/2 and nearest-neighbor interaction, which was first solved in 1931 by Hans Bethe [11]. The energy of the ferromagnetic spin chain is minimized when the spins are all aligned and we will therefore consider $|X^L\rangle \equiv |0\rangle$ to be the vacuum state and the Y-fields to be excitations[1]. To

[1] In Bethe's spin chain the X fields correspond to spin right and the Y-fields to spin left.

diagonalize the Hamiltonian, we start by considering one-body eigenstates or so-called magnon states being a superposition of all one-excitation states[2]. Since this system is translation invariant and free, the one-magnon solution to the eigenvalue equation $H|p\rangle = E|p\rangle$ is simply a superposition of plane waves with momentum p,

$$|p\rangle = \sum_{x=1}^{L} e^{ipx} a_x^\dagger |0\rangle \,, \tag{3.2}$$

where $a_x^\dagger = \frac{\sigma_x^1 - i\sigma_x^2}{2}$ is the creation operator acting on lattice site x. By imposing the periodic boundary condition on the wave function, $e^{ip(x+L)} = e^{ipx}$, we get the quantization condition of the momentum, $p = \frac{2\pi n}{L}$, $n = 0, \dots L-1$. By comparing coefficients in the eigenvalue equation, we see that the energy of one magnon is simply

$$E(p) = 2 - e^{ip} - e^{-ip} = 4\sin^2\frac{p}{2}. \tag{3.3}$$

Moving on to the two-body magnon states, one might suspect that the solution is again a simple Fourier transformation, but the system is no longer free. Instead, the two-body magnon states are written as superpositions of all two-body states parameterized by two momenta p_1, p_2

$$|p_1, p_2\rangle = \sum_{1\le x_1 < x_2 \le L} \psi(x_1, x_2) a_{x_1}^\dagger a_{x_2}^\dagger |0\rangle \,, \tag{3.4}$$

where $\psi(x_1, x_2)$ are the position wave functions or amplitudes which we are to determine. Then by comparing coefficients in the eigenvalue equation $E|p_1, p_2\rangle = H|p_1, p_2\rangle$ and considering the two cases, $x_2 = x_1 + 1$ and $x_2 > x_1 + 1$, we get

$$\begin{aligned} E\psi(x_1,x_2) &= 4\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1+1,x_2) \\ &\qquad - \psi(x_1,x_2-1) - \psi(x_1,x_2+1) \quad \text{for } x_2 > x_1+1 &(3.5a)\\ E\psi(x_1,x_2) &= 2\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1,x_2+1) \quad \text{for } x_2 = x_1+1. &(3.5b)\end{aligned}$$

Bethe's ansatz to the solution of this is then that the coordinate wave function is given by

$$\psi(x_1, x_2) = e^{ip_1x_1 + ip_2x_2} + s(p_2, p_1) e^{ip_2x_1 + ip_1x_2}, \tag{3.6}$$

which reflect the fact that the momenta should be separately conserved and the two particles should therefore either pass each other or exchange momenta with a probability amplitude given by the scattering matrix $s(p_1, p_2)$, which for now is just a number. The energy is then found by plugging (3.6) into (3.5a),

$$E\psi(x_1, x_2) = \left(4 - e^{-ip_1} - e^{-ip_2} - e^{ip_1} - e^{ip_2}\right)\psi(x_1, x_2), \tag{3.7}$$

and thus the energy is given by

$$E(p_1, p_2) = 4 - e^{ip_1} - e^{-ip_1} - e^{ip_2} - e^{-ip_2} = E(p_1) + E(p_2). \tag{3.8}$$

Likewise, the S-matrix is found by plugging (3.6) into (3.5b)

$$s(p_1, p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_1} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} + 1} = s(p_2, p_1)^{-1}, \tag{3.9}$$

[2]We could in principle write down the Hamiltonian as a $2^L \times 2^L$ block matrix with blocks of size $\binom{L}{M} \times \binom{L}{M}$ with M excitations. This would however quickly become computationally challenging since the diagonalization complexity goes like $\mathcal{O}(L^3)$ [22].

which for real momenta satisfies $|s|^2 = 1$, and the probability is thus preserved in scattering events. Then, by imposing the periodic boundary condition $\psi(x_1, x_2) = \psi(x_2, x_1 + L)$ the momenta becomes quantized and can be determined by the two Bethe equations

$$s(p_1, p_2) = e^{ip_1 L}, \qquad s(p_2, p_1) = e^{ip_2 L}. \tag{3.10}$$

So far, so good. Moving on to the M-body problem, we write the general M-body magnon state as a superposition of all M-excitation states specified by M momenta p_i,

$$|\{p_i\}\rangle = \sum_{1 \le x_1 < \ldots < x_M \le L} \psi(x_1, \ldots x_M) \prod_n^M a^\dagger_{x_n} |0\rangle . \tag{3.11}$$

The Bethe ansatz for the wave function then generalizes to[3]

$$\psi(x_1, \ldots, x_M) = \sum_{\{\tau\}} A(\tau) \prod_{i=1}^M e^{ip_{\tau_i} x_i}, \tag{3.12}$$

where $\{\tau\}$ is the set of all possible permutations of the M excitations. The principle of non-diffractive scattering states that the scattering factorizes into two-body scatterings and the amplitudes $A(\tau)$ are related to each other such that if two orderings τ_1 and τ_2 differ only by one exchange of two neighboring momenta p_1 and p_2, then their corresponding amplitudes are related by the two-body scattering matrix $s(p_1, p_2)$ [23]

$$\frac{A_2}{A_1} = s(p_1, p_2). \tag{3.13}$$

This then fixes A up to some overall normalization[4]

$$A(\tau) = \mathrm{sgn}(\tau) \prod_{i<j} (2 - e^{-ip_{\tau_i}} - e^{-ip_{\tau_j}}), \tag{3.14}$$

where $\mathrm{sgn}(\tau)$ is the signature of the permutation. The periodic boundary condition $\psi(x_1, \ldots x_M) = \psi(x_2, \ldots, x_M, x_1 + L)$ again quantizes the momenta such that the total phase factor picked up by a magnon when going around the chain is equal to the product of scattering with all of the other $M - 1$ magnons

$$e^{ip_k L} = \prod_{j \neq k}^M s(p_k, p_j). \tag{3.15}$$

Then by introducing Bethe roots $u_k = \frac{1}{2} \cot \frac{p_k}{2}$, we obtain the celebrated Bethe equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}. \tag{3.16}$$

Solving the eigenvalue equation $E |\{p_i\}\rangle = H |\{p_i\}\rangle$ again leads to an equation of the form (3.7) and the energy is thus given by

$$E = \sum_k^M E(p_k) = \sum_k^M 4 \sin^2 \frac{p_k}{2} = \sum_k^M \frac{4}{1 + \cot^2 \frac{p_k}{2}} = \sum_k^M \frac{1}{u_k^2 + \frac{1}{4}}. \tag{3.17}$$

[3] The two-body wave function has been normalized such that $A(1, 2) = 1$

[4] We see that since the wave function is anti-symmetric under the exchange of two momenta only solutions with distinct momenta are allowed.

The cyclicity of the trace demands us to identify spin chain configurations that are cyclic permutations of each other. We will thus only consider the subset of solutions to the Bethe equations that are invariant under discrete translations given by the shift operator

$$e^{iP} = U = \mathbb{P}_{1,2} \dots \mathbb{P}_{L-1,L}, \tag{3.18}$$

which moves all fields one lattice site to the right and the last field to the first site. It commutes with the Hamiltonian and has the property $U^L = \mathbb{1}$ and hence the possible eigenvalues of U are

$$u = e^{2\pi i n/L}, \qquad n = 0, 1, \dots, L-1. \tag{3.19}$$

The translation invariant solutions are those where the eigenvalue of the shift operator is one and the total momentum is zero. This leads to the additional requirement

$$1 = \prod_k^M e^{ip_k} = \prod_k^M \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}. \tag{3.20}$$

Instead of starting with the two scalar fields, it is possible to construct the $\mathfrak{su}(2)$ sector using two fermions instead. This would change all signs in the permutation operator and correspond to an antiferromagnetic spin chain. Doing this turns out to be equivalent in the sense that the Bethe ansatz still solves the problem and yields the same Bethe equations (3.16). The only difference is that the energies are now given by

$$E = 2L - \sum_i \sin \frac{p_i}{2}, \tag{3.21}$$

and the reference state $|X^L\rangle$ is thus no longer the ground state. As we shall see, this equivalence is also apparent in the other sectors of the theory.

3.2 The $\mathfrak{su}(1|1)$ Fermionic Sector

In the $\mathfrak{su}(1|1)$ sector of $\mathcal{N} = 4$ SYM theory, the composite operators consist of one scalar and one fermion. This also has a spin chain interpretation in which we consider the vacuum to consist only of scalars denoted by $|0\rangle = |\uparrow^L\rangle$ and fermions to be excitations denoted by $\downarrow$. The Hamiltonian is the same as before (3.1) but with a graded permutation operator $\Pi_{x,x+1}$ that permutes the fields at position x and $x+1$ but picks up a minus sign when permuting two fermions. This can again be written in terms of Pauli matrices

$$H = \sum_{x=1}^{L} \left(\mathbb{1}_2^{\otimes L} - \Pi_{x,x+1} \right), \quad \Pi_{x,x+1} = \frac{1}{2}(\sigma_x^3 \sigma_{x+1}^0 + \sigma_x^0 \sigma_{x+1}^3 + \sigma_x^1 \sigma_{x+1}^1 + \sigma_x^2 \sigma_{x+1}^2), \tag{3.22}$$

The solution to the one magnon states are again plane waves and the two-body states are again written as

$$|p_1, p_2\rangle = \sum_{1 \le x_1 < x_2 \le L} \psi(x_1, x_2) \, |\dots \uparrow\downarrow\uparrow \dots \uparrow\downarrow\uparrow \dots\rangle \, , \tag{3.23}$$

where the fermions are spin down. From the eigenvalue equation we still get (3.5a) from before, and the energy is thus the same, but now (3.5b) becomes

$$E\psi(x_1, x_2) = 4\psi(x_1, x_2) - \psi(x_1 - 1, x_2) - \psi(x_1, x_2 + 1) \quad \text{for } x_2 = x_1 + 1. \tag{3.24}$$

Consequently, the S-matrix is now given by

$$s^f(p_1, p_2) = -1, \tag{3.25}$$

and the fermions can therefore be interpreted as freely propagating in a bosonic background. Imposing the fermionic boundary condition $\psi(x_1, x_2) = -\psi(x_2, x_1 + L)$ the quantization condition for the momenta becomes

$$e^{ip_1 L} = -s^f(p_1, p_2) = 1 \Longrightarrow p_k = \frac{2\pi n_k}{L}, \quad n_k \in \mathbb{Z}. \tag{3.26}$$

This again generalizes to M-magnon solutions by the principle of non-diffractive scattering, and the wave functions are thus given by the Slater determinant

$$\psi(x_1, \dots x_M) = \sum_{\{\tau\}} \mathrm{sgn}(\tau) \prod_{i=1}^{M} e^{ip_{\tau_i} x_i}, \tag{3.27}$$

from which we again see that we need the M momenta to be distinct in order for the wave function not to vanish.

3.3 The Scattering Matrix of the $\mathfrak{su}(1|2)$ Spin Chain

The $\mathfrak{su}(1|2)$ sector of $\mathcal{N} = 4$ SYM theory is slightly more involved than the $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sector since the composite operators consist of two scalar fields Z and X and a fermionic field U. We consider operators of the form

$$\mathrm{tr}[ZXU \dots XZU] \to |ZXU \dots XZU\rangle \,, \tag{3.28}$$

with J_1 factors of X, J_2 factors of U and J_3 factors of Z. Again, the dilatation operator has a spin chain interpretation in which we consider the vacuum to be $|0\rangle = \mathrm{tr}[Z^L]$ and X to be a bosonic excitation and U to be a fermionic excitation. The Hamiltonian is then given by

$$H = \sum_{x=1}^{L} (\mathbb{1}_2^{\otimes L} - \Pi_{x,x+1}), \tag{3.29}$$

with the permutation operator Π being graded such that the exchange of two fermions picks up a minus sign. In a similar manner to what we did in the $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors, the permutation operator can be written as a 9×9 matrix using the eight Gell-Mann matrices with respect to the basis $\{ZZ, ZX, ZU, XZ, XX, XU, UZ, UX, UU\}$

$$\Pi_{x,x+1} = \frac{1}{2} \sum_{i=1}^{7} \lambda^i \otimes \lambda^i + \frac{1}{9} \lambda^0 \otimes \lambda^0 + \frac{2}{3\sqrt{3}} \left(\lambda^0 \otimes \lambda^8 + \lambda^8 \otimes \lambda^0 \right) - \frac{2}{3} \lambda^8 \otimes \lambda^8, \tag{3.30}$$

with $\lambda^0 = \mathbb{1}_3$. Once again the one-magnon solutions are simple plane waves. The two-magnon states can be written as a superposition of states with two excitations

$$|\xi\eta\rangle = \sum_{x_1 < x_2} \psi_{\xi\eta}(x_1, x_2) a^\dagger_{\xi,x_1} a^\dagger_{\eta,x_2} |0\rangle \,, \tag{3.31}$$

where ξ and η are either X or U and $\psi_{\xi\eta}(x_1, x_2)$ are the four wave functions we need to determine. In terms of Gell-Mann matrices the creation operators are

$$a^\dagger_{X,x} = \frac{\lambda^1_x - i\lambda^2_x}{2}, \qquad a^\dagger_{U,x} = \frac{\lambda^4_x - i\lambda^5_x}{2}. \tag{3.32}$$

The $|XX\rangle$ and $|UU\rangle$ states are already eigenstates of H with the wave functions found in the $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors,

$$\psi_{XX}(x_1,x_2) = e^{i(p_1x_1+p_2x_2)} + s(p_1,p_2)e^{i(p_1x_2+p_2x_1)}, \tag{3.33a}$$

$$\psi_{UU}(x_1,x_2) = e^{i(p_1x_1+p_2x_2)} + s^f(p_1,p_2)e^{i(p_1x_2+p_2x_1)}, \tag{3.33b}$$

where $s(p_1,p_2)$ is the scattering matrix for $\mathfrak{su}(2)$ given by (3.9) and $s^f(p_1,p_2) = -1$ the scattering matrix for $\mathfrak{su}(1|1)$. Since the permutation operator mixes the magnon states $|XU\rangle$ and $|UX\rangle$, the $\mathfrak{su}(1|2)$ sectors becomes a little more involved. The idea is to divide the states $|XU\rangle$ and $|UX\rangle$ into two parts each with opposite momenta

$$|XU\rangle = A_{\text{in}}f(x_1,x_2;p_1,p_2) + A_{\text{out}}f(x_1,x_2;p_2,p_1), \tag{3.34a}$$

$$|UX\rangle = B_{\text{in}}f(x_1,x_2;p_1,p_2) + B_{\text{out}}f(x_1,x_2;p_2,p_1). \tag{3.34b}$$

The two momenta should be separately conserved if the system is integrable and thus the first term in (3.34a) where the momenta is "ingoing" the momenta can either be reflected resulting in the second term of (3.34a) or they can pass each other resulting in the second term of (3.34b). Thus, the Bethe ansatz is that the wave functions are also divided into an ingoing and and outgoing part

$$\psi_{XU}(x_1,x_2) = A_{\text{in}}e^{i(p_1x_1+p_2x_2)} + A_{\text{out}}e^{i(p_1x_2+p_2x_1)}, \tag{3.35a}$$

$$\psi_{UX}(x_1,x_2) = B_{\text{in}}e^{i(p_1x_1+p_2x_2)} + B_{\text{out}}e^{i(p_1x_2+p_2x_1)}, \tag{3.35b}$$

which will mix under scattering. The two-body magnon state can be written

$$|\{XU\}\rangle = \sum_{x_1<x_2} \psi_{XU}(x_1,x_2)a^\dagger_{X,x_1}a^\dagger_{U,x_2}|0\rangle + \psi_{UX}(x_1,x_2)a^\dagger_{U,x_1}a^\dagger_{X,x_2}|0\rangle, \tag{3.36}$$

and by comparing the coefficients in the eigenvalue equation $H|\{XU\}\rangle = E|\{XU\}\rangle$ for $x_2 > x_1+1$ we obtain

$$\begin{aligned} E\psi_{XU}(x_1,x_2) = 4\psi_{XU}(x_1,x_2) - \psi_{XU}(x_1-1,x_2) - \psi_{XU}(x_1+1,x_2) \\ - \psi_{XU}(x_1,x_2-1) - \psi_{XU}(x_1,x_2+1), \end{aligned} \tag{3.37}$$

and a similar equation with $(X \leftrightarrow U)$. By inserting (3.35a) into (3.37), we obtain

$$E\psi_{XU}(x_1,x_2) = (4 - e^{-ip_1} - e^{-ip_2} - e^{+ip_1} - e^{+ip_2})\psi_{XU}(x_1,x_2), \tag{3.38}$$

and similarly for ψ_{UX}. This is the same we got for the $\mathfrak{su}(2)$ spin chain. Looking at the eigenvalue equation for $x_2 = x_1+1$, we get a mixing between ψ_{XU} and ψ_{UX}

$$E\psi_{XU}(x_1,x_2) = 3\psi_{XU}(x_1,x_2) - \psi_{UX}(x_1,x_2) - \psi_{XU}(x_1-1,x_2) - \psi_{XU}(x_1,x_2+1), \tag{3.39}$$

and a similar equation with $(X \leftrightarrow U)$. Inserting the ansatze (3.35a) and (3.35b) into (3.39) and using (3.7), we obtain the algebraic equations

$$A_{\text{in}}\left(e^{ip_2} - e^{i(p_2+p_1)} - 1\right) + A_{\text{out}}\left(e^{ip_1} - e^{i(p_2+p_1)} - 1\right) = B_{\text{in}}e^{ip_2} + B_{\text{out}}e^{ip_1}, \tag{3.40a}$$

$$B_{\text{in}}\left(e^{ip_2} - e^{i(p_2+p_1)} - 1\right) + B_{\text{out}}\left(e^{ip_1} - e^{i(p_2+p_1)} - 1\right) = A_{\text{in}}e^{ip_2} + A_{\text{out}}e^{ip_1}. \tag{3.40b}$$

By relating $A_{\text{out}} = tB_{\text{in}} + rA_{\text{in}}$ where t is the amplitude of transmission and r the amplitude of reflection, we obtain

$$t(p_1,p_2) = \frac{e^{ip_1} - e^{ip_2}}{1 - 2e^{ip_2} + e^{ip_1+ip_2}}, \qquad r(p_1,p_2) = -\frac{(1-e^{ip_1})(1-e^{ip_2})}{1 - 2e^{ip_1} + e^{ip_1+ip_2}}, \tag{3.41}$$

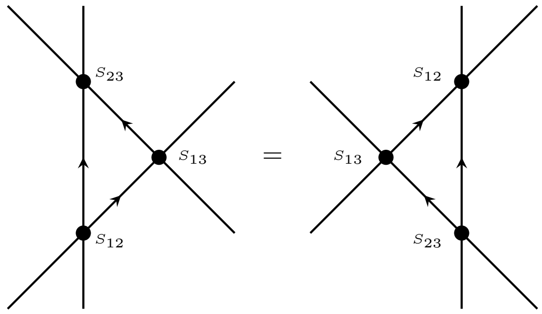


Figure 3.1: The S-matrices satisfy the Yang-Baxter Algebra stating that the order in which the magnons scatter can be reversed.

and because of the symmetry the same relation holds with $(A \leftrightarrow B)$ and (in $\leftrightarrow$ out). Consequently, the scattering matrix for the states $\{|XX\rangle\,, |XU\rangle\,, |UX\rangle\,, |UU\rangle\}$ is given by

$$S(p_1, p_2) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & r & 0 \\ 0 & r & t & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.42}$$

For the system to be integrable and for us to invoke the principle of non-diffractive scattering, a necessary condition is that three neighboring states satisfy the Yang-Baxter equation

$$\mathcal{S}_{1,2}(p_1,p_2)\mathcal{S}_{1,3}(p_1,p_3)\mathcal{S}_{2,3}(p_2,p_3) = \mathcal{S}_{2,3}(p_2,p_3)\mathcal{S}_{1,3}(p_1,p_3)\mathcal{S}_{1,2}(p_1,p_2). \tag{3.43}$$

This says that the order in which the scattering is taken can be reversed as depicted in fig (3.1). Because there are three magnons involved in this, we consider the scattering matrices to be acting on the eight dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and by using the basis

$$\{\ |\,XXX\rangle,\ |\,XXU\rangle,\ |\,XUX\rangle,\ |\,XUU\rangle,\ |\,UXX\rangle,\ |\,UXU\rangle,\ |\,UUX\rangle,\ |\,UUU\rangle\,\}\,, \tag{3.44}$$

the matrices are given by

$$\mathcal{S}_{1,2} = S_{1,2} \otimes \mathbb{1}_2, \quad \mathcal{S}_{1,3} = \begin{pmatrix} s' & & & & & & & \\ & t' & & & r' & & & \\ & & s' & & & & & \\ & & & t' & & & r' & \\ & r' & & & t' & & & \\ & & & & & -1 & & \\ & & & r' & & & t' & \\ & & & & & & & -1 \end{pmatrix}, \quad \mathcal{S}_{2,3} = \mathbb{1}_2 \otimes S_{2,3}, \tag{3.45}$$

with $s = s_{1,2}$, $s' = s_{1,3}$, $s'' = s_{2,3}$, and so forth, such that each matrix acts on two of the states while leaving the last alone. Then the Yang-Baxter equation yields

$$ts'r'' - st'r'' + rr't'' = 0, \tag{3.46a}$$
$$rs'r'' - sr's'' + tr't'' = 0, \tag{3.46b}$$
$$tr'r'' - rt's'' + rs't'' = 0, \tag{3.46c}$$
$$-tr'r'' - rt' + rt'' = 0, \tag{3.46d}$$
$$-tr't'' + r' + rr'' = 0, \tag{3.46e}$$
$$-r''t' - rr't'' + tr'' = 0, \tag{3.46f}$$

which can be verified by inserting (3.41) and (3.9).

3.4 The Nested Bethe Ansatz

So far we only know the two-body solution (3.36) to the $\mathfrak{su}(1|2)$ spin chain as well as the solutions corresponding to the subsectors $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$. To find solutions with an arbitrary number of either type of excitation, we employ the Nested Bethe Ansatz and follow [24]. Having shown that the S-matrix satisfy the Yang-Baxter equation, we employ the principle of non-diffractive scattering. The starting point is then that the total momentum of a magnon p_k should be equal to the phase picked up by scattering with all the other magnons around the spin chain. If we consider spin chains of length L with a total of M_1 excitations of which $M_1 - M_2$ are scalar excitations and M_2 are the fermionic excitations, the Bethe equation is then

$$e^{ip_k L} |M_1, M_2, \{p_j\}\rangle = S_{k,k+1} \dots S_{k,M_1} S_{k,1} \dots S_{k,k-1} |M_1, M_2, \{p_j\}\rangle\,, \tag{3.47}$$

where $S_{i,j}$ are no longer numbers but the matrices (3.42) acting on the sites i and j. The trick is then to introduce a "short" spin chain $|\Psi\rangle$ where all the vacuum fields Z have been omitted and where the $\mathfrak{su}(2)$ state $|X^{M_1}\rangle$ is considered as a new pseudo-vacuum described by the M_1 ordinary Bethe roots $\{u_i\}$ satisfying the $\mathfrak{su}(2)$ Bethe equations. This chain will then have length M_1 of which M_2 are U-fields which are excitations of this chain build upon the already excited states. The Bethe equation for this short spin chain then becomes

$$\lambda_k |\Psi\rangle = \tilde{S}_{k,k+1} \dots \tilde{S}_{k,M_1} \tilde{S}_{k,1} \dots \tilde{S}_{k,k-1} |\Psi\rangle\,, \tag{3.48}$$

where we have defined the reduced S-matrix

$$\tilde{S}_{i,j} = s_{i,j}^{-1} S_{i,j}, \tag{3.49}$$

such that the eigenvalue becomes

$$\lambda_k = e^{ip_k L} s_{k+1,k} \dots s_{M_1,k} s_{1,k} \dots . s_{k-1,k} = \left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \prod_{k\neq j}^{M_1} \frac{u_j - u_k - i}{u_j - u_k + i}. \tag{3.50}$$

So far this is just convenient notation, and we still have to find the states that satisfy (3.47). As usual, we start with the one-magnon short spin chain which is not translation invariant since the short spin chain is embedded in the original spin chain and the Fourier transformation will therefore not solve the problem. Instead, $|\Psi\rangle$ can be written in terms of coordinate space wave functions

$$|\Psi\rangle = \sum_{x=1}^{M_1} \psi_x |x\rangle\,, \tag{3.51}$$

where $|x\rangle$ is the state were one of the spins has been "flipped" from an X to a U excitation at site x. We can then determine the wave function ψ_x by a recurrence relation. Letting $\tilde{S}_{k,k-1}$ act on $|\Psi\rangle$ we obtain

$$\tilde{S}_{k,k-1} |\Psi\rangle = \left(\psi_k \tilde{t}_{k,k-1} + \psi_{k-1} \tilde{r}_{k,k-1}\right) |k\rangle + \left(\psi_{k-1} \tilde{t}_{k,k-1} + \psi_k \tilde{r}_{k,k-1}\right) |k-1\rangle + \sum_{x\neq k,k-1} \psi_x |x\rangle\,, \tag{3.52}$$

where $\tilde{t} = s^{-1}t$ and $\tilde{r} = s^{-1}r$ are the reduced transmission and reflection amplitudes. Since none of the remaining scattering matrices are going to affect the coefficient of $|k-1\rangle$, we see that

$$\lambda_k \psi_{k-1} = \psi_{k-1} \tilde{t}_{k,k-1} + \psi_k \tilde{r}_{k,k-1}. \tag{3.53}$$

To simplify notation, we introduce the iterated wave function

$$\psi_k^{(1)} \equiv \psi_k \tilde{t}_{k,k-1} + \psi_{k-1}\tilde{r}_{k,k-1}, \tag{3.54}$$

which will also be useful when we later need to flip more spins. By acting with the next scattering matrix $\tilde{S}_{k,k-2}$ we obtain

$$\tilde{S}_{k,k-2}\tilde{S}_{k,k-1}\left|\Psi\right\rangle = \left(\psi_k^{(1)}\tilde{t}_{k,k-2} + \psi_{k-2}\tilde{r}_{k,k-2}\right)\left|k\right\rangle + \left(\psi_{k-2}\tilde{t}_{k,k-2} + \psi_k^{(1)}\tilde{r}_{k,k-2}\right)\left|k-2\right\rangle + \ldots . \tag{3.55}$$

Again, no other S-matrix will affect the coefficient of $|k-2\rangle$, so we obtain

$$\lambda_k\psi_{k-2} = \psi_{k-2}\tilde{t}_{k,k-2} + \tilde{r}_{k,k-2}\psi_k^{(1)}. \tag{3.56}$$

This pattern continues and after the j^{th} iteration the coefficient of $|k\rangle$ is given by the iterated wave function

$$\psi_k^{(j)} = \psi_k^{(j-1)}\tilde{t}_{k,k-j} + \psi_{k-j}\tilde{r}_{k,k-j}, \tag{3.57}$$

which is ready for the next iteration while the coefficient of $|k-j\rangle$ will no longer change. This gives the eigenvalue equation

$$\lambda_k\psi_{k-j} = \psi_{k-j}\tilde{t}_{k,k-j} + \psi_k^{(j-1)}\tilde{r}_{k,k-j}. \tag{3.58}$$

The rest of the coefficients are left unchanged. By combining the preparation identity (3.57) with the eigenvalue equation (3.58), we get the recurrence relation

$$\frac{\psi_{k-j-1}}{\psi_{k-j}} = \frac{\tilde{r}^2_{k,k-j} - \tilde{t}^2_{k,k-j} + \lambda_k\tilde{t}_{k,k-j}}{\lambda_k - \tilde{t}_{k,k-j-1}}\frac{\tilde{r}_{k,k-j-1}}{\tilde{r}_{k-j}}, \tag{3.59}$$

for $k-j = 2\ldots M_1$. To clean up the expression, the scattering amplitudes can be rewritten in terms of Bethe roots,

$$s_{i,j} = \frac{u_i - u_j + i}{u_i - u_j - i}, \quad t_{i,j} = \frac{u_i - u_j}{u_i - u_j - i}, \quad r_{i,j} = \frac{i}{u_i - u_j - i}, \tag{3.60}$$

$$\Delta_{i,j} \equiv -s_{i,j}^{-1} = -\frac{u_i - u_j - i}{u_i - u_j + i}, \quad \tilde{t}_{i,j} = \frac{u_i - u_j}{u_i - u_j + i}, \quad \tilde{r}_{i,j} = \frac{i}{u_i - u_j + i}, \tag{3.61}$$

where $\Delta_{i,j}$ is the reduced scattering matrix of two fermions. The recurrence relation then becomes

$$\frac{\psi_{k-j-1}}{\psi_{k-j}} = \frac{u_k - u_{k-j} - \frac{i}{1-\lambda_k}}{u_k - u_{k-j-1} + \frac{i}{1-\lambda_k}} = \frac{v - u_{k-j} - \frac{i}{2}}{v - u_{k-j-1} + \frac{i}{2}}, \tag{3.62}$$

where we have defined the auxiliary Bethe root

$$v = u_k - \frac{i}{2}\left(\frac{1+\lambda_k}{1-\lambda_k}\right), \tag{3.63}$$

which must be a constant since the LHS only depends on the difference $k-j$. Choosing the normalization such that $\psi_1 = 1$, the solution to (3.62) is

$$\psi_k(v) = \prod_{j=1}^{k-1}\frac{u_j - v - \frac{i}{2}}{u_{j+1} - v + \frac{i}{2}} = \frac{u_1 - v + \frac{i}{2}}{u_k - v + \frac{i}{2}}\lambda_1\ldots\lambda_{k-1}, \tag{3.64}$$

where we used that the eigenvalues can be expressed in terms of the auxiliary root v

$$\lambda_k(v) = \frac{u_k - v - \frac{i}{2}}{u_k - v + \frac{i}{2}}. \tag{3.65}$$

Similarly, we can express the iterated wave function in terms of the eigenvalues

$$\psi_k^{(j)}(v) = \frac{u_{k-j-1} - v - \frac{i}{2}}{u_k - v + \frac{i}{2}} \psi_{k-j-1}(v) = \frac{\psi_k(v)}{\lambda_{k-1} \dots \lambda_{k-j}}, \tag{3.66}$$

which is just what we need in order to match the last amplitude of $|k\rangle$ with the eigenvalue equation such that

$$\lambda_k \psi_k = \psi_k^{(M_1-1)} = \frac{\psi_k}{\lambda_{k-1} \dots \lambda_{k-M_1+1}}. \tag{3.67}$$

This leads to the Bethe equation

$$1 = \prod_k^{M_1} \lambda_k = \prod_k^{M_1} \frac{u_k - v - \frac{i}{2}}{u_k - v + \frac{i}{2}} = \prod_k^{M_1} \frac{\tilde{v}_k + \frac{i}{2}}{\tilde{v}_k - \frac{i}{2}} = \prod_k e^{i\tilde{p}_k}, \tag{3.68}$$

where we introduced the M_1 shifted auxiliary Bethe roots $\tilde{v}_k = v - u_k$ as well as the shifted momentum $\tilde{v}_k = \frac{1}{2}\cot\frac{\tilde{p}_k}{2}$ which we see has a similar quantization to the Bethe root of the one-magnon problem of the $\mathfrak{su}(2)$ sector. This confirms the interpretation of the nested chain having magnons of its own propagating on top of the $\mathfrak{su}(2)$ spin chain. Inserting the eigenvalues into (3.50) we get the additional M_1 Bethe equations for our momentum-carrying roots

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L = \frac{u_j - v - \frac{i}{2}}{u_j - v + \frac{i}{2}} \prod_{k \neq j}^{M_1} \frac{u_j - u_k + i}{u_j - u_k - i}. \tag{3.69}$$

This completely solves the $M_2 = 1$ short spin chain.

Before moving on to the two-magnon problem on the short spin chain, it is instructive to recap what we did. After the j^{th} iteration the amplitudes of $|k-j\rangle, \dots, |k\rangle$ have been assigned an eigenvalue and therefore effectively been multiplied by λ_k applying the eigenvalue equation

$$\lambda_k \psi_{k-j} = \tilde{t}_{k,k-j} \psi_{k-j} + \tilde{r}_{k,k-j} \frac{\psi_k}{\lambda_{k-1} \dots \lambda_{k-j-1}}. \tag{3.70}$$

Meanwhile the repeated use of the recurrence relation,

$$\frac{\psi_k}{\lambda_{k-1} \dots \lambda_{k-j}} = \tilde{t}_{k,k-j} \frac{\psi_k}{\lambda_{k-1} \dots \lambda_{k-j+1}} + \tilde{r}_{k,k-j} \psi_{k-j}, \tag{3.71}$$

has prepared the amplitude of $|k\rangle$ for the $(j+1)^{\text{th}}$ iteration while the amplitudes to the left of $|k-j\rangle$ has been left unchanged. Going through all the scattering matrices the only thing left to do is match the coefficient of $|k\rangle$ with the eigenvalue equation.

We now move on to the two-magnon state of the short spin chain,

$$|\Psi\rangle = \sum_{1 \leq x_1 < x_2 \leq M_1} \psi_{x_1,x_2}(v, v') \, |x_1, x_2\rangle, \tag{3.72}$$

which is parameterized by two auxiliary roots v, v'. To solve this we try for a nested Bethe ansatz

$$\psi_{x_1,x_2}(v, v') = B\psi_{x_1}(v)\psi_{x_2}(v') - B'\psi_{x_1}(v')\psi_{x_2}(v) \equiv B\psi_{x_1}\psi'_{x_2} - \text{conj.}, \tag{3.73}$$

where we have introduced the short hand notations $\psi_{x_i}(v) = \psi_{x_i}$, $\psi_{x_i}(v') = \psi'_{x_i}$ as well as a conjugation operation that changes primed quantities into unprimed and vice versa. The hope is that ψ_{x_i} and ψ'_{x_i} satisfy the recurrence relations (3.71, 3.70) when acted upon by the scattering matrix such that $S_{k-j,k}$ prepares ψ_k for the $(j+1)^{\text{th}}$ iteration by dividing it with λ_{k-j} according to (3.71) while it fixes ψ_{k-j} by multiplying it with λ_k according to (3.70). After acting with the first scattering matrix of (3.48), $S_{k-1,k}$, only the amplitudes of states with U-fields at $k-1$ or k are affected. Using (3.70) the amplitude of the states $|k-x, k-1\rangle$ for $x > 1$ is

$$B\psi_{k-x}\psi'_{k-1}\lambda'_k - \text{conj.}, \tag{3.74}$$

and the amplitude of the state $|k-1, k\rangle$ is

$$\Delta_{k-1,k}B\psi_{k-1}\psi'_k - \text{conj.}, \tag{3.75}$$

due to the exchange of two fermions. However, we wish it had been

$$B\psi'_{k-1}\lambda'_k\frac{\psi_k}{\lambda_{k-1}} - \text{conj.}, \tag{3.76}$$

such that ψ_{k-1} and ψ'_{k-1} had been multiplied with λ_k and λ'_k respectively as in the case with one magnon while ψ_k and ψ'_k had been prepared for the next iteration by being divided by λ_{k-1} and λ'_{k-1} respectively according to (3.66). Asserting that this is the case, then after the n^{th} iteration the amplitude of $|k-n, k-n+1\rangle$ is

$$\lambda_k\lambda'_k\left(B\psi_{k-n}\psi'_{k-n+1} - \text{conj.}\right), \tag{3.77}$$

and we therefore have that the two-magnon eigenvalue factorizes

$$\lambda_k^{(2)}(v, v') = \lambda_k(v)\lambda_k(v'). \tag{3.78}$$

After the j^{th} iteration, the amplitude of the state $|k-j-x, k-j\rangle$ is

$$B\psi_{k-j-x}\psi'_{k-j}\lambda'_k - \text{conj.}, \tag{3.79}$$

and the amplitude of $|k-j, k\rangle$ is

$$\Delta_{k-j,k}B\psi_{k-j}\frac{\psi'_k}{\lambda'_{k-1}\ldots\lambda'_{k-j+1}} - \text{conj.} \tag{3.80}$$

Again, we want to be through with ψ_{k-j} and ψ'_{k-j} while preparing ψ_k and ψ'_k for the next iteration and therefore wish that the amplitude of $|k-j, k\rangle$ had been

$$B\psi'_{k-j}\lambda'_k\frac{\psi_k}{\lambda_{k-1}\ldots\lambda_{k-j}} - \text{conj.} \tag{3.81}$$

For this to be the case, we hope to satisfy the consistency equation

$$\frac{B'}{B} = \frac{\Delta_{k-j,k}\lambda'_{k-j}\psi'_k\psi_{k-j}\lambda_{k-1}\ldots\lambda_{k-j} - \psi_k\psi'_{k-j}\lambda'_k\ldots\lambda'_{k-j}}{\text{conj.}}. \tag{3.82}$$

By inserting the expression for ψ_k and λ_k and by reducing by the common factor $\lambda_1\ldots\lambda_{j-1}\lambda'_1\ldots\lambda'_{j-1}\left(u_1 - v + \frac{i}{2}\right)\left(u_1 - v' + \frac{i}{2}\right)$ which is invariant under conjugation, this becomes

$$\frac{B'}{B} = \frac{(u_{j-k} - u_j - i)\left(u_j - v - \frac{i}{2}\right)\left(u_{j-k} - v' + \frac{i}{2}\right) + (u_{j-k} - u_j + i)\left(u_j - v' + \frac{i}{2}\right)\left(u_{j-k} - v - \frac{i}{2}\right)}{\text{conj.}}. \tag{3.83}$$

The numerator turns out also to be invariant under conjugation and consequently we have that the consistency equation becomes

$$\frac{B'}{B} = 1. \tag{3.84}$$

A solution to the two-magnon wave function for the short spin chain is thus given by

$$\psi_{x_1,x_2}(v,v') = \psi_{x_1}(v)\psi_{x_2}(v') - \psi_{x_1}(v')\psi_{x_2}(v), \tag{3.85}$$

with the eigenvalue given by (3.78). The only thing left to do is to impose the fermionic periodic boundary condition $\psi_{x_1,x_2}(v,v') = -\psi_{x_2,x_{1+M_1}}(v,v')$ such that the amplitudes match in the ends. Again, this quantizes the auxiliary roots

$$\prod_{k=1}^{M_1} \lambda_k = \prod_{k=1}^{M_1} \frac{u_k - v - \frac{i}{2}}{u_k - v + \frac{i}{2}} = \prod_{k=1}^{M_1} e^{i\tilde{p}_{1,k}} = 1, \tag{3.86}$$

and similarly for λ'_k and v'. Here $\tilde{p}_{1,k}$ are the shifted momentum with respect to v. This procedure readily generalizes to M_2 nested excitations with the amplitudes of the short spin chain being quantized by M_2 Bethe roots v_i

$$\psi(v_1,\ldots,v_{M_2}) = \sum_{\{\tau\}} B(\tau) \prod_{i=1}^{M_2} \psi_{x_1}(v_{\tau_i}). \tag{3.87}$$

By once again requiring that $S_{k-j,k}$ multiplies ψ_{k-j} by λ_k while it prepares ψ_j for the next iteration, we get a consistency equation that is satisfied by the Slater determinant

$$\psi(v_1,\ldots,v_{M_2}) = \sum_{\{\tau\}} \operatorname{sgn}(\tau) \prod_{i=1}^{M_2} \psi_{x_1}(v_{\tau_i}) = \det \psi_{x_i}(v_j), \tag{3.88}$$

and again the eigenvalues will factorize

$$\lambda_k^{(M_2)} = \prod_{i=1}^{M_2} \lambda_k(v_i). \tag{3.89}$$

Writing this in terms of Bethe roots we get the $M_1 + M_2$ Bethe equations

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L = \prod_{k\neq j}^{M_1} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{M_2} \frac{u_j - v_k - \frac{i}{2}}{u_j - v_k + \frac{i}{2}}, \tag{3.90a}$$

$$1 = \prod_{k=1}^{M_1} \frac{v_j - u_k - \frac{i}{2}}{v_j - u_k + \frac{i}{2}}, \tag{3.90b}$$

which completely solve the $\mathfrak{su}(1|2)$ spin chain.

3.5 Dualities and Equivalent Sectors

When solving the $\mathfrak{su}(1|1)$ sector, we chose the scalars to constitute our vacuum while the fermions were considered as excitations upon these, but we might as well have chosen it the other way around. Likewise, there are multiple choices for the $\mathfrak{su}(1|2)$ sector and as we will see later this choice corresponds to the choice one has when choosing simple roots in the corresponding Lie superalgebra. The different choices come with different physical interpretations and lead to different sets of Bethe equations. For instance, our choice with fermionic excitations as the nested excitations is represented by the Dynkin diagram ⋯○⋯⊗— which leads to the Bethe equations (3.90a-3.90b). Had we instead chosen a fermionic field for our first level of excitations followed by a bosonic field for our second level corresponding to the Dynkin diagram —⊗—⊗⋯, we would have obtained the somewhat more simple Bethe equations

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L=\prod_{k=1}^{M_2}\frac{u_j-v_k+\frac{i}{2}}{u_j-v_k-\frac{i}{2}},\qquad 1=\prod_{k=1}^{M_1}\frac{v_j-u_k+\frac{i}{2}}{v_j-u_k-\frac{i}{2}}. \tag{3.91}$$

Here, the physical interpretation is that there are M_1 fermions living on the chain of which we excite M_2 to be free bosons living on our short spin chain. Although the two physical interpretations and their corresponding sets of Bethe roots are different they are dual in the sense that they both solve the same problem namely diagonalizing the dilatation operator. The two sets of Bethe equations are in fact closely related to each other and as we will see in chapter 6 the one set can be obtained through a duality transformation of the other.

When solving the $\mathfrak{su}(2)$ sector, we noted that constructing the operators with two fermions was equivalent to two constructing it with scalars in the sense that it led to the exact same eigenstates only with different energies. Likewise, the solutions to the $\mathfrak{su}(2|1)$ sector where operators consist of two fermions and one scalar can be obtained from the $\mathfrak{su}(1|2)$ sector by swapping fermions and scalars. This will again change the sign of the permutation operator $\Pi_{x,x+1}$ yielding different energies but the same eigenstates.

By this equivalence, the set of Bethe equations (3.90a-3.90b) represented by the Dynkin diagram ⋯○⋯⊗— then correspond to choosing the vacuum to consist of fermions on which we then make bosonic excitations[5] followed by nested fermionic excitations on the short spin chain. The dual set of Bethe equations (3.91) represented by —⊗—⊗⋯ correspond to choosing a bosonic vacuum and both the first and second level of excitations to be fermionic.

[5] A bosonic excitation excites a fermion to another fermion or a scalar to another scalar, unlike the a fermionic excitation which excites a fermion to a scalar or a scalar to a fermion.

Chapter 4

The Algebraic Bethe Ansatz and the Norm

While the CBA gives a very physical intuition to solving the spectral problem, it does not explain why the theory is solvable in the first place. In this chapter, we will review the Algebraic Bethe Ansatz (ABA) first introduced by Faddeev in [12] which proves the integrability of the $\mathfrak{su}(2)$ spin chain. We then use the tools developed in the ABA to derive a norm function for the Bethe states which will be useful later when determining one-point functions in a defect version of $\mathcal{N} = 4$ SYM.

4.1 The Lax Operator and Integrability

The $\mathfrak{su}(2)$ spin chain of length L is as before defined on the Hilbert space $\mathcal{H} = \mathbb{C}^{2\otimes L}$ with the Hamiltonian given by (3.1).

Instead of trying to figure out the eigenstates right away as we did with the CBA, the key element in the algebraic approach is the so-called Lax operator and the introduction of the auxiliary spaces $V = \mathbb{C}^2$. The ABA is not as intuitive as the CBA approach but as we will see the Lax operator can be used to generate the Hamiltonian along with the rest of the L commuting operators proving the integrability of the system.

The Lax operator is parameterized by the complex spectral parameter λ and acts in the local quantum space of lattice site n and the auxiliary space labeled by a,

$$\mathbb{L}_{n,a}(\lambda) = \left(\lambda - \frac{i}{2}\right) \mathbb{1}_{n,a} + i\mathbb{P}_{n,a} = \begin{pmatrix} \lambda + \frac{i}{2}\sigma_n^3 & \frac{i}{2}\sigma_n^- \\ \frac{i}{2}\sigma_n^+ & \lambda - \frac{i}{2}\sigma_n^3 \end{pmatrix}, \tag{4.1}$$

which in the RHS was written as a matrix acting on the auxiliary space with entries acting on the local quantum space $\mathcal{H}_n$. Here $\sigma_n^{\pm} = \sigma_n^1 \pm i\sigma_n^2$ are the spin ladder operators that raises or lowers the spin at site n.

The hope is then, that for two Lax operators with two different auxiliary spaces the products $L_{n,a_1} L_{n,a_2}$ and $L_{n,a_2} L_{n,a_1}$ are similar in the quantum space and connected by an intertwiner R such that the satisfy they Yang-Baxter equation

$$R_{a_1,a_2}(\lambda - \mu)\mathbb{L}_{n,a_1}(\lambda)\mathbb{L}_{n,a_2}(\mu) = \mathbb{L}_{n,a_2}(\mu)\mathbb{L}_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu). \tag{4.2}$$

A such intertwiner does exist and is given by

$$R_{a_1,a_2}(\lambda) = \lambda \mathbb{1}_{a_1} \otimes \mathbb{1}_{a_2} + i\mathbb{P}_{a_1,a_2}, \tag{4.3}$$

which can easily be checked by using the cyclic property of the permutation operators $\mathbb{P}_{i,j}\mathbb{P}_{j,k} = \mathbb{P}_{j,k}\mathbb{P}_{k,i} = \mathbb{P}_{k,i}\mathbb{P}_{i,j}$.

The next step is then to build a monodromy of our chain given by the ordered product of the Lax operator acting on all lattice sites,

$$T_a(\lambda) = \mathbb{L}_{L,a}(\lambda)\ldots\mathbb{L}_{1,a}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{4.4}$$

The monodromy also satisfies the Yang-Baxter equation

$$R_{a_1,a_2}(\lambda-\mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu), \tag{4.5}$$

which follows from the fact that (4.2) is satisfied at each lattice site. Taking the partial trace over both the auxiliary spaces yields the commuting operators

$$[F(\lambda), F(\mu)] = 0, \qquad F(\lambda) = \mathrm{tr}_a\, T(\lambda) = A(\lambda) + D(\lambda). \tag{4.6}$$

Since $T(\lambda)$ is a polynomial of order L in λ of which the trace of the λ^{L-1} term is zero, the expansion in λ of $F(\lambda)$ yields a set of $L-1$ commuting operators Q_i,

$$F(\lambda) = 2\lambda^L + \sum_i^{L-2} Q_i\lambda^i. \tag{4.7}$$

As we will show, the Hamiltonian belongs to this set. It is clear from (4.1) that the point $\lambda = \frac{i}{2}$ is special. Here the monodromy becomes a simple string of permutation operators

$$T(i/2) = i^L\mathbb{P}_{L,a}\mathbb{P}_{L-1,a}\ldots\mathbb{P}_{1,a} = i^L\mathbb{P}_{1,2}\mathbb{P}_{2,3}\ldots\mathbb{P}_{L,a} = i^L U\mathbb{P}_{L,a}, \tag{4.8}$$

where we again used the cyclic property of the permutation operator as well as introducing the shift operator U from (3.18). The expansion of the monodromy in the vicinity of $\lambda = \frac{i}{2}$ is likewise simple and we have that

$$\frac{d}{d\lambda}T(\lambda)\Big|_{\lambda=i/2} = i^{L-1}\sum_n \mathbb{P}_{1,2}\ldots\mathbb{P}_{n-1,n+1}\ldots\mathbb{P}_{L,a}, \tag{4.9}$$

where each term in the sum is skipping lattice site n. Taking the partial trace in auxiliary space a, we have

$$\frac{d}{d\lambda}F(\lambda)\Big|_{\lambda=i/2} = i^{L-1}\sum_n \mathbb{P}_{1,2}\ldots\mathbb{P}_{n-1,n+1}\ldots\mathbb{P}_{L-1,L}. \tag{4.10}$$

Then, by multiplying with the inverse shift operator U^{-1} we get a familiar sum of permutation operators

$$\frac{d}{d\lambda}F(\lambda)F(\lambda)^{-1}\Big|_{\lambda=i/2} = \frac{d}{d\lambda}\log F(\lambda)\Big|_{\lambda=i/2} = -i\sum_n \mathbb{P}_{n,n+1}, \tag{4.11}$$

which we can then use to rewrite the Hamiltonian (3.1) as

$$H = L - \sum_n \mathbb{P}_{n,n+1} = L - i\frac{d}{d\lambda}\log F(\lambda)\Big|_{\lambda=i/2}. \tag{4.12}$$

We have thus shown that the Hamiltonian belongs to a set of $L-1$ commuting operators that are generated from the partial trace of the monodromy. This family is completed by the operator measuring the total spin up and we thus have L commuting charges matching our L degrees of freedom. This proves the integrability of the system.

4.2 Diagonalizing the Operators

In order to diagonalize the family of operators $F(\lambda)$, we use the Yang-Baxter equation (4.5) and write out the Kronecker product of the intertwiner

$$R(\lambda) = \lambda \mathbb{1} \otimes \mathbb{1} + i\mathbb{P}_{a_1,a_2} = \begin{pmatrix} \lambda + i & 0 & 0 & 0 \\ 0 & \lambda & i & 0 \\ 0 & i & \lambda & 0 \\ 0 & 0 & 0 & \lambda + i \end{pmatrix}. \tag{4.13}$$

Likewise, the monodromies acting on each their auxiliary space becomes

$$T_{a_1}(\lambda) = \begin{pmatrix} A(\lambda) & 0 & B(\lambda) & 0 \\ 0 & A(\lambda) & 0 & B(\lambda) \\ C(\lambda) & 0 & D(\lambda) & 0 \\ 0 & C(\lambda) & 0 & D(\lambda) \end{pmatrix}, \quad T_{a_2}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) & 0 & 0 \\ C(\lambda) & D(\lambda) & 0 & 0 \\ 0 & 0 & A(\lambda) & B(\lambda) \\ 0 & 0 & C(\lambda) & D(\lambda) \end{pmatrix}, \tag{4.14}$$

and the product of the two becomes

$$T_{a_1}(\lambda)T_{a_2}(\mu) = \begin{pmatrix} A(\lambda)A(\mu) & A(\lambda)B(\mu) & B(\lambda)A(\mu) & B(\lambda)B(\mu) \\ A(\lambda)C(\mu) & A(\lambda)D(\mu) & B(\lambda)C(\mu) & B(\lambda)D(\mu) \\ C(\lambda)A(\mu) & C(\lambda)B(\mu) & D(\lambda)A(\mu) & D(\lambda)B(\mu) \\ C(\lambda)C(\mu) & C(\lambda)D(\mu) & D(\lambda)C(\mu) & D(\lambda)D(\mu) \end{pmatrix}, \tag{4.15}$$

with the other product $T_{a_2}(\mu)T_{a_1}(\lambda)$ being given by the same matrix with the exchange of λ and μ. Now, using the Yang-Baxter equation (4.5) we obtain

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0, \tag{4.16a}$$

$$A(\lambda)B(\mu) = f(\lambda, \mu)B(\mu)A(\lambda) + g(\lambda, \mu)B(\lambda)A(\mu), \tag{4.16b}$$

$$D(\lambda)B(\mu) = h(\lambda, \mu)B(\mu)D(\lambda) - g(\lambda, \mu)B(\lambda)D(\mu), \tag{4.16c}$$

$$[C(\lambda), B(\mu)] = \frac{1}{g(\lambda, \mu)} \left(A(\mu)D(\lambda) - A(\lambda)D(\mu)\right), \tag{4.16d}$$

where we have introduced

$$f(\lambda, \mu) = 1 - \frac{i}{\lambda - \mu}, \qquad h(\lambda, \mu) = 1 + \frac{i}{\lambda - \mu}, \qquad g(\lambda, \mu) = \frac{i}{\lambda - \mu}, \tag{4.17}$$

to lighten notation. We now look for a highest weight state or vacuum state such that $C(\lambda)|0\rangle = 0$ and note that the Lax operator becomes upper triangular in the auxiliary space when acting on $|\uparrow\rangle$

$$\mathbb{L}_n(\lambda)|\uparrow\rangle = \begin{pmatrix} \lambda + \frac{i}{2} & \frac{i}{2}\sigma_n^- \\ 0 & \lambda - \frac{i}{2} \end{pmatrix} |\uparrow\rangle. \tag{4.18}$$

Consequently, the monodromy is triangular on $|0\rangle = |\uparrow\rangle^{\otimes L}$

$$T(\lambda)|0\rangle = \begin{pmatrix} \alpha^L(\lambda) & * \\ 0 & \delta^L(\lambda) \end{pmatrix} |0\rangle, \tag{4.19}$$

where $*$ is some non-trivial operator that is irrelevant to us for now and

$$\alpha = \lambda + \frac{i}{2}, \qquad \delta = \lambda - \frac{i}{2}. \tag{4.20}$$

We see that $|0\rangle$ is indeed an eigenstate of $F(\lambda) = A(\lambda) + D(\lambda)$ which is no surprise since it is the exact same vacuum state we chose for the CBA in the previous chapter. We then hope to find the other eigenvectors by repeated use of $B(\lambda)$ as a "lowering" operator on $|0\rangle$. Introducing N spectral parameters $\{\lambda\}$, the eigenstate is then given by

$$\Phi(\{\lambda\}) = B(\lambda_1) \dots B(\lambda_N) |0\rangle . \tag{4.21}$$

The trick is now to act with $F(\lambda)$ on $\Phi(\{\lambda\})$ and require the resulting state to be an eigenstate. Using the commutation relations (4.16b) we obtain for $A(\lambda)$ on $\Phi(\{\lambda\})$

$$\begin{aligned} A(\lambda)B(\lambda_1) \dots B(\lambda_N) |0\rangle = & \prod_{k=1}^{N} f(\lambda, \lambda_k) \alpha^L(\lambda) B(\lambda_1) \dots B(\lambda_N) |0\rangle \\ & + \sum_{k=1}^{N} M_k(\lambda, \{\lambda\}) B(\lambda_1) \dots \hat{B}(\lambda_k) \dots B(\lambda_N) B(\lambda) |0\rangle . \end{aligned} \tag{4.22}$$

where the first term is found by using the first part of (4.16b) N times and $\hat{B}(\lambda_k)$ denotes that the factor is omitted from the sum. The unwanted terms

$$M_k(\lambda, \{\lambda\}) = g(\lambda, \lambda_k) \prod_{j \neq k}^{N} f(\lambda_k, \lambda_j) \alpha^L(\lambda_k), \tag{4.23}$$

are found by using the second part of (4.16b) once and then the first part $N-1$ times together with (4.16a) saying the B's commute. Similarly for $D(\lambda)$ on $\Phi(\{\lambda\})$ we get

$$\begin{aligned} D(\lambda)B(\lambda_1) \dots B(\lambda_N) |0\rangle = & \prod_{k=1}^{N} h(\lambda, \lambda_k) \delta^L(\lambda) B(\lambda_1) \dots B(\lambda_N) |0\rangle \\ & + \sum_{k=1}^{N} \tilde{M}_k(\lambda, \{\lambda\}) B(\lambda_1) \dots \hat{B}(\lambda_k) \dots B(\lambda_N) B(\lambda) |0\rangle , \end{aligned} \tag{4.24}$$

with h instead of f and the unwanted terms

$$\tilde{M}_k(\lambda, \{\lambda\}) = -g(\lambda, \lambda_k) \prod_{j \neq k}^{N} h(\lambda_k, \lambda_j) \delta^L(\lambda_k). \tag{4.25}$$

In order for Φ to be an eigenstate of $F(\lambda)$, we need each of the unwanted terms $M_j(\lambda, \{\lambda\})$ to cancel the other unwanted terms $\tilde{M}_j(\lambda, \{\lambda\})$

$$\prod_{k \neq j}^{N} f(\lambda_j, \lambda_k) \alpha^L(\lambda_j) = \prod_{k \neq j}^{N} h(\lambda_j, \lambda_k) \delta^L(\lambda_j), \tag{4.26}$$

for $j = 1, \dots N$. Writing this in terms of the spectral parameters we get

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \right)^L = \prod_{k \neq j}^{N} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}, \tag{4.27}$$

which is exactly the $\mathfrak{su}(2)$ Bethe equations from before if we identify the spectral parameters with the Bethe roots $\lambda_i = u_i$. The eigenstates (4.21) are proportional to the CBA eigenstates found in chapter 3, but their exact relation is far from trivial. However the two-body ABA states are calculated in Appendix (B.2) and the exact relationship was found in [25].

4.3 The Gaudin Matrix and its Relation to the Norm

The ABA seems like a rather abstract way to achieve the same results as the CBA, but as we will now see the framework we have introduced can be used to derive a useful relation for the norm of the Bethe states related to the Gaudin matrix [26]. In this framework, a general scalar product between two different N-body states that are not necessarily Bethe states is then

$$S_N \equiv \langle 0| \prod_j^N \mathbf{C}(\lambda_j^C) \prod_j^N \mathbf{B}(\lambda_j^B) |0\rangle \,, \tag{4.28}$$

where $\{\lambda_j^B\}$ and $\{\lambda_j^C\}$ are two independent sets of spectral parameters and

$$\mathbf{C}(\lambda) = \frac{C(\lambda)}{\delta^L(\lambda)}, \qquad \mathbf{B}(\lambda) = \frac{B(\lambda)}{\delta^L(\lambda)}, \tag{4.29}$$

are normalized versions of the quantum operators from before. In the following we will reparametrize the intertwiner (4.3)

$$R(\lambda) \to -iR(-\lambda^{-1}), \tag{4.30}$$

such that the commutation relation (4.16d) becomes

$$[C(\lambda), B(\mu)] = g(\lambda,\mu) \left(A(\mu)D(\lambda) - A(\lambda)D(\mu)\right), \tag{4.31}$$

while $\alpha^L(\lambda)$ and $\delta^L(\lambda)$ changes such that the ratio becomes the inverse of what it was before

$$r(\lambda_j^{B,C}) = r_j^{B,C} = \frac{\alpha^L(\lambda_j^{B,C})}{\delta^L(\lambda_j^{B,C})} = \left(\frac{\lambda_j^{B,C} - \frac{i}{2}}{\lambda_j^{B,C} + \frac{i}{2}}\right)^L. \tag{4.32}$$

We will then consider $\langle 0| \prod_j^N \mathbf{C}(\lambda_j^C)$ to be on-shell such that the parameters $\{\lambda_j^C\}$ satisfy the Bethe equation

$$r_j^C = \prod_{k\neq j}^N \frac{h(\lambda_k^C, \lambda_j^C)}{h(\lambda_j^C, \lambda_k^C)}, \tag{4.33}$$

whereas we will consider $\prod_j^N \mathbf{B}(\lambda_j^B)|0\rangle$ to be off-shell and thus treat λ_j^B and r_j^B as independent variables. We also introduce

$$s(\lambda,\mu) = \frac{h(\lambda,\mu)}{g(\lambda,\mu)} = 1 + \frac{\lambda-\mu}{i}, \tag{4.34}$$

to lighten notation. In order to derive a closed form expression for the scalar product, we will need to consider S_N as a function of one of the spectral parameters λ_k^C. As shown in [27] S_N is a function with simple poles at $\lambda_k^C = \lambda_m^B$ and in the limit $\lambda_k^C \to \lambda_m^B$ the factor S_{N-1} drops out

$$\begin{aligned}\lim_{\lambda_k^C\to\lambda_m^B} S_N(\{\lambda_j^C\},\{\lambda_j^C\},\{r_j^B\}) &= S_{N-1}(\{\lambda_j^C\}_{j\neq k},\{\lambda_j^C\}_{j\neq k},\{\tilde{r}_j^B\}_{j\neq k}) \\ &\times \lim_{\lambda_k^C\to\lambda_m^B} g(\lambda_k^C,\lambda_m^B)(r_k^C - r_m^B) \prod_{j\neq k}^N h(\lambda_k^C,\lambda_j^C) \prod_{j\neq m}^N h(\lambda_m^B,\lambda_j^B),\end{aligned} \tag{4.35}$$

where

$$\tilde{r}_j^B = r_j^B \frac{h(\lambda_j^B, \lambda_m^B)}{h(\lambda_m^B, \lambda_j^B)}, \tag{4.36}$$

is rescaled but is still considered as an independent variable. We are now ready to show that the scalar product S_N is identical to the function

$$\theta_N = G_N(\{\lambda_j^C\}, \{\lambda_j^B\}) \det M_{lk}(\{r_j^B\}, \{\lambda_j^C\}, \{\lambda_j^B\}), \tag{4.37}$$

where

$$G_N(\{\lambda_j^C\}, \{\lambda_j^B\}) = \prod_{j>k}^{N} g(\lambda_j^B, \lambda_k^B) g(\lambda_j^C, \lambda_k^C) \prod_{j,k=1}^{N} s(\lambda_j^C, \lambda_k^B), \tag{4.38}$$

is some prefactor and

$$M_{lk}(\{r_j^B\}, \{\lambda_j^C\}, \{\lambda_j^B\}) = \frac{g(\lambda_k^C, \lambda_l^B)}{s(\lambda_k^C, \lambda_l^B)} - r_l^B \frac{g(\lambda_l^B, \lambda_k^C)}{s(\lambda_l^B, \lambda_k^C)} \prod_m^N \frac{h(\lambda_l^B, \lambda_m^C)}{h(\lambda_m^C, \lambda_l^B)}, \tag{4.39}$$

is the Gaudin matrix. We will prove this by induction. For the base case, $N = 1$, we can drop the subscripts on λ and do a straight-forward calculation on (4.28) by using the commutator relation (4.31),

$$\begin{aligned} S_1 &= \frac{1}{\delta^L(\lambda^C)\delta^L(\lambda^B)} \langle 0|[C(\lambda^C), B(\lambda^B)]|0\rangle \\ &= \frac{1}{\delta^L(\lambda^C)\delta^L(\lambda^B)} g(\lambda^C, \lambda^B)(\alpha^L(\lambda^C)\delta^L(\lambda^B) - \alpha^L(\lambda^B)\delta^L(\lambda^C)) \\ &= g(\lambda^C, \lambda^B)(r^C - r^B) = g(\lambda^C, \lambda^B)(1 - r^B) \end{aligned} \tag{4.40}$$

where we used that λ^C is on-shell and thus $r^C = 1$. Similarly, using (4.38) and (4.39) we obtain

$$G_1 = s(\lambda^C, \lambda^B), \tag{4.41}$$

and

$$M_{11} = \frac{g(\lambda^C, \lambda^B)}{s(\lambda^C, \lambda^B)} - r^B \frac{g(\lambda^B, \lambda^C)}{s(\lambda^B, \lambda^C)} \frac{h(\lambda^B, \lambda^C)}{h(\lambda^C, \lambda^B)} = \frac{g(\lambda^C, \lambda^B)}{s(\lambda^C, \lambda^B)}(1 - r^B), \tag{4.42}$$

and the formula (4.37) thus holds in the base case.

For the inductive step, we will consider θ_N as a function of the last spectral parameter λ_N^C and assume that $\theta_{N-1} = S_{N-1}$. We note that θ_N has simple poles at $\lambda_N^C = \lambda_m^B$ due to the $g(\lambda_k^C, \lambda_l^B)$ part of M_{lk} and that $\theta_N \to 0$ as $|\lambda_N^C| \to \infty$.

We now aim to show that θ_N has the exact same residues as S_N and since both functions have the same asymptotic value, Louisville's theorem ensures that the two functions are indeed the same.

If we consider the first factor G_N in the limit $\lambda_N^C \to \lambda_m^B$ where θ_N has simple poles, we have that

$$\begin{aligned} \lim_{\lambda_N^C \to \lambda_m^B} G_N(\{\lambda_j^C\}, \{\lambda_j^B\}) &= G_{N-1}(\{\lambda_j^C\}_{j\neq N}, \{\lambda_j^B\}_{j\neq m}) \\ &\times \lim_{\lambda_N^C \to \lambda_m^B} (-1)^{N-m} s(\lambda_N^C, \lambda_m^B) \prod_{j\neq m} h(\lambda_m^B, \lambda_j^B) \prod_{j=1}^{N-1} h(\lambda_j^C, \lambda_N^C), \end{aligned} \tag{4.43}$$

with $(-1)^{N-m}$ arising form the fact that g is an odd function. We see that G_{N-1} factors out just as we had hoped. Similarly, by Laplace expanding M_{lk} along the m^{th} row, we see that only the N^{th} column contributes to the residue

$$\lim_{\lambda_N^C \to \lambda_m^B} \det M_{lk}(\{r_j^B\}, \{\lambda_j^C\}, \{\lambda_j^B\}) = \det M_{lk}(\{\tilde{r}_j^B\}_{j\neq m}, \{\lambda_j^C\}_{j\neq N}, \{\lambda_j^B\}_{j\neq m}) \quad (4.44)$$
$$\times \lim_{\lambda_N^C \to \lambda_m^B} (-1)^{N+m} \left(\frac{g(\lambda_N^C, \lambda_m^B)}{s(\lambda_N^C, \lambda_m^B)} - r_m^B \frac{g(\lambda_m^B, \lambda_N^C)}{s(\lambda_m^B, \lambda_N^C)} \prod_j^N \frac{h(\lambda_m^B, \lambda_j^C)}{h(\lambda_j^C, \lambda_m^B)} \right),$$

where $\tilde{r}_j^B$ again appears as independent rescaled parameters by absorbing the extra factors in the last term of (4.39). Putting these together we get that

$$\begin{aligned}\lim_{\lambda_N^C \to \lambda_m^B} \theta_N &= \theta_{N-1}(\{\tilde{r}_j^B\}_{j\neq m}, \{\lambda_j^C\}_{j\neq N}, \{\lambda_j^B\}_{j\neq m}) \prod_{j\neq m} h(\lambda_m^B, \lambda_j^B) \\ &\times \lim_{\lambda_N^C \to \lambda_m^B} g(\lambda_N^C, \lambda_m^B) \prod_{j=1}^{N-1} h(\lambda_j^C, \lambda_N^C) \left(1 + r_m^B \frac{s(\lambda_N^C, \lambda_m^B)}{s(\lambda_m^B, \lambda_N^C)} \prod_j^N \frac{h(\lambda_m^B, \lambda_j^C)}{h(\lambda_j^C, \lambda_m^B)}\right) \\ &= \theta_{N-1}(\{\tilde{r}_j^B\}_{j\neq m}, \{\lambda_j^C\}_{j\neq N}, \{\lambda_j^B\}_{j\neq m}) \prod_{j\neq m} h(\lambda_m^B, \lambda_j^B) \\ &\times \lim_{\lambda_N^C \to \lambda_m^B} g(\lambda_N^C, \lambda_m^B) \prod_{j=1}^{N-1} h(\lambda_N^C, \lambda_j^C) \left(\prod_j^{N-1} \frac{h(\lambda_j^C, \lambda_N^C)}{h(\lambda_N^C, \lambda_j^C)} - r_m^B\right),\end{aligned} \quad (4.45)$$

where we used that

$$\lim_{\lambda\to\mu} \frac{h(\lambda,\mu)}{h(\mu,\lambda)} = -1, \qquad \lim_{\lambda\to\mu} \frac{s(\lambda,\mu)}{s(\mu,\lambda)} = 1. \quad (4.46)$$

Since $\langle 0| \prod_j^N \mathbf{C}(\lambda_j^C)$ is on-shell, we see from the constraint (4.33) that θ_N and S_N indeed have the same residues because of the inductive hypothesis, $\theta_{N-1} = S_{N-1}$. The difference $\Delta = \theta_N - S_N$ is thus entire and bounded with the asymptotic value $\lim_{|\lambda_N^C|\to\infty} \Delta = 0$. Consequently, by Liouville's theorem, it is constantly zero and thus $\theta_N = S_N$ which completes the proof.

Having found a closed expression for the scalar product between one on-shell and one off-shell state we, can find the square of the norm of a Bethe state by simply considering the limit $\lambda_k^B \to \lambda_k^C$. By letting $\lambda_j^B = \lambda_j^C + \varepsilon$ we have that

$$r_j^B = r(\lambda_j^C + \varepsilon) = r_j^C + \varepsilon \frac{\partial}{\partial \lambda_j^C} r_j^C + \mathcal{O}(\varepsilon^2) = r_j^C \left(1 + \varepsilon \frac{\partial}{\partial \lambda_j^C} \log r_j^C\right) + \mathcal{O}(\varepsilon^2), \quad (4.47)$$

which is no longer an independent variable. Similarly, we have that

$$g(\lambda, \lambda+\varepsilon) = \frac{-i}{\varepsilon}, \quad s(\lambda, \lambda+\varepsilon) = 1 + i\varepsilon, \quad h(\lambda, \lambda+\varepsilon) = 1 - \frac{i}{\varepsilon} + \mathcal{O}(\varepsilon^2), \quad (4.48)$$

and

$$\prod_{m=1}^N \frac{h(\lambda_l - \lambda_m)}{h(\lambda_m - \lambda_l)} = \prod_{m\neq l} \frac{h(\lambda_l - \lambda_m)}{h(\lambda_m - \lambda_l)} \left(1 + \varepsilon \sum_{m\neq l} \left(\frac{1}{\lambda_l - \lambda_m + i} - \frac{1}{\lambda_l - \lambda_m - i}\right)\right) + \mathcal{O}(\varepsilon^2). \quad (4.49)$$

Combing these expressions the diagonal part of M_{lk} becomes

$$\begin{aligned} M_{ll} &= \frac{-i}{\varepsilon}\left[1 - i\varepsilon - r_l^B\,(1+i\varepsilon)\,(1-2i\varepsilon)\prod_{m\neq l}\frac{h(\lambda_l-\lambda_m)}{h(\lambda_m-\lambda_l)}\right. \\ &\quad\times\left.\left(1+\varepsilon\sum_{m\neq l}\left(\frac{1}{\lambda_l-\lambda_m+i}-\frac{1}{\lambda_l-\lambda_m-i}\right)\right)+\mathcal{O}(\varepsilon^2)\right] \\ &= i\frac{\partial}{\partial\lambda_l}\log r_l + \sum_{m\neq l}\frac{2}{(\lambda_l-\lambda_m)^2+1}+\mathcal{O}(\varepsilon), \end{aligned} \tag{4.50}$$

where we now dropped the superscripts of λ_j. For the off-diagonal part of M_{lk}, everything of order ε can be disregarded and we have

$$M_{lk} = \frac{-2}{(\lambda_k-\lambda_l)^2+1}+\mathcal{O}(\varepsilon). \tag{4.51}$$

The Gaudin matrix can then be written as

$$M_{lk} = \delta_{lk}\left(i\frac{\partial}{\partial\lambda_l}\log r_l+\sum_m\frac{2}{(\lambda_l-\lambda_m)^2+1}\right)-\frac{2}{(\lambda_k-\lambda_l)^2+1}, \tag{4.52}$$

or

$$M_{lk} = \frac{\partial\phi_k}{\partial\lambda_l}, \qquad \phi_k = i\log\left(r_k\prod_{j\neq k}\frac{h(\lambda_k,\lambda_j)}{h(\lambda_j,\lambda_k)}\right) = i\log\left(\left(\frac{\lambda_k-\frac{i}{2}}{\lambda_k+\frac{i}{2}}\right)^L\prod_{j\neq k}\frac{\lambda_k-\lambda_j+i}{\lambda_k-\lambda_j-i}\right). \tag{4.53}$$

The only thing left to consider is the prefactor G_N which becomes remarkably simple in the limit

$$G_N = \prod_j\prod_{k\neq j}h(\lambda_k,\lambda_j)+\mathcal{O}(\varepsilon). \tag{4.54}$$

Putting it all together and taking $\varepsilon\to 0$, we have a relation between the square of the norm of the on-shell Bethe states and the Gaudin matrix

$$S_N = \prod_j\prod_{j\neq k}h(\lambda_k,\lambda_j)\det\frac{\partial\phi_m}{\partial\lambda_l}. \tag{4.55}$$

We end this chapter by noting that both the ABA method and the relationship between the norm and the Gaudin matrix generalize to spin chains with larger symmetry groups [28]. In particular, it holds in the $\mathfrak{su}(2|1)$ sector in which we will see how a graded version of the Gaudin matrix appears in the overlaps between Bethe states and boundary states in a certain defect version of $\mathcal{N}=4$ SYM which we will discuss in the next chapter.

Chapter 5

Defect $\mathcal{N} = 4$ SYM and Boundary States

In this chapter, we briefly review a defect version of the $\mathcal{N} = 4$ SYM in which one-point functions are not trivial and can be written as overlaps between Bethe states and certain boundary states. These overlaps turn out to be expressible by the superdeterminant of the Gaudin matrix and specifically the overlaps with $\mathfrak{su}(2|1)$ states can be written in this way.

5.1 The Domain Wall

In the defect conformal field theory (dCFT) a domain wall separates space into two regions at $x_3 = 0$. The gauge group for $x_3 < 0$ is $U(N)$, while it for $x_3 \to \infty$ is $U(N + k)$. This defect is holographically dual to a D5-D3 probe brane living in $AdS_4 \times S^2$ and carrying k units of background gauge field flux in the S^2 part [14]. The $AdS_4 \times S^2$ space is again embedded in the $AdS_5 \times S^5$ of the string theory that is conjectured dual to our $\mathcal{N} = 4$ SYM.

The fields in the dCFT can be written with the block decomposition

$$A_\mu, \phi_i, \Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{5.1}$$

with A being $k \times k$ and D being $N \times N$ matrices. The D block propagates the whole space while the rest is confined to $x_3 > 0$. The classical scalar fields satisfy the equations of motion

$$\frac{d^2 \phi_i^{\text{cl}}}{dx_3^2} = [\phi_j^{\text{cl}}, [\phi_j^{\text{cl}}, \phi_i^{\text{cl}}]], \tag{5.2}$$

and for $k > 1$, the solution to this is

$$\phi_i^{\text{cl}} = -\frac{1}{x_3} \begin{pmatrix} (t_i)_{k\times k} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for } i = 1, 2, 3, \qquad \phi_i^{\text{cl}} = 0 \quad \text{for } i = 4, 5, 6, \tag{5.3}$$

with t_i being k-dimensional representations of $\mathfrak{su}(2)$ such that $[t_i, t_j] = i\varepsilon_{ijk} t_k$. The one-dimensional representation of $\mathfrak{su}(2)$ is trivial, so for $k = 1$ A has to be restricted to the half-space by hand by imposing Neumann bondary conditions for $i = 1, 2, 3$ and Dirichlet boundary conditions for $i = 4, 5, 6$. This turns out to simplify the one-point functions of interest. We will start by looking at the case with $k > 1$.

The domain wall breaks the conformal symmetry and the correlation functions are thus less restricted than before. In particular, the one-point correlation functions are no longer trivial but have the structure

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{C}{|x_3|^\Delta}. \tag{5.4}$$

We consider the most general operators in the $SO(6)$ sector build from L scalar fields,

$$\mathcal{O} = \Phi^{i_1 \dots i_L} \operatorname{tr} \phi_{i_1} \dots \phi_{i_L}, \tag{5.5}$$

with the coefficients $\Phi^{i_1 \dots i_L}$ being cyclically symmetric due to the trace. At tree-level the one-point function is given by inserting the classical value of the fields (5.3) into the operators

$$\langle \mathcal{O}(x) \rangle^{\text{cl}} = \frac{(-1)^L}{x_3^L} \Phi^{i_1 \dots i_L} \operatorname{tr} t_{i_1} \dots t_{i_L}. \tag{5.6}$$

This could in principle be straightforwardly calculated, but for operators corresponding to Bethe states, which are the ones we are interested in, we would have to write out the wave function which would be rather cumbersome.

5.2 Matrix Product States and Valence Bond States

Instead, we write (5.5) as an overlap between some Bethe state $|\Phi\rangle$ and the Matrix Product State (MPS) defined by

$$|\text{MPS}\rangle_L \equiv \operatorname{tr} \prod_{n=1}^{L} \left[t_1 \otimes |\uparrow\rangle_n + t_2 \otimes |\downarrow\rangle_n \right], \tag{5.7}$$

where the trace is to be taken over the color space of the t_i. The overlap then becomes

$$\Phi^{i_1 \dots i_L} \operatorname{tr} t_{i_1} \dots t_{i_L} \equiv \frac{\langle \text{MPS}|\Phi\rangle}{\langle \Phi|\Phi\rangle^{\frac{1}{2}}}. \tag{5.8}$$

The trace should be invariant under the automorphism $(t_1, t_2, t_3) \to (-t_1, -t_2, t_3)$ of $\mathfrak{su}(2)$ from which it follows that the overlap is only non-vanishing if L and M are even which we will henceforth take them to be. It is instructive to calculate

$$\operatorname{tr}\left(t_1^{n_1-1} t_2 \dots t_1^{m_2-n_1-1} t_2 \dots\right) = (-1)^{n_1+n_2+\dots} \operatorname{tr}(t_1^{L-M} t_2^M) = \frac{(-1)^{n_1+n_2+\dots}}{2^{L-1}}, \tag{5.9}$$

from which it is clear that the terms in the MPS with an even amount of excitations are invariant under parity ($n_i \to L - n_i$) and it follows that the Bethe state must also be invariant under parity in order for the overlap to be non-vanishing. One way to achieve this is to only consider states with paired rapidities $\{ u_i \} = \{ -u_i \}$ which we will henceforth do. One implication of this is that the determinant of the Gaudin matrix from our closed form expression for the norm factorize into two parts. To see this we write the Gaudin matrix on a block form

$$G = \partial_i \phi_j = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}, \quad A_1 = \partial_m \phi_n, \quad A_2 = \partial_{m+\frac{M}{2}} \phi_n, \quad \text{for } m, n \in \{ 0, \dots, \frac{M}{2} \}. \tag{5.10}$$

The equality of the off-diagonal blocks are seen by

$$A_2 = \frac{\partial \phi_{n+\frac{M}{2}}}{\partial u_m} = \frac{2}{(u_{n+\frac{M}{2}} - u_m)^2 + 1} = \frac{\partial \phi_n}{\partial u_{m+\frac{M}{2}}}, \tag{5.11}$$

while the diagonal parts are

$$A_1 = \frac{\partial\phi_n}{\partial u_m} = \delta_{nm}\left(L\frac{1}{u_n^2+\frac{1}{4}} + \sum_k^M \frac{-2}{(u_m-u_k)^2+1}\right) + \frac{2}{(u_n-u_m)^2+1} = \frac{\partial\phi_{n+\frac{M}{2}}}{\partial u_{m+\frac{M}{2}}}, \tag{5.12}$$

because of the pairing of the rapidities. Doing a column and a row operation, we see that the determinant factorize

$$\det\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} = \det\begin{pmatrix} A_1+A_2 & A_2 \\ 0 & A_1-A_2 \end{pmatrix} = \det G_+ \det G_-, \qquad G_\pm = A_1 \pm A_2. \tag{5.13}$$

Likewise, for the graded version of the Gaudin matrix, the superdeterminant also factorize

$$\operatorname{Sdet}\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} = \frac{\det G_+}{\det G_-} \equiv \mathbb{D}, \tag{5.14}$$

since the superdeterminant is also invariant under column and row additions.

As an example of an overlap in the defect theory, we will restrict ourselves to the case where $k=2$ where $t_1^2 = t_2^2 = \frac{1}{4}$ and $\{t_1, t_2\} = 0$. We will then do an explicit calculation of the overlap between the MPS and the simplest CBA Bethe state with two excitations

$$|u, -u\rangle = \sum_{n>m}\left(e^{ip(m-n-\frac{1}{2})} + e^{-ip(m-n-\frac{1}{2})}\right)|m,n\rangle\,. \tag{5.15}$$

The overlap becomes

$$\begin{aligned}\langle\text{MPS}|u,-u\rangle &= \sum_{n>m} e^{ip(m-n-\frac{1}{2})} + e^{-ip(m-n-\frac{1}{2})}\operatorname{tr}(t_1^{m-1}t_2t_1^{n-m-1}t_2t_1^{L-n}) \qquad (5.16)\\ &= -\frac{L}{2^L}\sum_{\Delta=1}^{L-1}\left(e^{ip(\Delta-\frac{1}{2})} + e^{-ip(\Delta-\frac{1}{2})}\right)(-1)^\Delta \\ &= \frac{L}{2^L}\left(\frac{e^{ip/2}}{1+e^{ip}} + \frac{e^{-ip/2}}{1+e^{-ip}}\right) = \frac{L}{2^{L-1}}\frac{1}{e^{ip/2}+e^{-ip/2}} \\ &= \frac{L}{2^{L-1}}\sqrt{\frac{u^2+\frac{1}{4}}{u^2}} = \frac{1}{2^{L-1}}\sqrt{\frac{\mathcal{Q}(\frac{i}{2})^3}{\mathcal{Q}(0)}}\det G_+,\end{aligned}$$

where we introduced $\Delta = m-n$ before evaluating the alternating geometric sum

$$\sum_{n=0}^N (-1)^n r^n = \frac{1+r^{N+1}}{1+r}, \qquad \text{for } N \text{ even}, \tag{5.17}$$

and used the Bethe equation $e^{ip(L-1)} = 1$. In the last line we expressed the result using the positive Gaudin matrix G_+ as well as the Baxter polynomials $\mathcal{Q}(u) = \prod(u_i - u)$ which turn out to be advantageous since the expression in this form holds for any number of excitations for $k=2$ [14]. Using our closed form expression from the previous chapter, we have that the norm of the two-body Bethe state is[1].

$$\langle u,-u|u,-u\rangle = \mathcal{Q}\left(\frac{i}{2}\right)^2\det G_+\det G_- = L(L-1), \tag{5.18}$$

[1] Here, we have used the normalization used in [21]

which also agrees with what we found by explicit calculation in appendix B. It turns out that this result holds for any number of excitations and consequently the overlap of interest

$$\frac{\langle \mathrm{MPS}|u,-u\rangle}{\langle u,-u|u,-u\rangle^{\frac{1}{2}}} = \frac{1}{2^{L-1}}\sqrt{\frac{u^2+\frac{1}{4}}{u^2}}\sqrt{\frac{L}{L-1}} = \frac{1}{2^{L-1}}\sqrt{\frac{\mathcal{Q}(\frac{i}{2})}{\mathcal{Q}(0)}}\sqrt{\mathbb{D}}, \tag{5.19}$$

will also hold for any number of excitations.

In the $\mathfrak{su}(2|1)$ sector the one-point functions become more involved but simplify for $k = 1$, where they can be expressed as overlaps between Bethe states and the Valence Bond State

$$|B\rangle = (|ZZ\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)^{\otimes L/2}. \tag{5.20}$$

Here the overlaps have been found numerically for spin chains up to length 10 where they can be expressed by [15]

$$\frac{\langle B|\mathbf{u},\mathbf{v}\rangle}{\langle \mathbf{u},\mathbf{v}|\mathbf{u},\mathbf{v}\rangle^{\frac{1}{2}}} \propto \sqrt{\frac{\mathcal{Q}_2(0)}{\mathcal{Q}_1(0)\mathcal{Q}_1(\frac{i}{2})}}\sqrt{\mathbb{D}}, \tag{5.21}$$

with $\mathcal{Q}_1$ encoding the momentum-carrying roots and $\mathcal{Q}_2$ the auxiliary roots. This is analog to what we found for $\mathfrak{su}(2)$ states in the sense that they are expressed as the square root of the superdeterminant of the Gaudin matrix decorated with some Baxter polynomials. One way of checking whether this is indeed the correct formula is by the use of the duality transformation we found in chapter 3. As we will see in the next chapter, the duality transformation will transform the auxiliary roots but the one-point functions should be independent of this transformation. The expression (5.21) should thus transform covariantly meaning that all dependence on the original auxiliary roots disappear and only the dual auxiliary roots will appear in the new formula.

Chapter 6

Duality Transformations

In this chapter we will take a closer look at the dualities we found in chapter 3 where different physical interpretations of the spin chains led to different but dual Bethe equations. We will start by considering bosonic duality transformations before moving on to consider fermionic duality transformation. We will then look closer at the fermionic duality transformation of the $\mathfrak{su}(2|1)$ spin chain and show how the superdeterminant of the Gaudin matrix appearing in the overlap formula for one-point functions transforms.

6.1 The Dual Bethe Roots

Having seen how the CBA gave dual interpretations and corresponding Bethe equations, we will now look closer into these dualities following [29] and starting with the $\mathfrak{su}(2)$ case. In order to do so, it is instructive to express the Bethe equations in terms of the Baxter $\mathcal{Q}$-functions

$$\mathcal{Q}_i(u) = \prod_{j=1}^{M_i} (u - u_j^i), \tag{6.1}$$

where the i index reefer to the set of Bethe roots while the j is to a specific root in that set. If we also introduce the notation

$$f^{\pm} = f\left(u \pm \frac{i}{2}\right), \tag{6.2}$$

then the $\mathcal{Q}\mathcal{Q}$ relation

$$\mathcal{Q}_1^+ \mathcal{Q}_2^- - \mathcal{Q}_1^- \mathcal{Q}_2^+ = u^L, \tag{6.3}$$

is equivalent to the $\mathfrak{su}(2)$ Bethe equations which is most easily seen by shifting the equation by $u \to u \pm \frac{i}{2}$

$$\mathcal{Q}_1^{++} \mathcal{Q}_2 - \mathcal{Q}_1 \mathcal{Q}_2^{++} = \left(u + \frac{i}{2}\right)^L, \qquad \mathcal{Q}_1 \mathcal{Q}_2^{--} - \mathcal{Q}_1^{--} \mathcal{Q}_2 = \left(u - \frac{i}{2}\right)^L. \tag{6.4}$$

If we then evaluate these equations at $u = u_j^1$ the $\mathcal{Q}_1$'s vanish, and we obtain the Bethe equations

$$-\frac{\mathcal{Q}_1^{++}}{\mathcal{Q}_1^{--}} = -\prod_{k=1}^{M_1} \frac{u_j^1 - u_k^1 + i}{u_j^1 - u_k^1 - i} = \left(\frac{u_j^1 + \frac{i}{2}}{u_j^1 - \frac{i}{2}}\right)^L. \tag{6.5}$$

If we consider the (6.3) relation we see that $\mathcal{Q}_2$ must be a polynomial of order $L - M + 1$ in u and we can then write it as

$$Q_2 = A \prod_{j=1}^{L-M_1+1} (u - u_j^2). \tag{6.6}$$

The roots of this polynomial are known as the dual Bethe roots and they also satisfy the $\mathfrak{su}(2)$ Bethe equations with $M_2 = L - M_1 + 1$ magnons. In this dual picture the reference state is $|0\rangle = |{\downarrow}^L\rangle$ and $\uparrow$ are the excitations instead of the other way around. As an example we will consider the case $L = 4$, $M_1 = 2$ with paired rapidities $u_1 = -u_2$. If we disregard the case where the roots are not distinct, the roots are uniquely determined to be

$$u_1^1 = \frac{1}{2\sqrt{3}} = -u_2^1. \tag{6.7}$$

Then, the M_2 dual roots u_i^2 must satisfy (6.3) and by expanding this in powers of u we have

$$\begin{aligned}\frac{1}{A}\left(\mathcal{Q}_1^+\mathcal{Q}_2^- - \mathcal{Q}_1^-\mathcal{Q}_2^+\right) = &- iu^4 + i\left(u^2 + \frac{1}{3}\right)\left(-\frac{1}{4} + u_1^2u_2^2 + u_1^2u_3^2 + u_2^2u_3^2\right) \\ &- \frac{iu}{2}\left(u_1^2u_2^2u_3^2 + \frac{1}{12}\left(u_1^2 + u_2^2 + u_3^2\right)\right).\end{aligned} \tag{6.8}$$

This however does not uniquely determine the roots. The reason for this is that the order of the $\mathcal{Q}_1$ polynomial is smaller than the order of the $\mathcal{Q}_2$ polynomial (2 and 3) and by shifting $\mathcal{Q}_2 \to \mathcal{Q}_2 + A\mathcal{Q}_1$ the $\mathcal{QQ}$ relation will still be satisfied but the roots of $\mathcal{Q}_2$ will be shifted. However, if we restrict ourselves to the zero momentum solutions the dual roots will be uniquely determined and we get that

$$u_1^2 = 0, \quad u_2^2 = \frac{i}{2}, \quad u_3^2 = -\frac{i}{2}. \tag{6.9}$$

Before moving on and discussing the fermionic dualities, we start by considering the somewhat more simple $\mathfrak{su}(3)$ dualities which can be nicely visualized in a Hasse diagram. With three distinct choices of vacua instead of two followed by two choices of excitations instead of one, there will be not one but four duality transformations. In order to encapsulate all the choices we need to introduce three $\mathcal{Q}$-functions for the momentum-carrying roots

$$\mathcal{Q}_a = \prod_{j=1}^{M_a}(u - u_j^a), \qquad a = 1, 2, 3 \tag{6.10}$$

and three $\mathcal{Q}$-functions for the auxiliary roots

$$\mathcal{Q}_A = \prod_{j=1}^{M_A}(u - u_j^A), \qquad A = (12), (13), (23), \tag{6.11}$$

as well as $\mathcal{Q}_\emptyset = u^L$ and $\mathcal{Q}_{123} = 1$. In order to visualize the choice of roots, we introduce the Hasse diagram (Fig 6.1) that connects $\mathcal{Q}$ functions labeled by subsets of $\{1, 2, 3\}$ from $\mathcal{Q}_\emptyset$ to $\mathcal{Q}_{123}$ corresponding to the choice of excitations. All the possible sets of Bethe equations are then contained in the generalized bosonic $\mathcal{QQ}$ relations

$$\mathcal{Q}_A\mathcal{Q}_{Aab} = \mathcal{Q}_{Aa}^+\mathcal{Q}_{Ab}^- - \mathcal{Q}_{Aa}^-\mathcal{Q}_{Ab}^+, \tag{6.12}$$

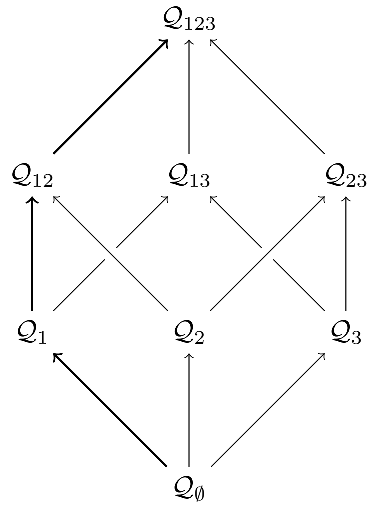


Figure 6.1: Hasse diagram for the $\mathfrak{su}(3)$ spin chain. Each path from $\mathcal{Q}_\emptyset$ to $\mathcal{Q}_{123}$ corresponds to a different set of Bethe equations. The duality transformations are encapsulated in the $\mathcal{QQ}$ relations connecting the Baxter polynomials of each face of the cube. The path $\mathcal{Q}_\emptyset \to \mathcal{Q}_1 \to \mathcal{Q}_{12} \to \mathcal{Q}_{123}$ corresponds to one system of Bethe equations and is connected to the path $\mathcal{Q}_\emptyset \to \mathcal{Q}_2 \to \mathcal{Q}_{12} \to \mathcal{Q}_{123}$ by a duality transformation.

of which we have one for each face in the Hasse diagram. For example the $\mathcal{QQ}$ relations

$$\mathcal{Q}_\emptyset \mathcal{Q}_{12} = \mathcal{Q}_1^+ \mathcal{Q}_2^- - \mathcal{Q}_1^- \mathcal{Q}_2^+, \tag{6.13}$$

and

$$\mathcal{Q}_1 \mathcal{Q}_{123} = \mathcal{Q}_{12}^+ \mathcal{Q}_{13}^- - \mathcal{Q}_{12}^- \mathcal{Q}_{13}^+, \tag{6.14}$$

correspond to two faces of the Hasse diagram and by shifting the two $\mathcal{QQ}$ relations by $u \to u \pm \frac{i}{2}$ and evaluating at roots $u = u_j^1$ and $u = u_j^{12}$ respectively we obtain the $\mathfrak{su}(3)$ Bethe equations

$$\frac{\mathcal{Q}_\emptyset^+}{\mathcal{Q}_\emptyset^-} = \frac{\mathcal{Q}_1^{++}}{\mathcal{Q}_1^{--}} \frac{\mathcal{Q}_{12}^-}{\mathcal{Q}_{12}^+}\Big|_{u=u_j^1}, \qquad \mathcal{Q}_{123} = 1 = \frac{\mathcal{Q}_{12}^{++}}{\mathcal{Q}_{12}^{--}} \frac{\mathcal{Q}_1^-}{\mathcal{Q}_1^+}\Big|_{u=u_j^{12}}, \tag{6.15}$$

corresponding to the path $\mathcal{Q}_\emptyset \to \mathcal{Q}_1 \to \mathcal{Q}_{12} \to \mathcal{Q}_{123}$. However, the two $\mathcal{QQ}$ relations contain one duality transformation each just like it did in the $\mathfrak{su}(2)$ case and we could, for instance, do a duality transformation on the first level resulting in the dual Bethe equations where $\mathcal{Q}_1$ is replaced by $\mathcal{Q}_2$, corresponding to the path $\mathcal{Q}_\emptyset \to \mathcal{Q}_2 \to \mathcal{Q}_{12} \to \mathcal{Q}_{123}$. This would not change the roots of $\mathcal{Q}_{12}$ but we would need (6.13) to find the roots of $\mathcal{Q}_2$. In this way, we can reach every set of Bethe equations by using the duality transformations contained in the $\mathcal{QQ}$ relations.

This easily generalizes to $\mathfrak{su}(N)$ spin chains where we have 2^N different $\mathcal{Q}$ functions all labeled by the powersets of $\{1, \ldots, N\}$ and containing roots that solve the Bethe equations corresponding to the level of nesting.

6.2 Fermionic Duality Transformations

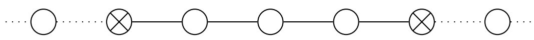
Having seen how the duality transformation works for the bosonic cases, we are now ready to investigate fermionic dualities found in the general Lie superalgebra where the Bethe equations are given by [19]

$$\left(\frac{u_p^j + \frac{i}{2}V_j}{u_p^j - \frac{i}{2}V_j}\right)^L = \prod_{j'=1}^{r} \prod_{\substack{q=1 \\ (j',q)\neq(j,p)}}^{M_{j'}} \frac{u_p^j - u_q^{j'} + \frac{i}{2}\mathcal{M}_{j,j'}}{u_p^j - u_q^{j'} - \frac{i}{2}\mathcal{M}_{j,j'}}. \tag{6.16}$$

Here $\mathcal{M}_{ij}$ is the Cartan Matrix of the Lie superalgebra, V_j are the Dynkin labels, r is the rank or number of nodes, and M_j is the number of Bethe roots for node j, where we have dropped the multi-index from the bosonic case. For any Lie superalgebra there are several different choices of simple roots and therefore several different corresponding Dynkin diagrams and Cartan matrices. An introduction to Lie superalgebras, Dynkin diagrams and Cartan matrices can be found in appendix A. The super conformal symmetry of our theory $\mathfrak{psu}(2,2|4)$ is of rank $r = 4$ and for one choice of simple roots the Cartan matrix is given by [19]

$$\mathcal{M} = \begin{pmatrix} -2 & +1 & & & & & \\ +1 & 0 & -1 & & & & \\ & -1 & +2 & -1 & & & \\ & & -1 & +2 & -1 & & \\ & & & -1 & +2 & -1 & \\ & & & & -1 & 0 & +1 \\ & & & & & +1 & -2 \end{pmatrix}, \tag{6.17}$$

which correspond to the Dynkin diagram



(6.18)

where we have included the Dynkin labels that are different from zero above its corresponding node. Here we note that the fermionic nodes must be connected with one dotted line and one solid line depending on the off-diagonal entries of the Cartan matrix while the bosonic nodes are connected by the same type of line. Since $V_4 \neq 0$ we say that the 4th node is momentum-carrying while the others are not. We now consider one of the fermionic nodes with M_u Bethe roots, u_p. The roots will appear in the Bethe equations belonging to the neighboring nodes

$$\ldots = \prod_{q=1}^{M_u} \frac{v_p - u_q + \frac{i}{2}}{v_p - u_q - \frac{i}{2}} \ldots \tag{6.19a}$$

$$\ldots = \prod_{q=1}^{M_u} \frac{w_p - u_q - \frac{i}{2}}{w_p - u_q + \frac{i}{2}} \ldots \tag{6.19b}$$

where v_p and w_p are the M_v and M_w roots of the neighboring nodes and they will appear in their own set of Bethe equations

$$\left(\frac{u_p + \frac{i}{2}V_u}{u_p - \frac{i}{2}V_u}\right)^L = \prod_{q=1}^{M_v} \frac{u_p - v_q + \frac{i}{2}}{u_p - v_q - \frac{i}{2}} \prod_{q=1}^{M_w} \frac{u_p - w_q + \frac{i}{2}}{u_p - w_q - \frac{i}{2}}. \tag{6.20}$$

In order to figure out the duality transformation, we introduce the polynomial

$$\begin{aligned} P(u) &= \left(u + \frac{i}{2}V_u\right)^L \prod_{q=1}^{M_v}\left(u - v_q - \frac{i}{2}\right) \prod_{q=1}^{M_w}\left(u - w_q + \frac{i}{2}\right) \\ &\quad - \left(u - \frac{i}{2}V_u\right)^L \prod_{q=1}^{M_v}\left(u - v_q + \frac{i}{2}\right) \prod_{q=1}^{M_w}\left(u - w_q - \frac{i}{2}\right) \\ &= i(V_u L - M_v + M_w) u^{L+M_v+M_w-1} + \ldots, \end{aligned} \tag{6.21}$$

from which we retrieve the Bethe equation (6.20) when $P(u_p) = 0$. We then see that $P(u)$ is a polynomial of degree $L + M_v + M_w - 1$ since the coefficient of $u^{L+M_v+M_w}$ vanishes and we can thus factorize it into

$$P(u) = i(V_u L - M_v + M_w) \prod_{p=1}^{M_u} (u - u_p) \prod_{q=1}^{M_{\tilde{u}}} (u - \tilde{u}_q), \tag{6.22}$$

where we have introduced $M_{\tilde{u}} = L + M_v + M_w - M_u - 1$ dual roots. Evaluating (6.22) and (6.21) in $P(v_p \pm \frac{i}{2})$ we get that

$$\begin{aligned} \frac{P(v_p + \frac{i}{2})}{P(v_p - \frac{i}{2})} &= \prod_{q=1}^{M_u} \frac{v_p - u_q + \frac{i}{2}}{v_p - u_q - \frac{i}{2}} \prod_{r=1}^{M_{\tilde{u}}} \frac{v_p - \tilde{u}_r + \frac{i}{2}}{v_p - \tilde{u}_r - \frac{i}{2}} \\ &= \left(\frac{v_p - \frac{i}{2}(V_u - 1)}{v_p + \frac{i}{2}(V_u - 1)} \right)^L \prod_{q \neq p}^{M_v} \frac{v_p - v_q + i}{v_p - v_q - i}, \end{aligned} \tag{6.23}$$

where w_r vanishes since all the fractions where it would appear become one. Likewise, if we evaluate the polynomial in $P(w_p \pm \frac{i}{2})$ we get that

$$\begin{aligned} \frac{P(w_p + \frac{i}{2})}{P(w_p - \frac{i}{2})} &= \prod_{q=1}^{M_u} \frac{w_p - u_q + \frac{i}{2}}{w_p - u_q - \frac{i}{2}} \prod_{r=1}^{M_{\tilde{u}}} \frac{w_p - \tilde{u}_r + \frac{i}{2}}{w_p - \tilde{u}_r - \frac{i}{2}} \\ &= \left(\frac{w_p + \frac{i}{2}(V_u - 1)}{w_p - \frac{i}{2}(V_u - 1)} \right)^L \prod_{q \neq p}^{M_w} \frac{w_p - w_q + i}{w_p - w_q - i}. \end{aligned} \tag{6.24}$$

Using this we can replace u_q by $\tilde{u}_q$ in (6.19a) and (6.19b)

$$\cdots \left(\frac{v_p + \frac{i}{2}(V_u - 1)}{v_p - \frac{i}{2}(V_u - 1)} \right)^L = \prod_{q \neq p}^{M_v} \frac{v_p - v_q + i}{v_p - v_q - i} \prod_{r=1}^{M_{\tilde{u}}} \frac{v_p - \tilde{u}_r - \frac{i}{2}}{v_p - \tilde{u}_r + \frac{i}{2}} \cdots, \tag{6.25a}$$

$$\cdots \left(\frac{w_p + \frac{i}{2}(V_u - 1)}{w_p - \frac{i}{2}(V_u - 1)} \right)^L = \prod_{q \neq p}^{M_w} \frac{w_p - w_q - i}{w_p - w_q + i} \prod_{r=1}^{M_{\tilde{u}}} \frac{w_p - \tilde{u}_r + \frac{i}{2}}{w_p - \tilde{u}_r - \frac{i}{2}} \cdots, \tag{6.25b}$$

and since the dual roots also satisfy the Bethe equation we can replace (6.20) by

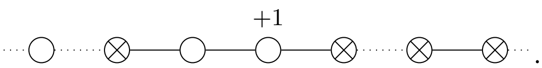
$$\left(\frac{\tilde{u}_p - \frac{i}{2} V_u}{\tilde{u}_p + \frac{i}{2} V_u} \right) = \prod_{q=1}^{M_v} \frac{\tilde{u}_p - v_q - \frac{i}{2}}{\tilde{u}_p - v_q + \frac{i}{2}} \prod_{q=1}^{M_w} \frac{\tilde{u}_p - w_q + \frac{i}{2}}{\tilde{u}_p - w_q - \frac{i}{2}}. \tag{6.26}$$

We have thus obtained a dual set of Bethe equations. The fact that $V_2 = V_6 = 0$ for our representation (6.18), and the fermionic nodes in question are therefore not momentum-carrying, changes the above considerations a bit. We will get L fewer roots in $P(u)$ and the number of dual roots will then be $M_{\tilde{u}} = M_v + M_w - M_u - 1$. The transformation rules (6.25a-6.26) stay the same except that the factor of $\left(\frac{v_p - \frac{i}{2}(V_u - 1)}{v_p + \frac{i}{2}(V_u - 1)} \right)^L$ will not be present.

The fermionic duality can be used to connect the different Dynkin diagrams of $\mathfrak{psu}(2,2|4)$ with each other. As an example we can now use the fermionic duality on the 6th node in order to transform the Cartan matrix (6.17) into

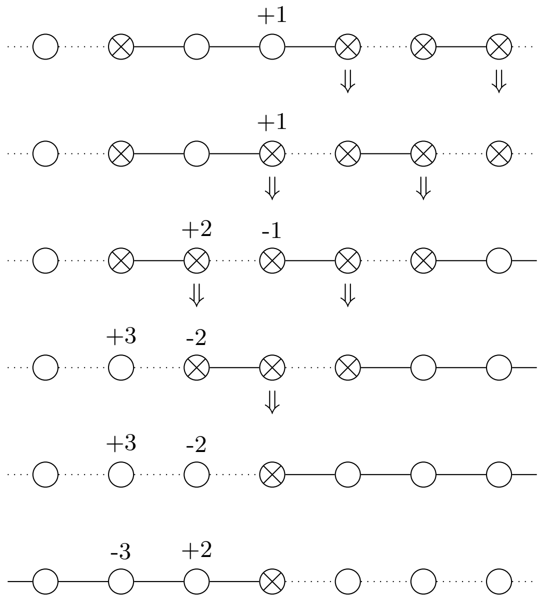
$$\mathcal{M} = \begin{pmatrix} -2 & +1 & & & & & \\ +1 & 0 & -1 & & & & \\ & -1 & +2 & -1 & & & \\ & & -1 & +2 & -1 & & \\ & & & -1 & 0 & +1 & \\ & & & & +1 & 0 & -1 \\ & & & & & -1 & 0 \end{pmatrix}, \tag{6.27}$$

and the Dynkin diagram (6.18) into



(6.28)

From this we see that the fermionic duality transformation changes the grade of the neighboring nodes while it stays fermionic itself. By using this newly acquired duality transformation we can continue to transform the Dynkin diagram



(6.29)

where the arrows indicate on which node we are doing the fermionic duality. The last transformation is not a duality transformation but simply amounts to inverting all the Bethe equations. One also has to be aware of how the Dynkin labels changes when doing the transformation on a momentum-carrying node by using (6.25a-6.25b). What we end up with is the distinguished Dynkin diagram where there is only one fermionic node, this also goes by the name Beast and the transformation is therefore from "Beauty" to the "Beast".

6.3 Fermionic Duality in the $\mathfrak{su}(2|1)$ Sector

The fermionic dualities can also be encoded in $\mathcal{QQ}$ relations and visualized by the use of Hasse diagrams. Instead of one multi-index, the Baxter polynomials $\mathcal{Q}_{A|I}$ now carry two types of multi-indices to distinguish between fermionic and bosonic excitations. Here A is the powerset of $\{1, \ldots, N\}$ with N being the fermionic degrees of freedom and I is the powerset of $\{1, \ldots, M\}$ with M being the bosonic degrees of freedom. Instead of just having the bosonic $\mathcal{QQ}$ relation

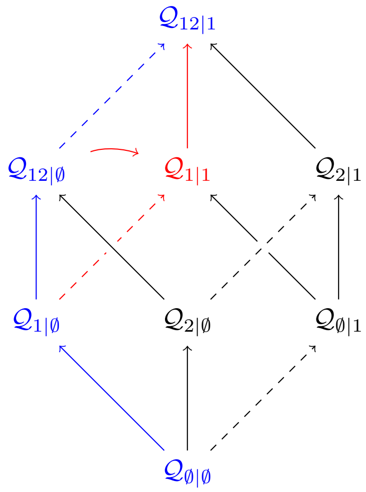


Figure 6.2: The Hasse diagram for the $\mathfrak{su}(2|1)$ spin chain. The blue path corresponds to the Dynkin diagram with the first excitation being bosonic and the second fermionic, while the red arrow represents the fermionic duality transformation that changes the Dynkin diagram to such that both the excitations are fermionic.

(6.12) which now can be used for swapping either two fermionic or two bosonic indices, we introduce a fermionic $\mathcal{QQ}$ relation which changes the grading of the neighboring nodes

$$\mathcal{Q}_{Aa|I}\mathcal{Q}_{A|Ii} = \mathcal{Q}^+_{Aa|Ii}\mathcal{Q}^-_{A|I} - \mathcal{Q}^-_{Aa|Ii}\mathcal{Q}^+_{A|I}. \tag{6.30}$$

As an example, we will consider the $\mathfrak{su}(2|1)$ spin chain where we already know two dual sets of Bethe equations from the Nested Bethe Ansatz of chapter 3. In total there are 6 different choices of Bethe equations which are represented by a path in the Hasse diagram in Fig. 6.2 from $\mathcal{Q}_{\emptyset|\emptyset}$ to $\mathcal{Q}_{12|1}$. Following blue path we get from the $\mathcal{Q}_{\emptyset|\emptyset}, \mathcal{Q}_{1|\emptyset}, \mathcal{Q}_{2|\emptyset}, \mathcal{Q}_{12|\emptyset}$ bosonic $\mathcal{QQ}$ relations

$$\mathcal{Q}_{\emptyset|\emptyset}\mathcal{Q}_{12|\emptyset} = \mathcal{Q}^+_{1|\emptyset}\mathcal{Q}^-_{2|\emptyset} - \mathcal{Q}^-_{1|\emptyset}\mathcal{Q}^+_{2|\emptyset} \Longrightarrow \frac{\mathcal{Q}^+_{\emptyset|\emptyset}}{\mathcal{Q}^-_{\emptyset|\emptyset}} = -\frac{\mathcal{Q}^{++}_{1|\emptyset}}{\mathcal{Q}^{--}_{1|\emptyset}}\frac{\mathcal{Q}^-_{12|\emptyset}}{\mathcal{Q}^+_{12|\emptyset}}\Big|_{u=u_j^{1|\emptyset}}, \tag{6.31}$$

while the fermionic $\mathcal{QQ}$ relation of the $\mathcal{Q}_{1|\emptyset}, \mathcal{Q}_{12|\emptyset}, \mathcal{Q}_{1|1}, \mathcal{Q}_{12|1}$ side gives us

$$\mathcal{Q}_{12|\emptyset}\mathcal{Q}_{1|1} = \mathcal{Q}^+_{12|1}\mathcal{Q}^-_{1|\emptyset} - \mathcal{Q}^-_{12|1}\mathcal{Q}^+_{1|\emptyset} \Longrightarrow 1 = \frac{\mathcal{Q}^-_{1|\emptyset}}{\mathcal{Q}^+_{1|\emptyset}}\Big|_{u=u_j^{12|\emptyset}}, \tag{6.32}$$

where we used that $\mathcal{Q}_{12|\emptyset} = 1$. These equations are exactly the $\mathfrak{su}(2|1)$ Bethe equations (3.90a) we found using the CBA in chapter 3 with momentum-carrying roots $u_j^{1|\emptyset} = u_j$ and auxiliary roots $u_j^{12|\emptyset} = v_j$. They solve the spin chain with a bosonic vacuum and a bosonic excitation followed by a nested fermionic excitation corresponding by the distinguished Dynkin diagram. Following instead the red path in Fig. 6.2, we get that the fermionic $\mathcal{QQ}$ relation of $\mathcal{Q}_{\emptyset|\emptyset}, \mathcal{Q}_{1|\emptyset}, \mathcal{Q}_{\emptyset|1}, \mathcal{Q}_{1|1}$ gives us

$$\mathcal{Q}_{1|\emptyset}\mathcal{Q}_{\emptyset|1} = \mathcal{Q}^+_{1|1}\mathcal{Q}^-_{\emptyset|\emptyset} - \mathcal{Q}^-_{1|1}\mathcal{Q}^+_{\emptyset|\emptyset} \Longrightarrow \frac{\mathcal{Q}^-_{\emptyset|\emptyset}}{\mathcal{Q}^+_{\emptyset|\emptyset}} = \frac{\mathcal{Q}^+_{1|1}}{\mathcal{Q}^-_{1|1}}\Big|_{u=u_j^{1|\emptyset}}, \tag{6.33}$$

while the same fermionic $\mathcal{QQ}$ relation from before of $\mathcal{Q}_{1|\emptyset}, \mathcal{Q}_{21|\emptyset}, \mathcal{Q}_{1|1}, \mathcal{Q}_{12|1}$ now gives us

$$\mathcal{Q}_{12|\emptyset}\mathcal{Q}_{1|1} = \mathcal{Q}^+_{12|1}\mathcal{Q}^-_{1|\emptyset} - \mathcal{Q}^-_{12|1}\mathcal{Q}^+_{1|\emptyset} \Longrightarrow 1 = \frac{\mathcal{Q}^-_{1|\emptyset}}{\mathcal{Q}^+_{1|\emptyset}}\Big|_{u=u_j^{1|1}}. \tag{6.34}$$

These equations are the dual $\mathfrak{su}(2|1)$ Bethe equations (3.91) with the same momentum-carrying roots $u_j^{1|\emptyset} = u_j$ which are now fermionic and the dual auxiliary roots $u_j^{1|1} = \tilde{v}_j$. They solve the spin chain with a fermionic vacuum and a bosonic excitation followed by a nested fermionic excitation corresponding to the Dynkin diagram $-\otimes - \otimes \cdots$. Having taken a closer look at the duality transformations first encountered in chapter 3 we are ready to see what this means for the overlap formulae discussed in chapter 5.

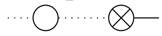
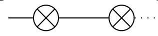
Chapter 7

Transformation of the Superdeterminant for $\mathfrak{su}(2|1)$

In chapter 5 we saw that the overlap between Bethe states and boundary states could be expressed as the superdeterminant of the Gaudin matrix decorated with some Baxter polynomials. The overlap in turn appeared in the one-point functions in the defect SYM at tree level. The overlap formulae depend on the choice of grading of our Lie superalgebra and the question is thus how the overlap formulae transform under our duality transformations. In this chapter, we will work out how the superdeterminant appearing in the overlap between a valence bond state and a $\mathfrak{su}(2|1)$ Bethe state transforms under the fermionic duality transformation discussed in the previous chapter.

We conjecture that the superdeterminant transforms covariantly meaning that the overlap formula will contain only the new dual auxiliary roots instead of the old ones.

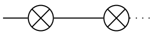
7.1 The Gaudin Matrices

In order to investigate the transformation, we will carefully set up our problem. We will be considering the transformation discussed in the previous chapter depicted in Fig. 6.2 taking us from the distinguished Dynkin diagram  to the Dynkin diagram . Furthermore, we will be considering the case where we initially have only paired roots of which we have $P_1 = M_1/2$ momentum-carrying pairs $\{u_i\}$ and $P_2 = M_2/2$ auxiliary pairs $\{v_i\}$. In the distinguished choice, the Cartan matrix and Dynkin labels are

$$\mathcal{M} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7.1}$$

yielding the Bethe equations

$$\left(\frac{u_p + \frac{i}{2}}{u_p - \frac{i}{2}}\right)^L = \prod_{q \neq p}^{2P_1} \frac{u_p - u_q + i}{u_p - u_q - i} \prod_{q=1}^{2P_2} \frac{u_p - v_q - \frac{i}{2}}{u_p - v_q + \frac{i}{2}}, \qquad 1 = \prod_{q=1}^{2P_1} \frac{v_p - u_q - \frac{i}{2}}{v_p - u_q + \frac{i}{2}}. \tag{7.2}$$

After the duality transformation we will have $\tilde{P}_2 = P_1 - P_2 - 1$ new pairs of dual auxiliary roots $\{\tilde{v}_i\}$ plus an extra zero root $\tilde{v}_0 = 0$ while the momentum-carrying roots, are the same roots as before though they are now fermionic. The transformation yields the dual Dynkin diagram  and the Cartan matrix and Dynkin labels become

$$\tilde{\mathcal{M}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{V} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7.3}$$

and the dual Bethe equations are given by

$$\left(\frac{u_p + \frac{i}{2}}{u_p - \frac{i}{2}}\right)^L = \prod_{q=1}^{2\tilde{P}_2} \frac{u_p - \tilde{v}_q + \frac{i}{2}}{u_p - \tilde{v}_q - \frac{i}{2}} \qquad 1 = \prod_{q=1}^{2P_1} \frac{\tilde{v}_p - u_q + \frac{i}{2}}{\tilde{v}_p - u_q - \frac{i}{2}}. \tag{7.4}$$

By relabeling the three Baxter polynomials of interest $\mathcal{Q}_{\emptyset|\emptyset} \to \mathcal{Q}_\emptyset$, $\mathcal{Q}_{1|\emptyset} \to \mathcal{Q}_1$, $\mathcal{Q}_{12|\emptyset} = \mathcal{Q}_2$ and $\mathcal{Q}_{1|1} = \tilde{\mathcal{Q}}_2$ the fermionic $\mathcal{Q}\mathcal{Q}$ relation encoding the duality transformation becomes

$$\mathcal{Q}_1^+ - \mathcal{Q}_1^- = 2iP_1 u \tilde{\mathcal{Q}}_2 \mathcal{Q}_2, \tag{7.5}$$

where the Baxter polynomials are now on reduced form such that they only contain paired roots and the zeroth auxiliary root thus appears outside the $\mathcal{Q}$ functions together with the factor of $2iP_1$ stemming from (6.21).

We conjecture that the superdeterminant appearing in (5.21) transforms according to

$$\tilde{\mathbb{D}} = 2P_1 \frac{\tilde{\mathcal{Q}}_2(0)\mathcal{Q}_2(0)}{\mathcal{Q}_1(\frac{i}{2})} \mathbb{D}, \tag{7.6}$$

with $\tilde{\mathbb{D}}$ being the dual superdeterminant. In this way, the overlap formula depends only on the ordinary auxiliary roots through $\mathbb{D}\mathcal{Q}_2(0)$ and will transform such that the dual overlap formula only depends on the dual auxiliary roots through $\tilde{\mathbb{D}}/\tilde{\mathcal{Q}}_2(0)$. We have found this to be true numerically semi-off-shell for $(2P_1, 2P_2) = (4,2), (4,0), (2,0)$ meaning that the momentum-carrying roots need not satisfy the Bethe equation while the ordinary and dual auxiliary roots have to satisfy (7.5). In order to prove it analytically, we will have to look at how the Gaudin matrix transforms.

In the distinguished choice of simple roots, the Gaudin matrix becomes

$$G = \frac{\partial \phi_i}{\partial u_j} = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_2 & A_1 & B_2 & B_1 \\ B_1^t & B_2^t & C_1 & C_2 \\ B_2^t & B_1^t & C_2 & C_1 \end{pmatrix}. \tag{7.7}$$

By a few row and column operations, this becomes

$$G \to \begin{pmatrix} A_+ & B_+ & A_2 & B_2 \\ B_+^t & C_+ & B_2^t & C_2 \\ 0 & 0 & A_- & B_- \\ 0 & 0 & B_-^t & C_- \end{pmatrix}, \tag{7.8}$$

with

$$A_\pm = A_1 \pm A_2, \qquad B_\pm = B_1 \pm B_2, \qquad C_\pm = C_1 \pm C_2. \tag{7.9}$$

The superdeterminant thus again factorizes into

$$\mathbb{D} \equiv \operatorname{Sdet} G = \frac{\det G_+}{\det G_-}, \qquad G_\pm = \begin{pmatrix} A_\pm & B_\pm \\ B_\pm^t & C_\pm \end{pmatrix}. \tag{7.10}$$

The m,n^{th} entries of the blocks of the Gaudin matrix are given by

$$A_1 = \frac{\partial \phi^1_n}{\partial u_m} = \frac{\partial \phi^1_{n+P_1}}{\partial u_{m+P_1}}, \qquad A_2 = \frac{\partial \phi^1_{n+P_1}}{\partial u_m} = \frac{\partial \phi^1_n}{\partial u_{m+P_1}}, \qquad m,n = 1,\dots P_1, \tag{7.11a}$$

$$C_1 = \frac{\partial \phi^2_n}{\partial v_m} = \frac{\partial \phi^2_{n+P_2}}{\partial v_{m+P_2}}, \qquad C_2 = \frac{\partial \phi^2_{n+P_2}}{\partial v_m} = \frac{\partial \phi^2_n}{\partial v_{m+P_2}}, \qquad m,n = 1,\dots P_2, \tag{7.11b}$$

$$B_1 = \frac{\partial \phi^2_n}{\partial u_m} = \frac{\partial \phi^2_{n+P_2}}{\partial u_{m+P_1}}, \qquad B_2 = \frac{\partial \phi^2_{n+P_2}}{\partial u_m} = \frac{\partial \phi^2_n}{\partial u_{m+P_1}}, \qquad m = 1,\dots P_1, n = 1,\dots P_2, \tag{7.11c}$$

$$B^t_1 = \frac{\partial \phi^1_n}{\partial v_m} = \frac{\partial \phi^1_{n+P_1}}{\partial v_{m+P_2}}, \qquad B^t_2 = \frac{\partial \phi^1_{n+P_1}}{\partial v_m} = \frac{\partial \phi^1_n}{\partial v_{m+P_2}}, \qquad m = 1,\dots P_2, n = 1,\dots P_1, \tag{7.11d}$$

where the norm functions are

$$\begin{aligned} \phi^1_p &= -iL\log\left(\frac{u_p - \frac{i}{2}}{u_p + \frac{i}{2}}\right) - i\sum_{q\neq p}^{2P_1}\log\left(\frac{u_p - u_q + i}{u_p - u_q - i}\right) - i\sum_{q=1}^{2P_2}\log\left(\frac{u_p - v_q - \frac{i}{2}}{u_p - v_q + \frac{i}{2}}\right), \\ \phi^2_p &= -i\sum_{q=1}^{2P_1}\log\left(\frac{v_p - u_q - \frac{i}{2}}{v_p - u_q + \frac{i}{2}}\right). \end{aligned} \tag{7.12}$$

Doing the explicit calculation, the m,n^{th} entries are given by

$$A_1 = \delta_{mn}\left(\frac{L}{u_m^2 + \frac{1}{4}} - 2\sum_q^{2P_1} f(u_m - u_q) + \sum_q^{2P_2} g(u_m - v_q)\right) + 2f(u_m - u_n), \tag{7.13a}$$

$$A_2 = 2f(u_m + u_n), \qquad B_1 = -g(v_n - u_m), \tag{7.13b}$$

$$B_2 = -g(v_n + u_m), \qquad C_1 = \delta_{nm}\sum_q^{2P_1} g(v_n - u_q), \qquad C_2 = 0, \tag{7.13c}$$

and we note that A_1, B_1 do indeed have an even grading while A_2, B_2 have an odd grading under parity. Here we have introduced the functions

$$g(x) = \frac{1}{x^2 + \frac{1}{4}}, \qquad f(x) = \frac{1}{x^2 + 1}, \tag{7.14}$$

to lighten the notation. These are not to be confused with the functions (4.17) introduced in chapter 4.

In the dual case the norm functions are

$$\tilde{\phi}^1_p = -iL\log\left(\frac{u_p - \frac{i}{2}}{u_p + \frac{i}{2}}\right) + i\sum_{q=1}^{2\tilde{P}_2+1}\log\frac{u_p - \tilde{v}_q - \frac{i}{2}}{u_p - \tilde{v}_q + \frac{i}{2}}, \qquad \tilde{\phi}^2_p = -i\sum_{q=1}^{2P_1}\log\frac{\tilde{v}_p - u_q + \frac{i}{2}}{\tilde{v}_p - u_q - \frac{i}{2}}, \tag{7.15}$$

and the dual Gaudin matrix factorizes into

$$\det\tilde{G} = \det\tilde{G}_+\det\tilde{G}_-, \quad \tilde{G}_+ = \begin{pmatrix} \tilde{A}_+ & \tilde{B}_+ & \sqrt{2}\tilde{D}_1 \\ \tilde{B}^t_+ & \tilde{C}_+ & \sqrt{2}\tilde{D}_2 \\ \sqrt{2}\tilde{D}^t_1 & \sqrt{2}\tilde{D}^t_2 & \tilde{D}_3 \end{pmatrix}, \quad \tilde{G}_- = \begin{pmatrix} \tilde{A}_- & \tilde{B}_- \\ \tilde{B}^t_- & \tilde{C}_- \end{pmatrix}. \tag{7.16}$$

Here, the auxiliary zeroth root has an even grading, hence the extra column and row of D's in $\tilde{G}_+$. The dual blocks are then given by

$$\tilde{A}_1 = \delta_{mn}\left(\frac{L}{u_m^2+\frac{1}{4}} - \sum_q^{2\tilde{P}_2+1} g(u_m-\tilde{v}_q)\right), \qquad \tilde{A}_2 = 0, \tag{7.17a}$$

$$\tilde{B}_1 = g(\tilde{v}_n - u_m), \qquad \tilde{B}_2 = g(\tilde{v}_n + u_m), \tag{7.17b}$$

$$\tilde{C}_1 = -\delta_{nm}\sum_q^{2P_1} g(\tilde{v}_n - u_q), \qquad C_2 = 0, \tag{7.17c}$$

$$\tilde{D}_1 = g(u_m), \qquad \tilde{D}_2 = 0, \qquad \tilde{D}_3 = \frac{\partial\phi_0^2}{\partial v_0} = -\sum_q^{2P_1} g(u_q), \tag{7.17d}$$

and we see that the dual Gaudin matrix is considerably simpler than the original one.

7.2 The Schur Complements

One way to further factorize $G_\pm$ and $\tilde{G}_\pm$ and help us prove our conjecture is by using their Schur complements. For a general block symmetric matrix we have that

$$\begin{pmatrix} A & B \\ B^t & D \end{pmatrix} = \begin{pmatrix} \mathbb{1} & B \\ 0 & D \end{pmatrix}\begin{pmatrix} A - BD^{-1}B^t & 0 \\ D^{-1}B^t & \mathbb{1} \end{pmatrix}, \tag{7.18}$$

where $A - BD^{-1}B^t$ is the Schur compliment. This is particularly nice in our case since $C_\pm, \tilde{C}_\pm, \tilde{A}_\pm$ are diagonal and therefore easily inverted. For the determinants of $G_\pm$ and $\tilde{G}_\pm$ we thus have that

$$\det G_+ = \det C_+ \det(W_+), \qquad \det G_- = \det C_- \det(W_-), \tag{7.19a}$$

$$\det \tilde{G}_+ = \det \tilde{\mathbb{C}}_+ \det(\tilde{W}_+), \qquad \det \tilde{G}_- = \det \tilde{C}_- \det(\tilde{W}_-), \tag{7.19b}$$

where the Schur complements are given by

$$W_\pm = A_\pm - B_\pm C_\pm^{-1} B_\pm^t, \qquad \tilde{W}_+ = \tilde{A}_+ - \tilde{\mathbb{B}}_+ \tilde{\mathbb{C}}_+^{-1} \tilde{\mathbb{B}}_+^t, \qquad \tilde{W}_- = \tilde{A}_- - \tilde{B}_- \tilde{C}_-^{-1} \tilde{B}_-^t, \tag{7.20}$$

where $\tilde{\mathbb{B}}_+$ and $\tilde{\mathbb{C}}_+$ now also include the D's of G_+ so they are $P_1 \times (P_2+1)$ and $(P_2+1)\times(P_2+1)$ matrices respectively,

$$\tilde{\mathbb{C}}_+ = -\delta_{nm}\sum_q^{2P_1} g(\tilde{v}_n - u_q), \qquad \tilde{\mathbb{B}}_{nm}^+ = g(\tilde{v}_m - u_n). \tag{7.21}$$

In the distinguished choice the superdeterminant can be written in terms of the Schur complements

$$\mathbb{D} = \frac{\det G_+}{\det G_-} = \frac{\det W_+}{\det W_-}, \tag{7.22}$$

since $C_+ = C_-$, while the dual superdeterminant becomes

$$\tilde{\mathbb{D}} = \frac{\det \tilde{G}_+}{\det \tilde{G}_-} = \frac{\det \tilde{\mathbb{C}}_+}{\det \tilde{C}_-}\frac{\det \tilde{W}_+}{\det \tilde{W}_-}. \tag{7.23}$$

For the first fraction of the dual superdeterminant, we have that

$$\frac{\det \tilde{\mathbb{C}}_+}{\det \tilde{C}_-} = \tilde{D}_3 = \sum_q^{2P_1} \frac{-1}{u_q^2 + \frac{1}{4}} = 2\sum_q^{P_1} \frac{-1}{u_q^2 + \frac{1}{4}} = -2\frac{\sum_j^{P_1}\prod_{k\neq j}^{P_1} u_k^2 + \frac{1}{4}}{\prod_k^{P_1} u_k^2 + \frac{1}{4}}$$
$$= 2(-1)^{P_1+1}\frac{\sum_j^{P_1}\prod_{k\neq j}^{P_1} u_k^2 + \frac{1}{4}}{\mathcal{Q}_1(\frac{i}{2})} = \frac{2P_1\mathcal{Q}_2(0)\tilde{\mathcal{Q}}_2(0)}{\mathcal{Q}_1(\frac{i}{2})}, \tag{7.24}$$

where we used the $\mathcal{QQ}$ relation (7.5) to find that

$$\begin{aligned}
\mathcal{Q}_2(0)\tilde{\mathcal{Q}}_2(0) &= \frac{i}{2P_1}\lim_{x\to 0}\frac{\mathcal{Q}_1^-(x) - \mathcal{Q}_1^+(x)}{x} \\
&= \frac{i}{2P_1}\lim_{x\to 0}\frac{\prod_j (x - u_j - \frac{i}{2}) - \prod_j (x - u_j + \frac{i}{2})}{x} \\
&= \frac{i}{2P_1}\sum_k^{2P_1}\left(\prod_{j\neq k}^{2P_1}(-u_j - \frac{i}{2}) - \prod_{j\neq k}^{2P_1}(-u_j + \frac{i}{2})\right) \\
&= -\frac{1}{2P_1}\sum_k^{2P_1}\frac{1}{u_k^2 + \frac{1}{4}}\prod_j^{2P_1}(-u_j - \frac{i}{2}) \\
&= -\frac{1}{P_1}\sum_k^{P_1}\prod_{j\neq k}^{P_1}(u_j^2 + \frac{1}{4}).
\end{aligned} \tag{7.25}$$

We have thus found the desired factor in the conjectured transformation of the superdeterminant. What we need now is to show that

$$\frac{\det W_+}{\det W_-} = \frac{\det \tilde{W}_+}{\det \tilde{W}_-}, \tag{7.26}$$

in order for the conjecture to be correct[1]. The idea is to prove this by induction over P_1 in analogy to what we did when proving the norm function for the Bethe states in chapter 4. In the base case, $P_1 = 1$, there are two paired momentum-carrying roots, $\{u_1, -u_1\}$, and only one auxiliary root which is a dual zero root by construction. We then have that

$$A_1 = \partial_{u_1}\phi_1 = -i\partial_{u_1}\left(\log\left(\frac{u_1 - \frac{i}{2}}{u_1 + \frac{i}{2}}\right)^L + \log\left(\frac{u_1 - u_2 + i}{-u_1 + u_2 + i}\right)\right) = \frac{L - \frac{1}{2}}{u_1^2 + \frac{1}{4}}, \tag{7.27}$$

and

$$A_2 = \partial_{u_2}\phi_1 = \frac{\frac{1}{2}}{u_1^2 + \frac{1}{4}}, \tag{7.28}$$

while $B_\pm$ and $C_\pm$ are empty and consequently

$$W_+ = A_1 + A_2 = \frac{L}{u_1^2 + \frac{1}{4}}, \qquad W_- = A_1 - A_2 = \frac{L-1}{u_1^2 + \frac{1}{4}}. \tag{7.29}$$

[1] Numerical computations with $(2P_1, 2P_2) = (4,2), (4,0), (2,0)$ indicate that not only are the fractions invariant but the Schur complements themselves are the same before and after the transformation. A stronger conjecture would therefore be $W_\pm = \tilde{W}_\pm$.

The dual set of roots are $\{\{u_1,-u_1\},\{0\}\}$ which yields

$$\tilde{A}_1=\partial_{u_1}\phi_1=-i\partial_{u_1}\left(\log\left(\frac{u_1-\frac{i}{2}}{u_1+\frac{i}{2}}\right)^L+\log\left(\frac{u_1-\tilde{v}+\frac{i}{2}}{u_1-\tilde{v}-\frac{i}{2}}\right)\right)=\frac{L-1}{u_1^2+\frac{1}{4}}, \tag{7.30}$$

while $\tilde{A}_2$ is zero. $\tilde{B}_\pm$ and $\tilde{C}_\pm$ are still empty blocks while

$$\tilde{D}_1=\partial_{\tilde{v}}\phi_1^1=-i\partial_{\tilde{v}}\log\left(\frac{u_1-\tilde{v}+\frac{i}{2}}{u_1-\tilde{v}-\frac{i}{2}}\right)=\frac{1}{u_1^2+\frac{1}{4}}, \tag{7.31}$$

and

$$\tilde{D}_3=\partial_{\tilde{v}}\phi^2=-\frac{2}{u_1^2+\frac{1}{4}}. \tag{7.32}$$

We therefore have that

$$\tilde{W}_-=\tilde{A}_1, \qquad \tilde{W}_+=\tilde{A}_1-2\tilde{D}_1\tilde{D}_3^{-1}\tilde{D}_1=\frac{L}{u_1^2+\frac{1}{4}}, \tag{7.33}$$

and thus

$$\frac{W_+}{W_-}=\frac{\tilde{W}_+}{\tilde{W}_-}=\frac{L}{L-1}. \tag{7.34}$$

Which concludes the base case. For the inductive step we need to show that

$$\frac{\det\Omega_+}{\det\Omega_-}-\frac{\det\tilde{\Omega}_+}{\det\tilde{\Omega}_-}=0 \tag{7.35}$$

where $\Omega_\pm$ and $\tilde{\Omega}_\pm$ are the same Schur complement as $W_\pm$ and $\tilde{W}_\pm$ but with an additional pair of momentum-carrying roots $\{u_m,-u_m\}$ and an additional pair of auxiliary roots $\{v_n,-v_n\}$. The idea is to consider (7.35) as a function of u_m. Then, by showing that the function vanishes in the limit $|u_m|\to\infty$ and has no residues it must in fact be zero. Since we want to satisfy the $\mathcal{QQ}$ relation (7.5) in the limit, we need to take the extra pair of dual roots to infinity as well. Specifically, we have that the leading order of the $\mathcal{QQ}$ relation with an extra pair of momentum-carrying and auxiliary roots are

$$\mathcal{Q}_1^+u_m^2-\mathcal{Q}_1^-u_m^2=i(2P_1+2)u\tilde{\mathcal{Q}}_2\mathcal{Q}_2v_n^2, \tag{7.36}$$

where v_n can be either ordinary or dual. Consequently, we need that

$$v_n^2=\alpha^2u_m^2, \qquad \alpha^2=\frac{P_1}{P_1+1}, \tag{7.37}$$

in order for the $\mathcal{QQ}$ relation to be satisfied by the old roots in the limit $|u_m|\to\infty$. Assuming first that v_n is dual, the ordinary Schur complements with the extra roots are

$$\begin{aligned}\Omega_\pm=\delta_{ij}&\left(\frac{L}{u_i^2+\frac{1}{4}}-2\sum_q^{2P_1+2}f(u_i-u_q)+\sum_q^{2P_2}g(u_i-v_q)\right)+2f(u_i-u_j)\pm2f(u_i+u_j)\\&-\sum_k^{2P_2/2}\left(g(u_i-v_k)\pm g(u_i+v_k)\right)\left(g(u_j-v_k)\pm g(u_j+v_k)\right)\left(\sum_q^{2P_1+2}g(v_k-u_q)\right)^{-1},\end{aligned} \tag{7.38}$$

where we used that

$$(B_\pm C_\pm^{-1} B_\pm^t)_{ij} = \sum_k (B_\pm)_{ik} (C_\pm^{-1})_{kk} (B_\pm)_{jk}, \tag{7.39}$$

since $C_\pm$ are diagonal and

$$\begin{aligned}\tilde{\Omega}_\pm = \delta_{ij} &\left(\frac{L}{u_i^2 + \frac{1}{4}} - \sum_q^{2\tilde{P}_2+2} g(u_i - \tilde{v}_q) \right) \\ &+ \sum_k^{\tilde{N}} \left(g(u_i - \tilde{v}_k) \pm g(u_i + \tilde{v}_k)\right)\left(g(u_j - \tilde{v}_k) \pm g(u_j + \tilde{v}_k)\right) \left(\sum_q^{2P_1+2} g(\tilde{v}_k - u_q) \right)^{-1},\end{aligned} \tag{7.40}$$

with $\tilde{N} = \tilde{P}_2 + 1$ for $\tilde{\Omega}_+$ and $\tilde{N} = \tilde{P}_2$ for $\tilde{\Omega}_-$. The behavior of f and g in the limit $\varepsilon = \frac{1}{u_m} \to 0$ is given by

$$g(\varepsilon^{-1}) = \frac{1}{\varepsilon^{-2} + \frac{1}{4}} \approx \varepsilon^2 \approx f(\varepsilon^{-1}), \tag{7.41}$$

and thus the m^{th} row and m^{th} column will go to zero for $\varepsilon \to 0$. Consequently, the leading order of the Laplace expansion in this limit is given by

$$\det \Omega_\pm \approx (\Omega_\pm)_{mm} \det W_\pm, \qquad \det \tilde{\Omega}_\pm \approx (\tilde{\Omega}_\pm)_{mm} \det \tilde{W}_\pm, \tag{7.42}$$

where $W_\pm$ and $\tilde{W}_\pm$ are the old Schur complements with P_1 momentum-carrying and P_2 auxiliary pairs since the contribution of u_m and $\tilde{v}_n$ vanishes in the limit. The behavior of Ω_{mm} in the limit $\varepsilon = \frac{1}{u_m} \to 0$ is

$$\begin{aligned}(\Omega_\pm)_{mm} &= \left(\frac{L}{\frac{1}{\varepsilon^2} + \frac{1}{4}} - 2\sum_q^{2P_1} f(\varepsilon^{-1} - u_q) + \sum_q^{2P_2} g(\varepsilon^{-1} - v_q) \right) + (-2 \pm 2) f(2\varepsilon^{-1}) \\ &\approx \varepsilon^2 \left(L - 4P_1 + 2P_2 - \frac{1}{2} \pm \frac{1}{2} \right).\end{aligned} \tag{7.43}$$

For the behavior of $\tilde{\Omega}_{mm}$, we need to consider the extra pair of dual roots in the limit $\varepsilon = \frac{1}{u_m} \to 0$ and we thus set $\tilde{v}_n = \alpha u_m$,

$$(\tilde{\Omega}_\pm)_{mm} \approx \varepsilon^2 \left(L - 2\tilde{P}_2 - \frac{1}{(1+\alpha)^2} - \frac{1}{(1-\alpha)^2} + \frac{\left(\frac{1}{(1+\alpha)^2} \pm \frac{1}{(1-\alpha)^2} \right)^2}{\frac{2P_1}{\alpha^2} + \frac{1}{(1+\alpha)^2} + \frac{1}{(1-\alpha)^2}} \right). \tag{7.44}$$

As argued before, we have to set $\alpha^2 = \frac{P_1}{P_1+1}$ and the terms of interest become

$$\frac{2P_1}{\alpha^2} + \frac{1}{(1+\alpha)^2} + \frac{1}{(1-\alpha)^2} = 4(P_1+1)^2, \tag{7.45a}$$

$$\frac{1}{(1+\alpha)^2} + \frac{1}{(1-\alpha)^2} = 2(P_1+1)(2P_1+1), \tag{7.45b}$$

$$\frac{1}{(1+\alpha)^2} - \frac{1}{(1-\alpha)^2} = 4\sqrt{\frac{P_1}{P_1+1}}(P_1+1)^2, \tag{7.45c}$$

which combines to

$$(\tilde{\Omega}_+)_{mm} \approx \varepsilon^2 \left(L - 4P_1 + 2P_2 \right), \qquad (\tilde{\Omega}_-)_{mm} \approx \varepsilon^2 \left(L - 4P_1 + 2P_2 - 1 \right). \tag{7.46}$$

Thus, the fractions, which we consider to be functions of u_m, behave like

$$\frac{\det\Omega_+}{\det\Omega_-} \to \frac{L-4P_1+2P_2}{L-4P_1+2P_2-1}\frac{\det W_+}{\det W_-}, \qquad \frac{\det\tilde{\Omega}_+}{\det\tilde{\Omega}_-} \to \frac{L-4P_1+2P_2}{L-4P_1+2P_2-1}\frac{\det\tilde{W}_+}{\det\tilde{W}_-}, \tag{7.47}$$

which is equal by the inductive hypothesis (7.26).

Assuming instead that v_n is an ordinary root, the terms including α in (47) moves to (46) and changes sign. Therefore, we have that

$$\begin{aligned}(\Omega_\pm)_{mm} &\approx \varepsilon^2\left(L-4P_1+2P_2-\frac{1}{2}\pm\frac{1}{2}+\frac{1}{(1+\alpha)^2}+\frac{1}{(1-\alpha)^2}-\frac{\left(\frac{1}{(1+\alpha)^2}\pm\frac{1}{(1-\alpha)^2}\right)^2}{\frac{2P_1}{\alpha^2}+\frac{1}{(1+\alpha)^2}+\frac{1}{(1-\alpha)^2}}\right)\\ &=\varepsilon^2\left(L-2P_1+2P_2+1\right),\end{aligned} \tag{7.48}$$

while for the dual Schur complements we simply get

$$(\tilde{\Omega}_\pm)_{mm} \approx \varepsilon^2\left(L-2\tilde{P}_2\right). \tag{7.49}$$

Consequently, in the case where the extra auxiliary root is ordinary, the ratio between the Schur complements behaves in the same way in the limit, with or without the extra dual roots

$$\frac{\det\Omega_+}{\det\Omega_-} \to \frac{\det W_+}{\det W_-}, \qquad \frac{\det\tilde{\Omega}_+}{\det\tilde{\Omega}_-} \to \frac{\det\tilde{W}_+}{\det\tilde{W}_-}, \tag{7.50}$$

In either case, whether the extra pair of roots are dual or not, we have that the difference (7.35) does indeed vanish in the limit. We therefore only need to show that the ratios $\frac{\det\Omega_+}{\det\Omega_-}$ and $\frac{\det\tilde{\Omega}_+}{\det\tilde{\Omega}_-}$ have the same residues such that the difference is an entire function. We will not give a full proof of this but will briefly sketch what needs to be shown in order to do so.

The possible poles we need to examine are located at $u_m = \pm i/2$, $u_m = \pm u_i \pm i$, $u_m = \pm v_i \pm \frac{i}{2}$ and when $\sum_q^{2P_1+2} g(\tilde{v}_k - u_q) = 0$ where the last factor in the sums of (7.38) and (7.40) diverge. Furthermore, we need to show that Ω_- being singular implies that either Ω_+ is singular such that the ratio $\frac{\det\Omega_+}{\det\Omega_-}$ is entire, or it implies that $\tilde{\Omega}_-$ is also singular in such a way that the ratios have the same residues and vice versa.

For the residues at $u = \pm i/2$, we see that they simply vanish

$$\mathrm{Res}\left(\frac{\det\Omega_+}{\det\Omega_-}, \pm\frac{i}{2}\right) = \lim_{u\to\pm i/2}\left(u\mp\frac{i}{2}\right)\frac{\frac{L}{(u+\frac{i}{2})(u-\frac{i}{2})}\det W_+}{\frac{L}{(u+\frac{i}{2})(u-\frac{i}{2})}\det W_-} = 0, \tag{7.51}$$

and similarly for $\frac{\det\tilde{\Omega}_+}{\det\tilde{\Omega}_-}$. Likewise, the other possible poles at $u_m = \pm u_i \pm i$ and $u_m = \pm v_i \pm \frac{i}{2}$ will also cancel since we get the same contribution in the nominator and the denominator. For the possible residues at $\sum_q^{2P_1+2} g(\tilde{v}_k - u_q) = 0$, we have to be a little more careful though. For most of the cases the residues will also simply vanish, but we see that $\tilde{\Omega}_+$ will contain a pole at $\sum_q^{2P_1+2} g(u_q) = 0$ which will not be canceled by $\tilde{\Omega}_-$. Instead, this will be canceled by $\tilde{D}_3$ which we have left out after realizing that it gave us our desired factor for our transformation.

For the cases where $\Omega_\pm$ is singular it is more difficult to see that the residues should vanish. If the assumption that $\frac{\det\Omega_+}{\det\Omega_-}$ is holomorphic was right we should in principle be able to use (7.47) to iteratively find the superdeterminant for all number of paired roots. However, numerical computations show that while this does indeed hold when the only dual auxiliary root is the zero root in which case the superdeterminant is given by

$$\mathbb{D} = \frac{L}{L-1} \prod_{n}^{P_1-1} \frac{L-4n}{L-4n-1}, \tag{7.52}$$

it does not hold when we start to add dual auxiliary roots. It thus seems likely that for our conjecture to be correct we need the residues of $\frac{\det \Omega_+}{\det \Omega_-}$ to cancel the residues of $\frac{\det \tilde{\Omega}_+}{\det \tilde{\Omega}_-}$ when $\Omega_\pm$ is singular. This however seems more difficult to show since the Schur complements are singular only when two rows become linearly dependent which numerically seems to happen only in highly non-trivial cases.

Assuming that the residues do cancel each other, by Liouville's theorem the two functions $\frac{\det \Omega_+}{\det \Omega_-}$ and $\frac{\det \tilde{\Omega}_+}{\det \tilde{\Omega}_-}$ are the same. Consequently, the superdeterminant transform according to

$$\tilde{\mathbb{D}} = 2P_1 \frac{\tilde{\mathcal{Q}}_2(0)\mathcal{Q}_2(0)}{\mathcal{Q}_1(\frac{i}{2})} \mathbb{D}, \tag{7.53}$$

for all number of paired roots. This completes our proof.

Chapter 8

Conclusion and Outlook

In this thesis, we have reviewed several aspects of the integrability of $\mathcal{N}=4$ SYM. In chapter 2, we showed how to derive the four-dimensional theory by dimensional reduction of the ten-dimensional $\mathcal{N}=1$ SYM theory and then discussed the symmetries and their consequences. In particular, we showed that the conformal symmetry restricted the structure of the two-point functions such that the space-time dependence was determined by the conformal dimension of the operators in question. We then showed that in the t' Hooft limit the gauge-invariant operators with a definite conformal dimension correspond to solutions of one-dimensional spin chains by doing an explicit Feynman diagram calculation at one-loop order in the $\mathfrak{so}(6)$ sector.

In chapter 3, we then showed that in the $\mathfrak{su}(2)$, $\mathfrak{su}(2|1)$ and $\mathfrak{su}(1|2)$ sectors the solutions to the spin chains could be found by the CBA, where the physical interpretation is that the M-body state consists of M magnons each of which is parameterized by a rapidity and travel around the spin chain scattering non-diffractively with the other magnons. With this physical interpretation, it came apparent that there were different dual ways to describe the spin chain in which different fields in the theory could be chosen to constitute the vacuum or a level of excitation.

As an interlude, we reviewed the ABA in chapter 4 which, though being a bit more abstract, formally proved the integrability of the theory. It also gave us the tools needed to prove a closed-form solution for the norm of a Bethe state in terms of the Gaudin matrix through induction and complex analysis.

In chapter 5, we then briefly discussed the dCFT in which a domain wall breaks the conformal symmetry such that one-point functions are no longer trivial. Instead, they can be expressed as overlaps between Bethe states and boundary states of the theory which in turn can be written as the square root of the superdeterminant of the Gaudin matrix decorated with some Baxter polynomials.

The overlap formulae, thus turned out to be an interesting playground to further investigate the duality transformations first encountered in the CBA. We therefore took a closer look on the nature of these dualities in chapter 6 where we showed that we could use a fermionic duality to transform between the different Bethe equations of our super spin chain and that these in turn corresponded to the different choices of simple roots of the Lie superalgebra.

Finally, we were ready to investigate the transformation of the superdeterminant appearing in the overlap formula between a valence bond state and a $\mathfrak{su}(2|1)$ Bethe state in chapter 7. Starting in the distinguished representation of the Lie superalgebra, ⋯○⋯⊗—, we attempted to prove that the overlap formula transformed covariantly with the duality transformation taking us to the —⊗—⊗⋯ representation. With the exception of possible poles when the Schur complements become singular we succeeded in doing so.

8.1 Outlook

It would have been nice to finish the proof of the transformation rule for the superdeterminant of the $\mathfrak{su}(2|1)$ sector, but the possible poles when the Schur complements become singular leaves room for further investigation. With the non-trivial way in which these singularities occur, it seems unlikely that there is an easy fix though.

As stated in chapter 6, the fermionic duality transformation of $\mathfrak{su}(2|1)$ is only part of a much bigger set of duality transformations of the superconformal algebra. In total, there are $2^8 = 256$ paths in the full Hasse diagram of $\mathfrak{psu}(2,2|4)$ which are all connected by the fermionic and bosonic dualities. Since all the known overlap formulae in the theory contain the superdeterminant of the Gaudin matrix, an obvious next step would be to generalize the proof given here to the rest of the fermionic duality transformations and prove the numerical findings in [20]. This includes duality transformations of momentum-carrying nodes which are a bit more complicated than non-momentum-carrying nodes since the superdeterminant becomes singular and needs regularization [20]. It would also be interesting to consider the bosonic dualities which would further restrict the form of the overlaps if the superdeterminant is to transform covariantly.

Having seen that both the overlaps between Bethe states and either matrix product states or valence bond state are proportional to $\det G_+$, an interesting question taking us in a different direction is whether there exist any boundary states where the overlap is proportional to $\det G_-$. This remains an open question as a such state is still to be found [21].

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Appendices

Appendix A

Lie Superalgebras

Following [30], we will briefly review the notion of a Lie superalgebra and determine the roots and generators of $\mathfrak{su}(2|1)$ to show how to arrive at the two dual representations extensively used throughout this thesis.

A.1 The Notion of a Lie Superalgebra

A Lie superalgebra g_s is a generalized Lie algebra where a grading has been introduced. To construct the algebra, we start out with a $\mathbb{Z}_2$ graded vector space

$$\mathfrak{g}_s = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \tag{A.1}$$

where homogeneous elements are either in $\mathfrak{g}_0$ and said to be even and have grade 0 or in $\mathfrak{g}_1$ and said to be odd and have grade 1. Just as with the regular Lie algebra we equip the vector space with additional structure, but instead of the usual Lie bracket we introduce a Lie superbracket

$$[\cdot,\cdot] : \mathfrak{g}_s \times \mathfrak{g}_s \to \mathfrak{g}_s. \tag{A.2}$$

For two elements $A, B \in \mathfrak{g}_s$ the superbracket satisfies

$$[aA + bB, C] = a[A, C] + b[B, C], \tag{A.3}$$

and for three homogeneous elements $A, B, C \in \mathfrak{g}_s$ it satisfies

$$\deg([A, B]) = (\deg A + \deg B) \qquad \mod 2. \tag{A.4}$$

Finally, any three elements $A, B, C \in \mathfrak{g}_s$ satisfy the generalized Jacobi identity

$$[A, [B, C]](-1)^{(\deg A)(\deg C)} + [B, [C, A]](-1)^{(\deg B)(\deg A))} + [C, [A, B]](-1)^{(\deg C)(\deg B)} = 0. \tag{A.5}$$

If the dimension of the even part is m and the dimension of the odd part is n a matrix representation is given by

$$E = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad O = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad A \in \mathbb{M}(m, m), \quad D \in \mathbb{M}(n, n), \ldots \tag{A.6}$$

such that the product of two even (odd) elements gives an even element and the product of an even and an odd element gives an odd element.

The Killing form is defined by

$$K(X,Y) = \mathrm{Str}(\mathrm{ad}\,X \circ \mathrm{ad}\,Y), \tag{A.7}$$

where the supertrace is defined by

$$\mathrm{Str}\,M = \mathrm{tr}\,A - \mathrm{tr}\,D, \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{A.8}$$

and the adjoint representation is such that

$$\mathrm{ad}(X)Y = [X,Y]. \tag{A.9}$$

A.2 The Generators and Roots of $\mathfrak{su}(2|1)$

The Lie superalgebra $A(1|0) = \mathfrak{sl}(2|1;\mathbb{C})$, which is the complexified version of $\mathfrak{su}(2|1)$, consists of elements which are 3×3 supertraceless matrices with grading $(2|1)$, i.e. the even dimension is 2 and the odd dimension is one. The exponential map will then result in a supergroup consisting of invertible supermatrices with superdeterminant one since the property

$$\det A = \exp(\mathrm{tr}\log A), \tag{A.10}$$

carries over to supermatrices.

The Cartan subalgebra consists as usual of the maximally diagonalizable number of generators

$$[H_i, H_j] = 0, \tag{A.11}$$

which we, in this case, choose to be

$$H = \mathrm{diag}(1,-1,0), \qquad C = \mathrm{diag}(-1,-1,-2). \tag{A.12}$$

The rest of the generators are then chosen to be the ladder operators that satisfy

$$[H_i, E_j^{\pm}] = \pm a_{ij} E_j^{\pm}, \qquad [E_i^+, E_j^-] = \delta_{ij} H_i. \tag{A.13}$$

In this case we get 6 pairs of ladder operators

$$(E_{ij})_{lm} = \delta_{il}\delta_{jm}, \tag{A.14}$$

i.e. all the matrices with one non-zero entry that is off-diagonal. We can then determine the commutators giving the roots

$$[C, E_{12}] = 0, \qquad [C, E_{23}] = E_{23}, \qquad [C, E_{13}] = E_{13}, \tag{A.15a}$$

$$[H, E_{12}] = 2E_{12}, \qquad [H, E_{23}] = -E_{23}, \qquad [H, E_{13}] = E_{13}, \tag{A.15b}$$

and the rest of the commutators are given by $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{jk}$. A root is even if its corresponding element is even and likewise odd if its corresponding element is odd, we thus have two even and four odd roots,

$$\alpha_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{A.16}$$

and their additive inverses. Now, using the basis $(C, H, E_{12}, E_{21}, E_{13}, E_{31}, E_{23}, E_{32})$ the adjoint representation is given by

$$\operatorname{ad} C = \operatorname{diag}(0,0,0,0,1,-1,1,-1), \qquad \operatorname{ad} H = \operatorname{diag}(0,0,2,-2,1,-1,-1,1), \tag{A.17}$$

and consequently the killing forms and the Killing metric are

$$K(H,H) = 4, \quad K(C,C) = -4, \quad K(H,C) = 0, \quad K(H^i, H^j) = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}, \tag{A.18}$$

giving the scalar products

$$\begin{aligned} &(\alpha_1, \alpha_1) = \alpha_1^i \alpha_1^j K(H^i, H^j)^{-1} = 1, \quad (\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -\frac{1}{2}, \\ &(\alpha_1, \alpha_3) = \frac{1}{2}, \quad (\alpha_2, \alpha_2) = (\alpha_3, \alpha_3) = 0. \end{aligned} \tag{A.19}$$

In order to determine the positive and negative roots, we need first to introduce the concepts of solvability and the Borel subalgebra. An algebra $\mathfrak{g}$ is set to be solvable if the iterative process $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}]$ with $\mathfrak{g}^o = \mathfrak{g}$ terminates at some point. A Borel subalgebra $\mathfrak{b}$ of an algebra $\mathfrak{g}$ is a maximally solvable subalgebra of $\mathfrak{g}$. For instance if $\mathfrak{g} = \mathfrak{sl}(2)$ then a Borel subalgebra consists of all the generators that are upper (lower) triangular matrices.

The Borel subalgebra $\mathfrak{b}$ can then be decomposed into the Cartan subalgebra h and a remainder $\mathcal{N}^+$, and the whole algebra then decomposes into the Borel algebra and a second remainder $\mathcal{N}^-$. The roots are then positive if the generator of the root is in $\mathcal{N}^+$ and negative if it is in $\mathcal{N}^-$. The simple roots are then the roots that cannot be written in terms of two positive roots.

Going back to $A(1|0)$ we can then chose the basis of $\mathfrak{b}$ to be $\{ H, C, E_{12}, E_{23}, E_{13} \}$ such that $\mathcal{N}^+ = \{ E_{12}, E_{23}, E_{13} \}$. Then $\alpha_1, \alpha_2, \alpha_3$ are positive roots. Since $\alpha_3 = \alpha_1 + \alpha_2$ only α_1, α_2 are simple roots, one even and one odd. Another choice for $\mathcal{N}^+$ could be $\mathcal{N}^+ = \{ E_{12}, E_{32}, E_{13} \}$ where the simple roots are now the two odd roots α_2, α_3.

The Cartan Matrix is given by the inner product of the simple roots

$$\begin{aligned} \mathcal{M}_{jk} &= \frac{2(\alpha_j, \alpha_k)}{(\alpha_j, \alpha_j)} && \text{if } (\alpha_j, \alpha_j) \neq 0 \\ , \mathcal{M}_{jk} &= \frac{(\alpha_j, \alpha_k)}{(\alpha_j', \alpha_j)} && \text{if } (\alpha_j, \alpha_j) = 0, \end{aligned} \tag{A.20}$$

where $\alpha_{j'}$ is another simple root such that $(\alpha_j, \alpha_{j'}) \neq 0$. For the case of $A(1|0)$ and with the simple roots α_1, α_2 we get the Cartan Matrix

$$\mathcal{M} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A.21}$$

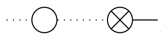
while with the choice of α_2, α_3 being simple we get that

$$\tilde{\mathcal{M}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.22}$$

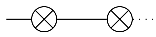
The generalized Dynkin diagram is given by the set of rules:

1. Draw a node for each simple root that is:
   - ○ if the root is even
   - ⊗ if the root is odd and has zero norm
   - ● if the root is odd and has a non-zero norm
2. Connect the nodes j and k by $\max\{|\mathcal{M}_{jk}|, |\mathcal{M}_{kj}|\}$ lines (and let the line be dotted if $\mathcal{M}_{jk}$ is -1)
3. Add an arrow from j to k if $|\mathcal{M}_{jk}| > 1$

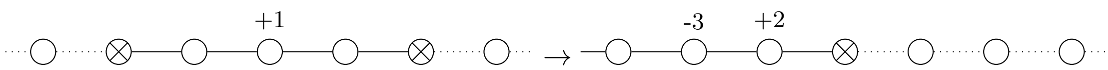
The Dynkin diagram for $A(1|0)$ with the choice of α_1, α_2 as simple roots is then


, (A.23)

while the choice of α_2, α_3 as simple roots gives the Dynkin diagram


. (A.24)

The two different choices og simple roots exactly corresponds to the two choices of vacuua and excitations for $\mathfrak{su}(2|1)$ considered in in this thesis. The duality transformations of the Bethe roots will, in general, correspond to the different choices of simple roots in the algebra [31]. One could, for example, change between the Beauty and the Beast representations of $\mathfrak{su}(2,2|4)$ using the fermionic duality



. (A.25)

A.3 Complexification of an Algebra

We usually deal with real Lie algebras, i.e. we deal with vector spaces of over the field of real numbers. We can then complexify the vector space of a real Lie algebra $\mathfrak{g}$ to get the complex vector space $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}$ and equip it with a new Lie bracket $[\cdot,\cdot]_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ defined by

$$[1 \otimes X_1 + i \otimes X_2, 1 \otimes Y_1 + i \otimes Y_2]_{\mathbb{C}} = 1 \otimes [X_1, Y_1] - 1 \otimes [X_2, Y_2] + i \otimes [X_1, Y_2] + i \otimes [X_2, Y_1], \tag{A.26}$$

with $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}$. With this in mind, the complexified algebra of $\mathfrak{su}(2)$ is can be shown to be isomorphic to $\mathfrak{sl}(2)$. If we consider the map $\phi : \mathfrak{su}(2)_{\mathbb{C}} \to \mathfrak{sl}(2,\mathbb{C})$ given by

$$1 \otimes X_1 + i \otimes X_2 \to X_1 + iX_2. \tag{A.27}$$

This spans all of $\mathfrak{sl}(2)$ since every matrix can be written in terms of a Hermitian plus an anti Hermitian matrix and the bracket is conserved since

$$[X_1 + iX_2, Y_1 + iY_2] = [X_1, Y_1] - [X_2, Y_2] + i[X_1, Y_2] + i[X_2, Y_1], \tag{A.28}$$

which is just the normal commutator of matrices and also what (A.26) maps to. $\mathfrak{su}(2)$ is thus said to be a real form of $\mathfrak{sl}(2,\mathbb{C})$. In the same way $\mathfrak{su}(2,2|4)_{\mathbb{C}} \cong \mathfrak{sl}(4|4,\mathbb{C})$ although in literature it is not atypical to use it interchangeably with $\mathfrak{su}(2,2|4)$.

A.4 Scalar Product with Killing Form

The Killing form is a symmetric bilinear form of two elements in a Lie algebra

$$K(X,Y) = \mathrm{tr}(\mathrm{ad}\,X \circ \mathrm{ad}\,Y), \tag{A.29}$$

where the adjoint representation is given by

$$\mathrm{ad}(X)Y = [X,Y], \tag{A.30}$$

such that all the algebra now acts on the generators. Now every root α in our algebra maps an element in our Cartan subalgebra to a number $\alpha : h \to h^*$

$$\alpha(H^i) = \alpha^i, \tag{A.31}$$

and the roots are therefore elements in the dual vector space to our Cartan subalgebra. The Killing form can then be used as an isomorphism between the Cartan subalgebra and its dual such that for an element in the dual space $\gamma \in h^*$ there is a corresponding element $H^\gamma \in h$

$$\gamma(H^i) = \gamma^i = K(H^i, H^\gamma). \tag{A.32}$$

For our roots α^i this implies that their connection to their covectors α_j are

$$\alpha(H^i) = \alpha^i = K(H^i, H^\alpha) = \alpha_j K(H^i, H^j) = \alpha_j K^{ij}, \tag{A.33}$$

where $H^\alpha = \alpha_i H^i$ and we can therefore use the Killing form to raise and lower indices and to define a scalar product in the dual space

$$(\alpha, \beta) = K(H^\alpha, H^\beta) = \alpha_i \beta_j K^{ij} = \alpha^i \beta^j K_{ij}, \tag{A.34}$$

with K_{ij} being the metric in the dual space and the inverse of K^{ij}.

Appendix B

Solutions to the Two-body Problem

The two-body problem is of special interest since it is the simplest non-trivial problem where we can enforce our zero momentum constraint. The problem can be investigated from both the CBA and ABA approaches which are very different in nature though they still give the same solutions. In order to see this we have explicitly made the calculations showing that the two-body CBA and ABA states are in fact the same up to some normalization. Furthermore, we have made an explicit calculation of the norm of the two-body states.

B.1 The Two-body CBA Eigenstate

Considering only solutions with paired rapidities, the two-body CBA wave function is given by

$$\begin{aligned}\Psi_2(x_1,x_2) &= \left(2-e^{-ip_1}-e^{ip_2}\right)e^{ip_1}e^{ip_1x_1+ip_2x_2}-\left(2-e^{-ip_2}-e^{ip_1}\right)e^{ip_2}e^{ip_2x_1+ip_1x_2}\\ &\propto e^{ip(x_2-x_1)}+e^{-ip(x_2-x_1+1)}\propto\cos\left(p\left(x_2-x_1+\frac{1}{2}\right)\right),\end{aligned} \tag{B.1}$$

where we used that $p=p_2=-p_1=\frac{2\pi n}{L-1}$. We then define

$$|\Psi\rangle_{\text{CBA}}\equiv 2\sum_{x_2>x_1}\cos\left(p\left(x_2-x_1+\frac{1}{2}\right)\right)S^-_{x_1}S^-_{x_2}|0\rangle \tag{B.2}$$

$$=\sum_{x_1=1}^{L}\sum_{\Delta=1}^{L-1}\cos\left(p\left(\Delta+\frac{1}{2}\right)\right)S^-_{x_1}S^-_{x_1+\Delta}|0\rangle, \tag{B.3}$$

where we in the last line used the periodicity $S_n=S_{n+L}$ of the chain and counted every term twice to simplify the sum. The norm of this state is then

$$\begin{aligned}\langle\Psi|\Psi\rangle_{\text{CBA}} &= 4\sum_{x_2>x_1}\cos^2\left(p\left(x_2-x_1+\frac{1}{2}\right)\right)=2\sum_{x_1=1}^{L}\sum_{\Delta=1}^{L-1}\cos^2\left(p\left(\Delta+\frac{1}{2}\right)\right)\\ &= L\sum_{\Delta=1}^{L-1}1+\cos\left(p\left(2\Delta+1\right)\right)=L(L-1)+\frac{L}{2}\sum_{\Delta=1}^{L-1}e^{ip(2\Delta+1)}+e^{-ip(2\Delta+1)},\end{aligned} \tag{B.4}$$

but

$$\sum_{\Delta=1}^{L-1}e^{2ip\Delta}=e^{2ip}\sum_{\Delta=0}^{L-2}e^{2ip\Delta}=e^{2ip\Delta}\frac{1-e^{2ip(L-1)}}{1-e^{2ip}}=0, \tag{B.5}$$

since $p=\frac{2\pi l}{L-1}$ for $l=0,\dots L-1$. Consequently the norm is simply given by

$$\langle\Psi|\Psi\rangle_{\text{CBA}}=L(L-1). \tag{B.6}$$

B.2 The Two-body ABA Eigenstate

In the ABA the two-body eigenstate is given by

$$|\Psi\rangle_{\mathrm{ABA}} \equiv B(-\lambda)B(\lambda)\,|0\rangle\,. \tag{B.7}$$

In order to find out what this is we need to consider the Lax operator acting on the vacuum state

$$\mathbb{L}_n\,|0\rangle = \begin{pmatrix} \lambda_+ & iS_n^- \\ 0 & \lambda_- \end{pmatrix}|0\rangle\,, \qquad \lambda_\pm = \lambda \pm \frac{i}{2}. \tag{B.8}$$

For two Lax operators we have that

$$\mathbb{L}_2\mathbb{L}_1\,|0\rangle = \begin{pmatrix} \lambda_+^2 & i\lambda_+S_1^- + i\lambda_-S_2^- \\ 0 & \lambda_-^2 \end{pmatrix}, \tag{B.9}$$

and for three we get

$$\mathbb{L}_3\mathbb{L}_2\mathbb{L}_1\,|0\rangle = \begin{pmatrix} \lambda_+^3 & \lambda_+(i\lambda_+S_1^- + i\lambda_-S_2^-) + i\lambda_-^2S_3^- \\ 0 & \lambda_-^3 \end{pmatrix}. \tag{B.10}$$

Following this pattern, we have that upper right part of the full monodromy on the vacuum state is given by

$$B(\lambda)\,|0\rangle = i\sum_{n=1}^{L}\lambda_+^{L-n}\lambda_-^{n-1}S_n^-\,|0\rangle = i\lambda_+^L\lambda_-^{-1}\sum_{n=1}^{L}e^{-ipn}S_n^-\,|0\rangle\,, \tag{B.11}$$

where we used the reparametrization of the rapidities to the momenta

$$\lambda_i = \frac{1}{2}\cot\frac{p_i}{2}, \qquad e^{ip_i} = \frac{\lambda_+^i}{\lambda_-^i}, \qquad \lambda_\pm^i = \pm\frac{i}{1-e^{\mp ip_i}}. \tag{B.12}$$

If we now consider the action of $B(\lambda)$ on $S_m^-\,|0\rangle$ we have that

$$\mathbb{L}_nS_m^-\,|0\rangle = S_m^-\begin{pmatrix} \lambda_+ & iS_n^- \\ 0 & \lambda_- \end{pmatrix}|0\rangle \qquad \text{for } n \neq m \tag{B.13}$$

$$= \begin{pmatrix} (\lambda + iS_m^3)S_m^- & 0 \\ iS_m^+S_m^- & (\lambda - iS_m^3)S_m^- \end{pmatrix}|0\rangle = \begin{pmatrix} \lambda_-S_m^- & 0 \\ i & \lambda_+S_m^- \end{pmatrix}|0\rangle \qquad \text{for } n = m. \tag{B.14}$$

Therefore we have that the full monodromy on $S_m^-\,|0\rangle$ is

$$\mathbb{L}_L\ldots\mathbb{L}_{m+1}\mathbb{L}_m\mathbb{L}_{m-1}\ldots\mathbb{L}_1S_m^-\,|0\rangle =$$
$$\begin{pmatrix} \lambda_+^{L-m} & i\sum_{n=m+1}^{L}\lambda_+^{L-n}\lambda_-^{n-m-1}S_n^- \\ 0 & \lambda_-^{L-m} \end{pmatrix}\begin{pmatrix} \lambda_-S_m^- & 0 \\ i & \lambda_+S_m^- \end{pmatrix}\begin{pmatrix} \lambda_+^{m-1} & i\sum_{n=1}^{m-1}\lambda_+^{m-n-1}\lambda_-^{n-1}S_n^- \\ 0 & \lambda_-^{m-1} \end{pmatrix}|0\rangle\,. \tag{B.15}$$

We thus have that $B(\lambda)$ on $S_m^- |0\rangle$ is

$$\begin{aligned} B(\lambda)S_m^- |0\rangle &= i\lambda_+^{L-1} S_m^- \sum_{n=1}^{m-1} e^{-ipn} S_n^- + i\lambda_+^{L+1}\lambda_-^{-2} S_m^- \sum_{n=m+1}^{L} e^{-ipn} S_n^- \\ &- i\lambda_+^{L+m-1}\lambda_-^{-m-2} \left(\sum_{n=m+1}^{L} e^{-ipn} S_n^- \right) \left(\sum_{n=1}^{m-1} e^{-ipn} S_n^- \right) |0\rangle \\ &= i\lambda_+^{L}\lambda_-^{-1} \left(S_m^- \left[\sum_{n=1}^{m-1} e^{-ip(n+1)} S_n^- + \sum_{n=m+1}^{L} e^{-ip(n-1)} S_n^- \right] \right. \\ &\left. -\lambda_+^{-1}\lambda_-^{-1} e^{ipm} \sum_{n=m+1}^{L} \sum_{l=1}^{m-1} e^{-ip(n+l)} S_n^- S_l^- \right) |0\rangle . \end{aligned} \tag{B.16}$$

Combining (B.11) with the first line in (B.16), we have

$$\begin{aligned} \frac{B(\lambda)B(-\lambda)}{(-1)^L \lambda_-^{L-1}\lambda_+^{L-1}} |0\rangle &= \sum_{m>n} e^{-ip(n-m+1)} S_m^- S_n^- |0\rangle + \sum_{m<n} e^{-ip(n-m-1)} S_m^- S_n^- |0\rangle \\ &- \lambda_+^{-1}\lambda_-^{-1} \sum_{1\le l<m<n\le L} e^{ip(2m-n-l)} S_n^- S_l^- |0\rangle . \end{aligned} \tag{B.17}$$

The first sum simply gives us

$$2\sum_{m>n} \cos\left(p(m-n+1)\right) S_m^- S_n^- |0\rangle , \tag{B.18}$$

while for the triple summation we have that

$$\begin{aligned} S &= \sum_{m=1}^{L} \sum_{n=m+1}^{L} \sum_{l=1}^{m-1} e^{ip(2m-n-l)} = \sum_{n>l+1} \sum_{\Delta=1}^{n-l-1} e^{ip(2\Delta+l-n)} \qquad (\Delta = m-l) \\ &= \sum_{n>l+1} e^{ip(l-n)} \sum_{\Delta=1}^{n-l-1} e^{2ip\Delta} = e^{ip} \sum_{n>l+1} \frac{e^{ip(l-n+1)} - e^{ip(n-l-1)}}{1-e^{2ip}} \\ &= e^{ip} \sum_{n>l} \frac{e^{ip(l-n-1)} - e^{-ip(l-n-1)}}{1-e^{2ip}} = 2i\frac{e^{ip}}{1-e^{2ip}} \sum_{m>n} \sin\left(p(m-n+1)\right). \end{aligned} \tag{B.19}$$

Furthermore, we have that

$$\begin{aligned} \lambda_-^{-1}\lambda_+^{-1}\frac{e^{ip}}{1-e^{2ip}} &= (1-e^{-ip})(1-e^{ip})\frac{1}{e^{-ip}-e^{ip}} = 4\sin^2\frac{p}{2}\frac{i}{2\sin p} \\ &= i\frac{\sin^2\frac{p}{2}}{\cos\frac{p}{2}\sin\frac{p}{2}} = i\frac{\sin\frac{p}{2}}{\cos\frac{p}{2}}. \end{aligned} \tag{B.20}$$

Then using the identity $\cos A \cos B + \sin A \sin B = \cos A - B$ we get that

$$\begin{aligned} \frac{B(\lambda)B(-\lambda)}{(-1)^L \lambda_-^{L-1}\lambda_+^{L-1}} |0\rangle &= \frac{2}{\cos\frac{p}{2}} \sum_{m>n} \left(\cos\left(p(m-n+1)\right)\cos\frac{p}{2} + \sin\left(p(m-n+1)\right)\sin\frac{p}{2} \right) S_m^- S_n^- |0\rangle \\ &= \frac{2}{\cos\frac{p}{2}} \sum_{m>n} \cos\left(p(m-n+\frac{1}{2}) \right) S_m^- S_n^- |0\rangle \\ &= \frac{1}{\cos\frac{p}{2}} |\Psi\rangle_{\mathrm{CBA}} , \end{aligned} \tag{B.21}$$

which is in perfect agreement with what we got from the CBA (B.3)

$$\begin{aligned} |\Psi\rangle_{\text{ABA}} \equiv B(\lambda)B(-\lambda)\,|0\rangle &= (-1)^L \lambda_-^{L-1}\lambda_+^{L-1}\frac{1}{\cos\frac{p}{2}}\,|\Psi\rangle_{\text{CBA}} \\ &= (-1)^L\left(\lambda^2+\frac{1}{4}\right)^{L-1}\frac{1}{\cos\frac{p}{2}}\,|\Psi\rangle_{\text{CBA}}\,. \end{aligned} \tag{B.22}$$

For the two-body case λ is real and having already calculated the norm for the CBA (B.6), we get that

$$\begin{aligned} \langle\Psi|\Psi\rangle_{\text{ABA}} &= \left(\lambda^2+\frac{1}{4}\right)^{2L-2}\frac{4}{\cos^2\frac{p}{2}}\frac{L(L-1)}{4} \\ &= \left(\lambda^2+\frac{1}{4}\right)^{2L-1}\lambda^{-2}L(L-1). \end{aligned} \tag{B.23}$$

B.3 The Norm From the ABA Gaudin Matrix

In the ABA the Gaudin matrix is given by

$$M_{lk} = \frac{\partial\phi_k}{\partial\lambda_l}, \qquad \phi_k = i\log\left(\left(\frac{\lambda_k-\frac{i}{2}}{\lambda_k+\frac{i}{2}}\right)^L\prod_{j\neq k}\frac{\lambda_k-\lambda_j+i}{\lambda_k-\lambda_j-i}\right), \tag{B.24}$$

and the norm is given by

$$S_N = \prod_j\prod_{j\neq k} h(\lambda_k,\lambda_j)\det\frac{\partial\phi_m}{\partial\lambda_l}. \tag{B.25}$$

For $M=1$, there is only one root, and the Gaudin matrix becomes

$$M_{11} = -i\frac{\partial}{\partial\lambda_1}\log\left(\frac{\lambda_1-\frac{i}{2}}{\lambda_1+\frac{i}{2}}\right)^L = -iL\left(\frac{1}{\lambda_1-\frac{i}{2}}-\frac{1}{\lambda_1+\frac{i}{2}}\right) = \frac{L}{\lambda_1^2+\frac{1}{4}}. \tag{B.26}$$

For $M=2$, there are two roots, and the Gaudin matrix becomes

$$M_{11} = -i\frac{\partial}{\partial\lambda_1}\left(L\log\left(\frac{\lambda_1-\frac{i}{2}}{\lambda_1+\frac{i}{2}}\right)+\log\frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i}\right) = \frac{L}{\lambda_1^2+\frac{1}{4}}-\frac{2}{(\lambda_1-\lambda_2)^2+1} \tag{B.27}$$

$$M_{12} = \frac{\partial\phi_1}{\partial\lambda_2} = -i\frac{\partial}{\partial\lambda_2}\log\frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i} = \frac{2}{(\lambda_1-\lambda_2)^2+1} = \frac{2}{4\lambda_1^2+1} = M_{21} \tag{B.28}$$

$$M_{22} = \frac{L}{\lambda_2^2+\frac{1}{4}}-\frac{2}{(\lambda_1-\lambda_2)^2+1} = \frac{L-\frac{1}{2}}{\lambda_1^2+\frac{1}{4}}. \tag{B.29}$$

where we used that the rapidities are paired, $\lambda=\lambda_1=-\lambda_2$. For the determinant of the Gaudin matrix, we then have that

$$\det M_{ij} = M_{11}^2-M_{12}^2 = \left(\frac{L-\frac{1}{2}}{\lambda^2+\frac{1}{4}}\right)^2-\left(\frac{2}{4\lambda^2+1}\right)^2 = \frac{L(L-1)}{\left(\lambda^2+\frac{1}{4}\right)^2}, \tag{B.30}$$

and prefactor becomes

$$h(\lambda,-\lambda)h(-\lambda,\lambda) = \frac{\lambda^2+\frac{1}{4}}{\lambda^2}. \tag{B.31}$$

The norm is thus given by

$$S_2 = h(\lambda, -\lambda)h(-\lambda, \lambda)\det M_{ij} = \frac{L(L-1)}{\lambda^2\left(\lambda^2 + \frac{1}{4}\right)}, \tag{B.32}$$

which is not the same as (B.23). Remembering the normalization used to find the norm formula (4.29), we get that

$$\begin{aligned}\langle\Psi|\Psi\rangle_{ABA} &\equiv \langle 0|\, C(\lambda)C(-\lambda)B(\lambda)B(-\lambda)\,|0\rangle = \delta^L(\lambda)^2\delta^L(-\lambda)^2\,\langle 0|\,\mathbf{C}(\lambda)\mathbf{C}(-\lambda)\mathbf{B}(\lambda)\mathbf{B}(-\lambda)\,|0\rangle \\ &= \left(\lambda - \frac{i}{2}\right)^{2L}\left(\lambda + \frac{i}{2}\right)^{2L}\frac{L(L-1)}{\lambda^2\left(\lambda^2+\frac{1}{4}\right)} = \left(\lambda^2 + \frac{1}{4}\right)^{2L-1}\lambda^{-2}L(L-1), \end{aligned} \tag{B.33}$$

which is precisely the same as (B.23)

Appendix C

Numerical Solutions

Because of the difficulty of finding analytical solutions to the Bethe equation, we will have to rely on numerical methods for doing anything but the simplest cases. Numerical methods also proved to be a great tool when developing intuition on how the superdeterminant in chapter 7 transformed.

C.1 Numerical Solutions to the Heisenberg Spin Chain

In the $\mathfrak{su}(2)$ sector of the theory corresponding to the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain, we have analytically found the two-body solution where the states are paired

$$u_1 = \frac{1}{2}\cot\frac{\pi p}{L-1} = -u_2. \tag{C.1}$$

In order to numerically find more solutions to the Bethe equations it is often easier to work with their logarithmic version

$$iL\log\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}} + 2\pi n_k = \sum_{j\neq k}^{M}\log\frac{u_k-u_j+i}{u_k-u_j-i}, \tag{C.2}$$

where n_k are called the modes and arises from the branch cut of the logarithm. In order to find numerical solutions we look at the limit where $L >> M$. Then the right hand side of the equations can be disregarded at first in order to show that

$$u_k \approx \frac{L}{2\pi n_k}, \tag{C.3}$$

for large L. Then, we introduce a term of order $\sqrt{L}$

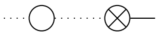
$$u_k \approx \frac{1}{2\pi n_k}\left(L + iz_k\sqrt{2L} + \mathcal{O}(1)\right), \tag{C.4}$$

where z_k is yet to be determined. Inserting this in the Bethe equations, we find that

$$z_k = \sum_{j\neq k}^{M}\frac{1}{z_k-z_j}. \tag{C.5}$$

This equation is satisfied by the Hermite polynomials. One way to numerically determine the solutions to the Bethe equations for finite L is therefore to use the approximated roots as starting points when doing a gradient decent.

C.2 Numerical Solutions to the $\mathfrak{su}(2|1)$ Bethe Equations

For the $\mathfrak{su}(2|1)$ Bethe equations of the  Dynkin diagram,

$$\left(\frac{u_p+\frac{i}{2}}{u_p-\frac{i}{2}}\right)^L=\prod_{q\neq p}^{K_1}\frac{u_p-u_q+i}{u_p-u_q-i}\prod_{q=1}^{K_2}\frac{u_p-v_q-\frac{i}{2}}{u_p-v_q+\frac{i}{2}}, \tag{C.6}$$

$$1=\prod_{q=1}^{K_1}\frac{v_p-u_q-\frac{i}{2}}{v_p-u_q+\frac{i}{2}}, \tag{C.7}$$

the simplest non trivial solution with paired rapidities is again $\{\{u_1,-u_1\},\{\}\}$, where the second set of roots is empty. The duality transformation encoded in the $\mathcal{QQ}$ relation

$$\mathcal{Q}_1^+-\mathcal{Q}_1^-=2iP_1u\tilde{\mathcal{Q}}_2\mathcal{Q}_2, \tag{C.8}$$

gives us the dual set of roots $\{\{u_1,-u_1\},\{0\}\}$. In order to find other solutions and test our conjecture for the transformation of the superdeterminant we again we resort to solving the equations numerically. Unlike the $\mathfrak{su}(2)$ sector, where we could find the roots in the large L limit we have not come up with any clever way of doing so for the $\mathfrak{su}(2|1)$ sector and therefore have only been able to solve for $L\leq 10$. The Mathematica code for solving the Bethe equations, doing the duality transformation, and comparing Schur complements is

```
(*Define the length of the spin chain, the number of pairs of bosonic and auxiliary roots and tolerance*)L = 8; M = 2; K = 0; t = 0.01;
(*The Bethe equations and norm functions are defined for each node*)

Bethe1 = -(Product[(I + u[i] - u[j])/(-I + u[i] - u[j]), {j, 2*M}]*Product[(-I/2 + u[i] - v[j])/(I/2 + u[i] - v[j]), {j, 2*K}]*((-I/2 + u[i])/(I/2 + u[i]))^L);

Bethe2 = Product[(-I/2 - u[j] + v[i])/(I/2 - u[j] + v[i]), {j, 2*M}];

phi1 := -I Log[Bethe1]; phi2 := -I Log[Bethe2]

(*Solves the coupled set of Bethe equations, its actually only half of the equations since the solutions we seek are paired*)

uv0 = Join[Table[u[i], {i, M}], Table[v[i], {i, K}]] /. NSolve[Join[Table[1 == Bethe1, {i, M}], Table[1 == Bethe2, {i, K}]] /. Join[Table[u[j + M] -> -u[j], {j, M}], Table[v[j + K] -> -v[j], {j, K}]], Join[Table[u[k], {k, M}], Table[v[k], {k, K}]], WorkingPrecision -> 30]; // Timing

EquiSet[x_, y_] := (perm = Permutations[Join[x, -x]];
com = Join[y, -y];
AnyTrue[Table[Norm[perm[[i]] - com] < t, {i, Length[perm]}], TrueQ])
(*Test wether two sets of paired roots are in fact the same*)

Sameroots[x_] := AnyTrue[Table[Abs[x[[i]] - x[[j]]] + KroneckerDelta[i, j] < t || Abs[x[[i]] + x[[j]]] < t, {i, Length[x]}, {j, Length[x]}], TrueQ, 2](*Test wether set contain non distinct roots*)
```

```
Num[x_] := AllTrue[Table[Abs[x[[i]] - I/2] > t && Abs[x[[i]] + I/2] > t && Abs[x[[i]]] > t, {i, Length[x]}], TrueQ](*Test wether set contain roots close to +-i/2 or 0*)

uv00 = DeleteDuplicates[Select[uv0, Num[#] && ! Sameroots[#[[1 ;; M]]] &], EquiSet[#1[[1 ;; M]], #2[[1 ;; M]]] &](*Removes duplicates and sets with non distinct roots or roots close to +-i/2 or 0*)

u0 = uv00[[1 ;;, 1 ;; M]]

v0 = uv00[[1 ;;, M + 1 ;;]]

GraphicsRow[{ComplexListPlot[u0, PlotRange -> 1.1 {{-2, 2}, {-1.5, 1.5}}, PlotStyle -> {PointSize[Large]}, PlotLabel -> "Bosonic"], ComplexListPlot[v0, PlotRange -> 1.1 {{-1.5, 1.5}, {-1.5, 1.5}}, PlotStyle -> {PointSize[Large]}, PlotLabel -> "Fermionic"]}](*Plots half of each set of roots*)

u00 = u0[[1]]
(*pick out a set*)

v00 = v0[[1]]

u00

(*Calculates all the submatrices of the Gaudin Matrix*)

A1 = Table[D[phi1, u[k]], {i, M}, {k, M}]; A2 = Table[D[phi1, u[k + M]], {i, M}, {k, M}];

B1 = Table[D[phi2, u[k]], {k, M}, {i, K}]; B2 = Table[D[phi2, u[k + M]], {k, M}, {i, K}];

C1 = Table[D[phi2, v[k]], {k, K}, {i, K}]; C2 = Table[D[phi2, v[k + K]], {k, K}, {i, K}];

(*Combining the submatrices to G+ and G-*)

Gp = Join[Join[A1 + A2, B1 + B2, 2], Join[Transpose[B1 + B2] /. {{} -> {{}}}, C1 + C2, 2], 1] /. Join[Table[u[i] -> u00[[i]], {i, M}], Table[u[i + M] -> -u00[[i]], {i, M}], Table[v[i] -> v00[[i]], {i, K}], Table[v[i + K] -> -v00[[i]], {i, K}], {{} -> Nothing}]

Gm = Join[Join[A1 - A2, B1 - B2, 2], Join[Transpose[B1 - B2] /. {{} -> {{}}}, C1 - C2, 2], 1] /. Join[Table[u[i] -> u00[[i]], {i, M}], Table[u[i + M] -> -u00[[i]], {i, M}], Table[v[i] -> v00[[i]], {i, K}], Table[v[i + K] -> -v00[[i]], {i, K}], {{} -> Nothing}]
```

```
(*Determines the superdeterminant*)

SD = Det[Gp]/Det[Gm]

Det[Gm]

(*Now for the dual equations*)

tK = M - K - 1 ;(*Number of pairs of dual auxiliary roots apart from the extra zeroth root which is placed at the end*)

(*Dual Bethe equations and norm functions*)

tBethe1 := Product[(I/2 - tv[j] + u[i])/(-I/2 - tv[j] + u[i]), {j, 1 + 2*tK}]*((-I/2 + u[i])/(I/2 + u[i]))^L;

tBethe2 := Product[(I/2 + tv[i] - u[j])/(-I/2 + tv[i] - u[j]), {j, 2*M}];

tphi1 := -I Log[tBethe1]; tphi2 := -I Log[tBethe2]

(*Solves for the dual auxiliary roots while keeping the old momentum carying roots *)

tv0 = Table[tv[i], {i, tK}] /. NSolve[Table[1 == tBethe2, {i, tK + 1}] /. Join[Join[Table[u[j] -> u00[[j]], {j, M}],
Table[u[j + M] -> -u00[[j]], {j, M}]], Join[Table[tv[j + tK] -> -tv[j], {j, tK}], {tv[2 tK + 1] -> 0}]], Table[tv[k], {k, tK}], WorkingPrecision -> 20];

tv00 = tv0[[1]](*Picks out a solution of *)

(*Determines the dualised Gaudin Matrix*)

tA1 = Table[D[tphi1, u[k]], {k, M}, {i, M}]; tA2 = Table[D[tphi1, u[k + M]], {k, M}, {i, M}];

tB1 = Table[D[tphi2, u[k]], {k, M}, {i, tK}]; tB2 = Table[D[tphi2, u[k + M]], {k, M}, {i, tK}];

tC1 = Table[D[tphi2, tv[k]], {k, tK}, {i, tK}]; tC2 = Table[D[tphi2, tv[k + tK]], {k, tK}, {i, tK}];

tD1 = Table[D[tphi2, u[k]], {k, M}, {i, 2 tK + 1, 2 tK + 1}]; tD2 = Table[D[tphi2, tv[k]], {k, tK}, {i, 2 tK + 1, 2 tK + 1}] /. {{} -> {{}}}; tD3 = Table[D[tphi2, tv[2 tK + 1]], {k, 1}, {i, 2 tK + 1, 2 tK + 1}];
```

```
tGp = Join[Join[tD3, Sqrt[2]*Transpose[tD1], Sqrt[2]*Transpose[tD2], 2], Join[Sqrt[2]*tD1, tA1 + tA2, tB1 + tB2, 2], Join[Sqrt[2]*tD2, Transpose[tB1 + tB2], tC1 + tC2, 2], 1] /. Join[Table[u[i] -> u00[[i]], {i, M}], Table[u[i + M] -> -u00[[i]], {i, M}], Table[tv[i] -> tv00[[i]], {i, tK}], Table[tv[i + tK] -> -tv00[[i]], {i, tK}], {tv[2 tK + 1] -> 0}, {{} -> Nothing}];

tGm = Join[Join[tA1 - tA2, tB1 - tB2, 2], Join[Transpose[tB1 - tB2] /. {{} -> {{}}}, tC1 - tC2, 2], 1] /. Join[Table[u[i] -> u00[[i]], {i, M}], Table[u[i + M] -> -u00[[i]], {i, M}], Table[tv[i] -> tv00[[i]], {i, tK}], Table[tv[i + tK] -> -tv00[[i]], {i, tK}], {tv[2 tK + 1] -> 0}, {{} -> Nothing}];

(*The dual superdeterminant*)

tSD = N[Det[tGp]/Det[tGm]]

(*The conjectured transformation of the superdeterminant*)

SD*2*M*Product[-v00[[i]]^2, {i, K}]*Product[-tv00[[i]]^2, {i, tK}]/Product[-u00[[i]]^2 - 1/4, {i, M}]

(*The schur compliments*)

tWm = tA1 - If[tK == 0, 0, (tB1 - tB2).Inverse[tC1].Transpose[tB1 - tB2]] /. Join[Table[u[i] -> u00[[i]], {i, M}], Table[u[i + M] -> -u00[[i]], {i, M}], Table[tv[i] -> tv00[[i]], {i, tK}], Table[tv[i + tK] -> -tv00[[i]], {i, tK}], {tv[2 tK + 1] -> 0}, {{} -> Nothing}] // MatrixForm

Wm = A1 - A2 - If[K == 0, 0, (B1 - B2).Inverse[C1].Transpose[B1 - B2]] /. Join[Table[u[i] -> u00[[i]], {i, M}], Table[u[i + M] -> -u00[[i]], {i, M}], Table[v[i] -> v00[[i]], {i, K}], Table[v[i + K] -> -v00[[i]], {i, K}], {{} -> Nothing}] // MatrixForm

Wp = A1 + A2 - If[K == 0, 0, (B1 + B2).Inverse[C1].Transpose[B1 + B2]] /. Join[Table[u[i] -> u00[[i]], {i, M}], Table[u[i + M] -> -u00[[i]], {i, M}], Table[v[i] -> v00[[i]], {i, K}], Table[v[i + K] -> -v00[[i]], {i, K}], {{} -> Nothing}] // MatrixForm

tWp = tA1 + tA2 - Join[tB1 + tB2, Sqrt[2] tD1, 2].Inverse[Join[Join[tC1 /. {{} -> {{}}}, Sqrt[2] tD2, 2], Join[Sqrt[2] Transpose[tD2] /. {{} -> {{}}}, tD3, 2], 1] /. {{} -> Nothing}].Transpose[Join[tB1 + tB2, Sqrt[2] tD1, 2]] /. Join[Table[u[i] -> u00[[i]], {i, M}], Table[u[i + M] -> -u00[[i]], {i, M}], Table[tv[i] -> tv00[[i]], {i, tK}], Table[tv[i + tK] -> -tv00[[i]], {i, tK}], {tv[2 tK + 1] -> 0}, {{} -> Nothing}] // MatrixForm
```