Master's Thesis

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Photonic Control-Phase gate based on emitters with chiral interactions

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Abstract

We study different factors that compromise the fidelity of a two-photon quantum gate implemented in a waveguide. In particular, the gate consists of a finite 1D array of N quantum emitters that implements a passive Control-Phase gate between counter propagating photons. The realization is based on chiral coupling in the waveguide. This means that when a photon interacts with a quantum emitter it retains a fixed polarization and propagation direction. When propagating through the waveguide, the photon is absorbed and reemitted by the quantum emitters in the array, and a π-phase is introduced by each them. The non-linear interactions that occur when two photons are simultaneously inside the waveguide, if engineered adequately, can result in the implementation of a Controlled Phase gate. Our attention in this work focuses on two effects. Firstly, non-perfect chirality. The decrease of the fidelity of the photon gate caused by it is quantified considering the simpler situation of a single incident photon at a time. The second effect is the necessary non-linear interaction between two photons. The role of the collision dynamics for the case of two interacting photons is studied in detail.

Acknowledgements

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Chapter 1

Introduction

Quantum computation emerged during the 80's opening the possibility to solve new problems unexplored until then exploiting the properties of quantum mechanics. It was thanks to pioneers as Paul Benioff [1], Richard Feynman [2] or David Deutsch [3]. During the 90's more algorithms that could take advantage of this systems were proposed by Peter Shor [4] or Lov Grover [5].

The scientific advances since the beginning of the century turned all of these theoretical proposals into a tangible possibility. The applications of quantum computation extend through countless fields: from biology and medicine to quantum cryptography.

The interest of private companies and investors across the world has increased remarkably during the last decade. Research groups and companies explore the different platforms where these algorithms could be implemented, seeking for the optimal one that allows to scale the system with high enough stability and fidelity. Two examples of such platforms could be superconducting qubits [6] or trapped ions [7].

Part of the inspiration for this project comes from the proposal of D. J. Brod and J. Combes in [8]. In our case, we will study how to implement a passive Controlled-Phase quantum gate exploiting recent advances in chiral quantum optics.

1.1 The quantum C-Phase gate

The most popular model for quantum computation is the quantum circuit model. The building blocks for such circuits are the quantum logic gates. In the same fashion that logic gates act upon bits in classical computation, the quantum gates are the basic circuits acting on a few qubits (quantum equivalent of bits). Quantum gates are reversible unitary operators.

The ultimate goal when implementing quantum gates in any of the platforms in which quantum computation is a possibility should always be to find a set of universal quantum gates. As the name indicates, such a set serves as a base for any unitary operation. Therefore, as building blocks for any quantum circuit one could design.

The Controlled-Phase quantum gate is a two-qubit gate that forms a set of universal quantum gates along with the single-qubit rotation operators $R_x(\theta)$, $R_y(\theta)$, $R_z(\theta)$. This is the gate we are trying to implement in this Master's thesis.

The matrix representation of the C-Phase gate is the following:

$$\text{C-Phase}(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{bmatrix} \tag{1.1}$$

When the phase introduced is $\phi = \pi$ the gate is often called Controlled-Z.

1.2 Chiral quantum optics

The advances in nanotechnology and nanofabrication have provided a new toolbox in the field of optics and photonics. The strong light confinement in nanostructures induces propagation-direction-dependent emission, scattering and absorption of photons by quantum emitters built in them. This is known as chiral quantum optics [9].

Our implementation of the C-Phase gate is based in this effect. We have an array of emitters and the coupling with light depends on its propagation direction. In the ideal and most extreme case, the propagation is unidirectional and the interaction of light with the emitters never results in a reflection back. This is what we will refer to as (completely) chiral interaction. In a more realistic case, all the light is not emitted into the original propagation mode and there is a small part that is scattered back. We will refer to this effect as partially chiral interaction.

In Figure 1.1 we can see our implementation of the gate. A 1D array of N emitters integrated in a nanophotonic waveguide. Our two qubits will be two counter-propagating photons. The coupling rates of the emitters will could different for each propagation direction.

To implement a two-qubit gate, it is necessary to have some kind of non-linear interaction between the two photons. This non-linear interaction will arise when considering that two photons cannot excite the same emitter at the same time.

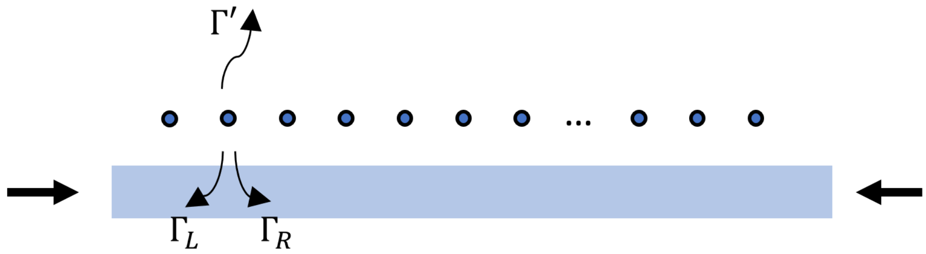

Figure 1.1: Finite 1D array of N quantum emitters integrated in a photonic waveguide. The qubits are two counter-propagating photons represented in this figures by an arroy entering the waveguide from the right and another doing the same from the left. The emission rates Γ will be different for each of them.

1.3 The two/three-level systems

Throughout this project, we will study separately the cases in which a single photon is propagating through the system and the case in which two photons simultaneously propagate in opposite directions. For the sake of simplicity, in the first case we will consider our emitters to be two-level systems, since the three levels effectively become two when having a single propagation direction.

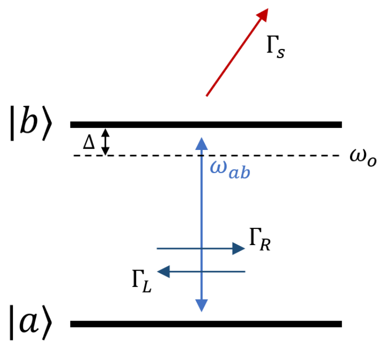

Figure 1.2: Two level system with partial chirality and decay to modes outside of the waveguide.

The Hamiltonian of a two level system with $\Gamma_L = \Gamma_R$ (non-chiral) and no decay to modes outside the waveguide is [10] :

$$\hat{H} = \hbar\omega_{ba}\hat{\sigma}_{bb} + \int dk\omega(k)\hat{a}_k^{\dagger}\hat{a}_k - \hbar g \int dk \left(\hat{\sigma}_{ba}\hat{a}_k e^{ikz_a} + h.c.\right). \qquad (1.2)$$

Here g is a coupling constant related to Γ_L and Γ_R. The first term represents the energy of being in the excited state $|b\rangle$. The second term is related to the electric field and the third one represents the light-matter interaction. The atomic operators are simply $\sigma_{ij} = |i\rangle\langle j|$.

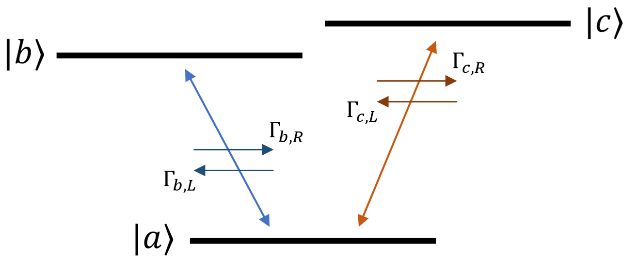

Figure 1.3: Three level system, partial chirality and no losses to other modes.

In the case of the three-level system, we will consider that photons propagate at infinite speed through the waveguide and therefore are only present in the system as excitations. The Hamiltonian in the case of complete chirality and $\Gamma_{b,R} = \Gamma_{c,L} = \Gamma_{1D}$ for an array of three-level emitters is:

$$\begin{aligned} H = & -i\Gamma_{1D}/2 \sum_j \left(\sigma^j_{ca}\sigma^j_{ac} + \sigma^j_{ba}\sigma^j_{ab}\right) \\ & -i\Gamma_{1D}/2 \sum_j \left[\sum_{i>j} e^{ik_o|z_j-z_i|}\sigma^j_{ca}\sigma^i_{ac} + \sum_{i<j} e^{ik_o|z_j-z_i|}\sigma^j_{ba}\sigma^i_{ab}\right]. \end{aligned} \tag{1.3}$$

The first term corresponds to the emission and re-absorption of a photon by the same atom j. In the second term, we have the two cases of moving to the left or to the right. In the first part a photon is emitted when decaying from c to a and the excitation moves to the left (to lower index j). The second part corresponds with the transition $b \to a$, the excitation moves to the right (higher index j). The exponential $e^{ik_o|z_j-z_i|}$ simply corresponds to the phase gained when traveling from one photon to the next.

1.4 Fidelity measurements

The fidelity between two quantum states is a measurement of how 'close' those states are [11]. Given two density operators ρ and σ, the fidelity is generally defined as $F(\rho,\sigma) = \left(\mathrm{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2$. If ρ and σ represent pure quantum states, this is, $\rho = |\psi_\rho\rangle\langle\psi_\rho|$ and $\sigma = |\psi_\sigma\rangle\langle\psi_\sigma|$, the definition reduces to: $F(\rho,\sigma) = |\langle\psi_\rho|\psi_\sigma\rangle|^2$.

This concept of fidelity for quantum states is well know and widely used, without ignoring other metrics that might be more suitable for specific cases. However, we are interested in the fidelity of the gate as a whole, and not the fidelity in the transmission of a single photon. The gate fidelity describes how noisy a quantum gate is.

It is possible to define a fidelity for channels analogous to the fidelity for states thanks to the Choi-Jamiołkowski isomorphism [12], [13]. To calculate the Choi-Jamiołkowski fidelity we will consider a 4-qubit state as in [14]. In particular, we take the following CJ input state:

$$|\Phi\rangle = \left|\Phi^+\right\rangle_{12} \otimes \left|\Phi^+\right\rangle_{34} = \frac{1}{\sqrt{2}}\left(|00\rangle_{12} + |11\rangle_{12}\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{34} + |11\rangle_{34}\right) \tag{1.4}$$

Now we evaluate the output of the gate in qubits 2 and 3. The output state $|\Phi'\rangle$ will be:

$$|\Phi\rangle = \frac{1}{2}\begin{bmatrix} |00\rangle_{12}|00\rangle_{34} \\ |00\rangle_{12}|11\rangle_{34} \\ |11\rangle_{12}|00\rangle_{34} \\ |11\rangle_{12}|11\rangle_{34} \end{bmatrix} \quad \longrightarrow \quad |\Phi'\rangle = \frac{1}{2}\begin{bmatrix} |00\rangle_{12}|00\rangle_{34} \\ t_R|00\rangle_{12}|11\rangle_{34} \\ t_L|11\rangle_{12}|00\rangle_{34} \\ t_{RL}|11\rangle_{12}|11\rangle_{34} \end{bmatrix} \tag{1.5}$$

Here, the coefficients t_R and t_L are the transmission coefficients when one photon propagates through the system and t_{RL} is the transmission coefficient when we have propagation and interaction of two photons.

The ideal state after the gate, looking at the definition of a C-Phase gate, would be:

$$|\Phi_{\text{ideal}}\rangle = \frac{1}{2}\begin{bmatrix} |00\rangle_{12}|00\rangle_{34} \\ |00\rangle_{12}|11\rangle_{34} \\ |11\rangle_{12}|00\rangle_{34} \\ -|11\rangle_{12}|11\rangle_{34} \end{bmatrix} \tag{1.6}$$

The CJ fidelity is just calculated by comparing the two outputs:

$$F = \left|\langle\Phi_{\text{ideal}}|\Phi'\rangle\right|^2 = \frac{1}{16}\left|1 + t_R + t_L - t_{RL}\right|^2 \tag{1.7}$$

1.5 Organization of this thesis

In this work we will study in detail two different effects involved in the functioning of the C-Phase gate. We will do it separately.

In Chapters 2 and 3 we focus on the effect that partial chirality has on the transmission of a single photon through the gate and comment on the resulting fidelity.

In Chapter 4 we study the collision dynamics that appear when we have two counter-propagating photons simultaneously inside the waveguide. We do so under the approximation of an infinite 1D array of emitters. In Chapter 5 we undo this approximation to consider what happens to a photon when it enters and leaves the waveguide. Finally, in Chapter 6 we study the effect all this considerations have on the fidelity.

Chapter 2

Transmission of a single photon

The first effect that we are interested in is how do scattering and partial-chirality decrease the fidelity of our gate when a single photon propagates through the waveguide.

Our starting point will be the Hamiltonian of the two level system [10]:

$$\hat{H} = \hbar\omega_{ba}\hat{\sigma}_{bb} + \int dk\omega(k)\hat{a}_k^\dagger\hat{a}_k - \hbar g \int dk \left(\hat{\sigma}_{ba}\hat{a}_k e^{ikz_a} + h.c.\right). \tag{2.1}$$

Here the first term corresponds to the excitation of the atom, the second term is related to the field and the third one is the interaction between them (absorption and emission of a photon), with g being the coupling constant.

We make the approximation that left- and right- propagating photons form completely separate quantum fields. This means:

$$\int_{-\infty}^{\infty} dk\omega(k)\hat{a}_k^\dagger\hat{a}_k \longrightarrow \int_0^{\infty} dk\omega(k)\left(\hat{a}_{R,k}^\dagger\hat{a}_{R,k} + \hat{a}_{L,-k}^\dagger\hat{a}_{L,-k}\right) \tag{2.2}$$

$$\hat{\sigma}_{ba}\hat{a}_k e^{ikz_a} \longrightarrow \hat{\sigma}_{ba}\left(\hat{a}_{R,k}e^{ikz_a} + \hat{a}_{L,-k}e^{-ikz_a}\right) \tag{2.3}$$

We also consider two different coupling constants, g_R and g_L, that describe the interaction of the atoms with photons propagating to the right or to the left respectively. The Hamiltonian is then:

$$\begin{aligned}\hat{H} =& \hbar\omega_{ba}\hat{\sigma}_{bb} + \int dk\omega(k)\left(\hat{a}_{R,k}^\dagger\hat{a}_{R,k} + \hat{a}_{L,-k}^\dagger\hat{a}_{L,-k}\right) \\ &- \hbar\int dk\left[\hat{\sigma}_{ba}\left(g_R\hat{a}_{R,k}e^{ikz_a} + g_L\hat{a}_{L,-k}e^{-ikz_a}\right) + h.c.\right].\end{aligned} \tag{2.4}$$

2.1 Expansion of $\omega(k)$ around k_o

Lets define the electric fields propagating to the right and to the left as:

$$E_R(z) = \frac{1}{\sqrt{2\pi}} \int_{k>0} dk e^{i(k-k_o)z} a_R(k) \tag{2.5}$$

$$E_L(z) = \frac{1}{\sqrt{2\pi}} \int_{k>0} dk e^{-i(k-k_o)z} a_L(-k) \tag{2.6}$$

Taking the first order expansion of the dispersion relation $\omega(k)$ around k_o one can write:

- For the photons propagating to the right:

$$\int_{k>0} dk\omega(k) \hat{a}^\dagger_{R,k} \hat{a}_{R,k} = \int_{k>0} dk\omega_o a^\dagger_R(k) a_R(k) - i2\pi v_g E^\dagger_R(z) \frac{\partial E_R(z)}{\partial z} \tag{2.7}$$

- For the photons propagating to the left:

$$\int_{k>0} dk\omega(k) \hat{a}^\dagger_{L,-k} \hat{a}_{L,-k} = \int_{k>0} dk\omega_o a^\dagger_L(-k) a_L(k) + i2\pi v_g E^\dagger_L(z) \frac{\partial E_L(z)}{\partial z} \tag{2.8}$$

If we substitute into the Hamiltonian we get:

$$\hat{H} = \sum_j \hbar\omega_{ba} \hat{\sigma}^j_{bb} + \hbar \int dk\omega_o \left[a^\dagger_R(k) a_R(k) + a^\dagger_L(-k) a_L(-k) \right] \tag{2.9}$$

$$+ i\hbar 2\pi v_g \left[E^\dagger_L(z) \frac{\partial E_L(z)}{\partial z} - E^\dagger_R(z) \frac{\partial E_R(z)}{\partial z} \right] \tag{2.10}$$

$$- \hbar\sqrt{2\pi} \int \sum_j \delta(z - z_j) \left\{ \hat{\sigma}^j_{ba} \left[g_R E_R(z) e^{ik_o z} + g_L E_L(z) e^{-ik_o z} \right] \right. \tag{2.11}$$

$$\left. + \left[g_R E^\dagger_R(z) e^{-ik_o z} + g_L E^\dagger_L(z) e^{ik_o z} \right] \hat{\sigma}^j_{ab} \right\} dz \tag{2.12}$$

2.2 Change to the interaction picture with respect to ω_o

Now we move to the interaction picture with respect to ω_o. We take H_o as follows:

$$H_o = \sum_j \hbar\omega_o\hat{\sigma}^j_{bb} + \hbar \int dk\omega_o \left[a^\dagger_R(k)a_R(k) + a^\dagger_L(-k)a_L(-k)\right] \tag{2.13}$$

The Hamiltonian in the interaction picture is given by [15]:

$$H_I = e^{iH_o t/\hbar} H e^{-iH_o t/\hbar} - H_o \tag{2.14}$$

The final Hamiltonian in the interaction picture is:

$$\begin{aligned} H_I =& - \hbar\Delta\hat{\sigma}^j_{bb} - \hbar\sqrt{2\pi} \int \sum_j \delta(z - z_j \left\{\hat{\sigma}^j_{ba} \left[g_R\hat{E}_R(z)e^{ik_o z} + g_L\hat{E}_L(z)e^{-ik_o z}\right]\right) \\ &+ \left[g_R\hat{E}^\dagger_R(z)e^{-ik_o z} + g_L\hat{E}^\dagger_L(z)e^{ik_o z}\right]\hat{\sigma}^j_{ab}\right\} dz \\ &+ i\hbar c \int \left[\hat{E}_L \dagger (z)\frac{\partial\hat{E}_L}{\partial z} - \hat{E}^\dagger_R(z)\frac{\partial\hat{E}_R}{\partial z}\right] dz. \end{aligned} \tag{2.15}$$

2.3 Equations of motion for the fields and the atomic operators

In the interaction picture, the time evolution of an operator $A_I(t)$ is given by [15]:

$$i\hbar\frac{d}{dt}A_I(t) = [A_I(t), H_I] \tag{2.16}$$

We can apply this to the electric fields propagating to the right and to the left:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\hat{E}_R(z) = \frac{ig_R\sqrt{2\pi}}{c}\sum_j \delta(z - z_j)\hat{\sigma}^j_{ab}e^{-ik_o z - i\omega_o t} \tag{2.17}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\hat{E}_L(z) = \frac{ig_L\sqrt{2\pi}}{c}\sum_j \delta(z - z_j)\hat{\sigma}^j_{ab}e^{-ik_o z - i\omega_o t} \tag{2.18}$$

After that, we apply the same equation to find the time evolution of the atomic operators:

$$\dot{\hat{\sigma}}^j_{ab} = i\Delta\hat{\sigma}^j_{ab} + i\sqrt{2\pi}\left(\hat{\sigma}^j_{aa} - \hat{\sigma}^j_{bb}\right)\left[g_R\hat{E}_R(z_j)e^{ik_o z_j} + g_L\hat{E}_L(z_j)e^{-ik_o z_j}\right] \tag{2.19}$$

The next step is to use the integrated equations of motion of the fields so that we can substitute them in (2.19). At this point, we need to make use of the slowly varying operator approximation. We can do this under the assumption of light traveling at infinite speed inside the waveguide. The time it takes a photon to get out of the atom, related to the decaying rate, is much larger than the time it takes the same photon to reach any other atom. This means that $\sigma_{ab}(t-\frac{|z_j-z_i|}{c}) \sim \sigma_{ab}(t)$.

Integrating the fields we get that:

$$\hat{E}_R(z_j) = \frac{1}{2}\left(\hat{E}_R(z_j^-) + \hat{E}_R(z_j^+)\right) = \frac{1}{2}\left(2\hat{E}_{R,in} + \frac{ig_R\sqrt{2\pi}}{c}\sum_{i\leq j}\hat{\sigma}_{ab}^i(t-\tfrac{z_j-z_i}{c})e^{-ik_oz_i}\right) \tag{2.20}$$

$$\hat{E}_L(z_j) = \frac{1}{2}\left(\hat{E}_L(z_j^-) + \hat{E}_L(z_j^+)\right) = \frac{1}{2}\left(2\hat{E}_{L,in} + \frac{ig_L\sqrt{2\pi}}{c}\sum_{i\leq j}\hat{\sigma}_{ab}^i(t-\tfrac{z_i-z_j}{c})e^{ik_oz_i}\right) \tag{2.21}$$

The resulting differential equation for the $\hat{\sigma}_{ab}^j$ operators is:

$$\begin{aligned}\dot{\hat{\sigma}}_{ab}^j =& \left[i\Delta - (\gamma_R+\gamma_L)\right]\hat{\sigma}_{ab}^j + i\sqrt{2\pi}\left[g_Re^{ik_oz_j}\hat{E}_{R,in} + g_LE^{-ik_oz_j}\hat{E}_{L,in}\right]\\ &-\gamma_R\sum_{i<j}\hat{\sigma}_{ab}^ie^{ik_o|z_j-z_i|} - \gamma_L\sum_{i>j}\hat{\sigma}_{ab}^ie^{ik_o|z_j-z_i|}\end{aligned} \tag{2.22}$$

We will consider the case of one photon traveling to the right (there is no input for the left-traveling field). We can rewrite this set of equations in a matrix form by defining a matrix form Hamiltonian H^{NH}:

$$\dot{\hat{\sigma}}_{ab}^j = -\sum_i H_{ij}^{NH}\hat{\sigma}_{ab}^i + \frac{ig_R\sqrt{2\pi}}{c}e^{ik_oz_j}\hat{E}_{R,in} \tag{2.23}$$

where we have defined $\gamma_{R,L} = g_{R,L}^2\pi/c$ and the matrix elements of the Hamiltonian are:

$$H_{ij}^{NH} = \begin{cases} -i\Delta + (\gamma_R+\gamma_L) & \text{if } i=j \\ \gamma_Re^{ik_o|z_j-z_i|} & \text{if } i<j \\ \gamma_Le^{ik_o|z_j-z_i|} & \text{if } i>j \end{cases} \tag{2.24}$$

Additionally, we should also consider the possibility of having decay to modes outside of the waveguide. To do so, one should consider an extra term in the interaction Hamiltonian in (2.15) in which emitters excited in level $|b\rangle$ decay with a decaying rate γ_s. By similarity with the Δ term we can trace this new scatering term into the matrix elements of the Hamiltonian. We will have:

$$H_{ij}^{NH} = \begin{cases} -i\Delta + (\gamma_s+\gamma_R+\gamma_L) & \text{if } i=j \\ \gamma_Re^{ik_o|z_j-z_i|} & \text{if } i<j \\ \gamma_Le^{ik_o|z_j-z_i|} & \text{if } i>j \end{cases} \tag{2.25}$$

Instead of considering γ_s, γ_R and γ_L, it is easier to work with the relative coefficients $\beta_s = \gamma_s/\gamma$, $\beta_R = \gamma_R/\gamma$ and $\beta_L = \gamma_L\gamma$, with $\gamma = \gamma_R + \gamma_L + \gamma_s$. We should also divide by γ the rest of the elements of our equations from now on. For the sake of clarity, this factor is sometimes ignored.

In the next pages we will always consider the case of no detuning, $\Delta = 0$.

2.4 Finding the transmission coefficient

To find a solution for the differential equation in (2.23) we apply the properties of the Fourier Transform:

$$-i\omega/\gamma\vec{\sigma}_{ab} = -H^{NH}\vec{\sigma}_{ab} + \vec{g}^{\dagger}E_{R,in} \tag{2.26}$$

$$\vec{\sigma}_{ab} = \left(-i\omega/\gamma + H^{NH}\right)^{-1}\vec{g}E_{R,in} \tag{2.27}$$

At this point we lose the hats over the operators for the sake of simplicity. Also, the new vector $\vec{g}$ includes all the constants and exponentials and has elements $g_j = \frac{ig_R\sqrt{2\pi}}{c}e^{ik_o z_j}$.

The output field will be given by:

$$E_{out} = E_{in} + \vec{g}^{\dagger}\vec{\sigma_{ab}} = E_{in} + \vec{g}^{\dagger}\left(-i\omega/\gamma + H^{NH}\right)^{-1}\vec{g}E_{R,in}. \tag{2.28}$$

If we rewrite this expression we can define our transmission coefficient $t(\omega)$:

$$E_{out} = t(\omega)E_{R,in} = \left[1 + \vec{g}^{\dagger}\left(-i\omega/\gamma + H^{NH}\right)^{-1}\vec{g}\right]E_{R,in} \tag{2.29}$$

Therefore the transmission coefficient is:

$$t(\omega) = 1 + \vec{g}^{\dagger}\left(-i\omega/\gamma + H^{NH}\right)^{-1}\vec{g} \tag{2.30}$$

The first thing we calculate is the case $N = 1$, this is, the transmission through a single atom.

$$t_1(\omega) = 1 - \frac{\beta_R}{-i\omega/\gamma + 1/2} \tag{2.31}$$

In Figure 2.1 we see how the transmission coefficient drops on resonance. Unfortunately, it is in this regime that we want to work at, since it is where we would expect the stronger non-linear interaction when two photons propagate simultaneously through the system. This indicates, even if it is just for a single atom, that we will need a system where $\beta_L, \beta_s \ll 1$.

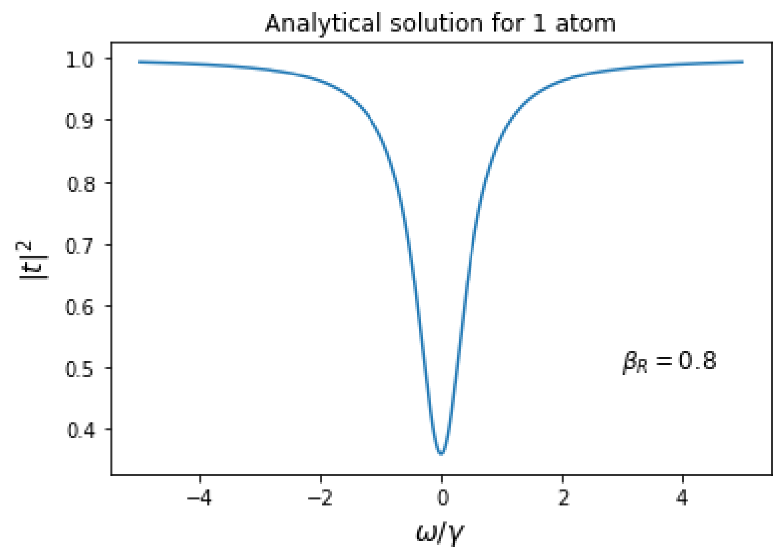

Figure 2.1: Transmission probability $|t|^2$ as a function of ω/γ. The transmission drops on resonance and has a minimum value of $(1-2\beta_R)^2$.

The main problem to calculate this coefficient for bigger values of N is inverting the matrix $\left(-i\omega + H^{NH}\right)$. This task becomes easier under certain approximations. First, we will assume a completely non-chiral system where $\beta_L = 0$. This transforms our problem of inverting a general matrix to a problem of inverting a lower triangular one. This is a much simpler problem that can be solved trivially.

The result is exactly what one would expect. If there is no scattering back, the field moves from one atom to the next with the only losses of the scattering to the side of the waveguide, in other words, with a reduced amplitude. The interaction with each atom is exactly the same N times in a row. The transmission coefficient is:

$$t(\omega) = (t_1(\omega))^N. \tag{2.32}$$

In this case, and as shown Figure 2.2, the drop of the transmission on resonance is even more pronounced, with a minimum value of $(1-2\beta_R)^{2N}$. The functioning of a gate that requires an array with $N \gg 1$ will we conditioned to have a strong chiral interaction ($\beta_R \sim 1$).

However, we want to see the effect that some partial chirality has on the efficiency of the gate. This partial chirality will be considered an imperfection. We have seen that any functioning gate will require $\beta_L \ll 1$. Therefore, it is justified to consider the approximation $\beta_L \ll \beta_R$. This allows us to rewrite the matrix we want to invert as $M = A + \beta_L B$. This way, it is decomposed into the sum of a lower and an upper diagonal matrix and one can calculate the inverse as $M^{-1} \approx A^{-1}[1 - \beta_L B A^{-1}]$. More details about this approximation can be found in Appendix A. After inverting the

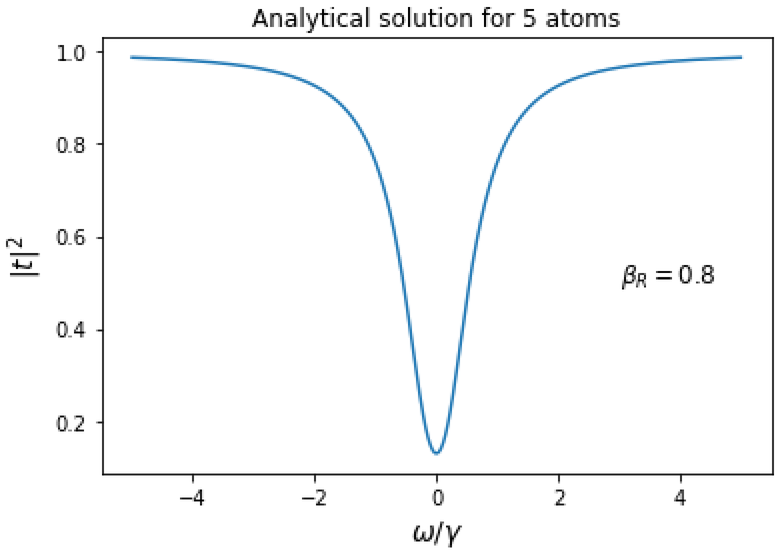

Figure 2.2: Transmission probability $|t|^2$ as a function of ω/γ for the case $N = 5$ and $\beta_L = 0$. The transmission drops on resonance and has a minimum value of $(1 - 2\beta_R)^{2N}$.

matrix and approximating to first order in β_L, the resulting transmission coefficient is:

$$t(\omega) = t_1^N(\omega) \left\{ 1 + e^{i2k_o d} \beta_R \beta_L (-i\omega/\gamma + 1/2)^{-2} \frac{N - 1 + N t_1 e^{i2k_o d} + t_1 e^{i2k_o d}}{(1 - t_1 e^{i2k_o d})^2} \right\} \tag{2.33}$$

We can further simplify this by considering $N \gg 1$. The final result is:

$$t(\omega) = [t_1(\omega)]^N \left\{ 1 + \beta_R \beta_L \left(-i\omega/\gamma + 1/2\right)^{-2} e^{i2k_o d} \frac{N}{1 - e^{i2k_o d} t_1(\omega)} \right\}, \tag{2.34}$$

with

$$t_1(\omega) = 1 - \beta_R \left(-i\omega/\gamma + 1/2\right)^{-1}. \tag{2.35}$$

2.5 Numerical results for the transmission of a single photon

In Figures 2.3 and 2.4 we can see the transmission and reflection probabilities in an a array with 30 atoms if the distance between them is set so that $k_o d = 0$ (or realistically $k_o d = 2\pi$) and $k_o d = \pi$. In this case the result is practically the same that when we had $\beta_L = 0$.

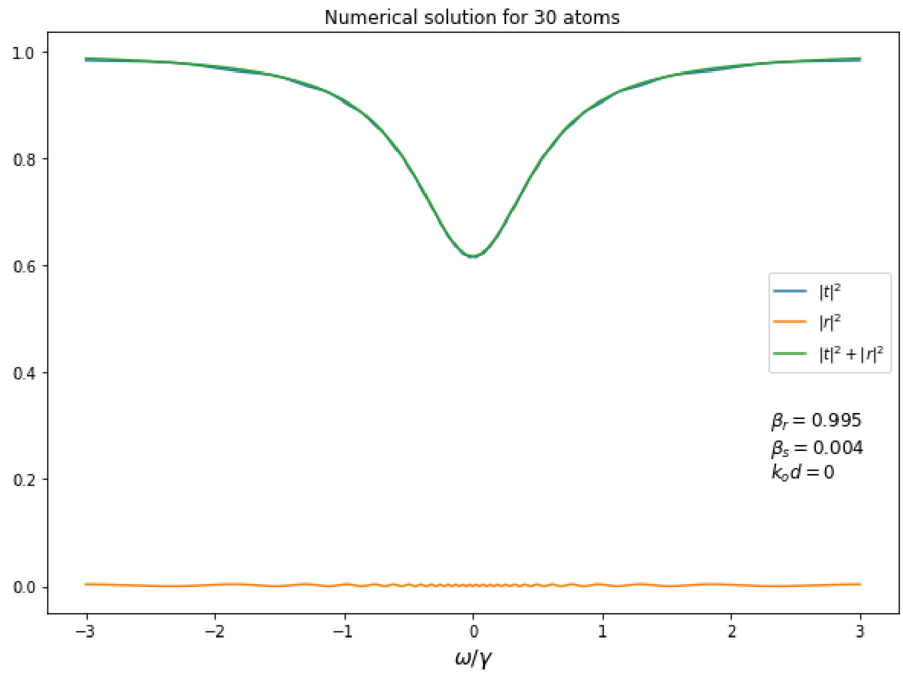

Figure 2.3: Transmission, $|t|^2$, and reflection, $|r|^2$, probability for the propagation to the right of a single photon over an array of 30 atoms with $k_o d = 0$.

If we set $k_o d \neq n\pi$ a sharp peak appears in the probabilities as we can see in Figures 2.5 and 2.6. The position of this peaks depends on the value of $k_o d$. This peak can be explained by a cavity-like behaviour at the ends of the waveguide that gets amplified for a given frequency.

For the implementation of the quantum gate, this effect should be avoided. Luckily, we will try to work in the vicinity of the values $k_o d = n\pi$. At this points, the transmission coefficient phase behaves as $\sim (-1)^N$, with non-existent imaginary part. If we set the number of atoms N to be en even number we will achieve the desired behaviour.

Finally, in Figure 2.8 we compare the results of our approximation in (2.34) with the numerical solution. The approximation to first order in β_L is good except around the peak of increased reflection. This fact is compatible with the idea of having a cavity like behaviour since it is in that case where the reflections described by β_L acquire the most importance.

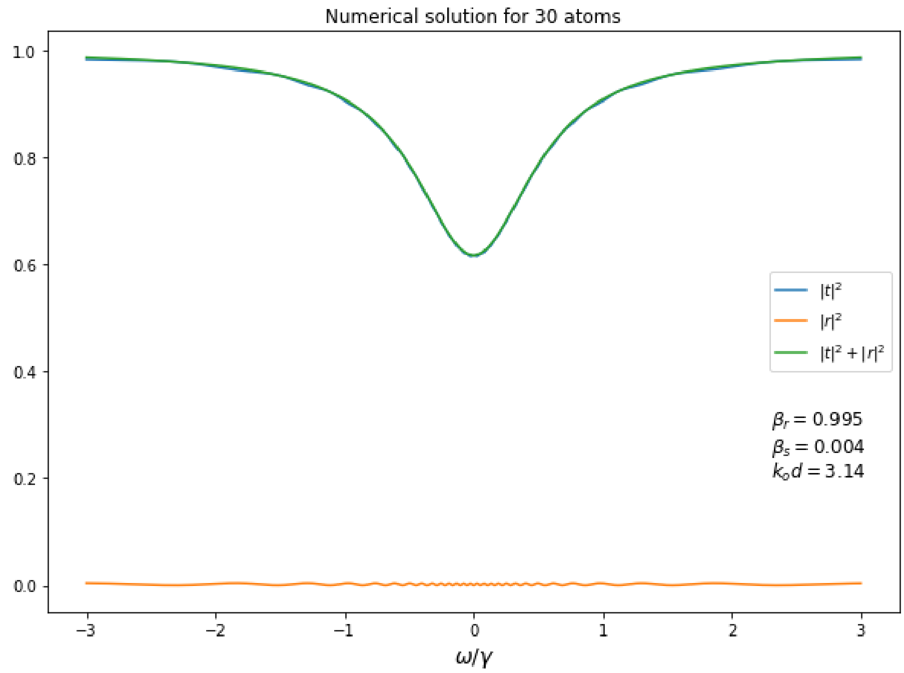

Figure 2.4: Transmission, $|t|^2$, and reflection, $|r|^2$, probability for the propagation to the right of a single photon over an array of 30 atoms with $k_o d = \pi$.

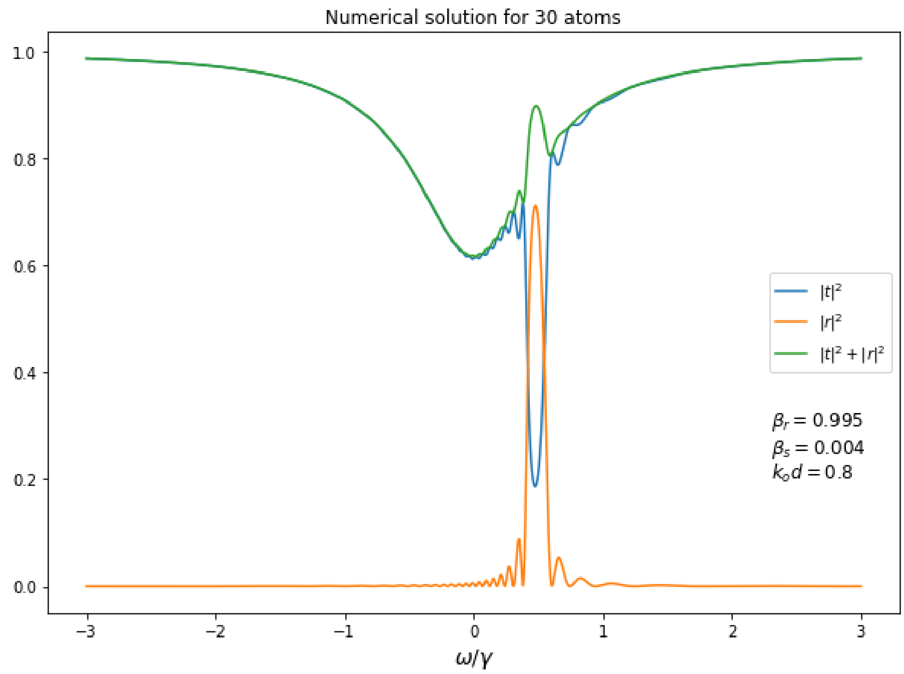

Figure 2.5: Transmission, $|t|^2$, and reflection, $|r|^2$, probability for the propagation to the right of a single photon over an array of 30 atoms with $k_o d = 0.8$.

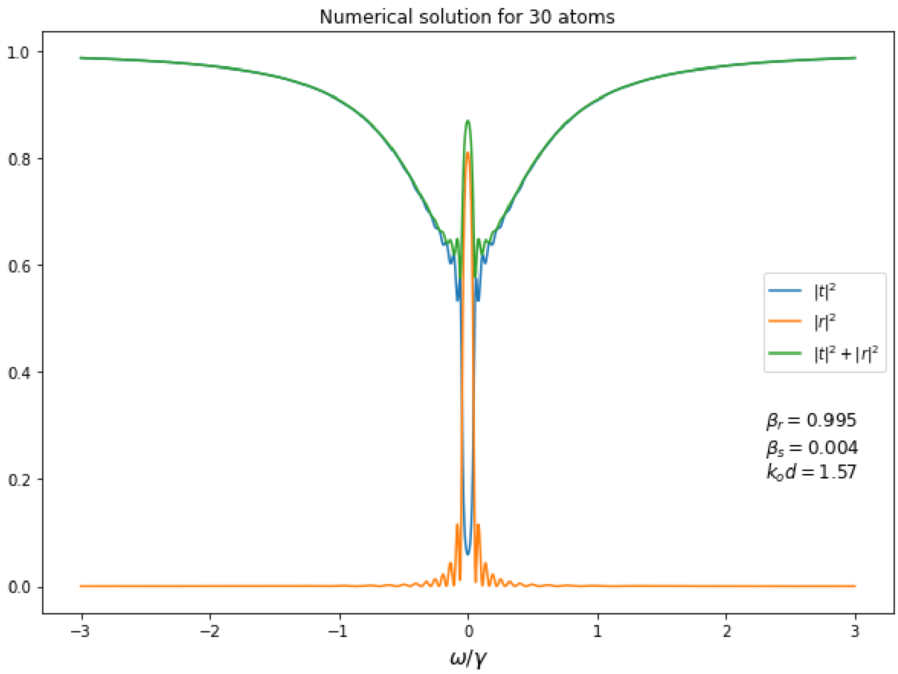

Figure 2.6: Transmission, $|t|^2$, and reflection, $|r|^2$, probability for the propagation to the right of a single photon over an array of 30 atoms with $k_o d = \pi/2$.

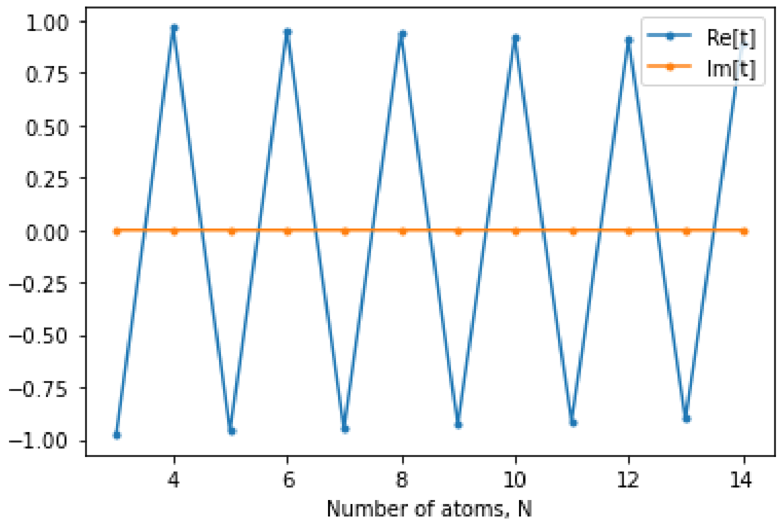

Figure 2.7: Real and imaginary parts of the transmission coefficient on resonance as a function of N for $k_o d = \pi$.

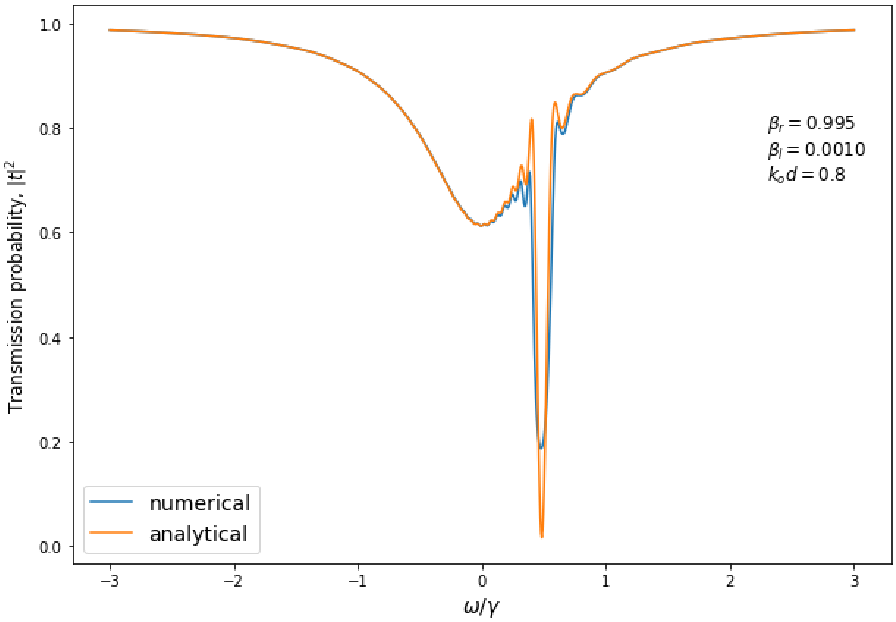

Figure 2.8: Comparison between the numerical (exact) and the analytical (approximated) solutions for 30 atoms.

Chapter 3

Fidelity measurements for the single photon case

We recover the expression we discussed in Chapter 1 for the fidelity of a quantum gate (1.7):

$$F = \frac{1}{16} \left|1 + t_R + t_L - t_{RL}\right|^2 \tag{3.1}$$

In Chapter 2, we have considered the case of a single photon propagating through our system. Now we want to calculate how this process in particular affects the fidelity of our gate. To do so, we will assume that everything else behaves as in an ideal gate. This means that we will take $t_{RL} = -1$.

The elements t_R and t_L will be given by:

$$t_{R,L} = \int \phi^*_{R,L}(\omega)\phi_{R,L}(\omega)t_{R,L}(\omega)d\omega \tag{3.2}$$

where $\phi_{R,L}(\omega)$ is the profile of the photon pulse. The transmission coefficient $t_{R,L}(\omega)$ is the one calculated in (2.34).

Assuming the propagation is the same in both propagation directions, the expression for the fidelity is:

$$F = \frac{1}{16} \left|2 + 2\int \phi^*_R(\omega)\phi_R(\omega)t_R(\omega)d\omega\right|^2 \tag{3.3}$$

We will assume both photons have a normalized Gaussian profile such as:

$$\phi_{L,R} = \sqrt[4]{\frac{2}{\pi\sigma^2}} e^{-\frac{(\omega-\omega_1)^2}{\sigma^2}} \tag{3.4}$$

3.1 Fidelity for one photon when $\beta_L, \sigma/\gamma \ll 1$

To find an expression for F we need to perform the following integral:

$$\int \phi_R^*(\omega)\phi_R(\omega)t_R(\omega)d\omega = \sqrt{\frac{2}{\pi(\sigma/\gamma)^2}} \int \exp\left[-2\frac{(\omega/\gamma-\omega_1/\gamma)^2}{(\sigma/\gamma)^2}\right] t_R(\omega)d\omega \quad (3.5)$$

If we assume a small deviation in the Gaussian profile ($\sigma/\gamma \ll 1$), we can expand $t(\omega)$ around ω_1, the central frequency.

$$t(\omega) = t(\omega_1) + \left.\frac{\partial t(\omega)}{\partial \omega}\right|_{\omega_1} (\omega-\omega_1) + \left.\frac{\partial^2 t(\omega)}{\partial \omega^2}\right|_{\omega_1} (\omega-\omega_1)^2 + ... \quad (3.6)$$

To facilitate the integration, we can also take the approximation $\beta_L \ll 1$ and simplify the derivatives of the transmission coefficient as:

$$\frac{\partial t(\omega)}{\partial \omega} = Nt_1(\omega)^{N-1}t_1'(\omega)\left\{1+\beta_L...\right\} + \beta_L... \approx Nt_1(\omega)^{N-1}t_1'(\omega)$$

$$\frac{\partial^2 t(\omega)}{\partial \omega^2} = N(N-1)t_1(\omega)^{N-2}t_1'(\omega)^2 + Nt_1(\omega)^{N-1}t_1''(\omega)$$

$$t_1'(\omega) = -i\beta_R\left(-i\omega/\gamma+1/2\right)^{-2} \qquad t_1''(\omega) = 2\beta_R\left(-i\omega/\gamma+1/2\right)^{-3}$$

Finally we use the well known results for the Gaussian integrals to find an expression for the fidelity:

$$\int_{-\infty}^{+\infty} x^{2n}e^{-\frac{1}{2}ax^2}dx = \left(\frac{2\pi}{a}\right)^{1/2}\frac{1}{a^n}(2n-1)!! \qquad \int_{-\infty}^{+\infty} x^{2n+1}e^{-\frac{1}{2}ax^2}dx = 0 \quad (3.7)$$

The fidelity of the C-Phase gate is:

$$F = \frac{1}{16}\left|2+2t_1(\omega_1)^N\left\{1+\beta_R\beta_L\left(-i\omega_1/\gamma+1/2\right)^{-2}e^{i2k_od}\frac{N}{1-e^{i2k_od}t_1(\omega_1)}\right.\right.$$
$$\left.\left. -\frac{\beta_R N(N-1)\left(-i\omega_1/\gamma+1/2\right)^{-4}}{t_1(\omega_1)^2}\frac{(\sigma/\gamma)^2}{8} + \frac{2\beta_R N\left(-i\omega_1/\gamma+1/2\right)^{-3}}{t_1(\omega_1)}\frac{(\sigma/\gamma)^2}{8}\right\}\right|^2 \quad (3.8)$$

In Figures 3.1 and 3.2 we can see how does F behave as a function of σ/γ and the central frequency ω_1.

In the first case, the fidelity oscillates and reaches 0 for the cases in which the phase gained propagating through the medium is π. When far from resonance, this oscillation happens between 0 and 1. When closer to resonance, the envelope of the oscillation behaves as the probability of transmission $|t|^2$.

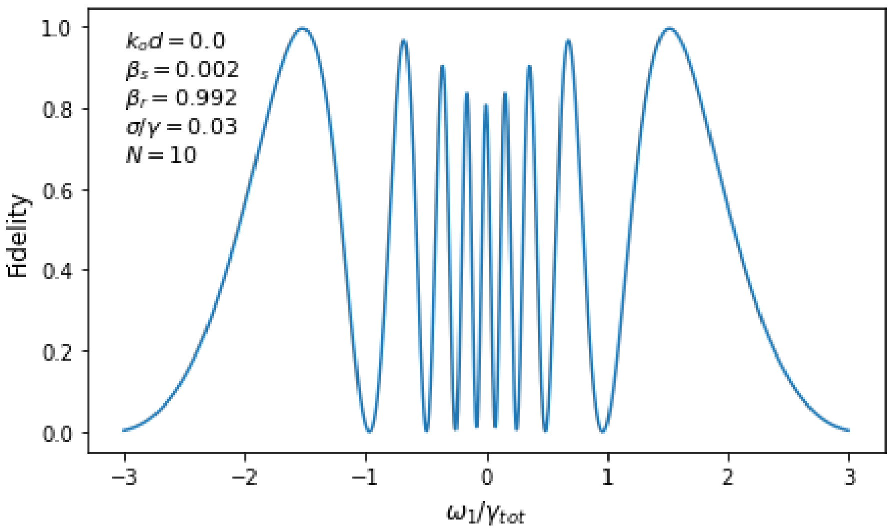

Figure 3.1: Fidelity as a function of the central frequency of a pulse with $\sigma/\gamma = 0.03$.

In Figure 3.2 we can see how the fidelity on resonance $\omega = 0$ drops if the pulse is too wide.

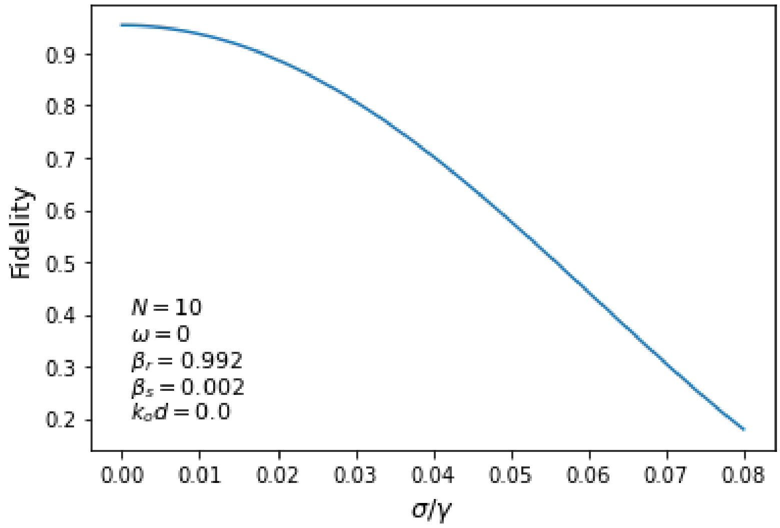

Figure 3.2: Fidelity for $\omega = 0$ as a function of the standard deviation of the Gaussian pulse.

Chapter 4

Transmission of two photons

In Chapter 4, we will focus on the case of two photons travelling simultaneously through the waveguide in opposite directions. This is the last situation missing to have a complete description of the operation of the C-Phase gate.

The two level system considered in the previous chapter becomes now a three-level system such as the one in Figure 4.1. In this case, we will focus in the ideal case of perfect chirality. One of the transitions will be excited by the photon traveling to the left and the other one by the photon travelling to the right.

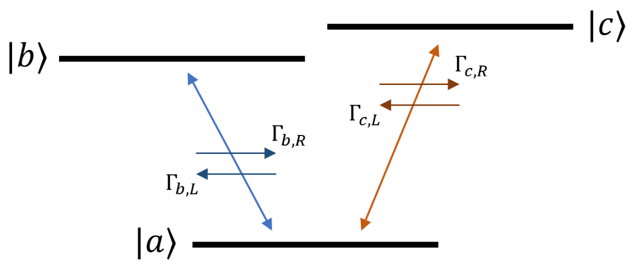

Figure 4.1: Scheme of the energy levels of each of the emitters of the 1D array. In this general representation both interactions are partially-chiral and correspond to different excitation energies.

The formalism used in Chapter 2 becomes too complicated and the impossibility of solving the system analytically deprives us of information about what we are really interested in, the interaction between the two photons. The approximation we will consider simplifies the analysis by focusing only on the interaction. We will assume that the transmission of each of the photons through the waveguide is not affected by the presence of the other and study what happens when they cross paths.

4.1 Interaction inside the waveguide

As explained before, we will now focus on what happens when the photons' trajectories cross. We are working under the assumption of infinite speed of light inside the waveguide. This means that the photons only appear in our formalism as excitations of the atoms, since they travel instantaneously from one to the next. Moreover, we are considering a perfectly chiral system in which each atom can only be excited by a photon at a time.

In order to simplify the analysis, we will for the time being also consider that this interaction occurs inside an infinite 1D array of atoms extending from $-\infty$ to ∞.

The Hamiltonian of such system reads as follows:

$$\begin{aligned} H = &- i\Gamma_{1D}/2 \sum_j \left(\sigma^j_{ca}\sigma^j_{ac} + \sigma^j_{ba}\sigma^j_{ab}\right) \\ &- i\Gamma_{1D}/2 \sum_j \left[\sum_{i>j} e^{ik_o|z_j - z_i|}\sigma^j_{ca}\sigma^i_{ac} + \sum_{i<j} e^{ik_o|z_j - z_i|}\sigma^j_{ba}\sigma^i_{ab}\right]. \end{aligned} \tag{4.1}$$

The first term corresponds to the emission and re-absorption of a photon by the same atom j. In the second term, we have the two cases of moving to the left or to the right. In the first part a photon is emitted when decaying from c to a and the excitation moves to the left (to lower index j). The second part corresponds with the transition $b \to a$, the excitation moves to the right (higher index j). The exponential $e^{ik_o|z_j - z_i|}$ simply corresponds to the phase gained when traveling from one photon to the next.

To proceed with the analysis, we define a basis of eigenstates with two excitations defined by the distance Δ between them:

$$|\Delta\rangle = \sum_s e^{ik\left(z_{s+\Delta/2} + z_{s-\Delta/2}\right)} \sigma^{s-\Delta/2}_{ba} \sigma^{s+\Delta/2}_{ca} |0\rangle . \tag{4.2}$$

Here, k is just the sum of the momentum of the individual photons, $k = k_1 + k_2$.

A general state using this basis will be:

$$|\Phi\rangle = \sum_\Delta \sum_s e^{ik\left(z_{s+\Delta/2} + z_{s-\Delta/2}\right)} f(\Delta) \sigma^{s-\Delta/2}_{ba} \sigma^{s+\Delta/2}_{ca} |0\rangle = \sum_\Delta f(\Delta) |\Delta\rangle . \tag{4.3}$$

Here, s is the middle position between the two excitations (centre of mass) and Δ is the difference between the two. We will assume that the two excitations can not occur at the same atom at the same time, so this does not exist for $\Delta = 0$ and $f(0) = 0$.

One can find the effect the Hamiltonian in (4.1) has when applied to each element of this set of states $|\Delta\rangle$. It is useful to remember the commutation relation for the atomic operators: $[\sigma^i_{ba}, \sigma^j_{ab}] = \delta_{ij}(\sigma^i_{bb} - \sigma^i_{aa})$.

$$\begin{aligned} H\,|\Delta\rangle =& - i\Gamma_{\mathrm{1D}}/2\,|\Delta\rangle \\ &- i\Gamma_{\mathrm{1D}}/2 \sum_s e^{ik\left(z_{s+\Delta/2}+z_{s-\Delta/2}\right)} \sum_j e^{ik_o|z_j - z_{s+\Delta/2}|}\theta(s+\Delta/2-j)\sigma_{ba}^{s-\Delta/2}\sigma_{ca}^{j}\,|0\rangle \\ &- i\Gamma_{\mathrm{1D}}/2 \sum_s e^{ik\left(z_{s+\Delta/2}+z_{s-\Delta/2}\right)} \sum_j e^{ik_o|z_j - z_{s-\Delta/2}|}\theta(j-s+\Delta/2)\sigma_{ba}^{j}\sigma_{ca}^{s+\Delta/2}\,|0\rangle \end{aligned} \tag{4.4}$$

This expression can be simplified with a change of variables that will allow us to rewrite the Hamiltonian in matrix form. We introduce two new indexes s' and Δ' that relate to s, j and Δ differently in the first and in the second sum. In the first sum we perform the change:

$$s - \Delta/2 = s' - \Delta'/2 \qquad j = s' + \Delta'/2. \tag{4.5}$$

Meanwhile, in the second sum, the change we need is:

$$s + \Delta/2 = s' + \Delta'/2 \qquad j = s' - \Delta'/2 \tag{4.6}$$

We also change the way we refer to the position of each atom z_j and consider the distance between them is d so that $z_j = dj$ and $\Delta \longrightarrow \Delta d$. With all this considerations, we can rewrite the expression in (4.4) as:

$$H\,|\Delta\rangle = -i\Gamma_{\mathrm{1D}}\,|\Delta\rangle - i\Gamma_{\mathrm{1D}}/2 \sum_{\Delta'} \left\{ e^{i(k+k_o)d(\Delta-\Delta')} + e^{-i(k-k_o)d(\Delta-\Delta')} \right\} \theta(\Delta-\Delta')\,|\Delta'\rangle\,. \tag{4.7}$$

Finally, we write the Hamiltonian in matrix form:

$$H = \sum_{\Delta,\Delta'} H_{\Delta\Delta'}\,|\Delta\rangle\,\langle\Delta'|\,, \tag{4.8}$$

with matrix elements

$$H_{\Delta\Delta'} = -i\Gamma_{1D}/2\;\delta(\Delta-\Delta') - i\Gamma_{1D}/2 \left\{ e^{i(k+k_o)d(\Delta-\Delta')} + e^{-i(k-k_o)d(\Delta-\Delta')} \right\} \theta(\Delta-\Delta'). \tag{4.9}$$

Now we apply the Schrödinger equation to the general state in (4.3):

$$H\,|\Phi\rangle = E\,|\Phi\rangle = E\sum_{\Delta} f(\Delta)\,|\Delta\rangle = \sum_{\Delta,\Delta'} H_{\Delta'\Delta} f(\Delta)\,|\Delta'\rangle \tag{4.10}$$

$$Ef(\Delta) = -i\frac{\Gamma_{1D}}{2}\,f(\Delta) - i\frac{\Gamma_{1D}}{2}\sum_{\Delta'}\left[e^{i(k+k_o)d(\Delta-\Delta')} + e^{-i(k-k_o)d(\Delta-\Delta')}\right]\theta(\Delta-\Delta')f(\Delta') \tag{4.11}$$

The only thing left to do to solve the system is find $f(\Delta)$.

4.1.1 First ansatz: the naive approach

Our first guess for the function $f(\Delta)$ is the simplest we can make. We are considering two photons propagating through a waveguide in opposite directions, unaffected by each other. Therefore, before they cross ($\Delta < 0$), $f(\Delta)$ just corresponds with the phase gained propagating over the distance of the waveguide. However, when they cross ($\Delta = 0$), an additional phase t is gained and conserved as they move far apart ($\Delta > 0$.)

Our ansatz is:

$$f(\Delta) = \begin{cases} e^{iqd\Delta} & \Delta < 0 \\ te^{iqd\Delta} & \Delta > 0, \end{cases} \tag{4.12}$$

where q the difference in momentum between the two photons $q = k_2 - k_1$.

We can perform the sum in (4.11) distinguishing between the cases $\Delta < 0$ and $\Delta > 0$ to find expression for E and t.

- For $\Delta < 0$:

  The resulting energy is:

  $$E = -i\frac{\Gamma_{1D}}{2} + i\frac{\Gamma_{1D}}{2}\left\{\frac{1}{1-e^{i(q-k-k_o)d}} + \frac{1}{1-e^{i(q+k-k_o)}}\right\}. \tag{4.13}$$

  As expected, it is real and does not depend on Δ, as it should be conserved over time and constant no matter where the excitations are placed. The fact that this is real can be seen considering that:

  $$\frac{1}{1-e^{i\theta}} = \frac{1}{2} + i\frac{\sin\theta}{2(1-\cos\theta)} \tag{4.14}$$

  It is worth mentioning that this energy is just the sum of the energy of two single excitations.

- For $\Delta > 0$:

  The expression in this case is much more complex:

$$\begin{aligned} Et =& -i\frac{\Gamma_{1D}}{2}\, t - i\frac{\Gamma_{1D}}{2}\left[e^{i(k+k_o)d\Delta}\frac{e^{-i(q-k-k_o)d}}{1-e^{-i(q-k-k_o)d}} + e^{-i(k-k_o)d\Delta}\frac{e^{-i(q+k-k_o)d}}{1-e^{-i(q+k-k_o)d}}\right] \\ & -\frac{\Gamma_{1D}}{2}t\left[e^{i(k+k_o)d\Delta}\frac{e^{i(q-k-k_o)}}{1-e^{i(q-k-k_o)}} + e^{-i(k-k_o)d\Delta}\frac{e^{i(q+k-k_o)d}}{1-e^{i(q+k-k_o)d}}\right] \\ & + i\frac{\Gamma_{1D}}{2}t\left[\frac{1}{1-e^{i(q-k-k_o)d}} + \frac{1}{1-e^{i(q+k-k_o)}}\right] \end{aligned} \tag{4.15}$$

Using (4.13) and (4.15) one cannot find a general expression for t (with no dependence on Δ).

$$\begin{aligned} & t\left[e^{i(k+k_o)d\Delta}\frac{1}{1-e^{-i(q-k-k_o)d}} + e^{-i(k-k_o)d\Delta}\frac{1}{1-e^{-i(q+k-k_o)d}}\right] \\ & = e^{i(k+k_o)d\Delta}\frac{e^{-i(q-k-k_o)d}}{1-e^{-i(q-k-k_o)d}} + e^{-i(k-k_o)d\Delta}\frac{e^{-i(q+k-k_o)d}}{1-e^{-i(q+k-k_o)d}}. \end{aligned} \tag{4.16}$$

This means our ansatz in (4.12) is not correct and we need to try a new one.

4.1.2 Second ansatz: inelastic collision

The new ansatz is quite similar, and still assumes that before the crossing ($\Delta < 0$) there is no interaction between photons. Looking at the energy in (4.13), one can realize it is degenerated, and therefore, there are two different valid values of q for a given E. We can also see this in Figure 4.2.

Our new hypothesis for the interaction is that some sort of inelastic collision occurs:

$$f(\Delta) = \begin{cases} e^{iq_1 d\Delta} & \Delta < 0 \\ t_1 e^{iq_1 d\Delta} + t_2 e^{iq_2 d\Delta} & \Delta > 0. \end{cases} \tag{4.17}$$

After the photons cross, there are two contributions with different q, and weights t_1 and t_2. This means the total momentum k of the pair does not change but the difference between k_1 and k_2 does, this is why we talk about an inelastic collision. Both the total energy and the energy of the individual photons are conserved.

We proceed in the same way as before and perform the sum in (4.11) distinguishing between the cases $\Delta < 0$ and $\Delta > 0$.

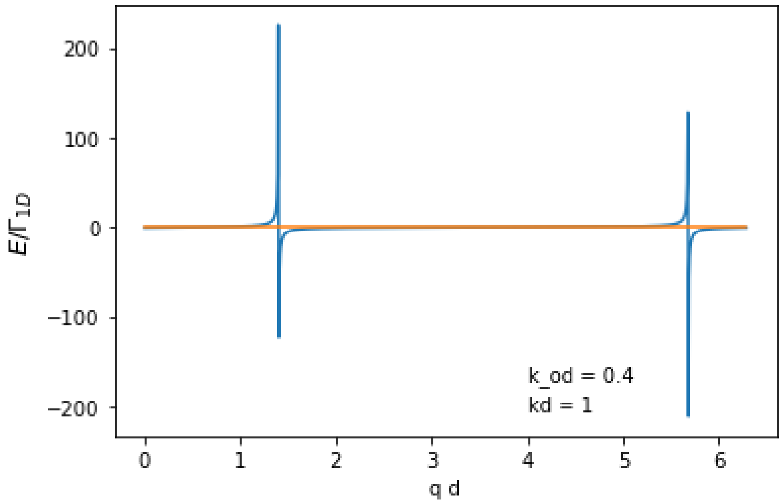

Figure 4.2: Total energy of the system with two excitations. One can clearly see that there two values of qd corresponding to each value of E. In blue we have the real part and in orange the imaginary part (always zero).

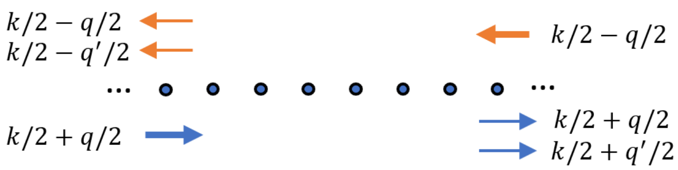

Figure 4.3: Scheme of the situation before and after the photons interact with each other.

- For $\Delta < 0$:

  The result is exactly the same as before (as so is the ansatz). Once again, the energy is the sum of the energies of two single excitations.

$$E = -i\Gamma_{1D}/2 + i\Gamma_{1D}/2 \left\{ \frac{1}{1 - e^{i(q_1 - k - k_o)d}} + \frac{1}{1 - e^{i(q_1 + k - k_o)}} \right\} \tag{4.18}$$

- For $\Delta > 0$:

$$\begin{aligned} E\left[t_1 e^{iq_1 d\Delta} + t_2 e^{iq_2 d\Delta}\right] &= -i\Gamma_{1D}/2 \left[t_1 e^{iq_1 d\Delta} + t_2 e^{iq_2 d\Delta}\right] \\ &- i\Gamma_{1D}/2 \sum_{\Delta'=-\infty}^{-1} \left[e^{i(k+k_o)d(\Delta-\Delta')} + e^{-i(k-k_o)d(\Delta-\Delta')}\right] \theta(\Delta - \Delta') e^{iq_1 d\Delta'} \\ &- i\Gamma_{1D}/2 \sum_{\Delta'=1}^{\Delta-1} \left[e^{i(k+k_o)d(\Delta-\Delta')} + e^{-i(k-k_o)d(\Delta-\Delta')}\right] \theta(\Delta - \Delta') \left[t_1 e^{iq_1 d\Delta'} + t_2 e^{iq_2 d\Delta'}\right] \end{aligned} \tag{4.19}$$

Using (4.18) and (4.19) we find the following equation:

$$\begin{aligned} t_2 &\left[e^{i(k+k_o)d\Delta} \frac{e^{i(q_2-k-k_o)}}{1-e^{i(q_2-k-k_o)}} + e^{-i(k-k_o)d\Delta} \frac{e^{i(q_2+k-k_o)d}}{1-e^{i(q_2+k-k_o)d}} \right] \\ &= e^{i(k+k_o)d\Delta} \frac{1-t_1 e^{i(q_1-k-k_o)}}{1-e^{i(q_1-k-k_o)}} + e^{-i(k-k_o)d\Delta} \frac{1-t_1 e^{i(q_1+k-k_o)d}}{1-e^{i(q_1+k-k_o)d}} \end{aligned} \tag{4.20}$$

In order to have a solution that does not depend on Δ we need to find a solution for the following system of equations:

$$\begin{cases} t_2 = \frac{\left(1-t_1 e^{i(q_1-k-k_o)d}\right)\left(1-e^{i(q_2-k-k_o)d}\right)}{\left(1-e^{i(q_1-k-k_o)d}\right) e^{i(q_2-k-k_o)d}} \\ t_2 = \frac{\left(1-t_1 e^{i(q_1+k-k_o)d}\right)\left(1-e^{i(q_2+k-k_o)d}\right)}{\left(1-e^{i(q_1+k-k_o)d}\right) e^{i(q_2+k-k_o)d}} \end{cases} \tag{4.21}$$

In view of the difficulties to find an analytical solution to the system in (4.21), finding a numerical solution seems a good approach. We should not forget to check that probability is normalized and adjusted with the group velocity for each of the contributions after the collision. This condition translates into:

$$1 = |t_1|^2 + \frac{v_{g,2}}{v_{g,1}} |t_2|^2, \tag{4.22}$$

where the group velocity is defined as:

$$v_g \equiv \frac{\partial \omega}{\partial k} = id\Gamma_{1D}/2 \left\{ \frac{e^{i(q_1-k-k_o)d}}{\left(1-e^{i(q_1-k-k_o)d}\right)^2} - \frac{e^{i(q_1+k-k_o)}}{\left(1-e^{i(q_1+k-k_o)d}\right)^2} \right\}. \tag{4.23}$$

In the following figures we can see the transmission coefficients t_1 and t_2 and the probabilities $|t_1|^2$ and $|t_2|^2 |v_1|/|v_2|$. In particular, in Figure 4.4 we have $kd = \pi$ and $k_o d = 0$ (or equivalently $k_o d = 2\pi$). We see that for $q_1 d = 0, 2\pi$, the system behaves exactly as we want it to, introducing a π phase for the elastic contribution. There is no inelastic contribution since $t_2 = 0$ for all values of $q_1 d$. The C-Phase gate would have an ideal behavior when the two photons enter the waveguide with the same momentum (but opposite sign) so that $q_1 = 0$.

For any set of k, k_o and q_1 we see that both t_1 and t_2 have nonexistent imaginary part for $q_1 d = k_o + \pi$. This is marked with the vertical dashed line in all the figures.

In Figure 4.4 we have $kd = \pi/2$ and $k_o d = 0$. In this case, even though it is possible to have a completely elastic collision $|t_1|^2 = 1$ for some values, we never gain a π phase. In Figure 4.6 we observe the same behaviour but displaced by $k_o d = 0.2$.

Finally, Figure 4.7 shows a more general case in which we cannot operate the C-Phase gate either.

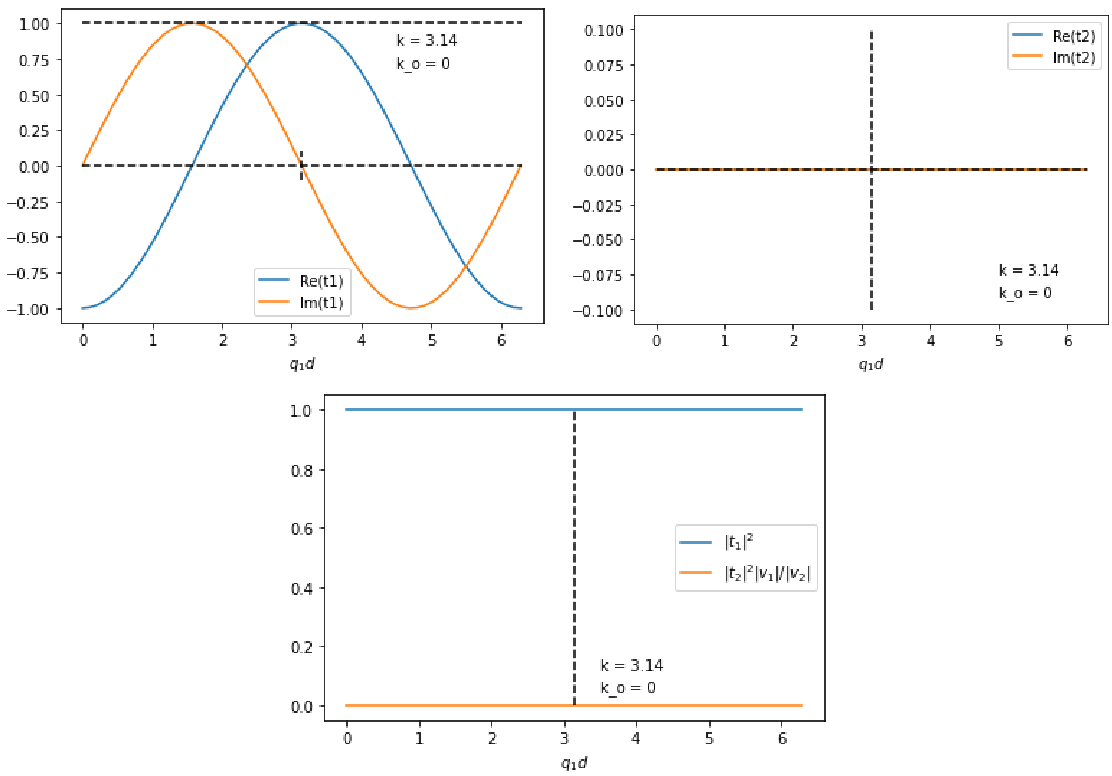

Figure 4.4: In the figures on top we can observe the real and imaginary part of the coefficients of the non-linear interaction t_1 and t_2 as a function of q_1d for $kd = \pi$ and $k_od = 0$. In the plot below we have the probabilities $|t_1|^2$ and $|t_2|^2|v_1|/|v_2|$ of having an elastic or an inelastic collision.

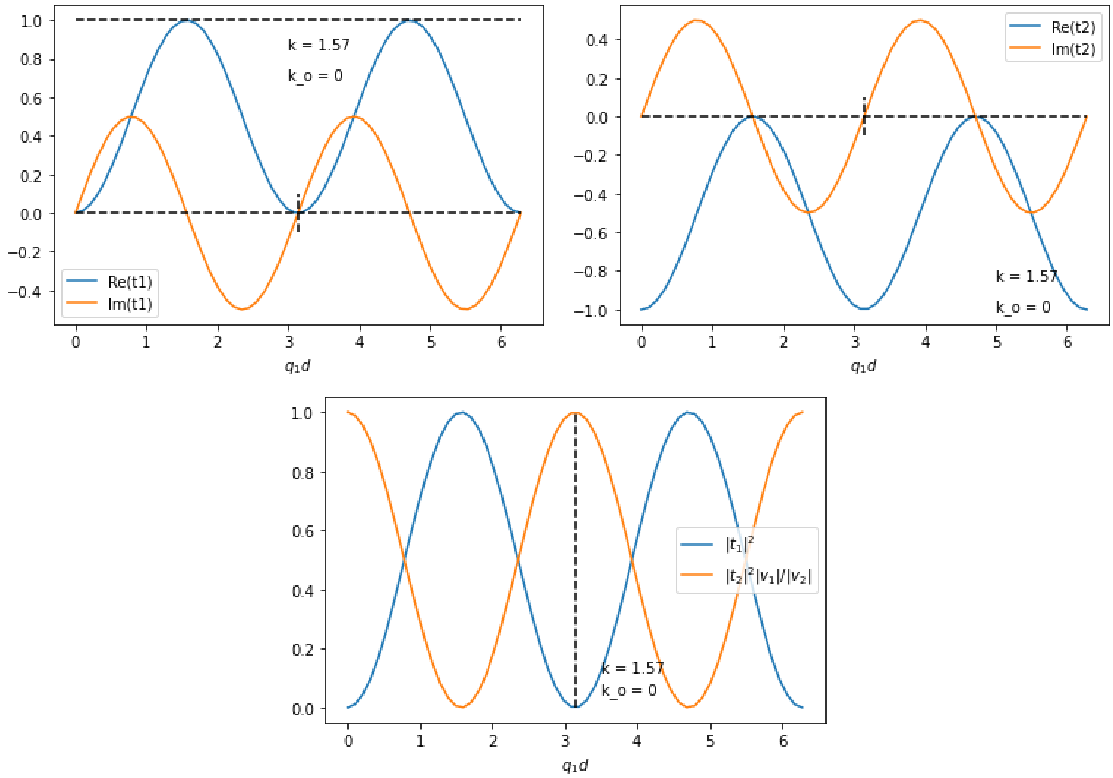

Figure 4.5: In the figures on top we can observe the real and imaginary part of the coefficients of the non-linear interaction t_1 and t_2 as a function of q_1d for $kd = \pi/2$ and $k_od = 0$. In the plot below we have the probabilities $|t_1|^2$ and $|t_2|^2|v_1|/|v_2|$ of having an elastic or an inelastic collision.

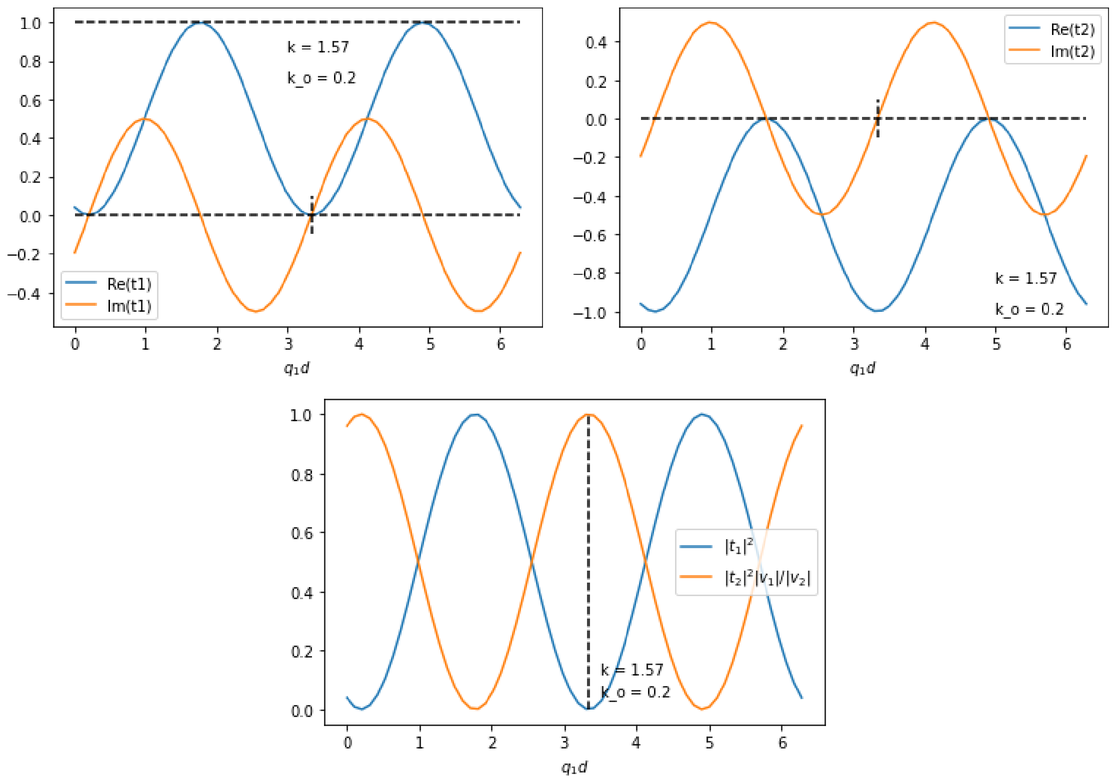

Figure 4.6: In the figures on top we can observe the real and imaginary part of the coefficients of the non-linear interaction t_1 and t_2 as a function of $q_1 d$ for $kd = \pi/2$ and $k_o d = 0.2$. In the plot below we have the probabilities $|t_1|^2$ and $|t_2|^2|v_1|/|v_2|$ of having an elastic or an inelastic collision.

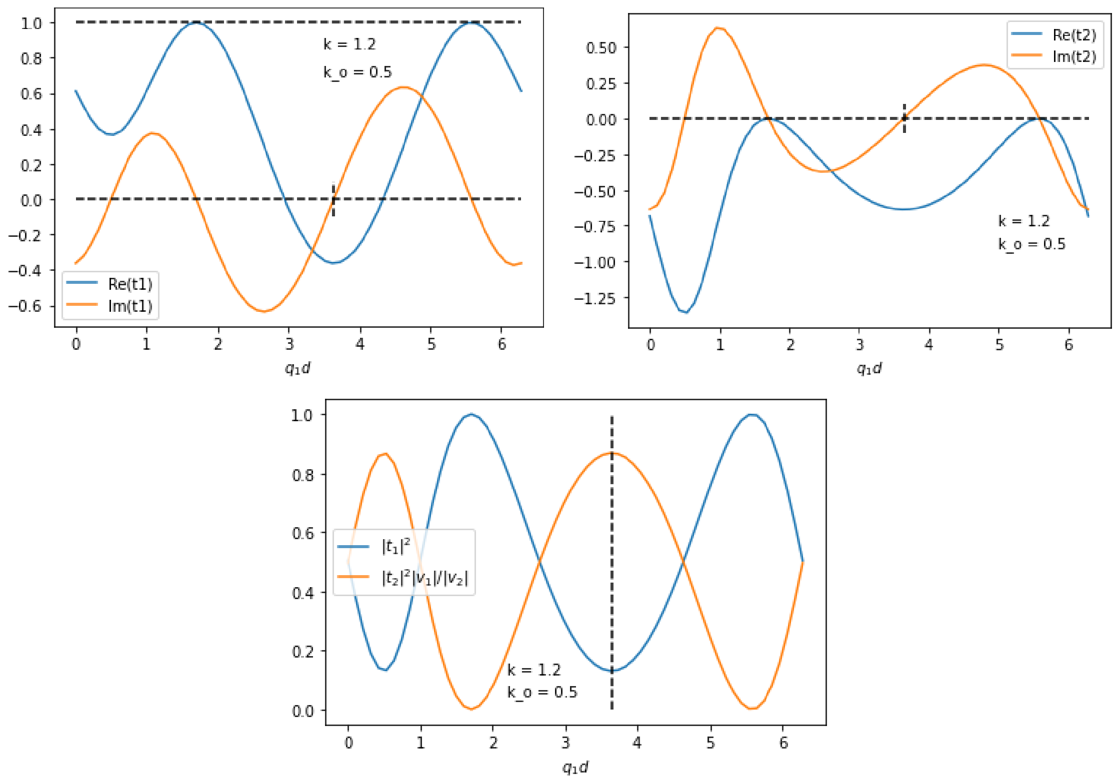

Figure 4.7: In the figures on top we can observe the real and imaginary part of the coefficients of the non-linear interaction t_1 and t_2 as a function of q_1d for $kd = \pi$ and $k_od = 0$. In the plot below we have the probabilities $|t_1|^2$ and $|t_2|^2|v_1|/|v_2|$ of having an elastic or an inelastic collision.

Chapter 5

Input/Output relation

For the moment, we have found the key part of the two-photon case, the non-linear interaction. However, in order to have a full description we still need to figure out what happens in a finite waveguide when the photons enter and leave the system.

In particular, we will consider both situations (entering and leaving) separately and change from the infinite array approximation to a semi-infinite one. We will continue to work in the completely chiral case so that we can put this together with the collision dynamics results.

5.1 Input relation

Our starting point will be the equation of motion that we calculated in Chapter 2. In particular, we take an expression analogous to (2.23) but now the atomic operators σ_{ab} are represented by c_n and the index has changed from j to n.

$$\dot{c}_n(t) = \tilde{g} e^{ik_o dn} \mathcal{E}_{in}(t) + i \sum_m H_{mn} c_m(t) \tag{5.1}$$

The Hamiltonian in the completely chiral case with $\beta_R = 1$ is simply:

$$H_{mn} = \begin{cases} 1 & \text{if } m = n \\ e^{ik_o|z_n - z_m|} & \text{if } m < n \\ 0 & \text{if } m > n \end{cases} \tag{5.2}$$

The Z-transform relates the coefficients c_n in space time with the coefficients c_k in momentum space.

$$c_n(t) = \frac{1}{d}\frac{1}{2\pi}\int_{-\pi/d-i\eta}^{\pi/d+i\eta} e^{ikdn} c_k(t) dk \qquad\qquad c_k(t) = \sum_{n=0}^{\infty} e^{-ikdn} c_n(t) \tag{5.3}$$

For convergence reasons we need to introduce the parameter η and then take the limit $\eta \to 0$.

Applying this relations to (5.1) and knowing the matrix elements of the Hamiltonian H_{mn} we get:

$$\dot{c}_k(t) = \sum_{n=0}^{\infty} \tilde{g} e^{ik_o dn} e^{-ikdn} \mathcal{E}_{in}(t) + \sum_{n=0}^{\infty}\sum_{m=0}^{n-1} e^{-ikdn} e^{ik_o d(n-m)} c_m(t) + \tfrac{1}{2} c_k(t) \tag{5.4}$$

For the input, we are considering a semi-infinite array of atoms that goes from 0 to $+\infty$. This defines the limits of the sums above.

If we perform the sums, this expression turns into:

$$\dot{c}_k(t) = G(k)\mathcal{E}_{in}(t) - i\omega_k c_k(t) \tag{5.5}$$

with:

$$G(k) = \frac{\tilde{g}}{1 - e^{i(k_o - k)d}} \qquad \text{and} \qquad \omega_k = \frac{\sin\left[(k_o - k)d\right]}{2(1 - \cos\left[(k_o - k)d\right])}. \tag{5.6}$$

We also introduce the Laplace transform that relates the coeffients c_k in time and frequency space:

$$c_k(t) = \int_{-\infty}^{\infty} e^{-i\omega t} c_k(\omega) d\omega \qquad\qquad c_k(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} c_k(t) e^{i\omega t} dt \tag{5.7}$$

Using the the Laplace transform properties for time derivatives, we can solve the differential equation in (5.5):

$$-i\omega c_k(\omega) = G(k)\mathcal{E}_{in}(\omega) - i\omega_k C_k(\omega) \qquad \longrightarrow \qquad c_k(\omega) = \frac{G(k)\mathcal{E}_{in}(\omega)}{i(\omega_k - \omega)} \tag{5.8}$$

Now we have to transform back into time space:

$$c_k(t) = G(k)\int_{-\infty}^{\infty} e^{-i\omega t} \frac{\mathcal{E}_{in}(\omega)}{i(\omega_k - \omega)} \tag{5.9}$$

Using Jordan's Lemma and the Residue Theorem applied to a pole at $\omega = \omega_k - i\epsilon$ (and then taking $\epsilon \to 0$) we end up with our final expression for the input:

$$C_k(t) = -2\pi G(k) e^{-i\omega_k t} \mathcal{E}_{in}(\omega_k) \tag{5.10}$$

5.2 Output relation

The process to find the output relation is very similar. Now we only have the contributions from the atoms and not an input field. Also, our semi-infinite array will have atoms from $-\infty$ to $N-1$. The starting point is:

$$\dot{c}_n(t) = i\sum_m H_{mn}c_m(t). \tag{5.11}$$

To facilitate the calculation and make it as similar as possible to the previous, we flip the axis and consider the array going from $-(N-1)$ to ∞. This affects our differential equation and the definition of the Z-transform:

$$\dot{c}_n(t) = -i\sum_m H_{mn}c_m(t) \tag{5.12}$$

$$c_k(t) = \sum_{n=0}^{\infty} e^{ikdn}c_n(t). \tag{5.13}$$

The output field is given by:

$$\mathcal{E}_{out}(t) = \sum_{n=-(N-1)}^{\infty} \tilde{g}e^{ik_o dn}c_n(t). \tag{5.14}$$

Using (5.13) this turns into:

$$\begin{aligned}\mathcal{E}_{out}(t) &= \frac{1}{2\pi}\frac{1}{d}\tilde{g}\sum_{n=0}^{\infty}\int e^{i(k_o-k)d(N-1)}e^{ik_o dn}e^{-ikdn}c_k(t)dk \\ &= \frac{1}{2\pi}\frac{1}{d}\int dk e^{i(k_o-k)d(N-1)}G(k)c_k(t),\end{aligned} \tag{5.15}$$

where $G(k)$ is the same as in (5.6).

We now have:

$$\dot{c}_k(t) = \sum_{n=0}^{\infty} e^{ikdn}c_n(t) = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} e^{ik_o d(m-n)}e^{ikdn}c_m(t) + 1/2c_k(t). \tag{5.16}$$

This, after performing the sums, can be written as:

$$\dot{c}_k(t) = -i\omega_k c_k(t), \tag{5.17}$$

where ω_k has the same dispersion relation as before, found in (5.6).

Once again, we use the properties of the Laplace transform to solve this differential equation.

$$- i\omega c_k(\omega) = \frac{1}{2\pi}C_k(t=0) - i\omega_k c_k(\omega) \qquad \longrightarrow \qquad c_k(\omega) = -\frac{C_k(t=0)}{i2\pi(\omega-\omega_k)} \tag{5.18}$$

We insert this expression into the output field:

$$\mathcal{E}_{out}(\omega) = \frac{1}{d}\frac{-1}{(2\pi)^2}\int dk e^{i(k_o-k)d(N-1)}G(k)c_k(t=0)\frac{1}{i(\omega-\omega(k))}. \tag{5.19}$$

We perform the substitution $k \rightarrow \omega(k)$ so that we can integrate over the frequency:

$$\mathcal{E}_{out}(\omega) = \frac{1}{d}\frac{-1}{(2\pi)^2}\int d\omega_k e^{i(k_o-k)d(N-1)}\left|\frac{\partial k}{\partial \omega_k}\right|G(k)c_k(t=0)\frac{1}{i(\omega-\omega(k))}. \tag{5.20}$$

One more time, we apply Jordan's lemma and the Residue theorem to a pole situated now in $\omega_k = \omega + i\epsilon$ and then take $\epsilon \rightarrow 0$. The final expression for the output is:

$$\mathcal{E}_{out}(\omega) = -\frac{1}{d}\frac{1}{2\pi}G(k)\frac{1}{v_g(\omega)c_k(t=0)}e^{i(k_o-k)d(N-1)}. \tag{5.21}$$

Here we have introduced the group velocity v_g which is defined as:

$$v_g(\omega) = \frac{d}{4\sin^2\left[(k-k_o)d/2\right]}. \tag{5.22}$$

5.3 Input/Output relation for a single photon

Now we have to put together the two expressions from above, (5.10) and (5.21). We start with the simplest case, just one photon traveling to the right, entering and leaving the waveguide.

We consider a general input state:

$$|\Psi\rangle = \int d\omega \Psi(\omega)\mathcal{E}^{\dagger}_{in}(\omega)\,|vac\rangle \tag{5.23}$$

Using (5.10) for $t = 0$ we get the state just after it enters the waveguide:

$$|\Psi\rangle = \int d\omega \Psi(\omega)\frac{-1}{2\pi G^*(k)}c^{\dagger}_{k(\omega)}\,|vac\rangle \tag{5.24}$$

Now the pulse leaves the waveguide after interacting with the N atoms. We use the output expression in (5.21) to describe how this frequencies inside the waveguide are translated into frequencies outside of it:

$$|\Psi\rangle = \int d\omega \Psi(\omega) \frac{-v_g^*(\omega)}{(G^*(k))^2} e^{i(k-k_o)d(N-1)} c_{k(\omega)}^\dagger |vac\rangle \tag{5.25}$$

We want to rewrite this as a relation of the form

$$|\Psi\rangle = \int d\omega \Psi(\omega) \mathcal{E}_{in}^\dagger(\omega) |vac\rangle \longrightarrow |\Psi\rangle = \int d\omega \Psi(\omega) f(\omega) \mathcal{E}_{in}^\dagger(\omega) |vac\rangle . \tag{5.26}$$

We can do so considering the function

$$f(\omega) = e^{i(k(\omega)-k_o)N}, \tag{5.27}$$

since everything else inside the waveguide cancel out.

Since we are considering a perfect chiral system, there are no losses (no reflections and no losses to other modes) and the only thing that happens to our photon is that it gains an energy-dependent phase as it propagates through the waveguide. This is what we see in (5.27).

5.4 Input/Output relation for the two photon case

At this point, we focus again on the semi-infinite chain approximation under which we are calculating the transmission of the photons. In our perfectly non-chiral system we can consider that the input and output of each photon are not affected by the presence of the other one. This allows us to simply apply the relation calculated above to each photon separately.

The additional component that refers to the non-linear interaction between the photons is the result of the inelastic collision that we studied earlier. We have to be careful with the propagation direction, one to the right and the other one to the left (opposite sign of k).

The two-photon initial state is:

$$|\Psi\rangle = \int \int d\omega d\omega' \Psi_R(\omega) \Psi_L(\omega') \mathcal{E}_{in}^{R\dagger}(\omega) \mathcal{E}_{in}^{L\dagger}(\omega') |vac\rangle . \tag{5.28}$$

Right after entering the waveguide this becomes

$$|\Psi\rangle = \int \int d\omega d\omega' \Psi_R(\omega) \Psi_L(\omega') \frac{-1}{2\pi G(k_R)} \frac{-1}{2\pi G(-k_L)} {c_{k(\omega)}^R}^\dagger {c_{k(\omega')}^L}^\dagger |vac\rangle , \tag{5.29}$$

where we have applied (5.10) twice.

As it was discussed before, we are interested in working in a range for which the contribution from the inelastic collision can be neglected. Either because the probability approximates to zero or because the individual energies of the resulting pair of photons are so different from the initial ones that they can easily be filtered out in a hypothetical measurement device. Therefore, in this input/output relation, we will only consider the elastic collision component. This corresponds with the coefficient t_1 in (4.17). From now on we will refer to it as $t(k_R(\omega), k_L(\omega'))$.

The interaction just adds a term to the previous state.

$$|\Psi\rangle = \int\int d\omega d\omega' t(k_R(\omega), k_L(\omega'))\Psi_R(\omega)\Psi_L(\omega')\frac{-1}{2\pi G(k_R)}\frac{-1}{2\pi G(-k_L)}c^{R}_{k(\omega)}{}^{\dagger}c^{L}_{k(\omega')}{}^{\dagger}|vac\rangle\,. \tag{5.30}$$

Now we apply once again the output relation (5.21) to each of the photons and get the final expression:

$$\begin{aligned}|\Psi\rangle = &\int\int d\omega d\omega' t(k_R(\omega), k_L(\omega'))\Psi_R(\omega)\Psi_L(\omega')\frac{-v_g(\omega)}{G^2(k_R)}e^{i(k_R-k_o)d(N-1)} \\ &\times \frac{-v_g(\omega')}{G^2(-k_L)}e^{i(-k_L-k_o)d(N-1)}c^{R}_{k(\omega)}{}^{\dagger}c^{L}_{k(\omega')}{}^{\dagger}|vac\rangle\end{aligned} \tag{5.31}$$

In an equivalent way to (5.26) we can write:

$$f(\omega,\omega') = e^{i(k_R(\omega)-k_o)N}e^{i(-k_L(\omega')-k_o)N}t(k_R(\omega), k_L(\omega')) \tag{5.32}$$

Chapter 6

Fidelity measurements for the two photon case

One more time, we recover the formula for the fidelity as discussed in Chapter 1.

$$F = \frac{1}{16} \left| 1 + t_L + t_R - t_{RL} \right|^2 \tag{6.1}$$

We now assume that the transmission of a single photon through the waveguide is perfect. In this case we would have $t_L = t_R = 1$. The formula for the fidelity turns into:

$$F = \frac{1}{16} \left| 3 - t_{RL} \right|^2 \tag{6.2}$$

Notice that, also in the ideal case, the optimal phase introduced between the two photons when they interact is π. The fidelity will be maximum when $t_{LR} = -1$.

In the same way we calculated the coefficients t_R and t_L in Chapter 3, we calculate t_{RL} considering a Gaussian spectrum for each of the photons:

$$t_{RL} = \int \int d\omega d\omega' \phi_R^*(\omega) \phi_R(\omega) \phi_L^*(\omega') \phi_L(\omega') f(\omega, \omega') \tag{6.3}$$

The normalized Gaussian spectrum is:

$$\phi_{L,R} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5(\omega-\omega_1)^2/\sigma^2} e^{i(\omega-\omega_1)\tau} \tag{6.4}$$

The transmission function we just calculated with the input/output relation was:

$$f(\omega, \omega') = e^{i(k-k_o)dN} e^{i(-k'-k_o)dN} t(\omega, \omega') \tag{6.5}$$

where $t(\omega, \omega')$ is the coefficient of the elastic collision between the two photons and the dispersion relation is:

$$\omega(k) = -\frac{\cos\left[(k-k_o)d/2\right]}{2\sin\left[(k-k_o)d/2\right]} \qquad \longrightarrow \qquad k(\omega)d - k_o d = 2\arctan(-2/\omega) \tag{6.6}$$

However, we will leave the exponentials out of the calculation, since they are related to the propagation over the distance of the waveguide, one would also find those oscillations in the ideal case with which we are comparing our output photon. Therefore, the transmission function is simply:

$$f(\omega, \omega') = t(\omega, \omega'), \tag{6.7}$$

and we can focus in the effects of the collision only.

Figures 6.1 and 6.2 show the result of numerically performing the integral in (6.3) and introducing it into the fidelity.

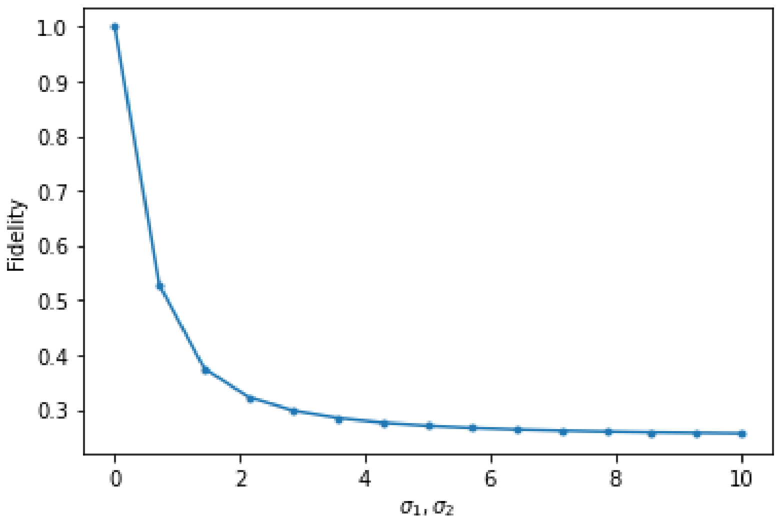

Figure 6.1: Fidelity of the gate considering only the inelastic collision of two photons against the standard deviation of the Gaussian profile of the two incident photons. The two pulses are centered on resonance ($\omega_1 = 0$) and have the same standard deviation.

Because of the way I defined F, the minimum value is 0.25. One can see this looking at (6.2). At the worst possible scenario $t_{RL} = 1$, the C-Phase gate introduces no phase at all.

In Figure 6.2 we can see how we get a fidelity of over 90% for reasonable values of σ_1 and σ_2. We can afford to send relatively broad pulses and still get the desired non-linear interaction. However, we must be careful, if we reduce the width of the

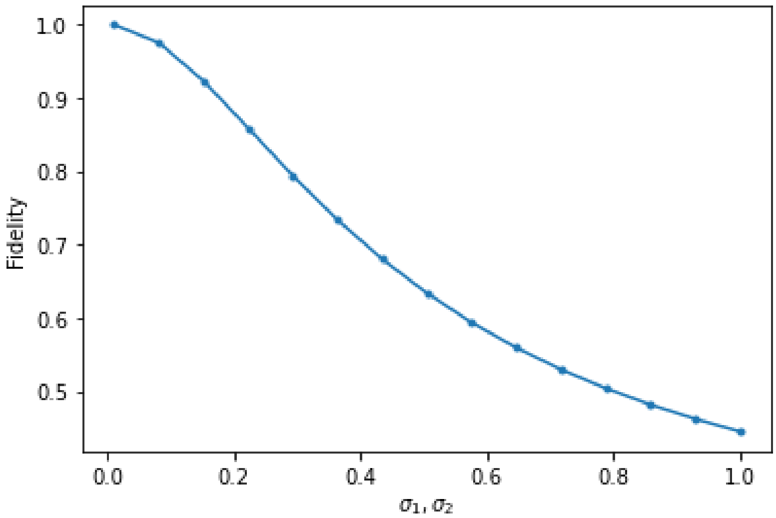

Figure 6.2: Closer look at the fidelity shown in Figure 6.1.

pulse in frequency space to achieve higher fidelities, we would be taking a very broad pulse in the time domain. In such case, one should be careful and ensure that the waveguide is long enough to fit in the whole pulse.

Chapter 7

Conclusions and outlook

7.1 Conclusions

In this Master's thesis, we have studied two key aspects in the operation of a quantum C-Phase gate. The results show that the implementation of such a gate in a setup like the one proposed is possible. The ideal behaviour of the system in all the possible situations is that of a C-Phase gate.

For the case of propagation of a single photon, we have seen that we need to be very close to an ideal system with $\beta_R \sim 1$ in order to mitigate the losses and achieve a high enough fidelity. Moreover, we have observed this cavity like behaviour in which the reflection is enhanced. This could be interesting for some applications but is also avoidable for our implementation.

For the interaction of two photons we have observed the most relevant results. The inelastic collision that occurs when two photons find each other inside the waveguide was an unexpected and interesting result that could also give rise to many interesting applications. It is in fact, a way of changing the wavelength of two photons by simply sending both through a waveguide. However, in our case, we are interested in avoiding it. There is a regime in which we can achieve a completely elastic collision that introduces the desired relative phase between our two qubits.

7.2 Outlook

During the course of this project, simplicity and analytical resolution is prioritized over a more realistic study. Of course, this has allowed us to successfully and roughly determine that a C-Phase quantum gate can be implemented with this setup. But moving forward we should get rid of several approximations.

The obvious next step would be to study how the non-linear interaction occurs in a system with partial-chirality and losses to outside of the waveguide ($\beta_R \neq 0$). This would allow us to combine the results found in Chapter 2 and 4 for the cases of transmitting one and two photons respectively.

A second interesting thing to do would be solving the whole system for the case of two photons, and get rid of the infinite waveguide approximation. The input/output relation calculated in Chapter 5 will be of much more relevance in the case $\beta_R \neq 0$ and it is possible then that, in a shorter array of atoms, the two effects cannot be considered completely independent.

Finally, a generalization of the formalism, in which we can consider different coupling rates for the propagation in each direction and different energy levels, would give us a complete and realistic description of the system.

As for the experimental implementation, it seems far from becoming a reality soon. It is specially difficult to have long and stable 1D arrays of quantum emitters and it is even harder doing so controlling the exact distance between them.

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Appendix A

Matrix inversion

A.1 Approximation to first order of M^{-1}

One can show that given a matrix $M = A + \epsilon B$ with $\epsilon \ll 1$, the inverse matrix can be approximated to first order in ϵ by using $M^{-1} = A^{-1}[1 - \epsilon BA^{-1}]$.

We check this is correct calculating $MM^{-1} = 1$.

$$\begin{aligned} MM^{-1} &= [A + \epsilon B]A^{-1}[1 - \epsilon BA^{-1}] = [AA^{-1} + \epsilon BA^{-1}][1 - \epsilon BA^{-1}] \\ &= 1 + \epsilon BA^{-1} - \epsilon BA^{-1} - \epsilon^2 BA^{-1}BA^{-1} \end{aligned}$$

If we consider the term with ϵ^2 negligible, we have that $MM^{-1} = 1$.

A.2 Inverse of a lower diagonal matrix

To calculate the previous approximation we still need to invert the matrix A, which we know is of the form:

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & ... \\ a_{21} & a_{22} & 0 & 0 & ... \\ a_{31} & a_{32} & a_{33} & 0 & ... \\ a_{14} & a_{42} & a_{43} & a_{44} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.1}$$

An inverse matrix must fulfill that $AA^{-1} = 1$. The inverse matrix of a lower diagonal matrix is always also lower diagonal. This means:

$$AA^{-1} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & ... \\ a_{21} & a_{22} & 0 & 0 & ... \\ a_{31} & a_{32} & a_{33} & 0 & ... \\ a_{41} & a_{42} & a_{43} & a_{44} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} l_{11} & 0 & 0 & 0 & ... \\ l_{21} & l_{22} & 0 & 0 & ... \\ l_{31} & l_{32} & l_{33} & 0 & ... \\ l_{41} & l_{42} & l_{43} & l_{44} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & ... \\ 0 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & ... \\ 0 & 0 & 0 & 1 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.2}$$

The elements in the diagonal will always fulfill:

$$a_{ii}\ l_{ii} = 1 \qquad \longrightarrow \qquad l_{ii} = a_{ii}^{-1} \tag{A.3}$$

The rest of the elements can be found using forward substitution and then we can find a generalized expression:

$$\begin{aligned}
a_{21}\ l_{11} + a_{22}\ l_{21} = 0 \qquad &\longrightarrow \qquad l_{21} = -\frac{a_{21}\ l_{11}}{a_{22}} \\
a_{31}\ l_{11} + a_{32}\ l_{21} + a_{33}\ l_{31} = 0 \qquad &\longrightarrow \qquad l_{31} = -\frac{a_{31}\ l_{11} + a_{32}\ l_{21}}{a_{33}} \\
a_{32}\ l_{22} + a_{33}\ l_{32} = 0 \qquad &\longrightarrow \qquad l_{32} = -\frac{a_{32}\ l_{22}}{a_{33}} \\
a_{42}\ l_{22} + a_{43}\ l_{32} + a_{44}\ l_{42} = 0 \qquad &\longrightarrow \qquad l_{42} = -\frac{a_{42}\ l_{22} + a_{43}\ l_{32}}{a_{44}} \\
a_{41}\ l_{11} + a_{42}\ l_{21} + a_{43}\ l_{31} + a_{44}\ l_{41} = 0 \qquad &\longrightarrow \qquad l_{41} = -\frac{a_{41}\ l_{11} + a_{42}\ l_{21} + a_{43}\ l_{31}}{a_{44}} \\
&\vdots
\end{aligned}$$

The final expression for the elements below the diagonal of the inverse matrix is:

$$l_{ij} = -\frac{\sum_{m=j}^{i-1} a_{im} l_{mj}}{a_{ii}} \tag{A.4}$$