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Collective modes of superconductor ferromagnet hybrid materials

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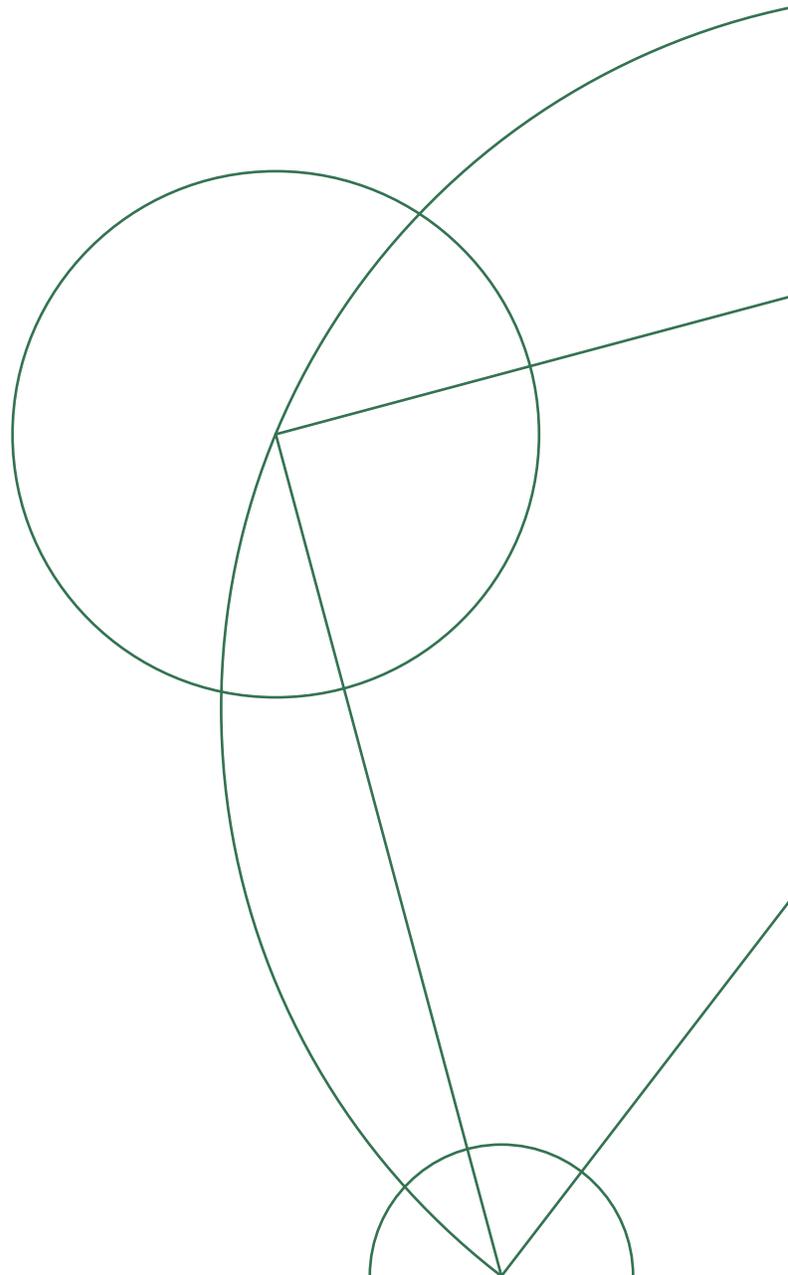
Abstract

In this thesis I study the system of 1-D and 2-D superconductors and it's collective mode excitation the plasmon. I study the system of a 1-D ferromagnet and it's collective mode excitation the magnon in a linearized regime where the Landau-Lifshitz equation is analytically solvable. I verify the results of the magnon calculations using a numerical approach. I study a superconducting system with spin-orbit coupling and a magnetic exchange field. Using this last system as a method of coupling magnetization to current I calculate the dispersion relation of a coupled plasmon/magnon system and solve for its eigenstates and eigenvalues resulting in a gapped magnon/plasmon hybrid state. I discuss potential implementation of this work in a magnon to plasmon converter that would flip magnons into plasmons without the need for measurement devices or signal creating devices.

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Contents

1 Introduction

The intention of this project was to investigate the properties of collective wave modes of superconductor ferromagnetic hybrid materials.

Collective wave modes can be several things but the two pictures I draw the most from are the phenomena of plasmons and magnons. Plasmons are the oscillation of electron density inside a sea of electrons in a metal, or superconducting electrons in a superconductor.

The ground state of the electron gas is one of equal density of electrons everywhere, as the electrons obtain minimal free energy by creating this state. If however a disturbance in this state were to occur, causing a gathering of electrons that increase the density in a particular area, that same repulsive force would force out a wave of electron density that propagates through the material as the area of high density and low density changes.

Magnons in ferromagnets are also a collective mode I draw from. One can imagine a ferromagnet as a collection of sites separated by equal distances which have a spin. Each spin wishes to align with its neighbouring spin to minimize the free energy (just like the electric density wished to be evenly distributed to minimize free energy). These sites make up the lattice of a solid crystalline material. If one imagines a deviation in the lined up spins happening spontaneously at a single site, what would happen is that it would affect the adjacent sites and those adjacent sites would affect their adjacent sites thus cascading through the material. This would again propagate a wave throughout the material as all the sites would adjust due to the original disturbance. These phenomena are called collective modes, because one can identify an energy and velocity associated with a given wave propagating through the materials. It is an excited state of the system, but it is not an excitation on an individual atom or electron, it is an excitation that requires the consideration of the whole material to exist and describe, thus the collective.

A superconductor ferromagnet hybrid is a material that is on one side of an interface ferromagnet, and on the other side superconductor. What happens in between is also of consideration as any real material would have a transitional period from being magnet or superconductor as magnetic fields are incompatible with superconduction. Superconduction requires the expulsion (or penetration) of magnetic fields to not interfere with the Cooper pairs that make up the superconducting electron sea.

1.1 Approach

We will split the analysis of the hybrid material into an analysis of 3 different parts explaining and getting deep into the calculation and analysis of other researchers work on these 3 parts. Then finally we will combine the results of these 3 parts in an original analysis.

In Ch. 2 we will explore the properties of the superconducting side of the hybrid material and what properties, collective modes of the bulk superconductor have in 1 dimension and to a lesser extent 2 dimensions. In this chapter we derive the plasmon dispersion relation following the 1973 Kulik paper on plasmons [1]. We also outline an equation governing the motion of the electric current to be used in Ch. 5. Additionally we expand the original analysis of Kulik's 1 dimensional system to one of 2 dimensions.

In Ch. 3 we will explore a system laid out by 2015 Pershoguba [2]. This system is a mixed system of superconductor and ferromagnet. The system incorporates superconductivity and spin-orbit coupling. The system consists of a 2-D metal plane with a macroscopic magnetic impurity placed on top of it providing a magnetic exchange field. Inside this system we derive the form of the current and see how it depends on the magnetization of the magnetic

exchange term.

In Ch. 4 we will look at a completely ferromagnetic system with no superconductivity, thus completing the transition from superconductor to ferromagnet. Here we derive equations for the dispersion of the magnons using a reasonable linearization of the magnons assuming only small deviations of magnetization. We then use numerical simulations to assess how accurate this dispersion relation is. The numerical part of this chapter follows some of the numerical methods outlaid in Xiao-Ping 2001 [3], while the linearized analysis is original. At the end we touch upon the non linear regime and how the dispersion relation breaks down and becomes incompatible with the analytics.

In Ch. 5 we combine the free energy for the magnons with that of the mixed state material that makes up the spin-orbit-magnetic exchange area and calculate a closed set of equations for the current in the material (plasmons) based upon the pure superconductor material and the magnetization of the material (magnons). The Pershoguba article [2] and material deals with a constant macroscopic magnetic impurity which generates their magnetic exchange field and we are dealing with a variable magnetization based upon a ferromagnet in this section. We make the leap here to consider the magnetization to be proportional to the magnetic exchange field and as such use their analysis of an external ferromagnet on top of a superconductor to serve as our transition regime between magnon and plasmon.

While the 3 different areas (superconductor, superconductor with magnetic exchange field, ferromagnet) are analyzed employing different geometries for their respective systems, we are looking towards using a 1D geometry taking the form of a wire in our final analysis as this is the sort of material we are most interested in. As such we will be reducing the dimensionality effectively down to 1D upon the combination of our equations from each individual area. This geometry of a wire that starts off as superconductor, then eventually turns into ferromagnet might be useful to keep in mind while reading the dissertation.

2 Plasmons in 1-D and 2-D superconductors

In this section we wish to characterize the motion of the oscillatory modes of superconducting electrons We do so by following the calculations of Kulik 1973 "Surface-charge oscillations in superconductors" [1]. We will first follow closely the derivation of Kulik in 1-D then perform a quick 2-D derivation assuming a very low temperature superconductor, where it can be assumed that the non superconducting part of the material is inactive.

We will start by outlining the equations of motion for 1-D and derive all further results from this starting point. The goals are to derive the effective coulomb interaction in 1-D, to acquire an equation for the current based on the electric field for further use in Ch. 5 where we will couple the plasmon and magnons through a unique current expression. This part of the thesis thus creates the basis for our current equation that makes up the plasmon part of our system. Additionally we will also derive the dispersion relation for a heavily superconductive system and a less heavily superconducting one, which represents the temperature regime where superconductivity starts to break down and mixed superconductor, normal conductor dynamics come into play. The actual temperature dependence will not be analyzed and will be left as and outside variable that affects our parameters, such as conductivity and superconductive electron density.

2.1 Equations of motion

We start by examining the general equations for superconducting electrons motion inside a superconducting 1D filament (3D wire with very small cross section). The filament will be along the x-direction. The basic equations we will need are the following: The continuity equation for charge, where $I(x,t)$ is the current along the filament and $Q(x,t)$ is the linear charge density at a given position x. We ignore the possibility of charge escaping the filament, so no further conservation equation is needed for the dimensions that make up the cross section. We also ignore the possibility of charge circling around the wire creating motion in the 2D layer that makes up the surface.

$$\frac{\partial I(x,t)}{\partial x} + \frac{\partial Q(x,t)}{\partial t} = 0 \tag{1}$$

The definition of the E-field as the only force ($\boldsymbol{F}$) present. Where $\boldsymbol{p}$ is the momentum, $\boldsymbol{v}_s$ is the superconducting electron velocity, m is the electron mass and e is the electron charge. The sign is due to the particles in motion being negatively charged.

$$\boldsymbol{E} = -\frac{1}{e}\boldsymbol{F} = -\frac{d\boldsymbol{p}}{dt}\frac{1}{e} = -\frac{\partial \boldsymbol{v}_s}{\partial t}\frac{m}{e} \tag{2}$$

Gauss's law with $\delta(\boldsymbol{\rho})$ being a delta function that keeps us confined to the filament. This term together with the linear charge density forms a density of charge in 3D

$$\nabla \cdot \boldsymbol{E} = 4\pi Q(x,t)\delta(\boldsymbol{\rho}). \tag{3}$$

Additionally we need an expression for the current in the superconductor. This is the normal current, applicable for charge in a normal metal with the addition of a supercurrent term, with a modification that takes into account the existence of a critical current above which no superconducting current exists [1]. v_s is the velocity of the superconducting electrons along the filament, while v_c is the critical velocity of the superconducting electrons. σ_n

is the normal conductivity of the material. Even while superconducting, part of the material will be normal and be affected by ohms law, this part is proportional to the temperature of the material. For $T = 0$ this term completely disappears but for any finite temperature we would have normally conductive electrons that would contribute a term like this. N_s is the density of superconducting electrons and S is the surface area that makes up the cross section of the filament. In total this leaves our current as the following:

$$I = \left(N_s e v_s \left(1 - \frac{v_s^2}{v_c^2} \right) + \sigma_n E \right) S. \tag{4}$$

We will use this equation for the current in our 1-D analysis but will assume to be in a strongly superconducting temperature regime and low enough current regime for our 2-D analysis so that we can ignore the effects of critical currents and normal electrons.

2.2 Calculating the potential $\phi(x)$

We start from Gauss's law written in Gaussian units

$$\nabla \cdot \boldsymbol{E} \left(\boldsymbol{r}, t \right) = 4\pi Q(x,t) \delta(\boldsymbol{\rho}). \tag{5}$$

We wish to calculate the potential $\phi(\boldsymbol{r})$ so we use the following equation for the calculation of the electric field based on the electric potential, as the electric force is a conservative force:

$$\boldsymbol{E} \left(\boldsymbol{r}, t \right) = -\nabla \phi \left(\boldsymbol{r}, t \right). \tag{6}$$

Inserting this into Gauss law (Eq 3) we get the following:

$$-\nabla^2 \phi \left(\boldsymbol{r}, t \right) = 4\pi Q \left(x, t \right) \delta \left(\boldsymbol{\rho} \right). \tag{7}$$

We write $\phi(\boldsymbol{r}, t)$, $Q(x, t)$ and $\delta(\boldsymbol{\rho})$ as their Fourier transforms. Using convention:

$$f(r) = \frac{1}{2\pi} \int dq f(q) e^{iqr} \tag{8}$$

for the Fourier transform. We will continue to use this same convention for the Fourier transform throughout the thesis to stay consistent with the placement of $\frac{1}{2\pi}$ and the sign of the exponentials e^{ikx}. This yields us

$$-\nabla^2 \phi \left(\boldsymbol{r}, t \right) = -\nabla^2 \frac{1}{(2\pi)^3} \int d^3k \phi \left(\boldsymbol{k} \right) e^{i\boldsymbol{k} \cdot \boldsymbol{r}} = \frac{1}{(2\pi)^3} \int d^3k 4\pi Q \left(k_x \right) \delta \left(\boldsymbol{\kappa} \right) e^{i\boldsymbol{k} \cdot \boldsymbol{r}}. \tag{9}$$

Where the integral over d^3k is over k_x and $\boldsymbol{\kappa}$, where $\boldsymbol{\kappa}$ is the wave vector in the y and z direction. The Fourier transform of the delta function is easily computed

$$\delta \left(\boldsymbol{\kappa} \right) = \int d^2\rho e^{i\boldsymbol{\kappa} \cdot \boldsymbol{\rho}} \delta \left(\boldsymbol{\rho} \right) = 1. \tag{10}$$

Likewise the Laplacian of the exponential function is also easily computed and we end up with the following expression:

$$\frac{1}{(2\pi)^3} \int d^3k \phi(\boldsymbol{k}) |\boldsymbol{k}|^2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \frac{1}{(2\pi)^3} \int d^3k 4\pi Q(k_x) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{11}$$

For this expression to hold true for all possible values of $\boldsymbol{r}$, the integrands must be identical. This can be seen by imagining a small value for $\boldsymbol{r}$, such that each part of the sum that makes up the integral has a different phase. By continuously making $\boldsymbol{r}$ arbitrarily smaller and smaller, we can make sure that each term has a different phase and none of the phases go above 2π. For such an expression, it is easy to see that the integrands in front of the phases must be equal for the expression to be true. This gives us a much simpler set of equations

$$\phi(k_x, \boldsymbol{\kappa}) = \frac{4\pi Q(k_x)}{k_x^2 + \kappa^2}. \tag{12}$$

We can now calculate the potential in real space by taking the Fourier transform of $\phi(k_x, \boldsymbol{\kappa})$

$$\phi(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int dk_x \int d^2\kappa 4\pi Q(k_x) e^{ik_x x + i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}} \frac{1}{k_x^2 + \kappa^2}. \tag{13}$$

Writing $Q(k_x)$ as its Fourier transform, we get the following

$$\phi(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int dk_x \int d^2\kappa 4\pi \int dx' e^{ik_x(x-x') + i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}} Q(x') \frac{1}{k_x^2 + \kappa^2}. \tag{14}$$

Deciding to only consider $\boldsymbol{\rho} = 0$ for $\phi(\boldsymbol{r}, t)$ we get the following expression:

$$\phi(x, \boldsymbol{\rho} = 0) = \phi(x) = \frac{4\pi^2}{(2\pi)^2} \int dx' Q(x') \frac{1}{2\pi} \int dk_x e^{ik_x(x-x')} \left(\frac{1}{\pi} \int d^2\kappa \frac{1}{k_x^2 + \kappa^2} \right). \tag{15}$$

By limiting ourselves to $\boldsymbol{\rho} = 0$ we cut ourselves off from all information about the E-field in the $y - z$ plane. We might have rotations on the outside or inside of the filament in this plane, but we will no longer be able to analyze or detect them in our solutions.

We can define a function $\alpha(k)$ to simplify as the following:

$$\alpha(k_x) = \frac{1}{\pi} \int d^2\kappa \frac{1}{k_x^2 + \kappa^2}. \tag{16}$$

Which yields the equation:

$$\phi(x) = \frac{4\pi^2}{(2\pi)^2} \int dx' Q(x') \frac{1}{2\pi} \int dk_x e^{ik_x(x-x')} \alpha(k_x.) \tag{17}$$

Seeing that we have the Fourier transform of $\alpha(k_x)$, we proceed to write the potential in real space

$$\phi(x) = \int dx' Q(x') \alpha(x - x'). \tag{18}$$

We see now that $\alpha(x)$ is a localizing function that serves much the same role as $1/r$ in the coulomb potential. This function holds the sum knowledge of how charge affects the electric potential in our system.

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2.3 Calculating $\alpha(x)$

From the previous section we found a potential given in terms of $\alpha(x)$. We now take upon us the task of calculating $\alpha(k)$ and from that inverse Fourier transforming and getting $\alpha(x)$ so that we can understand how this potential looks in space and compare it to other potentials such as the coulomb potential. The integral over the plane can easily be computed

$$\alpha(k) = \frac{1}{\pi} \int d^2\kappa \frac{1}{k^2 + \kappa^2} = \frac{1}{\pi} \int_0^\infty d\kappa \int_0^{2\pi} d\theta \frac{\kappa}{k^2 + \kappa^2} = 2 \int_0^\infty d\kappa \frac{\kappa}{k^2 + \kappa^2}. \tag{19}$$

We then perform a substitution $\kappa' = \frac{\kappa}{k}$, such that we can integrate using only unitless variables

$$\alpha(k) = 2 \int_0^\infty d\kappa \frac{\kappa}{k^2 + \kappa^2} = 2 \int_0^\infty \frac{d\kappa}{k} \frac{\frac{\kappa}{k}}{1 + \frac{\kappa^2}{k^2}} = 2 \int_0^\infty d\kappa' \frac{\kappa'}{1 + \kappa'^2}. \tag{20}$$

We then do a substitution $u = 1 + \kappa'^2$. The integral becomes trivial, we solve it and insert the old variables

$$\alpha(k) = \int_1^\infty du \frac{1}{u} = \ln\left(1 + \kappa_m'^2\right) = \ln\left(\frac{k^2 + \kappa_m^2}{k^2}\right). \tag{21}$$

In this case we have to evaluate $\ln(1 + \kappa')$ at $\kappa' = \infty$, to avoid this divergence we implement a cutoff κ_m, which then goes on to define a length scale $d = \kappa_m^{-1}$.

We can now go on to evaluate $\alpha(x)$

$$\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \ln\left(\frac{k^2 + \kappa_m^2}{k^2}\right) e^{ikx}. \tag{22}$$

We use the property of the logarithm to split the equation into two terms

$$\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \left(-2\ln(k) + \ln\left(k^2 + \kappa_m^2\right)\right) e^{ikx}. \tag{23}$$

We then perform partial integration:

$$\alpha(x) = \frac{-1}{2\pi} \int_{-\infty}^\infty dk \left(-2\frac{1}{k} + \frac{2k}{k^2 + \kappa_m^2}\right) \frac{1}{ix} e^{ikx} + \frac{1}{2\pi} \left[\left(-2\ln(k) + \ln\left(k^2 + \kappa_m^2\right)\right) e^{ikx}\right]_{-\infty}^\infty. \tag{24}$$

Looking at the surface term we see that we get no contribution. Our original expression contained k^2, and by abusing the logarithm further and restoring it to its original form we see that at $-\infty$ and at ∞ we get the same value and both values are zero. Turning ourselves back to the integral at hand, we rearrange the equation to clearly show the parts dependent on k and then perform contour integration in the upper half of the imaginary plane, assuming x to be positive [4]

$$\alpha(x) = \frac{1}{\pi} \frac{1}{ix} \int_{-\infty}^\infty dk \left(\frac{e^{ikx}}{k} - \frac{e^{ikx} k}{k^2 + \kappa_m^2}\right). \tag{25}$$

Since e^{ikx} becomes zero as $k \to i\infty$, the contribution from the connecting contour is zero and the resulting contour integration is equal to the integral along the real axis. Via

the residue theorem we then know that $\alpha(x)$ is equal to the sum of the residues within the contour multiplied by $2i\pi$

$$\alpha(x) = \frac{1}{\pi}\frac{1}{ix}2i\pi\sum_i a_{-1,i} = \frac{2}{x}\sum_i a_{-1,i}. \tag{26}$$

There are two residues, one at $k = 0$ for the first term and one for $k = i\kappa_m$ for the second term. Since the $k = 0$ residue is on the contour line and it passes straight through it, a common solution is to count half of its value as the residue inside the contour. This gives the following residues

$$a_{-1,1} = \frac{1}{2}\lim_{k\to 0}\frac{e^{ikx}}{k}k = \frac{1}{2}, \tag{27}$$

$$a_{-1,2} = \lim_{k\to i\kappa_m}\frac{e^{ikx}k}{(k - i\kappa_m)(k + i\kappa_m)}(k - i\kappa_m) = \frac{e^{-\kappa_m x}i\kappa_m}{2i\kappa_m} = \frac{e^{-\kappa_m x}}{2}. \tag{28}$$

Inserting this into Eq. 26 we get the following expression for $\alpha(x)$ for positive x. Remembering the length scale $d = 1/\kappa_m$

$$\alpha(x) = \frac{1 - e^{-x/d}}{x}. \tag{29}$$

We see that if one were to do the same calculation using the residues on the lower half, one would get the following complete equation

$$\alpha(x) = \frac{1 - e^{-|x|/d}}{|x|}. \tag{30}$$

The length scale d which was incorporated as a neat math trick to help us complete an infinite integral turn out to define a length scale proportional with the size of the the cross section for our filament. As $\rho \to 0$ we have κ take on values $\kappa \sim \frac{2\pi}{\rho}$. We see then that the maximum value of κ is defined by a length scale that is the total length of the material in that direction. This ends up being proportional to d in our calculations.

2.4 Inverting the electric field equation

The task is to invert Equation 18. We wish to have the linear charge density $Q(x,t)$ expressed in terms of $E(x,t)$ and $\alpha(x)$ so that we can incorporate our knowledge of how the electric field is being generated by our charges into the continuity equation (Eq. 1). We start from the electric field along the filament, which is given as minus the gradient of the potential

$$E = -\int_{-\infty}^{\infty} dx' \partial_x \alpha(x - x')Q(x',t). \tag{31}$$

We then rewrite both $Q(x,t)$ and $\alpha(x)$ in the form of their Fourier transforms, so that the differential of $\alpha(x)$ can easily be executed

$$E = \frac{-1}{(2\pi)^2}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk \int_{\infty}^{\infty} dk' e^{i(k'-k)x'}e^{ikx}ik\alpha(k)Q(k',t). \tag{32}$$

We use the integral over x' to create a delta function and compute one of the integrals over k or k' to simplify the expression. In this case we choose k'.

$$E = \frac{-1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \delta \left(k - k'\right) e^{ikx} ik\alpha \left(k\right) Q \left(k', t\right) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} dk\alpha \left(k\right) Q \left(k, t\right) e^{ikx} ik.$$
(33)

Next we integrate $E(x', t)$ over the x' dimension while multiplying it with a yet to be determined function $\lambda(x - x')$

$$\int_{-\infty}^{\infty} dx' \lambda \left(x - x'\right) E \left(x', t\right) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx' \lambda \left(x - x'\right) e^{ikx'} ik\alpha \left(k\right) Q \left(k, t\right).$$
(34)

Writing $\lambda(x - x')$ as its Fourier transform,

$$\lambda(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk' \lambda(k') e^{ik'(x - x')}$$
(35)

and inserting it into Eq. 34 we get:

$$\int_{-\infty}^{\infty} dx' \lambda \left(x - x'\right) E \left(x', t\right) = \frac{-1}{(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dx' e^{i(k-k')x'} e^{ik'x} ik\lambda \left(k'\right) \alpha \left(k\right) Q \left(k, t\right).$$
(36)

Using the integral over x' we obtain another delta function and can simplify again to the following:

$$= \frac{-1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \delta \left(k - k'\right) e^{ik'x} ik\lambda \left(k'\right) \alpha \left(k\right) Q \left(k, t\right) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} ik\lambda \left(k\right) \alpha \left(k\right) Q \left(k, t\right).$$
(37)

We now choose $\lambda(x - x')$ such that its Fourier transform looks like the following:

$$\lambda(k) = \frac{-1}{ik\alpha(k)}.$$
(38)

Making our integral yield the following results

$$\int_{-\infty}^{\infty} dx' \lambda \left(x - x'\right) E \left(x', t\right) = \int_{-\infty}^{\infty} dk e^{ikx} Q \left(k, t\right) = Q \left(x, t\right).$$
(39)

This is just the Fourier transform form of the linear charge density $Q(x, t)$. The elegance of this inversion is really that it is applicable for any $\alpha(x - x')$. Even the functions that would make the charge density otherwise unsolvable analytically, one could numerically solve for $\lambda(x)$ and still apply these mathematics to calculate $Q(x, t)$. To sum up our final result, our reversal of the integral leads us to the following expression

$$Q(x, t) = \int_{-\infty}^{\infty} dx' \lambda(x - x') E(x', t).$$
(40)

We would here like to draw attention to a deficit in our analysis that we were not able to locate the origin of. Somewhere along the way a sign error was performed that still eludes us. This sign can be remedied by taking

$$\lambda(k) \to -\lambda(k) \tag{41}$$

We suspect that this sign error is caused by a missed consideration of the charge being negative somewhere but were unable to find the source.

2.5   Deriving the dispersion relation of the plasmon

We insert Eq. 40 for the charge into the continuity equation (Eq. 1) to get the following equation:

$$\frac{\partial I(x,t)}{\partial x} + \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx' \lambda (x - x') E (x',t) = 0. \tag{42}$$

We could already work with an equation of current assuming it to take a plane wave form here, but we choose to look at the form given by using Eq. 4 for the current. Thus yielding us something nonlinear in v_s that still includes the addition of the terms contributed by normal conductive electrons and the critical supercurrent. The analysis is done in v_s instead of I and yields additional interesting results that while not relevant for our combined analysis in Ch. 5 are independently interesting. We proceed by inserting the current as written in Eq. 4 into Eq. 42

$$\frac{\partial}{\partial x} \left(N_s e v_s (x,t) \left(1 - \frac{v_s(x,t)^2}{v_c{}^2} \right) + \sigma_n E (x,t) \right) S + \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx' \lambda (x - x') E (x',t) = 0. \tag{43}$$

All the dependencies on x and t are written out explicitly in this equation. Despite the fact that we are looking for density waves, we keep N_s a constant, presuming it to vary very little with space. We insert Eq. 2 for the force into this to get a closed expression for $v_s (x,t)$

$$\frac{\partial}{\partial x} \left(N_s \frac{e^2}{m} v_s (x,t) \left(1 - \frac{v_s(x,t)^2}{v_c{}^2} \right) + \sigma_n \frac{\partial v_s (x,t)}{\partial t} \right) S - \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx' \lambda (x - x') \frac{\partial v_s (x',t)}{\partial t} = 0. \tag{44}$$

From here on, we can insert particular solutions for $v_s (x,t)$. We attempt to insert the following function:

$$v_s (x,t) = v_0 + w e^{ikx - i\omega t}. \tag{45}$$

Where v_0 is a large macroscopic average speed and w is the amplitude of a small oscillation. Inserting this into our closed expression and looking at the highest orders of w we get the following equation:

$$w S e^{ikx - i\omega t} \left(i k N_s \frac{e^2}{m} \left(1 - \frac{v_0{}^2}{v_m{}^2} \right) - i\omega i k \sigma_n \right) - w \omega^2 e^{-i\omega t} \int_{-\infty}^{\infty} dx' \lambda (x - x') e^{ikx'} = 0. \tag{46}$$

Where we have defined a new constant $v_m = v_c/\sqrt{3}$. The factor of 3 comes from v_s being of third order but only the first order being considered significant. Using that

$$\int_{-\infty}^{\infty} dx' \lambda \left(x - x'\right) e^{ikx'} = -e^{ikx} \int_{\infty}^{-\infty} du \lambda \left(u\right) e^{-iku} = e^{ikx} \int_{-\infty}^{\infty} du \lambda \left(u\right) e^{-iku} = e^{ikx} \lambda \left(k\right).$$
(47)

We can simplify the equation for our wave to the following dispersion relation

$$S \left(ikN_s \frac{e^2}{m} \left(1 - \frac{v_0{}^2}{v_m{}^2}\right) + \omega k \sigma_n \right) - \omega^2 \lambda \left(k\right) = 0.$$
(48)

Defining

$$\eta = 1 - \frac{v_0{}^2}{v_m{}^2}$$
(49)

and using the definition of $\lambda(k)$ we can write the dispersion relation as the following:

$$\omega^2 + k^2 \alpha \left(k\right) \left(i\omega \sigma_n - N_s \frac{e^2}{m} \eta \right) S = 0.$$
(50)

2.6 Analyzing the frequency spectrum of the plasmon

We are now left with a dispersion relation that is related to the value of $\eta(I_0)$, which controls whether our specimen remains superconductive or not, depending on the bulk average velocity of our particles v_0. Despite writing η as a function that can take negative values, we have no intentions of letting it take on negative values. This was simply a case of being imprecise in exchange for the ability to write our equations without cutoffs, fully continuous and differentiable. Once $v_0 = v_m$ we reach I_c, the critical current and we set $\eta = 0$. Considering such a case where $v_0 > v_m$ we get the following dispersion:

$$\omega = -ik^2 \alpha \left(k\right) \sigma_n S.$$
(51)

Since $\alpha(k)$ is always positive this leaves us with a purely imaginary and negative frequency. What this means can be clarified by inserting it into our original expression for v_s.

$$v_s \left(x, t\right) = v_0 + we^{ikx - i\omega t} = v_0 + we^{ikx} e^{\text{Im}(\omega)t}$$
(52)

We now see that $\text{Im}\{(\omega)\} < 0$ corresponds to an exponential collapse of the fluctuations with characteristic timescale comparable to the energy in those fluctuations ω. This leads us to the conclusion that these fluctuations in v_s are something intrinsically connected to superconductive mediums as opposed to normal plasma oscillations in non superconductive material, since they break down when superconductivity breaks down. We now wish to investigate how these oscillations behave for $v_0 < v_m$. Doing this makes the frequency ω complex, so we choose to write it as the following

$$\omega = \bar{\omega} - i\gamma,$$
(53)

splitting the real and the imaginary part into two. Inserting this into our dispersion relation gives us a real part and an imaginary part that both have to be equal to 0

$$\left(\bar{\omega} - i\gamma\right)^2 + k^2 \alpha \left(k\right) \left(i \left(\bar{\omega} - i\gamma\right) \sigma_n - N_s \frac{e^2}{m} \eta \right) S = 0.$$
(54)

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Written out explicitly it looks like the following:

$$\left(\bar{\omega}^2 - \gamma^2 + k^2 \alpha\left(k\right)\gamma\sigma_n S - k^2 \alpha\left(k\right)N_s S\frac{e^2}{m}\eta S\right) + i\left(k^2 \alpha\left(k\right)\bar{\omega}\sigma_n S - 2\gamma\bar{\omega}\right) = 0. \qquad (55)$$

From this we can identify the imaginary part of ω (γ) and the real part of ω ($\bar{\omega}$) by solving the two equation with two unknowns

$$\bar{\omega}^2 = k^2 \alpha\left(k\right)N_s S\frac{e^2}{m}\eta + k^4 \frac{\alpha(k)^2 \sigma_n^2 S^2}{2} \qquad (56)$$

and

$$\gamma = \frac{k^2 \alpha\left(k\right)\sigma_n S}{2}. \qquad (57)$$

Looking closer at $\alpha(k)$, and limiting ourselves to small k to get the long wavelength, large scale effects into play

$$\alpha(k) = \ln\left(1 + \frac{1}{k^2 d^2}\right). \qquad (58)$$

For small values of k we have large arguments of the logarithm, the 1 in the function $\alpha(k)$ becomes comparatively small, we can thus consider

$$\alpha(k) \sim \ln\left(\frac{1}{k^2 d^2}\right) = 2\ln\left(\frac{1}{kd}\right). \qquad (59)$$

Considering that k in any physical system takes on a minimum size of $2\pi/L$ where L is the size of the system in the x direction, we can write $\alpha(k)$ as the following

$$\alpha(k) = 2\ln\left(\frac{L}{d}\frac{1}{2\pi n}\right) \qquad (60)$$

Where n is the multiples of the minimum wave vector. Since d is considered minuscule compared to L, the change in the value of $\alpha(k)$ is small under changing integer n because the logarithm is such a slowly changing function at high values. Because of this we can at small k regard $\alpha(k) = \alpha$ to be a constant. Doing so gives us the following $\bar{\omega}$, ignoring higher orders of k.

$$\bar{\omega} = k\sqrt{\alpha\left(k\right)N_s S\frac{e^2}{m}\eta} = vk \qquad (61)$$

So we see that at small k we get a linear spectrum with decay that is proportional to k^2. Since the frequency given by $\bar{\omega}$ is proportional to k and the decay given by γ is proportional to k^2, one could consider small enough k and the decay would be negligible. The same goes for dirty systems with poor conductivity $\sigma_n \to 0$ or small filaments with smaller surfaces $S \to 0$, since the decay scales with S more powerfully than $\bar{\omega}$.

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2.7 2D Plasmon

The same procedure can be completed in 2D with many of the same broad strokes. Since many of our interests are in the lower number of dimensions understanding how our expression changes from 1D to 2D is of interest to us. There might be phenomena we are missing in 1D but can be explained by expanding very subtlety to a 2D model with a limited scope in the second dimension. Due to this we will in this section outline how to derive the plasmon in 2D in a complimentary fashion as to the 1D approach shown previously in this chapter. In 2D our continuity equation is the following:

$$\nabla \cdot \boldsymbol{j}(\boldsymbol{x}, t) + \frac{\partial \rho(\boldsymbol{x})}{\partial t}. \tag{62}$$

Where we use $\boldsymbol{x} = (x, y)$ and $\boldsymbol{r} = (\boldsymbol{x}, z) = (x, y, z)$. Equivalently we will in this section use $\boldsymbol{k} = (\boldsymbol{\kappa}, k_z) = (k_x, k_y, k_z)$ where κ spans the 2D plane which we are limiting ourselves to. Our connective equation is:

$$\boldsymbol{E}(\boldsymbol{x}, t) = -\frac{m}{n_s} \frac{\partial \boldsymbol{j}(\boldsymbol{x}, t)}{\partial t}. \tag{63}$$

Gauss's law tells us the following:

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t) = 4\pi \rho(\boldsymbol{x}, t)\delta(z). \tag{64}$$

In this approach we follow identically with what we did in 1D up to Eq. 12. Where we get the following result for the Fourier transform of the electric potential:

$$\phi(\boldsymbol{k}) = \frac{4\pi\rho(\boldsymbol{\kappa})}{k^2}. \tag{65}$$

We write the electric potential, but take the z component to be 0 as anything out of the plane is irrelevant

$$\phi(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int dk_z \int d^2\kappa 4\pi\rho(\boldsymbol{\kappa}) \frac{1}{k_z^2 + \kappa^2} e^{i\boldsymbol{\kappa}\cdot\boldsymbol{x}} e^{ik_z z} \tag{66}$$

$$\phi(\boldsymbol{x}) = \phi(\boldsymbol{x}, z = 0) = \frac{1}{(2\pi)^3} \int d^2\kappa 4\pi\rho(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa}\cdot\boldsymbol{x}} \int dk_z \frac{1}{k_z^2 + \kappa^2}. \tag{67}$$

We go on to examine the integral over k_z

$$\int_{-\infty}^{\infty} dk_z \frac{1}{k_z^2 + \kappa^2} = \frac{1}{\kappa} \int_{-\infty}^{\infty} dq \frac{1}{q^2 + 1} = \frac{\pi}{\kappa}. \tag{68}$$

By making the integral unitless using the variable change $q = \frac{k_z}{\kappa}$ we solve it.
Inserting this back into Eq. 67 and writing $\rho(\boldsymbol{\kappa})$ as its Fourier transform we get:

$$\phi(\boldsymbol{x}) = \int d^2x' \rho(\boldsymbol{x}') \frac{1}{(2\pi)^2} \int d^2\kappa \frac{2\pi}{\kappa} e^{i\boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{x}')}. \tag{69}$$

Once again we take the last part of this term to be the function $\alpha_{2D}(\kappa) = \frac{2\pi}{\kappa}$ analogous to the one dimension case and arrive at

$$\phi(\boldsymbol{x}) = \int d^2x' \rho(\boldsymbol{x}')\alpha_{2D}(\boldsymbol{x} - \boldsymbol{x}'). \tag{70}$$

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The derivation now follows exactly as it did for 1D but the function $\lambda(k) \to \boldsymbol{\lambda}_{2D}(\kappa)$ is now a vector as it needs to fulfill the equation

$$\rho(\boldsymbol{x}, t) = \int d^2 x' \boldsymbol{\lambda}_{2D}(\boldsymbol{x} - \boldsymbol{x}') \cdot \boldsymbol{E}(\boldsymbol{x}', t). \tag{71}$$

It takes on the following form, again completely analogous to 1D but now simply directed in the $\boldsymbol{\kappa}$ direction

$$\boldsymbol{\lambda}_{2D}(\boldsymbol{\kappa}) = \frac{-\boldsymbol{\kappa}}{i\kappa^2 \alpha_{2D}(\kappa)}. \tag{72}$$

Inserting this into our continuity equation together with our connective equation we get the following closed expression for the current

$$\nabla \cdot \boldsymbol{j}(\boldsymbol{x}, t) - \frac{m}{n_s} \frac{\partial}{\partial t} \int d^2 x' \boldsymbol{\lambda}_{2D}(\boldsymbol{x} - \boldsymbol{x}') \cdot \frac{\partial \boldsymbol{j}(\boldsymbol{x}, t)}{\partial t} = 0. \tag{73}$$

We insert the following 2D plane wave to derive a dispersion relation, where $\boldsymbol{j}_0$ is a constant vector

$$\boldsymbol{j}(\boldsymbol{x}, t) = \boldsymbol{j}_0 e^{i\boldsymbol{\kappa} \cdot \boldsymbol{x} - i\omega t}. \tag{74}$$

This yields the following dispersion relation

$$\omega = \pm \sqrt{2\pi \frac{n_s}{m}} \sqrt{\kappa} \tag{75}$$

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3 Spin-orbit interactions and coupling of supercurrents to magnetic exchange field

In this section we will be looking at a superconductive system with spin orbit coupling in the vicinity of a ferromagnet that generates a Zeeman term. We will be following the analysis of the article "Currents Induced by Magnetic Impurities in Superconductors with Spin-Orbit Coupling" by Pershoguba et al [2]. Specifically we will be diving deeper into the calculations covered in the supplemental material of the article and the calculations skipped in between lines in this supplemental material. Justify the omission of terms left out of their calculations and hopefully come to a precise formulation of how to perform analogous calculations. The goal is to derive a Ginzburg-Landau free energy term that describes the energy associated with the magnetic exchange field and the spin orbit coupling of the superconductor. We also wish to derive a current expression using this same energy term.

Using this we will couple the area of magnon supporting material and the area of plasmon supporting material, using this superconductive spin-orbit magnetic exchange area as a sort of glue between the two. The hope is that a material that supports superconductivity while being affected by a magnetic exchange field will act a lot like a material that would be the transition between a pure ferromagnetic slowly turning into a full superconductor, this will help us realize how a collective mode of a hybrid material of these two components would need to act in order to be compatible.

In practice the focus is on the free energy and current because we need the current expression to couple the plasmon equation which are based around current. And we need the free energy to couple the magnon equations which are based around the effective local field, derived from the free energy. The article that we draw from is dealing with a large magnetic impurity applied directly to a 2-D superconductor. It is a macroscopic ferromagnet in a hetero structure with the superconductor, not to be confused with impurity states that are based around small single atom impurities.

We will first analyze the Hamiltonian presented in Pershoguba [2], then we will switch the focus towards a more general formulation of the Hamiltonian that has yet to be written in Nambu-formalism and apply minimal coupling, we do this with the aim of acquiring a full Hamiltonian that contains information about the electromagnetic fields interaction with the material and thus the current that flows in it. From this we go into the derivation of the partition function $\mathcal{Z}$, with the goal of using it to derive the free energy F and then the current $\boldsymbol{J}$. This section is long and cumbersome but also necessary. It requires the calculation of several action terms, Matsubara frequency sums, gauge transformations and traces. After this we finish off by finding the additional free energy terms generated by the spin-orbit coupling and magnetic exchange field interactions (F_{extra}) and the current terms those free energy contributions create ($\boldsymbol{J}_{extra}$).

3.1 The Hamiltonian

The Hamiltonian for the system as written in [2] is the following:

$$H = \frac{1}{2} \int d^2\boldsymbol{r} \Psi^\dagger(\boldsymbol{r}) \left[h(\boldsymbol{p})\tau_z + \tau_x\Delta - \boldsymbol{S}(\boldsymbol{r}) \cdot \boldsymbol{\sigma} \right] \Psi(\boldsymbol{r}) \tag{76}$$

$$h(\boldsymbol{p}) = \frac{p^2}{2m} + \lambda(\boldsymbol{\sigma} \times \boldsymbol{p})_z - \mu, \quad \boldsymbol{p} = -i\left(\nabla_x, \nabla_y\right). \tag{77}$$

Where if we consider $\boldsymbol{S}(\boldsymbol{r}) = 0$ the system is translationally invariant in the x and y direction and consists of a bulk material ranging infinitely in the x and y directions, but ends at $z = 0$. We consider a system consisting of a bulk superconductor with small magnetic impurity described by $\boldsymbol{S}(\boldsymbol{r})$ at $z = 0$ placed on top of the material, where there would otherwise be vacuum. By only looking at the system at this top surface layer and not further outside or inside the material we can consider the superconducting gap Δ to be finite both inside and outside the superconducting material. We thus consider a kind of melding of magnetic and superconductive properties in this layer, where we consider the superconducting terms of the Hamiltonian to exist even inside the magnetic impurity.

$\boldsymbol{\sigma}$ is a vector of Pauli-matrices representing the spin. τ_a is also a set of Pauli-matrices, but in this case they describe the duality between the holes and the particles that exist as this Hamiltonian has already been through a Hubert-Stratonovich transformation and exhibits a Nambu-space structure.

The interaction term in the Hamiltonian should actually take the form

$$\tau_x \rightarrow \tau_+ \bar{\Delta} + \tau_- \Delta. \tag{78}$$

Where $\bar{\Delta}$ is the complex conjugate field of Δ and $\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In the article by Pershoguba they have chosen Δ to be real and constant to make the calculations easier. We will keep it as a variable field for a time more, letting it vary across space, but impose restrictions when necessary.

$\Psi(\boldsymbol{r}) = \left(\psi_\uparrow, \psi_\downarrow, \psi_\downarrow^\dagger, -\psi_\uparrow^\dagger \right)^\mathrm{T}$ is a vector of the fermionic operators in second quantization. They are the fermionic creation and annihilation operators for the electrons of spin up and down. Looking at the Hamiltonian we can clearly see the hole particle duality when looking at the single particle energy $h(\boldsymbol{p})$. As τ_z takes on the value 1 or -1, we see that particles and those particles corresponding holes have opposite energies as expected.

3.2 Minimal coupling with the electromagnetic field

We wish to minimally couple the Hamiltonian of our system with the electromagnetic field by taking

$$-i\nabla \rightarrow -i\nabla - e\boldsymbol{A}. \tag{79}$$

We however cannot use Eq. 76 to perform this minimal coupling as certain manipulations have been made upon the base Hamiltonian of the actual system before this equation was reached. To accurately achieve the correct minimal coupling we need to apply it at the very first step. We therefore start with the Hamiltonian of the uncoupled single particle system with $\hbar = 1$ for ease of reading

$$H_s = \sum_{\boldsymbol{k}} \psi^\dagger(\boldsymbol{k}) \left[\frac{k^2}{2m} - \mu + \lambda (\boldsymbol{\sigma} \times \boldsymbol{k})_z \right] \psi(\boldsymbol{k}). \tag{80}$$

This equation uses the right annihilation operator and the left creation operator to preserve states that have particles of momentum $\boldsymbol{k}$ and output their energy. These are vectors composed of both the up and down spin momentum annihilation fermionic operators

$$\psi^\dagger(\boldsymbol{k}) = \begin{bmatrix} \psi_\uparrow^\dagger(\boldsymbol{k}) & \psi_\downarrow^\dagger(\boldsymbol{k}) \end{bmatrix}, \quad \psi(\boldsymbol{k}) = \begin{bmatrix} \psi_\uparrow(\boldsymbol{k}) \\ \psi_\downarrow(\boldsymbol{k}) \end{bmatrix}. \tag{81}$$

We now do minimal coupling in momentum space and consider $\boldsymbol{A}$ to be constant in space

$$H_s = \sum_{\boldsymbol{k}} \psi^\dagger(\boldsymbol{k}) \left[\frac{k^2}{2m} - \mu + \lambda(\boldsymbol{\sigma} \times \boldsymbol{k})_z \right] \psi(\boldsymbol{k}) + \psi^\dagger(\boldsymbol{k}) \left[-\frac{\{\boldsymbol{k}, e\boldsymbol{A}\}}{2m} - \lambda e(\boldsymbol{\sigma} \times \boldsymbol{A})_z + \frac{e^2 \boldsymbol{A}^2}{2m} \right] \psi(\boldsymbol{k}).$$
(82)

Setting $e = 1$ and rewriting this in terms of a 4x4 matrix and 1x4 Nambu-spinors we get

$$H_s = \sum_{\boldsymbol{k}} \frac{1}{2} \begin{bmatrix} \psi^\dagger_\uparrow(\boldsymbol{k}) \\ \psi^\dagger_\downarrow(\boldsymbol{k}) \\ \psi_\downarrow(-\boldsymbol{k}) \\ -\psi_\uparrow(-\boldsymbol{k}) \end{bmatrix} \begin{pmatrix} h(\boldsymbol{k}) + C(\boldsymbol{A}) + \gamma(\boldsymbol{k}, \boldsymbol{A}) & 0 \\ 0 & -h(\boldsymbol{k}) - C(\boldsymbol{A}) + \gamma(\boldsymbol{k}, \boldsymbol{A}) \end{pmatrix} \begin{bmatrix} \psi_\uparrow(\boldsymbol{k}) \\ \psi_\downarrow(\boldsymbol{k}) \\ \psi^\dagger_\downarrow(-\boldsymbol{k}) \\ -\psi^\dagger_\uparrow(-\boldsymbol{k}) \end{bmatrix}$$
$$+ \left(\frac{k^2}{2m} - \mu - \frac{\{\boldsymbol{k}, \boldsymbol{A}\}}{2m} + \frac{\boldsymbol{A}^2}{2m} \right)$$
(83)

where

$$h(\boldsymbol{k}) = \frac{k^2}{2m} - \mu + \lambda(\boldsymbol{\sigma} \times \boldsymbol{k})_z,$$
(84)

$$\gamma(\boldsymbol{k}, \boldsymbol{A}) = - \left(\frac{\{\boldsymbol{k}, \boldsymbol{A}\}}{2m} + \lambda(\boldsymbol{\sigma} \times \boldsymbol{A})_z \right),$$
(85)

$$C(\boldsymbol{A}) = \frac{\boldsymbol{A}^2}{2m}.$$
(86)

This can be shown by explicit calculation of Eq. 83. The preferred approach to verify this is by multiplying everything in Eq. 83 out and flipping the sign of the momentum for the terms generated by the lower half of the matrix ($\boldsymbol{k} \to -\boldsymbol{k}$) and exchanging position of the fermionic operators. Doing so one can find that the expression for H_s in Eq. 82 and Eq. 83 are equal. The last term comes from the commutation of the annihilation operators and can be simplified, if we consider what we are actually interested in. We wish to derive the current in the material from these equations, which means we only care about the derivative of H_s with regards to $\boldsymbol{A}$. This leaves us with a term of the form

$$\boldsymbol{J}_0 = \frac{-}{2m} \frac{\delta}{\delta \boldsymbol{A}} \left(\boldsymbol{k} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \boldsymbol{k} - \boldsymbol{A}^2 \right).$$
(87)

Which summed over all values of $\boldsymbol{p}$, gives us nothing from the first two terms. From the last term we get the following current

$$\boldsymbol{J}_0 = \frac{e^2}{m} \boldsymbol{A}.$$
(88)

This will become part of the term for our normal superconductive current, unrelated to the spin-orbit coupling.

Defining the 4 by 1 vectors of fermionic operators as $\Psi^\dagger(\boldsymbol{k})$ and $\Psi(\boldsymbol{k})$

$$\Psi^\dagger(\boldsymbol{k}) = \begin{bmatrix} \psi_\uparrow^\dagger(\boldsymbol{k}) & \psi_\downarrow^\dagger(\boldsymbol{k}) & \psi_\downarrow(-\boldsymbol{k}) & -\psi_\uparrow(-\boldsymbol{k}) \end{bmatrix}, \qquad \Psi(\boldsymbol{k}) = \begin{bmatrix} \psi_\uparrow(\boldsymbol{k}) \\ \psi_\downarrow(\boldsymbol{k}) \\ \psi_\downarrow^\dagger(-\boldsymbol{k}) \\ -\psi_\uparrow^\dagger(-\boldsymbol{k}) \end{bmatrix}. \tag{89}$$

We can write H_s as the following compact expression

$$H_s = \frac{1}{2} \sum_{\boldsymbol{k}} \Psi^\dagger(\boldsymbol{k}) \begin{pmatrix} h(\boldsymbol{k}) + C(\boldsymbol{A}) + \gamma(\boldsymbol{k}, \boldsymbol{A}) & 0 \\ 0 & -h(\boldsymbol{k}) - C(\boldsymbol{A}) + \gamma(\boldsymbol{k}, \boldsymbol{A}) \end{pmatrix} \Psi(\boldsymbol{k}). \tag{90}$$

We can now inverse Fourier transform this back to space to insert it into the Hamiltonian that also includes the interaction between particles and holes and the interaction with the ferromagnet. Inserting the Fourier transform of $\Psi(\boldsymbol{k})$ and its complex conjugate $\Psi^\dagger(\boldsymbol{k})$ we get the following expression

$$H_s = \frac{1}{2} \sum_{k} \int d\boldsymbol{x}' d\boldsymbol{x} \Psi^\dagger(\boldsymbol{x}) e^{i\boldsymbol{x}\cdot\boldsymbol{k}} \left(h(\boldsymbol{k})\tau_z + C(\boldsymbol{A})\tau_z + \gamma(\boldsymbol{k}, e\boldsymbol{A}))\tau_I \right) \Psi(\boldsymbol{x}') e^{-i\boldsymbol{x}'\cdot\boldsymbol{k}}. \tag{91}$$

Where τ_I is the identity matrix in the hole-particle space. Performing the sum over $\boldsymbol{k}$ we are granted a delta function which when integrated over x' yields us the following equation in realspace for the single particle energy:

$$H_s = \frac{1}{2} \int d\boldsymbol{x} \Psi^\dagger(\boldsymbol{x}) \left[h(i\nabla_x)\tau_z + C(\boldsymbol{A})\tau_z + \gamma(i\nabla_x, e\boldsymbol{A})\tau_I \right] \Psi(\boldsymbol{x}). \tag{92}$$

We are now able to plug this single particle energy back into the Hamiltonian. If we had naively done minimal coupling on the modified Hamiltonian from Eq. 76 we would have ended up with τ_z for both the γ term and the normal single particle energy.

3.3 Use of the Feynmann path integral formalism for the partition function $\mathcal{Z}$

We will attempt to calculate the free energy by using the Feynmann path integral formalism for the partition function and then integrating out the fermions. After making suitable approximations, we can arrive at an action which will serve as our free energy expression using

$$F = -\frac{1}{\beta} \ln \mathcal{Z}. \tag{93}$$

We will then calculate the current by taking the functional derivative with regards to the vector potential and evaluating the result at $\boldsymbol{A} = 0$, which means no magnetic fields present

$$\boldsymbol{J} = \frac{\delta F}{\delta \boldsymbol{A}} \bigg|_{\boldsymbol{A}=0}. \tag{94}$$

We know this is technically wrong, as the presence of a current would create a magnetic field, but we choose to ignore this and consider the generated current to be small enough that we can safely ignore it.

We start by writing the partition function $\mathcal{Z}$ [5]

$$\mathcal{Z} = \int \mathcal{D}\left(\Psi, \Psi^\dagger, \Delta, \bar{\Delta}\right) \exp\left\{-S\left[\Psi, \Psi^\dagger, \Delta, \bar{\Delta}\right]\right\}. \tag{95}$$

Where the action S is defined as the following

$$S\left[\Psi, \Psi^\dagger, \Delta, \bar{\Delta}\right] = \frac{1}{2}\int_0^\beta d\tau \int d\boldsymbol{r}\Psi^\dagger(\boldsymbol{r})\left[\partial_\tau + i\phi + h(-\boldsymbol{p})\tau_z + C(\boldsymbol{A})\tau_z + \gamma(-\boldsymbol{p}, \boldsymbol{A}) + \tau_+\bar{\Delta} + \tau_-\Delta - \boldsymbol{S}(\boldsymbol{r})\cdot\boldsymbol{\sigma}\right]\Psi(\boldsymbol{r}). \tag{96}$$

Where τ without the index is imaginary time and ϕ is the electric field introduced through minimal coupling. It is important to keep in mind that in this equation $\boldsymbol{p}$ is still given by $\boldsymbol{p} = -i\left(\nabla_x, \nabla_y\right)$ and is thus completely $\boldsymbol{r} = (x, y)$ dependant. As it stands this nomenclature is needlessly confusing, as it obfuscates what is momentum dependant and what is space dependant, nonetheless it serves to make the expression more readable and is illustrative as we will eventually Fourier transform in all spacial dimensions that have translational invariance.

Using our freedom of U(1) transformation

$$\psi_j \to e^{i\eta}\psi_j, \quad \psi_j^\dagger \to e^{-i\eta}\psi_j^\dagger, \quad \phi \to \phi - \partial_\tau\eta, \quad \boldsymbol{A} \to \boldsymbol{A} + \nabla\eta \tag{97}$$

where η is an arbitrary space and time dependant phase configuration of the superconductive wave function.

By choosing the phase η to be the following:

$$\eta(\boldsymbol{r}, \tau) = \theta(\boldsymbol{r}, \tau)/2. \tag{98}$$

Where $\theta(\boldsymbol{r}, \tau)$ is the phase of the gap $\bar{\Delta} = |\Delta|e^{i\theta}$, $\Delta = |\Delta|e^{-i\theta}$. We can eliminate imaginary phase elements from the gap terms and make the coupling between holes and particles completely real. While this phase transformation doesn't include any physics it does simplify our expression and calculations going forward and let us replace the vector potential with the gauge invariant vector potential. Doing so we get the full action of the following form

$$S\left[\Psi, \Psi^\dagger, \Delta, \theta\right] = \frac{1}{2}\int_0^\beta d\tau \int d\boldsymbol{r}\Psi^\dagger(\boldsymbol{r})\left[\partial_\tau + i\phi + h(i\nabla)\tau_z + C(\boldsymbol{A})\tau_z + \gamma(i\nabla, \boldsymbol{A}) + \Delta\tau_x - \boldsymbol{S}(\boldsymbol{r})\cdot\boldsymbol{\sigma}\right]\Psi(\boldsymbol{r}). \tag{99}$$

We wish to drag from this expression the inverse Greens function of the non interacting holes and particles respectively, but to do so we must first analyze the Fourier transform of the γ term.

Defining the gauge invariant vector potential $\boldsymbol{\mathcal{A}} = \boldsymbol{A} + \nabla\theta/2$, we can write the term that includes γ as the following:

$$f_1 = -\int_0^\beta d\tau \int d\boldsymbol{r}\Psi^\dagger(\boldsymbol{r})\left[\frac{(i\nabla)(\boldsymbol{\mathcal{A}})}{2m} + \frac{(\boldsymbol{\mathcal{A}})(i\nabla)}{2m} + \lambda(\boldsymbol{\sigma}\times\boldsymbol{\mathcal{A}})_z\right]\Psi(\boldsymbol{r}). \tag{100}$$

Treating $\Psi^\dagger$ and Ψ as independent variables we can draw a general conclusion for all the terms involving the single particle energy, whether they be holes or particles. Going forth however we will need to use abbreviated nomenclature to fit the equations on the page

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$$\int_0^\beta d\tau \int d\boldsymbol{r} \sum_{\omega_p} \int d\boldsymbol{p} \sum_{\omega_q} \int d\boldsymbol{q} \sum_{\omega_\kappa} \int d\boldsymbol{\kappa} \rightarrow \int_{r,p,q,\kappa} . \tag{101}$$

We will also combine the time and space coordinates into one 3-vector (since we only have 2 space dimensions), and reuse the label of the space dimension 2-vector to encompass it

$$\boldsymbol{p} \cdot \boldsymbol{r} - \omega_p \tau \rightarrow \boldsymbol{p} \cdot \boldsymbol{r}. \tag{102}$$

Doing this we can now work out the first part of f_1, now dubbed $f_{1,1}$ using the Fourier transform of everything

$$f_{1,1} = -\int_{r,p,q,\kappa} \Psi^\dagger(\boldsymbol{p},\omega_p) e^{-i\boldsymbol{p}\cdot\boldsymbol{r}} \frac{(i\nabla)(\boldsymbol{A}(\boldsymbol{\kappa},\omega_\kappa)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}}) + (\boldsymbol{A}(\boldsymbol{\kappa},\omega_\kappa)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}})(i\nabla)}{2m} \Psi(\boldsymbol{q},\omega_q) e^{i\boldsymbol{q}\cdot\boldsymbol{r}}. \tag{103}$$

The only $\boldsymbol{r}$ dependence is in the exponentials which trivializes the evaluation of the nabla operators giving us

$$f_{1,1} = -\int_{r,p,q,\kappa} \Psi^\dagger(\boldsymbol{p},\omega_p) e^{-i\boldsymbol{p}\cdot\boldsymbol{r}} \frac{(-\boldsymbol{q}-\boldsymbol{\kappa})\cdot(\boldsymbol{A}(\boldsymbol{\kappa},\omega_\kappa)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}}) + (\boldsymbol{A}(\boldsymbol{\kappa},\omega_\kappa)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}})\cdot(-\boldsymbol{q})}{2m} \Psi(\boldsymbol{q},\omega_q) e^{i\boldsymbol{q}\cdot\boldsymbol{r}}. \tag{104}$$

Using the delta function provided to us by the complex exponential functions and the integral of $\boldsymbol{r}$ and τ, we can simplify this to the following more compact form

$$\begin{aligned} f_{1,1} &= -\int_{p,q,\kappa} \Psi^\dagger(\boldsymbol{p},\omega_p) \frac{(-2\boldsymbol{q}-\boldsymbol{\kappa})\cdot(\boldsymbol{A}(\boldsymbol{\kappa},\omega_\kappa))}{2m} \Psi(\boldsymbol{q},\omega_q)\delta(\boldsymbol{\kappa}+\boldsymbol{q}-\boldsymbol{p}) \\ &= \int_{p,q} \Psi^\dagger(\boldsymbol{p}) \frac{(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{A}(\boldsymbol{p}-\boldsymbol{q})}{2m} \Psi(\boldsymbol{q}). \end{aligned} \tag{105}$$

Where in the last line we drop the frequency index and compress it into the momentum variable. The second part of f_1, $f_{1,2}$ is easily calculated using the same method.

$$f_{1,2} = -\int_{p,q} \Psi^\dagger(\boldsymbol{p})\lambda(\boldsymbol{\sigma}\times\boldsymbol{A}(\boldsymbol{p}-\boldsymbol{q}))_z\Psi(\boldsymbol{q}) \tag{106}$$

This gives us a combined expression for f_1

$$f_1 = f_{1,1} + f_{1,2} = \int_{p,q} \Psi^\dagger(\boldsymbol{p}) \left[\frac{(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{A}(\boldsymbol{p}-\boldsymbol{q})}{2m} - \lambda(\boldsymbol{\sigma}\times\boldsymbol{A}(\boldsymbol{p}-\boldsymbol{q}))_z\right] \Psi(\boldsymbol{q}). \tag{107}$$

We now return to Eq. 99, now knowing what f_1 is. Considering that our purpose is to generate a free energy suitable for calculating the current in a system with no magnetic or electric fields, we see that the term containing ϕ (the electric potential) can be safely discarded due to the restriction on electric fields.

We thus have an action of the form

$$S = \frac{1}{2} \int_0^\beta d\tau \int d\boldsymbol{r}\, \Psi^\dagger(\boldsymbol{r}) \left[\partial_\tau + h(i\nabla)\tau_z + C(\boldsymbol{A})\tau_z + \gamma(i\nabla, \boldsymbol{A}) + \Delta\tau_x - \boldsymbol{S}(\boldsymbol{r}) \cdot \boldsymbol{\sigma} \right] \Psi(\boldsymbol{r}). \tag{108}$$

We now write the above action using the Fourier transform of above terms. We trivially get Dirac delta functions from most terms as they contain no differential operators. We have already evaluated the Fourier transform of the only term that proves to be complicated yielding us the function f_1. The action is thus:

$$\begin{aligned}
S = \frac{1}{2} \int_{p,q} \Psi^\dagger(\boldsymbol{p}) \Big[&- i\omega_q \delta(\boldsymbol{q} - \boldsymbol{p}) + h(-\boldsymbol{q})\delta(\boldsymbol{q} - \boldsymbol{p})\tau_z + \Delta(\boldsymbol{p} - \boldsymbol{q})\tau_x \\
&+ \frac{(\boldsymbol{p} + \boldsymbol{q}) \cdot \boldsymbol{A}(\boldsymbol{p} - \boldsymbol{q})}{2m} - \lambda(\boldsymbol{\sigma} \times \boldsymbol{A}(\boldsymbol{p} - \boldsymbol{q}))_z \\
&- \boldsymbol{S}(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{\sigma} \\
&+ \int_k \frac{\boldsymbol{A}(\boldsymbol{k}) \cdot \boldsymbol{A}(\boldsymbol{p} - \boldsymbol{q} - \boldsymbol{k})}{2m}\tau_z \Big] \Psi(\boldsymbol{q}).
\end{aligned} \tag{109}$$

We now define the above four lines as four separate functions. One includes all the single particle and superconductive action of the system. One includes all the electromagnetic dependence of the action given by $\boldsymbol{A}$ to the first order. One includes all the ferromagnetic effects given by the dependence on $\boldsymbol{S}$ and finally one includes all the second order $\boldsymbol{A}$ dependence:

$$\begin{aligned}
G_0^{-1}(\boldsymbol{q}, \boldsymbol{p}) &= -i\omega_q \delta(\boldsymbol{q} - \boldsymbol{p}) + h(-\boldsymbol{q})\delta(\boldsymbol{q} - \boldsymbol{p})\tau_z + \Delta(\boldsymbol{p} - \boldsymbol{q})\tau_x, \\
\gamma(\boldsymbol{q}, \boldsymbol{p}) &= \frac{(\boldsymbol{p} + \boldsymbol{q}) \cdot \boldsymbol{A}(\boldsymbol{p} - \boldsymbol{q})}{2m} - \lambda(\boldsymbol{\sigma} \times \boldsymbol{A}(\boldsymbol{p} - \boldsymbol{q}))_z, \\
S_\sigma(\boldsymbol{q}, \boldsymbol{p}) &= -\boldsymbol{S}(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{\sigma}, \\
C(\boldsymbol{q}, \boldsymbol{p}) &= \int_k \frac{\boldsymbol{A}(\boldsymbol{k}) \cdot \boldsymbol{A}(\boldsymbol{p} - \boldsymbol{q} - \boldsymbol{k})}{2m}\tau_z.
\end{aligned} \tag{110}$$

This yields us a more clean looking action

$$S = \frac{1}{2} \int_{p,q} \Psi^\dagger(\boldsymbol{p}) \left[G_0^{-1}(\boldsymbol{q}, \boldsymbol{p}) + \gamma(\boldsymbol{q}, \boldsymbol{p}) + S_\sigma(\boldsymbol{q}, \boldsymbol{p}) + C(\boldsymbol{q}, \boldsymbol{p}) \right] \Psi(\boldsymbol{q}). \tag{111}$$

We are now ready to treat Ψ as a vector, where $\boldsymbol{q}$ and $\boldsymbol{p}$ are the generalization of the indexes for the vector taken to infinitely large space and integrate out Ψ from our original expression of the partition function given in equation 95

$$\mathcal{Z} = \int \mathcal{D}(\theta, \Delta) \det\left(G_0^{-1}(\boldsymbol{q}, \boldsymbol{p}) + \gamma(\boldsymbol{q}, \boldsymbol{p}) + S_\sigma(\boldsymbol{q}, \boldsymbol{p}) + C(\boldsymbol{q}, \boldsymbol{p}) \right)^{-1}. \tag{112}$$

Taking the exponential and the logarithm of this expression we can get it back on the form of an action, now just over the fields θ and Δ as Ψ has been removed through integration

$$\mathcal{Z} = \int \mathcal{D}(\theta, \Delta) \exp\left\{ -\ln\left(\det\left(G_0^{-1}(\boldsymbol{q}, \boldsymbol{p}) + \gamma(\boldsymbol{q}, \boldsymbol{p}) + S_\sigma(\boldsymbol{q}, \boldsymbol{p}) + C(\boldsymbol{q}, \boldsymbol{p}) \right) \right) \right\}. \tag{113}$$

We now use that $\ln \det = \operatorname{Tr} \ln$ and we have the following action for the θ and Δ field

$$\mathcal{S}[\theta, \Delta] = \frac{1}{2} \operatorname{Tr} \ln \big(G_0^{-1}(\boldsymbol{q}, \boldsymbol{p}) + \gamma(\boldsymbol{q}, \boldsymbol{p}) + S_\sigma(\boldsymbol{q}, \boldsymbol{p}) + C(\boldsymbol{q}, \boldsymbol{p})\big). \tag{114}$$

Using the logarithm we split up the action and discard the component that has no $\boldsymbol{A}$ dependence

$$\mathcal{S} = \operatorname{Tr} \ln \big(G_0^{-1}(1 + G_0(\gamma + S_\sigma + C))\big) = -\operatorname{Tr} \ln(G_0) + \operatorname{Tr} \ln(1 + G_0(\gamma + S_\sigma + C)) \\ \to \operatorname{Tr} \ln(1 + G_0(\gamma + S_\sigma + C)). \tag{115}$$

We now expand the logarithm to second order getting the following terms

$$\mathcal{S} = \operatorname{Tr}(G_0(\gamma + S_\sigma + C)) \\ -\frac{1}{2} \operatorname{Tr}(G_0 \gamma G_0 \gamma) - \frac{1}{2} \operatorname{Tr}(G_0 S_\sigma G_0 S_\sigma) - \frac{1}{2} \operatorname{Tr}(G_0 C G_0 C) - \operatorname{Tr}(G_0 \gamma G_0 S_\sigma) - \operatorname{Tr}(G_0 \gamma G_0 C) - \operatorname{Tr}(G_0 C G_0 S_\sigma). \tag{116}$$

We see that the first trace term is the adjustment made by the electromagnetic field and ferromagnet directly interfering with the unencumbered particles given by G_0. The second order term without γ or C can be ignored for the purposes of identifying terms that contribute to the current.

We now name all of the above terms and later we will identify which are necessary to calculate the added current that exists due to the spin-orbit coupling and magnetic exchange field.

$$\mathcal{S}_{1.1} = \operatorname{Tr}(G_0 \gamma) \tag{117}$$

$$\mathcal{S}_{1.2} = \operatorname{Tr}(G_0 S_\sigma) \tag{118}$$

$$\mathcal{S}_{1.3} = \operatorname{Tr}(G_0 C) \tag{119}$$

$$\mathcal{S}_{2.1} = -\operatorname{Tr}(G_0 \gamma G_0 S_\sigma) \tag{120}$$

$$\mathcal{S}_{2.2} = -\operatorname{Tr}(G_0 \gamma G_0 C) \tag{121}$$

$$\mathcal{S}_{2.3} = -\operatorname{Tr}(G_0 C G_0 S_\sigma) \tag{122}$$

$$\mathcal{S}_{2.4} = -\frac{1}{2} \operatorname{Tr}(G_0 \gamma G_0 \gamma) \tag{123}$$

$$\mathcal{S}_{2.5} = -\frac{1}{2} \operatorname{Tr}(G_0 S_\sigma G_0 S_\sigma) \tag{124}$$

$$\mathcal{S}_{2.6} = -\frac{1}{2} \operatorname{Tr}(G_0 C G_0 C) \tag{125}$$

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When evaluating the trace over our expression we remember that our frequency and momentum integrals can simply be treated as sums over a vector space in which case we treat functions like $G_0(\boldsymbol{q}, \boldsymbol{p})$ as matrices being multiplied together as shown below for $\mathcal{S}_{2.1}$

$$\mathcal{S}_{2.1} = -\int_{q,p,l,m,n} \mathrm{Tr}_\sigma \, \mathrm{Tr}_\tau \, G_0(\boldsymbol{q}, \boldsymbol{p}) \gamma(\boldsymbol{p}, \boldsymbol{l}) G_0(\boldsymbol{l}, \boldsymbol{m}) S_\sigma(\boldsymbol{m}, \boldsymbol{n}) \delta(\boldsymbol{q} - \boldsymbol{n}). \tag{126}$$

The last delta function is equivalent to taking the last and first index of a set of multiplied matrices and contracting them ($\mathrm{Tr}(ABCD) = A_{i,j} B_{j,k} C_{k,l} D_{l,i}$). In addition to the trace over our momentum variables, we also have the trace over the spin space given by σ_j and our particle hole Nambu-space given by τ_i.

3.4   Finding the Greens function

In the interest of actually calculating the current associated with the spin orbit coupling terms we now choose to restrict ourselves to a constant gap parameter

$$\Delta(x) = \Delta. \tag{127}$$

Considering the gap constant in space gives us the following Fourier transform of the gap energy

$$\Delta(q) = \Delta \delta(q). \tag{128}$$

This simplifies our propagator to the following

$$G_0^{-1}(\boldsymbol{q}, \boldsymbol{p}) = (-i\omega_q + h(-\boldsymbol{q})\tau_z + \Delta\tau_x)\delta(\boldsymbol{q} - \boldsymbol{p}). \tag{129}$$

We see that the inverse Greens function is now only dependant on one momentum variable and we can define that new function in front of the delta function as $\underline{G}_0^{-1}(\boldsymbol{q})$

$$G_0^{-1}(\boldsymbol{q}, \boldsymbol{p}) = \underline{G}_0^{-1}(\boldsymbol{q})\delta(\boldsymbol{q} - \boldsymbol{p}). \tag{130}$$

We now turn to the problem of identifying $G_0(\boldsymbol{q}, \boldsymbol{p})$. This is the inverse of a term that includes a delta function. Since the delta function is a distribution and not just another function we have to be a bit careful about this inverse. It is also not just an inverse of the function or matrices that comprise the spin and Nambu-space but also an inverse in terms of the space created by the momentum variable. Due to these complications we need to return to the root of what we have implicitly assumed about this function to find its form

$$\mathrm{Tr}\big(G_0^{-1}G_0 f\big) = \mathrm{Tr}(f). \tag{131}$$

The above is the property we assumed when we so casually included G_0 into our expressions. We can now break both the RH and LH side down to see what this necessitates for G_0. Starting with the RHS:

$$\mathrm{Tr}(f) = \int_{l,n} f(\boldsymbol{l}, \boldsymbol{n})\delta(\boldsymbol{l} - \boldsymbol{n}) = \int_n f(\boldsymbol{n}, \boldsymbol{n}). \tag{132}$$

And then the LHS:

$$\text{Tr}\left(G_0^{-1}G_0 f\right) = \int_{n,p,l} G_0^{-1}(\boldsymbol{n}, \boldsymbol{p}) G_0(\boldsymbol{p}, \boldsymbol{l}) f(\boldsymbol{l}, \boldsymbol{n}) = \int_{n,p,l} \underline{G}_0^{-1}(\boldsymbol{n}) \delta(\boldsymbol{n} - \boldsymbol{p}) G_0(\boldsymbol{p}, \boldsymbol{l}) f(\boldsymbol{l}, \boldsymbol{n}).$$

(133)

From this we gather that

$$\underline{G}_0^{-1}(\boldsymbol{n}) G_0(\boldsymbol{n}, \boldsymbol{l}) = \delta(\boldsymbol{l} - \boldsymbol{n}).$$

(134)

Since $\underline{G}_0^{-1}$ lacks any special properties of being a distribution, we can easily take the inverse of it in regards to the Nambu-space. By doing so we get the following:

$$G_0(\boldsymbol{n}, \boldsymbol{l}) = \delta(\boldsymbol{l} - \boldsymbol{n}) \frac{-i\omega_n - [\xi(\boldsymbol{n}) - \lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z]\tau_z - \Delta\tau_x}{(i\omega_n)^2 - [\xi(\boldsymbol{n}) - \lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z]^2 - \Delta^2}.$$

(135)

We have here defined the following as $\xi(\boldsymbol{p})$

$$\xi(\boldsymbol{p}) = \frac{p^2}{2m} - \mu.$$

(136)

In the same vein as with $\underline{G}_0^{-1}$ we now define a $\underline{G}_0$

$$\underline{G}_0(\boldsymbol{n}) = \frac{-i\omega_n - [\xi(\boldsymbol{n}) - \lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z]\tau_z - \Delta\tau_x}{(i\omega_n)^2 - [\xi(\boldsymbol{n}) - \lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z]^2 - \Delta^2} -$$

(137)

We simplify it further by defining two new functions:

$$\xi_\pm(\boldsymbol{q}) = \xi(\boldsymbol{q}) \pm \lambda(\boldsymbol{\sigma} \times \boldsymbol{q})_z,$$

(138)

$$E_\pm(\boldsymbol{q}) = \sqrt{\xi_\pm(\boldsymbol{q})^2 + \Delta^2}$$

(139)

.

This allows us to write $\underline{G}_0(\boldsymbol{q})$ as the following

$$\underline{G}_0(\boldsymbol{q}) = \frac{-i\omega_q - \xi_-(\boldsymbol{q})\tau_z - \Delta\tau_x}{(i\omega_q)^2 - E_-(\boldsymbol{q})^2}$$

(140)

With a well defined G_0 in hand we can return to our action $\mathcal{S}$ and calculate most of the terms without any further problems.

3.5 Evaluating $\mathcal{S}$

We start by looking at our terms from equations 117 to 125. We are not going to calculate the term that normally appears when dealing with a pure superconductor with no spin orbit coupling or magnetic exchange field as we know it to take the form

$$\boldsymbol{j_s} = \frac{n_s}{2m}\boldsymbol{\mathcal{A}}.$$

(141)

Instead we are going to focus on the lowest order term that incorporates both the spin orbit coupling term γ and the magnetic exchange term S_σ.

We immediately see that the two terms $\mathcal{S}_{1.2}$ and $\mathcal{S}_{2.5}$ have no contribution to the current as they have no factors of $\boldsymbol{\mathcal{A}}$. Terms such as $\mathcal{S}_{1.3}$ and $\mathcal{S}_{2.6}$ include only C and are therefore terms relating to normal superconductors, we refrain from calculating this addition.

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Since we are going to take a functional derivative with regards to $\boldsymbol{A}$ and then evaluate the current for $\boldsymbol{A} = 0$, every order of $\boldsymbol{A}$ will give us current terms. The difference will be the degree of $\nabla\theta$ that they are proportional to. First order $\boldsymbol{A}$ terms are proportional to $(\nabla\theta)^0$, second order $\boldsymbol{A}$ terms are proportional to $(\nabla\theta)^1$ and third order $\boldsymbol{A}$ terms are proportional to $(\nabla\theta)^2$ and so on. Since $\nabla\theta$ is taken to be rather small, we will disregard higher orders of $\boldsymbol{A}$ on this pretense.

Due to this we ignore $\mathcal{S}_{2.2}$ which is of third order, $\mathcal{S}_{2.3}$ which is of second order and has no spin orbit coupling and $\mathcal{S}_{2.4}$ which is of second order and has no magnetic exchange coupling.

The terms $\mathcal{S}_{1.1}$ and $\mathcal{S}_{2.1}$ are first order in $\boldsymbol{A}$ and are therefore of great interest as they will be proportional to the current directly. The term $\mathcal{S}_{2.1}$ is of particularly high interest as it is the lowest order term with both spin orbit coupling and magnetic exchange effects. If this term happens to be 0 we must go to higher orders, fortunately we will spoil that it happens to be finite.

We start off by calculating the spin orbit correction to the action given by $\mathcal{S}_{1.1}$

$$\mathcal{S}_{1.1} = \frac{1}{2}\operatorname{Tr}_{\sigma,\tau}\int_{q,p}\underline{G_0}(\boldsymbol{q})\delta(\boldsymbol{q}-\boldsymbol{p})\gamma(\boldsymbol{p},\boldsymbol{q}) = \operatorname{Tr}_{\sigma,\tau}\int_p \underline{G_0}(\boldsymbol{p})\gamma(\boldsymbol{p},\boldsymbol{p}). \tag{142}$$

Inserting $\gamma(\boldsymbol{p},\boldsymbol{p})$ we get the following:

$$\mathcal{S}_{1.1} = \frac{1}{2}\operatorname{Tr}_{\sigma,\tau}\int_p \frac{-i\omega_p}{(i\omega_p)^2 - E_-(\boldsymbol{p})^2}\left(\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}(\boldsymbol{0}) - \lambda(\boldsymbol{\sigma}\times\boldsymbol{\mathcal{A}}(\boldsymbol{0}))_z\right), \tag{143}$$

where the trace over the τ domain eliminates the τ_x and τ_z dependant part of $\underline{G_0}$. Writing out the shorthand formalism inside the integral explicitly we see that the Matsubara frequency sum over $i\omega_p$ goes from $-\infty$ to ∞ and that $\mathcal{S}_{1.1}$ is linear in $i\omega_p$

$$\mathcal{S}_{1.1} = \operatorname{Tr}_\sigma \sum_{i\omega_p=-\infty}^{\infty}\int d\boldsymbol{p}\frac{-i\omega_p}{(i\omega_p)^2 - E_-(\boldsymbol{p})^2}\left(\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}(\boldsymbol{0}) - \lambda(\boldsymbol{\sigma}\times\boldsymbol{\mathcal{A}}(\boldsymbol{0}))_z\right) = 0. \tag{144}$$

We see that there is no first order spin orbit correction and turn our attention to $\mathcal{S}_{2.1}$ immediately. Writing it out and evaluating the delta functions:

$$\begin{aligned}\mathcal{S}_{2.1} &= \frac{1}{2}\operatorname{Tr}_{\sigma,\tau}\int_{q,p,l,m,n}\underline{G_0}(\boldsymbol{q})\delta(\boldsymbol{q}-\boldsymbol{p})\gamma(\boldsymbol{p},\boldsymbol{l})\underline{G_0}(\boldsymbol{l})\delta(\boldsymbol{l}-\boldsymbol{m})\boldsymbol{S}(-\boldsymbol{m}-\boldsymbol{n})\cdot\boldsymbol{\sigma}\delta(\boldsymbol{q}-\boldsymbol{n}) \\ &= \frac{1}{2}\operatorname{Tr}_{\sigma,\tau}\int_{q,m}\underline{G_0}(\boldsymbol{q})\gamma(\boldsymbol{q},\boldsymbol{m})\underline{G_0}(\boldsymbol{m})\boldsymbol{S}(-\boldsymbol{m}-\boldsymbol{q})\cdot\boldsymbol{\sigma},\end{aligned} \tag{145}$$

switching to a Einstein summation notation for the dot product of $\boldsymbol{S}$ and $\boldsymbol{\sigma}$ and inserting γ we get the following equation:

$$\mathcal{S}_{2.1} = \operatorname{Tr}_{\sigma,\tau}\int_{q,m}\underline{G_0}(\boldsymbol{q})\left[\frac{(-\boldsymbol{m}+\boldsymbol{q})\cdot\boldsymbol{\mathcal{A}}(\boldsymbol{m}+\boldsymbol{q})}{2m} - \lambda(\boldsymbol{\sigma}\times\boldsymbol{\mathcal{A}}(\boldsymbol{m}+\boldsymbol{q}))_z\right]\tau_z\underline{G_0}(-\boldsymbol{m})S_a(-\boldsymbol{m}-\boldsymbol{q})\sigma_a. \tag{146}$$

Where S_a is the component of the $\boldsymbol{S}(\boldsymbol{q})$ vector. One can play around with variable transformations and eventually get the following expression:

$$\mathcal{S}_{2.1} = \frac{1}{2} \operatorname{Tr}_{\sigma,\tau} \int_{n,k} -\underline{G}_0(-\boldsymbol{k} - \frac{\boldsymbol{n}}{2}) \left[\frac{\boldsymbol{k}}{m} \cdot \boldsymbol{\mathcal{A}}(-\boldsymbol{n}) + \lambda(\boldsymbol{\sigma} \times \boldsymbol{\mathcal{A}}(-\boldsymbol{n}))_z \right] \tau_z \underline{G}_0(-\boldsymbol{k} + \frac{\boldsymbol{n}}{2}) S_a(\boldsymbol{n}) \sigma_a.$$
(147)

Defining the following velocity operator which is equal to $\frac{\boldsymbol{p}}{m}$ with a spin-orbit correction

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{m} + \lambda(\hat{z} \times \boldsymbol{\sigma}),$$
(148)

we can write the action as the following:

$$\mathcal{S}_{2.1} = \int_q S_a(\boldsymbol{q}) K_{ab}(\boldsymbol{q}) \mathcal{A}_b(-\boldsymbol{q})$$

$$K_{ab}(\boldsymbol{q}) = \frac{1}{2} \int_p \operatorname{Tr}_{\sigma,\tau} \sigma_a \underline{G}_0(-\boldsymbol{p} - \frac{\boldsymbol{q}}{2}) v_b \tau_z \underline{G}_0(-\boldsymbol{p} + \frac{\boldsymbol{q}}{2}).$$
(149)

To simplify our expression we define the following function

$$g_n = -\underline{G}_0(-\boldsymbol{n}) = \frac{i\omega_n + [\xi(\boldsymbol{n}) + \lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z] \tau_z + \Delta \tau_x}{(i\omega_n)^2 - [\xi(\boldsymbol{n}) + \lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z]^2 - \Delta^2},$$
(150)

which gives us the following short and neat expression for K_{ab}:

$$K_{ab}(\boldsymbol{q}) = \frac{1}{2} \int_p \operatorname{Tr}_{\sigma,\tau} \sigma_a g_{p+\frac{q}{2}} v_b \tau_z g_{p-\frac{q}{2}}.$$
(151)

One can trivially show that the term $\lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z$ takes on the eigenvalues of $\pm \lambda n$. We thus introduce a set of polarizers $\Pi_\pm$:

$$\Pi_\pm = \frac{1}{2} \left[1 \pm (\sigma \times \hat{p})_z \right], \hat{p} = \frac{p}{p}.$$
(152)

These polarizers sum to 1 and split a state into the two polarizations that yield eigenstates $\pm n$

$$\lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z \Pi_\pm |\psi\rangle = \lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z c_\pm |\psi_\pm\rangle = \pm \lambda n c_\pm |\psi_\pm\rangle.$$
(153)

By multiplying from the right side by identity in the form of two polarizers to our Greens function g_n we can simply write the eigenvalues in place of $\lambda(\boldsymbol{\sigma} \times \boldsymbol{n})_z$ as the polarizers will make sure that no component goes through that does not yield that value from the matrix

$$g_n = \frac{i\omega_n + [\xi(\boldsymbol{n}) + \lambda n] \tau_z + \Delta \tau_x}{(i\omega_n)^2 - [\xi(\boldsymbol{n}) + \lambda n]^2 - \Delta^2} \Pi_+ + \frac{i\omega_n + [\xi(\boldsymbol{n}) - \lambda n] \tau_z + \Delta \tau_x}{(i\omega_n)^2 - [\xi(\boldsymbol{n}) - \lambda n]^2 - \Delta^2} \Pi_-.$$
(154)

This makes the task at hand a lot more direct. There are no more matrix elements inside denominators and the calculation of $\mathcal{S}_{2.1} \propto F_{extra}$ is a matter of solving integrals and taking traces.

The approach from here is to expand $K_{ab}(\boldsymbol{q})$ in powers of $\boldsymbol{q}$ and select the lowest ones to incorporate

$$K_{ab}(\boldsymbol{q}) = K_{ab}(0) + q^c \partial_{q_c} K_{ab}(0) + \mathcal{O}(q^2).$$
(155)

This will yield both a first order $\boldsymbol{q}$ term and a zeroth order $\boldsymbol{q}$ term. We will only be covering the derivation of the zeroth order term as in our geometry it will be the only one that matters in Ch 5. We will add the first order term at the end and showcase later on how it does not affect our selected problem. If you wish to see the derivation of the first order term it can be found elsewhere in the Pershoguba articles supplemental material [2].

3.6   Calculating $K_{ab}(0)$

To zeroth order the equation for K_{ab} is as following

$$K_{ab}(0) = \frac{1}{2} \int_p \text{Tr}_{\sigma,\tau}[\sigma_a g_p v_b g_p]. \tag{156}$$

We start by completing the trace over the spin matrices

$$K_{ab}(0) = \frac{1}{2} \int_p \text{Tr}_\tau \text{Tr}_\sigma [\sigma_a (g_{p+}\Pi_+ + g_{p-}\Pi_-) v_b (g_{p+}\Pi_+ + g_{p-}\Pi_-)]. \tag{157}$$

Where g_{p+} and g_{p-} refer to the spin up or down Greens function in the basis created by $\lambda(\sigma \times \hat{p})_z$

$$g_{p\pm} = \frac{i\omega_n + [\xi(\boldsymbol{n}) \pm \lambda n]\tau_z + \Delta\tau_x}{(i\omega_n)^2 - [\xi(\boldsymbol{n}) \pm \lambda n]^2 - \Delta^2}. \tag{158}$$

Since Π polarizers include 2 terms and v_b does the same, there are collectively 32 terms of which the trace needs to be calculated for all variations of a and b but our work can be made a bit easier. If we only consider terms with the spin orbit coupling λ of order 1 or less then we can limit ourselves to only needing to analyze 16 terms of which 4 are of zeroth order in λ, these zeroth order terms trivially have trace 0 as they include only the σ_a matrix in the spin domain.

By writing out the polarizers we get

$$\frac{1}{4} \text{Tr}_\sigma [\sigma_a(g_{p+}(1+(\sigma \times \hat{p})_z)+g_{p-}(1-(\sigma \times \hat{p})_z))v_b(g_{p+}(1+(\sigma \times \hat{p})_z)+g_{p-}(1-(\sigma \times \hat{p})_z),)]. \tag{159}$$

and can go through it from the lowest order up.

We start with the easiest term, the one that includes only the unity part of the polarizers and the λ term from v_b:

$$v_{\text{Tr}} = \frac{1}{4} \text{Tr}_\sigma [\sigma_a(g_{p+} + g_{p-})\lambda(\hat{z} \times \sigma)_b(g_{p+} + g_{p-})]. \tag{160}$$

This trivially solves to the following

$$v_{\text{Tr}} = (g_{p+} + g_{p-})^2 \frac{\lambda}{2}\epsilon_{abz}. \tag{161}$$

Where ϵ_{abc} is the antisymmetric symbol. The trace over the other terms ends up giving 8 terms that can be ordered as 2 terms with a sum over 4 states. These 4 states will be distinguished between using the indexes $\pm_f$ and $\pm_l$ which are plus or minus respectively but independently. This would give us a state such as $(+_f, +_l)$ or $(+_f, -_l)$, while the notation is not ideal it serves the purpose of distinguishing these states and it is short lived

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$$\Pi_{\text{Tr}} = \text{Tr}_\sigma \left(\sum_{\pm_f \pm_l} \frac{\sigma_a}{4} g_{\pm_f} \lambda(\pm_f)(\sigma \times \hat{p})_z \frac{p_b}{m} g_{\pm_l} + \frac{\sigma_a}{4} g_{\pm_l} \frac{p_b}{m} g_{\pm_f} \lambda(\pm_l)(\sigma \times \hat{p})_z \right). \tag{162}$$

Collecting the two terms together and using that $g_{p\pm}$ only has one object τ that is position sensitive inside it and this term can be circularly shifted due to the trace

$$\Pi_{\text{Tr}} = \frac{\lambda}{4} \frac{p_x p_y}{mp} \epsilon_{abz} \sum_{\pm_f, \pm_l} \pm_f g_{p\pm_f} g_{p\pm_l} \pm_l g_{p\pm_l} g_{p\pm_f} = \frac{\lambda}{2} \frac{p_x p_y}{mp} \epsilon_{abz} (g_{p+} g_{p+} - g_{p-} g_{p-}). \tag{163}$$

Doing the momentum integration over this we find that the Π_{Tr} term is 0 due to the angular momentum integral over p_x and p_y

$$\int_0^{2\pi} d\theta \int_0^\infty dp(...) p_x p_y = \int_0^{2\pi} \sin(\theta) \cos(\theta) \int_0^\infty dp(...) = 0. \tag{164}$$

We thus need only worry about the terms from v_{Tr}

$$K_{ab}(0) = \frac{\lambda \epsilon_{abz}}{4} \int_p \text{Tr}_\tau [(g_{p+} + g_{p-})^2]. \tag{165}$$

We will start by showcasing how the integral over g_{p+}^2 and g_{p-}^2 are both zero. The Matsubara frequency sum will be enough to showcase this with complete freedom in values of $E_{+/-}$ as long as they remain real. As $E_{+/-}$ are energies they are of course also real. The following is the Matsubara frequency sum of g_{p+}^2 the proof for g_{p-}^2 is identical:

$$\mathcal{S}_{++} = \sum_{i\omega_n} \frac{\text{Tr}_\tau \{ i\omega_n + \xi_+(\boldsymbol{p})\tau_z + \Delta\tau_x \} \{ i\omega_n + \xi_+(\boldsymbol{p})\tau_z + \Delta\tau_x \}}{(i\omega_n + E_+(\boldsymbol{p})^2 (i\omega_n - E_+(\boldsymbol{p})^2}. \tag{166}$$

This can be simplified to be written in terms of only $E_+(\boldsymbol{p})$

$$\mathcal{S}_{++} = 2 \sum_{i\omega_n} \frac{(i\omega_n)^2 + E_+(\boldsymbol{p})^2}{(i\omega_n + E_+(\boldsymbol{p})^2 (i\omega_n - E_+(\boldsymbol{p})^2}. \tag{167}$$

Motivated by our knowledge of complex plane integrals [4], we start elsewhere by looking at the following integral over the complex plane denounced by z

$$0 = \oint \frac{dz}{2\pi i} \frac{z^2 + E_+^2}{(z - E_+)^2 (z + E_+)^2} n_f(z) e^{\eta z}. \tag{168}$$

Where the integral over z takes the path as shown in Fig. 1 and η takes on a minute positive value infinitesimally close to 0

$$0 < \eta < \beta, \tag{169}$$
$$\eta = 0^+.$$

We let η go to 0 after we evaluate the result of the integral. $n_f(z)$ is the Fermi-Dirac distribution function
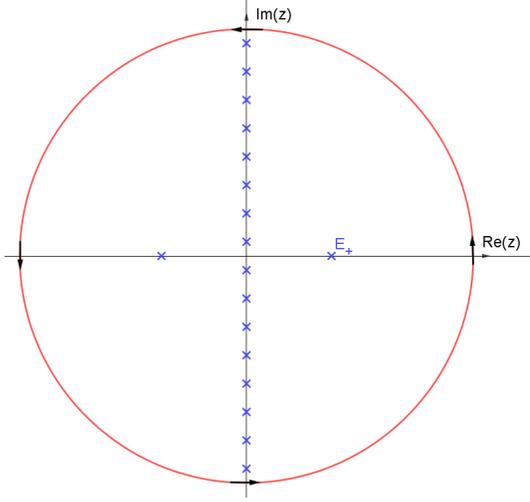
Figure 1: *The integral path in the complex plane follows the circle along a positive trajectory. It extends practically all the way to infinity and encompasses all residues. There are an infinite number of residues along the imaginary-axis and 2 on the real-axis at $\pm E_+$.*

$$n_f(z) = \frac{1}{e^{\beta z} + 1}. \tag{170}$$

At $z \to \infty$ the Fermi function suppresses the integrand and makes it 0. At $z \to -\infty$ the $e^{\eta z}$ suppresses the integrand making it 0. We thus know that the integral in total must be equal to 0

$$\lim_{z \to \infty} \frac{e^{\eta z}}{e^{\beta z} + 1} = e^{(\eta - \beta)z} = 0,$$
$$\lim_{z \to -\infty} \frac{e^{\eta z}}{e^{\beta z} + 1} = e^{\eta z} = 0. \tag{171}$$

We can compute the integral in Eq. 167 using the residue theorem, which dictates that the value of a closed line integral in the complex plane is equal to the sum of the residues enclosed within the closed integral multiplied by $2\pi i$

$$0 = 2\pi i \sum_N Res(N). \tag{172}$$

The Fermi function takes on simple poles at exactly $z = i\omega_n$ for all values of n. Additionally there are 2 poles located at $\pm E_+$

$$i\omega_n = i\frac{(2n + 1)\pi}{\beta},$$
$$n_f(i\omega_n) = \frac{1}{e^{2\pi i + i\pi} + 1} = \frac{1}{-1 + 1}. \tag{173}$$

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This means that the the complicated Matsubara frequency sum is equal to the sum of the 2 poles at residues $\pm E_+$

$$\sum_{i\omega_n} \frac{(i\omega_n)^2 + E_+(\boldsymbol{p})^2}{(i\omega_n + E_+(\boldsymbol{p})^2(i\omega_n - E_+(\boldsymbol{p})^2} e^{\eta i \omega_n} = -2\pi i \sum Res(E_+) + Res(-E_+). \tag{174}$$

Residues at E_+ and $-E_+$ are calculated to be the following:

$$
\begin{aligned}
Res(E_+) &= -\frac{1}{2\pi i} \frac{\beta}{2} n_f(E_+)^2 e^{\beta E_+}, \\
Res(-E_+) &= -\frac{1}{2\pi i} \frac{\beta}{2} n_f(-E_+)^2 e^{-\beta E_+}.
\end{aligned}
\tag{175}
$$

Adding up the two results we get the following value for $\mathcal{S}_{++}$:

$$\mathcal{S}_{++} = \frac{-1}{2} \left(n_f(E_+)^2 e^{\beta E_+} + n_f(-E_+) e^{-\beta E_+} \right). \tag{176}$$

Letting $\beta \to \infty$ we see what the result is if E_+ is negative or positive. If β is infinite then the Fermi-Dirac distribution function turns into a Heaviside step function that is either 1 if the argument is below 0 and 0 if the argument is above 0. We see that whether E_+ is positive or negative the remainder is an exponentially suppressed function, suppressed with the power of β. This thus leaves us with the result that

$$\mathcal{S}_{++/--} = \sum_{i\omega_n} g_{+/-}^2 = 0. \tag{177}$$

We are thus left only with the cross term:

$$\mathcal{S}_{+-} = 2 \sum_{i\omega_n} \frac{\mathrm{Tr}_\tau \{ i\omega_n + \xi_+ \tau_z + \Delta \tau_x \} \{ i\omega_n + \xi_- \tau_z + \Delta \tau_x \}}{(i\omega_n - E_+)(i\omega_n + E_+)(i\omega_n - E_-)(i\omega_n + E_-)}. \tag{178}$$

We go through the same procedure as before but this time use the following contour integral

$$0 = \oint \frac{dz}{2\pi i} \frac{z^2 + \xi_+ \xi_- + \Delta^2}{(z - E_+)(z + E_+)(z - E_-)(z + E_-)} n_f(z) e^{\eta z}. \tag{179}$$

We find the residues at E_+, $-E_+$, E_- and $-E_-$ to be the following:

$$
\begin{aligned}
Res(E_+) &= \frac{(E_+^2 + \xi_+ \xi_- + \Delta^2)}{E_+(E_+ - E_-)(E_+ + E_-)} \frac{n_f(E_+)}{2}, \\
Res(-E_+) &= \frac{(E_+^2 + \xi_+ \xi_- + \Delta^2)}{E_+(E_+ - E_-)(E_+ + E_-)} \frac{-n_f(-E_+)}{2}, \\
Res(E_-) &= \frac{(E_-^2 + \xi_+ \xi_- + \Delta^2)}{E_-(E_+ - E_-)(E_+ + E_-)} \frac{-n_f(E_-)}{2}, \\
Res(-E_-) &= \frac{(E_-^2 + \xi_+ \xi_- + \Delta^2)}{E_-(E_+ - E_-)(E_+ + E_-)} \frac{n_f(-E_-)}{2}.
\end{aligned}
\tag{180}
$$

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Summing these up we find that the Matsubara frequency sum is equal to

$$\sum_{i\omega_n} \frac{(i\omega_n)^2 + \xi_+\xi_- + \Delta^2}{(i\omega_n - E_+)(i\omega_n + E_+)(i\omega_n - E_-)(i\omega_n + E_-)} = \frac{1}{2}\frac{E_+E_- - \xi_+\xi_- - \Delta^2}{E_+E_-(E_+ + E_-)} + u(\boldsymbol{p}), \quad (181)$$

where $u(\boldsymbol{p})$ is equal to

$$u(\boldsymbol{p}) = \frac{-(\xi_+\xi_- + \Delta^2 + E_+^2)E_-n_f(E_+) + (\xi_+\xi_- + \Delta^2 + E_-^2)E_+n_f(E_-)}{4E_+E_-\xi\lambda p}. \quad (182)$$

Neglecting the function $u(\boldsymbol{p})$ as for λ very small compared to ξ the two terms of the numerator are opposite and cancel out. $K_{ab}(0)$ takes the following form

$$K_{ab}(0) = \frac{\lambda\epsilon_{abz}}{2}\int d\boldsymbol{p}\frac{E_+E_- - \xi_+\xi_- - \Delta^2}{E_+E_-(E_+ + E_-)}. \quad (183)$$

Completing this integral in the limit of $\lambda p_f \gg m\lambda^2 \gg \Delta$ we get

$$K_{ab}(0) = \epsilon_{zba}\frac{\lambda m}{2\pi} = \epsilon_{zba}\alpha, \quad (184)$$

where

$$\alpha = \frac{\lambda m}{2\pi}. \quad (185)$$

Completing this integral is a nontrivial and unfortunately not one we were able to get a satisfying result for.

3.7   Results for Free energy and Current

Finally we will get to the fruits of our labor, we now seek to find the additional free energy term and current term spawning from the spin-orbit and magnetic exchange field interaction. Inserting the results of Eq. 184 back into Eq. 149 and then inserting that into Eq. 113 we get:

$$\mathcal{Z} = \int \mathcal{D}(\theta, \Delta)e^{\int_q S_a(\boldsymbol{q})\epsilon_{zba}\alpha\mathcal{A}_b(-\boldsymbol{q})}. \quad (186)$$

As this expression is completely independent of Δ and not directly dependant on θ but only on its derivative $\nabla\theta$ through $\mathcal{A}$ we can reduce the integral to a single term

$$\mathcal{Z} = \exp\left\{\int_q S_a(\boldsymbol{q})\epsilon_{zba}\alpha\mathcal{A}_b(-\boldsymbol{q})\right\}. \quad (187)$$

Inserting the Fourier transform for $S_a(\boldsymbol{q})$ and $\mathcal{A}_b(-\boldsymbol{q})$ we can produce a delta function by taking the integral over q

$$\mathcal{Z} = \exp\left\{\int_{q,x,l} S_a(\boldsymbol{x})\epsilon_{zba}\alpha\mathcal{A}_b(-\boldsymbol{q})e^{i\boldsymbol{q}\cdot(\boldsymbol{l}-\boldsymbol{x})}\right\}. \quad (188)$$

This leaves us with a real space integral for $\mathcal{Z}$

$$\mathcal{Z} = \exp\left\{ \int_0^\beta d\tau \int d^2x S_a(\boldsymbol{x})\epsilon_{zba}\alpha\mathcal{A}_b(\boldsymbol{x}) \right\}. \tag{189}$$

Going all the way back to Eq. 93. We find the free energy to be

$$F_{extra} = -\frac{1}{\beta} \int_0^\beta d\tau \int d^2x S_a(\boldsymbol{x})\epsilon_{zba}\alpha\mathcal{A}_b(\boldsymbol{x}). \tag{190}$$

$S_a\epsilon_{zba}\mathcal{A}_b$ can be written as $-(\hat{\boldsymbol{z}}\times\boldsymbol{S}(\boldsymbol{x}))\cdot\mathcal{A}$. The term is independant of τ so the integral over imaginary time simply gives β

$$F_{extra} = \alpha \int d^2x (\hat{\boldsymbol{z}} \times \boldsymbol{S}) \cdot \mathcal{A}. \tag{191}$$

From there one can easily calculate the extra contributing term to the current

$$\boldsymbol{J}_{extra} = \frac{\delta}{\delta\boldsymbol{A}}F = \alpha(\hat{\boldsymbol{z}} \times \boldsymbol{S}). \tag{192}$$

The total current includes a term that was ignored in this derivation as it had nothing to do with either spin orbit coupling or the impurity and exists in regular superconductors. It also includes a second order $K_{ab}(\boldsymbol{q})$ term which we will add here but will not derive. When it is time to use this current in Ch 5. we will show that this term proves irrelevant in our setup

$$\boldsymbol{J} = \frac{n_s}{2m}\nabla\theta + \alpha(\hat{\boldsymbol{z}} \times \boldsymbol{S}) + \beta(\nabla\boldsymbol{S}_z \times \hat{\boldsymbol{z}}). \tag{193}$$

For a in depth derivation of the β term consult the supplementary material of [2].

4 Magnons derived from linearized Landau-Lifshitz equation

The final system we wish to examine independently is the magnon system. A magnon is a quasiparticle made out of the collective spin of electrons across a magnetic lattice. To get a simply intuition for magnons, imagine a 1-D or 2-D material on a set of sites that make up a line (1-D) or a square (2-D) lattice. Each site has one electron bound that has freedom in its spin. Each site is affected by a effective magnetic field from its neighbours dependent upon their spin which results in a net effective local magnetic field affecting each site. The question is now how this material would evolve over time and what kind of low energy excitations it would hold.

If we zoom out from the microscopic site by site view and look at the system on a larger scale we can no longer see the spin of each individual electron. What we can however see and measure is the magnetization at a given point. Consider an experimentalist measuring the collective magnetic field generated by a small area of a magnetic material or the response of the material to a magnetic field. While it would encompass 1000's of individual spins we get a single reading for its collectively generated magnetic field inside this area. By examining such areas side by side we can consider the sum of the electron spins after normalization as a fluid measurable parameter called the magnetization. If we are to limit ourselves to systems where the magnetic spins of electrons is not rapidly changing from one site to another then this becomes a useful parameter to characterize the whole system as no important information is lost by summing over several very similar areas of magnetic spins. We call this the magnetization and it measures the average spin polarization inside an area when seen from a macroscopic point of view.

The system we are trying to characterize is a ferromagnet, this means that the spins find themselves in a minimum energy configuration when aligned with their neighbours. We wish to understand the excitations that might exist in such a material when the magnetization deviates from being the same across the entire material.

We will start off by explaining what the Landau Lifshitz equation is and what goes into it. Then we will shortly discuss anisotropy and why one would expect a ferromagnet to have the ground state be a polarization in one direction, this also builds the basis for why linearization is justified. We will then linearize the Landau Lifshitz equation (LLE) to make it analytically solvable and then solve it. We will then compare our results in the linear regime to a numerical approach inspired by the methods presented in "A Gauss–Seidel Projection Method for Micromagnetics Simulations" by Xiao-Ping Wang, Carlos J. García-Cerveria and Weinan E [3]. using a implicit Gauss-Seidal projection scheme for simulating the magnetic system instead of a straightforward Euler scheme. Lastly we will look at the breakdown of the linearized model and how the numerics fail to take the same form as the analytics in the extremely non linearized domain.

4.1 Landau Lifshitz Equation

The equation that governs the evolution of the magnetization in time is called the Landau-Lifshitz equation [6] and looks like the following:

$$\frac{d\boldsymbol{m}}{dt} = -\gamma \boldsymbol{m} \times \mathcal{H} - \frac{\gamma \epsilon}{m_s} \boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}). \tag{194}$$

Where $\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{x}, t)$ is the magnetization at a given time and place in the system.

$\mathcal{H} = \mathcal{H}(\boldsymbol{x}, t)$ is effective local field. γ is the gyromagnetic ratio. ϵ is a dimensionless damping coefficient. The effective local field is calculated from the free energy of the system as the following:

$$\mathcal{H} = -\frac{\delta F}{\delta \boldsymbol{m}}. \tag{195}$$

Without considering the fact that the effective local field $\mathcal{H}$ can be dependent on the magnetization $\boldsymbol{m}$ it is already clear from the Landau-Lifshitz equation that these equations are non linear and cannot be solved analytically. This serves as the motivation for following through both with a linearization of the magnetization around a presumed ground state of constant magnetization and also a numerical simulation approach using the full machinery of the Landau-Lifshitz equation.

To begin solving for the magnetization we first need a free energy to compute the local field $\mathcal{H}$. The free energy terms that are dependent upon the magnetization are the only relevant ones to our analysis of the ferromagnets magnons. So we will not be concerning ourselves with terms that have no dependence on the magnetization $\boldsymbol{m}$. The following is our free energy for the ferromagnet when only including $\boldsymbol{m}$ dependent terms

$$F_{magnon} = \frac{1}{2} \int_{\Omega} \{\Phi\left(\frac{\boldsymbol{m}}{m_s}\right) + \frac{A'}{m_s^2}|\nabla \boldsymbol{m}|^2 - 2\mu_0 \boldsymbol{B} \cdot \boldsymbol{m}\} dx. \tag{196}$$

The integral over Ω is an integral over the whole system, whether it be 1, 2 or 3 dimensional and whether it be infinite or limited. Φ is the anisotropic energy that might be present due to a particular preference of magnetization by the material. A' is the exchange constant and serves to modulate the strength of the interaction between neighboring spins. μ_0 is the vacuum permeability and $\boldsymbol{B}$ is proportional to the applied magnetic field onto the material. m_s is the saturated magnetization and is given by $m_s = |\boldsymbol{m}|$.

4.2 Anisotropy of the ferromagnet

We will not be covering a complete description of all anisotropy terms here but will briefly touch on the two that are most important to us. This being systems that showcase easy-axis or easy-plane anisotropy [7].

Easy axis and easy plane are names for anisotropy schemes that promote either magnetization along the axis or plane. This is the simplest kind of anisotropy that differs only in either 1 plane or along one axis. Given that we consider only this form of anisotropy we can take the anisotropy term $\Phi\left(\frac{\boldsymbol{m}}{m_s}\right)$ to look like the following:

$$\Phi\left(\frac{\boldsymbol{m}}{m_s}\right) = \Phi_z m_z^2 + \Phi_{xy}(m_x^2 + m_y^2) = m_z^2(\Phi_z - \Phi_{xy}) + \Phi_{xy} m_s^2. \tag{197}$$

Where $\Phi_{x/y}$ is the cost of magnetization along the $x - y$ plane and Φ_z is the cost of magnetization along the z-axis. Both these energy costs can be collected into a single parameter Φ_{xyz}, ignoring the constant contribution:

$$\Phi\left(\frac{\boldsymbol{m}}{m_s}\right) = m_z^2 \Phi_{xyz}. \tag{198}$$

The sign of Φ_{xyz} now governs whether it is an easy plane or easy axis anisotropy in the system. If Φ_{xyz} is negative then it is easy-axis and promotes magnetization along the z-axis, correspondingly if positive it is easy plane.

This yields a free energy of the form

$$F_{magnon} = \frac{1}{2} \int_{\Omega} \{m_z^2 \Phi_{xyz} + \frac{A'}{m_s^2} |\nabla \boldsymbol{m}|^2 - 2\mu_0 \boldsymbol{B} \cdot \boldsymbol{m}\} dx. \tag{199}$$

4.3 Analytical linearized treatment of magnons

While we can already tell that the Lindau-Lifshifz equation will not yield us a linear answer, we can go forward with the equation and see how far we can get before needing to make approximations or delve into numerical solutions. We start by calculating the local field $\mathcal{H}$ as given by Eq. 195 and Eq. 199:

$$\mathcal{H} = -\frac{\delta}{\delta \boldsymbol{m}} \left(\frac{1}{2} \int_{\Omega} \{\Phi_{xyz} m_z^2 + \frac{A'}{m_s^2} |\nabla \boldsymbol{m}|^2 - 2\mu_0 \boldsymbol{B} \cdot \boldsymbol{m}\} dx \right). \tag{200}$$

Treating $|\nabla \boldsymbol{m}|^2$ as $\nabla \boldsymbol{m} \cdot \nabla \boldsymbol{m}$, we can perform partial integration and discard the surface term due to the functional integral. Keep in mind here that $\boldsymbol{m}$ is a real physical observable and must be real. When we consider it a complex wave later on, it is done with the implication that only the real part of the final result is physical

$$\mathcal{H} = -\frac{\delta}{\delta \boldsymbol{m}} \left(\frac{1}{2} \int_{\Omega} \{\Phi_{xyz} m_z^2 - \frac{A'}{m_s^2} \nabla^2 \boldsymbol{m} \cdot \boldsymbol{m} - 2\mu_0 \boldsymbol{B} \cdot \boldsymbol{m}\} dx \right). \tag{201}$$

Evaluating this we find the following effective field:

$$\mathcal{H} = -\Phi_{xyz} m_z \hat{\boldsymbol{z}} + \frac{A'}{2m_s^2} \nabla^2 \boldsymbol{m} + \mu_0 \boldsymbol{B}. \tag{202}$$

We see now that $\mathcal{H}$ is of linear order of $\boldsymbol{m}$ and therefore all our terms with the exception of the applied magnetic field Zeeman term is nonlinear. To sidestep this issue and still find analytical solutions, even if it is inside a smaller regime, we linearize $\boldsymbol{m}$ around $\boldsymbol{m} = m_s \hat{\boldsymbol{z}}$. This is in accordance with what would be expected of the ground state of an easy axis system along the z-axis where the energy contribution from the interaction term is zero and energy is minimized purely based upon the anisotropy term and the applied magnetic field is considered small or non existent. This mirrors what in the real world the material would look like if we put a piece of it in a cooler and just let it be and as such is a good starting condition for the material to start exploring excitations from. We consider $\boldsymbol{m}$ the following in our linearized approximation:

$$\boldsymbol{m}(\boldsymbol{x},t) = \begin{bmatrix} m_x(\boldsymbol{x},t) \\ m_y(\boldsymbol{x},t) \\ m_z(\boldsymbol{x},t) \end{bmatrix} = \begin{bmatrix} m_x(\boldsymbol{x},t) \\ m_y(\boldsymbol{x},t) \\ m_s \end{bmatrix}. \tag{203}$$

Now that we have acquired the local field $\mathcal{H}$ we dive into the calculation of the Landau-Lifshitz equation. First we calculate what $\boldsymbol{m} \times \mathcal{H}$ is equal to as it shows up twice in the Landau-Lifshitz equation:

$$\boldsymbol{m} \times \mathcal{H} = (m_y \mathcal{H}_z - m_s \mathcal{H}_y)\hat{\boldsymbol{x}} + (m_s \mathcal{H}_x - m_x \mathcal{H}_z)\hat{\boldsymbol{y}} + (m_x \mathcal{H}_y - m_y \mathcal{H}_x)\hat{\boldsymbol{z}}. \tag{204}$$

To first order in m_x or m_y the only remaining terms in $\boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H})$ are the following:

$$\boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}) = m_s(m_x\mathcal{H}_z - m_s\mathcal{H}_x)\hat{\boldsymbol{x}} + m_s(m_y\mathcal{H}_z - m_s\mathcal{H}_y)\hat{\boldsymbol{y}} + m_s(m_x\mathcal{H}_x + m_y\mathcal{H}_y)\hat{\boldsymbol{z}}. \quad (205)$$

Combining the two we find the Landau-Lifshitz equation to take the following form:

$$\begin{aligned}
\frac{d\boldsymbol{m}}{dt} = &- \gamma(m_y\mathcal{H}_z - m_s\mathcal{H}_y)\hat{\boldsymbol{x}} - \gamma\epsilon(m_x\mathcal{H}_z - m_s\mathcal{H}_x)\hat{\boldsymbol{x}} \\
&- \gamma(m_s\mathcal{H}_x - m_x\mathcal{H}_z)\hat{\boldsymbol{y}} - \gamma\epsilon(m_y\mathcal{H}_z - m_s\mathcal{H}_y)\hat{\boldsymbol{y}} \\
&- \gamma(m_x\mathcal{H}_y - m_y\mathcal{H}_x)\hat{\boldsymbol{z}} - \gamma\epsilon(m_x\mathcal{H}_x + m_y\mathcal{H}_y)\hat{\boldsymbol{z}}.
\end{aligned} \quad (206)$$

We now have 3 equations to solve and the first one we look at is the simplest. The equation for the $\hat{\boldsymbol{z}}$ direction:

$$0 = -\gamma(m_x\mathcal{H}_y - m_y\mathcal{H}_x) - \gamma\epsilon(m_x\mathcal{H}_x + m_y\mathcal{H}_y). \quad (207)$$

Inserting our function for the local field $\mathcal{H}$ and simplifying we get the following equation:

$$0 = m_x(B_x\epsilon + B_y) + m_y(B_y\epsilon - B_x). \quad (208)$$

A solution with $m_x = 0$ or $m_y = 0$ would naturally lead us to the other being 0 as well as our freedom to adjust B_x, B_y, ϵ is preserved. This would be a trivial solution. This means that both the terms in the parenthesis must be 0 simultaneously. Since ϵ and $\boldsymbol{B}$ are independent parameters that can be adjusted independently, the only way to make the equation true consistently for non-zero values of ϵ is by having $B_x = B_y = 0$.

Moving onto the two equations pertaining to the time derivative of m_x and m_y narrowing our search to magnetic fields $\boldsymbol{B} = B_z\hat{\boldsymbol{z}}$, we see that every term in the equation is linear in m_x and m_y. We insert plane waves for both functions as we are looking for collective modes

$$\begin{aligned}
m_x &= \delta m_x e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t} \\
m_y &= \delta m_y e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}.
\end{aligned} \quad (209)$$

Inserting into Eq. 206 we get the following two equations:

$$\begin{aligned}
\delta m_x\left(-i\omega + \gamma\epsilon\mathcal{H}_z + \gamma\epsilon\frac{A'}{2m_s}k^2\right) &= \delta m_y\left(-\gamma\mathcal{H}_z - \gamma\frac{A'}{2m_s}k^2\right), \\
\delta m_y\left(-i\omega + \gamma\epsilon\mathcal{H}_z + \gamma\epsilon\frac{A'}{2m_s}k^2\right) &= \delta m_x\left(\gamma\mathcal{H}_z + \gamma\frac{A'}{2m_s}k^2\right).
\end{aligned} \quad (210)$$

Substituting one equation into the other and cancelling the factor of the magnitude $\delta m_x/\delta m_y$ on either side gives the following equation relating ω and $k = |\boldsymbol{k}|$

$$\omega^2 + \omega 2i\gamma\epsilon\left(\mathcal{H}_z + \frac{A'}{2m_s}k^2\right) - (1 + \epsilon^2)\gamma^2\left[\mathcal{H}_z + \frac{A'}{2m_s}k^2\right]^2 = 0. \quad (211)$$

Solving for ω in Eq. 211 gives us the following dispersion relation for the magnons:

$$\omega = -i\gamma\epsilon\left(\mathcal{H}_z + \frac{A'}{2m_s}k^2\right) \pm \gamma\left(\mathcal{H}_z + \frac{A'}{2m_s}k^2\right). \quad (212)$$

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We see that $\mathcal{H}_z$

$$\mathcal{H}_z = -\Phi_{xyz} m_s + \mu_0 B_z \tag{213}$$

is the effective magnetic field in the z-direction. Given that Φ_{xyz} is negative due to the easy axis assumption, it is also fair to assume that B_z would be positive or at least that the net effect of the two would be positive to such a degree that we can still consider the system in an easy axis state.

This creates a dispersion that is parabolic and gapped. Since the plasmon dispersion relation found in the plasmon chapter was linear these two are highly distinguishable which will be handy when we try to mix the two states in Ch 5.

As expected the imaginary part creates a decay function

$$e^{-\gamma \epsilon \left(\mathcal{H}_z + \frac{A'}{2m_s} k^2\right) t} \tag{214}$$

that scales with the damping coefficient ϵ.

4.4 Numerical dispersion relation

As our analytical approach is only skin deep requiring heavy modification in the form of linearization, we also approach using a numerical approach. The simulation will be one that takes an input magnetization on a one dimensional finite lattice with periodic boundary conditions and simulates forward in time using the Landau-Lifshitz equation.

We will input into it a plane wave with a given wavenumber k and see how it simulates forward in time. By Fourier transform of the magnetization data over the course of the simulation, we can identify a frequency spectrum for the wave. Picking out the top frequency we can plot the dispersion.

The specifics of the simulation are covered in Appendix A. We will only touch on the method by which we obtain the dispersion relation from the simulation, the results of the simulation and how they compare to the analytical linearized approach in this section.

The simulation yields after each iteration over all the time steps a data set of magnetizations in each spacial direction on each site for each time step.

By numerical Fourier transformation at any given time step we can determine what wavenumber k the function inhabits beyond what we originally inserted. Doing so we find that k varies very little and is stable over the entire simulation, this tells us that the simulation does indeed showcase eigenstates as a function of k.

The above procedure can be done for any one of the 3 spacial directions. As we are looking for spin-waves that live in the $x - y$ plane we naturally choose to look at either the x magnetization or the y magnetization.

To find the dispersion relation we Fourier transform the y magnetization across time on a arbitrary site. Since we have already confirmed that we have stable wave vector k, we know that the simulation correctly incorporates translational invariance and the site we choose is thus not relevant. For the purposes of calculating the dispersion relation, we choose a site in the middle of our simulations chain and the magnetization in the y-direction. After this we search through the ω spectrum created by our Fourier transform in time and pick out the value of ω that has the highest amplitude. This predictably turns out to be a very spiked value thus ensuring us that a single frequency indeed can be linked to a single wavevector.

Plotting ω as a function of input wave vector k with an applied effective field of 10 in the natural units of the system, we get the graph seen on Fig. 2. The first thing to note
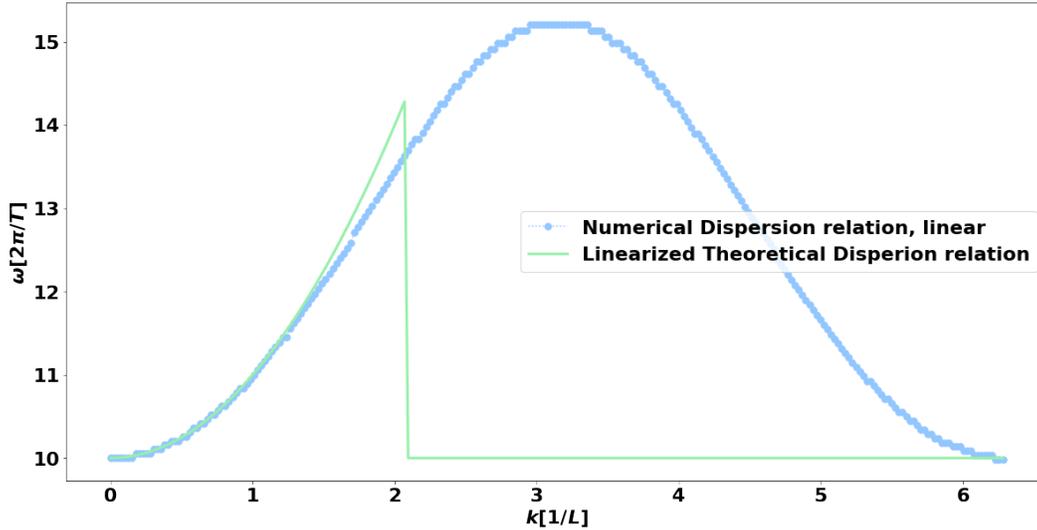
Figure 2: *Dispersion relation in the natural units of 1/L where L is the length of the system and 1/T where T is the total time period the system ran through. The blue dots are numerical results for $m_z = 0.95m_s$, while the green line is the theoretical results derived from our linearization previously in this chapter. We have cut off the linearized result at 1/3 of the spectrum as the result is only appropriate for low k. In both cases there is an effective applied magnetic field of 10 natural energy units of the system.*

is that the numerical dispersion relation knows about the total system size and periodic boundary conditions of the system and therefore provides a dispersion relation across all k. The theoretical result is based of the local Hamiltonian that has no knowledge about periodic boundary conditions. The theoretical results are therefor only applicable for low values of k and has graphically been cut off for higher values in the plot for clarity sake. It obviously follows Eq. 212 and continues upwards with a k^2 growth.

We see that the low k results match rather well, giving us the k^2 dispersion relation we had expected with an offset equal to the magnetic field. This confirms that in the linear regime our magnon simulation is accurate. Emboldened by this confirmation of the validity of our simulation we now explore a non linear regime in Fig. 4. In this regime instead of having $m_z = 0.95m_s$ we have $m_z = 0.23m_s$ resulting in a plot that looks a bit troublesome. The reason for this has to do with the capture method for the dispersion relation and the actual spectrum. The dispersion relation was derived by capturing the value of ω that had the highest amplitude in a Fourier transform, assuming there to be only one dominant value of ω. Inside the non linear regime this however is not the case anymore. On Fig. 3 one can see a Fourier transform picked out for $k = 1.73/L$ and one for $k = 1.75/L$ in the non linear simulation. One can clearly see that there is more than one spike inside the Fourier spectrum and the ideal one k to one ω assumption breaks down. Additionally because of the wild fluctuation of the maximum amplitude of ω in this regime, not even the dominant frequency is stable.

In fact even inside the "well behaved" areas of Fig. 4, such as right in the middle of it at $k = \pi$, we see that there are multiple values of ω with high representation in the Fourier
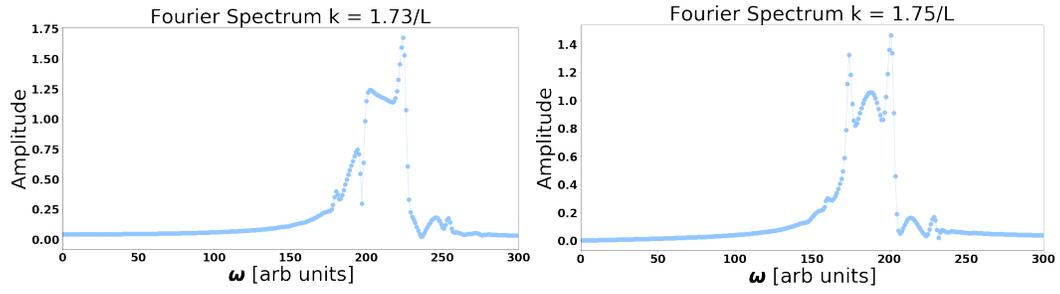
(a) Non linearized Fourier spectrum showing the splitting of the ω spectrum at $k = 1.73/L$.

(b) Non linearized Fourier spectrum showing the splitting of the ω spectrum at $k = 1.75/L$.

Figure 3: Even with small variance in k we see a significant deviation in the Fourier spectrum in the non linearized simulation, showcasing the instability of the solutions in this area of k values.

spectrum. A Fourier spectrum at $k = \pi$ for the non linear simulation can be seen on Fig. 5.
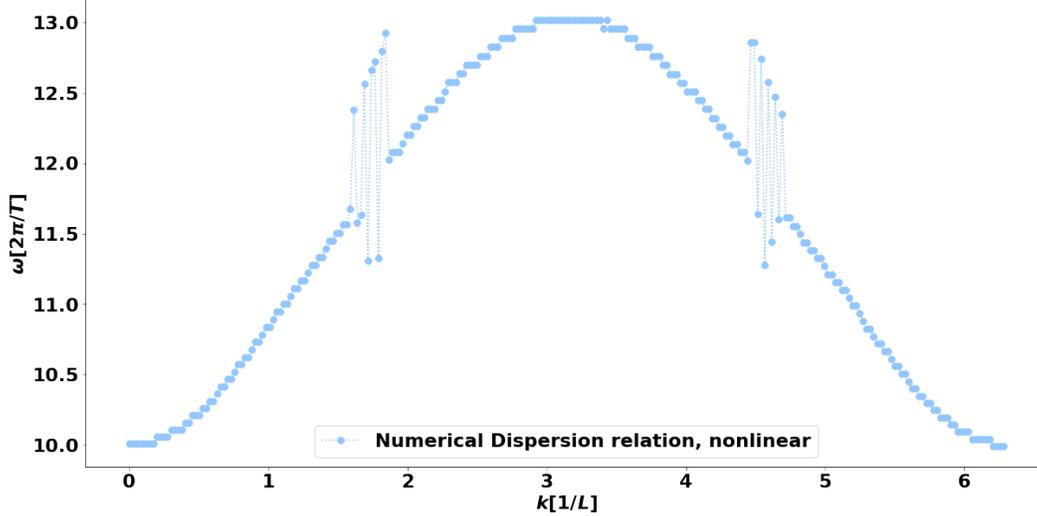
Figure 4: *This simulation showcases what happens to the "dispersion" with an applied magnetic field in the z-direction outside the linearized regime. The simulation is done with initial conditions having $m_z = 0.23 m_s$ and equal amplitude in waves propagating in x-y directions.*

5 Plasmon-Magnon hybrid modes in superconductor-ferromagnet heterostructure

From our previous investigation of the ferromagnetic system that is inhabited by magnon collective modes and the superconductor system that is inhabited by plasmon collective modes we now wish to use the free energy and current equations from the spin-orbit-magnetic exchange interaction system to couple these systems equations. This will be done with the goal of finding a dispersion relation that agrees with all the restrictions of each environment.

We will do this by first listing the relevant results of the past three sections. These will be the basis for the analysis. From them we will derive a set of coupled equations for m_y, m_x and I. This will require us to couple the current equation to the magnetization through the electric field and the calculation of a new effective local field $\mathcal{H}$ stemming from the new terms in the free energy that our spin-orbit-magnetic exchange analysis adds. Finally we will solve the coupled equations using an eigenvalue equation. As this turns out to be a fourth order polynomial root equation we will do so numerically and analyze it graphically.

5.1 Derivation of coupled equations

We start from the continuity equation as presented in Eq. 42 in Ch. 2 on plasmons

$$\frac{\partial I}{\partial x} + \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx' \lambda \left(x - x' \right) E \left(x', t \right) = 0. \tag{215}$$

We have here set e the electron charge to unity as is done in the calculations pertaining to the spin-orbit magnetic exchange system, as we will be using both equations in tandem. Additionally the electron mass m^* now carries an asterisk to differentiate it from the magnetization.
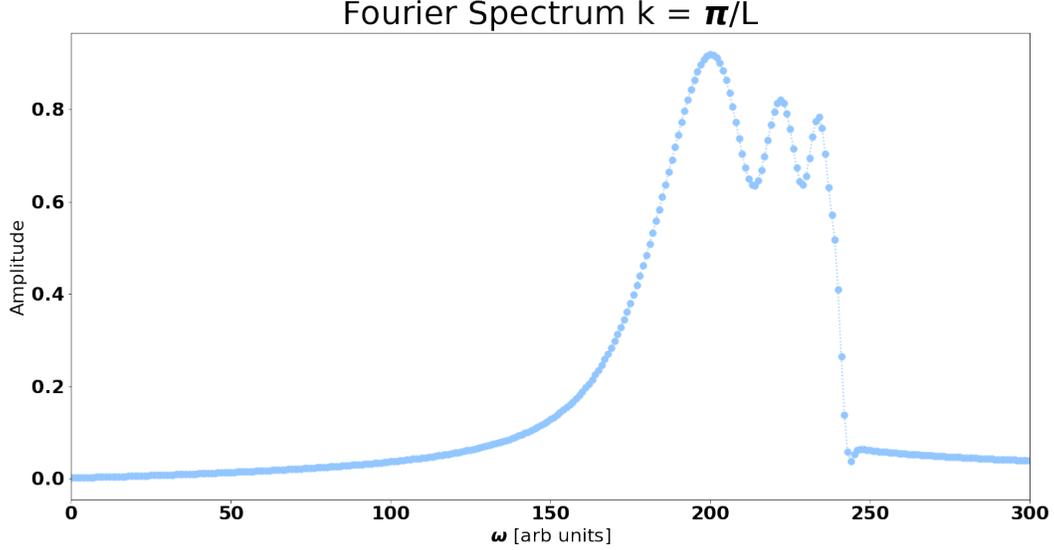
Figure 5: *Fourier transform of $m_z = 0.23 m_s$ simulation for $k = \pi$.*

The current will be dictated by the result that was derived in Eq. 193 in Ch 3. on the spin-orbit magnetic exchange coupling[2]

$$\boldsymbol{j} = \frac{n_s}{m^*}\boldsymbol{\mathcal{A}} + \alpha(\hat{\boldsymbol{z}} \times \boldsymbol{m}) + \beta(\nabla \boldsymbol{m_z} \times \hat{\boldsymbol{z}}), \tag{216}$$

where $\boldsymbol{\mathcal{A}}$ is the gauge invariant vector potential

$$\boldsymbol{\mathcal{A}} = \boldsymbol{A} + \frac{\theta}{2}. \tag{217}$$

We have here extended the magnetic exchange field to be the magnetization and absorbed any relevant factors between the two into the constants α and β.

The equation governing the evolution of the magnetization is still the Landau–Lifshitz equation, but our free energy is governed by the free energy of the magnon system with the addition of the terms provided from the spin-orbit magnetic exchange system, thus coupling the current to the magnetization in the resulting local field $\mathcal{H}$. This results in the equations of motion for the magnetization being the following:

$$\frac{d\boldsymbol{m}}{dt} = -\gamma \boldsymbol{m} \times \mathcal{H} - \frac{\gamma \epsilon}{m_s} \boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}) \qquad \mathcal{H} = -\frac{\delta F}{\delta \boldsymbol{m}}, \tag{218}$$

$$F_{coupling} = \int d^2 r \frac{n_s}{2m^*} \boldsymbol{\mathcal{A}}^2 + \alpha(\hat{\boldsymbol{z}} \times \boldsymbol{m}) \cdot \boldsymbol{\mathcal{A}} + \beta(\nabla m_z \times \hat{\boldsymbol{z}}) \cdot \boldsymbol{\mathcal{A}}, \tag{219}$$

$$F_{magnon} = \frac{1}{2} \int_\Omega \{\Phi\left(\frac{\boldsymbol{m}}{m_s}\right) + \frac{A'}{m_s^2}|\nabla \boldsymbol{m}|^2 - 2\mu_0 \boldsymbol{B} \cdot \boldsymbol{m}\} dx. \tag{220}$$

The ferromagnetic, spin-orbit magnetic exchange, and plasmon system we investigated prior were of different dimension number. To unify all these systems into one we consider the 1-dimensional geometry of a wire as was done in the Ch 2. This restricts our current

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to only flow along the x-axis $\boldsymbol{j} = j\hat{\boldsymbol{x}}$. Additionally we drop the inclusion of any applied external magnetic field, removing the last term from F_{magnon}. The combined free energy of our system is the following:

$$F = \int_{-\infty}^{\infty} dx \frac{n_s}{2m^*} \boldsymbol{\mathcal{A}}^2 + \alpha(\hat{\boldsymbol{z}} \times \boldsymbol{m}) \cdot \boldsymbol{\mathcal{A}} + \beta \left(\frac{\partial m_z}{\partial x} \hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}} \right) \cdot \boldsymbol{\mathcal{A}} + \frac{1}{2} \Phi \left(\frac{\boldsymbol{m}}{m_s} \right) + \frac{A'}{2m_s^2} \left(\frac{\partial}{\partial x} \boldsymbol{m} \right)^2 . \tag{221}$$

In limiting ourselves to this one dimensional geometry we trap the current to fit inside our wire of cross section S thus making the equation for the current the following:

$$\frac{I}{S} = \boldsymbol{j} \cdot \hat{\boldsymbol{x}} = \frac{n_s}{m^*} \mathcal{A}_x + \alpha(\hat{\boldsymbol{z}} \times \boldsymbol{m}) \cdot \hat{\boldsymbol{x}} + \beta(\partial_x m_z \hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}}) \cdot \hat{\boldsymbol{x}}. \tag{222}$$

Using these equations as our starting point will yield us a set of coupled equations. The equations will be the equations that governs the evolution of magnetization with regards to time based upon the Landau-Lifshitz equation and the equation that govern the evolution of current with regards to time based upon the continuity equation. Inserting plane waves into these equations leads us to a set of solutions where we can identify the dispersion relation of the plasmons and magnons of the system.

5.1.1 Relating E to I and $\boldsymbol{m}$

We start by connecting the electric field E to the gauge invariant vector potential $\boldsymbol{\mathcal{A}}$. The logic goes as following, consider the electric potential on a small scale, where it can be thought to be approximately linear, such that $\nabla\phi$ is a constant

$$\phi = \phi_0 + \nabla\phi x. \tag{223}$$

The phase of a wave function would evolve with the energy

$$\psi e^{-iEt} = \psi e^{-i\frac{\theta}{2}}. \tag{224}$$

Since there are no other sources of energy than the electric potential the energy would be given by the potential at a given position

$$-\frac{\theta}{2} = -\phi_0 t - \nabla\phi x t. \tag{225}$$

Taking the derivative and adding the vector potential $\boldsymbol{A}$ to make the left side gauge invariant

$$-\frac{\nabla\theta}{2} - \boldsymbol{A} = -\nabla\phi t - \boldsymbol{A}. \tag{226}$$

Then taking the time derivative and using that $\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$ per definition of the vector potential and electric potential and we get

$$-\frac{\partial}{\partial t} \left(\boldsymbol{A} + \frac{\theta}{2} \right) = \boldsymbol{E} = -\frac{\partial \boldsymbol{\mathcal{A}}}{\partial t}. \tag{227}$$

In Eq. 215 we have only the E-field in the x direction due to our limiting geometry. Inserting our findings we find the following continuity equation

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$$\frac{\partial I}{\partial x} - \int_{-\infty}^{\infty} dx' \lambda \left(x - x'\right) \frac{\partial^2 \mathcal{A}_x}{\partial t^2} \left(x', t\right) = 0. \tag{228}$$

We now simplify the equation for the current in our 1D system as shown in Eq. 222. Using that the triple products appearing in the equation can be circularly shifted we can manipulate it into a simpler form

$$\frac{I}{S} = \frac{n_s}{m^*} \mathcal{A}_x + \alpha(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}}) \cdot \boldsymbol{m} + \beta \partial_x m_z (\hat{\boldsymbol{x}} \times \hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{z}} = \frac{n_s}{m^*} \mathcal{A}_x - \alpha m_y. \tag{229}$$

Inserting this into Eq. 228 we can eliminate the dependence on $\mathcal{A}_x$ and make the equation one that only relates to the magnetization $\boldsymbol{m}$ and the current I

$$\frac{\partial I}{\partial x} - \int_{-\infty}^{\infty} dx' \lambda (x - x') \frac{m^*}{n_s} \left(\frac{1}{S} \frac{\partial^2 I}{\partial t^2} + \alpha \frac{\partial^2 m_y}{\partial t^2}\right) = 0. \tag{230}$$

5.1.2 Derivation of $\mathcal{H}$

To bring the Landau–Lifshitz equation into a solvable form we first need to derive the local field $\mathcal{H}$. We will obtain terms from both the coupled part of the free energy $F_{coupled}$ and the magnon part F_{magnon}. We will split these into two different contributions to $\mathcal{H}$. $\mathcal{H}_{coupled}$ and $\mathcal{H}_{magnon}$.

$$\mathcal{H}_{magnon} = \frac{A'}{2m_s^2} \frac{\partial^2 \boldsymbol{m}}{\partial x^2} - \frac{1}{2} \frac{\delta \Phi}{\delta \boldsymbol{m}} \tag{231}$$

is the magnon part of the effective local field and it is the same as in the magnon system, except the explicitly applied B-field is set to 0.

Given the geometry of the spin-orbit magnetic exchange system which has a bias towards the $\hat{\boldsymbol{z}}$ direction, as in that system the magnetism comes from a macroscopic impurity in the z-direction, our anisotropy term would take on the form:

$$\Phi \left(\frac{\boldsymbol{m}}{m_s}\right) = \Phi_z m_z^2 + \Phi_{xy}(m_x^2 + m_y^2) = m_z^2(\Phi_z - \Phi_{xy}) + \Phi_{xy} m_s^2. \tag{232}$$

This would yield a effective magnetic field $\mathcal{H}_{magnon}$:

$$\mathcal{H}_{magnon} = \frac{A'}{2m_s^2} \frac{\partial^2 \boldsymbol{m}}{\partial x^2} - m_z \hat{\boldsymbol{z}} \Phi_{xyz}, \tag{233}$$

where the microscopic nature of the anisotropy can be collected into a single parameter Φ_{xyz}. This retroactively justifies our use of an easy axis scheme. The coupled term is given by the following:

$$\mathcal{H}_{coupled} = -\frac{\delta}{\delta \boldsymbol{m}} \left(\int_{-\infty}^{\infty} dx \frac{n_s}{2m^*} \mathcal{A}^2 + \alpha(\hat{\boldsymbol{z}} \times \boldsymbol{m}) \cdot \boldsymbol{\mathcal{A}} + \beta \left(\frac{\partial m_z}{\partial x} \hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}}\right) \cdot \boldsymbol{\mathcal{A}}\right). \tag{234}$$

Just like before the functional derivative will pick out the function $\boldsymbol{m}$ at the matching variable x from the integral. This equation can be simplified by circularly shifting the triple product as to make $\boldsymbol{m}$ the vector that is dotted onto the cross product. Additionally since $\mathcal{A}$ is independent of $\boldsymbol{m}$, it is considered a constant with regards to the functional derivative

$$\mathcal{H}_{coupled} = \frac{\delta}{\delta \boldsymbol{m}} \left(-\alpha(\boldsymbol{A} \times \hat{\boldsymbol{z}}) \cdot \boldsymbol{m} + \beta \frac{\partial m_z}{\partial x} \mathcal{A}_y \right). \tag{235}$$

Performing partial integration on $\partial_x m_z$ we get the following:

$$\mathcal{H}_{coupled} = -\alpha(\boldsymbol{A} \times \hat{\boldsymbol{z}}) - \beta \frac{\partial A_y}{\partial x} \hat{\boldsymbol{z}} = -\alpha(\boldsymbol{A} \times \hat{\boldsymbol{z}}) = \alpha \mathcal{A}_x \hat{\boldsymbol{y}}, \tag{236}$$

where the last two equalities are given by the restriction to the one dimensional material

$$\mathcal{H}_{coupled} = \alpha \frac{m^*}{n_s} \left(\frac{I}{S} + \alpha m_y \right) \hat{\boldsymbol{y}}. \tag{237}$$

This yields us a combined $\mathcal{H}$ field of the following form and will let us write the Landau–Lifshitz equation in a form only dependant on I and $\boldsymbol{m}$

$$\mathcal{H} = \frac{A'}{2m_s^2} \frac{\partial^2 \boldsymbol{m}}{\partial x^2} - m_z \hat{\boldsymbol{z}} \Phi_{xyz} + \alpha \frac{m^*}{n_s} \left(\frac{I}{S} + \alpha m_y \right) \hat{\boldsymbol{y}}. \tag{238}$$

5.2 Solving the coupled equations for the magnons and plasmons

By inserting $\mathcal{H}$ into Eq. 218 and linearizing in m_z assuming an easy axis system we can arrive at a first order differential equation that together with the coupled equation for I will yield us the dispersion for the magnons and plasmons.

First we calculate what $\boldsymbol{m} \times \mathcal{H}$ is equal to as it shows up twice in the Lindau-Lifshitz equation

$$\boldsymbol{m} \times \mathcal{H} = (m_y \mathcal{H}_z - m_s \mathcal{H}_y)\hat{\boldsymbol{x}} + (m_s \mathcal{H}_x - m_x \mathcal{H}_z)\hat{\boldsymbol{y}} + (m_x \mathcal{H}_y - m_y \mathcal{H}_x)\hat{\boldsymbol{z}}. \tag{239}$$

Given that $\mathcal{H}_z$ is proportional with unity and that $\mathcal{H}_x$ and $\mathcal{H}_y$ are proportional with $I/m_x/m_y$ the cross product can be simplified to first order as:

$$\boldsymbol{m} \times \mathcal{H} = (m_y \mathcal{H}_z - m_s \mathcal{H}_y)\hat{\boldsymbol{x}} + (m_s \mathcal{H}_x - m_x \mathcal{H}_z)\hat{\boldsymbol{y}}. \tag{240}$$

To first order the only remaining terms in $\boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H})$ are the following:

$$\boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}) = -m_s(m_s \mathcal{H}_x - m_x \mathcal{H}_z)\hat{\boldsymbol{x}} + m_s(m_y \mathcal{H}_z - m_s \mathcal{H}_y)\hat{\boldsymbol{y}}. \tag{241}$$

Combining the two we find the Landau-Lifshitz equation to take the following form:

$$\begin{aligned}
\frac{d\boldsymbol{m}}{dt} =& -\gamma(m_y \mathcal{H}_z - m_s \mathcal{H}_y)\hat{\boldsymbol{x}} - \gamma(m_s \mathcal{H}_x - m_x \mathcal{H}_z)\hat{\boldsymbol{y}} \\
& - \gamma\epsilon(m_x \mathcal{H}_z - m_s \mathcal{H}_x)\hat{\boldsymbol{x}} - \gamma\epsilon(m_y \mathcal{H}_z - m_s \mathcal{H}_y)\hat{\boldsymbol{y}}.
\end{aligned} \tag{242}$$

Inserting the the relevant component of $\mathcal{H}$ we get the following two equations for m_x and m_y:

$$\begin{aligned}
\frac{dm_x}{dt} =& \gamma \left\{ m_y m_s \Phi_{xyz} + m_s \left[\frac{-A'}{2m_s^2} k^2 m_y + \alpha \frac{m^*}{n_s} \left(\frac{I}{S} + \alpha m_y \right) \right] \right\} \\
& + \gamma\epsilon \left\{ m_x m_s \Phi_{xyz} + m_s \left[\frac{-A'}{2m_s^2} k^2 m_x \right] \right\},
\end{aligned} \tag{243}$$

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$$\frac{dm_y}{dt} = \gamma \left\{ -m_s \left[\frac{-A'}{2m_s^2} k^2 m_x \right] - m_x m_s \Phi_{xyz} \right\}$$
$$+ \gamma \epsilon \left\{ m_y m_s \Phi_{xyz} + m_s \left[\frac{-A'}{2m_s^2} k^2 m_y + \alpha \frac{m^*}{n_s} \left(\frac{I}{S} + \alpha m_y \right) \right] \right\}. \tag{244}$$

Inserting plane waves for I, m_x, m_y, and completing the Fourier transform of $\lambda(x - x')$ we get a third equation of the form:

$$ikI - \frac{m^*}{n_s} \lambda(k) \frac{\partial^2}{\partial t^2} \left(\frac{I}{S} + \alpha m_y \right) = 0. \tag{245}$$

We resolve the differentials with regards to x as we have already used a plane wave form of I in completing the Fourier transform of $\lambda(x - x')$, there is therefore no choice in what form with regards to x that I, m_x, m_y must take.

We solve this problem by creating a matrix that can represent all our coupled equations as a single eigenvalue equation. Assuming now an exponential form of I, m_x, m_y with the notation as following:

$$m_x = \delta m_x e^{ikx - i\omega t},$$
$$m_y = \delta m_y e^{ikx - i\omega t}, \tag{246}$$
$$I = \delta I e^{ikx - i\omega t}.$$

The eigenvalue equation we seek looks like the following:

$$\frac{\partial}{\partial t} \boldsymbol{V} = -i\omega \boldsymbol{V} = M \boldsymbol{V}. \tag{247}$$

Our set of equations unfortunately carry second order derivatives with regards to time, thus not being directly compatible with an eigenvalue equation matrix representation of this form. We also notice that the term $I + S\alpha m_y$ shows up more often than I itself and thus serves as a better basis for our eigenvalue equation. We solve this issue by choosing a basis of m_x, m_y, g and h, with g and h being the following:

$$g = I + S\alpha m_y$$
$$h = \frac{\partial g}{\partial t}. \tag{248}$$

The eigenvalue equation can now be written with all time derivatives on the left side, with all mentions of second order derivatives being instead first order derivatives of h:

$$-i\omega \begin{pmatrix} m_x \\ m_y \\ g \\ h \end{pmatrix} = \begin{pmatrix} \gamma\epsilon m_s \Phi_{xyz} - \gamma\epsilon\Pi k^2 & \gamma m_s \Phi_{xyz} - \gamma\Pi k^2 & \gamma m_s \alpha \frac{m^*}{n_s S} & 0 \\ -\gamma m_s \Phi_{xyz} + \gamma\Pi k^2 & \gamma\epsilon m_s \Phi_{xyz} - \gamma\epsilon\Pi k^2 & \gamma\epsilon m_s \alpha \frac{m^*}{n_s S} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{n_s}{m^*} S^2 v' \alpha k^2 & -\frac{n_s}{m^*} S v' k^2 & 0 \end{pmatrix} \begin{pmatrix} m_x \\ m_y \\ g \\ h \end{pmatrix}. \tag{249}$$

Introducing the following shorthand for the spin to spin coupling

$$\Pi = \frac{A'}{2m_s}, \tag{250}$$

and writing the function $\lambda(k)$ as

$$\lambda(k) = \frac{1}{ikv'}. \tag{251}$$

Where v' is equal to the approximately constant Fourier transform of the localizing function that makes up the potential function of the 1D wire from Ch. 2, for low k values (Eq. 58). In that section it was called $\alpha(x)$, but since we found it to be proportional to the velocity of the plasmon we here call it v'.

Solving this eigenvalue equation numerically we get the graph found in Fig. 6. Comparing this to the same eigenvalue equation but with $\alpha = 0$, seen on Fig. 7 making the magnon and plasmon modes not coupled to each other we see a distinct repelling of plasmon and magnon dispersion relations at the crossover points creating a gap. The states are unmarked in Fig. 6 due to the mixing of the states near the crossing point. The green solution for example starts out as a clearly plasmon solution that then evolves into a mixture and finally ends up becoming the magnon dispersion.
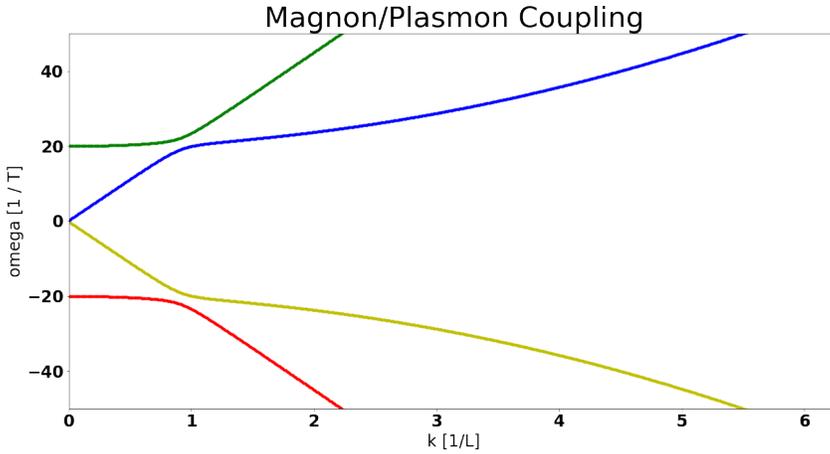
Figure 6: *Numerical solution for ω for Eq. 249. The real part of ω is plotted against the value of k. A small offset offset from $\omega = 0$ is given via an anisotropy term which affects the parabolic magnon modes. The plasmon mode is unaffected and linear. This plot has $\epsilon = 0$ so that the magnon mode does not decay naturally, this also results in no decay as ω remains real. A small coupling is present, melding the plasmon with the magnon when the dispersion relations cross.*
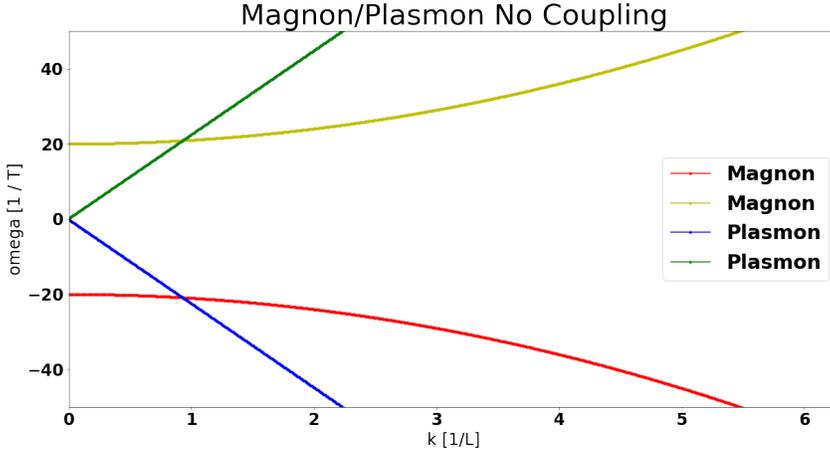
Figure 7: *Magnon/Plasmon dispersion relation without without coupling $\alpha = 0$ and without decay $\epsilon = 0$*

6   Conclusions and Outlook

We started by looking at plasmons independently, then plasmons interacting with a magnetic exchange field via spin-orbit coupling, then magnons independently and finally a combined system that showcases both plasmon and magnon behaviour and their coupling between each. The result culminating in the results of the eigenvalue equation given by Eq. 249.

From this eigenvalue equation we solve for the eigenstates of the coupled system and find Fig. 6. We see by this figure that a gap forms between the magnons and plasmon states where they would have crossed in their dispersion relation. We keep the magnon and plasmon properties for low values of k and high values of k, where the plasmons and magnons are relatively unaffected by the coupling and the eigenstates remain pure magnons and plasmons, however close to their crossing point, where the splitting appears the eigenstates take on that of magnon/plasmon hybrid modes.

This has interesting applications. If one were to create for example a magnon at a momentum close to the crossing point and send it into the material we would find that it would no longer be an eigenstate of the system. Due to this the magnon would not be conserved over time as it would be compromised of different eigenstates that would evolve at different rates in time. One could imagine sending in a magnon and by engineering the material to have a specific length and as such a specific exposure time for the wave packet to this areas modified Hamiltonian. One could flip the magnon into a plasmon. This would require precise knowledge of the energy of the different eigenstates of the coupled systems and also of their composition in the basis of plasmons/magnons. This however is exactly what our analysis in the final chapter of this thesis provides. Analytical form for the composition and energy of plasmon/magnon hybridized eigenstates.

One would have to take into consideration that the group velocity for the different parts of the input wave packet would differ based upon which eigenstate they belong to as the dispersion relations are different. This knowledge however helps us understand what kind of drift we would see of the wave packet as it passes through the medium and help us understand what fidelity our magnon to plasmon converter would have. If the states

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inside the material move away from each other to a large extent we would obviously have a breakdown of our methods as we could no longer ensure a cancellation of magnon on the output side by addition of multiple eigenstates. This would leave the magnon to plasmon converter flawed. Thankfully the knowledge needed to further work on this issue is at hand as it is included in the dispersion relation. For each eigenstate the group velocity at any given wavevector k can be computed and from it we would understand how to tweak the parameters of our system to optimize for this issue and increase fidelity.

In fact with this knowledge one could even send in multiple packets with the express goal of matching up different ones at the end of the material to get the desired output.

There is however still further work to be done. Without even considering the work of explicitly working out the calculations for magnon to plasmon flopping or the spreading of wave packets there is one glaring issue that has yet to be addressed. In the consideration of a magnon to plasmon converter that starts off with a ferromagnetic domain then turns into a superconducting domain one would obviously have to consider scattering. When the collective modes reach borders we would have parts of the packet reflect back and parts of the wave transfer through. No calculations or estimates of this has been made in this thesis and much work could still be done on this subject. In fact it would probably prove vital to the application of this work. Both analytical and numerical methods could be used to provide results in this department yet there was not time to do either in this work.

Appendices

A Numerical Treatment of Magnons

The most straightfoward approach to simulating the magnetization would be one that simulates the next term based upon the previous one as following.

$$\boldsymbol{m}_{n+1} = \boldsymbol{m}_n + \Delta t \frac{d\boldsymbol{m}_n}{dt} \tag{252}$$

Where $\frac{d\boldsymbol{m}_n}{dt}$ would be sourced from the Landau-Lifshitz equation as written in 194. Using the effective local field as found in Eq. 202. This approach is called the forward Euler scheme and has been shown to lack "stability" (what does that mean?) [3]. Due to this we will be using a implicit Gauss-Seidel scheme as presented in Xiao-Ping 2001 [3].

The easiest way to showcase the principal of this scheme is to do it for a simple differential equation for the magnetization with regards to time. So an example using the following undamped application of just a B-field will be used.

$$\frac{d\boldsymbol{m}}{dt} = \boldsymbol{a} \times \boldsymbol{m} \tag{253}$$

The implicit Gauss-Seidel scheme would in this case look like the following

$$\begin{pmatrix} m_1^{n+1} \\ m_2^{n+2} \\ m_3^{n+3} \end{pmatrix} = \begin{pmatrix} m_1^n + \Delta t(a_y m_3^n - a_z m_2^n) \\ m_2^n + \Delta t(a_z m_x^{n+1} - a_x m_3^n) \\ m_3^n + \Delta t(a_x m_2^{n+1} - a_y m_1^{n+1}) \end{pmatrix} \tag{254}$$

As you can see at the heart of the scheme is the use of the newly calculated x magnetization to calculate the y magnetization and the newly calculated y magnetization to calculate the z magnetization. This procedure has been shown to lead to more precise results that is less sensitive to larger time steps Δt [3]. Because of this we will be using this implicit Gauss-Seidal scheme instead of the straightforward forward Euler scheme.

A.1 Numerical Laplacian

In our equation for $\mathcal{H}$ we have to deal with more complicated terms than constant magnetic fields. One of these terms that is trivial in analytical calculations but not quite so straight forward in numerics is the laplacian ∇^2. In our calculations we follow the five point approximation making the laplacian out to be a matrix of the following form.

$$\nabla^2 m(x) = \frac{1}{12h^2}\left(-m(x-2h) + 16m(x-h) - 30m(x) + 16m(x+h) - m(x+2h)\right) \tag{255}$$

Where h is lattice distance by which the points are spaced. Since we set this lattice distance to be 1 our laplacian is as following

$$\nabla^2 m(x) = \frac{1}{12}\left(-m(x-2) + 16m(x-1) - 30m(x) + 16m(x+1) - m(x+2)\right) \tag{256}$$

When written as a matrix that takes the laplacian of an entire set of magnetizations defined at all the lattice points it takes the form seen in Fig. 8.

```
%precision 2
LAP
```

```
array([[-2.5 ,  1.33, -0.08, ...,  0.  , -0.08,  1.33],
       [ 1.33, -2.5 ,  1.33, ...,  0.  ,  0.  , -0.08],
       [-0.08,  1.33, -2.5 , ...,  0.  ,  0.  ,  0.  ],
       ...,
       [ 0.  ,  0.  ,  0.  , ..., -2.5 ,  1.33, -0.08],
       [-0.08,  0.  ,  0.  , ...,  1.33, -2.5 ,  1.33],
       [ 1.33, -0.08,  0.  , ..., -0.08,  1.33, -2.5 ]])
```

Figure 8: *The Laplacian matrix*

Additionally we will take advantage of a fractional step procedure to handle the term

$$\frac{d\boldsymbol{m}}{dt} = -\boldsymbol{m} \times \nabla^2 \boldsymbol{m} \tag{257}$$

We consider a middle step $\boldsymbol{m}^*$ that follows the following.

$$\frac{\boldsymbol{m}^* - \boldsymbol{m}^n}{\Delta t} = \nabla^2 \boldsymbol{m}^* \tag{258}$$

and let our scheme take the following form

$$\boldsymbol{m}^{n+1} = \boldsymbol{m}^n - \boldsymbol{m}^n \times \boldsymbol{m}^* \tag{259}$$

Reverse engineering we get the following for $\boldsymbol{m}^*$

$$\boldsymbol{g}_n = (I - \Delta t \nabla^2)^{-1} \boldsymbol{m}^n \tag{260}$$

Taking on an implicit Gauss-Seidel form this looks like the following

$$\begin{pmatrix} m_1^{n+1} \\ m_2^{n+2} \\ m_3^{n+3} \end{pmatrix} = \begin{pmatrix} m_1^n - (m_2^n g_3^n - m_3^n g_2^n) \\ m_2^n - (m_3^n g_1^{n+1} - m_1^{n+1} g_3^n) \\ m_3^n - (m_1^{n+1} g_2^{n+1} - m_2^{n+1} g_1^{n+1}) \end{pmatrix} \tag{261}$$

Substituting this with $\mathcal{H}$ from our analytics we get the following for the local field in our simulation. Incorporating the time step into it and updating it component wise after every component of $\boldsymbol{m}^n$ has been calculated

$$\mathcal{H}^n = A\boldsymbol{g}^n - \Delta t \boldsymbol{B} + \Delta t \Phi_{xyz} m_z \hat{\boldsymbol{z}} \tag{262}$$

Where A and $\boldsymbol{B}$ absorb all the physical constants required. The total scheme is thus the following

$$\begin{pmatrix} m_1^{n+1} \\ m_2^{n+2} \\ m_3^{n+3} \end{pmatrix} = \begin{pmatrix} m_1^n - (m_2^n \mathcal{H}_3^n - m_3^n \mathcal{H}_2^n) \\ m_2^n - (m_3^n \mathcal{H}_1^{n+1} - m_1^{n+1} \mathcal{H}_3^n) \\ m_3^n - (m_1^{n+1} \mathcal{H}_2^{n+1} - m_2^{n+1} \mathcal{H}_1^{n+1}) \end{pmatrix} \tag{263}$$

Using the same numerical $\mathcal{H}$ we also calculate decay after every component is calculated and subtract decay component wise. We run a low value for ϵ such that we have decay to $1/10$ over hundreds of oscillations.

A.2 variable time step procedure

We initialize the simulation with a given k across the sites, trying to follow inline with our analytics as best. This simulation is able to work for any values of $\boldsymbol{m}$ but since we can only compare it to the linearized model we try to run it in a regime where the linearized model works so that the two can be compared.

$$
\begin{aligned}
m_z(l) &= 1 \\
m_y(l) &= 0.05\cos(kl) \\
m_x(l) &= 0.05\sin(kl) \\
\boldsymbol{m}(l) &= \frac{\boldsymbol{m}(l)}{|\boldsymbol{m}(l)|}
\end{aligned}
\tag{264}
$$

We initialize with the large majority of the magnetization in the z-direction and small amounts rotated around as we go along the sites denounced by l here with a wavevector k. Finally we normalize the magnetization at each site.

To decrease runtime of the simulation we vary the timestep Δt after each iteration of k we initialized it with. We want to run it over a spectrum of initial k plane waves to grab a full dispersion relation. After each iteration we measure the dispersion relation and then tune the timestep to optimize the runtime of the next k value. We do this inversely so that $\Delta t^{k+1} \propto \frac{1}{\omega^k}$. The reasoning being that if the frequency is high we need a smaller timestep to get the granularity in the time domain to measure variances in the magnetization.

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