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Master Thesis, Physics

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Dephasing of Majorana Box Qubits

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Abstract

The Majorana box qubit [2] is an exciting proposal for realising a qubit using Majorana bound states. Majorana bound states are delocalised over potentially big distances, and for that reason the box qubit is expected to have a long coherence time in the face of local noise. As of yet there has been no concrete study of how long the coherence time is.
After a review of Majorana physics and in particular the box qubit, the main work in this thesis is to provide the study of this. This is done modelling the noisy environment as a capacitative coupling from box qubit to an environment impedance, resulting in a fluctuating potential. It is then demonstrated that these fluctuations lead to decoherence of the box qubit. The main result is an expression for the instantaneous Majorana propagator, the decay of which is shown to be a measure of information loss in the system. For systems where the gap energy in the Majorana system is much larger than the characteristic energy scale in the electric circuit we find an expression for the dephasing rate of the qubit, which suggests long life time for the qubit.

Referat

En af de helt store mål med forskning i moderne faststoffysik er at undersøge muligheden for at konstruere kvantebits, såkaldte qubits, som danner grundstenene i fremtidens kvantecomputere. En af udfordringerne ved at bygge en god qubit er, at den skal være godt isoleret fra støj fra omgivelserne, som ellers kan udviske informationen og lede til fejl i kvantecomputerens algoritmer. Et forslag på en qubit, der forventes at være modstandsdygtig over for sådan støj kaldes for en Majorana box qubit [2]. Denne udnytter bestemte delokaliserede bundne tilstande, kaldet Majorana fermioner, som kan findes i mikroskopiske nanowires, under særlige omstændigheder. Majorana fermionerne forventes ikke at blive let påvirket af lokale vekselvirkninger og støj. Det er ikke før undersøgt, hvor lang levetiden er for en Majorana box qubit, når den udsættes for støj. I dette speciale bliver de første skridt taget mod at forstå dette.
Hoved resultatet består i et udtryk for hvor hurtigt information lagret i en Majorana box qubit går tabt. Den udledes ved at antage, at box qubiten er koblet kapacitativt til en omgivelsesimpedans, hvilket leder til de potentialfluktuationer der driver informationstabet . Når gabsenergien i nanowiren er meget større end de karakteristiske energier i det elektriske kredsløb, bliver der fundet et udtryk for levetiden, som indikerer en lang levetid for Majorana box qubiten.

Contents

1 Introduction

The last decades have seen a big shift of attention in the condensed matter community towards the emerging field of topological phases. The novelty of the phases lies both at a fundamental level, since they are phases that cannot be described by order parameters and broken symmetries, but also in the potential technological applications of them. One of the topological phenomena that has undergone remarkable progress in the last 10 years is the field of Majorana bound states. These exotic states appear as edge modes in certain one dimensional systems, and they have the exciting property that two such Majorana bound states together can form a fermionic state at zero energy, which is potentially delocalised over very long distances.
For about 30 years now one of the holy grails of condensed matter research has been to construct a reliable quantum computer, and many are now hopeful that Majorana bound states may hold the key to doing so.Though theoretically well understood, the quantum computer has hitherto escaped realisation because of the problem of decoherence. As information is saved and manipulated in microscopic "qubits", the macroscopic world around them can easily disrupt the fragile quantum states, corrupting the information. The new hope that has sparked major interest in Majorana research within the industry is that the non-local nature of the Majorana bound states provide a natural protection from such decoherence effects.
A particularly useful application of Majorana bound states is the so-called Majorana box qubit, first proposed by Plugge et al. [2]. The box qubit is appealing experimentally since it is probably the simplest way to set up a qubit made out of Majorana bound states. Other proposals include complicated wire networks where the Majoranas are braided around one another in order to manipulate them. But the box qubit involves only parallel nanowires with quantum dots coupled to the vaious ends, and manipulating it involves tuning gates at the dots.
When two Majoranas are well seperated, they are more resilient to local noise, but there seems to be a widespread misconception that Majoranas are completely immune to local noise. This is not the case, however such noise will need to be sufficiently energetic, since it will involve exciting the zero energy state into highly energetic continuum state. In any realistic implementation of the box qubit there will be electric noise from the gates and other electronic components in the device. This thesis studie how such electric potential fluctuations in the box qubit can lead to decoherence. To the best of my knowledge this has not been done yet, and thus can help shed light on which time scales the box qubit has to be operated to prevent erroneous computations, and whether the box qubit is viable as a basic constituent of a quantum computer. The fluctuations are included classically by adding a time dependent function $\phi(t)$ to the chemical potential h. Figure 1 shows a schematic plot of the zero energy wave function in the nanowire. At different times the wave function is shifted ever so slightly. If $\phi(t)$ varies slowly enough, the adiabatic theorem tells us that nothing interesting happens; the state just follows the instantaneos zero energy state. If however there is a quick diabatic change, then the two states will not overlap completely anymore, and therefore the new state must have amplitudes in the above-gap states. If the zero mode was initially occupied and one wanted later to do a readout of the instantaneous zero mode occupancy, then there is a small probability that the state will collapse to a vacant zero energy state is vacant and a continuum state is occupied instead, leading to an error in the readout. As time passes all these small diabatic changes in $\phi(t)$ add up, leading eventually to a considerable probability for errors to occur at the readout. The function $\phi(t)$ is unknown, but in the end we will assume that it is distributed as a Gaussian so that we can average over it. In order to do so we will model the potential $\phi(t)$ as potential flucutations in a capacitor which is coupled to an environment, modelled simply as an impedance. The box qubit feels the potential fluctuations, but dephasing actually depends not on ϕ but on the time derivative $\dot{\phi}$, since it is a non-adiabatic process. So in the end what we need to average over turns out to be $\dot{\phi}$.

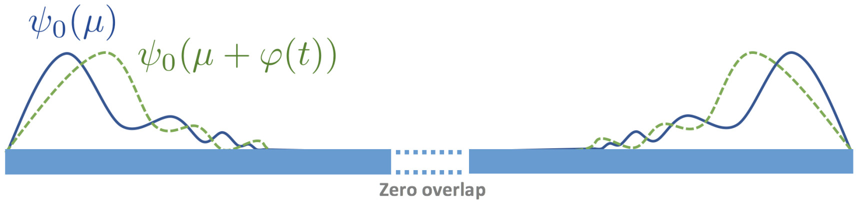


Figure 1: Schematic plot of the zero energy wave functions in a very long wire. As the potential fluctuations are added the zero mode shifts about. Diabatic changes therefore entails non-zero overlaps with continuum states.

The question we want to answer is the following: If we initially prepare some state what is the chance of finding the same state at some later time t. In order to make the question tractable we assume that it is possible to measure the the state of the Majorana box qubit on a time scale which is much shorter than the decoherence time, so we may ignore the fluctuations during the measurement procedure.
In order to answer this question we will need to understand several things about the box qubit and the effect of adding the time dependence to it. We are going to use the Hamiltonian for the so-called p-wave supersuperconductor as our model for the topological nanowire. This Hamiltonian is introduced in Section 2 along with a brief introduction to the properties of Majorana bound states. In this section we also go through how to obtain the p-wave superconductor from the actual physical ingredients that are used in the lab, namely semiconductors with a big spin-orbit coupling, magnetic fields and proximitisation by ordinary s-wave superconductors. The calculation used to conclude how the addition of $\phi(t)$ to the chemical potential manifests itself in the Hamiltonian for the p-wave superconductor. Lastly in this section will be a short introduction to the concept of Coulomb blockade, which plays a central role in the physics of the box qubit. In Section 3 we will see how one may read read out the state of two Majorana bound states by coupling a dot to them. The trick with reading out the Majorana states is always to break the degeneracy between the occupied and the empty state of the fermionic state they constitute. With the dot the way it works is that depending on whether the fermionic state is occupied or not an electron on the dot can either tunnel into the state or not, thus potentially winning kinetic energy. Once we understand the read out protocol we will continue by seeing in Section 4 what happens when we add the time dependent potential $\phi(t)$ to the Hamiltonian for the p-wave superconductor. As expected this turns out to generate new terms connecting the zero mode to the continuum. In this section we will also discuss the relevant quantity we should calculate to get a good measure of the decoherence. Maybe not surprisingly the quantity turns out to involve the instantaneous zero modes. More precisely we will see that we need to calculate the propagator for the instantaneous Majorana operators. The instantaneous Majorana operators are addressed and derived in Section 5. After that in Section 6 we will find the eigenstates of the p-wave superconductor Hamiltonian H_{pw} and calculate explicitly the new term connecting the zero mode to the continuum. Then in Section 7 we calculate $G_{\dot{\phi}\dot{\phi}}(t)$ by choosing a particular model for capacitatively coupling the box qubit to the environment. Finally in Section 8 we put all the pieces together and calculate the propagator for the instantaneous zero mode, which then gives us an expression for how the box qubit dephases.
So without further preamble let's get to it and start with a brief introduction to Majorana bound state.

2 Majorana bound states and the properties of Coulomb blockaded nanowires

Majorana bound states are even chargeless mixtures of electrons and holes at zero energy, and as such are their own antiparticle. In this way they are the condensed matter analogues of the Majorana fermions which have long been sought in high energy physics. It is maybe not surprising that they can be found in a type of superconductors, since quasiparticle excitations in superconductors are mixtures of electrons and holes. The trick is just to find a pairing that allows for an exactly equal mixture. Since the Majorana bound states are their own antiparticles, they have have hermitian annihilation/creation operators, which we denote by γ. Given any fermion annihilation operator c, we may form such hermitian Majorana operators as

$$\begin{aligned} \gamma_1 &= c^\dagger + c \\ \gamma_2 &= i(c^\dagger - c). \end{aligned} \tag{2.1}$$

Importantly the Majorana operators satisfy the following algebra:

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j} \tag{2.2}$$

One can think of Majorana operators as just a basis change for regular Dirac fermions. Inverting (2.1) one sees

$$\begin{aligned} c &= \frac{\gamma_1 + i\gamma_2}{2} \\ c^\dagger &= \frac{\gamma_1 - i\gamma_2}{2}. \end{aligned} \tag{2.3}$$

and this can be done for any Dirac fermion. This means that a system of N fermionic states can equivalently be thought of as a system of $2N$ Majoranas. It is not always useful to use this basis. But when two Majoranas are spacially well seperated, and their combined Dirac fermion is at zero energy the writing is relevant. This is because if the Majorana operators commute with the Hamiltonian, then they have trivial dynamics. If they are not well seperated, then the degeneracy between the occupied and empty state of their combined Dirac fermion is split.
If the two are brought completely together they may then "fuse" into either nothing or a regular Dirac fermion. Thus bringing together two Majorana bound states constitutes a measurement of their combined Dirac fermion state. In general the degeneracy has to be lifted some way or another in order to do a readout of their state.

Majorana bound states generally occurs only in systems where an odd number of bands have been filled. The reason for this is that each such filled band may provide two Majorana bound state in either side of the system. If an even number of bands do so, then the Majoranas at either end will have a tendency to simply fuse. For this reason, Majorana bound states tend to appear only in spinless superconductors.

The occupation operator for a fermionic state may be written in terms of its Majorana operators:

$$c^\dagger c = \frac{1}{4}(\gamma_1 - i\gamma_2)(\gamma_1 + i\gamma_2) = \frac{1}{2}(1 + i\gamma_1\gamma_2). \tag{2.4}$$

We denote $i\gamma_1\gamma_2$ by σ_z in the occupation basis of the Dirac fermion, and it has the eigenvalues $+1$ for the empty state $|0\rangle$ and the eigenvalue -1 for the occupied state $|1\rangle$. Thus it counts the parity of the state, and it is often called the parity operator of the two Majoranas. For a system consisting of multiple Majoranas, a state cannot be both an eigenstate of $i\gamma_i\gamma_j$ and $i\gamma_i\gamma_k$ with $j \neq k$, since

$$i\gamma_i\gamma_j i\gamma_i\gamma_k = -i\gamma_i\gamma_k i\gamma_i\gamma_j.$$

It is a useful fact that Majorana operators anticommute with Dirac fermions which are composed only of different Majoranas.

2.1 The 1D p-wave superconductor

As mentioned above Majorana bound states appear in spinless superconductors. Since fermions are odd under exchange, when there is no spin the combined spatial wavefunctions of Cooper pair electrons has to be odd. This implies that the superconducting pairing will need to be odd in momentum. An example of such a Hamiltonian is the following which we will study throughout this thesis (setting $\hbar = 1$):

$$H_{\text{pw}} = \begin{pmatrix} \frac{p^2}{2m} - \mu & \Delta p \\ \Delta p & -\frac{p^2}{2m} + \mu \end{pmatrix} = (\frac{p^2}{2m} - \mu)\tau_z + \Delta p \tau_x. \tag{2.5}$$

This Hamiltonian is written in particle-hole space, so τ_z is the operator that switches a hole to an electron and vice-versa. μ is the gap energy, m is the effective mass of the electrons and Δ is a pairing parameter between electrons and holes with dimensions of velocity, which we can take to be real. In Section 6 we will solve for the eigenstates of H_{pw}, taking the system to be half infinite, with the left end of the wire at $x = 0$ and the right end at $x = \infty$. As we will see, for certain ranges of parameters H_{pw} has a single zero energy eigenstate localised near $x = 0$. This corresponds to the wavefunction $\psi_L(x)$ for the left most Majorana bound state. But how does one convince oneself that the zero energy solutions correspond to Majorana bound states? In order to see this, let us note that H_{pw} has a chiral symmetry. That is, there exists an anti-unitary operator C that anticommutes with H_{pw}. That anti-unitary operator is

$$C \equiv \tau_x K, \tag{2.6}$$

where K is complex conjugation and $\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. To see that it anti-commutes with H_{pw}, we use that that $Kp = -p$, and get

$$\begin{aligned} H_{\text{pw}}C &= (\frac{p^2}{2m} - \mu)i\tau_y K + \Delta p K \\ CH_{\text{pw}} &= (\frac{p^2}{2m} - \mu)(-i)\tau_y K - \Delta p K, \end{aligned} \tag{2.7}$$

whereby $\{H_{\text{pw}}, C\} = 0$. This means that if ψ is an eigenstate with energy E, then $\psi' \equiv C\psi$ will be an eigenstate too with energy $-E$:

$$H_{\text{pw}}\psi' = -CH_{\text{pw}}\psi = -E\psi'. \tag{2.8}$$

Now consider the zero energy solution ψ_L. From the above we see that $\psi'_L = C\psi_L$ should also be a solution at zero energy, but as only one solution at zero energy is found we must have that $\psi'_L \propto \psi_L$, being equivalent up to a gauge factor. Let us denote the electron annihilation operator by $\hat{\Psi}$ and write $\psi_L = \begin{pmatrix} \psi_L^e \\ \psi_L^h \end{pmatrix}$. Then $\psi'_L = \begin{pmatrix} \psi_L^{h*} \\ \psi_L^{e*} \end{pmatrix}$, which means that

$$\begin{aligned} \psi_L^{h*} &= g\psi_L^e \\ \psi_L^{e*} &= g\psi_L^h. \end{aligned} \tag{2.9}$$

If we write the field operators related to ψ_L as $\hat{\Psi}_L$, then

$$\hat{\Psi}_L = \int_0^\infty dx \begin{pmatrix} \psi_L^e(x) & \psi_L^h(x) \end{pmatrix} \begin{pmatrix} \hat{\Psi}(x) \\ \hat{\Psi}^\dagger(x) \end{pmatrix} = \int_0^\infty dx (\psi_L^e(x)\hat{\Psi}(x) + \psi_L^h(x)\hat{\Psi}^\dagger(x)), \tag{2.10}$$

and on the other hand

$$\hat{\Psi}'_L = \int_0^\infty dx (\psi_L^{h*}(x)\hat{\Psi}(x) + \psi_L^{e*}(x)\hat{\Psi}^\dagger(x)) = \hat{\Psi}_L^\dagger. \tag{2.11}$$

But then we may use (2.9) to write

$$\hat{\Psi}_L^\dagger = g\hat{\psi}_L, \tag{2.12}$$

and after doing a gauge transformation we would have that the field operator related to $\psi_L(x)$ is hermitian, and therefore is a Majorana. Henceforth I will denote $\hat{\Psi}_L = \gamma_1$ and $\hat{\Psi}_R = \gamma_2$.

A neat property of the zero modes is that they topologically protected. What this means is that if we do continuous deformations of H_{pw}, then there is no way to get rid of the zero mode without also closing the gap. Converserly if the parameters are such that there is no zero mode, then it also cannot appear without closing the gap. The reason for this is that as H_{pw} deformed continuously, the energy of all individual eigenstates are deformed continuously as well. During the deformation the dimension of the Fock space cannot change, but if the zero mode were to move away from zero and acquire some energy ϵ, then by the chiral symmetry there also had to be a new state at energy $-\epsilon$. This is impossible as it adds a new state to the Fock space. The only way to add or remove the existence of the zero mode is for a state to come down from the continuum and come up again.

H_{pw} may be diagonalised in terms of quasi particle operators α_k, and doing so it takes the form

$$H_{\text{pw}} = \sum_k E_k \alpha_k^\dagger \alpha_k. \tag{2.13}$$

The Majorana operators are absent in (2.13), since they are at zero energy. It will be useful to express the original electron operators $\hat{\Psi}$ in terms of the γ and the α_k operators. I won't need the precise expansion cofficients, and I will only need it in terms of the semi-infinite wire, where γ_2 is absent. In principle the coefficients can be found by solving for the wavefunctions $\psi_k(x)$ above the gap and then formally inverting the following two equations

$$\begin{aligned} \gamma_1 &= \int_0^\infty \psi_L(x) \begin{pmatrix} \hat{\Psi}(x) \\ \hat{\Psi}^\dagger(x) \end{pmatrix} \\ \alpha_k &= \int_0^\infty \psi_k(x) \begin{pmatrix} \hat{\Psi}(x) \\ \hat{\Psi}^\dagger(x) \end{pmatrix}. \end{aligned} \tag{2.14}$$

The result is written as

$$\psi(x) = A_1(x)\gamma_1 + \sum_k (u_k(x)\alpha_k + v_k(x)\alpha_k^\dagger). \tag{2.15}$$

2.2 Realising a p-wave superconductor

One may rightly how (2.5) may be realised in practice. Oreg [8] and Lutchyn [9] independently came up with a good answer. They proposed to use semiconductor wires with a strong spin-orbit coupling and Zeeman splitting which are proximitised by regular s-wave superconductors. If one projects on the two lowest energy levels in this system, then one obtains the Hamiltonian (2.5). As was mentioned in the introduction, we want to study the effect of potential fluctuations on the Majoranas in the p-wave superconductor. The effect of this is shifting the chemical potential h to $h+\phi(t)$. In Section 4 when we start studying the effects of this we need to know how to add this to the Hamiltonian, which is not immediately clear when starting from (2.5). Because of this we will now go through the story of how the p-wave superconductor emerges as an effective low energy theory from the above mentioned ingredients. We consider a system of electrons in 1D, with Rashba spin-orbit coupling parameter λ, Zeeman splitting B and real proximitised pairing δ.[1] Let σ_i be Pauli matrices in spin space and let τ_i be Pauli matrices in particle/hole space. Then the BdG Hamiltonian becomes

$$\begin{aligned} H &= (\frac{p^2}{2} - h)\sigma_0\tau_z + B\sigma_z\tau_0 + \lambda(\mathbf{p}\times\hat{\mathbf{z}})\bar{\sigma}\tau_z + \delta\sigma_0\tau_x \\ &= \begin{pmatrix} (\frac{p^2}{2m} - h)\sigma_0 + B\sigma_z + \lambda p\sigma_y & \delta\sigma_0 \\ \delta\sigma_0 & -(\frac{p^2}{2m} - h)\sigma_0 + B\sigma_z - \lambda p\sigma_y \end{pmatrix}. \end{aligned} \tag{2.16}$$

The basis that the above Hamiltonian is written in has

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |\psi^e_\uparrow\rangle, \qquad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = |\psi^e_\downarrow\rangle \qquad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = |\psi^h_\downarrow\rangle \qquad \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |\psi^h_\uparrow\rangle. \tag{2.17}$$

The eigenstates of H are plane waves so we write them as

$$\psi(x) = e^{ikx}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}, \tag{2.18}$$

and letting H act on $\psi(x)$ yields a k-dependent Hamiltonian H_k given by

$$H_k = \begin{pmatrix} (\frac{k^2}{2m} - h)\sigma_0 + B\sigma_z + \lambda k\sigma_y & \delta\sigma_0 \\ \delta\sigma_0 & -(\frac{k^2}{2m} - h)\sigma_0 + B\sigma_z - \lambda k\sigma_y \end{pmatrix} \tag{2.19}$$

The strategy is next to find a unitary transformation that diagonalizes the block diagonal

$$\begin{aligned} H_{0k} &\equiv H_k - \begin{pmatrix} 0 & \delta\sigma_0 \\ \delta\sigma_0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{k^2}{2m} - h + B & -i\lambda k & 0 & 0 \\ i\lambda k & \frac{k^2}{2m} - h - B & 0 & 0 \\ 0 & 0 & -\frac{k^2}{2m} + h + B & i\lambda k \\ 0 & 0 & -i\lambda k & -\frac{k^2}{2m} + h - B \end{pmatrix} \end{aligned} \tag{2.20}$$

$$\equiv \begin{pmatrix} H^e_{0k} & 0 \\ 0 & H^h_{0k} \end{pmatrix}. \tag{2.21}$$

The eigenvalues $E^e_\pm$ of H^e_{0k} satisfy

[1] The reason that I choose these symbols is that I want to reserve μ to mean the gap and Δ to be the pairing in (2.5) and α will be a dimensionless parameter later.

$$\Big(\frac{k^2}{2m} - h - E^e_\pm\Big)^2 - B^2 - \lambda^2 k^2 = 0$$
$$\Rightarrow E^e_\pm = \frac{k^2}{2m} - h \pm K, \tag{2.22}$$

where $K = \sqrt{B^2 + \lambda^2 k^2}$. Similarly the eigenvalues $E^h_\pm$ of H^h_{0k} satisfy

$$\Big(-\frac{k^2}{2m} + h - E^h_\pm\Big)^2 - B^2 - \lambda^2 k^2 = 0$$
$$\Rightarrow E^e_\pm = -\frac{k^2}{2m} + h \mp K. \tag{2.23}$$

The eigenstates of H^e_{0k} and H^h_{0k} are denoted $\psi^e_{k\pm}$ and $\psi^h_{k\pm}$ respectively. Writing $\psi^e_{k-} = \begin{pmatrix} a \\ b \end{pmatrix}$, we get

$$\Big(\frac{k^2}{2m} - h + B\Big)a - i\lambda k b = E^e_- a$$
$$\Rightarrow b = \frac{1}{i\lambda k}\Big(\frac{k^2}{2m} - h + B - E^e_-\Big)a = \frac{1}{i\lambda_k}(B + K\Big)a, \tag{2.24}$$

and so

$$\psi^e_{k-} = \frac{1}{\mathcal{N}^e}\begin{pmatrix} i\lambda k \\ B + K \end{pmatrix}$$
$$\psi^e_{k+} = \frac{1}{\mathcal{N}^e}\begin{pmatrix} B + K \\ i\lambda k \end{pmatrix}, \tag{2.25}$$

where $\mathcal{N}^e = \sqrt{(B+K)^2 + \lambda^2 k^2}$. Likewise we write $\psi^h_{k-} = \begin{pmatrix} c \\ d \end{pmatrix}$ and we see

$$\Big(-\frac{k^2}{2m} + h + B\Big)c + i\lambda k d = E^h_- a$$
$$\Rightarrow d = \frac{1}{-i\lambda k}\Big(-\frac{k^2}{2m} + h + B - E^h_-\Big)c = \frac{1}{-i\lambda k}(B - K\Big)c, \tag{2.26}$$

which gives

$$\psi^h_{k-} = \frac{1}{\mathcal{N}^h}\begin{pmatrix} -i\lambda k \\ B - K \end{pmatrix}$$
$$\psi^h_{k+} = \frac{1}{\mathcal{N}^h}\begin{pmatrix} B - K \\ -i\lambda k \end{pmatrix}, \tag{2.27}$$

with $\mathcal{N}^h = \sqrt{(B-K)^2 + \lambda^2 k^2}$. Now we can construct the desired unitary matrix

$$U \equiv \begin{pmatrix} U_e & 0 \\ 0 & U_h \end{pmatrix}, \tag{2.28}$$

where $U_e = \begin{pmatrix} \psi^e_{k-} \psi^e_{k+} \end{pmatrix}$ and $U_h = \begin{pmatrix} \psi^h_{k+} \psi^h_{k-} \end{pmatrix}$. Let us define

$$H_\delta = \begin{pmatrix} 0 & \delta {U_e}^\dagger U_h \\ \delta {U_h}^\dagger U_e & 0 \end{pmatrix}, \tag{2.29}$$

which can be evaluated using the wave functions from above. The rotated Hamiltonian becomes

$$\begin{aligned} U^\dagger H_k U &= \begin{pmatrix} E^e_- & 0 & 0 & 0 \\ 0 & E^e_+ & 0 & 0 \\ 0 & 0 & E^h_+ & 0 \\ 0 & 0 & 0 & E^h_- \end{pmatrix} + H_\delta \\ &= \frac{1}{K} \begin{pmatrix} KE^e_- & 0 & -iB\delta & -\lambda\delta k \\ 0 & KE^e_+ & -\lambda\delta k & -iB\delta \\ iB\delta & -\lambda\delta k & KE^h_+ & 0 \\ -\lambda\delta k & iB\delta & 0 & KE^h_- \end{pmatrix} \end{aligned} \tag{2.30}$$

We want to arrive at an effective model for the lowest electron and highest hole band, since these are the bands closest to zero energy. That is, we want to find an effective Hamiltonian for the E^e_- and E^h_- bands. Following the philosophy in Appendix A, we take

$$\begin{aligned} P &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ Q = \mathbb{1} - P &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{2.31}$$

and we can use (A.5) to write an energy-dependent effective Hamiltonian.

$$H_k(E)^{\text{eff}} = PU^\dagger H_k UP + PH_\delta Q(EQ - QU^\dagger H_k UQ)\Big|_Q^{-1} QH_\delta P. \tag{2.32}$$

The second term in (2.32) is ostensibly complicated. But if we remove the zero block and take the limit $\lambda k \ll B$ it simplifies, and we end up with

$$\begin{aligned} H^{\text{eff}}_k &\approx \begin{pmatrix} \frac{k^2}{2m} - h - B + \frac{\delta^2}{B} & \frac{\lambda\delta}{B} k \\ \frac{\lambda\delta}{B} k & -\frac{k^2}{2m} + h + B - \frac{\delta^2}{B} \end{pmatrix} \\ &= \begin{pmatrix} \frac{k^2}{2m^*} - \mu & \Delta k \\ \Delta k & -\frac{k^2}{2m^*} + \mu \end{pmatrix}, \end{aligned} \tag{2.33}$$

where for clarity I have replaced k with $-k$. We finally se that we indeed recover the p-wave superconductor (2.33) wth

$$\mu = h + B - \frac{\delta^2}{B} \tag{2.34}$$

$$\Delta = \frac{\lambda\delta}{B}. \tag{2.35}$$

Crucially only μ depends on h, and it does so linearly. The take-away message from this calculation is then, that when we want to add time dependent fluctuations $\phi(t)$ to the chemical potential h, then in the Hamiltonian H_{pw} for p-wave superconductor we should simply add them linearly to the gap μ.

2.3 Coulomb blockade and the Majorana box qubit

In a closed system there is an energy associated with the Coulomb repulsion of the electrons with each other, and it has the form $E_C\hat{N}^2$, where $\hat{N}$ is the total electron number operator for the system and the charging energy $E_C = \frac{e^2}{2C}$. If there is a gate voltage U present as well, then there is an extra added energy of $-eU\hat{N}$. Together, up to a constant, this gives what is called the charging energy term

$$H_C = E_C(\hat{N} - \mathcal{N}_g)^2. \tag{2.36}$$

In (2.36) the dimensionless parameter $\mathcal{N}_g = \frac{eU}{2E_C}$ can be tuned as we wish. The charging energy can be very big; so big in fact that it can seriously restrict the dynamics of the system. If it is big, and $\mathcal{N}_g$ is tuned close, to an integer, then the system is said to be Coulomb blockaded, and no electrons can enter or leave the system without paying the big energy cost E_c. This effect is very useful when dealing with Majoranas since it can diminish the probability that an electron enters or leaves the system, potentially switches the parity in the process. With a big Coulomb blockade the system is essentially isolated.
Let us first consider a system consisting of a single Coulomb blockaded nanowire, and assume that $\mathcal{N}_g$ has been tuned close to an even integer. We should note a subtle detail. H_{pw} is the Hamiltonian for the single-particle spectrum of the superconductor, and it leaves out the Cooper pairs. These are however counted also by $\hat{N}$, which counts all the electrons on the island. If we denote the Cooper pair counting operator by $\hat{N}_{\text{cp}}$, we can write

$$\hat{N} = \int_0^\infty dx\big(\psi_1^\dagger(x)\psi_1(x) + \psi_2^\dagger(x)\psi_2(x)\big) + 2\hat{N}_{\text{cp}}. \tag{2.37}$$

The combined Hamiltonian for the Coulomb blockaded nanowire is

$$H = H_{\text{pw}} + E_C(\hat{N} - \mathcal{N}_g)^2. \tag{2.38}$$

If we think of the single nanowire as two level system, then the logical Hilbert space $\mathcal{H}_L$ is given by the subspace spanned by the Majorana degrees of freedom, and has a basis of the even parity state $|0\rangle_L$ and the odd parity state $|1\rangle_L$. If $\mathcal{N}_g$ is tuned to an integer, then the ground state is $|0\rangle_L$, and there are two ways one can get excited states. The first is by paying the energy cost μ of splitting a cooper pair and putting one of the quasiparticles in the zero energy state and the other in the continuum. The second is to tunnel in an electron or a hole from the environment, paying the energy cost E_C by increasing or decreasing the number of electrons on the island. Since both the gap energy μ and the chargin energy are assumed to be big, the lone Majorana pair on the island cannot be efficiently used as a qubit. If a superposition of $|0\rangle_L$ and $|1\rangle_L$ is prepared, then they are at very different energies, so they will acquire different dynamical phases as time goes, mixing them.

This leads to the idea of the box qubit [2]: Putting two wires instead of one on the same Coulomb blockaded island as in Figure 2. $\hat{N}$ is now the total number of electrons in both wires. When E_C is very big and $\mathcal{N}_g$ is tuned to an integer, the charging term suppresses processes where electrons leave or enter the island, and this effectively enforces the parity constraint $-\gamma_1\gamma_2\gamma_3\gamma_4 = (-1)^{\mathcal{N}_g}$ in the the ground state subspace. It is thus possible to have two different

Figure 2: Sketch of the Majorana box qubit. Two nanowires are put on the same Coulomb blockaded island.

states, both at zero energy, namely the ones given by the states of the two pairs of Majoranas, which I will denote $|P_1\rangle|P_2\rangle$. If for example we again take $\mathcal{N}_g$ to be an even integer then we have the following basis for the logical Hilbert space $|0\rangle_L = |0\rangle|0\rangle$ and $|1\rangle_L = |1\rangle|1\rangle$. They are both at zero energy since it is possible to supply the fermions to fill the zero energy states by breaking a single cooper pair without having to put any particles in the continuum.

The ground state space of the box qubit constitute a two-level system, and one of the beautiful aspects of the box qubit is that Pauli operators may theoretically be easily implemented. For instance if we take

$$\sigma_x = i\gamma_1\gamma_3 \tag{2.39}$$

$$\sigma_y = i\gamma_3\gamma_2 \tag{2.40}$$

$$\sigma_z = i\gamma_1\gamma_2, \tag{2.41}$$

then they satisfy the Pauli algebra $[\sigma_i, \sigma_j] = \epsilon_{ijk}\sigma_k$, and thus they can be represented as Pauli matrices on the the logical two-level Hilbert space. If we ignore the continuum, then (2.15) tells us, if we tunnel an operator near the end of the wire, it effectively implements that particular Majorana operator. For example in Figure 3, a σ_x operation is implemented by coupling an occupied dot with very low orbital energy to γ_1 and an empty dot with high orbital energy to γ_3. When the energy of the leftmost dot is adiabatically increased while the dot energy to the right is adiabatically decreased, then the electron will tunnel through the Majoranas from the left to the right, and by the above arguments the process amounts to acting with $\gamma_1\gamma_3$ on the logical Hilbert space, which is proportional to a σ_x operation.
In Section 3 we will get into details with how the state of the box qubit can be measured.

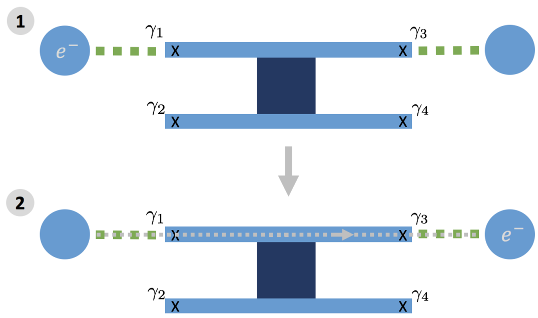


Figure 3: Illustration of the implementation of a the Pauli $\sigma_x = i\gamma_1\gamma_3$. The left most dot is initially occupied while the right most is initially vacant. Increasing energy on the left most dot while simultaneously lowering the energy on the right most dot makes the electron tunnels through the wire, affecting the state of the dot qubit by applying the product of the two Majoranas that it tunneled through.

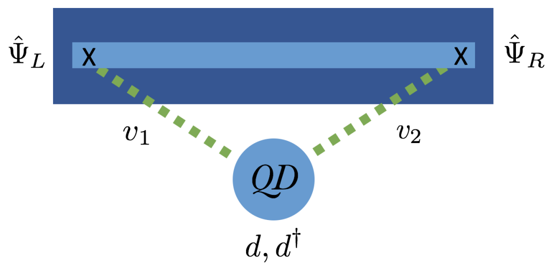


Figure 4: A dot is coupled to two Majoranas at the same end of the two different wires in the box qubit.

3 Reading out the Majorana state using a dot

In this section we will see how it is possible to read out the logical state of the Majorana box qubit. We will start with the simpler problem of reading out the state of a single Coulomb blockaded nanowire, by coupling a quantum dot to the two Majorana edge modes as shown on Figure 4. The nanowires are intended to be long, so it is not very practical to do this, but it is a little clearer when working with the single wire, and most of the calculation generalises to the box qubit.

We ignore the spin on the quantum dot state, gapping it out using the same magnetic field as the one required to drive the wire into the topological regime. Our starting point is the following Hamiltonian

$$H = H_{\text{pw}} + \epsilon d^\dagger d + E_C(\hat{N} - \mathcal{N}_g)^2 + d^\dagger\left(v_1\hat{\Psi}_L + v_2\hat{\Psi}_R\right) + \text{h.c.}, \tag{3.1}$$

where d and $d^\dagger$ are the annihilation and creation operators for the electronic state on the dot $\hat{\Psi}_1(0)$ and $\hat{\Psi}_2(0)$ are the annihilation operators for electrons at the end of the wires, as shown on Figure 4. We expand $\Psi_i(0)$ in terms of the Majorana operator γ_1 and the continuum operators α_k by using equation (2.15). Now that we are including the charging energy term we need to make a small modification and include the phase operator $\hat{\varphi}$, which is canonically conjugate to $\hat{N}$. We write [4]

$$\hat{\Psi}_i = A_i(0)e^{-i\frac{\hat{\varphi}}{2}}\gamma_i + \sum_k(u_{i,k}\alpha_{i,k} + v_{i,k}\alpha^\dagger_{i,k}). \tag{3.2}$$

Exponentiating $\hat{\varphi}$ gives the translation operator for $\hat{N}$. That is $e^{i\varphi/2}\hat{N}e^{-i\varphi/2} = \hat{N} - 1$. The Majoranas have zero charge, but in the tunneling terms we still need to keep track of how the charge on the island changes, which is the reason that we add the exponentiated phase factors. For the purposes of this section, we want to ignore the continuum, and drop all terms with an α_k operator. The concrete logic is detailed in Appendix A, taking P to project on states with no occupancy of the continuum states, and Q to project onto states with at least one continuum state occupied. Equation (A.9) then gives that the effective low energy Hamiltonian is equal to PHP plus a term which is proportional to $\frac{1}{\mu}$. Thus neglecting the continuum operators is an approximation to order $\frac{1}{\mu}$. For notational clarity I will keep denoting the total electron operator by $\hat{N}$ after projecting out the continuum states. Taking $\lambda_i \equiv v_iA_i(0)$ we find

$$H \approx \epsilon d^\dagger d + E_C(\hat{N} - \mathcal{N}_g)^2 + d^\dagger e^{-i\frac{\hat{\varphi}}{2}}(\lambda_1\gamma_1 + \lambda_2\gamma_2) + \text{h.c.} \tag{3.3}$$

Next we transform the Hamiltonian to $\tilde{H} \equiv U^\dagger H U$, where $U = e^{-i\hat{\varphi}d^\dagger d/2}$. The BCS ground state wave functions is a superposition of different numbers N_C of cooper pairs, with a specific parity p of the Majorana degrees of freedom [3]

$$|\phi, p\rangle = \frac{1}{N}\sum_{N_C} e^{-i\hat{\varphi}N_C}|N_C, p\rangle. \tag{3.4}$$

γ_1 acts on this by flipping p, so we see that $[\hat{\varphi}, \gamma] = 0$. Furthermore we can use that

$$\begin{aligned} e^{i\hat{\varphi}d^\dagger d} d^\dagger e^{-i\hat{\varphi}d^\dagger d} &= d^\dagger e^{i\hat{\phi}} \\ e^{i\hat{\varphi}d^\dagger d} d e^{-i\hat{\varphi}d^\dagger d} &= d e^{-i\hat{\phi}}, \end{aligned} \tag{3.5}$$

which can be checked by applying the left hand side operators to the basis states $|0\rangle_d$ and $|1\rangle_d$. Using that $e^{i\frac{\varphi d^\dagger d}{2}}$ is the translation operator by $d^\dagger d$ of $\hat{N}$, the transformation yields

$$\begin{aligned} \tilde{H} &= \epsilon d^\dagger d + E_C(e^{i\frac{\hat{\varphi}d^\dagger d}{2}}\hat{N}e^{-i\frac{\hat{\varphi}d^\dagger d}{2}} - \mathcal{N}_g)^2 + e^{i\frac{\hat{\varphi}d^\dagger d}{2}} d^\dagger e^{-i\frac{\hat{\varphi}}{2}}(\lambda_1\gamma_1 + \lambda_2\gamma_2)e^{-i\frac{\hat{\varphi}d^\dagger d}{2}} + \text{h.c.} \\ &= \epsilon d^\dagger d + E_C(\hat{N} - d^\dagger d - \mathcal{N}_g)^2 + d^\dagger(\lambda_1\gamma_1 + \lambda_2\gamma_2) + \text{h.c.} \\ &= \epsilon d^\dagger d + E_C\Big(\hat{N}^2 + (1 + 2\mathcal{N}_g)d^\dagger d - 2\mathcal{N}_g\hat{N} - 2\hat{N}d^\dagger d\Big) + d^\dagger(\lambda_1\gamma_1 + \lambda_2\gamma_2) + \text{h.c.} + E_C + \mathcal{N}_g^2, \end{aligned} \tag{3.6}$$

where it was used that $(d^\dagger d)^2 = d^\dagger d$. The number of electrons on the island is conserved, since $\hat{N}$ commutes with all the terms in the Hamiltonian (3.6). Hence we may replace $\hat{N}$ with a number N. Using this equation (3.6) becomes up to a constant

$$\tilde{H} = \tilde{\epsilon} d^\dagger d + d^\dagger(\lambda_1\gamma_1 + \lambda_2\gamma_2) + \text{h.c.}, \tag{3.7}$$

with $\tilde{\epsilon} = \epsilon - 2E_C N + E_C(1 + 2\mathcal{N}_g)$. We choose the following basis of $\{|n_d, P\rangle\}$ for the subspace of the Hilbert space we have restricted to, where $n_d \in \{0, 1\}$ denotes the dot occupancy and $P \in \{0, 1\}$ is the occupancy of the zero mode made out of γ_1 and γ_2 of the Majorana nanowire ground state. This means $\frac{1 - i\gamma_1\gamma_2}{2}|n_d, P\rangle = P|n_d, P\rangle$. With this definition we may represent the γ operators as Paulis on the P-space:

$$\begin{aligned} \sigma_x &= \gamma_1 \\ \sigma_y &= -\gamma_2 \\ \sigma_z &= i\gamma_1\gamma_2. \end{aligned} \tag{3.8}$$

Thus we may write (3.7) as

$$\tilde{H} = \begin{pmatrix} \tilde{\epsilon} d^\dagger d & d^\dagger(\lambda_1 + i\lambda_2) + d(\lambda_1^* + i\lambda_2^*) \\ d^\dagger(\lambda_1 - i\lambda_2) + d(\lambda_1^* - i\lambda_2^*) & \tilde{\epsilon} d^\dagger d \end{pmatrix}. \tag{3.9}$$

We may then represent d and $d^\dagger$ as the following

$$\begin{aligned} d &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ d^\dagger &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ d^\dagger d &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.10}$$

This means that (3.9) becomes

$$\tilde{H} = \begin{pmatrix} 0 & 0 & 0 & \lambda_1^* + i\lambda_2^* \\ 0 & \tilde{\epsilon} & \lambda_1 + i\lambda_2 & 0 \\ 0 & \lambda_1^* - i\lambda_2^* & 0 & 0 \\ \lambda_1 - i\lambda_2 & 0 & 0 & \tilde{\epsilon} \end{pmatrix}. \tag{3.11}$$

Noticing that Hamiltonian is block diagonal, we find the energies E as

$$\begin{aligned} 0 &= \det\begin{pmatrix} -E & \lambda_1^* + i\lambda_2^* \\ \lambda_1 - i\lambda_2 & \tilde{\epsilon} - E \end{pmatrix} \det\begin{pmatrix} -E & \lambda_1 + i\lambda_2 \\ \lambda_1^* - i\lambda_2^* & \tilde{\epsilon} - E \end{pmatrix} \\ &= \Big(- E(\tilde{\epsilon} - E) - \lambda^2 - i(\lambda_1\lambda_2^* - \lambda_1^*\lambda_2)\Big)\Big(- E(\tilde{\epsilon} - E) - \lambda^2 + i(\lambda_1\lambda_2^* - \lambda_1^*\lambda_2)\Big) \\ &= \Big(E^2 - \tilde{\epsilon}E - \lambda^2 + 2\mathrm{Im}\lambda_1\lambda_2^*\Big)\Big(E^2 - \tilde{\epsilon}E - \lambda^2 - 2\mathrm{Im}\lambda_1\lambda_2^*\Big), \end{aligned} \tag{3.12}$$

where $\lambda^2 \equiv |\lambda_1|^2 + |\lambda_2|^2$. Evidently

$$\begin{aligned} 0 &= E^2 - \tilde{\epsilon}E - \lambda^2 + 2p\mathrm{Im}\lambda_1\lambda_2^* \\ \Rightarrow E &= \frac{\tilde{\epsilon}}{2} \pm \sqrt{\frac{\tilde{\epsilon}^2}{4} + \lambda^2 - 2p\mathrm{Im}\lambda_1\lambda_2^*}. \end{aligned} \tag{3.13}$$

where $p = \pm 1$. The two split ground state energies are

$$E_{g,p} = \frac{\tilde{\epsilon}}{2} - \sqrt{\frac{\tilde{\epsilon}^2}{4} + \lambda^2 - 2p\mathrm{Im}\lambda_1\lambda_2^*}, \tag{3.14}$$

and for later use the excited state energies are denoted $E_{e,p}$. The Hamiltonian (3.11) has two invariant subspaces, namely Span$\{|0,0\rangle, |1,1\rangle\}$ and Span$\{|1,0\rangle, |0,1\rangle\}$, corresponding to even and odd total parity, respectively. Let's first consider the even total parity ground state state, which we will denote by $|E_{g,+}\rangle = \frac{\alpha_+|0,0\rangle + \beta_+|1,1\rangle}{\mathcal{N}_+} = \frac{1}{\mathcal{N}_+}\begin{pmatrix} \alpha_+ \\ 0 \\ 0 \\ \beta_+ \end{pmatrix}$. We get the two following eigenvalue equations:

$$(\lambda_1^* + i\lambda_2^*)\beta_+ = E_{g,p}\alpha_+ \tag{3.15}$$

$$(\lambda_1 - i\lambda_2)\alpha_+ + \tilde{\epsilon}\beta_+ = E_{g,p}\beta_+ \tag{3.16}$$

Plugging (3.15) into (3.16) we obtain

$$\begin{aligned} \frac{\lambda_1^* + i\lambda_2^*}{E_{g,p}} &= \frac{E_{g,p} - \tilde{\epsilon}}{\lambda_1 - i\lambda_2} \\ \Rightarrow (\lambda_1^* + i\lambda_2^*)(\lambda_1 - i\lambda_2) &= \Big(\frac{\tilde{\epsilon}}{2} - \sqrt{\frac{\tilde{\epsilon}^2}{4} + \lambda^2 - 2p\mathrm{Im}\lambda_1\lambda_2^*}\Big)\Big(- \frac{\tilde{\epsilon}}{2} - \sqrt{\frac{\tilde{\epsilon}^2}{4} + \lambda^2 - 2p\mathrm{Im}\lambda_1\lambda_2^*}\Big) \\ &= \frac{\tilde{\epsilon}^2}{4} + \lambda^2 - 2p\mathrm{Im}\lambda_1\lambda_2^* - \frac{\tilde{\epsilon}^2}{4} = (\lambda_1 - ip\lambda_2)(\lambda_1^* + ip\lambda_2^*), \end{aligned} \tag{3.17}$$

and evidently $p = 1$. If the total parity is odd, we write the state as $|E_{g,-}\rangle = \frac{\beta_-|1,0\rangle + \alpha_-|0,1\rangle}{\mathcal{N}_-} = \frac{1}{\mathcal{N}_-}\begin{pmatrix} 0 \\ \beta_- \\ \alpha_- \\ 0 \end{pmatrix}$ eigenvalue equations read

$$
\begin{aligned}
\tilde{\epsilon}\beta_- + (\lambda_1 + i\lambda_2)\alpha_- &= E_{g,p}\beta_- \\
(\lambda_1^* - i\lambda_2^*)\beta_- &= E_{g,p}\alpha_-,
\end{aligned} \tag{3.18}
$$

which combine to give

$$
(\lambda_1 + i\lambda_2)(\lambda_1^* - i\lambda_2^*) = (E_{g,p} - \tilde{\epsilon})E_{g,p} = (\lambda_1 - ip\lambda_2)(\lambda_1^* + ip\lambda_2^*), \tag{3.19}
$$

which implies $p = -1$. We thus see that p is equal to the total parity of the system, or $p = (-1)^{n_d+P}$, and crucially this is preserved in time evolution of the system.

Now we are set to propose a readout protocol. First assume that the nanowire is initially decoupled from the dot and is initialized in some unknown pure state $|P\rangle$. The dot is initially unoccupied, so the combined state is $|0, P\rangle$. After the initialisation, the coupling is turned on and for simplicity we will allow the state to relax to the ground state, equilibrating at close to zero temperature. Since the Hamiltonian conserves the total parity, the state will equilibrate to $|E_{g,p}\rangle$. Hence at equilibrium we find

$$
\langle n_d \rangle = \begin{cases} \big(\alpha_+^* \langle 0,0| + \beta_+^* \langle 1,1|\big) d^\dagger d \big(\alpha_+ |0,0\rangle + \beta_+ |1,1\rangle\big) & : p = 1 \\ \big(\beta_-^* \langle 1,0| + \alpha_-^* \langle 0,1|\big) d^\dagger d \big(\beta_- |1,0\rangle + \alpha_- |0,1\rangle\big) & : p = -1 \end{cases}, \tag{3.20}
$$

which means

$$
\langle n_d \rangle = |\beta_p|^2. \tag{3.21}
$$

We can find the two eigenstates from (3.18) and (3.15). We get

$$
|E_{g,+}\rangle = \frac{1}{\mathcal{N}_+} \begin{pmatrix} \lambda_1^* + i\lambda_2^* \\ 0 \\ 0 \\ E_{g,+} \end{pmatrix} \tag{3.22}
$$

$$
|E_{g,-}\rangle = \frac{1}{\mathcal{N}_-} \begin{pmatrix} 0 \\ E_{g,-} \\ \lambda_1^* - i\lambda_2^* \\ 0 \end{pmatrix}, \tag{3.23}
$$

where the normalisation is given by

$$
\mathcal{N}_p = \sqrt{\lambda^2 - 2p\mathrm{Im}\lambda_1\lambda_2^* + E_{g,p}^2}. \tag{3.24}
$$

This gives us the values of β_p which can then be plugged into (3.21).

At this point if one could just measure the instantaneous occupation of the dot, it wouldn't necessarily be enough to determine what p is. However a typical way of measuring the dot occupancy is to make use of another dot, which we will call a test dot. The test dot is coupled capaciatively to the first dot, let's call it the target dot. The capacitative coupling is very strong and the test dot is gated so the conductance through it is as sensitive to the gate voltage as possible. It can be sensitive enough that the conductance changes measurably when the target dot is occupied. When measuring the conductance through the test dot, one would run a current through it for a time while the occupancy on the target dot fluctuates around the mean value $\langle n_d \rangle$. The total current that has run through the test dot in that time can be used to infer $\langle n_d \rangle$. This constitutes a measurement of p, and I will refer to it as measuring $\langle n_d \rangle_p$.

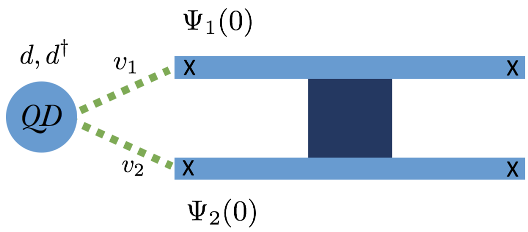


Figure 5: A dot is coupled to two Majoranas at the same end of the two different wires in the box qubit.

Once p is known the readoff is done, since p determines the original Majorana state.

Now we turn our attention to the slightly more subtle problem of doing projective measurements of a Box qubit. We adapt the strategy from the case of one wire and the setup in mind is shown in Figure 5. It is much more feasible to couple a dot in this way, since the two wires may be close to each other. The starting point is now the Hamiltonian

$$H = H_{\text{pw},1} + H_{\text{pw},2} + \epsilon d^\dagger d + E_C(\hat{N} - \mathcal{N}_g)^2 + d^\dagger\big(v_1\hat{\Psi}_1(0) + v_2\hat{\Psi}_0(0)\big) + \text{h.c.} \tag{3.25}$$

All the steps from above can be taken, ignoring now γ_3 and γ_4 in the expansion of $\hat{\Psi}_i(0)$ since they are far away. The state of the two other Majoranas do not affect the energy, and so the Hamiltonian will end up looking just like (3.11), but the Hilbert space is a little different, now that the other wire has been added. Let us use the basis $|P\rangle_a \otimes |P'\rangle_b$ for the four Majoranas. The first tensor factor denotes the occupation of the Dirac fermion composed of γ_1 and γ_2. The logical space of the two Majoranas is spanned by

$$\begin{aligned} |0\rangle_{L+} &= |00\rangle \\ |1\rangle_{L+} &= |11\rangle \end{aligned} \tag{3.26}$$

when the parity is even, and it is spanned by

$$\begin{aligned} |0\rangle_{L-} &= |01\rangle \\ |1\rangle_{L-} &= |10\rangle \end{aligned} \tag{3.27}$$

when the parity is odd. In the following r will denote the overall parity, given by the parity of the integer that $\mathcal{N}_g$ is tuned to. When including the dot we use the basis $|n_d, PP'\rangle$. Let us assume that the initial state $|\psi_{0r}\rangle$ that we want to projectively measure is coupled to an empty dot and write it as

$$|\psi_{0r}\rangle = \zeta_0|0\rangle \otimes |0\rangle_{Lr} + \zeta_1|0\rangle \otimes |1\rangle_{Lr}. \tag{3.28}$$

As above we want the state to equilibrate and measure the dot occupancy. Again we will couple the system to the environment in such a fashion that no particles are exchanged. This entails that the total parity is conserved, but moreover, since the dot is coupled only to two of the four Majoranas, the state of the two other ones is unchanged as well. Let us be a little more precise about this and define the projection operators on the two parity sectors as

$$R_+ = |0,00\rangle\langle 0,00| + |0,11\rangle\langle 0,11| + |1,10\rangle\langle 1,10| + |1,01\rangle\langle 1,01| \tag{3.29}$$

$$R_- = |1,00\rangle\langle 1,00| + |0,10\rangle\langle 0,10| + |0,01\rangle\langle 0,01| + |1,11\rangle\langle 1,11|, \tag{3.30}$$

as well as the projection operators onto the two different states $|P\rangle_b$

$$Q_0 = |0,00\rangle\langle 0,00| + |1.00\rangle\langle 1,00| + |0,10\rangle\langle 0,10| + |1,10\rangle\langle 1,10| \tag{3.31}$$
$$Q_1 = |0,01\rangle\langle 0,01| + |1,01\rangle\langle 1,01| + |0,11\rangle\langle 0,11| + |1,11\rangle\langle 1,11|, \tag{3.32}$$

so their products $\Pi_{r,P_b} \equiv R_r Q_{P_b}$ are

$$\Pi_{+0} = |0,00\rangle\langle 0,00| + |1,10\rangle\langle 1,10| \tag{3.33}$$
$$\Pi_{+1} = |0,11\rangle\langle 0,11| + |1,01\rangle\langle 1,01| \tag{3.34}$$
$$\Pi_{-0} = |1,00\rangle\langle 1,00| + |0,10\rangle\langle 0,10| \tag{3.35}$$
$$\Pi_{-1} = |0,01\rangle\langle 0,01| + |1,11\rangle\langle 1,11|. \tag{3.36}$$

Notice that the projectors in (3.36) also project onto states with definite total parity of the dot and the coupled Majoranas. Let us refer to this as the subparity and denote it p. The projector Π_{r,P_b} projects onto a subspace with subparity $p = (-1)^{\frac{r-1}{2}+P_b}$. When coupling the system to an environment the total Hamiltonian is

$$H_{\text{tot}} = H + H_{\text{env}} + H_c, \tag{3.37}$$

where H_{env} is the Hamiltonian for the isolated environment and H_c is the coupling term, which by assumption doesn't connect the two different parity sectors and doesn't couple different $|P\rangle_b$. In other words

$$H_{\text{tot}} = \sum_{r,P_b} \Pi_{r,P_b} H_{\text{tot}} \Pi_{r,P_b} \equiv \sum_{r,P_b} H_{r,P_b}. \tag{3.38}$$

This then also means that time evolution does not mix states with different subparities. More concretely the time evolution operator on the combined system satisfies

$$U_{\text{tot}}(t) = e^{-iH_{\text{tot}}t} = e^{-iH_{+0}t} e^{-iH_{+1}t} e^{-iH_{-0}t} e^{-iH_{-1}t} \equiv U_{+0}(t) U_{+,1}(t) U_{-0}(t) U_{-1}(t). \tag{3.39}$$

If $|\phi_{\text{env}}\rangle$ i some state in the environment Hilbert space this means for instance

$$U(t)|0,00\rangle \otimes |\phi_{\text{env}}\rangle = U_{+,0}(t)|0,00\rangle \otimes |\phi_{\text{env}}\rangle. \tag{3.40}$$

The initial density matrix for the combined system is $\rho_{0r} \otimes \rho_{\text{env}}$, where (3.28) gives us in the even parity sector

$$\rho_{0+} = |\psi_{0+}\rangle\langle\psi_{0+}| = |\zeta_0|^2 |0,00\rangle\langle 0,00| + |\zeta_1|^2 |0,11\rangle\langle 0,11| + \zeta_0\zeta_1^* |0,00\rangle\langle 0,11| + \zeta_0^*\zeta_1 |0,11\rangle\langle 0,00|. \tag{3.41}$$

Using (3.39) we find

$$\begin{aligned}\rho_+(t) &= \text{tr}_{\text{env}}\Big(U(t)\rho_{0+} \otimes \rho_{\text{env}} U^\dagger(t)\Big) \\ &= \text{tr}_{\text{env}}\Big(|\zeta_0|^2 U_{+0}(t)|0,00\rangle\langle 0,00| \otimes \rho_{\text{env}} U_{+0}^\dagger(t) + |\zeta_1|^2 U_{+1}(t)|0,11\rangle\langle 0,11| \otimes \rho_{\text{env}} U_{+1}^\dagger(t) \\ &\qquad + \zeta_0\zeta_1^* U_{+0}(t)|0,00\rangle\langle 0,11| \otimes \rho_{\text{env}} U_{+1}^\dagger(t) + \zeta_0^*\zeta_1 U_{+1}(t)|0,11\rangle\langle 0,00| \otimes \rho_{\text{env}} U_{+0}^\dagger(t)\Big).\end{aligned} \tag{3.42}$$

$$\equiv \begin{pmatrix} |\zeta_0|^2 \rho_{+0}(t) & \zeta_0\zeta_1^* r_+(t) \\ \zeta_0^*\zeta_1 r_+^\dagger(t) & |\zeta_1|^2 \rho_{+1}(t) \end{pmatrix}. \tag{3.43}$$

In a similar way we can compute the density matrix for the odd sector:

$$\rho_-(t) = \mathrm{tr}_{\mathrm{env}}\Big(|\zeta_0|^2 U_{-1}(t)|0,01\rangle\langle 0,01| \otimes \rho_{\mathrm{env}} U^\dagger_{-1}(t) + |\zeta_1|^2 U_{-0}(t)|0,10\rangle\langle 0,10| \otimes \rho_{\mathrm{env}} U^\dagger_{-0}(t)$$
$$+ \zeta_0\zeta_1^* U_{-1}(t)|0,01\rangle\langle 0,10| \otimes \rho_{\mathrm{env}} U^\dagger_{-0}(t) + \zeta_0^*\zeta_1 U_{-0}(t)|0,10\rangle\langle 0,01| \otimes \rho_{\mathrm{env}} U^\dagger_{-1}(t)\Big). \tag{3.44}$$

$$\equiv \begin{pmatrix} |\zeta_0|^2\rho_{-1}(t) & \zeta_0\zeta_1^* r_-(t) \\ \zeta_0^*\zeta_1 r_-^\dagger(t) & |\zeta_1|^2\rho_{-0}(t) \end{pmatrix}. \tag{3.45}$$

At times later than the equilibration time the off diagonals $r_r(t)$ go to zero for the usual reasons: If the bath has a macroscopic number of degrees of freedom then $r(t)$ will include a macroscopic number of different phases, cancelling the term. The diagonals however equilibrate to two different Boltzmann distributions with different subparities. The energies are given solely by the state of the dot and the Majoranas it is coupled to, since the uncoupled Majoranas are at zero energy. So the uncoupled Majoranas are from here on irrelevant, and from here on I will leave them out of the notation. We do this by using $p = (-1)^{\frac{r-1}{2}+P_b}$ to relabel ρ_{rP_b}. This time, for greater accuracy we include the excited states $|E_{e,p}\rangle$. The dependence of time in (3.43) is dropped, assuming that the state has equilibrised in this sense described above. So we have

$$\rho_{\mathrm{eq},+} = \begin{pmatrix} |\zeta_0|^2\rho_{+0} & 0 \\ 0 & |\zeta_1|^2\rho_{+1} \end{pmatrix} = \begin{pmatrix} |\zeta_0|^2\rho_{+} & 0 \\ 0 & |\zeta_1|^2\rho_{-} \end{pmatrix}$$
$$\rho_{\mathrm{eq},-} = \begin{pmatrix} |\zeta_0|^2\rho_{-1} & 0 \\ 0 & |\zeta_1|^2\rho_{-0} \end{pmatrix} = \begin{pmatrix} |\zeta_0|^2\rho_{+} & 0 \\ 0 & |\zeta_1|^2\rho_{-} \end{pmatrix}, \tag{3.46}$$

with

$$\rho_p = \frac{e^{-\beta E_{g,p}}|E_{g,p}\rangle\langle E_{g,p}| + e^{-\beta E_{e,p}}|E_{e,p}\rangle\langle E_{e,p}|}{Z_p}, \tag{3.47}$$

where the normalisation is given by

$$Z_p = e^{-\beta E_{g,p}} + e^{-\beta E_{e,p}}. \tag{3.48}$$

Equation (3.46) tells us that the overall parity is immaterial for reading out the logical states. It also tells us that at equilibrium the quantum state has decohered to ρ_+ with probability $|\zeta_0|^2$ and to ρ_- with probability $|\zeta_0|$. This means that doing a measurement of $\langle n_d\rangle$ is going to give $\mathrm{tr}(d^\dagger d\rho_+)$ with probability $|\zeta_0|^2$ and $\mathrm{tr}(d^\dagger d\rho_-)$ with probability $|\zeta_1|^2$. We can read out which one it is by doing a measurement of $\langle n_d\rangle$, which then tells us which state the Majoranas were in initially. If one wants to complete the projective measurement on the box qubit state, leaving it in the state corresponding to the measurement outcome, then this can be achieved by taking $\tilde\epsilon \to \infty$ adiabatically and letting it equilibrate. Then the dot winds up empty and as a result the logical state is determined by subparity. Decoupling the dot in the end the final state is $|0\rangle_{Lr}$ if $p = 1$ and $|1\rangle_{Lr}$ if $p = -1$.

The protocol takes the Majorana state $\zeta_0|0\rangle_{Lr} + \zeta_1|1\rangle_{Lr}$ and outputs $|0\rangle$ with probability $|\zeta|^2$ and $|1\rangle$ with probability $|1\rangle$, meaning that it is a projective measurement. What is left is to calculate $\langle n_d\rangle$ for the two subparity sectors. Let us denote the two expectation values $\langle n_d\rangle_p$. I haven't yet checked that $E_{e,p}$, given by

$$E_{e,p} = \frac{\tilde\epsilon}{2} + \sqrt{\frac{\tilde\epsilon^2}{4} + \lambda^2 - 2p\mathrm{Im}\lambda_1\lambda_2^*}, \tag{3.49}$$

is the excited energy for the state corresponding to the subparity p. The excited states $|E_{e,p}\rangle$ are orthorgonal to $|E_{g,p}\rangle$ inside the respective parity subspaces. So

$$|E_{e,+}\rangle = \frac{1}{\mathcal{N}_+}\begin{pmatrix} -\beta_+^* \\ 0 \\ 0 \\ \alpha_+^* \end{pmatrix} = \frac{1}{\mathcal{N}_+}\begin{pmatrix} E_{e,+1} - \tilde{\epsilon} \\ 0 \\ 0 \\ \lambda_1 - i\lambda_2 \end{pmatrix} \tag{3.50}$$

$$|E_{e,-}\rangle = \frac{1}{\mathcal{N}_-}\begin{pmatrix} 0 \\ -\alpha_-^* \\ \beta_-^* \\ 0 \end{pmatrix} = \frac{1}{\mathcal{N}_-}\begin{pmatrix} 0 \\ \lambda_1 + i\lambda_2 \\ E_{e,-1} - \tilde{\epsilon} \\ 0 \end{pmatrix}, \tag{3.51}$$

since $E_{g,p} = -E_{e,p} + \tilde{\epsilon}$. It is straightforward to check that these are eigenstates with eigenvalue $E_{e,p}$, for instance

$$H|E_{e,+1}\rangle = \frac{1}{\mathcal{N}_+}\begin{pmatrix} (\lambda_1^* + i\lambda_2^*)(\lambda_1 - i\lambda_2) \\ 0 \\ 0 \\ (\lambda_1 - i\lambda_2)(E_{e,+} - \epsilon + \epsilon) \end{pmatrix} = \frac{E_{e,+}}{\mathcal{N}_+}\begin{pmatrix} \frac{\lambda^2 - 2\mathrm{Im}\lambda_1\lambda_2^*}{E_{e,+}} \\ 0 \\ 0 \\ \lambda_1 - i\lambda_2 \end{pmatrix} = E_{e,+}|E_{e,p1}\rangle, \tag{3.52}$$

where for the last equality it was used that

$$\begin{aligned} E_{e,+} - \epsilon &= \frac{1}{E_{e,+}}E_{e,+}(E_{e+} - \epsilon) \\ &= \frac{1}{E_{e,+}}\Big(\frac{\epsilon}{2} + \lambda^2 - 2\mathrm{Im}(\lambda_1\lambda_2^*) + \epsilon\sqrt{\frac{\epsilon}{4} + \lambda^2 - 2\mathrm{Im}\lambda_1\lambda_2^*} - \frac{\epsilon}{2} - \epsilon\sqrt{\frac{\epsilon}{4} + \lambda^2 - 2\mathrm{Im}\lambda_1\lambda_2^*}\Big) \\ &= \frac{\lambda^2 - 2\mathrm{Im}\lambda_1\lambda_2^*}{E_{e,+}}. \end{aligned} \tag{3.53}$$

With this we can calculate $\langle n_d\rangle_p$ and distinguish the two subparity sectors. The diagonal of ρ_+ is

$$\frac{\big(|\alpha_+|^2 e^{-\beta E_{g,+}} + |\beta_+|^2 e^{-\beta E_{e,+}}\big)|00\rangle\langle 00|\big(|\beta_+|^2 e^{-\beta E_{g,+}} + |\alpha_+|^2 e^{-\beta E_{e,+}}\big)|11\rangle\langle 11|}{|\mathcal{N}_+|^2 Z_+}, \tag{3.54}$$

so

$$\langle n_d\rangle_+ = \mathrm{tr}(d^\dagger d\rho_+) = \frac{|\beta_+|^2 e^{-E_{g,+}} + |\alpha_+|^2 e^{-E_{e,+}}}{|\mathcal{N}_+|^2 Z_+}. \tag{3.55}$$

Likewise

$$\frac{\big(|\beta_-|^2 e^{-\beta E_{g,-}} + |\alpha_-|^2 e^{-\beta E_{e,-}}\big)|10\rangle\langle 10|\big(|\alpha_-|^2 e^{-\beta E_{g,-}} + |\beta_-|^2 e^{-\beta E_{e,-}}\big)|01\rangle\langle 01|}{|\mathcal{N}_-|^2 Z_-}, \tag{3.56}$$

so

$$\langle n_d\rangle_- = \mathrm{tr}(d^\dagger d\rho_-) = \frac{|\beta_-|^2 e^{-E_{g,-}} + |\alpha_-|^2 e^{-E_{e,-}}}{|\mathcal{N}_-|^2 Z_-}, \tag{3.57}$$

and so we see that the expectation value of the the occupation of the dot in the p-parity sector thus becomes

$$\langle n_d\rangle_p = \mathrm{tr}(d^\dagger d\rho_p) = \frac{|\beta_p|^2 e^{-\beta E_{g,p}} + |\alpha_p|^2 e^{-\beta E_{e,p}}}{|\mathcal{N}_p|^2 Z_p}. \tag{3.58}$$

Let us conclude the section by examining what the maximum contrast is in equation (3.58) between positive and negative subparity. This is found when we choose

$$\lambda_2 = -i\lambda_1 \tag{3.59}$$
$$\lambda_1 \in \mathbb{R} \tag{3.60}$$
$$\tilde{\epsilon} \gg \lambda. \tag{3.61}$$

With these choices $\lambda_1\lambda_2^* = i\lambda_1^2 = i\frac{\lambda^2}{2}$, and

$$\begin{aligned} E_{g,p} &= \frac{\tilde{\epsilon}}{2} - \sqrt{\frac{\tilde{\epsilon}^2}{4} + \lambda^2 - 2p\frac{\lambda^2}{2}} = \frac{\tilde{\epsilon}}{2} - \sqrt{\frac{\tilde{\epsilon}^2}{4} + (1-p)\lambda^2}, \\ &\approx -(1-p)\frac{\lambda^2}{\tilde{\epsilon}} \end{aligned} \tag{3.62}$$

so since $\beta_p = E_{g,p}$ we find

$$\beta_p \begin{cases} 0: & p = + \\ -2\frac{\lambda^2}{\tilde{\epsilon}}: & p = - \end{cases}. \tag{3.63}$$

Furthermore we see

$$\lambda_1^* + i\lambda_2^* = \lambda_1 - \lambda_1 = 0,$$

and

$$\lambda_1^* - i\lambda_2^* = 2\lambda_1 = \sqrt{2}\lambda,$$

so since $\alpha_p = \lambda_1^* + ip\lambda_2^*$ we have

$$\alpha_p = \begin{cases} 0: & p = + \\ \sqrt{2}\lambda: & p = - \end{cases}. \tag{3.64}$$

We immediately see $\langle n_d \rangle_+ = 0$. In the negative subparity sector normalisation is

$$\mathcal{N}_- = \sqrt{2\lambda^2 + E_{g,-}^2} \approx \sqrt{2\lambda^2 + 4\frac{\lambda^4}{\tilde{\epsilon}^2}} \approx \sqrt{2}\lambda. \tag{3.65}$$

Putting everything together we have

$$\begin{aligned} \langle n_d \rangle_- &\approx \frac{4\frac{\lambda^4}{\tilde{\epsilon}^2}e^{-\beta E_{g,-}} + 2\lambda^2 e^{-\beta E_{e,-}}}{2\lambda^2(e^{-\beta E_{g,-}} + e^{-\beta E_{e,-}})} \\ &\approx \frac{2\lambda^2 e^{-\beta E_{e,-}}}{2\lambda^2(e^{-\beta E_{g,-}} + e^{-\beta E_{e,-}})} = \frac{e^{-\beta E_{e,-}}}{(e^{-\beta E_{g,-}} + e^{-\beta E_{e,-}})} \\ &= \frac{1}{(e^{\beta(E_{e,-} - E_{g,-})} + 1)} = n_F(E_{g,-} - E_{e,-}) \approx n_F\Big(4\frac{\lambda^2}{\tilde{\epsilon}}\Big) \approx 1 \end{aligned} \tag{3.66}$$

where n_F is the Fermi function at temperature $T = \frac{1}{\beta}$ and it was assumed $\beta\frac{\lambda^2}{\tilde{\epsilon}} \ll 1$. As advertised we have maximum contrast:

$$\langle n_d \rangle_p = \begin{cases} 0: & p = + \\ 1: & p = - \end{cases}. \tag{3.67}$$

To summarise the readout protocol consists of the following steps:

1. The state is initialised with the dot empty and the Majoranas in $\zeta_0|0\rangle_L + \zeta_0|1\rangle_L$.

2. The dot is coupled to two of the Majoranas by turning on λ_1 and λ_2.

3. The system is brought to equilibrium without exchanging particles with the environment.

4. Measure $\langle n_d\rangle$. $\langle n_d\rangle_+$ is measured with probability $|\zeta_0|^2$ and $\langle n_d\rangle_-$ with probability $|\zeta_1|^2$. $\langle n_d\rangle_p$ is given by equation (3.58). This effectively reads out the state.

The above procedure assumes that there is no noise in the system. When we add the noise the zero energy subspace gets entangled with the continuum, so tracing the continuum away leaves a mixed state. Therefore we should close this section off with discussing how the readout protocol handles mixed states. It turns out to work in completely the same way as with coherent superpositions, the only extra difficulty is that now the state may include mixtures of different overall parities. But since $\langle n_d\rangle$ only depends on the subparity, this doesn't end up posing a real concern.
Let us be more specific and write the initial mixed state and the empty dot as

$$\rho_0 = |0\rangle\langle 0|_d \otimes \Big(a|00\rangle\langle 00| + b|01\rangle\langle 01| + c|10\rangle\langle 10| + d|11\rangle\langle 11| + r_0\Big), \tag{3.68}$$

where r_0 is off-diagonal. We now again couple it to an environment where no particles are exhanged, and which we assume do not excite states in the continuum. As before the coupling to the bath preserves the total parity r, the parity of the state of the uncoupled Majoranas, P_b and therefore also the subparity p. Otherwise, again, the coupling is assumed ergodic so the state thermalises. This means that the block off-diagonal parts go to zero, and each of the block-diagonal parts equilibrates to a Boltzmann distribution ρ_p^r, with total parity r and subparity p. The ground and excited states in each (p, r) sector again are equal to the states described above, but with the state of the uncoupled Majoranas compensating to agree with the overall parity. They are denoted as $|E_{g/e,p}\rangle_r$, so for instance

$$|E_{g,+}\rangle_+ = \frac{\alpha_+}{\mathcal{N}_p}|0,00\rangle + \frac{\beta_+}{\mathcal{N}_p}|1,10\rangle,$$

while

$$|E_{e,+}\rangle_- = \frac{\alpha_+}{\mathcal{N}_p}|0,01\rangle + \frac{\beta_+}{\mathcal{N}_p}|1,11\rangle.$$

Measuring $\langle n_d\rangle$ corresponds to measuring p, as discribed above, and this cannot distinguish between the two states above. We may write

$$\rho_p^r = \frac{1}{Z_p}\Big(e^{-\beta E_{g,p}}|E_{g,p}\rangle\langle E_{g,p}|_r + e^{-\beta E_{e,p}}|E_{e,p}\rangle\langle E_{e,p}|_r\Big), \tag{3.69}$$

and after thermalisation the equilibrium density matrix is

$$\rho_{\text{eq}} = \begin{pmatrix} \rho_+^+ & 0 & 0 & 0 \\ 0 & \rho_+^- & 0 & 0 \\ 0 & 0 & \rho_-^+ & 0 \\ 0 & 0 & 0 & \rho_-^- \end{pmatrix}. \tag{3.70}$$

So it is a classical mixture of states with different p and r. Measuring the dot occupancy now yields $\langle n_d\rangle_+$ with probability $a + b$ and $\langle n_d\rangle_-$ with probability $c + d$. So we have managed to do a read out of $i\gamma_1\gamma_2$.
If we wanted to read out also r we would have to couple a new dot to the previously uncoupled

Majorana pair and measure the occupation on that dot. Together this constitutes a read out of the total state.

Now that we have a feeling for what the Majorana box qubit is and how it may be manipulated and read out, it is time to turn our attention to the main topic of this thesis, namely the effect of the time dependent potentials.

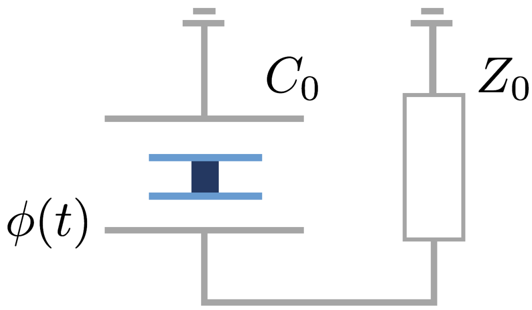


Figure 6: The Majorana box qubit is subject to a classical potential $\phi(t)$ which arises from a circuit a seen in the figure. The fluctuations arise from a parallel coupling to the environment, modeled as an impedance Z_0.

4 Adding potential fluctuations

Now we turn our attention to the effects of fluctuating potentials. We start by considering a single Majorana nanowire on a Coulomb blockaded island with uniform voltage fluctuations $\phi(t)$ over the whole system, taking $\phi(0) = 0$. The idea is outlined in Figure 6: We consider the system capacitatively coupled to the environment, which is modelled as an impedance Z_0. For now we just take $\phi(t)$ to be some arbitrary function, but later we will average it.
As we saw in equation (2.35) effect of this is $\phi(t)$ to the chemical potential h. The Hamiltonian for a p-wave superconductor with such time dependence is

$$H_t = \begin{pmatrix} \frac{p^2}{2m} - \mu + \phi(t) & \Delta p \\ \Delta p & -\frac{p^2}{2m} + \mu + \phi(t) \end{pmatrix} = H_{\text{pw}} + \phi(t)\tau_z. \tag{4.1}$$

We need $\phi(t) \ll \mu$ at all times to ensure that we are not pushed out of the topological regime. We denote the original zero energy wave function by $|\psi_0\rangle$ and the wave function of the state with wavenumber k as $|\psi_k\rangle$. Since the states $|\psi_0\rangle$ and $|\psi_k\rangle$ diagonalise H_{pw}, but are not eigenstates of τ_z, if $\phi(t)$ changes fast enough that the change is not adiabatic, if a zero energy state is initially prepared, then at a later time t there will be nonzero amplitutes also for continuum states. So doing a readout of e.g. $i\gamma_1\gamma_2$ leads to potential errors. We denote the matrix element between the zero energy state and the continuum by

$$\langle\psi_0|\phi(t)\tau_z|\psi_k\rangle = \phi(t)\delta H_{0,k}. \tag{4.2}$$

Using this we may write the term connecting the zero energy mode and the continuum in second quantisation as

$$\phi(t) \sum_{i=1,2} \Big(\delta H_{0,k}\gamma_i\alpha_k + \phi(t)\delta H^*_{0,k}\alpha^\dagger_k\gamma_i\Big), \tag{4.3}$$

and so the Hamiltonian is

$$H_t = \sum_k E_k\alpha^\dagger_k\alpha_k + \sum_{i=1,2}\sum_k \Big(V_{kt}\alpha^\dagger_k - V^*_{kt}\alpha_k\Big)\gamma_i + \dots, \tag{4.4}$$

where

$$V_{kt} = \phi(t)\delta H^*_{0,k}. \tag{4.5}$$

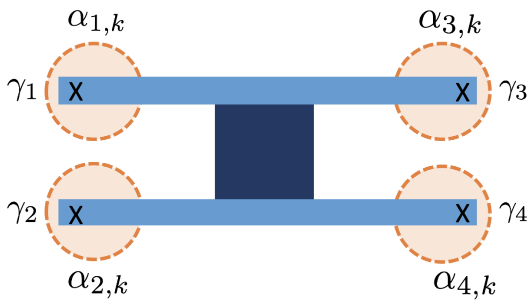


Figure 7: In view of the big system size, the Majoranas γ_i are modeled as only interacting with a set of local continuum states $\alpha_{i,k}$.

The terms that are left out in (4.4) have the form $\sum_{k,k'} \langle k|\phi(t)\tau_z|k'\rangle \alpha_k^\dagger \alpha_{k'}$. In the scope of this project we leave out this, expecting processes that involves this to not drastically change our results. I will discuss this approximation more in the Section 9.
In the previous section we saw how to readout the parity of two Majoranas by coupling a dot them. Going through these calculations would be quite difficult if including the new time dependence. For instance, it is not immediately clear how to thermalise the state while including the fluctuations. So let us make life easier for us self and assume that it is possible to do a readout on a time scale where the dephasing from $\phi(t)$ is negligible.

Now let us return to the box qubit and assume that initialisation is perfect in the sense that the Majorana states are completely disentangled from continuum. Denoting the initial state of the continuum by the density matrix ρ_c we may use (B.7) to write the combined initial state as

$$\rho(0) = \frac{1}{2}(\mathbb{1} + \bar{r} \cdot \bar{\sigma}) \otimes \rho_c. \tag{4.6}$$

Before we continue we will make a new assumption, namely that each Majorana only interact with continuum states localised around that it. The concept is sketched on Figure 7 These localised continuum states are not eigenstates of H_{pw}, but rather like wavepackets formed by linear combinations of the actual continuum eigenstates. This is a good approximation if the time scales we are looking at are much smaller than the time it takes for such a wave packet to travel from one end of the wire to the other. The way we implement this assumption is by writing new continuum operators $\alpha_{i,k}$ and consider the Hamiltonian (henceforth using the H_t to denote only the following Hamiltonian

$$H_t = \sum_{i=1,4} \sum_k E_k \alpha_{i,k}^\dagger \alpha_{i,k} + \sum_{i=1,4} \sum_k \Big(V_{kt} \alpha_{i,k}^\dagger - V_{kt}^* \alpha_{i,} \Big) \gamma_i. \tag{4.7}$$

Time evolving the initial state (4.6) using the Hamiltonian in (4.7) gives us a state entangled with the continuum, so at time t the state is

$$\rho(t) = \frac{1}{2} U(t) \big(\mathbb{1} + \bar{r} \cdot \bar{\sigma}\big) \rho_c U^\dagger(t). \tag{4.8}$$

Let's now consider a measurement of the parity of two Majoranas. In general the probability $P_{\psi,\rho}$ for measuring a state $|\psi\rangle$ given a density matrix ρ is $P_{\psi,\rho} = \langle\psi|\rho|\psi\rangle$. Using that the trace of a number is the identity along with the cyclicity of the trace, we can rewrite it to

$$P_{\psi,\rho} = \operatorname{tr}\Big(\langle\psi|\rho|\psi\rangle\Big) = \operatorname{tr}\Big(|\psi\rangle\langle\psi|\rho\Big). \tag{4.9}$$

We want to know the probability of measuring the pure logical state which in the logical two-level system Hilbert space is

$$\rho_{\tilde{r}} = \frac{1}{2}(\mathbb{1} + \tilde{r} \cdot \tilde{\sigma}), \tag{4.10}$$

with $|\tilde{r}| = 1$ and $\tilde{\sigma}_t$ is comprised of the instantaneous zero modes $\tilde{\gamma}_{i,t}$, for instance $\tilde{\sigma}_z = i\tilde{\gamma}_{1,t}\tilde{\gamma}_{2,t}$. The logical space is spanned by the instantaneous zero modes $\tilde{\gamma}_{i,t}$, subject to the parity constraints $\tilde{\gamma}_{1,t}\tilde{\gamma}_{2,t}\tilde{\gamma}_{3,t}\tilde{\gamma}_{4,t} = \pm 1$ from the Coulomb blockade. Given the state in (4.8) we can infer to a mixed state on the instantaneous logical space, namely

$$\rho_{\text{inst}}(t) = \text{tr}_{\tilde{c}}\rho(t). \tag{4.11}$$

where $\text{tr}_{\tilde{c}}$ is the trace over all degrees of freedom but the instantaneous zero mode. By writing the trace over the degrees of freedom of the instantaneous zero mode by $\text{tr}_{\tilde{\gamma}}$ we find the probability $P_{\tilde{r},\bar{r}}$ of measuring the state $\rho_{\tilde{r}}$ as

$$\begin{aligned}
P_{\tilde{r},\bar{r}} &= \text{tr}_{\tilde{\gamma}}\bigg(\frac{1}{2}(\mathbb{1} + \tilde{r} \cdot \tilde{\sigma})\text{tr}_{\tilde{c}}\Big(U(t)\frac{1}{2}(\mathbb{1} + \tilde{r} \cdot \tilde{\sigma})\rho_c\Big)\bigg) \\
&= \frac{1}{4}\text{tr}\Big(\big(\mathbb{1} + \tilde{r} \cdot \tilde{\sigma}_t\big)U(t)\big(\mathbb{1} + \bar{r} \cdot \bar{\sigma}\big)\rho_c U^\dagger(t)\Big) \\
&= \frac{1}{4}\text{tr}\Big(U^\dagger(t)\big(\mathbb{1} + \tilde{r} \cdot \tilde{\sigma}_t\big)U(t)\big(\mathbb{1} + \bar{r} \cdot \bar{\sigma}\big)\rho_c\Big) \\
&= \frac{1}{4}\text{tr}(\rho_c) + \frac{1}{4}\sum_{i=x,y,z} \tilde{r}_i\bar{r}_i\text{tr}\Big(U^\dagger(t)\tilde{\sigma}_{i,t}U(t)\sigma_i\big)\rho_c\Big) + \dots \\
&= \frac{1}{2} + \frac{1}{4}\sum_{i=x,y,z} \tilde{r}_i\bar{r}_i\text{tr}\Big(U^\dagger(t)\tilde{\sigma}_{i,t}U(t)\sigma_i\big)\rho_c\Big).
\end{aligned} \tag{4.12}$$

In the above we have made use of the fact that trace is basis independent. The terms that were left out are terms proportional to either σ_i or $\tilde{\sigma}_i$. Since there is no term coupling the different Majoranas in H_t, these types of terms, that are odd in a particualar Majorana operator, be it the initial or an instantaneous one, are off diagonal, and thus their trace vanish.
We are ultimately interested in learning how rapidly information leaks out of the system, so let us initialise the system in the $+1$ eigenstate of $\sigma_z = i\gamma_1\gamma_2$. We then want to evaluate the probability of measuring $+1$ of the corresponding later instantaneous operator $\tilde{\sigma}_{z,t} = i\tilde{\gamma}_{1,t}\tilde{\gamma}_{2,t}$. This means we also take $\tilde{r} = \hat{z}$ and find

$$P_{\tilde{z},z}(t) = \frac{1}{2} - \frac{1}{4}\langle U^\dagger(t)\tilde{\gamma}_{1,t}\tilde{\gamma}_{2,t}U(t)\gamma_1\gamma_2\rangle_c, \tag{4.13}$$

where the subscript c indicates that the average is just with respect to the density matrix ρ_c. If we assume that this initial continuum density matrix seperates, so $\rho_c = \bigotimes_i \rho_{c,i}$. Because of our assumption that each Majorana only interacts with its own continuum, that is we take $\{\alpha_{i,k}^\dagger, \alpha_{j,k}\} = 0$, the time evolution operator also factors to $U(t) = \bigotimes_i U_i(t)$, and $[U_i(t), \gamma_j] = 0$ for $i \neq j$. This means that the average in (4.13) becomes

$$\begin{aligned}
\langle U^\dagger(t)\tilde{\gamma}_{1,t}\tilde{\gamma}_{2,t}U(t)\gamma_1\gamma_2\rangle_c &= \mathrm{tr}\Big(\prod_{i=1}^{4}\Big(U_i^\dagger(t)\Big)\tilde{\gamma}_{1,t}\tilde{\gamma}_{2,t}\prod_{i=1}^{4}\Big(U_i(t)\Big)\gamma_1\gamma_2\prod_{i=1}^{4}\Big(\rho_{c,i}\Big)\Big)\\
&= \mathrm{tr}\Big(U_1^\dagger(t)U_2^\dagger(t)\tilde{\gamma}_{1,t}\tilde{\gamma}_{2,t}U_1(t)U_2(t)\gamma_1\gamma_2\rho_{c,1}\rho_{c,2}\Big)\mathrm{tr}_{c,3,4}\Big(\rho_{c,3}\rho_{c,4}\Big)\\
&= -\mathrm{tr}_{\mathrm{maj}}\Big(\mathrm{tr}_{c,1}\Big(U_1^\dagger(t)\tilde{\gamma}_{1,t}U_1(t)\gamma_1\rho_{c,1}\Big)\mathrm{tr}_{c,2}\Big(U_2^\dagger(t)\tilde{\gamma}_{2,t}U_2(t)\gamma_2\rho_{c,2}\Big)\Big)\\
&= -\sum_{p=0,1}\langle p|\mathrm{tr}_{c,1}\Big(U_1^\dagger(t)\tilde{\gamma}_{1,t}U_1(t)\gamma_1\rho_{c,1}\Big)\mathrm{tr}_{c,2}\Big(U_2^\dagger(t)\tilde{\gamma}_{2,t}U_2(t)\gamma_2\rho_{c,2}\Big)|p\rangle\\
&= -\sum_{p,p'=0,1}\langle p|\mathrm{tr}_{c,1}\Big(U_1^\dagger(t)\tilde{\gamma}_{1,t}U_1(t)\gamma_1\rho_{c,1}\Big)|p'\rangle\langle p'|\mathrm{tr}_{c,2}\Big(U_2^\dagger(t)\tilde{\gamma}_{2,t}U_2(t)\gamma_2\rho_{c,2}\Big)|p\rangle\\
&\equiv -\sum_{p,p'=0,1}G_{\tilde{\gamma}_1}(t,p,p')G_{\tilde{\gamma}_2}(t,p',p) \qquad (4.14)
\end{aligned}$$

From its definition we have

$$G_{\tilde{\gamma}_i}(0,0,0) = G_{\tilde{\gamma}_i}(0,1,1) = 1 \qquad (4.15)$$
$$G_{\tilde{\gamma}_i}(0,0,1) = G_{\tilde{\gamma}_i}(0,1,0) = 0. \qquad (4.16)$$

In the next section we are going to use an equations of motion analysis to derive an expression for the instantaneous Majorana propagator. Let us make an educated guess and assume that $G_{\tilde{\gamma}_i}(t,p,p') \propto \delta_{p,p'}$ and that $G_{\tilde{\gamma}_i}(t,p,p) \equiv G_{\tilde{\gamma}_i}(t)$ is independent of p.In principle we should do the equations of motion analysis with $G_{\tilde{\gamma}_i}(t,p,p')$. However, since our Hamiltonian is quadratic this won't change anything until the inital conditions are inserted (this happens in equation (5.47)), and at that stage we would indeed find the guess to be correct. Using this we find

$$\begin{aligned}
\langle U^\dagger(t)\tilde{\gamma}_{1,t}\tilde{\gamma}_{2,t}U(t)\gamma_1\gamma_2\rangle_c &= -\sum_{p,p'}G_{\tilde{\gamma}_1}(t,p,p)G_{\tilde{\gamma}_2}(t,p',p)\delta_{p,p'}\\
&= -\sum_{p}G_{\tilde{\gamma}_1}(t)G_{\tilde{\gamma}_2}(t) = -2G_{\tilde{\gamma}_1}(t)G_{\tilde{\gamma}_2}(t). \qquad (4.17)
\end{aligned}$$

Thus we find

$$P_{\tilde{z},z}(t) = \frac{1}{2} + \frac{1}{2}G_{\tilde{\gamma}_1}(t)G_{\tilde{\gamma}_2}(t). \qquad (4.18)$$

If we assume that $G_{\tilde{\gamma}_i}(0)$ decays to zero for large times, then we see that $P_{\tilde{z},z}(0) = 1$, and decays to complete randomness at $\frac{1}{2}$ for $t \to \infty$. Note that in the way we set up the problem there is "rotational symmetry" in the sense of states on the Bloch sphere. We chose to take $\tilde{r} = \bar{r} = \hat{z}$, but the exact same functional form as (4.18) would be found for $\tilde{r} = \bar{r} = \hat{x}$ or $\tilde{r} = \bar{r} = \hat{y}$.
So in order to estimate the effect of dephasing due to the potential fluctuations we apparently need to estimate the propagator $G_{\tilde{\gamma}_i}(t)$. In the next section we turn to an equations of motion analysis for $G_{\tilde{\gamma}_i}(t)$, but before we can do that we need to find expressions for the instantaneous Majorana operators $\tilde{\gamma}_{i,t}$.

5 Instantaneous Majorana operator

In this section we will derive an expression for the instantaneous Majorana propagator $\langle\tilde{\gamma}_{i,t}(t)\gamma_i\rangle$ using an equations of motion approach. Before we can do this, we need to know more about the instanteous zero mode, so we will start by tackling this problem. The Hamiltonian is

$$H_t = \sum_{i=1,2}\Big(\sum_k E_k \alpha^\dagger_{i,k}\alpha_{i,k} + \Omega_{i,t}\gamma_i\Big), \tag{5.1}$$

with

$$\Omega_{i,t} \equiv \sum_k (V_{kt}\alpha^\dagger_{i,k} - V^*_{kt}\alpha_{i,k}). \tag{5.2}$$

Since we take the fluctuations $\phi(t)$ to be small compared to μ, the topological protection of the zero modes implies that there must be unitary operators S_t, such that $\tilde{\gamma}_{t,i} = S^\dagger_t\gamma_i S_t$ obeys

$$[H_t, \tilde{\gamma}_{i,t}] = 0. \tag{5.3}$$

In other words $\tilde{\gamma}_{t,i}$ are the instantaneous zero modes of the system. Note that (5.3) is equivalent to

$$[\tilde{H}_t, \gamma_i] = 0, \tag{5.4}$$

with

$$\tilde{H}_t \equiv S_t H_t S^\dagger_t. \tag{5.5}$$

Equation (5.4) provides a more convenient equation to solve. We start with the ansatz

$$S_t = e^{\Gamma_{1,t}\gamma_1 + \Gamma_{2,t}\gamma_2}, \tag{5.6}$$

with

$$\Gamma_{i,t} = \sum_k (F_{kt}\alpha^\dagger_{i,k} + F^*_{kt}\alpha_{i,k}), \tag{5.7}$$

which means that

$$S_t = e^{\Gamma_{1,t}\gamma_1}e^{\Gamma_{2,t}\gamma_2} \equiv S_{t,1}S_{t,2}. \tag{5.8}$$

$\Gamma_{i,t}$ satisfies

$$\begin{aligned}
\Gamma^2_{i,t} &= \sum_{k,k'}\Big(F_{kt}F_{k't}\alpha^\dagger_{i,k}\alpha^\dagger_{i,k'} + F^*_{kt}F^*_{k't}\alpha_{i,k}\alpha^\dagger_{i,k'} + F_{kt}F^*_{k't}\{\alpha^\dagger_{i,k},\alpha_{i,k'}\}\Big)\\
&= \sum_{k,k'}\Big(\frac{1}{2}F_{kt}F_{k't}\{\alpha^\dagger_{i,k},\alpha^\dagger_{i,k'}\} + \frac{1}{2}F^*_{kt}F^*_{k't}\{\alpha_{i,k},\alpha^\dagger_{i,k'}\} + F_{kt}F^*_{k't}\delta_{k,k'}\Big)\\
&= \sum_k |F_{kt}|^2 \equiv a^2_t,
\end{aligned} \tag{5.9}$$

which means that $(\Gamma_{i,t}\gamma_i)^2 = -\Gamma^2_{i,t} = -a^2_t$. This means that

$$\begin{aligned}
S_{i,t} &= \sum_{n=0}^{\infty}\frac{1}{n!}(\Gamma_{i,t}\gamma_i)^n = \sum_{n\text{ even}}\frac{1}{n!}(-1)^{n/2}a^n_t + \sum_{n\text{ odd}}\frac{1}{n!}(-1)^{(n-1)/2}a^{n-1}_t\\
&= \cos a_t + \frac{\sin a_t}{a_t}\Gamma_{i,t}\gamma_i.
\end{aligned} \tag{5.10}$$

Transforming the second term in (5.1) gives

$$\begin{aligned}
S_t\Omega_{i,t}\gamma_i S_t^\dagger &= S_{i,t}\Omega_{i,t}\gamma_i S_{i,t}^\dagger = \big(\cos a_t + \frac{\sin a_t}{a_t}\Gamma_{i,t}\gamma_i\big)\Omega_{i,t}\gamma_i\big(\cos a_t - \frac{\sin a_t}{a_t}\Gamma_{i,t}\gamma_i\big)\\
&= \cos^2 a_t\Omega_{i,t}\gamma_i - \frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t}\gamma_i\Omega_{i,t}\gamma_i\Gamma_{i,t}\gamma_i + \frac{\cos a_t \sin a_t}{a_t}[\Gamma_{i,t}\gamma_i, \Omega_{i,t}\gamma_i]\\
&= \cos^2 a_t\Omega_{i,t}\gamma_i + \frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t}\Omega_{i,t}\Gamma_{i,t}\gamma_i - \frac{\sin 2a_t}{2a_t}[\Gamma_{i,t}, \Omega_{i,t}]. \qquad (5.11)
\end{aligned}$$

Notice that

$$\Gamma_{i,t}\Omega_{i,t}\Gamma_{i,t} = \Gamma_{i,t}\Omega_{i,t}\Gamma_{i,t} + \Gamma_{i,t}^2\Omega_{i,t} - a_t^2\Omega_{i,t} = \Gamma_{i,t}\{\Omega_{i,t}, \Gamma_{i,t}\} - a_t^2\Omega_{i,t}, \qquad (5.12)$$

and that

$$\{\Omega_{i,t}, \Gamma_{i,t}\} = \sum_{k,k'}\{F_{kt}\alpha_{i,k}^\dagger + F_{kt}^*\alpha_{i,k}, V_{k't}\alpha_{i,k'}^\dagger - V_{k't}^*\alpha_{i,k'}\} = \sum_k(F_{kt}^*V_{kt} - F_{kt}V_{kt}^*). \qquad (5.13)$$

We make the choice that F_{kt} and V_{kt} have the same phase, so that (5.13) vanishes, which we will later find to be self-consistent choice. This means that (5.12) becomes

$$\Gamma_{i,t}\Omega_{i,t}\Gamma_{i,t} = -a_t^2\Omega_{i,t}. \qquad (5.14)$$

Using this (5.11) becomes

$$\begin{aligned}
S_t\Omega_{i,t}\gamma_i S_t^\dagger &= \cos^2 a_t\Omega_{i,t}\gamma_i - \frac{\sin^2 a_t}{a_t^2}a_t^2\Omega_{i,t}\gamma_i - \frac{\sin 2a_t}{2a_t}[\Gamma_{i,t}, \Omega_{i,t}]\\
&= \cos 2a_t\Omega_{i,t}\gamma_i - \frac{\sin 2a_t}{2a_t}[\Gamma_{i,t}, \Omega_{i,t}]. \qquad (5.15)
\end{aligned}$$

Using $[\alpha_{i,k}^\dagger, \alpha_{i,k'}] = 2\alpha_{i,k}^\dagger\alpha_{i,k'} - \delta_{k,k'}$, the commutator may be evaluated:

$$\begin{aligned}
[\Gamma_{i,t}, \Omega_{i,t}] &= \sum_{k,k'}\big[(F_{kt}\alpha_{i,k}^\dagger + F_{kt}^*\alpha_{i,k}), (V_{k't}\alpha_{i,k'}^\dagger - V_{k't}^*\alpha_{i,k'})\big]\\
&= \sum_{k,k'}\Big(2F_{kt}V_{k't}\alpha_{i,k}^\dagger\alpha_{i,k'}^\dagger - 2F_{kt}^*V_{k't}^*\alpha_{i,k}\alpha_{i,k'} - F_{kt}V_{kt}^*[\alpha_{i,k}^\dagger, \alpha_{i,k'}] + F_{kt}^*V_{kt}[\alpha_{i,k}, \alpha_{i,k'}^\dagger]\Big)\\
&= \sum_{k,k'}\Big(2F_{kt}V_{k't}\alpha_{i,k}^\dagger\alpha_{i,k'}^\dagger - 2F_{kt}^*V_{k't}^*\alpha_{i,k}\alpha_{i,k'} - 2F_{kt}V_{kt}^*\alpha_{i,k}^\dagger\alpha_{i,k'} + 2F_{kt}^*V_{kt}\alpha_{i,k}\alpha_{i,k'}^\dagger\\
&\quad - \delta_{k,k'}(F_{kt}^*V_{kt} - F_{kt}V_{kt}^*)\Big), \qquad (5.16)
\end{aligned}$$

and the last term is zero from the assumption $F_{kt}^*V_{kt} \in \mathbb{R}$, so

$$\begin{aligned}
S_t\Omega_{i,t}\gamma_i S_t^\dagger &= \sum_{k,k'}\Big(2F_{kt}V_{k't}\alpha_{i,k}^\dagger\alpha_{i,k'}^\dagger - 2F_{kt}^*V_{k't}^*\alpha_{i,k}\alpha_{k',i} - F_{kt}V_{kt}^*[\alpha_{i,k}^\dagger, \alpha_{i,k'}] + F_{kt}^*V_{kt}[\alpha_{i,k}, \alpha_{i,k'}^\dagger]\Big)\\
&= \sum_{k,k'}\Big(2F_{kt}V_{k't}\alpha_{i,k}^\dagger\alpha_{k',i}^\dagger - 2F_{kt}^*V_{k't}^*\alpha_{i,k}\alpha_{i,k'} - 2F_{kt}V_{kt}^*\alpha_{i,k}^\dagger\alpha_{i,k'} + 2F_{kt}^*V_{kt}\alpha_{i,k}\alpha_{i,k'}^\dagger\Big)\\
&= 2\Gamma_{i,t}\Omega_{i,t}, \qquad (5.17)
\end{aligned}$$

This means

$$S_t\Omega_{i,t}\gamma_i S_t^\dagger = \cos 2a_t \Omega_{i,t}\gamma_i - \frac{\sin 2a_t}{a_t}\Gamma_{i,t}\Omega_{i,t}. \tag{5.18}$$

Next we define $n_{i,k} \equiv \alpha_{i,k}^\dagger \alpha_{i,k}$ and transform the first term in (5.1), obtaining

$$\begin{aligned}
S_t n_{i,k} S_t^\dagger = S_{t,i} n_{i,k} S_{t,i}^\dagger &= \big(\cos a_t + \frac{\sin a_t}{a_t}\Gamma_{i,t}\gamma_i\big) n_{i,k}\big(\cos a_t - \frac{\sin a_t}{a_t}\Gamma_{i,t}\gamma_i\big) \\
&= \cos^2 a_t n_{i,k} - \frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t}\gamma_i n_{i,k}\Gamma_{i,t}\gamma_i - \frac{\sin 2a_t}{2a_t}[n_{i,k}, \Gamma_{i,t}\gamma_i] \\
&= \cos^2 a_t n_{i,k} + \frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t} n_{i,k}\Gamma_{i,t} + \frac{\sin 2a_t}{2a_t}\gamma_i [n_{i,k}, \Gamma_{i,t}] \\
&= \cos^2 a_t n_{i,k} + \frac{\sin^2 a_t}{a_t^2}\big(\Gamma_{i,t}[n_{i,k}, \Gamma_{i,t}] + \Gamma_{i,t} n_{i,k}\big) + \frac{\sin 2a_{i,t}}{2a_{i,t}}\gamma_i [n_{i,k}, \Gamma_{i,t}] \\
&= n_{i,k} + \Big(\frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t} + \frac{\sin 2a_t}{2a_t}\gamma_i\Big)[n_{i,k}, \Gamma_{i,t}].
\end{aligned} \tag{5.19}$$

This may be simplified, since

$$\begin{aligned}
[n_{i,k}, \Gamma_{i,t}] &= \sum_{k'}\Big(F_{k't}[\alpha_{i,k}^\dagger\alpha_{i,k}, \alpha_{i,k'}^\dagger] + F_{k't}^*[\alpha_{i,k}^\dagger\alpha_{i,k}, \alpha_{i,k'}]\Big) \\
&= \sum_{k'}\Big(F_{k't}\big(\alpha_{i,k}^\dagger\{\alpha_{i,k}, \alpha_{i,k'}^\dagger\} - \alpha_{i,k}^\dagger\{\alpha_{i,k}^\dagger, \alpha_{i,k'}^\dagger\}\alpha_{i,k}\big) + F_{k't}^*\big(\alpha_{i,k}^\dagger\{\alpha_{i,k}, \alpha_{i,k'}\} - \{\alpha_{i,k}^\dagger, \alpha_{i,k'}\}\alpha_{i,k}\big)\Big) \\
&= F_{kt}\alpha_{i,k}^\dagger - F_{kt}^*\alpha_{i,k},
\end{aligned} \tag{5.20}$$

so

$$S_t n_{i,k} S_t^\dagger = n_{i,k} + \Big(\frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t} + \frac{\sin 2a_t}{2a_t}\gamma_i\Big)\big(F_{kt}\alpha_{i,k}^\dagger - F_{kt}^*\alpha_{i,k}\big). \tag{5.21}$$

Combining (5.18) and (5.21) we thus find

$$\begin{aligned}
S_t H_t S_t^\dagger = \sum_i \Bigg[&- \frac{\sin 2a_t}{a_t}\Gamma_{i,t}\Omega_{i,t} + \sum_k E_k\Big(n_{i,k} + \frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t}\big(F_{kt}\alpha_{i,k}^\dagger - F_{kt}^*\alpha_{i,k}\big)\Big) \\
&+ \sum_k \Big(\cos 2a_t\Omega_{i,t} - \frac{\sin 2a_t}{2a_t}E_k\big(F_{kt}\alpha_{i,k}^\dagger - F_{kt}^*\alpha_{i,k}\big)\Big)\gamma_i\Bigg].
\end{aligned} \tag{5.22}$$

Equation (5.4) is satisfied when the second term vanishes, so we choose

$$\cos 2a_t\big(V_{kt}\alpha_{i,k}^\dagger - V_{kt}^*\alpha_{i,k}\big) - \frac{\sin 2a_t}{2a_t}E_k\big(F_{kt}\alpha_{i,k}^\dagger - F_{kt}^*\alpha_{i,k}\big) = 0, \tag{5.23}$$

which is satisfied when

$$\begin{aligned}
\cos 2a_t V_{kt} &= \frac{\sin 2a_t}{2a_t}E_k F_{kt} \\
\Rightarrow F_{kt} &= 2a_t\frac{V_{kt}}{E_k}\cot 2a_t,
\end{aligned} \tag{5.24}$$

and (5.9) gives

$$
\begin{aligned}
\sum_k |F_{kt}|^2 = a_t^2 = 4a_t^2 \sum_k \left(\frac{|V_{kt}|}{E_k}\right)^2 \cot^2 2a_t \\
\Rightarrow \tan^2 2a_t = 4\sum_k \left(\frac{|V_{kt}|}{E_k}\right)^2. \qquad (5.25)
\end{aligned}
$$

Note that (5.24) satisfies the previous assumption of $F_{kt}^* V_{kt} \in \mathbb{R}$. Using (5.24) we can write the remaining terms left in the Hamiltonian as

$$
\begin{aligned}
\tilde{H}_t = S_t H_t S_t^\dagger &= \sum_{i,k} E_k n_{i,k} + \sum_i \Big(-\frac{\sin 2a_t}{a_t}\Gamma_{i,t}\Omega_{i,t} + \sum_k E_k \frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t} 2a_t \frac{1}{E_k}\cot 2a_t \big(V_{kt}\alpha_{i,k}^\dagger - V_{kt}^*\alpha_{i,k}\big)\Big) \\
&= \sum_{i,k} E_k n_{i,k} + \sum_i \frac{1}{a_t}\Big(2\sin^2 a_t \cot 2a_t - \sin 2a_t\Big)\Gamma_{i,t}\Omega_{i,t}. \qquad (5.26)
\end{aligned}
$$

This could now be diagonalized by doing a Bogoliubov transformation, so that

$$
\tilde{H}_t = \sum_k \tilde{E}_{k,t}\beta_k^\dagger \beta_k. \qquad (5.27)
$$

We won't need these details, but we are now in shape to calculate the instantaneous zero mode:

$$
\begin{aligned}
\tilde{\gamma}_{i,t} &= S_t^\dagger \gamma_i S_t = S_{i,t}^\dagger \gamma_i S_{i,t} \\
&= \big(\cos a_t + \frac{\sin a_t}{a_t}\Gamma_{i,t}\gamma_i\big)\gamma_i\big(\cos a_t - \frac{\sin a_t}{a_t}\Gamma_{i,t}\gamma_i\big) \\
&= \cos^2 a_t \gamma_i - \frac{\sin^2 a_t}{a_t^2}\Gamma_{i,t}\gamma_i\gamma_i\Gamma_{i,t}\gamma_i + \frac{\sin 2a_t}{2a_t}[\Gamma_{i,t}\gamma_i, \gamma_i] \\
&= \cos^2 a_t \gamma_i - \sin^2 a_t \gamma_i + \frac{\sin 2a_t}{2a_t}(\Gamma_{i,t}\gamma_i\gamma_i - \gamma_i\Gamma_{i,t}\gamma_i) \\
&= \cos 2a_t \gamma_i + \frac{\sin 2a_t}{a_t}\Gamma_{i,t}. \qquad (5.28)
\end{aligned}
$$

Since we want the non adiabatic corrections to the time evolution of the zero mode under the fluctuations, we want to have an expansion in $\dot{\phi} \ll \mu^2$, or $\frac{V_{i,kt}}{E_k^2} \ll 1$, and in order to do this it turns out to be much more useful to stay to the original basis, and instead transform the time evolution operator. The time evolution operator is written out by using infinitesimal time slices Δt and transforming H_t at each step using S_t:

$$
\begin{aligned}
U(t) = \mathcal{T}_t e^{-i\int_0^t dt' H_{t'}} &= e^{-i\Delta t H_t} e^{-i\Delta t H_{t-\Delta t}} \dots e^{-i\Delta t H_0} + \mathcal{O}(\Delta t^2) \\
&= S_t^\dagger e^{-i\Delta t \tilde{H}_t} S_t S_{t-\Delta t}^\dagger e^{-i\Delta t \tilde{H}_{t-\Delta t}} \dots S_0^\dagger e^{-i\Delta t \tilde{H}_0} S_0. \qquad (5.29)
\end{aligned}
$$

We assume that the initialisation is perfect with the fluctuations turned off at $t = 0$ and $\phi(0) = 0$. This means that $S_0 = \mathbb{1}$. Baker-Hausdorff gives

$$
\begin{aligned}
S_t S^\dagger_{t-\Delta t} &= \prod_i e^{\Gamma_{i,t}\gamma_i} e^{-\Gamma_{i,t-\Delta t}\gamma_i} \approx \prod_i e^{(\Gamma_{it}-\Gamma_{i,t-\Delta t})\gamma_i} e^{\frac{1}{2}[\Gamma_{i,t}\gamma_i,\Gamma_{i,t-\Delta t}\gamma_i]} \\
&= \prod_i e^{(\Gamma_{it}-\Gamma_{i,t-\Delta t})\gamma_i} e^{\mathcal{O}(a^2)} \\
&\approx \prod_i e^{\Delta t \dot{\Gamma}_{i,t}\gamma_i},
\end{aligned} \tag{5.30}
$$

as well as

$$
\begin{aligned}
e^{-i\Delta t \tilde{H}_t} e^{\Delta t \dot{\Gamma}_{i,t}\gamma_i} &\approx e^{-i\Delta t(\tilde{H}_t + i\dot{\Gamma}_{i,t}\gamma_i)} e^{-\frac{1}{2}[-i\tilde{H}_t\Delta t, \dot{\Gamma}_{i,t}\gamma_i\Delta t]} \\
&= e^{-i\Delta t(\tilde{H}_t + i\dot{\Gamma}_{i,t}\gamma_i)}
\end{aligned} \tag{5.31}
$$

to order Δt. In total we see

$$
U_i(t) = S_t^\dagger \mathcal{T}_t e^{-i\int_0^t dt'(\tilde{H}_{t'} + i\dot{\Gamma}_{i,t'}\gamma_i)} \equiv S_t^\dagger \tilde{U}_i(t). \tag{5.32}
$$

The only approximation happened when deriving (5.32) happened in equation (5.30). In this step we collected linear order in $\dot{a}_{i,t}$ and dropped high orders of $a_{i,t}$ afterwards. In principle higher order derivatives might appear. In this sense equation (5.32) is an approximation only including some aspects of non-adiabaticity.
Now we are all set to calculate the instantaneous Majorana propagator

$$
G_{\tilde{\gamma}_i}(t) \equiv \langle \tilde{\gamma}_{i,t}(t)\gamma_i\rangle, \tag{5.33}
$$

using an equations of motion approach. Since $\tilde{\gamma}_{i,t}(t) = U^\dagger(t)\tilde{\gamma}_{i,t}U(t)$ we may rewrite equation (5.33) by using equation (5.32), and this gives us

$$
G_{\tilde{\gamma}_i}(t) \approx \langle \tilde{U}^\dagger(t) S_t S_t^\dagger \gamma_i S_t S_t^\dagger \tilde{U}\rangle_c = \langle \tilde{U}^\dagger(t)\gamma_i\tilde{U}(t)\gamma_i\rangle_c. \tag{5.34}
$$

Since $\frac{d\gamma_i}{dt} = 0$, we can use the Heisenberg equations of motion to get

$$
\begin{aligned}
i\partial_t G_{\tilde{\gamma}_i} &= -\langle \tilde{U}^\dagger(t)[\tilde{H}_t + i\dot{\Gamma}_{i,t}\gamma_i, \gamma_i]\tilde{U}(t)\gamma_i\rangle_c + 0 \\
&= 0 - i\langle \tilde{U}^\dagger(t)[\dot{\Gamma}_{i,t}\gamma_i, \gamma_i]\tilde{U}(t)\gamma_i\rangle_c = -2i\langle \tilde{U}^\dagger(t)\dot{\Gamma}_{i,t}\tilde{U}(t)\gamma_i\rangle_c.
\end{aligned} \tag{5.35}
$$

To lowest order in $\frac{V_{kt}}{E_k}$ we have

$$
\begin{aligned}
\dot{\Gamma}_{i,t} &= \frac{d}{dt}\sum_k \left(2a_t\cot(2a_t)\frac{V_{kt}}{E_k}\alpha^\dagger_{i,k} + 2a_t\cot(2a_t)\frac{V^*_{kt}}{E_k}\alpha_{i,k}\right) \\
&\approx \sum_k \Big(\frac{\dot{V}_{kt}}{E_k}\alpha^\dagger_{i,k} + \frac{\dot{V}^*_{kt}}{E_k}\alpha_{i,k}\Big).
\end{aligned} \tag{5.36}
$$

Using this (5.35) becomes

$$
i\partial_t G_{\tilde{\gamma}_i} = -2i\sum_k \Big(\frac{\dot{V}_{kt}}{E_k}G_{\alpha^\dagger_{i,k},\gamma_i}(t) + \frac{\dot{V}^*_{kt}}{E_k}G_{\alpha_{i,k},\gamma_i}(t)\Big), \tag{5.37}
$$

where

$$G_{\tilde{\alpha}^{(\dagger)}_{i,k},\gamma_i} \equiv \langle \tilde{U}^\dagger(t)\alpha^{(\dagger)}_{i,k}\tilde{U}(t)\gamma_i\rangle_0. \tag{5.38}$$

Since $\frac{d\alpha_{i,k}}{dt} = 0$ the Heisenberg equation of motion for $G_{\alpha_{i,k},\gamma_i}$ gives

$$\begin{aligned} i\partial_t G_{\alpha_{i,k},\gamma_i} &= -\langle \tilde{U}^\dagger(t)[\tilde{H}_t + i\dot{\Gamma}_{i,t}\gamma_i, \alpha_k]\tilde{U}(t)\gamma_i\rangle_0 + 0 \\ &= -\langle \tilde{U}^\dagger(t)[\tilde{H}_t, \alpha_k]\tilde{U}(t)\gamma_i\rangle_0 - i\langle \tilde{U}^\dagger(t)(0 - \{\dot{\Gamma}_{i,t}, \alpha_k\}\gamma_i)\tilde{U}(t)\gamma_i\rangle_0. \end{aligned} \tag{5.39}$$

Using (5.26) we see that to order $\mathcal{O}(a_{i,t}^2)$

$$[\tilde{H}_t, \alpha_{i,k}] = -E_k\alpha_{i,k}. \tag{5.40}$$

Using again (5.36) we see

$$\{\dot{\Gamma}_{i,t}, \alpha_{i,k}\} \approx \frac{\dot{V}_{kt}}{E_k}. \tag{5.41}$$

Hence (5.39) becomes

$$(i\partial_t - E_k)G_{\alpha_{i,k},\gamma_i} = i\frac{\dot{V}_{kt}}{E_k}G_{\tilde{\gamma}_i}. \tag{5.42}$$

The solution to this differential equation, with the boundary condition $G_{\alpha_{i,k},\gamma_i}(0) = 0$ is

$$G_{\alpha_{i,k},\gamma_i}(t) = \int_0^t e^{-iE_k(t-s)}\frac{\dot{V}_{ks}}{E_k}G_{\tilde{\gamma}_i}(s), \tag{5.43}$$

which we can verfiy by differentating:

$$\begin{aligned} i\partial_t G_{\alpha_{i,k},\gamma_i}(t) &= ie^{-iE_k(t-t)}\frac{\dot{V}_{kt}}{E_k}G_{\tilde{\gamma}_i}(t) - i^2E_k\int_0^t e^{-iE_k(t-s)}\frac{\dot{V}_{ks}}{E_k}G_{\tilde{\gamma}_i}(s) \\ &= i\frac{\dot{V}_{kt}}{E_k}G_{\tilde{\gamma}_i}(t) + E_kG_{\alpha_{i,k},\gamma_i}(t). \end{aligned} \tag{5.44}$$

Going through the same calculation for $G_{\alpha^\dagger_{i,k},\gamma_i}$ amounts to exhanging $\alpha_{i,k}$ with $\alpha^\dagger_{i,k}$, and using $[\tilde{H}_t, \alpha^\dagger_{i,k}] = E_k\alpha^\dagger_{i,k}$ in contrast to (5.40), as well as using $\{\dot{\Gamma}_{i,t}, \alpha^\dagger_{i,k}\} \approx \frac{\dot{V}^*_{kt}}{E_k}$ instead of (5.41). This means

$$G_{\alpha^\dagger_{i,k},\gamma_i}(t) = \int_0^t e^{iE_k(t-s)}\frac{\dot{V}^*_{ks}}{E_k}G_{\tilde{\gamma}_i}(s), \tag{5.45}$$

Plugging (5.43) and (5.45) into (5.37) we get

$$i\partial_t G_{\tilde{\gamma}_i} = -2i\sum_k\int_0^t ds'\Big(\frac{\dot{V}_{kt}}{E_k}e^{iE_k(t-s')}\frac{\dot{V}^*_{ks'}}{E_k} + \frac{\dot{V}^*_{kt}}{E_k}e^{-iE_k(t-s')}\frac{\dot{V}_{ks'}}{E_k}\Big)G_{\tilde{\gamma}_i}(s'). \tag{5.46}$$

We can integrate this using $G_{\tilde{\gamma}_i}(0) = 1$, and we obtain

$$G_{\tilde{\gamma}_i}(t) = 1 - 2\sum_k\int_0^t ds\int_0^s ds'\Big(\frac{\dot{V}_{ks}}{E_k}e^{iE_k(s-s')}\frac{\dot{V}^*_{ks'}}{E_k} + \frac{\dot{V}^*_{ks}}{E_k}e^{-iE_k(s-s')}\frac{\dot{V}_{ks'}}{E_k}\Big)G_{\tilde{\gamma}_i}(s'), \tag{5.47}$$

so in conclusion

$$G_{\tilde{\gamma}_i}(t) = 1 + \int_0^t ds \int_0^s ds' K(s,s') G_{\tilde{\gamma}_i}(s') \tag{5.48}$$

$$K(s,s') = -2 \sum_k \Big(\frac{\dot{V}_{ks}}{E_k} e^{iE_k(s-s')} \frac{\dot{V}^*_{ks'}}{E_k} + \frac{\dot{V}^*_{ks}}{E_k} e^{-iE_k(s-s')} \frac{\dot{V}_{ks'}}{E_k} \Big). \tag{5.49}$$

6 The eigenfunctions of the p-wave superconductor and matrix elements

In this section we will calculate the matrix elements (4.2) in order to obtain an expression for V_{kt}, as given by (4.5). We use the Hamiltonian H_{pw} as in (2.5) and want to calculate $\delta H_{0,k}$ with respect to the eigenstates of H_{pw}. For simplicity I will find these under the assumption $0 < \mu < \frac{m\Delta^2}{2}$, and we take the system as half invinite, with its left edge at $x = 0$. First we will calculate the zero energy solution $\psi_0(x)$, and then afterwards we will look at the solutions $\psi_E(x)$ with $|E| \geq \mu$. Note that in this section I'm labeling the above gap eigenstates by the energy E. Elsewhere I'm using k, namely the wavenumber, but these are equivalent, and changing from a sum over k to an integral over E as usual involves the inclusion of the density of states $\rho(E) = \frac{dk}{dE}$.
The zero energy eigenstate ψ_0 solves the time independent Schrödinger equation

$$\begin{pmatrix} \frac{p^2}{2m} - \mu & \Delta p \\ \Delta p & -\frac{p^2}{2m} + \mu \end{pmatrix} \psi = 0. \tag{6.1}$$

I look for a solution of the form $\psi_0(x) = \sum_z \psi_z(x) = \sum_z \begin{pmatrix} u_z \\ v_z \end{pmatrix} e^{izx} = \sum_z \chi_z e^{izx}$ satisfying the boundary conditions that $\psi(0) = \lim_{x\to\infty} \psi(x) = 0$. Each $\psi_z(x)$ satisfies (6.1) and are linearly independent. This gives us

$$\begin{pmatrix} \frac{z^2}{2m} - \mu & \Delta z \\ \Delta z & -\frac{z^2}{2m} + \mu \end{pmatrix} \chi_z = 0, \tag{6.2}$$

which implies

$$\begin{aligned} 0 &= -(\frac{z^2}{2m} - \mu)^2 - \Delta^2 z^2 \\ \Rightarrow \frac{z^2}{2m} &- s_1 i\Delta z - \mu = 0 \\ \Rightarrow z &= is_1 m\Delta + s_2\sqrt{-m^2\Delta^2 + 2m\mu} = im\Delta\Big(s_1 + s_2\sqrt{1 - 2\frac{\mu}{m\Delta^2}}\Big), \end{aligned} \tag{6.3}$$

where $s_1, s_2 = \pm 1$. In order to have a normalisable solution the imaginary part of z must på positive. Assuming $0 < \mu < \frac{m\Delta^2}{2}$ we have $1 > \sqrt{1 - 2\frac{\mu}{m\Delta^2}}$ and so

$$z_\pm = im\Delta\Big(1 \pm \sqrt{1 - 2\frac{\mu}{m\Delta^2}}\Big). \tag{6.4}$$

This means that

$$\psi_0(x) = e^{iz_+x}\chi_+ + e^{iz_-x}\chi_-. \tag{6.5}$$

The boundary condition $\psi(0) = 0$ gives $\chi_+ = -\chi_- \equiv \chi$. Noting that both components of χ must be non-zero for a non-trivial solution, we write $\chi = \frac{1}{N}\begin{pmatrix} 1 \\ b \end{pmatrix}$. Plugging this into (6.2) yields

$$\frac{z_\pm^2}{2m} - \mu + \Delta z_\pm b = 0. \tag{6.6}$$

This means

$$b = \frac{-z_\pm^2 + 2m\mu}{2m\Delta z_\pm} = \frac{m^2\Delta^2(1 \pm \sqrt{1-\frac{2\mu}{m\Delta^2}})^2 - 2m\mu}{i2m^2\Delta^2(1 \pm \sqrt{1-\frac{2\mu}{m\Delta^2}})}$$
$$= -1\frac{m^2\Delta^2\big(1+1-\frac{2\mu}{m\Delta^2} \pm 2\sqrt{1-\frac{2\mu}{m\Delta^2}}\big) - 2m\mu}{2m^2\Delta^2(1 \pm \sqrt{1-\frac{2\mu}{m\Delta^2}})} = -i. \tag{6.7}$$

Thus the wavefunction for the Majorana zero mode is

$$\psi_0(x) = \frac{1}{\mathcal{N}_0}\begin{pmatrix} 1 \\ -i \end{pmatrix}\Big(e^{-m\Delta\big(1+\sqrt{1-\frac{2\mu}{m\Delta^2}}\big)x} - e^{-m\Delta\big(1-\sqrt{1-\frac{2\mu}{m\Delta^2}}\big)x}\Big). \tag{6.8}$$

Letting $l = \sqrt{1-\frac{2\mu}{m\Delta^2}}$ then the normalisation $\mathcal{N}_0$ is

$$|\mathcal{N}_0|^2 = \int_0^\infty dx 2\Big(e^{-m\Delta(1+l)x} - e^{-m\Delta(1-l)x}\Big)^2 = 2\int_0^\infty dx\Big(e^{-2m\Delta(1+l)x} + e^{-2m\Delta(1-l)x} - 2e^{-2m\Delta x}\Big)$$
$$= 2\Big(\frac{-1}{-2m\Delta(1+l)} + \frac{-1}{-2m\Delta(1-l)} - \frac{-2}{-2m\Delta}\Big)$$
$$= \frac{1}{m\Delta}\Big(\frac{1}{1+l} + \frac{1}{1-l} - 2\Big) = \frac{1}{m\Delta}\frac{1-l+1+l-2(1+l)(1-l)}{(1+l)(1-l)}$$
$$= \frac{2}{m\Delta}\frac{l^2}{1-l^2} = \frac{2}{m\Delta}\frac{1-\frac{2\mu}{m\Delta^2}}{\frac{2\mu}{m\Delta^2}}, \tag{6.9}$$

so

$$\mathcal{N}_0 = \sqrt{\frac{2}{m\Delta}\frac{1-\frac{2\mu}{m\Delta^2}}{\frac{2\mu}{m\Delta^2}}}. \tag{6.10}$$

Above the gap the wavefunctions still have the same form $\psi_E(x) = \sum_k \chi_k e^{ikx}$, where k now is a a continuous function of the energy E. The Schrödinger equation for each term is now

$$\begin{pmatrix} \frac{k^2}{2m} - \mu & \Delta k \\ \Delta k & -\frac{k^2}{2m} + \mu \end{pmatrix}\chi_k = E\chi_k. \tag{6.11}$$

Let us find an expression of k as a function of energy. The eigenvalue equation for (6.11) gives us

$$0 = (\frac{k^2}{2m} - \mu - E)(\frac{-k^2}{2m} + \mu - E) - \Delta^2 k^2 = E^2 - (\frac{k^2}{2m} - \mu)^2 - \Delta^2 k^2, \tag{6.12}$$

or

$$E = \pm\sqrt{(\frac{k^2}{2m} - \mu)^2 + \Delta^2 k^2}. \tag{6.13}$$

This shows that $E \geq \mu$. Squaring the above yields

$$E^2 = (\frac{k^2}{2m} - \mu)^2 + \Delta^2 k^2 = \frac{1}{4m^2}k^4 + (\Delta^2 - \frac{\mu}{m})k^2 + \mu^2, \tag{6.14}$$

which we can solvce for

$$k^2 = 2m^2\Big(-\Delta^2 + \frac{\mu}{m} \pm \sqrt{\big(\Delta^2 - \frac{\mu}{m}\big)^2 + \frac{E^2-\mu^2}{m^2}}\Big), \tag{6.15}$$

whereby we find

$$k = s_1\sqrt{2}m\Delta\sqrt{-1+\frac{\mu}{m\Delta^2}+s_2\sqrt{\big(1-\frac{\mu}{m\Delta^2}\big)^2+\frac{E^2-\mu^2}{m^2\Delta^4}}} \equiv s_1 k_{s_2}, \tag{6.16}$$

where $s1, s2 = \pm 1$. Since $E \geq \mu$ we see k_+ is real. Furthermore we always have $\big(1-\frac{\mu}{m\Delta^2}\big)^2 + \frac{E^2-\mu^2}{m^2\Delta^4} > 0$. Because we assumed $\mu < \frac{m\Delta^2}{2}$ we now see that k_- is purely imaginary. The corresponding unnormalized wave functions are found from (6.11):

$$\psi_{\pm k_+}(x) = e^{\pm ik_+x}\begin{pmatrix} \pm\Delta k_+ \\ E-\frac{k_+^2}{2m}+\mu \end{pmatrix} \tag{6.17}$$

Since k_- is imaginary the only normalisable solution is

$$\psi_{+k_-}(x) = e^{ik_-x}\begin{pmatrix} \Delta k_- \\ E-\frac{k_-^2}{2m}+\mu \end{pmatrix}. \tag{6.18}$$

For future convenience let us rename $k_+ \equiv k$ and $k_- \equiv i\kappa$. Imposing the boundary conditions we have the wave function at energy E

$$\psi_E(x) = \frac{1}{\mathcal{N}_E}\Big(Ae^{ikx}\begin{pmatrix} \Delta k \\ E-\frac{k^2}{2m}+\mu \end{pmatrix} + Be^{-ikx}\begin{pmatrix} -\Delta k \\ E-\frac{k^2}{2m}+\mu \end{pmatrix} + Ce^{-\kappa x}\begin{pmatrix} i\Delta\kappa \\ E+\frac{\kappa^2}{2m}+\mu \end{pmatrix}\Big), \tag{6.19}$$

which should satisfy $\psi_E(0) = \lim_{x\to\infty}\psi_E(x) = 0$, and we can take $A = 1$.
We can use the continuity equation [6] to continue [6]. It says that $\nabla j = 0$, where the probability current is proportional to $\psi^*\nabla\psi - \psi\nabla\psi^*$. Let us evaluate this for very large x where the last term in (6.19) has completely decayed. Here

$$\nabla\psi_E(x) = \frac{1}{\mathcal{N}_E}\Big(ike^{ikx}\begin{pmatrix} \Delta k \\ -\frac{k^2}{2m} \end{pmatrix} - ikBe^{-ikx}\begin{pmatrix} -\Delta k \\ -\frac{k^2}{2m} \end{pmatrix}\Big), \tag{6.20}$$

so

$$\psi_E^*(x)\nabla\psi_E(x) = ik\big(1-|B|^2\big)\big(\Delta^2k^2+\frac{k^4}{4m^2}\big) + ik\big(B^*e^{2ikx}-Be^{-2ikx}\big)\big(-\Delta^2k^2+\frac{k^4}{4m^2}\big), \tag{6.21}$$

which means that

$$\psi_E^*(x)\nabla\psi_E(x) - \psi_E(x)\nabla\psi_E^*(x) = 2ik\big(1-|B|^2\big)\big(\Delta^2k^2+\frac{k^4}{4m^2}\big)+0. \tag{6.22}$$

Plugging this into the continuity equation gives $|B| = 1$, and we write $B = e^{i\delta}$. Evaluating (6.19) at $x = 0$ gives us two equations for C:

$$0 = \begin{pmatrix} \Delta k \\ E-\frac{k^2}{2m}+\mu \end{pmatrix} + e^{i\delta}\begin{pmatrix} -\Delta k \\ E-\frac{k^2}{2m}+\mu \end{pmatrix} + C\begin{pmatrix} i\Delta\kappa \\ E+\frac{\kappa^2}{2m}+\mu \end{pmatrix}. \tag{6.23}$$

The first one yields

$$C = i\frac{k}{\kappa}(1-e^{i\delta}), \tag{6.24}$$

and the second gives

$$C = -\epsilon(1+e^{i\delta}), \tag{6.25}$$

where

$$\epsilon \equiv \frac{E - \frac{k^2}{2m} + \mu}{E + \frac{\kappa^2}{2m} + \mu}. \tag{6.26}$$

Equating (6.24) and (6.25) we obtain

$$\tan\frac{\delta}{2} = \frac{\kappa}{k}\epsilon, \tag{6.27}$$

and so the wave function above the gap is

$$\psi_E(x) = \frac{1}{\mathcal{N}_E}\Big(e^{ikx}\begin{pmatrix}\Delta k \\ E - \frac{k^2}{2m} + \mu\end{pmatrix} + e^{-ikx+i\delta}\begin{pmatrix}-\Delta k \\ E - \frac{k^2}{2m} + \mu\end{pmatrix} - \epsilon(1+e^{i\delta})e^{-\kappa x}\begin{pmatrix}i\Delta\kappa \\ E + \frac{\kappa^2}{2m} + \mu\end{pmatrix}\Big). \tag{6.28}$$

Now we need to determine the normalisation $\mathcal{N}_E$. This is regularised by cutting off x at L which will later be taken to infinity. Note that when taking the norm of (6.28), the x-independent terms will be proportional to L after integration and thus will dominate all other terms. Thus the normalisation satisfies

$$\mathcal{N}_E \approx \sqrt{L\Big(2\Delta^2 k^2 + 2(E - \frac{k^2}{2m} + \mu)^2\Big)}. \tag{6.29}$$

Now we have the wave functions we can calculate the matrix elements of the perturbing term in (4.2). For later convenience I am expressing everything in terms of dimensionless quantities $\xi \equiv \frac{E}{\mu}$ and $\alpha \equiv \frac{\mu}{m\Delta^2}$. With this we have

$$k = \sqrt{2}m\Delta\sqrt{-1 + \alpha + \sqrt{(1-\alpha)^2 + \alpha^2(\xi^2 - 1)}} \equiv m\Delta\tilde{k}, \tag{6.30}$$

$$\kappa = \sqrt{2}m\Delta\sqrt{1 - \alpha + \sqrt{(1-\alpha)^2 + \alpha^2(\xi^2 - 1)}} \equiv m\Delta\tilde{\kappa}, \tag{6.31}$$

and

$$\begin{aligned}
|\mathcal{N}_0\mathcal{N}_E| &= \sqrt{\frac{2}{m\Delta}\frac{1-2\alpha}{2\alpha}2L\Big(m^2\Delta^4\tilde{k}^2 + \mu^2\big(\xi - \frac{m^2\Delta^2\tilde{k}^2}{2m\mu} + 1\big)^2\Big)} \\
&= \sqrt{\frac{4L\mu^2}{m\Delta}\frac{1-2\alpha}{2\alpha}\Big(\frac{\tilde{k}^2}{\alpha^2} + \big(\xi - \frac{\tilde{k}^2}{2\alpha} + 1\big)^2\Big)} \\
&= \sqrt{\mu\Delta L}\,2\sqrt{(1-2\alpha)\Big(\frac{\tilde{k}^2}{\alpha^2} + \big(\xi - \frac{\tilde{k}^2}{2\alpha} + 1\big)^2\Big)} \\
&\equiv \sqrt{\mu\Delta L}\tilde{\mathcal{N}}.
\end{aligned}$$

Expressing ϵ and δ in terms of the dimensionless quantities gives

$$\epsilon = \frac{\xi - \frac{\tilde{k}^2}{2\alpha} + 1}{\xi + \frac{\tilde{\kappa}^2}{2\alpha} + 1} \tag{6.32}$$

$$\delta = 2\arctan\Big(\frac{\tilde{\kappa}}{\tilde{k}}\epsilon\Big). \tag{6.33}$$

Now let us calculate $\delta H_{0,\xi} = \langle\psi_0|\tau_z|\psi_\xi\rangle$:

$$
\begin{aligned}
\delta H_{0,\xi} &= \frac{1}{\mathcal{N}_0^* \mathcal{N}_\xi} \int_0^L dx \begin{pmatrix} 1 & -i \end{pmatrix} \left(e^{-m\Delta(1+l)x} - e^{-m\Delta(1-l)x} \right) \times \\
&\quad \times \left(e^{ikx} \begin{pmatrix} \Delta k \\ -E + \frac{k^2}{2m} - \mu \end{pmatrix} + e^{-ikx+i\delta} \begin{pmatrix} -\Delta k \\ -E + \frac{k^2}{2m} - \mu \end{pmatrix} - \epsilon(1+e^{i\delta}) e^{-\kappa x} \begin{pmatrix} i\Delta\kappa \\ -E - \frac{\kappa^2}{2m} - \mu \end{pmatrix} \right) \\
&= \frac{1}{\sqrt{\mu \Delta L \tilde{\mathcal{N}}}} \int_0^L dx e^{-m\Delta x} \big(-2\sinh(m\Delta l x) \big) \Big(e^{ikx} \Big(\Delta k + i(E - \frac{k^2}{2m} + \mu) \Big) \\
&\quad + e^{-ikx+i\delta} \Big(-\Delta k + i(E - \frac{k^2}{2m} + \mu) \Big) - \epsilon(1+e^{i\delta}) e^{-\kappa x} \Big(i\Delta\kappa + i(E + \frac{\kappa^2}{2m} + \mu) \Big) \Big) \\
&= \frac{-2e^{i\frac{\delta}{2}}}{\sqrt{\mu \Delta L \tilde{\mathcal{N}}}} \int_0^L dx e^{-m\Delta x} \sinh(m\Delta l x) \Big(2i\Delta k \sin\big(kx - \frac{\delta}{2}\big) + 2i(E - \frac{k^2}{2m} + \mu) \cos\big(kx - \frac{\delta}{2}\big) \\
&\quad - 2i\epsilon \cos\big(\frac{\delta}{2}\big) e^{-\kappa x} (\Delta\kappa + E + \frac{\kappa^2}{2m} + \mu) \Big) \\
&= \frac{-4ie^{i\frac{\delta}{2}}}{\sqrt{\mu \Delta L \tilde{\mathcal{N}}}} \frac{1}{m\Delta} \int_0^{m\Delta L} d\tilde{x} e^{-\tilde{x}} \sinh(l\tilde{x}) \Big(\mu \frac{\tilde{k}}{\alpha} \sin\big(\tilde{k}\tilde{x} - \frac{\delta}{2}\big) + \mu(\xi - \frac{\tilde{k}^2}{2\alpha} + 1) \cos\big(\tilde{k}\tilde{x} - \frac{\delta}{2}\big) \\
&\quad - \epsilon \cos\big(\frac{\delta}{2}\big) e^{-\tilde{\kappa}\tilde{x}} \mu (\frac{\tilde{\kappa}}{\alpha} + \xi + \frac{\tilde{\kappa}^2}{2\alpha} + 1) \Big) \\
&= \frac{-4i\alpha e^{i\frac{\delta}{2}}}{\tilde{\mathcal{N}}} \sqrt{\frac{\Delta}{\mu L}} \int_0^{m\Delta L} d\tilde{x} e^{-\tilde{x}} \sinh(l\tilde{x}) \Big(\frac{\tilde{k}}{\alpha} \sin\big(\tilde{k}\tilde{x} - \frac{\delta}{2}\big) + (\xi - \frac{\tilde{k}^2}{2\alpha} + 1) \cos\big(\tilde{k}\tilde{x} - \frac{\delta}{2}\big) \\
&\quad - \epsilon \cos\big(\frac{\delta}{2}\big) e^{-\tilde{\kappa}\tilde{x}} (\frac{\tilde{\kappa}}{\alpha} + \xi + \frac{\tilde{\kappa}^2}{2\alpha} + 1) \Big) \\
&\approx \frac{-4i\alpha e^{i\frac{\delta}{2}}}{\tilde{\mathcal{N}}} \sqrt{\frac{\Delta}{\mu L}} \int_0^{\infty} d\tilde{x} e^{-\tilde{x}} \sinh(l\tilde{x}) \Big(\frac{\tilde{k}}{\alpha} \sin\big(\tilde{k}\tilde{x} - \frac{\delta}{2}\big) + (\xi - \frac{\tilde{k}^2}{2\alpha} + 1) \cos\big(\tilde{k}\tilde{x} - \frac{\delta}{2}\big) \\
&\quad - \epsilon \cos\big(\frac{\delta}{2}\big) e^{-\tilde{\kappa}\tilde{x}} (\frac{\tilde{\kappa}}{\alpha} + \xi + \frac{\tilde{\kappa}^2}{2\alpha} + 1) \Big),
\end{aligned}
$$

where in the last line it is used that the $L \to \infty$ will be taken later. I can use the following integrals

$$
\begin{aligned}
\int_0^\infty dx e^{-\alpha x} \sinh(\beta x) e^{-\kappa x} &= \frac{\beta}{(\alpha+\kappa)^2 - \beta^2} \\
\int_0^\infty dx e^{-\alpha x} \sinh(\beta x) \sin(\gamma x + \delta) &= \frac{\beta}{(\alpha^2 - \gamma^2 - \beta^2)^2 + 4\alpha^2\gamma^2} \big((\alpha^2 - \gamma^2 - \beta^2) \sin\delta + 2\alpha\gamma\cos\delta \big) \\
\int_0^\infty dx e^{-\alpha x} \sinh(\beta x) \cos(\gamma x + \delta) &= \frac{\beta}{(\alpha^2 - \gamma^2 - \beta^2)^2 + 4\alpha^2\gamma^2} \big((\alpha^2 - \gamma^2 - \beta^2) \cos\delta - 2\alpha\gamma\sin\delta \big),
\end{aligned}
$$

which hold for $\alpha > \beta$ when $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Hence

$$
\begin{aligned}
\delta H_{0,\xi} = {} & \frac{-4i\alpha e^{i\frac{\delta}{2}}}{\tilde{\mathcal{N}}}\sqrt{\frac{\Delta}{\mu L}}\Big(\frac{\tilde{k}}{\alpha}\frac{l}{(1-\tilde{k}^2-l^2)^2+4\tilde{k}^2}\big((1-\tilde{k}^2-l^2)\sin\frac{\delta}{2}+2\tilde{k}\cos\frac{\delta}{2}\big) \\
& + (\xi-\frac{\tilde{k}^2}{2\alpha}+1)\frac{l}{(1-\tilde{k}^2-l^2)^2+4\tilde{k}^2}\big((1^2-\tilde{k}^2-l^2)\cos\frac{\delta}{2}+2\tilde{k}\sin\frac{\delta}{2}\big) \\
& - \epsilon\cos\big(\frac{\delta}{2}\big)\big(\frac{\tilde{\kappa}}{\alpha}+\xi+\frac{\tilde{\kappa}^2}{2\alpha}+1\big)\frac{l}{(1+\tilde{\kappa})^2-l^2}\Big) \\
= {} & -i\sqrt{\frac{\Delta}{\mu L}}\frac{4}{\tilde{\mathcal{N}}}e^{i\frac{\delta}{2}}\alpha\sqrt{1-2\alpha}\Bigg(\frac{1}{(1-\tilde{k}^2-1+2\alpha)^2+4\tilde{k}^2}\Bigg(\cos\frac{\delta}{2}\Big(2\frac{\tilde{k}^2}{\alpha}+(\xi-\frac{\tilde{k}^2}{2\alpha}+1)(1-\tilde{k}^2-1+2\alpha)\Big) \\
& + \sin\frac{\delta}{2}\Big(\frac{\tilde{k}}{\alpha}(1-\tilde{k}^2-1+2\alpha)+2\tilde{k}(\xi-\frac{\tilde{k}^2}{2\alpha}+1)\Big)\Bigg) \\
& - \epsilon\cos\big(\frac{\delta}{2}\big)\big(\frac{\tilde{\kappa}}{\alpha}+\xi+\frac{\tilde{\kappa}^2}{2\alpha}+1\big)\frac{1}{(1+\tilde{\kappa})^2-1+2\alpha}\Bigg)
\end{aligned}
$$

So we obtain the following expression for the matrix element between the zero energy state and a state with energy $E=\mu\xi$:

$$
\begin{aligned}
\delta H_{0,\xi} = {} & -i\sqrt{\frac{\Delta}{\mu L}}\frac{4}{\tilde{\mathcal{N}}}e^{i\frac{\delta}{2}}\alpha\sqrt{1-2\alpha}\Bigg(\frac{1}{(2\alpha-\tilde{k}^2)^2+4\tilde{k}^2}\Bigg(\cos\frac{\delta}{2}\Big(2\frac{\tilde{k}^2}{\alpha}+(\xi-\frac{\tilde{k}^2}{2\alpha}+1)(2\alpha-\tilde{k}^2)\Big) \\
& + \sin\frac{\delta}{2}\Big(\frac{\tilde{k}}{\alpha}(2\alpha-\tilde{k}^2)+2\tilde{k}(\xi-\frac{\tilde{k}^2}{2\alpha}+1)\Big)\Bigg) \\
& - \epsilon\cos\big(\frac{\delta}{2}\big)\big(\frac{\tilde{\kappa}}{\alpha}+\xi+\frac{\tilde{\kappa}^2}{2\alpha}+1\big)\frac{1}{(\tilde{\kappa})^2+2\alpha}\Bigg),
\end{aligned}
\tag{6.34}
$$

where

$$
\begin{aligned}
\xi &= \frac{E}{\mu} \\
\alpha &= \frac{\mu}{m\Delta^2} \\
\tilde{\mathcal{N}} &= 2\sqrt{(1-2\alpha)\Big(\frac{\tilde{k}^2}{\alpha^2}+\big(\xi-\frac{\tilde{k}^2}{2\alpha}+1\big)^2\Big)} \\
\tilde{k} &= \sqrt{2}\sqrt{-1+\alpha+\sqrt{(1-\alpha)^2+\alpha^2(\xi^2-1)}} \\
\tilde{\kappa} &= \sqrt{2}\sqrt{1-\alpha+\sqrt{(1-\alpha)^2+\alpha^2(\xi^2-1)}} \\
\epsilon &= \frac{\xi-\frac{\tilde{k}^2}{2\alpha}+1}{\xi+\frac{\tilde{\kappa}^2}{2\alpha}+1} && (6.35) \\
\delta &= 2\arctan\Big(\frac{\tilde{\kappa}}{\tilde{k}}\epsilon\Big), && (6.36)
\end{aligned}
$$

and the computation assumed $0<\mu<\frac{m\Delta^2}{2}$, or $0<\alpha<\frac{1}{2}$.

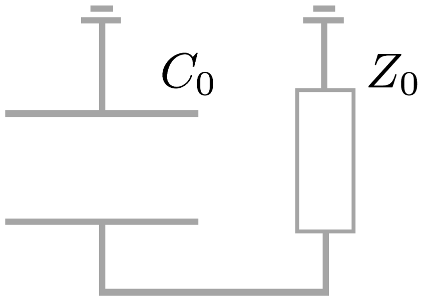


Figure 8: A sketch of the circuit giving rise to the potential $\phi(t)$, which is modelled as the voltage drop over a capacitance C coupled in parallels with ab environment impedance Z_0, which is here taken to be frequency independent and real.

7 Calculating the potential propagator

There is only one thing left to study before we are able to calculate $G_{\tilde{\gamma}_i}$, and so find out how the box qubit decoheres. We need to be able to average over the as of yet unspecified function $\dot{\phi}(t)$. To do this we assume that it is distributed as a gaussian, so that all we need to know is the variance $\langle\dot{\phi}(t)\dot{\phi}(t')\rangle$. Determining this is the subject of this sections. $\phi(t)$ is a classical field, and we model the fluctuations by describing $\phi(t)$ as the potential on a capacitance C coupled in parallels with an impedance Z_0, which emulates a dissipative environment. The setup is shown on Figure 8. The environment impedance is for simplicity assumed to be frequency independent. We will calculate the correlation function by taking a detour through quantum mechanics and using the fluctuation dissipation theorem, which for completeness we derive in this section. In the end we will then take the classical limit of the correlation function and assume that the classical limit of the quantum mechanical correlation function conicides with the classical correlation function. The total impedance Z_{tot} of the RC-circuit in Figure 8 is

$$Z_{\text{tot}} = \frac{1}{\frac{1}{Z_C} + \frac{1}{Z_0}} = \frac{1}{i\omega C + \frac{1}{Z_0}} = \frac{1}{C}\frac{1}{i\omega + \omega_0}, \tag{7.1}$$

where $\omega_0 = \frac{1}{Z_0 C}$. Taking I to refer to the electric current through the circuit, linear response gives [1]

$$\text{Re}\Big(\frac{1}{Z_{\text{tot}}(\omega)}\Big) = \text{Re}\Big(\frac{i}{\omega}G^R_{II}(\omega)\Big) = -\frac{1}{\omega}\text{Im}\Big(G^R_{II}(\omega)\Big), \tag{7.2}$$

where the retarded current-current correlation function is given by $G^R_{II}(t_1 - t_2) = -i\theta(t_1 - t_2)\langle[I(t_1), I(t_2)]\rangle$. Remember that ϕ is the potential energy, so the electric potential is $\frac{\phi}{e}$. We want to massage (7.2) into an expression for $G_{\phi\phi}$. In order to do this we will make the assumption that Ohm's law holds for the correlation function, so

$$G^R_{II}(\omega) = \frac{1}{e^2}|Z_{\text{tot}}(\omega)|^{-2}G^R_{\phi\phi}(\omega). \tag{7.3}$$

If we had taken Z_0 to be an inductance with no dissipation, then the system would be quadratic, and so the classical equations of motion would hold also for the quantum operators, and we would have $\hat{\phi}(\omega) = Z(\omega)\hat{I}(\omega)$. In our case, where Z_0 is a resistance, so there is energy loss, and wether the relation still holds on an operator level depends on the model of the bath. When we use (7.3) however, it may at least be thought of as a classical approximation, which is consistent

with taking the classical limit later.
To proceed we will need to use the fluctuation dissipation theorem, which states that

$$\mathrm{Im}\Big(G^R_{\phi\phi}(\omega)\Big) = \frac{i}{2}\tanh\big(\frac{\beta\omega}{2}\big)G_{\phi\phi}(\omega). \tag{7.4}$$

The proof of this uses the Lehman representation, which amounts to expressing the Green's functions in the basis $\{|n\rangle\}$ that diagonalises the Hamiltonian H. The arguments follow closely those of Bruus and Flensberg [1]. First let us define the Green's function by

$$G_{\phi\phi}(t) = -i\langle\Big(\phi(t)\phi(0) + \phi(0)\phi(t)\Big)\rangle. \tag{7.5}$$

In thermal equilibrium at termperature $T = \frac{1}{k_B\beta}$ we may use that ϕ is hermitian to get

$$\begin{aligned}
G_{\phi\phi}(\omega) &= -\frac{i}{Z}\int dte^{-i\omega t}\sum_n e^{-\beta E_n}\Big(\langle n|\phi(t)\phi(0)|n\rangle + \langle n|\phi(0)\phi(t)|n\rangle\Big) \\
&= -\frac{i}{Z}\int dte^{-i\omega t}\sum_{nn'} e^{-\beta E_n}|\langle n|\phi|n'\rangle|^2\Big(e^{i(E_n-E_{n'})t} + e^{-i(E_n-E_{n'})t}\Big) \\
&= -\frac{2\pi i}{Z}\sum_{nn'} e^{-\beta E_n}|\langle n|\phi|n'\rangle|^2\Big(\delta(\omega + E_n - E_{n'}) + \delta(\omega - E_n + E_{n'})\Big),
\end{aligned} \tag{7.6}$$

where Z is the partition function and not to be confused with the impedance. The retarded potential correlation function is

$$\begin{aligned}
G^R_{\phi\phi}(\omega) &= \int_{-\infty}^{\infty} dte^{-i(\omega-i\eta)t}\Big(-i\theta(t)\langle[\phi(t),\phi(0)]\rangle\Big) = -i\int_0^{\infty} dte^{-i(\omega+i\eta)t}\Big(\langle\phi(t)\phi(0)\rangle - \langle\phi(0)\phi(t)\rangle\Big) \\
&= -\frac{i}{Z}\int_0^{\infty} dte^{-i(\omega-i\eta)t}\sum_{nn'} e^{-\beta E_n}|\langle n|\phi|n'\rangle|^2\Big(e^{i(E_n-E_{n'})t} - e^{-i(E_n-E_{n'})t}\Big) \\
&= -\frac{i}{Z}\sum_{nn'} e^{-\beta E_n}|\langle n|\phi|n'\rangle|^2\Big(\frac{-1}{i(-\omega + E_n - E_{n'} + i\eta)} - \frac{-1}{i(-\omega - E_n + E_{n'} + i\eta)}\Big) \\
&= \frac{1}{Z}\sum_{nn'} e^{-\beta E_n}|\langle n|\phi|n'\rangle|^2\Big(\frac{1}{-\omega + E_n - E_{n'} + i\eta} - \frac{1}{-\omega - E_n + E_{n'} + i\eta}\Big) \\
&= \frac{1}{Z}\sum_{nn'} e^{-\beta E_n}|\langle n|\phi|n'\rangle|^2\Big(\frac{1}{\omega + E_n - E_{n'} - i\eta} - \frac{1}{\omega - E_n + E_{n'} - i\eta}\Big).
\end{aligned} \tag{7.7}$$

Next we use that $\frac{1}{x-i\eta} = \mathcal{P}\frac{1}{x} + i\pi\delta(x)$ in the limit $\eta \to 0^+$ and we see that

$$
\begin{aligned}
\mathrm{Im}\Big(G^R_{\phi\phi}(\omega)\Big) &= \frac{\pi}{Z}\sum_{nn'} e^{-\beta E_n}|\langle n|\phi|n'\rangle|^2\Big(\delta(\omega+E_n-E_{n'})-\delta(\omega-E_n+E_{n'})\Big)\\
&= \frac{\pi}{Z}\sum_{nn'}|\langle n|\phi|n'\rangle|^2\Big(e^{-\beta E_n}-e^{-\beta E_{n'}}\Big)\delta(\omega+E_n-E_{n'})\\
&= \frac{\pi}{2Z}\sum_{nn'}|\langle n|\phi|n'\rangle|^2\Big(\Big(e^{-\beta E_n}-e^{-\beta E_{n'}}\Big)\delta(\omega+E_n-E_{n'})+\Big(e^{-\beta E_{n'}}-e^{-\beta E_n}\Big)\delta(\omega-E_n+E_{n'})\Big)\\
&= \frac{\pi}{2Z}\sum_{nn'}|\langle n|\phi|n'\rangle|^2\Big(e^{-\beta E_n}\big(1-e^{-\beta\omega}\big)\delta(\omega+E_n-E_{n'})\\
&\qquad\qquad + e^{-\beta E_n}\big(e^{\beta\omega}-1\big)\delta(\omega-E_n+E_{n'})\Big)\\
&= \frac{\pi}{2Z}\big(1-e^{-\beta\omega}\big)\sum_{nn'}|\langle n|\phi|n'\rangle|^2 e^{-\beta E_n}\Big(\delta(\omega+E_n-E_{n'})+e^{\beta\omega}\delta(\omega-E_n+E_{n'})\Big)\\
&= \frac{\pi}{2Z}\tanh\Big(\frac{\beta\omega}{2}\Big)\sum_{nn'}|\langle n|\phi|n'\rangle|^2 e^{-\beta E_n}\Big(\big(1+e^{-\beta\omega}\big)\delta(\omega+E_n-E_{n'})\\
&\qquad\qquad + \big(1+e^{\beta\omega}\big)\delta(\omega-E_n+E_{n'})\Big)\\
&= \frac{i}{4}\tanh\Big(\frac{\beta\omega}{2}\Big)G_{\phi\phi}(\omega)\\
&\qquad + \frac{\pi}{2Z}\tanh\Big(\frac{\beta\omega}{2}\Big)\sum_{nn'}|\langle n|\phi|n'\rangle|^2 e^{-\beta E_n}\Big(e^{-\beta\omega}\delta(\omega+E_n-E_{n'})+e^{\beta\omega}\delta(\omega-E_n+E_{n'})\Big)\\
&= \frac{i}{4}\tanh\Big(\frac{\beta\omega}{2}\Big)G_{\phi\phi}(\omega)\\
&\qquad + \frac{\pi}{2Z}\tanh\Big(\frac{\beta\omega}{2}\Big)\sum_{nn'}|\langle n|\phi|n'\rangle|^2 e^{-\beta E_n}\Big(e^{\beta(E_n-E_{n'})}\delta(\omega+E_n-E_{n'})\\
&\qquad\qquad + e^{\beta(E_n-E_{n'})}\delta(\omega-E_n+E_{n'})\Big)\\
&= \frac{i}{2}\tanh\Big(\frac{\beta\omega}{2}\Big)G_{\phi\phi}(\omega), \qquad (7.8)
\end{aligned}
$$

where in the third and the seventh line the delta function was used to modify the exponents. Now that we have derived (7.4) let us put it to use along with (7.3). Putting them together we get

$$
\begin{aligned}
&\mathrm{Re}\Big(\frac{1}{Z_{\mathrm{tot}}(\omega)}\Big) = -\frac{1}{e^2\omega}|Z_{\mathrm{tot}}(\omega)|^{-2}\mathrm{Im}\Big(G^R_{\phi\phi}(\omega)\Big) = -\frac{1}{e^2\omega}|Z_{\mathrm{tot}}(\omega)|^{-2}\frac{i}{2}\tanh\big(\frac{\beta\omega}{2}\big)G_{\phi\phi}(\omega)\\
&\Rightarrow G_{\phi\phi}(\omega) = 2i|Z_{\mathrm{tot}}(\omega)|^2\mathrm{Re}\Big(\frac{1}{Z_{\mathrm{tot}}(\omega)}\Big)e^2\omega\coth\big(\frac{\beta\omega}{2}\big) = 2i\frac{\omega_0}{C}\frac{1}{\omega^2+\omega_0^2}e^2\omega\coth\big(\frac{\beta\omega}{2}\big). \qquad (7.9)
\end{aligned}
$$

Lastly we use that $G_{\dot\phi\dot\phi}(\omega)=\omega^2 G_{\phi\phi}(\omega)$, since it is the Fourier transform of a double derivative.Thus in the end we have

$$
G_{\dot\phi\dot\phi}(\omega) = \frac{2ie^2\omega_0}{C}\frac{\omega^3\coth\big(\frac{\beta\omega}{2}\big)}{\omega^2+\omega_0^2}. \qquad (7.10)
$$

In the next section we will again treat ϕ as classical fields, and so for consistency we take the classical limit of (7.10), $T\to 0$ (or equivalently reinstating $\hbar$ and setting $\hbar\to 0$), and get

$$
G_{\dot\phi\dot\phi}(\omega) \approx i\frac{4e^2\omega_0 k_B T}{C}\frac{\omega^2}{\omega^2+\omega_0^2}. \qquad (7.11)
$$

8 Evaluating the Majorana propagator

With the work of the last sections we are at last in postition to calculate the instantaneous Majorana propagator $G_{\tilde{\gamma}_i}(t)$. Section 5 concluded the equations of motion analysis with the integral equation (5.48) for $G_{\tilde{\gamma}_i}(t)$ and the expression (5.49) for the integral kernel $K(s,s')$. Now that we know the relevant matrix elements we can write it as[2]

$$\begin{aligned} G_{\tilde{\gamma}_i}(t) &= 1 + \frac{2\mathcal{L}\mu}{\pi\mu^2}\frac{\Delta}{\mu\mathcal{L}}\frac{m\Delta}{\mu}\int_0^t ds \int_0^s ds' \dot{\phi}(s)\dot{\phi}(s')A(s-s')G_{\tilde{\gamma}_i}(s'), \\ &= 1 + \frac{2m\Delta^2}{\pi\mu^3}\int_0^t ds \int_0^s ds' \dot{\phi}(s)\dot{\phi}(s')A(s-s')G_{\tilde{\gamma}_i}(s') \end{aligned} \tag{8.1}$$

where

$$A(t) = \int_1^\infty d\xi m(\xi)\cos\mu\xi t, \tag{8.2}$$

and

$$m(\xi) = \frac{|\delta\tilde{H}_{0\xi}|^2\rho(\xi)}{\xi^2}. \tag{8.3}$$

$\delta\tilde{H}_{0\xi}$ is the dimensionless part of the matrix element, and the density of states is given by $\rho(\xi) = \frac{d\tilde{k}}{d\xi}$. Repeated insertion in (8.1) into the right hand side yields

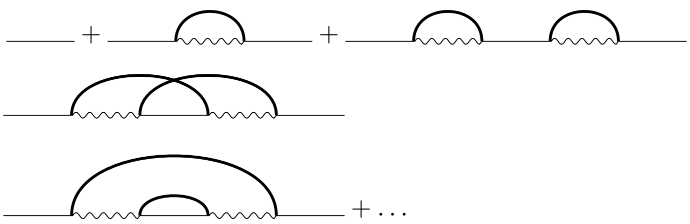
$$\begin{aligned} G_{\tilde{\gamma}_i}(t) = 1 &+ \frac{2m\Delta^2}{\pi\mu^3}\int_0^t ds_1 \int_0^{s_1} ds_2 \dot{\phi}(s_1)\dot{\phi}(s_2)A(s_1-s_2) \\ &+ \Big(\frac{2m\Delta^2}{\pi\mu^3}\Big)^2\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_0^{s_3} ds_4 \dot{\phi}(s_1)\dot{\phi}(s_2)\dot{\phi}(s_3)\dot{\phi}(s_4)A(s_1-s_2)A(s_3-s_4) + \dots \end{aligned} \tag{8.4}$$

Now we need to get rid of the pesky unknown $\dot{\phi}$ function, and we do this by averaging over it, assuming that it follows a Gaussian distribution. We now treat ϕ as a classical field, distributed after a Gaussian with variance $\langle\dot{\phi}(s_i)\dot{\phi}(s_j)\rangle = \frac{i}{2}(-i)\langle\dot{\phi}(s_i)\dot{\phi}(s_j) + \dot{\phi}(s_j)\dot{\phi}(s_i)\rangle = \frac{i}{2}G_{\dot{\phi}\dot{\phi}}(s_i - s_j)$. Thus armed we can average (8.4) over $\dot{\phi}$ and use Wick's theorem on the right hand side. This gives

$$\begin{aligned} \overline{G_{\tilde{\gamma}_i}}(t) =& 1 + i\frac{m\Delta^2}{\pi\mu^3}\int_0^t ds_1 \int_0^{s_1} ds_2 G_{\dot{\phi}\dot{\phi}}(s_1-s_2)A(s_1-s_2) \\ &- \Big(\frac{m\Delta^2}{\pi\mu^3}\Big)^2\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_0^{s_3} ds_4 \Big(G_{\dot{\phi}\dot{\phi}}(s_1-s_2)G_{\dot{\phi}\dot{\phi}}(s_3-s_4) + G_{\dot{\phi}\dot{\phi}}(s_1-s_3)G_{\dot{\phi}\dot{\phi}}(s_2-s_4) \\ &\qquad + G_{\dot{\phi}\dot{\phi}}(s_1-s_4)G_{\dot{\phi}\dot{\phi}}(s_2-s_3)\Big)A(s_1-s_2)A(s_3-s_4) \\ &+ \dots \end{aligned} \tag{8.5}$$

This equation may be represented in terms of the following diagrams:

[2]The first of the dimensionfull factors in front of the integral comes from changing from k to E and then to ξ. The second one is the dimensions of $\delta H_{0,\xi}$ and the last is the dimensions of $\rho(\xi)$.

$\overline{G_{\tilde{\gamma}_i}}(t) =$  $+ \dots$

(8.6)

In the above diagrams one should imagine time variables, of the following sort, which has been left out for clarity:

$$0 \,\text{—}\, t_4 \sim t_3 \,\text{—}\, t_2 \sim t_1 \,\text{—}\, t \ , \tag{8.7}$$

with the constraint $t \geq t_1 \geq t_2 \geq t_3 \geq t_4 \geq 0$, which means that the three fourth order diagrams in (8.6) are distinct. The diagrammatics are the following:

$$\begin{aligned} t_i \,\rule[0.5ex]{1.5em}{0.4pt}\, t_j \ &= 1 \\ t_i \,\rule[0.5ex]{1.5em}{1.5pt}\, t_j \ &= i\frac{m\Delta^2}{\pi\mu^3} G_{\dot{\phi}\dot{\phi}}(t_j - t_i) \\ t_i \sim t_j \ &= A(t_j - t_i) \end{aligned} \tag{8.8}$$

As in (8.5) integration is implied over over all the internal times t_i from 0 to t_{i-1} and t_1 from 0 to t. I will refer to these integral limits as being "nested". We cannot solve equation (8.6) as it is, but looking at it a little bit closer we will realise that there is a reason to only keep the diagrams that don't cross. This really depends on the details of $A(t)$ as well as $G_{\dot{\phi}\dot{\phi}}$ and here I will sketch an argument of why it is, leaving the finer details to Appendix C.
Notice first that $A(t)$ is a rapidly oscillating function, with the mininum frequency equal to the gap energy μ, which is big compared to all other energy scales in our setup. Because of this there are only contributions around sharp peaks of $G_{\dot{\phi}\dot{\phi}}$ for each time integral. But because the integral limits are nested, if $G_{\dot{\phi}\dot{\phi}}$ is sharply peaked however crossing diagrams will drop out. Consider for instance the following diagram:

$$0 \,\text{—}\, t_4 \sim t_3 \,\text{—}\, t_2 \sim t_1 \,\text{—}\, t \ . \tag{8.9}$$

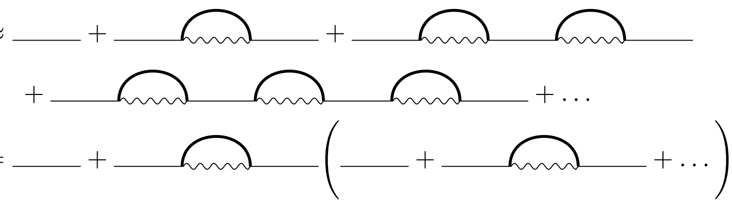
To be precise let us assume $G_{\dot{\phi}\dot{\phi}}$ is sharply peaked, with a narrow width of ϵ at $t = 0$, and that its integral is independent of ϵ. We then only have contributions if $t_1 \approx t_3$ and if $t_2 \approx t_4$. Since the integral limits are nested, they all bunch up, and in particular the integral over t_3 has the form $\int_{t_2-\epsilon}^{t_2} dt_3(\dots)$, where the integrand is not sharply peaked, so it will be of order ϵ. This is potentially very small compared to the direct terms of the form

$$0 \,\text{—}\, t_4 \sim t_3 \,\text{—}\, t_2 \sim t_1 \,\text{—}\, t \ , \tag{8.10}$$

which are independent of ϵ. It will turn out that for our case $G_{\dot{\phi}\dot{\phi}}$ consists of both a very sharply peaked term and one which isn't. By the above argument the sharply peaked term only contributes when it connects neighbouring times. The other term tends to give very fast oscillating

contributions in crossing diagrams, oscillating with a frequency of μ. These fast oscillations are in general not measurable, as they average out of any measurement conceivable.These fast oscillating diagrams are not necessarily of a much smaller magnitude than the direct diagrams. I give a more thorough discussion for this in Appendix C.
Dropping the crossing diagrams in 8.6 we may recombine the the sum as

$$\overline{G_{\tilde{\gamma}_i}}(t) \approx$$



$$= 1 + i\frac{m\Delta^2}{\pi\mu^3}\int_0^t ds_1 \int_0^{s_1} ds_2 G_{\dot{\phi}\dot{\phi}}(s_1 - s_2)A(s_1 - s_2)\overline{G_{\tilde{\gamma}_i}}(s_2), \tag{8.11}$$

which means that

$$\partial_t \overline{G_{\tilde{\gamma}_i}}(t) \approx i\frac{m\Delta^2}{\pi\mu^3}\int_0^t ds G_{\dot{\phi}\dot{\phi}}(t-s)A(t-s)\overline{G_{\tilde{\gamma}_i}}(s). \tag{8.12}$$

Equation (8.12) may be solved by Laplace transformation. Let $\mathscr{G}_i(z) = \mathscr{L}[\overline{G_{\tilde{\gamma}_i}}](z)$ and $R(z) = \mathscr{L}[G_{\dot{\phi}\dot{\phi}}(t)A(t)]$. Then we have, since $\overline{G_{\tilde{\gamma}_i}}(0) = 1$,

$$\begin{aligned} z\mathscr{G}_i(z) - 1 &= i\frac{m\Delta^2}{\pi\mu^3}R(z)\mathscr{G}_i(z) \\ \Rightarrow \mathscr{G}_i(z) &= \frac{1}{z - i\frac{m\Delta^2}{\pi\mu^3}R(z)}. \end{aligned} \tag{8.13}$$

Let's us now try and evaluate (8.13). The first ingredient we need is the fourier transform (7.11). We find

$$\begin{aligned} G_{\dot{\phi}\dot{\phi}}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega t} G_{\dot{\phi}\dot{\phi}}(\omega) = i\frac{2e^2\omega_0 k_B T}{\pi C}\int_{-\infty}^{\infty} d\omega e^{i\omega t}\frac{\omega^2}{\omega^2 + \omega_0^2} \\ &= i\frac{2e^2\omega_0 k_B T}{\pi C}\int_{-\infty}^{\infty} d\omega e^{i\omega t}\Big(1 - \frac{\omega_0^2}{\omega_2 + \omega_0^2}\Big) \\ &= i\frac{2e^2\omega_0 k_B T}{\pi C}\left(2\pi\delta(t) - \omega_0^2 \int d\omega \frac{e^{i\omega t}}{(\omega + i\omega_0)(\omega - i\omega_0)}\right) \\ &= i\frac{2e^2\omega_0 k_B T}{\pi C}\left(2\pi\delta(t) - \pi\omega_0 e^{-\omega_0|t|}\right) \\ &= i\frac{4e^2\omega_0 k_B T}{C}\left(\delta(t) - \frac{\omega_0}{2}e^{-\omega_0|t|}\right) \end{aligned} \tag{8.14}$$

Next we simplify $A(t)$ using the Saddle point approximation. What this will accomplish is to pick out a single frequency component for $A(t)$, thus greatly simplifying the succeeding analysis. Writing the cosine in terms of exponentials we have

$$A(t) = \frac{1}{2}\int_1^{\infty} d\xi m(\xi)\Big(e^{i\mu\xi t} + e^{-i\mu\xi t}\Big) = \frac{1}{2}\int_1^{\infty} d\xi\Big(e^{i\mu\xi t + \log(m(\xi))} + e^{-i\mu\xi t + \log(m(\xi))}\Big) \equiv I_+(t) + I_-(t).$$

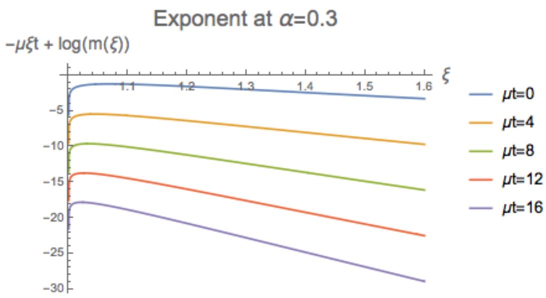


Figure 9: The exponent of the integrand in equation (8.15) at $\alpha = 0.3$ at different times. Similar behaviour is present at all $\alpha \in (0, \frac{1}{2})$.

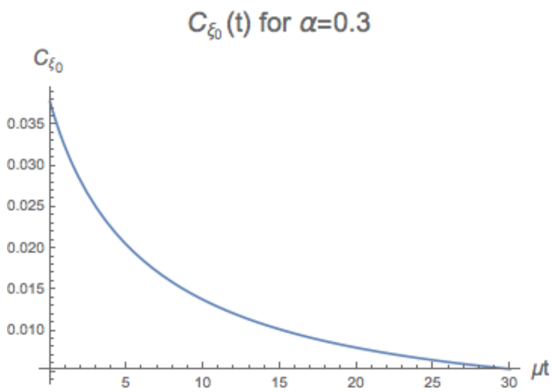


Figure 10: Plot of $C_{\xi_0}(t)$.

The trick now is to do Wick rotations, defined by $\mathcal{W}_\pm(f(t)) = f(\mp it)$. Notice that $\mathcal{W}_\pm \circ \mathcal{W}_\mp = \mathbb{1}$. Thus we can rewrite $A(t)$ as

$$A(t) = \mathcal{W}_+\Big(\mathcal{W}_-\big(I_+(t)\big)\Big) + \mathcal{W}_-\Big(\mathcal{W}_+\big(I_-(t)\big)\Big) = \mathcal{W}_+\Big(\mathcal{I}(t)\Big) + \mathcal{W}_-\Big(\mathcal{I}(t)\Big),$$

where

$$\mathcal{I}(t) = \frac{1}{2}\int_1^\infty d\xi e^{-\mu\xi t + \log(m(\xi))}. \tag{8.15}$$

At any time t the exponent $-\mu\xi t + \log(m(\xi))$ has maximum at $\xi = \xi_0(t) \gtrsim 1$ this illustrated on Figure 9, where $-\mu\xi t + \log(m(\xi))$ is plotted at a few different times and with $\alpha = 0.3$. Now we can do the saddle point approximation to the lowest order which gives

$$\mathcal{I}(t) \approx \frac{1}{2}\sqrt{\frac{2\pi}{\mu t}} \frac{e^{-\mu\xi_0 t + \log(m(\xi_0))}}{\sqrt{-\frac{d^2}{d\xi^2}\Big|_{\xi=\xi_0} \frac{\log m(\xi)}{\mu t}}} = \frac{m(\xi_0)}{\sqrt{\frac{2}{\pi}\left(\frac{m'(\xi_0)}{m^2(\xi_0)} - \frac{m''(\xi_0)}{m(\xi_0)}\right)}} e^{-\mu\xi_0 t} \equiv C_{\xi_0} e^{-\mu\xi_0 t}. \tag{8.16}$$

This means that

$$A(t) \approx C_{\xi_0}(t)\cos(\mu\xi_0 t). \tag{8.17}$$

The function $C_{\xi_0}(t)$ is plotted on Figure 10 with $\alpha = 0.3$. The exact analytic form of $C_{\xi_0}(t)$ is quite complicated, and it is not tractable to calculate $R(z)$ without some sort of approximation. In the following, for simplicity, I'm going to simply replace $C_{\xi_0}(t)$ with $C_{\xi_0}(0)$. This will give us an overestimate for the decay of $\overline{G_{\tilde{\gamma}_i}}(t)$, which can be seen by inserting the saddle-point approximation already into (8.1). Looking at Figure 10 we see that $C_{\xi_0}(\mu t = 5) \approx \frac{1}{2} C_{\xi_0}(0)$, and this is roughly the case for all α. Another option is to fit a decaying exponential function to $C_{\xi_0}(t)$. If we do this with the standard least squares method however, the value of $C_{\xi_0}(t)$ is greatly underestimated at early times, and so we would have to be more careful. If we could find a good way to fit an expontential to $C_{\xi_0}(t)$, then the following analysis would run the exact same way. The only difference is that ω_0 would be shifted an amount equal to the fitted decay rate of $C_{\xi_0}(t)$. But let us assume $C_{\xi_0}(t)$ is constant and calculate $R(z)$. For notational convenience I write $\mu\xi_0 \equiv \omega_m$. We get

$$
\begin{aligned}
R(z) &= C_{\xi_0} \int_0^\infty dt e^{-tz} G_{\dot\phi\dot\phi}(t) \cos(\omega_m t) = i \frac{4e^2 \omega_0 k_B T C_{\xi_0}}{C} \int_0^\infty dt e^{-zt} \Big(\delta(t) - \frac{\omega_0}{2} e^{-\omega_0 t} \Big) \cos(\omega_m t) \\
&= i \frac{2e^2 \omega_0 k_B T C_{\xi_0}}{C} \sum_{d=\pm 1} \int_0^\infty dt e^{-t(z + id\omega_m)} \Big(\delta(t) - \frac{\omega_0}{2} e^{-\omega_0 t} \Big) \\
&= i \frac{2e^2 \omega_0 k_B T C_{\xi_0}}{C} \sum_{d=\pm 1} \Big(\frac{1}{2} - \frac{\omega_0}{2} \int_0^\infty dt e^{-t(z + id\omega_m + \omega_0)} \Big) \\
&= i \frac{2e^2 k_B T \omega_0 C_{\xi_0}}{C} \sum_{d=\pm 1} \Big(\frac{1}{2} - \frac{\omega_0}{2} \frac{-1}{-(z + \omega_0 + i\omega_m d)} \Big) \\
&= i \frac{e^2 \omega_0 k_B T C_{\xi_0}}{C} \sum_{d=\pm 1} \frac{z + id\omega_m}{z + \omega_0 + id\omega_m} = i \frac{e^2 \omega_0 k_B T C_{\xi_0}}{C} \sum_{d=\pm 1} \frac{(z + id\omega_m)(z + \omega_0 - id\omega_m)}{(z+\omega_0)^2 + \omega_m^2} \\
&= i \frac{2e^2 k_B T C_{\xi_0}}{C} \omega_0 \frac{z(z+\omega_0) + \omega_m^2}{(z+\omega_0)^2 + \omega_m^2}
\end{aligned}
\tag{8.18}
$$

Using this we get

$$
\begin{aligned}
\mathscr{G}_i(z) &= \frac{1}{z - i\frac{m\Delta^2}{\pi\mu^3} i \frac{2e^2 k_B T C_{\xi_0}}{C} \omega_0 \frac{z(z+\omega_0)+\omega_m^2}{(z+\omega_0)^2+\omega_m^2}} = \frac{(z+\omega_0)^2 + \omega_m^2}{z(z+\omega_0)^2 + \omega_m^2 z + \frac{m\Delta^2}{\pi\mu^3} \frac{2e^2 k_B T C_{\xi_0}}{C} \omega_0 (z(z+\omega_0) + \omega_m^2)} \\
&= \frac{(z+\omega_0)^2 + \omega_m^2}{z^3 + \Big(p+2\Big)\omega_0 z^2 + \Big((1+p)\omega_0^2 + \omega_m^2\Big) z + p\omega_0\omega_m^2},
\end{aligned}
\tag{8.19}
$$

where

$$
p = 2\frac{m\Delta^2}{\pi\mu^3} \frac{e^2}{C} k_B T C_{\xi_0}. \tag{8.20}
$$

We can rewrite equation (8.19) as

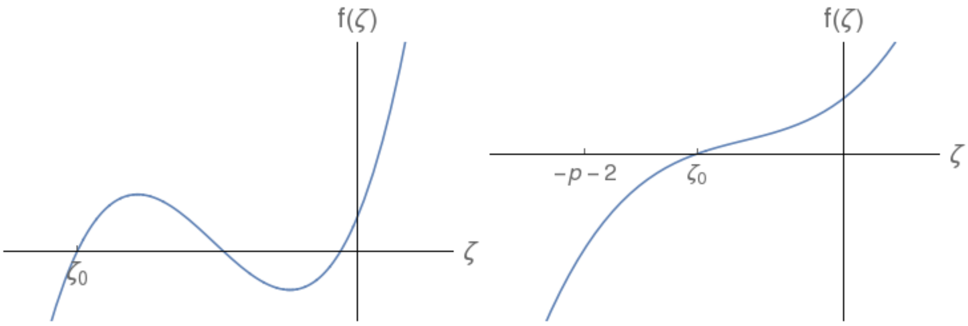
$$
\mathscr{G}_i(z) = \frac{(z+\omega_0)^2 + \omega_m^2}{(z-z_1)(z-z_2)(z-z_3)}, \tag{8.21}
$$

where z_i are the solutions to the cubic equation

$$
z^3 + (p+2)\omega_0 z^2 + \big((1+p)\omega_0^2 + \omega_m^2\big) z + p\omega_0\omega_m^2 = 0. \tag{8.22}
$$

To do the inverse Laplace transformation we need to pick a real $\lambda > \max\{\mathrm{Re}(z_i)\}$ and then we have

$$
\overline{G_{\tilde{\gamma}_i}}(t) = \frac{1}{2\pi i} \lim_{A\to\infty} \int_{\lambda - iA}^{\lambda + iA} dz e^{zt} \frac{(z+\omega_0)^2 + \omega_m^2}{(z-z_1)(z-z_2)(z-z_3)}. \tag{8.23}
$$



(a) $d_f > 0$ (b) $d_f < 0$

Figure 11: Generic third degree polynomials with positive discriminant in (a) and a negative discriminant in (b). In the case of negative discriminant we will need to check whether $\zeta_0 > -2-p$.

Deforming the contour to an infinite radius half-circle of centered in $z = \lambda$, and noting that the integrant have only simple poles, we finally get the main result of this thesis

$$\boxed{\overline{G_{\tilde{\gamma}_i}}(t) = \frac{(z_1+\omega_0)^2+\omega_m^2}{(z_1-z_2)(z_1-z_3)}e^{z_1 t} + \frac{(z_2+\omega_0)^2+\omega_m^2}{(z_2-z_1)(z_2-z_3)}e^{z_2 t} + \frac{(z_3+\omega_0)^2+\omega_m^2}{(z_3-z_1)(z_3-z_2)}e^{z_3 t} \qquad (8.24)}$$

The expression (8.24) only makes sense for all times if all the roots z_i satisfy $\mathrm{Re}(z_i) \leq 0$ (one or two of the roots may potentially be 0). To figure out when this condition fails let us rewrite (8.22) by dividing by ω_0 and writing $\zeta \equiv \frac{z}{\omega_0}$ as well as $\nu \equiv \frac{\omega_m}{\omega_0}$:

$$\zeta^3 + (p+2)\zeta^2 + \big(1+p+\nu^2\big)\zeta + p\nu^2 = 0, \tag{8.25}$$

or, for notational brevity,

$$f(\zeta) \equiv \zeta^3 + a\zeta^2 + b\zeta + c = 0. \tag{8.26}$$

Notice that $a, b, c > 0$. This means that $f(\zeta) > 0$ for all $\zeta \geq 0$. We also have $f'(\zeta) = 3\zeta^2 + 2a\zeta + b > 0$ when $\zeta \geq 0$. Because of this, all real roots need to be negative or equal to 0. In particular it means that there is always at least one negative real root ζ_0. The discriminant d_f of f is given by

$$d_f = a^2b^2 - 4b^3 - 4a^3c - 27c^2 - 18abc. \tag{8.27}$$

Depending on the sign of d_f the behaviour of f is indicated on Figure 11 . If $d_f > 0$ then all roots are real as in 11a and as explained above they need to be negative and one of them can be 0. This means that in the case where $d_f > 0$ equation (8.24) decays as it should for all t. In the case $d_f = 0$ then besides ζ_0 there is a real double root, which again needs to be negative or equal to 0. Equation (8.24) doesn't hold in this case however, since the inverse Laplace transformation now has a second order pole. This is not a computational problem but this case will be excluded since it only matters in a measure 0 part of the parameter space.
The last case has $d_f < 0$, as shown on Figure 11b. In this case there is the negative solution ζ_0 as well as two conjugate complex solutions $\zeta_\pm = r \pm is$ (this is necessary since $f(\zeta)$ is real).

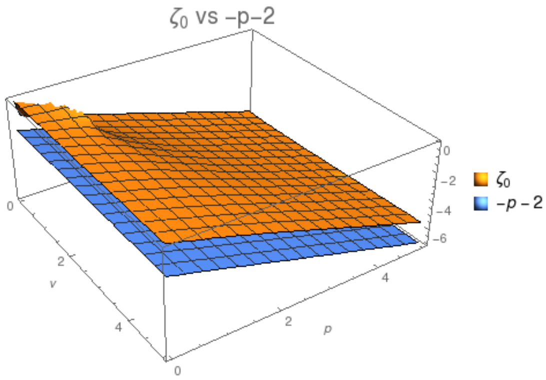


Figure 12: Plot of ζ_0 vs $-p-2$. As can be seen they don't cross in the region plotted here. This continues seemingly indefinitely.

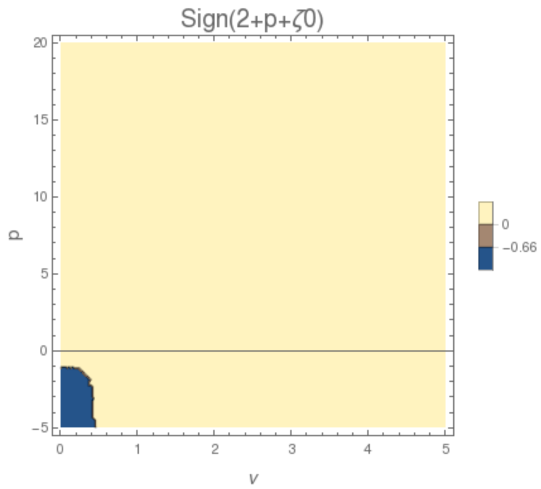


Figure 13: Plot of $\mathrm{sign}(2+p+\zeta_0)$ as a function of ν and p. As can be seen for all physical, positive values $\zeta_0 > -2-p$.

Thus we need to check when $r \leq 0$. By the fundamental theorem of algebra we may write

$$\begin{aligned} f(\zeta) &= (\zeta-\zeta_0)(\zeta-r-is)(\zeta-r+is) \\ &= \zeta^3+\zeta^2(-\zeta_0-2r)+\zeta(r^2+s^2+2r\zeta_0)-(r^2+s^2)\zeta_0. \end{aligned} \tag{8.28}$$

From this we see that $a = -\zeta_0 - 2r$ or $r = \frac{-\zeta_0 - a}{2}$ which implies that $r < 0 \Leftrightarrow -a = -p-2 < \zeta_0$. So we need to check whether this condition holds. Figure 12 shows a plot of ζ_0 as a function of ν and p. The behaviour continues in a similar fashion for all positive ν and p outside the plotted region, and so we can conclude that $\zeta_0 > -2-p$ is always the case. Figure 13 shows a plot of $\mathrm{sign}(2+p+\zeta_0)$ demonstrating the same point.

When $d_f < 0$ we have the negative roots and thus expect an oscillatory decay of $\overline{G_{\tilde{\gamma}_i}}$. Figure 14 shows a contour plot of d_f as a function of ν and p. For physical parameters we expect ν to be large, so in most relevant cases d_f will be negative and we thus expect to see an oscillatory decay of $\overline{G_{\tilde{\gamma}_i}}$.

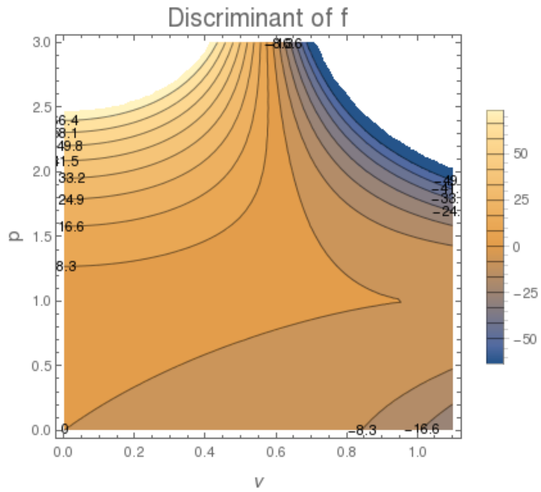


Figure 14: Plot of the discriminant d_f. The 0 line continues in a similar fashion for values outside the shown region indicating $d_f < 0$ for all $\nu > 1$.

9 Discussion

At this stage we have done many different things, and we can now put the pieces together and see what we have achieved. We saw in equation (4.18) that a measure for how well information is stored in the Majorana states is $P_{\tilde{z},z}(t) = \frac{1}{2} + \frac{1}{2}G_{\tilde{\gamma}_{1,t}}(t)G_{\tilde{\gamma}_{2,t}}(t)$. In the preceding section we obtained an approximate expression for the propagator in equation (8.24), where we saw that it is a sum of exponentially decaying functions. With our assumptions $G_{\tilde{\gamma}_{1,t}} = G_{\tilde{\gamma}_{2,t}}$, so this means that

$$P_{\tilde{z},z}(t) = \frac{1}{2} + \frac{1}{2}G^2_{\tilde{\gamma}_t}(t). \tag{9.1}$$

We want to know the time scales on which information leaks out of the system, which means finding the most negative decay rate $z_{\max} = \min\{\mathrm{Re} z_i\}$, unless the term decaying at this rate is much smaller than the other terms. Since we need to square $G_{\tilde{\gamma}_i}(t)$ in (9.1), the shortest decay time then is $t_{\min} \equiv -\frac{1}{2z_{\max}}$. If we assume $\omega_m \gg \omega_0$ so $\nu \gg 1$, then we can obtain an expression for $t_{\min}$ by Taylor expanding the exact solutions of (8.25) in ν to lowest order. There is one real root

$$\zeta_0 \approx -p, \tag{9.2}$$

and two complex roots

$$\zeta_\pm \approx -1 \pm i\nu. \tag{9.3}$$

Since in general p will be less than one the terms with the complex roots decay the fastest. But looking closer these terms turn out to be of order ν^{-2} so they are negligible. Thus in this limit we have

$$t_{\min} = \frac{1}{2\omega_0 p} = \frac{\pi\mu^3 Z_0 C^2}{4C_{\xi_0} m\Delta^2 e^2 k_B T}. \tag{9.4}$$

Using the relations in equations (2.35) we found between μ and Δ on the one hand and the chemical potential h, Zeeman splitting B, induced pairing δ and the spin-orbit parameter λ, we can write (9.4) as

$$t_{\min} = \frac{\pi Z_0 C^2 B^2 (h + B - \frac{\delta^2}{B})^3}{4C_{\xi_0} m\Delta^2 e^2 k_B T}. \tag{9.5}$$

Throughout this thesis we have treated the case where B is the larger energy scale in the problem. This means that the dephasing time in (9.5) should be long in this regime. Since $\omega_m \approx \mu \sim B$, it is a consistent approximation that leads us to (9.5). The opposite limit where μ is small compared to ω_0 is less interesting, since here the natural frequency of the RC-circuit can excite continuum states in the nanowires, leading to many errors. Therefore in that limit the dephasing time should be on the order of $\frac{1}{\omega_0}$, which would be small.

The path to calculating $G_{\tilde{\gamma}_i}(t)$ was long, and we had to do a few assumptions along the way. The consequences of some of these assumptions are currently not well understood, and there are plenty of ways to expand upon this project in future research. Some of the assumptions cut away certain channels for dephasing, which suggests that there the actual dephasing time could be shorter than $t_{\min}$. But other assumptions are conservative, so all in all it's not clear at this stage whether the actual dephasing time is shorter or longer than the one suggested in our model.

One of the very first assumptions that was made in this project is that the dephasing doesn't happen during the readout of the Majorana box qubit. If an experiment is intending to use the readout protocol that was proposed in Section 3, then the equilibration time has to be much shorter than the dephasing time $t_{\min}$. If this equilibration time is comparable to the dephasing time, then one would have to consider if there are faster methods of reading out the state.

Another important assumption that made the problem tractable was the splitting of the continua, assuming that the Majoranas only interact with local wavepacket-like combinations of continuum quasiparticles. The actual physical mechanism for dephasing is exchanges of photons between the environment and the electrons in the system. This will have a predominant tendency to happen locally, which would excite these localised combinations of quasi particles, but in principle it is perfectly for the photons to have a wavelength on the order of the nanowire length, even though such photons would tend to have lower energy, and they still have to surmount the gap energy in order to make the excitations.
An even more interesting possible source of correlation between the well seperated Majorana operators happens if the quasi particle wave packets can travel down the length of the wire on the time scales we are considering. Besides ruining the assumptions that the continuum may be split up into local independent continua, this can also lead to logical Pauli errors within the ground state space. Assume for instance the overall parity is odd and initially the fermionic state associated to γ_1 and γ_2 is occupied. At some time the fermionic state is kicked into the contiuum from where it travels and drops into the before empty fermionic state comprised of γ_3 and γ_4. As far as I have been able to determine the group velocity of wave packets of continuum quasi particles in the p-wave superconductor is at this point an open question, and it is the plan to investigate this next. This would also be interesting from a transport point of view, since it seems likely that transport through a Majorana nanowire should happen at roughly the same speed. Neglecting the propagation of quasi particles means ignoring a likely source of decoherence. However it will be suppressed with the length of the wire, and so if the wires can be assumed long enough, one could theoretically get rid of this source of decoherence. Therefore it doesn't necessarily change the dephasing time drastically from what we have found in this work.
The least controlled assumption was dropping the term in H_t that couples different continuum states. This could arguably be a fine thing to do if the matrix element $\delta H_{\xi\xi'}$ is small, but it could still integrate up to a sizeable effect. Currently I don't know what $\delta H_{\xi,\xi'}$ is, but it is straight forward, albeit tedious, to calculate it using the eigenstates i found in Section 6. I don't think including this term can qualitatively change the result by much however. First of all with every jump comes a factor $\phi(t)$ which is assumed to be small. So a process will be heavily suppressed if it jumps around too many times in the continuum. All processes start and end below the gap, and thus is at least to order ϕ^2. If a process involves a single jump above the gap, say $\delta H_{\xi,\xi'}$, then also involves both $\delta H_{0\xi}$ and $\delta H_{0,\xi'}$. In Figure 15 is shown a plot of $|\delta H_{0,\xi}|$, which is a fairly peaked function, and thus jumps in the continuum cannot go too far away without either being suppressed by $\delta H_{0,\xi}$ or by a many factors of ϕ. Still it would be useful to get a more quantitative understanding of this assumption.

In the analysis in Section 8 we made three approximations. The first was neglecting the crossing diagrams. We did this arguing that any measurement we could think of wouldn't be able to measure such fast oscillating functions anyway. I don't have a formal proof, but the calculations in Appendix C seem to indicate, that to the order that we are considering, the fast oscillating terms are zero initially. If this is not the case, then in a sense we are overestimating the dephasing effects, getting an instantaneous dephasing effect, since it leads to $P_{\tilde{z}z}$ being less than one immediately. I don't think that is the case however.
The second approximation was perhaps the least controlled, which was the saddle-point approximation which was done in equation (8.17). But can be seen on Figure 9 the exponents peaks are not sharp at all times. It eventually becomes a good approximation at around $\mu t = 8$, but initially it is not great. The function which is the subject of the approximation is the integral kernel in the integral equation (8.1). This means that the error done by doing the saddle-point approximation at some early time is accumulated to later times. After the saddle-point approx-

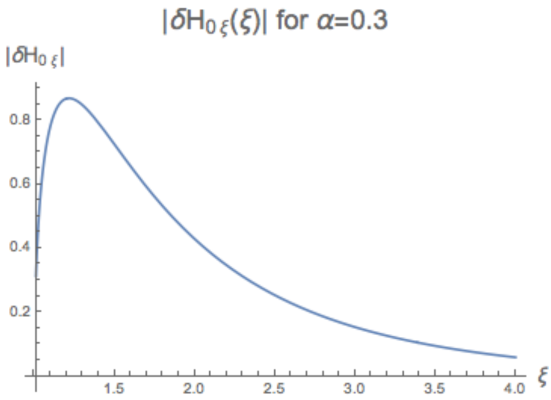


Figure 15: Plot of the matrix element $\delta H_{0,\xi}$ between the zero energy ground state and the excited continuum states. ξ is defined as $\frac{E}{\mu}$, where E is the energy of the continuum state and μ is the gap.

imation we did the last approximation, namely taking the function $C_{\xi_0(t)}$ to be constant. As is seen on Figure 10 this approximation starts leading to an overestimation of the dephasing effect, until after $\mu t = 10$ the actual value value of $C_{\xi_0(t)}$ is less than a third of its initial value. As $C_{\xi_0(t)}$ approaches 0, so does the change in $G_{\tilde{\gamma}_i}(t)$, which can be seen for instance in equation (8.12) indicating either that the information has completely dissipated or that the approximations break down at this point.
Lastly it should maybe be repeated that the setup depends on the form of the time evolution operator in equation (5.32), which as was mentioned might potentially break down if higher derivatives of $\phi(t)$ can potentially be big.

10 Conclusions

In this thesis we set out to study how the state of a Majorana box qubit decoheres in the face of potential fluctuations. The box qubit was capacitatively coupled to the environment, and the potential fluctuactions were modelled as the fluctuating voltage drop over a capactitance C coupled to the environment impedance Z_0. We have seen that this decoherence happens as an exponential decay of the information. Concretely if a particular state on the box qubit is prepared at time $t = 0$, then the probability of finding the same state at a later time t decays exponentially. If we assume gap energy μ dominates the characteristic frequency of the RC circuit, $\frac{1}{Z_0 C}$, then the decay time $t_{\min}$ is found to be proportional to $\frac{\mu^3 Z_0 C^2}{m \Delta^2 e^2 k_B T}$, where m is the effective mass of the electrons and Δ is the p-wave pairing parameter of the system. As such the lifetime of the qubit is expected to be long when subjected to this type of noise.

In future studies I will investigate the impact of the assumption that transitions within the continuum were neglected. It is not expected that including these should drastically change the result, but as of yet it is an open question. Another open question is how quickly quasi particles propagate in the system. This has been neglected in this thesis, suggesting that the lifetime of the box qubit is a bit shorter than what I have found. Decoherence processes involving quasi particle propagation are suppressed in very long wires, but it is unknown whether the fact that they can lead to logical Pauli errors compensates for this suppression.

Appendices

A Finding effective Hamiltonians projectively

Let H be a time-independent Hamiltonian acting on a Hilbert space $\mathcal{H}$ and let P and Q be projection operators, that is $P^2 = P$ and $Q^2 = Q$, satisfying $P + Q = \mathbb{1}$ and $PQ = QP = 0$. In this section we will review how to find an effective Hamiltonian H_{eff} for the subset $P\mathcal{H}$ of $\mathcal{H}$ by projecting out the rest. The approach is useful for finding effective low-energy Hamiltonians, and is used several times in this thesis. The stratey is to massage the time-independent Schrödinger equation $E\psi = H\psi$ into an equation for just $P\psi$. This equation will be non-linear, so in order to identify an effective Hamiltonian, we will need either to do a Taylor expansion or argue that the non-linear term may be dropped. The time independent Shrödinger equation can be cast into the form

$$E\begin{pmatrix}\psi_P \\ \psi_Q\end{pmatrix} = \begin{pmatrix}H_{PP} & H_{PQ} \\ H_{QP} & H_{QQ}\end{pmatrix}\begin{pmatrix}\psi_P \\ \psi_Q\end{pmatrix}, \tag{A.1}$$

where $\psi_O = O\psi$ and $H_{O_iO_j} = O_i H O_j$. From the second component of (A.1) we have

$$(EQ - H_{QQ})\psi_Q = H_{QP}\psi_P. \tag{A.2}$$

Both the left-hand side and the right-hand have non-zero components only in $Q\mathcal{H}$. Therefore we may invert it for an expression for ψ_Q, noting that the inversion is only taken on the Q block[3]. We denote this type of inversion by $M_{QQ}\Big|_Q^{-1}$, for some matrix M. Since Q is the identity on its own projected subspace we get

$$\psi_Q = (EQ - H_{QQ})\Big|_Q^{-1} H_{QP}\psi_P. \tag{A.3}$$

Inserting this into the first components yields

$$E\psi_P = \Big(H_{PP} + H_{PQ}(EQ - H_{QQ})\Big|_Q^{-1} H_{QP}\Big)\psi_P, \tag{A.4}$$

which is an exact non-linear Schrödinger equation equation for just the projection of ψ onto the subspace $P\mathcal{H}$, with the Hamiltonian

$$H_P(E) = H_{PP} + H_{PQ}(EQ - H_{QQ})\Big|_Q^{-1} H_{QP}. \tag{A.5}$$

Let us now assume that the Hamiltonian H describes a gapped system with an energy gap μ and a continuum of states above the gap. The Hilbert space $\mathcal{H}$ then is a many-particle Fock space. We take P to be the projector onto the sub-gap subspace of $\mathcal{H}$ and Q to be the projector onto the continuum. Let us write $H = H_0 + H'$, where H_0 is diagonal and H' connects subgap states to the continuum, as well as continuum states to other continuum states. We assume that $H' \propto \phi$, where $\phi \ll \mu$, just as is the case in the thesis' main text. The idea will be to eventually do a Taylor expansion in $\frac{\phi}{\mu}$ to second order. We have $PH_0Q = QH_0P = 0$. The idea is not to use (A.5) to obtain an effective low-energy Hamiltonian for $P\mathcal{H}$.
Since H' is assumed to only connect the sub gap states to the continuum we have $PH'P = 0$. Thus (A.5) becomes

[3] It is not invertible in the full Hilbert space since part of it is projected away. Note that this type of inversion is only meaningful on block-diagonal matrices

$$
\begin{aligned}
H_P &= PH_0P + PH'Q(EQ - QHQ)\Big|_Q^{-1} QH'P \\
&= PH_0P + PH'Q(EQ - QH_0Q - QH'Q)\Big|_Q^{-1} QH'P.
\end{aligned} \tag{A.6}
$$

The next step is to obtain our effective Hamiltonian by expanding (A.6) to first order in $\frac{\phi}{\mu}$. Since a term $(EQ - QH_0Q)\Big|_Q^{-n} \sim \frac{1}{\mu^n}$ and $(H')^n \sim \phi^n$ we find

$$
H_{\text{eff}}(E) \approx PH_0P + PH'Q(EQ - QH_0Q)\Big|_Q^{-1} QH'P. \tag{A.7}
$$

so that

$$
E\psi_P = H_{\text{eff}}(E)\psi_P. \tag{A.8}
$$

Note that (A.8) is an non-linear equation for the low-energy part of the wave functions ψ_P. Since the mixing term H' is small we don't expect the energies E to be shifted far away from the unperturbed ones. The projector P ensures that we are projecting states away that live primarily in the continuum, and thus we take $E \ll \mu$ and drop E on the right hand side of (A.7) getting

$$
H_{\text{eff}} \approx PH_0P + PH'QH_0^{-1}QH'P. \tag{A.9}
$$

Note that to this order only terms in H' that connects the subgap states to the continuum are included. Terms that connect different continuum states are projected to zero.

B A useful fact about two level systems

Any pure state $|\psi\rangle$ in a two level system may be written as a coherent state :

$$
|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \tag{B.1}
$$

with $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$. The density matrix $\rho(\theta, \phi)$ for such a state is

$$
\begin{aligned}
\rho = |\psi\rangle\langle\psi| &= \cos^2\frac{\theta}{2}|0\rangle\langle 0| + \sin^2\frac{\theta}{2}|1\rangle\langle 1| + \underbrace{\cos\frac{\theta}{2}\sin\frac{\theta}{2}}_{=\frac{1}{2}\sin\theta}\big(e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0|\big) \\
&= \begin{pmatrix} \cos\frac{\theta}{2} & \frac{\sin\theta e^{-i\phi}}{2} \\ \frac{\sin\theta e^{i\phi}}{2} & \sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \frac{\sin\theta e^{-i\phi}}{2} \\ \frac{\sin\theta e^{i\phi}}{2} & 1-\cos\frac{\theta}{2} \end{pmatrix} \\
&= \frac{1}{2}\sigma_0 + \frac{\sin\theta\cos\phi}{2}\sigma_x + \frac{\sin\theta\sin\phi}{2} + \big(\underbrace{\cos^2\frac{\theta}{2} - \frac{1}{2}}_{=\frac{\cos\theta}{2}}\big)\sigma_z,
\end{aligned} \tag{B.2}
$$

or

$$
\begin{aligned}
\rho(\theta, \phi) &= \frac{1}{2}\tilde{r}\cdot\tilde{\sigma} & \text{(B.3)} \\
\tilde{r} &\equiv (1, \bar{r}) \equiv (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi) & \text{(B.4)} \\
\tilde{\sigma} &\equiv (\sigma_0, \bar{\sigma}). & \text{(B.5)}
\end{aligned}
$$

For mixed states we have a similar form as (B.3). For instance for a statistical mix ρ_{mix} of two pure coherent states we find

$$\rho_{\text{mix}} \equiv p_1\rho(\theta,\phi) + p_2\rho(\theta',\phi') = \frac{1}{2}(p_1\tilde{r} + p_2\tilde{r}')\cdot\tilde{\sigma}, \tag{B.6}$$

so the most general density matrix on a two level system has the form

$$\rho = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\bar{\sigma}) = \frac{1}{2}\tilde{\mathbf{r}}\cdot\tilde{\sigma}, \tag{B.7}$$

where $|\mathbf{r}| \leq 1$ and $\tilde{\mathbf{r}} = (1, \mathbf{r})$.

C Dropping the crossing diagrams

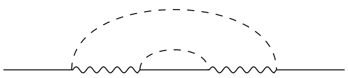
In this section we will examine the approximation of dropping the crossing diagrams in (8.6). For simplicity we will use the form of $A(t) = C_{\xi_0}\cos(\omega_m t)$ from the saddle point approximation. We will see that dropping the crossing diagrams ignores corrections of order ω_m^{-2} as well as terms at order ω_m^{-2} which oscillate very fast, and we thus neglect since they have unmeasurable contributions.
$G_{\dot\phi\dot\phi}$ is given by (8.14) and to distinguish the two terms I will use

$$t_i \;\textbf{---}\; t_j \;\equiv\; t_i \text{ - - - } t_j \;+\; t_i \sim t_j\;, \tag{C.1}$$

where (again $t_j \geq t_i$)

$$\begin{aligned} t_i \text{ - - - } t_j &= -D\delta(t_j - t_i) \\ t_i \sim t_j &= -D\frac{\omega_0}{2}e^{-\omega_0(t_j - t_i)}, \end{aligned} \tag{C.2}$$

where $D \equiv \frac{m\Delta^2}{\pi\mu^3}\frac{4e^2\omega_0 k_B T}{C}$. The first claim is that diagrams where the dashed lines corresponding to delta functions, cross are zero. For example the following diagram vanishes:



$$\begin{aligned}
&\propto \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \delta(t_1 - t_4)\delta(t_2 - t_3)\cos(\omega_m(t_1 - t_2))\cos(\omega_m(t_3 - t_4)) \\
&= \frac{1}{4}\sum_{d_1,d_2=\pm1}\int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 \int_0^t dt_4 \delta(t_1 - t_4)\delta(t_2 - t_3)\theta(t_1 - t_2)\theta(t_2 - t_3)\theta(t_3 - t_4) \\
&\quad \times e^{i\omega_m d_1(t_1 - t_2)}e^{i\omega_m d_2(t_3 - t_4)} \\
&= \frac{1}{4}\sum_{d_1,d_2=\pm1}\int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 \delta(t_2 - t_3)\theta(t_1 - t_2)\theta(t_2 - t_3)\theta(t_3 - t_1)e^{i\omega_m d_1(t_1 - t_2)}e^{i\omega_m d_2(t_3 - t_1)} \\
&= \frac{1}{4}\sum_{d_1,d_2=\pm1}\int_0^t dt_1 \int_0^t dt_2 \theta(t_1 - t_2)\theta(t_2 - t_1)\theta(0)e^{i\omega_m d_1(t_1 - t_2)}e^{i\omega_m d_2(t_2 - t_1)} \\
&= \frac{1}{8}\sum_{d_1,d_2=\pm1}\int_0^t dt_1 \int_{t_1}^{t_1} dt_2 e^{i\omega_m(t_1 - t_2)(d_1 - d_2)} = 0.
\end{aligned} \tag{C.3}$$

Likewise diagrams of the following type

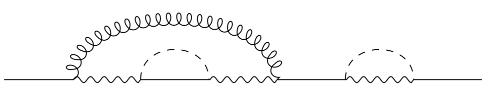
(C.4)

are also zero since the delta function collapses one of the integrals to be over just a single point. So regarding the non-zero diagrams we can conclude two things:

– Dashed lines cannot cross curly lines.

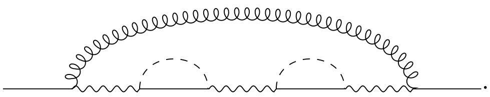
– Two dashed lines cannot cross one another, and one cannot lie underneath the other.

In the following I will assume all diagrams to be non-zero, obeying the above two rules.
Let us start by considering diagrams consisting only of dashed lines. This has to be a direct diagram where the dashed lines match up with the wavy lines. This means that argument of the delta functions and the cosines match, and so the dependence on ω_m drops out of all these diagrams. If we then replace one of the dashed lines with a curly one we have some freedom to put it in different places, for instance



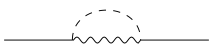
(C.5)

or



(C.6)

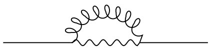
Let us introduce som nomenclauture for the following discussion. I will call diagrams of the follwing sort



(C.7)

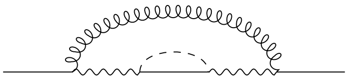
for delta diagrams, while I will call diagrams with a curly line crossing l of the wavy lines an l-sheep. This means that an l-sheep has $l-1$ delta diagrams inside of it. So for instance (C.6) is 3-sheep while (C.5) is a 2-sheep followed by a delta diagram. The fact that the integral limits are nested means that it is not immediately trivial what happens when one diagram precedes or follows another. For the rest of this section I am going to demonstrate that all diagrams including a single l-sheep are of order ω_m^{-2}, and that if $l > 1$, then it is a fast oscillating functions to this order. Then I'll show that if a diagram includes more than one l-sheep, these potentially crossing, then it is of at least order ω_m^{-4}. The conclusion to this is that to order ω_m^{-2} we keep all non-oscillating contributions to $\overline{G_i}(t)$ when we sum up only the direct diagrams.
In order to reach the above conclusions we are going to have to evaluate a few of the diagrams and then notice some patterns. We don't need to calculate the diagrams in their fullness however, we just need to find the terms to leading order in $\frac{1}{\omega_m}$. Let us start by evaluating a

1-sheep:



$$
\begin{aligned}
&\propto \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-\omega_0(t_1-t_2)} \cos(\omega_m(t_1-t_2)) \\
&= \frac{1}{2}\sum_{d=\pm1} \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-\omega_0(t_1-t_2)} e^{i\omega_m d(t_1-t_2)} \\
&= \frac{1}{2}\sum_{d=\pm1} \frac{1}{\omega_0 - i\omega_m d} \int_0^t dt_1 e^{(-\omega_0+i\omega_m d)t_1}\left(e^{(\omega_0-i\omega_m d)t_1}-1\right) \\
&= \frac{1}{2}\sum_{d=\pm1} \frac{1}{\omega_0 - i\omega_m d} t + \mathcal{O}(\omega_m^{-4}) = \frac{1}{2}\sum_{d=\pm1} \frac{\omega_0 + i\omega_m d}{\omega_0^2+\omega_m^2} t + \mathcal{O}(\omega_m^{-4}) = \frac{\omega_0}{\omega_0^2+\omega_m^2} t + \mathcal{O}(\omega_m^{-4}),
\end{aligned}
\tag{C.8}
$$

where in the third line we icluded only the first term in the parenthesis where the exponential cancels. Notice that if we go to higher orders we would still have the exponential, and so would see oscillatory behaviour. Next let us take a closer look at the 2-sheep:



$$
\begin{aligned}
&\propto \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \delta(t_2-t_3) e^{-\omega_0(t_1-t_4)} \cos(\omega_m(t_1-t_2)) \cos(\omega_m(t_3-t_4)) \\
&= \frac{1}{4}\sum_{d_1,d_2=\pm1} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \delta(t_2-t_3) e^{-\omega_0(t_1-t_4)} e^{i\omega_m d_1(t_1-t_2)} e^{i\omega_m d_2(t_3-t_4)} \\
&= \frac{1}{4}\sum_{d_1,d_2=\pm1} \frac{1}{\omega_0 - i\omega_m d_2} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \delta(t_2-t_3) e^{(-\omega_0+i\omega_m d_1)t_1} e^{-i\omega_m d_1 t_2} e^{i\omega_m d_2 t_3} \left(e^{(\omega_0-i\omega_m d_2)t_3}-1\right) \\
&= \frac{1}{8}\sum_{d_1,d_2=\pm1} \frac{1}{\omega_0 - i\omega_m d_2} \int_0^t dt_1 \int_0^{t_1} dt_2 e^{(-\omega_0+i\omega_m d_1)t_1} e^{-i\omega_m (d_1-d_2) t_2} \left(e^{(\omega_0-i\omega_m d_2)t_2}-1\right).
\end{aligned}
\tag{C.9}
$$

Now comes an important step. Since the propagators are not aligned with the wavy lines we don't see a term where the exponentials flat out cancel. However when $d_1 = d_2$ and we take the second term in the parenthesis of (C.9), then the dependence on t_2 drops out giving

$$
\begin{aligned}
&-\frac{1}{8}\sum_{d=\pm1} \frac{1}{\omega_0 - i\omega_m d} \int_0^t dt_1 e^{(-\omega_0+i\omega_m d)t_1} t_1 \\
&= -\frac{1}{8}\sum_{d=\pm1} \frac{1}{\omega_0 - i\omega_m d} \frac{1}{-\omega_0 + i\omega_m d} \lim_{\lambda\to1} \partial_\lambda \int_0^t dt_1 e^{\lambda(-\omega_0+i\omega_m d)t_1}. \\
&= -\frac{1}{8}\sum_{d=\pm1} \frac{1}{\omega_0 - i\omega_m d} \frac{1}{-\omega_0 + i\omega_m d} \lim_{\lambda\to1} \partial_\lambda \frac{e^{\lambda(-\omega_0+i\omega_m d)t}-1}{\lambda(-\omega_0+i\omega_m d)}.
\end{aligned}
\tag{C.10}
$$

The leading contribution is found when differentiating the exponential, which yields

$$
\begin{aligned}
&-\frac{1}{8}\sum_{d=\pm1} \frac{1}{\omega_0 - i\omega_m d} \frac{(-\omega_0+i\omega_m d)t}{(-\omega_0+i\omega_m d)^2} e^{(-\omega_0+i\omega_m d)t} = \frac{1}{4}\sum_{d=\pm1} \frac{1}{(-\omega_0+i\omega_m d)^2} t e^{(-\omega_0+i\omega_m d)t} \\
&= \frac{1}{8}\sum_{d=\pm1} \frac{(\omega_0+i\omega_m d)^2}{(\omega_0^2+\omega_m^2)^2} t e^{(-\omega_0+i\omega_m d)t} = \frac{1}{8}\sum_{d=\pm1} \frac{\omega_0^2-\omega_m^2+2i\omega_0\omega_m d}{(\omega_0^2+\omega_m^2)^2} t e^{(-\omega_0+i\omega_m d)t},
\end{aligned}
\tag{C.11}
$$

and so to the leading order in $\frac{1}{\omega_m}$ the 2-sheep is proportional to

$$\frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2} t e^{-\omega_0 t}\cos(\omega_m t), \tag{C.12}$$

which is indeed of order $\omega_m^{-2}\cos(\omega_m t)$. It may seem like a bit of a coincidence so let's take a look at what happened along the way to lead us to this result, and what would happen if there had been more delta diagrams inside the sheep. The very first integration brought down a factor of $\frac{1}{\omega_0-i\omega_m d}$. The crucial step came after integrating over the delta function, where there was a term whose dependence on the next integration variable dropped out. This happened because the very first integral didn't cancel the contribution from the cosine. This left the final integral to the leading order to be over $e^{(-\omega_0+i\omega_m d)t_1}t_1$, which to the lowest order in $\frac{1}{\omega_m}$ gives both the cosine contribution and a factor of $\frac{1}{-\omega_0+i\omega_m d}$. The first and final integration factors then combined to give the result. Now let's consider how this would change if we had a 3-sheep instead. After the integration of the first delta function we would again have the leading term whose integrand would be proportional to $e^{i\omega_m d_1 t_2}e^{i\omega_m d_2)t_3}t_3$. The next delta function would evaluate this in $t_3 = t_2$ and when $d_1 = d_2$ the exponentional is equal to one and integration would simply make the final integrand proportional to $e^{(-\omega_0+i\omega_m d)t_1}t_1^2$. In general for an l-sheep the final integrand is proportional to $e^{(-\omega_0+i\omega_m d)t_1}t_1^{l-1}$. This means that to the leading order in $\frac{1}{\omega_m}$ the l-sheep is proportional to

$$\sum_d \frac{1}{\omega_0-i\omega_m d}\int_0^t dt_1 e^{(-\omega_0+i\omega_m d)t_1}t_1^{l-1} = \sum_d \frac{1}{(-\omega_0+i\omega_m d)^{l-1}}\lim_{\lambda\to 1}\partial_\lambda^{l-1}\int_0^t dt_1 e^{\lambda(-\omega_0+i\omega_m d)t_1}.$$
$$= \sum_{d=\pm 1}\frac{1}{\omega_0-i\omega_m d}\frac{1}{(-\omega_0+i\omega_m d)^{l-1}}\lim_{\lambda\to 1}\partial_\lambda^{l-1}\frac{e^{\lambda(-\omega_0+i\omega_m d)t}-1}{\lambda(-\omega_0+i\omega_m d)}. \tag{C.13}$$

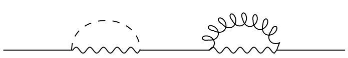
As before the leading contribution comes from acting with the derivative only on the exponential, yielding

$$\sum_{d=\pm 1}\frac{1}{\omega_0-i\omega_m d}\frac{(-\omega_0+i\omega_m d)^{l-1}t^{l-1}}{(-\omega_0+i\omega_m d)^l}e^{(-\omega_0+i\omega_m d)t}.$$
$$= -\sum_{d=\pm 1}\frac{\omega_0^2-\omega_m^2+2i\omega_0\omega_m d}{(\omega_0^2+\omega_m^2)^2}t^{l-1}e^{(-\omega_0+i\omega_m d)t}, \tag{C.14}$$

and so the leading order in $\frac{1}{\omega_m}$ is proportional to

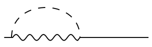
$$l-\text{sheep} \propto \frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2}t^{l-1}e^{-\omega_0 t}\cos(\omega_m t)+\mathcal{O}(\omega_m^{-3}). \tag{C.15}$$

Now that we understand that the l-sheep for $l > 1$ is of the order $\omega_m^{-2}\cos(\omega_m t)$ we need to consider what happens if the l-sheep is followed or preceded by delta diagrams. Let's first check the case of $l = 1$. In this the total diagram will still be of order $\frac{1}{\omega_m^2}$ with no oscillations. In the case where the delta diagram follows the 1-sheep this is clear, since no oscillating behaviour is picked up after the 1-sheep, and the following deltafunction kills all further dependence on ω_m, so the final diagram to leading order is proportional to $\frac{t^2}{\omega_0^2+\omega_m^2}$. If on the other hand the sheep follows after the delta function, then the sheep is to be evaluated with an extra factor of t. Let's write the integrand out after the delta diagram:



$$\propto \sum_{d=\pm 1}\int_0^t e^{(-\omega_0+i\omega_m d)t_1}dt_1\int_0^{t_1}dt_2 t_2 e^{(\omega_0-i\omega_m d)t}. \tag{C.16}$$

After performing the t_2 integral, the dominant contribution will have a factor of $\frac{1}{\omega_0 - i\omega_m d}$, but the time dependence drops out in the next integral. In this way the full diagram is still of order $\frac{1}{\omega_0^2+\omega_m^2}$ and there is no oscillating behaviour. If there were now multiple delta diagrams before and after the 1-sheep the argument would run the same way, and we find that the leading order of these diagrams are non oscillating and of order $\frac{1}{\omega_m^2}$.
Now we should turn our attention to what happens to l-sheep for $l > 1$ when they are preceded or followed by delta diagrams. First let us examine a general l-sheep with $l > 1$ followed by a single delta diagram. Then the leading order after integrating the sheep is going to be proportional to the expression in (C.15). After evaluating the l-sheep the whole diagram will to order $\frac{1}{\omega_m^2}$ be proportional to

- l − sheep 

$$
\begin{aligned}
&\propto \int_0^t dt_1 \int_0^{t_1} dt_2 \delta(t_1 - t_2) \frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2} t_2^{l-1} e^{-\omega_0 t_2} \cos(\omega_m t_2) \cos(\omega_m (t_1 - t_2)) \\
&= \frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2} \int_0^t dt_1 t_1^{l-1} e^{-\omega_0 t_1} \cos(\omega_m t_1) = \frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2} \sum_{d=\pm 1} \int_0^t dt_1 t_1^{l-1} e^{(-\omega_0 + i\omega_m d)t_1} \\
&= \frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2} \sum_{d=\pm 1} \frac{1}{(-\omega_0 + i\omega_m d)^{l-1}} \lim_{\lambda\to 1} \partial_\lambda^{l-1} \int_0^t dt_1 e^{\lambda(-\omega_0+i\omega_m d)t_1} \\
&= \frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2} \sum_{d=\pm 1} \frac{1}{(-\omega_0 + i\omega_m d)^{l-1}} \lim_{\lambda\to 1} \partial_\lambda^{l-1} \frac{e^{\lambda(-\omega_0+i\omega_m d)t} - 1}{\lambda(-\omega_0 + i\omega_m d)}.
\end{aligned} \tag{C.17}
$$

Acting with the derivatives on only the exponential yields the dominant contribution:

$$
\frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2} \sum_{d=\pm 1} \frac{t^{l-1} e^{\lambda(-\omega_0+i\omega_m d)t}}{-\omega_0 + i\omega_m d} = -\frac{2\omega_m^2}{(\omega_0^2+\omega_m^2)^3} t^{l-1} e^{-\omega_0 t} (\omega_0 \cos(\omega_m t) - \omega_m \sin(\omega_m t)). \tag{C.18}
$$

The conclusion is that l-sheep with $l \geq 2$ are even more strongly suppressed when they are followed by delta diagrams.
Next let us see what happens when an l-sheep follows a delta diagram. After the delta diagram the integrand will have picked up a factor of t, which will propagate through to the end, so the full diagram is still oscillating and of order ω_m^{-2}. This means that if we had instead multiple delta diagrams before the l-sheep the conclusion is the same, it changes the power of t.

Now we are left with one single challenge, which is to figure out what happens when we have multiple sheep. Let us start by considering when diagrams include one l-sheep and one k-sheep. There are three ostensibly different cases:

1. The two sheep are "product sheep", meaning that they follow one another without the curly lines crossing the same times.

2. One sheep is "embedded" in the other, meaning that it lies completely underneath the outer curly line of the other.

3. The sheep "cross", meaning that the curly lines intersect one another once.

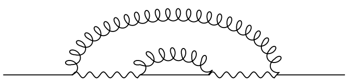
It turns out though that the embedded and crossing case is equivalent. This is simply because the curly lines only enter in the exponential function as $e^{-\omega_0(t_i - t_j)} e^{-\omega_0(t_k - t_l)}$ which is equal to

$e^{-\omega_0(t_i-t_l)}e^{-\omega_0(t_k-t_j)}$. If a diagram with the former is an embedded sheep, then the diagram with the latter is a crossing sheep.

Since we are going to order ω_m^2 our aim will be to show that whichever way one arranges the two sheep, the leading order is going to be at least $\frac{1}{\omega_m^3}$, so we safely can ignore these diagrams.

Let's start with the case of product sheep and put the l-sheep before the k-sheep. By (C.15) after evaluating the l-sheep to leading order will have an overall factor of $\frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2}$ and the will have an additional time dependence of $t^{l-1}\cos(\omega_m t)$. The factor of t^{l-1} doesn't influence whether the rest of the diagram gets additional factors of $\frac{\omega_m^2}{(\omega_0^2+\omega_m^2)^2}$ from the k-sheep. The extra factor of $\cos(\omega_m t)$ doesn't prevent it either, and the reason for this is that while it gives a contribution where the first cosine in the k-sheep drops out, the last cosine is still around and gives another factor of $\frac{1}{\omega_m^2}$.

The last case we need to check is when the k-sheep be embedded inside l-sheep. Let's simplify matters and evaluate the following diagram



$$
\begin{aligned}
&\propto \sum_{d_1,d_2=\pm1}\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\int_0^{t_3}dt_4 e^{(-\omega_0+i\omega_m d_1)t_1}e^{(-\omega_0-i\omega_m d_1)t_2}e^{(\omega_0+i\omega_m d_2)t_3}e^{(\omega_0-i\omega_m d_2)t_4}\\
&=\sum_{d_1,d_2=\pm1}\frac{1}{\omega_0-i\omega_m d_1}\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3 e^{(-\omega_0+i\omega_m d_1)t_1}e^{(-\omega_0-i\omega_m d_1)t_2}e^{(\omega_0+i\omega_m d_2)t_3}\Big(e^{(\omega_0-i\omega_m d_2)t_3}-1\Big)\\
&=\sum_{d_1,d_2=\pm1}\frac{1}{\omega_0-i\omega_m d_1}\int_0^t dt_1\int_0^{t_1}dt_2 e^{(-\omega_0+i\omega_m d_1)t_1}e^{(-\omega_0-i\omega_m d_1)t_2}\Big(\frac{e^{2\omega_0 t_2}-1}{2\omega_0}-\frac{e^{(\omega_0+i\omega_m d_2)t_2}-1}{\omega_0+i\omega_m d_2}\Big)\\
&=\sum_{d_1,d_2=\pm1}\frac{1}{\omega_0-i\omega_m d_1}\int_0^t dt_1\int_0^{t_1}dt_2 e^{(-\omega_0+i\omega_m d_1)t_1}e^{(-\omega_0-i\omega_m d_1)t_2}\Big(\frac{e^{2\omega_0 t_2}-1}{2\omega_0}-\frac{e^{(\omega_0+i\omega_m d_2)t_2}-1}{\omega_0+i\omega_m d_2}\Big)\\
&=\sum_{d_1,d_2=\pm1}\frac{1}{\omega_0-i\omega_m d_1}\int_0^t dt_1 e^{(-\omega_0+i\omega_m d_1)t_1}\Bigg(\frac{1}{2\omega_0}\Big(\frac{e^{(\omega_0-i\omega_m d_1)t_1}}{\omega_0-i\omega_m d_1}-\frac{e^{(-\omega_0-i\omega_m d_1)t_1}}{-\omega_0-i\omega_m d_1}\Big)\\
&\qquad-\frac{1}{\omega_0+i\omega_m d_2}\Big(\frac{e^{i\omega_m(-d_1+d_2)t_1}-1}{i\omega_m(-d_1+d_2)}\delta_{-d_1,d_2}+t_2\delta_{d_1,d_2}-\frac{e^{(-\omega_0-i\omega_m d_1)t_1}-1}{-\omega_0-i\omega_m d_1}\Big)\Bigg)
\end{aligned}
\tag{C.19}
$$

This is clearly at least of order $\frac{1}{\omega_m^4}$. If delta diagrams where inserted undernearth the outer curly line, then it only contributes to higher orders of the time variables at various stages in the embedded sheep. This however doesn't affect the number of factors of e.g $\frac{1}{\omega_0-i\omega_m d}$ that gets pulled down under the integral, and so all embedded sheep, and by that token also crossing sheep, are negligible.
If more sheep are added to this the calculations run in analogous ways, and we find for each sheep an overall additional factor of $\frac{1}{\omega_m^2}$ to leading order.
Now finally for the punch line: We have seen that diagrams of order ω_m^{-2} can have at most a single sheep, and if that is an l-sheep of $l>1$, the diagram evaluates to a fast oscillating

function which we may drop, since it will fall out anyway later when we insert the result into equation. If we leave out all crossing diagrams, then we are left with direct diagrams with any number of 1-sheep, but in doing so we havce kept exactly all the diagrams of order $\frac{1}{\omega_m^2}$ which don't oscillate fast.

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