Effective Negative Mass and Frequency Down-Conversion in Bichromatic Optomechanics

Author	Nikolaj Aagaard Larsen
Primary Supervisor	Assoc. Prof. Emil Zeuthen
Co-Supervisor	Prof. Eugene S. Polzik

UNIVERSITY OF COPENHAGEN

QUANTOP

The Danish Center for Quantum Optics
The Niels Bohr Institute

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Abstract

With a diverse range of mesoscopic and macroscopic physical implementations and quantum-level control, cavity optomechanics is now a mature research direction in experimental and theoretical quantum optics alike. Recently in the article “Trajectories without quantum uncertainties in composite systems with disparate energy spectra” Zeuthen et. al. made a theoretical proposal which can further extend the set of physical platforms that can implement such quantum-backaction evasion that rely on the concept of a negative-mass reference frame. In particular, their work showed how an intrinsically positive-mass mechanical oscillator can be converted to an effective negative-mass oscillator with a down-converted frequency using two-tone driving and feedback.

Abstract In this thesis we generalize the work by Zeuthen et. al. by going beyond the unresolved sideband regime and including variable detuning and relative strength of the two drive tones, leading to a number of new effects relevant for the practical implementation of the scheme. Firstly we make considerations on the consequences of dynamical back-action arising from a non-zero detuning between the cavity resonance and the relative drive frequency, and how such effects affect the effective oscillator parameters and our ability to remove the extraneous noise induced by the two-tone driving. Unlike much previous work on quantum back-action evasion in cavity optomechanics, we do not limit our theory to neither the resolved nor unresolved sideband regime. We find that it is possible to obtain a simple analytical theory in the limit where the light-oscillator interaction generates sidebands that are well-separated and narrow compared to the cavity susceptibility. Within this simple theory we consider two noise removal schemes: We first show that it is possible to completely suppress the extraneous light noise by using a homodyne measurement, provided the power of the two drive tones is used to equalize the response from the corresponding sidebands. Secondly we show that in absence of such balancing, only the amplitude or phase part of the extraneous noise can be removed using feedback based on a homodyne measurement, with the residual noise leading to an effective elevated thermal bath occupancy of the mechanical oscillator. The present work establishes that an optomechanical system with arbitrary degree of sideband resolution can realize an effective negative-mass oscillator.

Table of Contents

CHAPTER 1

INTRODUCTION

> "A long time ago in a galaxy far, far away...."
>
> \- *Star Wars*, George Lucas

One of the major consequences of quantum mechanics is the fundamental limits imposed on the precision of measurements, for example simultaneous tracking of a pair of canonically conjugate variables such as position $\hat{X}(t)$ and momentum $\hat{P}(t)$. These limits all stem from the non-commutative nature of the given observables of interest, for example the canonical commutation relation $\left[\hat{X}, \hat{P}\right] = i$ results in the famous Heisenberg uncertainty relation $\mathrm{Var}(\hat{X})\mathrm{Var}(\hat{P}) \geq \frac{1}{4}$. In experiments such limitation on measurement precision is enforced by the quantum back-action imposed on a system by the probe [2]. If e.g. you measure the position of a moving mirror by probing it with a laser, the quantum fluctuations of the laser field leads to a stochastic radiation pressure on the mirror.

One way to circumvent the limitations set by the uncertainty principle is by only measuring commuting operators; for a set of commuting operators, usually referred to as a *qunatum-mechanics free subspace* (QMFS) [8], there is no uncertainty relation, given the right engineering, the set of operators can be free of quantum back-action. One particular instance of such a QMFS for measurement of position and momentum of a quantuum oscillator, can be constructed by measuring the system of interest relative to an effective negative mass oscillator [7]. For definiteness say we have two quantum harmonic oscillators with position and momentum $\hat{X}_1$, $\hat{P}_1$ and $\hat{X}_2, \hat{P}_2$, their Heisenberg equations of motion (EOMs) are:

$$\hat{X}_i(t) = \cos(\Omega_i t)\hat{\hat{X}}_i(t) + \sin(\Omega_i t)\hat{\hat{P}}_i(0), \tag{1.1}$$

$$\hat{P}_i(t) = \cos(\Omega_i t)\hat{\hat{P}}_i(0) - \sin(\Omega_i t)\hat{\hat{X}}_i(0) \tag{1.2}$$

These equations show that in the typical scenario of a weak, continuous measurement of an individual oscillator position $\hat{X}_i(t)$ over several periods $2\pi/|\Omega_i|$ amounts to a simultaneous measurement of the non-commuting pair$\hat{\hat{X}}_i(t)$ and $\hat{\hat{P}}_i(t)$, resulting in both being contaminated by quantum backaction. To remedy this, we now define a composite set of so called EPR variables:

$$\hat{X}_{\text{EPR}}(t) = \frac{\hat{X}_1(t) + \hat{X}_2(t)}{\sqrt{2}} \tag{1.3}$$

$$\hat{P}_{\text{EPR}}(t) = \frac{\hat{P}_1(t) - \hat{P}_2(t)}{\sqrt{2}} \tag{1.4}$$

For $\Omega_1 = \Omega_2 = \Omega_0$ the EPR variables for the two harmonic oscillators are.

$$\hat{X}_{\text{EPR}}(t) = \left[\frac{\hat{\tilde{X}}_1(t) + \hat{\tilde{X}}_2(t)}{\sqrt{2}}\right]\cos(\Omega_0 t) + \left[\frac{\hat{\tilde{P}}_1(t) + \hat{\tilde{P}}_2(t)}{\sqrt{2}}\right]\sin(\Omega_0 t), \tag{1.5}$$

$$\hat{P}_{\text{EPR}}(t) = \left[\frac{\hat{\tilde{P}}_1(t) - \hat{\tilde{P}}_2(t)}{\sqrt{2}}\right]\cos(\Omega_0 t) - \left[\frac{\hat{\tilde{X}}_1(t) - \hat{\tilde{X}}_2(t)}{\sqrt{2}}\right]\sin(\Omega_0 t) \tag{1.6}$$

A weak, continuous measurement of, e.g., $\hat{X}_{\text{EPR}}$ still cannot be QBA-free seeing as its decomposition into slowly-varying operator combinations also consists of non-commuting variables:

$$\left[\frac{\hat{\tilde{X}}_1(t) + \hat{\tilde{X}}_2(t)}{\sqrt{2}}, \frac{\hat{\tilde{P}}_1(t) + \hat{\tilde{P}}_2(t)}{\sqrt{2}}\right] = i \tag{1.7}$$

However consider the case of two counter rotating oscillators $\Omega_1 = -\Omega_2 = \Omega_0$ with $\Omega_0 > 0$, then something interesting happens, namely we now have:

$$\hat{X}_{\text{EPR}}(t) = \left[\frac{\hat{\tilde{X}}_1(t) + \hat{\tilde{X}}_2(t)}{\sqrt{2}}\right]\cos(\Omega_0 t) + \left[\frac{\hat{\tilde{P}}_1(t) - \hat{\tilde{P}}_2(t)}{\sqrt{2}}\right]\sin(\Omega_0 t), \tag{1.8}$$

$$\hat{P}_{\text{EPR}}(t) = \left[\frac{\hat{\tilde{P}}_1(t) - \hat{\tilde{P}}_2(t)}{\sqrt{2}}\right]\cos(\Omega_0 t) - \left[\frac{\hat{\tilde{X}}_1(t) + \hat{\tilde{X}}_2(t)}{\sqrt{2}}\right]\sin(\Omega_0 t) \tag{1.9}$$

in which case a continuous measurement of $\hat{X}_{\text{EPR}}(t)$ is a simultaneous measurement of the commuting pair consisting of $\frac{1}{\sqrt{2}}\left(\hat{\tilde{X}}_1(t) + \hat{\tilde{X}}_2(t)\right)$ and $\frac{1}{\sqrt{2}}\left(\hat{\tilde{P}}_1(t) - \hat{\tilde{P}}_2(t)\right)$ (which also determines $\hat{P}_{\text{EPR}}(t)$).

So we see that if one of the oscillators has an effective negative frequency, also referred to as an effective negative mass oscillator(1), and we measure the composite variables rather than a single subsystem we can in principle do so we now limitations imposed by quantum mechanics. A futher intersting point, is that the EPR variables satisfy the so-called Duan criterion:

$$\text{Var}\left[\hat{X}_{\text{EPR}}(t)\right] + \text{Var}\left[\hat{P}_{\text{EPR}}(t)\right] < 2 \tag{1.10}$$

and consequently the two subsystems, i.e. the two harmonic oscillators are entangled. The takeaway is that constructing a QMFS by utilizing an effective negative frequency can be used both for quantum enhanced sensing and entanglement generation.

The consideration so far now begs the question, are there such systems which display an effective negative mass? Indeed there are! The prototypical example is the collective spin degree of freedom for a polarized ensemble of atoms precessing in a constant bias magnetic field can display a negative Larmor frequency [6].

(1) QBA evasion utilizing an effective negative frequency are often referred to as "measurements in the effective negative mass reference frame", the reason being the the sign of the frequency can equally well be assigned to the mass. However this is more of a slight of hand than anything profound. Personally I prefer to refer to the susceptibility of the system at hand, since a $\frac{\pi}{2}$ phase shift in the susceptibility is what causes this "negative mass" behavior, however the negative mass usually grabs more attention.

Recently in [10] a proposal was made for generating such an effective negative mass in an optomechanical system, which would allow for even broader applications of the QMFS based QBA evasion. This proposal introduces a scheme for generating an effective mechanical oscillator with a tunable resonance and an effective negative frequency. The basic idea of the scheme is based on the following consideration. In a canonical single-tone driven optomechanical system, a single laser tone is used to drive a cavity with a mechanical oscillator embedded in it. The mechanical motion of the oscillator is imprinted on the light in the cavity as two sidebands, the lower sideband is due to a two-mode-squeezing interaction and the upper sideband due to s beam-splitter interaction. These two sidebands are seperated from the drive frequency by exactly the mechanical frequency. If you instead drive a cavity with two tones, the optomechanical interaction leads to four sidebands. By suitable tuning of the two tones, the central sidebands can be made to mimic the response of the single-tone-driven system, except with the sidebands resulting from two-mode-squeezing and beam-splitter interaction, respectively, having swapped positions. This is the characteristic of negative-frequency oscillator seeing as, e.g., cooling via the beam-splitter interaction requires extracting a negative amount of energy from the oscillator. However, the two processes associated with the additional, outermost sidebands must be somehow suppressed. In the article by Zeuthen et. al they show that it is possible to simply measure the light near the outer sidebands which allows one to remove the extraneous noise.

However their proposal is limited to the case where the linewidth of the cavity is large compared to the sideband seperation, the so called unresolved sideband regime, and they do not account for the dynamical back-action arising when the mean frequency of the two drive tones is detuned from the cavity resonance.

In this thesis we will generalize the work started by Zeuthen et. al. by accounting for the dynamical back-action arising from detuning, and by making no assumption on the cavity resolution: our work captures both the resolved - and unresolved sidebands regime. Our guiding principle will be to find a simple analytical theory which can be used as a starting point for further investigation.

1.1 Outline

- In Chapter 2 we discuss how to measure quadratures of light. Being able to measure quadratures is an essential part of the noise suppression scheme we need to generate a use full effective negative mass oscillator
- In Chapter 3 we introduce single-tone optomechanics. This is essential as our end goal is to engineer an effective single-tone driven oscillator.
- In Chapter 4 we introduce two-tone driven optomechanics by deriving the Heisenberg-Langevin equations of motion which will lay the foundation for the remainder of the thesis.
- In Chapter 5 we use the contents of the previous chapters to derive simple theory for two-tone optomechanics which allows us to examine the possibility of generating an effective negative mass oscillator.

CHAPTER 2

Measurement of Light Quadratures

“ And the Lord spake, saying, ”First shalt thou take out the Holy Pin. Then shalt thou count to three, no more, no less. Three shall be the number thou shalt count, and the number of the counting shall be three. Four shalt thou not count, neither count thou two, excepting that thou then proceed to three. Five is right out. Once the number three, being the third number, be reached, then lobbest thou thy Holy Hand Grenade of Antioch towards thy foe, who, being naughty in My sight, shall snuff it.”

BROTHER MAYNARD

\- *Monty Python and the Holy Grail*, the Pythons

In this chapter we discuss various methods for measuring quadratures of light. This is crucial for the present work which aim is to engineer the interaction between light and a localized oscillator by means of feedback loops based exactly on the measurement of light In particular we focus on so called homodyne measurements, where a signal of interest is mixed on a 50/50 beam splitter with a strong coherent field, referred to as the local oscillator (LO). We will first discuss homodyne measurements for an arbitrary signal field and then using the so called two-photon formalism which is natural when the signal of interest is contained in two sidebands.

2.1 Balanced Homodyne Detection

2.1.1 Homodyning with a Generic Signal

In cavity optomechanics the degrees of freedom we usually are interested in are the position and momentum of the mechanical oscillator or the phase and amplitude quadrature of the intra-cavity field. The information about the optomechanical dynamics are imprinted in the quadratures of the light exiting the cavity and thus it is of great interest to measure these quadratures. Let us refer to the light we wish to measure as the signal field represented by the annihilation operator[1]

(1) In this section we work in a frame rotating at the carrier frequency ω_s of the signal field

$\hat{a}_s(t)$. We assume that the signal field is weak in the sense that it is dominated by quantum fluctuations. From the signal field annihilation operator we can define an amplitude quadrature $\hat{q}_s(t)$ and a phase quadrature $\hat{y}_s(t)$ as follows[3]:

$$\hat{q}_s(t) = \frac{\hat{a}_s(t) + \hat{a}_s^\dagger(t)}{\sqrt{2}} \tag{2.1a}$$

$$\hat{y}_s(t) = \frac{\hat{a}_s(t) - \hat{a}_s^\dagger(t)}{\sqrt{2}\, i} \tag{2.1b}$$

These quadratures satisfy the canonical commutation relation:

$$[\hat{q}_s(t), \hat{y}(t')] = i\delta(t - t') \tag{2.2}$$

Another important observation can be made by writing the quadrutures in Fourier space(2):

$$\hat{q}_s(\Omega) = \frac{\hat{a}(\Omega) + \hat{a}^\dagger(-\Omega)}{\sqrt{2}}, \tag{2.3a}$$

$$\hat{y}_s(\Omega) = \frac{\hat{a}(\Omega) - \hat{a}^\dagger(-\Omega)}{\sqrt{2}\, i}, \tag{2.3b}$$

Namely, we see that the quadratures are two-photon quadratures in the sense that they each depend on two frequencies $\pm\Omega$ relative to the LO frequency. We will elaborate further on the two-photon nature later. Our goal for now is to describe how to measure these observables. Measuring light will at the end of the day boil down to counting photons with a photo detector, so we need to devise a scheme which will relate the quadratures of interest to photon counting. The first step is to introduce a so-called local oscillator $\hat{a}_{\mathrm{LO}}^{(\mathrm{hom})}(t)$. For homodyne detection we assume that the local oscillator is resonant with the carrier tone of signal field, $\omega_{\mathrm{LO}} = \omega_o$, and that it is strong; it is in a coherent state with a large displacement $\alpha_{\mathrm{LO}}^{(\mathrm{hom})}$ from vacuum:

$$\hat{a}_{\mathrm{LO}}^{(\mathrm{hom})}(t) = \alpha_{\mathrm{LO}}^{(\mathrm{hom})} + \delta\hat{a}_{\mathrm{Lo}}(t), \tag{2.4}$$

where we have written the LO field in a frame rotating at the signal frequency ω_o, and hence the coherent amplitude is constant. The large classical part can be written in terms of a constant amplitude and a phase:

$$\alpha_{\mathrm{LO}}^{(\mathrm{hom})} = |\alpha_{LO}| e^{i\phi_{\mathrm{LO}}}, \tag{2.5}$$

which will be very important later. We now combine the signal field $\hat{a}_s$ and the LO field $\hat{a}_{\mathrm{LO}}$ on a 50/50 beam splitter, which produces two output fields $\hat{a}_1$ and $\hat{a}_2$. We choose a convention such that the beam-splitter transformation is real:

$$\begin{pmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} \hat{a}_s(t) \\ \hat{a}_{\mathrm{LO}}(t) \end{pmatrix} \tag{2.6}$$

We then detect the number of photons in each output channel, i.e. measure $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ for $i = 1, 2$, and subtract the photon flux(3) results we have (dropping the (hom) superscript for ease of notation)

(2) For the Fourier convention used in this thesis see Appendix A. Essentially we choose a convention such that $\hat{a}^\dagger(\Omega) = (\hat{a}(\Omega))^\dagger$, and consequently the $\pm\Omega$ must appear in the quadrature expressions Eq. (2.3)

(3) Or photon number if we integrate over a given time interval

$$\hat{n}_1 - \hat{n}_2 = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2 \tag{2.7}$$
$$= \hat{a}_{\mathrm{LO}} \hat{a}_s^\dagger + \hat{a}_{\mathrm{LO}}^\dagger \hat{a}_s \tag{2.8}$$

If we denote the subtracted photon numbers by:

$$\hat{I}^{(\mathrm{hom})}(t) \equiv \hat{n}_1(t) - \hat{n}_2(t) \Rightarrow$$
$$\hat{I}^{(\mathrm{hom})}(t) = \hat{a}_{\mathrm{LO}}(t) \hat{a}_s^\dagger(t) + \hat{a}_{\mathrm{LO}}^\dagger(t) \hat{a}_s(t)$$

We refer to $\hat{I}^{(\mathrm{hom})}$ as the measurement current operator. If we insert the expression Eq. (2.3) for the LO in the measurement current operator we get:

$$\hat{I}^{(\mathrm{hom})} = (\alpha_{LO} + \delta\hat{a}_{\mathrm{LO}})\, \delta\hat{a}_{\mathrm{s}}^\dagger + (\alpha_{LO} + \delta\hat{a}_{\mathrm{LO}})^\dagger\, \delta\hat{a}_{\mathrm{s}} \tag{2.9}$$
$$= \alpha_{LO} \delta\hat{a}_{\mathrm{s}}^\dagger + \alpha_{LO}^* \delta\hat{a}_{\mathrm{s}} + \delta\hat{a}_{\mathrm{LO}} \hat{a}_{\mathrm{s}}^\dagger + \delta\hat{a}_{\mathrm{LO}} \hat{a}_{\mathrm{s}} \tag{2.10}$$

We now neglect the terms bilinear in the light operators, such as $\delta\hat{a}_{\mathrm{LO}} \hat{a}_{\mathrm{s}}^\dagger$ as we assume they are small:

$$\hat{I}^{(\mathrm{hom})}(t) \approx \alpha_{LO} \hat{a}_{\mathrm{s}}^\dagger(t) + \alpha_{LO}^* \hat{a}_{\mathrm{s}}(t) \tag{2.11}$$

By explicitly writing the LO in terms of its amplitude and phase Eq. (2.5) we find that:

$$\hat{I}^{(\mathrm{hom})}(t) = \sqrt{2}\,|\alpha_{\mathrm{LO}}| \left(\cos(\phi_{\mathrm{LO}}) \frac{\hat{a}_{\mathrm{s}} + \hat{a}_{\mathrm{s}}^\dagger}{\sqrt{2}} + \sin(\phi_{\mathrm{LO}}) \frac{\hat{a}_{\mathrm{s}} - \hat{a}_{\mathrm{s}}^\dagger}{\sqrt{2}\, i} \right), \tag{2.12}$$

where we recognize the amplitude and phase quadrature Eq. (2.5) of the signal field:

$$\hat{I}^{(\mathrm{hom})}(t) = \sqrt{2}\,|\alpha_{\mathrm{LO}}| \left(\cos(\phi_{\mathrm{LO}}) \hat{q}_s(t) + \sin(\phi_{\mathrm{LO}})\, \hat{y}_s(t) \right) \tag{2.13}$$

There are two important things to note. Firstly the homodyne detection current scales with the large amplitude $|\alpha_{\mathrm{LO}}|$ of the LO field, and thus we are able to detect even weak quantum signals. Secondly we can control which quadrature we measure by adjusting the LO phase, the canonical choices are:

$$\frac{1}{\sqrt{2}\,|\alpha_{\mathrm{LO}}|}\, \hat{I}^{(\mathrm{hom})}(t) \Big|_{\phi_{\mathrm{LO}=0}} = \hat{q}_s(t), \tag{2.14a}$$
$$\frac{1}{\sqrt{2}\,|\alpha_{\mathrm{LO}}|}\, \hat{I}^{(\mathrm{hom})}(t) \Big|_{\phi_{\mathrm{LO}=\frac{\pi}{2}}} = \hat{y}_s(t), \tag{2.14b}$$

However we can more generally define a set of arbitrarily rotated quadratures:

$$\hat{q}_\phi(t) \equiv \frac{\hat{a}_{\mathrm{s}}(t) e^{-i\phi} + \hat{a}_{\mathrm{s}}^\dagger e^{i\phi}}{\sqrt{2}} \tag{2.15a}$$
$$\hat{y}_\phi(t) \equiv \hat{q}_{\phi+\frac{\pi}{2}}(t) = \frac{\hat{a}_{\mathrm{s}}(t) e^{-i\phi} - \hat{a}_{\mathrm{s}}^\dagger(t) e^{i\phi}}{\sqrt{2}\, i} \tag{2.15b}$$

Such that they satisfy the canonical commutation relation:

$$\left[\hat{q}_\phi(t), \hat{y}_\phi(t') \right] = i\delta(t - t') \tag{2.16}$$

2.1.2 Homodyning In the Two-Photon Formalism

In the description above we considered a completely generic signal, which we will now denote $\hat{a}_s^{(\text{lab})}(t)$, with carrier frequency ω_o. However when using homodyne detection we usually consider a situation where the light we wish to measure consists of two distinct, i.e. localized and well-separated, sidebands. We assume the sidebands are localized near $\omega_o \pm \Omega_0$, and for definiteness we assume $\omega_o \gg \Omega_0 \geq 0$, as this fixes what we refer to as upper and lower sideband. We now formally introduce the two-photon formalism which we will use extensively. We already saw a hint of the two-photon nature of a homodyne measurement in the Fourier space quadratures Eq. (2.5), but we will now make this more explicit. First we imagine a signal with carrier frequency ω_o, such that we can define a slowly varying field $\hat{a}_s(t)$ by:

$$\hat{a}_s^{(\text{lab})}(t) = \hat{a}_s(t)e^{-i\omega_o t} \tag{2.17}$$

In the lab frame the local oscillator now oscillates at $\omega_{\text{LO}} = \omega_o$:

$$\hat{a}_{\text{LO}}^{(\text{lab})}(t) = \alpha_{\text{LO}}(t) + \delta\hat{a}_{\text{LO}}^{(\text{lab})} \tag{2.18}$$

$$\alpha_{\text{LO}}(t) = |\alpha_{\text{LO}}|e^{-i(\omega_o t - \phi_{Lo})} \tag{2.19}$$

We now define the upper sideband operator $\hat{a}_+(t)$ and the lower sideband operator $\hat{a}_-(t)$, in frame rotating at ω_o as:

$$\hat{a}_\pm(t) \equiv \hat{a}_s(t)e^{\pm i\Omega_0 t}, \tag{2.20}$$

Within a RWA that neglects dynamics at time scales shorter than $\lesssim 1/\Omega_0$ sideband operators represent distinct modes and satisfy the commutation relations:

$$\left[\hat{a}_\pm(t), \hat{a}_\mp^\dagger(t')\right] = e^{\pm 2i\Omega_0 t}\delta(t-t') \approx 0$$

From these sideband operator we can define non-Hermitian quadratures (Bogoliubov modes) $\hat{Q}$ and $\hat{Y}$ which are given by quadrature like combinations of the upper and lower sideband light:

$$\hat{Q}(t) = \frac{\hat{a}_+(t) + \hat{a}_-^\dagger(t)}{\sqrt{2}}, \tag{2.21a}$$

$$\hat{Y}(t) = \frac{\hat{a}_+(t) - \hat{a}_-^\dagger(t)}{\sqrt{2}i} \tag{2.21b}$$

with the inverse relation:

$$\hat{a}_+ = \frac{\hat{Q} + i\hat{Y}}{\sqrt{2}}, \tag{2.22a}$$

$$\hat{a}_- = \frac{\hat{Q}^\dagger + i\hat{Y}^\dagger}{\sqrt{2}} \tag{2.22b}$$

Within the same RWA The non-Hermitian quadratures satisfy the commutation relations(4):

$$\left[\hat{Q}(t), \hat{Q}(t')\right] = \left[\hat{Y}(t), \hat{Y}(t')\right] = 0 \tag{2.23}$$

$$\left[\hat{Q}(t), \hat{Q}^\dagger(t')\right] = \left[\hat{Y}(t), \hat{Y}^\dagger(t')\right] = 0 \tag{2.24}$$

$$\left[\hat{Q}(t), \hat{Y}(t')\right] = 0 \tag{2.25}$$

$$\left[\hat{Q}(t), \hat{Y}^\dagger(t')\right] = i\delta(t-t') \tag{2.26}$$

(4) So e.g. $[\hat{Q}(t), \hat{Y}(t')] = ie^{2i\Omega_0 t}\delta(t-t') \approx 0$

Take special note of the fact that $\hat{Q}(t)$ commutes with its Hermitian conjugate, while it doesn't commute with $\hat{Y}^\dagger(t')$.

We now write the signal of interest in terms of two sidebands operators:

$$\hat{a}_s^{(\text{lab})}(t) = \hat{a}_+(t)e^{-i(\omega_o+\Omega_0)t} + \hat{a}_-(t)e^{-i(\omega_o-\Omega_0)t} \tag{2.27}$$

In terms of the non-Hermitian quadratures this becomes:

$$\hat{a}_s^{(\text{lab})}(t) = e^{-i\omega_o t}\left(\frac{\hat{Q}e^{-i\Omega_0 t} + e^{i\Omega_0 t}\hat{Q}^\dagger}{\sqrt{2}} - i\frac{e^{-i\Omega_0 t}\hat{Y} + e^{i\Omega_0 t}\hat{Y}^\dagger}{\sqrt{2}}\right) \tag{2.28}$$

We note that the Hermitian part of the signal only depends $\hat{Q}$ and its Hermitian conjugate:

$$\frac{\hat{a}_s(t) + \hat{a}_s^\dagger(t)}{\sqrt{2}} = \frac{\hat{Q}e^{-i\Omega_0 t} + e^{i\Omega_0 t}\hat{Q}^\dagger}{\sqrt{2}}$$

while the Anti-Hermitian component of the signal only depends on $\hat{Y}$ and its Hermitian conjugate:

$$\frac{\hat{a}_s(t) - \hat{a}_s^\dagger(t)}{\sqrt{2}} = \frac{e^{-i\Omega_0 t}\hat{Y} + e^{i\Omega_0 t}\hat{Y}^\dagger}{\sqrt{2}i}.$$

If we now homodyne the two signal sidebands, i.e. mix the signal field with the LO field on a 50/50 BS we find that the measurement current operator is given by:

$$\hat{I}_\theta^{(\text{hom})} = \sqrt{2}|\alpha_{\text{LO}}|\left(\cos(\Omega_0 t)\left[\hat{q}_{\theta,-}(t) + \hat{q}_{\theta,+}(t)\right] + \sin(\Omega_0 t)\left[\hat{y}_{\theta,+}(t) - \hat{y}_{\theta,-}(t)\right]\right), \tag{2.29}$$

where we have defined the upper and lower sideband phase and amplitude quadratures:

$$\hat{q}_\theta = \frac{e^{-i\phi_{\text{LO}}}\hat{a}(t) + e^{i\phi_{\text{LO}}}\hat{a}^\dagger(t)}{\sqrt{2}} \tag{2.30}$$

$$\hat{y}_\theta = q_{\theta+\frac{\pi}{2}} = \frac{e^{-i\phi_{\text{LO}}}\hat{a}(t) - e^{i\phi_{\text{LO}}}\hat{a}^\dagger(t)}{\sqrt{2}i} \tag{2.31}$$

We see that the signal consists of a cosine and sine part, and crucially these two commute:

$$\left[\hat{q}_{\theta,-}(t) + \hat{q}_{\theta,+}(t), \hat{y}_{\theta,+}(t) - \hat{y}_{\theta,-}(t)\right] = 0 \tag{2.32}$$

Meaning that the cosine and sine component are compatible observables. We can also write the expression in terms of sine and cosine quadratures, which turn out to be the EPR-like combination of individual sideband quadratures:

$$\hat{q}_{\cos}(t) = \frac{1}{\sqrt{2}}\left[\frac{\hat{Q}(t) + \hat{Q}^\dagger(t)}{\sqrt{2}}\right] = \frac{\hat{q}_+(t) + \hat{q}_-(t)}{\sqrt{2}}, \tag{2.33a}$$

$$\hat{q}_{\sin}(t) = \frac{1}{\sqrt{2}}\left[\frac{\hat{Q}(t) - \hat{Q}^\dagger(t)}{\sqrt{2}i}\right] = \frac{\hat{y}_+(t) - \hat{y}_-(t)}{\sqrt{2}}, \tag{2.33b}$$

$$\hat{y}_{\cos}(t) = \frac{1}{\sqrt{2}}\left[\frac{\hat{Y}(t) + \hat{Y}^\dagger(t)}{\sqrt{2}}\right] = \frac{y_+(t) + \hat{y}_-(t)}{\sqrt{2}}, \tag{2.33c}$$

$$\hat{y}_{\sin}(t) = \frac{1}{\sqrt{2}}\left[\frac{\hat{Y}(t) - \hat{Y}^\dagger(t)}{\sqrt{2}i}\right] = \frac{-\hat{q}_+(t) + \hat{q}_-(t)}{\sqrt{2}}, \tag{2.33d}$$

with the inverse transformation given by:

$$\hat{q}_{\pm}(t) = \frac{\hat{q}_{\cos}(t) \pm \hat{y}_{\sin}(t)}{\sqrt{2}}, \tag{2.34}$$

$$\hat{y}_{\pm}(t) = \frac{\hat{y}_{\cos}(t) \pm \hat{q}_{\sin}(t)}{\sqrt{2}}, \tag{2.35}$$

and the single sidebands operators given by:

$$\hat{q}_{\pm}(t) = \frac{\hat{a}_{\pm}(t) + \hat{a}^{\dagger}_{\pm}(t)}{\sqrt{2}}$$
$$\hat{y}_{\pm}(t) = \frac{\hat{a}_{\pm}(t) - \hat{a}^{\dagger}_{\pm}(t)}{\sqrt{2}\,i}$$

The sine and cosine operators satisfy the commutation relations:

$$[\hat{q}_n(t), \hat{y}_n(t')] = i\delta_{nm}\delta(t - t') \tag{2.36}$$

$$[\hat{q}_n(t), \hat{q}_m(t')] = [\hat{y}_n(t), \hat{y}_m(t')] = 0 \tag{2.37}$$

These sine and cosine quadratures are defined such that they are exactly the sine and the cosine components of the homodyne detection current for the phase choices:

$$\frac{\hat{I}^{(\mathrm{hom})}_{\theta=0}}{\sqrt{2}|\alpha_{\mathrm{LO}}|} = \cos(\Omega_0 t)\hat{q}_{\cos}(t) + \sin(\Omega_0 t)\hat{q}_{\sin}(t)$$
$$\frac{\hat{I}^{(\mathrm{hom})}_{\theta=\frac{\pi}{2}}}{\sqrt{2}|\alpha_{\mathrm{LO}}|} = \cos(\Omega_0 t)\hat{y}_{\cos}(t) + \sin(\Omega_0 t)\hat{y}_{\sin}(t)$$

From the definition Eq. (2.33) we clearly see that compatibility of the cosine and sine component of a given quadrature cam be traced back to the the fact that the non-Hermitian quadratures commute with their respective Hermitian conjugates. With this knowledge we need only consider the non-Hermitian quadratures to determine if a given quadrature can be measured with a Homodyne detection. This is great in practice as it will allow us to perform calculations with only the creation and annihilation operators and non-Hermitian operators, rather than having to rewrite everything in terms of quadratures.

This also allow us to gauge the limitations of Homodyne detection. If we e.g. wish to measure a non-Hermitian quadrature which is an unequally weighted combination of the upper and lower sideband, or equivalently a non-equal combination of the non-Hermitian quadratures:

$$\hat{Q}'(t) = \frac{\alpha_{+}\hat{a}_{+}(t) + \alpha_{-}\hat{a}^{\dagger}_{-}(t)}{\sqrt{2}} =$$
$$\frac{\alpha_{-} + \alpha_{+}}{\sqrt{2}}\hat{Q} + \frac{\alpha_{-} - \alpha_{+}}{\sqrt{2}\,i}\hat{P}$$

If we now consider the commutator with its Hermitian conjuagate we find:

$$\left[\hat{Q}'(t), \hat{Q}'^{\dagger}(t)\right] = \left[\left(|\alpha_{+}|^2 - |\alpha_{-}|^2\right)\delta(t - t')\right] \neq 0,$$

and thus the Hermitian quadrature corresponding to $\hat{Q}$ won't be measurable using a homodyne detection. We can also see this by the fact that an unbalanced combination of the quadratures lead to a As we shall see in Eq. (2.33), such unequally weighted quantities will be of interest once we start discussing the removal of unwanted noise in a two-tone driven optomechanical system.

CHAPTER 3

Cavity Optomechanics with a Single-tone Drive

“ wha-what’s wrong Rick is it the Quantum Carburetor or something?”

”Quantum Carburetor? Jesus, Morty you can’t just add a sci-fi word to a car word and hope it means something. Huh. Looks like something is wrong with the microverse battery.”

Rick Sanchez and Morty Smith

- Rick and Morty: The Ricks Must Be Crazy, Justin Roiland *and* Dan Harmon

The aim of the two-tone scheme we wish to examine, is to achieve an effective oscillator behaving like a single-tone driven oscillator with an effective negative mass and a down-converted resonance frequency.

For this reason we need a thorough understanding of single-tone optomechanics, and therefor we dedicate this chapter to discussing the canonical optomechanical system: a cavity with one movable mirror, driven by a single coherent tone. Using a transfer matrix approach will see how the dynamics of the mechanical oscillations of the mirror are imprinted on the light, and how this light can be detected. Importantly we will consider the effects of driving the cavity off-resonantly, and discuss the how the dynamics are modified by such a detuning. We will later encounter similar effects in the two-tone driving case.

3.1 A Setup to Have in Mind

Coupling of a mechanical degree of freedom to light can be realized in a plethora of ways. In particular we will consider so called *cavity optomechanics [1],* where a cavity is used to enhance the light-matter coupling. While the calculations we will be doing apply to a wide range of different physical implementations of optomechanical systems, it is beneficial to discuss at least one actual example.

For definiteness we will consider a setup with a Fabry-Perot cavity with one fixed mirror and one movable mirror (see Fig. 3.1). More precisely we imagine that the first mirror is fixed and has a transmission coefficient $T_1 < 1$, such that the cavity can be driven by an external field. The second mirror is assumed to have $T_2 = 0$, and is attached to a spring with spring constant k. This allows the mirror to oscillate when it is perturbed by the light in the cavity. The coupling of the light in the cavity and the motion of the mirror can now be understood as follows: The resonance frequencies of the cavity are given by

$$\omega_{c,n} = n\frac{c}{2\ell}, \tag{3.1}$$

where c is the speed of light, ℓ is the length of the cavity and n is the mode number. If the length of the cavity is ℓ_0 when the spring is in equilibrium, the length is dependent on the displacement $\hat{x}$:

$$\ell(\hat{x}) = \ell_0 + \hat{x} \tag{3.2}$$

Consequently, the resonance frequencies of the cavity, now depend on the displacement of the second mirror:

$$\omega_{c,n}(\hat{x}) = n\frac{c}{2(\ell_0 + \hat{x})} = \frac{nc}{2\ell_0}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\hat{x}}{\ell_0}\right)^m$$

where we emphasize that the displacement is assumed to be much smaller than the equilibrium cavity length, so that the geometric series converges(1) , and as we shall see, we usually consider only position dependence to first order.

When the optomechanical coupling arises due to the position dependence of the cavity resonance frequency, we refer to it as *dispersive*. The optomechanical coupling can however arise through other mechanisms. For example we could imagine that the decay rate κ of the intra-cavity field depends on position so we have $\kappa(\hat{x})$, leading instead to a *dissipative* coupling. From this point on we only consider dispersive optomechanics.

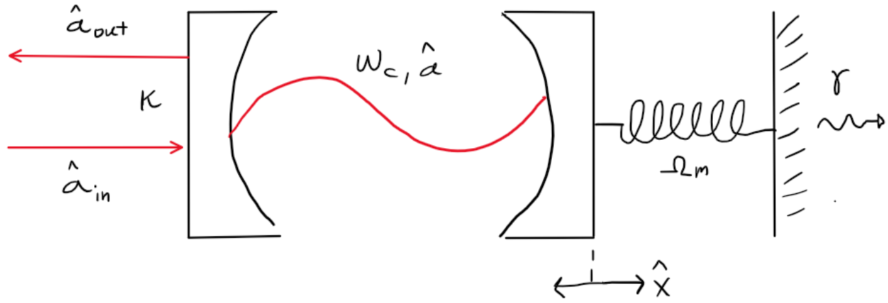

Figure 3.1: Sketch of a mirror-at-the-end setup. The intracavity field leaks out at a rate κ but is also driven at the same rate by a single laser tone. The mechanical oscillator decays at a rate γ.

3.2 Hamiltonian Formulation of Dispersive Cavity Optomechanics

The most generic form of the Hamiltonian can be split into three contributions, a system Hamiltonian $\hat{H}_S$, a reservoir Hamiltonian $\hat{H}_R$ and a system-reservoir coupling Hamiltonian $\hat{H}_{SR}$:

$$\hat{H}_{\text{tot}} = \underbrace{\hat{H}_c + \hat{H}_m + \hat{H}_d}_{\hat{H}} + \underbrace{\hat{H}_\gamma + \hat{H}_\kappa}_{\hat{H}_R} + \underbrace{\hat{H}_{m-\gamma} + \hat{H}_{c-\kappa}}_{\hat{H}_{SR}} \tag{3.3}$$

(1) The geometric series $\frac{1}{1-x} = \sum_{m=0}^{\infty} x^n$ only converges for $|x| < 1$. If for some reason you should be inclined to consider a cavity where displacement of one mirror is comparable or larger than the equilibrium length of the cavity, then you would have to use the spectral representation of the frequency. The second quantized spectral representation of a function $f(\hat{x})$ is $f(\hat{x}) = \int_{-\infty}^{\infty} f(x)\hat{\psi}^\dagger(x)\hat{\psi}(x)$, where $\hat{\psi}^\dagger(x)$ is the creation operator for a position eigenstate.

Using the Heisenberg-Langevin Formalism [4], we can eliminate the reservoir degrees of freedom and obtain a theory in terms of the system operators of interest and stochastic input noise operators arising from the interaction with the reservoir. For now all we need to know is that we denote the cavity decay rate by κ and mechanical decay rate by γ, we will cover the rest of the necessary details in the coming sections.

3.2.1 The Intra-Cavity light Hamiltonian

Firstly we have the Hamiltonian for the intra-cavity light field. We consider a single optical mode with frequency $\omega_c(\hat{x})$. In the Heisenberg picture, such a single mode is described by a simple harmonic oscillator Hamiltonian:

$$\hat{H}_c = \hbar\omega_c(\hat{x}(t))\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right), \tag{3.4}$$

where intra-cavity photons are described by the bosonic creation and annihilation operators $\hat{a}^\dagger(t)$ and $\hat{a}(t)$ satisfying the usual synchronous commutation relations:

$$\left[\hat{a}(t), \hat{a}^\dagger(t)\right] = 1 \tag{3.5}$$

3.2.2 The Mechanical Oscillator Hamiltonian

The free Hamiltonian $\hat{H}_m$ for mechanical degree of freedom, is assumed to be a single vibrational mode, which is just another harmonic oscillator, with frequency Ω_m:

$$\hat{H}_m = \hbar\Omega_m\left(\hat{b}^\dagger(t)\hat{b}(t) + \frac{1}{2}\right) \tag{3.6}$$

The bosonic operators $\hat{b}^\dagger(t)$ and $\hat{b}(t)$ are the phonon creation and annihilation operators:

$$\left[\hat{b}(t), \hat{b}^\dagger(t)\right] = 1 \tag{3.7}$$

These are related to the position and momentum of the mechanical mode through the relations:

$$\begin{cases} \hat{b}(t) = \frac{1}{x_{\text{zpf}}\sqrt{2}}\left(\hat{x}(t) + \frac{i}{m\Omega_m}\hat{p}(t)\right) \\ b^\dagger(t) = \frac{1}{x_{\text{zpf}}\sqrt{2}}\left(\hat{x}(t) - \frac{i}{m\Omega_m}\hat{p}(t)\right) \end{cases} \tag{3.8}$$

Or if we invert them:

$$\begin{cases} \hat{x}(t) = x_{\text{zpf}}\left(\frac{\hat{b}(t)+\hat{b}^\dagger(t)}{\sqrt{2}},\right) \\ \hat{p}(t) = p_{\text{zpf}}\left(\frac{\hat{b}(t)-\hat{b}^\dagger(t)}{i\sqrt{2}}\right) \end{cases} \tag{3.9}$$

The factors x_{xpf} and p_{zpf} set the scale of the zero point fluctuations of the position $\hat{x}$ and the momentum $\hat{p}$, more precisely they are defined by the standard deviation of the position and momentum operator when the mechanical oscillator is in the vacuum state:

$$x_{\text{zpf}} \equiv \sqrt{\text{Var}_{\text{vac}}\left[\hat{x}(0)\right]} = \sqrt{\frac{\hbar}{m\Omega_m}} \tag{3.10}$$

$$p_{\text{zpf}} \equiv \sqrt{\text{Var}_{\text{vac}}\left[\hat{p}(0)\right]} = m\Omega_m x_{\text{zpf}} \tag{3.11}$$

where the actual expressions follow from writing the mechanical Hamiltonian as:

$$\hat{H}_m = \frac{\hat{p}^2(t)}{2m} + \frac{1}{2}m\Omega_m^2\hat{x}^2(t) \tag{3.12}$$

and we have defined the variance of an operator $\hat{O}$ as:

$$\mathrm{Var}\left[\hat{O}\right] \equiv \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2 \tag{3.13}$$

It is often convenient to define a dimensionless position $\hat{X}(t)$ and momentum $\hat{P}(t)$ using:

$$\hat{X}(t) = \frac{\hat{x}(t)}{x_{\mathrm{zpf}}} \tag{3.14a}$$

$$\hat{P}(t) = \frac{\hat{p}(t)}{p_{\mathrm{zpf}}} \tag{3.14b}$$

These dimensionless operators then satisfy the canonical (synchronous) commutation relation on the form:

$$\left[\hat{X}(t), \hat{P}(t)\right] = i$$

And the Hamiltonian can be written:

$$\hat{H}_m = \hbar\Omega_m\left(\hat{X}^2(t) + \hat{P}^2(t)\right) \tag{3.15}$$

Let us have a small interlude to discuss what we mean by effective negative mass or frequency. If we consider the Hamiltonian Eq. (3.15), we could in principle have $\Omega_m$take on negative values. On the other hand, in the Hamiltonian Eq. (3.15) the frequency is squared, but instead we could imagine m taking on negative values. In the end these two Hamiltonians are equivalent, its all a mater of assigning the sign to either the mass or frequency. That is we can either write $\Omega_m = \mu|\Omega_m|$ and $m > 0$ or equally well $\Omega_m > 0$ and $m = \mu|m|$, where $\mu = \pm 1$ depending on the sign of the mass or frequency. We must also remember that the effective mass-sign /frequency-sign μ always is an effective property, it has nothing to do with the actual mass of the object, rather it is about the response of the object to an external force. For example for the two-tone optomechanics we will examine, the effective negative mass simply means that the response of effective mechanical oscillator to an external optical force, is $\frac{\pi}{2}$out of phase with a normal mechanical oscillator. So if the positive mass oscilator has suscptibility $\chi_{m>0}(\Omega)$ the susceptibility of a negative mass oscillator is simple $\chi_{m<0} = e^{i\frac{\pi}{2}}\chi_{m>0}$.

3.2.3 The Drive Hamiltonian

The optical reservoir which the intra-cavity field couples to is effectively at temperature $T = 0$. This means that all modes in the optical reservoir are in the vacuum state. Turning on a laser now means that we activate one mode in the sense that it is now in a coherent state. Hence, to include a coherent laser drive, we formally apply a displacement operator to the photon reservoir, to activate a single mode at frequency ω_o. This amounts to transforming to an interaction picture:

$$\hat{H} \xrightarrow{\text{turn on laser}} \hat{D}^\dagger(\alpha_d(t))\hat{H}\hat{D}(\alpha_{\mathrm{d}}(t)) + i\hbar\hat{D}^\dagger(\alpha_{\mathrm{d}}(t))\frac{\partial}{\partial t}\hat{D}(\alpha_{\mathrm{d}}(t)) \tag{3.16}$$

$$= \hat{H} + \hat{H}_d \tag{3.17}$$

where the displacement operator is given by:

$$\hat{D}(\alpha_{\mathrm{d}}(t)) = e^{\alpha_d(t)\hat{a}_R^\dagger(\omega_o) - \alpha_d^*(t)\hat{a}_R(\omega_o)}, \quad \alpha_{\mathrm{d}}(t) \propto \sqrt{2\kappa}\,\alpha_{\mathrm{in}}e^{-i\omega_o t}, \tag{3.18}$$

where $\hat{a}_R^\dagger(\omega_o)$ is the vacuum mode we displace by turning on the laser, i.e.

$$\hat{D}^\dagger(\alpha_{\rm d}(t))\hat{a}_R^\dagger(\omega_o)\hat{D}(\alpha_{\rm d}(t)) = \hat{a}_R^\dagger(\omega_o) + \delta(\omega - \omega_0)\alpha_{\rm drive}(t) \tag{3.19}$$

The resulting drive Hamiltonian is given by:

$$\hat{H}_d = \hbar\sqrt{2\kappa}\left(i\alpha_{\rm in}e^{-i\omega_o t}\hat{a}^\dagger + \text{H.c}\right) \tag{3.20}$$

To show this we need to write down the actual reservoir Hamiltonian, however as we leave out formal derivation of the Heisenberg-Langevin equations, we will not do so here.

3.2.4 Rotating frame Hamiltonian

The full Hamiltonian, excluding the reservoir and system-reservoir coupling contributions is:

$$\hat{H} = \hbar\omega_c(\hat{x})\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\Omega_m\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) \tag{3.21}$$

$$+ \hbar\sqrt{2\kappa}\left(i\alpha_{\rm in}e^{-i\omega_o t}\hat{a}^\dagger + \text{H.c}\right) \tag{3.22}$$

We now wish to transform to a rotating frame, rotating at the laser frequency ω_0, using the unitary transformation:

$$\hat{R}(t) = e^{i\omega_o\hat{a}^\dagger\hat{a}t} \tag{3.23}$$

We now arrive at a key point: In this rotating frame the explicit time dependence of the drive Hamiltonian is removed:

$$\hat{H} \longrightarrow \hat{R}^\dagger(t)\hat{H}\hat{R}(t) + i\hbar\hat{R}^\dagger(t)\frac{\partial}{\partial t}\hat{R}(t) \Rightarrow \tag{3.24}$$

$$\hat{H} = \hbar\left(\omega_c(\hat{x}) - \omega_o\right)\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\Omega_m\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) \tag{3.25}$$

$$+ \hbar\sqrt{2\kappa}\left(i\alpha_{\rm in}\hat{a}^\dagger + \text{H.c}\right)$$

As we shall see, the elimination of any explicit time dependence is not possible for a two-tone drive.

3.2.5 The Optomechanical Coupling & Radiation Pressure

Intuitively we expect that the optomechanical coupling should be realized by a radiation pressure from the intra-cavity photon on the movable mirror, and the consequent modulation of the intra-cavity field by the motion of the mechanical oscillator. To see this we expand[(2)] the dispersion relation for the light around the

(2) For example, for a Fabry-perot resonator, the frequency-position dispersion relation can be written

$$\omega_{c,n}(\hat{x}) = \frac{nc}{2}\frac{1}{\ell_0 + \hat{x}} = \frac{nc}{2(\bar{x} + \ell_0)}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\hat{x} - \bar{x}}{\ell_0 + \bar{x}}\right)^m, \tag{3.26}$$

From which we can formally define the derivative:

$$\left.\frac{\partial^m\omega_{c,n}(\hat{x})}{\partial x^m}\right|_{\bar{x}} \equiv (-1)^m\frac{nc}{2(\bar{x} + \ell_0)^{m+1}} \tag{3.27}$$

And thus find the optomechancical coupling per displacement to be, for mode n, to be

$$G_n = \frac{nc}{2(\bar{x}_{\rm ss} + \ell_0)} \tag{3.28}$$

classical steady state value[(3)] $\bar{x}_{\mathrm{ss}}$ of the mechanical oscillator:

$$\omega_c(\hat{x}) = \omega_c(\bar{x}_{\mathrm{ss}}) + \left.\frac{\partial \omega_c}{\partial x}\right|_{\bar{x}_{\mathrm{ss}}} \hat{x} + O(\hat{x}^2) \tag{3.29}$$

We will only consider position dependence to first order. In this case we define the frequency change per displacement as:

$$G = -\left.\frac{\partial \omega_c}{\partial x}\right|_{\bar{x}_{\mathrm{ss}}} \tag{3.30}$$

If we rewrite the cavity frequency in terms of phonon operators or the dimension less position $\hat{X}$ we instead define the vacuum optomechanical coupling strength as:

$$g_0 = x_{\mathrm{zpf}} G \tag{3.31}$$

Using the expression Eq. (3.15) for the position and momentum in terms of the phonon operators we now have:

$$\omega_c \approx \omega_c(\bar{X}_{\mathrm{ss}}) - \frac{\hbar g_0}{\sqrt{2}}(\hat{b}^\dagger(t) + \hat{b}(t)) \tag{3.32}$$

The intra-cavity field Hamiltonian now becomes:

$$\hat{H}_c \approx \left(\omega_c(\bar{X}_{\mathrm{ss}}) - \hbar g_0 \hat{X}\right)\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right) \tag{3.33}$$

$$= \underbrace{\omega_c(\bar{x}_{\mathrm{ss}})\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right)}_{\hat{H}_c} \underbrace{- \hbar g_0 \hat{X}\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right)}_{-\hat{H}_{\mathrm{om}}} \tag{3.34}$$

The first contribution is just harmonic oscillator describing the free evolution of the intra-cavity field for a fixed position of the mechanical oscillator. The second term is now the interaction term describing how the light and the mechanics interact. We can evaluate the force on the mechanical oscillator due to this interaction using the Heisenberg equation of motion:

$$\dot{P}(t) = \frac{1}{i\hbar}\left[\hat{P}(t), \hat{H}_{\mathrm{om}}\right] \Rightarrow \tag{3.35}$$

$$\dot{P}(t) = g_0\left(\hat{n}_c(t) + \frac{1}{2}\right) \tag{3.36}$$

where we have defined the cavity photon number operator:

$$\hat{n}_c(t) = \hat{a}^\dagger(t)\hat{a}(t) \tag{3.37}$$

So the force $\hat{F}_{\mathrm{rad}}(t) = \dot{P}(t)$ is indeed a radiation pressure:

$$\hat{F}_{\mathrm{rad}}(t) = g_0\left(\hat{n}_c(t) + \frac{1}{2}\right) \tag{3.38}$$

As we see the mechanical coupling strength is essentially the vacuum radiation pressure $\hat{F}^{(\mathrm{vac})}_{\mathrm{rad}}(t) = \frac{g}{2}$ on the mechanical oscillator. Henceforth we will assume that the cavity population is large:

$$n_c(t) = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \gg 1, \tag{3.39}$$

and we can thus neglect the vacuum contribution to the optomechanical interaction[(4)], so we simply have

(3) For now we assume that such a constant steady state exists. Using this assumption we can derive the classical equations of motion, which we must then solve in steady state to ensure everything is consistent.

(4) Eventually the vacuum force term would drop out anyways, once we define the position of the oscillator relative to its equilibrium.

$$\hat{H}_{\text{om}} = -\hbar g \hat{n}_c(t)\hat{X} = -\frac{\hbar g_0}{\sqrt{2}}\hat{a}^\dagger(t)\hat{a}(t)\left(\hat{b}^\dagger(t)+\hat{b}(t)\right) \tag{3.40}$$

The Hamiltonian can then be written[5]:

$$\hat{H} \approx -\hbar\Delta_0\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \frac{\hbar g_0}{\sqrt{2}}\hat{a}^\dagger\hat{a}(\hat{b}+\hat{b}^\dagger) + \hbar\sqrt{2\kappa}\left[i\alpha_{\text{in}}\hat{a}^\dagger + \text{H.c.}\right] \tag{3.41}$$

where we have define the detuning between the laser drive frequency ω_o and the equilibrium cavity frequency $\omega_c(\bar{X})$ as:

$$\Delta_0 = \omega_o - \omega_c(\bar{X}_{\text{ss}}) \tag{3.42}$$

Note that at this point we have also neglected the zero point energies $\frac{\hbar\Omega_m}{2}$ and $\frac{\hbar\Delta_0}{2}$, as these do not contribute to the dynamics[6]. Alternatively we can write the Hamiltonian in terms of the position and momentum of the mechanical oscillator:

$$\hat{H} \approx -\hbar\Delta_0\hat{a}^\dagger\hat{a} + \hbar\Omega_m\left(\hat{X}^2+\hat{P}^2\right) - \hbar g_0\hat{a}^\dagger\hat{a}\hat{X} + \hbar\sqrt{2\kappa}\left[i\alpha_{\text{in}}\hat{a}^\dagger + \text{H.c.}\right] \tag{3.43}$$

3.3 Equations of motion in Optomechanics

3.3.1 Heisenberg-Langevin Equations For Light and Mechanics

The Heisenberg-langevin equation for the mechanical degrees of freedom $\hat{X}$, $\hat{P}$ and the intra-cavity light field $\hat{a}$ are:

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}\left[\hat{H},\hat{a}(t)\right] - \kappa\hat{a}(t) + \sqrt{2\kappa}\,\delta\hat{a}_{\text{in}}(t), \tag{3.44}$$

$$\dot{\hat{X}}(t) = \frac{i}{\hbar}\left[\hat{H},X(t)\right], \tag{3.45}$$

$$\dot{\hat{P}}(t) = \frac{i}{\hbar}\left[\hat{H},X(t)\right] - \gamma\hat{P}(t) + \hat{f}(t), \tag{3.46}$$

where we have used the RWA on the photon reservoir, which results in a damping term in both quadratures. Meanwhile, the phonon reservoir only leads to a drag-like damping term in the momentum. The Langevin force acting on the mechanical oscillator is given by:

$$\hat{f}(t) = \frac{\sqrt{2\gamma}}{p_{zpf}}p_{\text{in}}(t), \quad p_{\text{in}}(t) = p_{\text{zpf}}\left(\frac{\hat{b}_{\text{in}}(t)+\hat{b}^\dagger_{\text{in}}(t)}{\sqrt{2}}\right) \tag{3.47}$$

Importantly the input operators, which arise from integrating out the resevoir degrees of freedom satisfy the commutation relations:

$$\left[\delta\hat{a}_{\text{in}}(t),\delta\hat{a}^\dagger_{\text{in}}(t')\right] = \left[\hat{b}_{\text{in}}(t),\hat{b}^\dagger_{\text{in}}(t')\right] = \delta(t-t') \tag{3.48}$$

$$\left[\delta\hat{a}_{\text{in}}(t),\delta\hat{a}_{\text{in}}(t')\right] = \left[\delta\hat{a}^\dagger_{\text{in}}(t),\delta\hat{a}^\dagger_{\text{in}}(t')\right] = \left[\hat{b}_{\text{in}}(t),\hat{b}_{\text{in}}(t')\right] = \left[\hat{b}^\dagger_{\text{in}}(t),\hat{b}^\dagger_{\text{in}}(t')\right] = 0 \tag{3.49}$$

(5) This Hamiltonian is the same in the Heisenberg interaction picture and the Schrödinger interaction picture, except for the time dependence of the operators.

(6) Remember, that for an operator $\hat{O}(t)$ the evolution is determined by taking its commutator with the Hamiltonian. If we shift the energy of a system by E_0, i.e. $\hat{H} \to \hat{H} + E_0$, then the commutator is unchanged since E_0 is just a number and $\left[\hat{O}(t),\hat{H}+E_0\right] = \left[\hat{O}(t),\hat{H}\right] + \underbrace{\left[\hat{O}(t),E_0\right]}_{0}$.

Commutators of phonon and photon resevoir operators all vanish. The correlation function of the input noise, in the Markov approximation, are:

$$\left\langle \delta\hat{a}_{\mathrm{in}}(t)\delta\hat{a}^{\dagger}_{\mathrm{in}}(t')\right\rangle = \delta(t-t') \tag{3.50}$$

$$\left\langle \delta\hat{a}^{\dagger}_{\mathrm{in}}(t')\delta\hat{a}_{\mathrm{in}}(t)\right\rangle = 0 \tag{3.51}$$

$$\left\langle \hat{b}_{\mathrm{in}}(t)\hat{b}^{\dagger}_{\mathrm{in}}(t')\right\rangle = (\bar{n}_b+1)\delta(t-t') \tag{3.52}$$

$$\left\langle \hat{a}^{\dagger}_{\mathrm{in}}(t')\hat{a}_{\mathrm{in}}(t)\right\rangle = \bar{n}_b\delta(t-t') \tag{3.53}$$

All photon and phonon reservoir operators are uncorrelated. From these input commutators and correlation function, we can eventually determine all commutators and correlation functions we desire. Note that at room temperature, the thermal occupation of optical photons is vanishing $\bar{n}_a = 0$, and hence we see no contribution from the reservoir photon occupation. Evaluating the commutators, leads to the following set of equations of motion:

$$\dot{\hat{a}}(t) = (i\Delta_0 - \kappa)\,\hat{a}(t) + ig_0\hat{a}(t)\hat{X}(t) + \sqrt{2\kappa}\,\delta\hat{a}_{\mathrm{in}}(t) \tag{3.54}$$

$$\dot{\hat{X}}(t) = \Omega_m\hat{P} \tag{3.55}$$

$$\dot{\hat{P}}(t) = g_0\hat{a}^{\dagger}(t)\hat{a}(t) - 2\gamma\hat{P}(t) - \Omega_m\hat{X}(t) + \hat{f}(t) \tag{3.56}$$

where we have written the input noise in terms of the small quantum fluctuations $\delta\hat{a}_{\mathrm{in}}(t)$ arising from all reservoir modes, and the large coherent component from the drive tone:

$$\hat{a}_{\mathrm{in}}(t) = \delta\hat{a}_{\mathrm{in}}(t) + \alpha_{\mathrm{in}}$$

We can combine the $\hat{X}$ and $\hat{P}$ equations so that we only have to solve for $\hat{X}$ and $\hat{a}$:

$$\dot{\hat{a}}(t) = \sqrt{2\kappa}\,\delta\hat{a}_{\mathrm{in}}(t) + (i\Delta_0 - \kappa)\,\hat{a}(t) + ig_0\hat{a}(t)\hat{X}(t) \tag{3.57}$$

$$\frac{1}{\Omega_m}\left[\ddot{\hat{X}}(t) + 2\gamma\dot{\hat{X}} + \Omega_m\hat{X}(t)\right] = g_0\hat{a}^{\dagger}(t)\hat{a}(t) + \hat{f}(t) \tag{3.58}$$

This system of coupled equations is unfortunately non-linear in the operators, and consequently they do not generally admit an analytical solution.

3.3.2 Linearized Optomechanics

We now assume that the fields can be expanded as a large classical contribution and a contribution from quantum fluctuations[7]:

$$\hat{a}(t) = \underbrace{\langle\hat{a}\rangle}_{\bar{\alpha}} + \delta\hat{a}(t) = \bar{\alpha}(t) + \delta\hat{a}(t) \tag{3.59a}$$

$$X(t) = \underbrace{\langle\hat{X}\rangle}_{\bar{X}} + \delta\hat{X}(t) = \bar{x}(t) + \delta\hat{X}(t) \tag{3.59b}$$

$$P(t) = \underbrace{\langle\hat{X}\rangle}_{\bar{P}} + \delta\hat{P}(t) = \bar{p}(t) + \delta\hat{P}(t) \tag{3.59c}$$

We eventually make the assumption that this large classical value is a constant steady state value of the field, but let us first ensure that such a steady state exists. The Hamiltonian with this linear ansatz is:

(7) We can formally do these by acting with appropiate displacements operators $\hat{D}(\bar{\alpha})$ and $\hat{D}(\bar{x})$

$$\begin{aligned}\hat{H}_S &= -\hbar\Delta_0\left(\delta\hat{a}^\dagger\delta a + |\bar{\alpha}|^2 + \bar{\alpha}\delta\hat{a}^\dagger + \bar{\alpha}^*\delta\hat{a}\right) \\ &+ \hbar\Omega_m\left(\delta\hat{X}^2 + \bar{X}^2 + 2\bar{X}\delta X + \delta\hat{P}^2 + \bar{P}^2 + 2\bar{P}\delta P\right) \\ &- \hbar g_0\left(\delta\hat{a}^\dagger\delta a + |\bar{\alpha}|^2 + \bar{\alpha}\delta\hat{a}^\dagger + \bar{\alpha}^*\delta\hat{a}\right)\left(\bar{X} + \delta\hat{X}\right) \\ &+ \hbar\sqrt{2\kappa}\left[i\alpha_{\text{in}}\left(\bar{\alpha}^* + \delta\hat{a}^\dagger\right) + \text{H.c.}\right]\end{aligned} \tag{3.60}$$

Which can be divided into four contributions according to the order of the quantum operators:

$$\hat{H}_S = \underbrace{-\hbar\Delta_0|\bar{\alpha}|^2 + \hbar\Omega_m\left(\bar{X}^2 + \bar{P}^2\right) - \hbar g_0|\bar{\alpha}|^2\bar{X} + \hbar\sqrt{2\kappa}\left[i\alpha_{\text{in}}\bar{\alpha}^* + \text{C.C.}\right]}_{\text{constant}} \tag{3.61}$$

$$\underbrace{-\hbar\Delta_0\left(\bar{\alpha}\delta\hat{a}^\dagger + \bar{\alpha}^*\delta\hat{a}\right) - \hbar g_0|\bar{\alpha}|^2\delta\hat{X} + \hbar\sqrt{2\kappa}\left[i\alpha_{\text{in}}\delta\hat{a}^\dagger + \text{H.c.}\right]}_{\text{linear (classical)}}$$

$$\underbrace{+2\hbar\Omega_m\left(\bar{X}\delta X + \bar{P}\delta P\right) - \hbar g_0\left(\bar{\alpha}\delta\hat{a}^\dagger + \text{H.C.}\right)\bar{x}}_{\text{linear (classical)}} \tag{3.62}$$

$$\underbrace{-\hbar\Delta_0\delta\hat{a}^\dagger\delta a + \hbar\Omega_m\left(\delta\hat{X}^2 + \delta\hat{P}^2\right) - \hbar g_0\left(\bar{\alpha}\delta\hat{a}^\dagger + \bar{\alpha}^*\delta\hat{a}\right)\delta\hat{X}}_{\text{quadratic(quantum)}}$$

$$\underbrace{-\hbar g_0\delta\hat{a}^\dagger\delta a\delta\hat{X}}_{\text{cubic}}$$

We assume that the quantum fluctuations are small, and so we neglect all terms cubic in the quantum fluctuations. Since constant terms do not contribute to the dynamics we also throw these away:

$$\begin{aligned}\hat{H}_S = -\hbar\Delta_0\left(\bar{\alpha}\delta\hat{a}^\dagger + \bar{\alpha}^*\delta\hat{a}\right) - \hbar g_0|\bar{\alpha}|^2\delta\hat{X} + \hbar\sqrt{2\kappa}\left[i\alpha_{\text{in}}\delta\hat{a}^\dagger + \text{H.c.}\right] \\ +2\hbar\Omega_m\left(\bar{X}\delta X + \bar{P}\delta P\right) - \hbar g_0\left(\bar{\alpha}\delta\hat{a}^\dagger + \text{H.C.}\right)\bar{X} \\ -\hbar\Delta_0\delta\hat{a}^\dagger\delta a + \hbar\Omega_m\left(\delta\hat{X}^2 + \delta\hat{P}^2\right) - \hbar g_0\left(\bar{\alpha}\delta\hat{a}^\dagger + \bar{\alpha}^*\delta\hat{a}\right)\delta\hat{X}\end{aligned} \tag{3.63}$$

Classical equation of motion From the Hamiltonian to first order in the quantum fluctuations, we simple get classical optomechanics(8), the equations motions for this are(9):

$$\dot{\bar{\alpha}}(t) = \sqrt{2\kappa}\,\bar{\alpha}_{\text{in}} + \left(i\Delta(t) - \kappa\right)\bar{\alpha}(t), \tag{3.64}$$

$$\frac{1}{\Omega_m}\left[\ddot{\bar{X}}(t) + 2\gamma\dot{\bar{X}} + \Omega_m^2\bar{X}(t)\right] = g_0|\bar{\alpha}(t)|^2 \tag{3.65}$$

where we have redefined the detuning:

(8) In this thesis we will not be concerned with the classical part of the fields beyond the justification of expanding ω_c about some steady state.
(9) The classical EOM could equally well be obtained form the classical Hamiltonian $H_{\text{cl}}(\bar{x}, \bar{p}, \bar{\alpha}, \bar{\alpha}^*)$ by e.g. $\dot{\bar{\alpha}}(t) = \{\bar{\alpha}(t), H_{\text{cl}}\} + \frac{\partial\bar{\alpha}(t)}{\partial t}$,where $\{\cdot,\cdot\}$denotes the Poisson bracket. Recall for two functions $f(x_i, p_i)$ and $g(x_i, p_i)$,which are functions of a set of generalized positions x_i and canonically conjugate momenta p_i, the Poisson bracket defined by $\{f(x_i, p_i), g(x_i, p_i)\} = \sum_i\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}\right)$.

$$\Delta(t) = \omega_0 - \omega_c(\bar{X}) + g_0\bar{X}(t) \tag{3.66}$$

Since there is no explicit time dependence we can solve steady state using

$$\frac{\mathrm{d}\bar{\alpha}_{ss}}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\bar{X}_{ss}}{\mathrm{d}t} = 0, \tag{3.67}$$

we find(10) the steady state relations:

$$\bar{\alpha}_{ss} = \frac{\sqrt{2\kappa}}{(\kappa - i\Delta)}\bar{\alpha}_{in} \tag{3.68}$$

$$\bar{X}_{ss} = \frac{g_0}{\Omega_m}|\bar{\alpha}_{ss}|^2 \tag{3.69}$$

This shows us that once all transient behavior has died out, the classical part of the intra-cavity field and the motion of the membrane remain stationary. It is this stationary steady state value $\bar{X}_{ss}$ we expanded the cavity frequency ω_c about in Eq. (3.29). If we now measure all displacements relative to the steady state displacement of the membrane $X \to X - \bar{X}_{ss}$ we simply have

$$\bar{X}_{ss} = 0 \tag{3.70}$$

It immediately follows form the equations of motion that classical part of the mechanical momentum is:

$$\frac{\mathrm{d}\bar{X}_{ss}}{\mathrm{d}t} = \Omega_m\bar{P}_{ss} \Rightarrow \tag{3.71}$$

$$\bar{P}_{ss} = 0 \tag{3.72}$$

We also note that this implies that the detuning is stationary:

$$\Delta = \omega_0 - \omega_c(0)$$

The quantum equation of motion If we assume that enough time has passed for the classical part of the system to be in steady state, and we measure displacements relative to the steady state displacement of the mechanical oscillator(11), then the quantum part of the linearized Hamiltonian reduces to:

$$\hat{H}_S = -\hbar\Delta_0\delta\hat{a}^\dagger\delta a + \hbar\Omega_m\left(\delta\hat{X}^2 + \delta\hat{P}^2\right) - \hbar g_0\left(\bar{\alpha}\delta\hat{a}^\dagger + \bar{\alpha}^*\delta\hat{a}\right)\delta\hat{X} \tag{3.73}$$

where we have dropped the ss subscript from α for brevity. The equations of motion are:

$$\dot{\delta\hat{a}}(t) = \sqrt{2\kappa}\,\delta\hat{a}_{in}(t) + (i\Delta_0 - \kappa)\,\delta\hat{a}(t) + ig_0|\bar{\alpha}|\delta\hat{X}(t) \tag{3.74}$$

$$\frac{1}{\Omega_m}\left[\delta\ddot{\hat{X}}(t) + 2\gamma\delta\dot{\hat{X}} + \Omega_m^2\delta\hat{X}(t)\right] = g_0\left(\alpha^*\delta\hat{a} + \alpha\delta\hat{a}^\dagger\right) + \hat{f}(t) \tag{3.75}$$

We can eliminate the phase of the classical intra-cavity field:

(10) Note that this isn't the full solution for the intra-cavity field $\bar{\alpha}_{ss}$ and the potion $\bar{x}_{ss}$, since Δ depends on $\bar{x}_{ss}$. One can rewrite the equations as a cubic equation in $\bar{x}_{ss}$: $\bar{x}_{ss}\left((\Delta_0 + G\bar{x}_{ss})^2 + \kappa^2\right) = \frac{G2\kappa n_{c,ss}}{m\Omega_m}$,where the steady state photon population is $\bar{n}_{c,ss} = |\alpha_{ss}|^2$.The cubic equation for $\bar{x}_{ss}$ implies the existence of several possible steady state values, but we will assume that the system is locked to one particular solution.

(11) So we can also freely use $\Delta_0 = \Delta$

$$\bar{\alpha} = |\bar{\alpha}|e^{i\mathrm{Arg}[\bar{\alpha}]}, \tag{3.76}$$

by making the $U(1)$ gauge transformation[12]

$$\begin{cases} \delta\hat{a} \longrightarrow \delta\hat{a}e^{i\mathrm{Arg}[\bar{\alpha}]} \\ \delta\hat{a}_{\mathrm{in}} \longrightarrow \delta\hat{a}_{\mathrm{in}}e^{i\mathrm{Arg}[\bar{\alpha}]} \end{cases}$$

By doing so we have a set of equations where we can factor out the dependence on phase of the classical intra-cavity field:

$$\delta\dot{\hat{a}}(t) = \sqrt{2\kappa}\,\delta\hat{a}_{\mathrm{in}}(t) + (i\Delta - \kappa)\,\delta\hat{a}(t) + ig_0|\bar{\alpha}|\delta\hat{X}(t) \tag{3.77}$$

$$\frac{1}{\Omega_m}\left[\delta\ddot{\hat{X}}(t) + 2\gamma\delta\dot{\hat{X}} + \Omega_m^2\delta\hat{X}(t)\right] = g_0|\bar{\alpha}|\left(\delta\hat{a} + \delta\hat{a}^\dagger\right) + \hat{f}(t) \tag{3.78}$$

This allows us to define the photon-enhanced opto-mechanical coupling

$$g \equiv \sqrt{\bar{n}_c}\,g_0$$

We then finally arrive at the canonical optomechanical equations:

$$\delta\dot{\hat{a}}(t) = \sqrt{2\kappa}\,\delta\hat{a}_{\mathrm{in}}(t) + (i\Delta - \kappa)\,\delta\hat{a}(t) + ig\delta\hat{X}(t) \tag{3.79}$$

$$\frac{1}{\Omega_m}\left[\delta\ddot{\hat{X}}(t) + 2\gamma\delta\dot{\hat{X}} + \Omega_m^2\delta\hat{X}(t)\right] = g\left(\delta\hat{a} + \delta\hat{a}^\dagger\right) + \hat{f}(t) \tag{3.80}$$

We could also write them only in terms of quadratures:

3.3.3 Quadrature Equations of Motion

We will often prefer to work with quadrature operators[13]. This gives us three equations of motion:

$$\dot{\hat{q}}(t) = \sqrt{2\kappa}\,\hat{q}_{\mathrm{in}}(t) - \kappa\hat{q}(t) - \Delta\hat{y}(t) \tag{3.81a}$$

$$\dot{\hat{y}}(t) = \sqrt{2\kappa}\,\hat{y}_{\mathrm{in}}(t) - \kappa\hat{y}(t) + \Delta\hat{q}(t) + \sqrt{\frac{\kappa\Gamma_0}{2}}\,\hat{X} \tag{3.81b}$$

$$\frac{1}{\Omega_m}\left[\ddot{\hat{X}}(t) + 2\gamma\dot{\hat{X}} + \Omega_m^2\hat{X}(t)\right] = \sqrt{\frac{\kappa\Gamma_0}{2}}\,\hat{q}(t) + \hat{f}(t) \tag{3.81c}$$

where we have defined the readout rate:

$$\Gamma_0 \equiv \frac{4g^2}{\kappa} \tag{3.82}$$

Even without solving the equations of motion we can see the presence of a coherent feedback loop: The mechanical oscillator Eq. (3.81c) is driven by the amplitude quadrature of the light. This disturbance of the mechanical oscillator is imprinted on the phase quadrature in Eq. (3.81b). Through the rotation of the phase quadrature into the amplitude quadrature, which arises due to the detuning Δ, the mechanical signal enters into the amplitude quadrature with a delay. This now means that the radiation pressure on the mechanical oscillator depends on the position of the oscillator at an earlier time, constituting a coherent retarded feedback loop. We will later see that this feedback loop results in dynamical back action.

(12) The transformation is valid, as it preserve the canonical commutation relations between the creation and annihilation operators.
(13) For brevity, we now drop the δ on all quantum operators.

3.3.4 Equations of Motion in Fourier space

Since the equations of motion Eq. (3.81) are linear in the operators we can obtain a mathematically simpler description of the dynamic by going to Fourier space, where all EOM are simply linear algebraic equations. Furthermore it is convenient to arrange the light degrees of freedom in a vector , so we have:

$$\begin{bmatrix} (\kappa - i\Omega) & \Delta \\ -\Delta & (\kappa - i\Omega) \end{bmatrix} \begin{pmatrix} \hat{q}(\Omega) \\ \hat{y}(\Omega) \end{pmatrix} = \sqrt{2\kappa} \begin{pmatrix} \hat{q}_{\text{in}}(\Omega) \\ \hat{y}_{\text{in}}(\Omega)) \end{pmatrix} + \sqrt{\frac{\kappa\Gamma}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{X}(\Omega) \tag{3.83a}$$

$$\chi_{m,0}^{-1}(\Omega)\hat{X}(\Omega) = \sqrt{\frac{\kappa\Gamma}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{q}(\Omega) \\ \hat{y}(\Omega)) \end{pmatrix} + \hat{f}(\Omega) \tag{3.83b}$$

where we have defined the bare mechanical susceptibility as:

$$\chi_{m,0}(\Omega) = \frac{\Omega_m}{\Omega_m^2 - \Omega^2 - 2i\Omega\gamma} \tag{3.84}$$

As we will see, the resonance frequency Ω_m and the linewidth γ will be modified by the dynamical back-action.

3.4 Solving the EOM

In this following we will show how to solve the Fourier space EOM Eq. (3.83). As it turns out, it is beneficial to start by considering the prolbem of a cavity with no mechanical DOF, and then use this solution to obtain the solution to the optomechanical problem.

3.4.1 The Cavity susceptibility

We start by considering the the problem of a cavity driven by an external field. The EOM for the intra-cavity ahnihilation operator is the same as Eq. (3.79) with $g \to 0$:

$$\chi_c(\Omega)^{-1}\hat{a}(\Omega) = \sqrt{2\kappa}\,\hat{a}_{\text{in}}(\Omega) \Leftrightarrow \tag{3.85}$$

$$\hat{a}(\Omega) = \sqrt{2\kappa}\,\chi_c(\Omega)\hat{a}_{\text{in}}(\Omega) \Leftrightarrow \tag{3.86}$$

where we have defined the cavity susceptibility:

$$\chi_c(\Omega) = \frac{1}{\kappa - i\,(\Omega + \Delta)} \tag{3.87}$$

We will often write the cavity susceptibility on polar form:

$$\chi_c(\Omega) = |\chi_c(\Omega)|e^{i\theta(\Omega)} \tag{3.88a}$$

$$|\chi_c(\Omega)| = \frac{1}{\sqrt{\kappa^2 + (\Omega + \Delta)^2}} \tag{3.88b}$$

$$\theta(\Omega) = \text{Arg}\left[\chi_c(\Omega)\right] = \arctan\left(\frac{\Omega + \Delta}{\kappa}\right) \tag{3.88c}$$

The above definitions the following relation holds:

$$\chi_c^*(-\Omega) = |\chi_c(-\Omega)|\, e^{-i\theta(-\Omega)} \tag{3.89}$$

It is worth noting that the cavity susceptibility for the annihilation operator isn't skew-Hermitian:

$$\chi_c^*(\Omega) \neq \chi_c(-\Omega)$$

Once we start discussing the optical spring effect and dynamical broadening, it is useful to know that the real and imaginary part of the cavity susceptibility can be written:

Note that we have the two useful relations:

$$\mathrm{Re}\left[\chi_c(\Omega)\right] = \kappa|\chi_c(\Omega)|^2 \tag{3.90a}$$

$$\mathrm{Im}\left[\chi_c(\Omega)\right] = \kappa\tan\left(\theta_c\right)|\chi_c(\Omega)|^2 \tag{3.90b}$$

3.4.2 Transfer Matrix with Optomechanical Coupling

It is convinient to solve for the intra-cavity fields using a transfer matrix approach. The intra-cavity field EOM Eq. (B.9) has the solution (see Appendix B for the detailed derivation of the transfer matrix):

$$\begin{pmatrix}\hat{q}(\Omega)\\ \hat{y}(\Omega)\end{pmatrix} = \boldsymbol{T}_0\sqrt{2\kappa}\begin{pmatrix}\hat{q}_{\mathrm{in}}(\Omega)\\ \hat{y}_{\mathrm{in}}(\Omega)\end{pmatrix} + \sqrt{\frac{\kappa\Gamma_0}{2}}\boldsymbol{T}_0\begin{pmatrix}0\\ 1\end{pmatrix}\hat{X}(\Omega) \tag{3.91}$$

By factoring out $\frac{|\chi_c(\Omega)|+|\chi_c(-\Omega)|}{2}$ from the cavity transfer matrix $\boldsymbol{T}_0$ we can defines an effective read-out rate:

$$\Gamma(\Omega) = \Gamma_0\left(\kappa\frac{|\chi_c(\Omega)|+|\chi_c(-\Omega)|}{2}\right)^2, \tag{3.92}$$

and a sideband asymmetry factor:

$$\zeta(\Omega) = \frac{|\chi_c(\Omega)|-|\chi_c(-\Omega)|}{|\chi_c(\Omega)|+|\chi_c(-\Omega)|} \tag{3.93}$$

The intra-cavity field can then be written:

$$\begin{aligned}\begin{pmatrix}\hat{q}(\Omega)\\ \hat{y}(\Omega)\end{pmatrix} &= \boldsymbol{T}_0\sqrt{2\kappa}\begin{pmatrix}\hat{q}_{\mathrm{in}}(\Omega)\\ \hat{y}_{\mathrm{in}}(\Omega))\end{pmatrix}\\ &+ e^{i\frac{\theta(\Omega)-\theta(-\Omega)}{2}}\boldsymbol{\mathcal{R}}_{\frac{\theta(\Omega)+\theta(-\Omega)}{2}}\sqrt{\frac{\Gamma(\Omega)}{2\kappa}}\begin{pmatrix}i\zeta(\Omega)\\ 1\end{pmatrix}\hat{X}(\Omega)\end{aligned} \tag{3.94}$$

Using this the input-output relation can be written:

$$\begin{aligned}\begin{pmatrix}\hat{q}_{\mathrm{out}}(\Omega)\\ \hat{y}_{\mathrm{out}}(\Omega))\end{pmatrix} &= e^{i2\phi_-(\Omega)}\boldsymbol{\mathcal{R}}_{2\phi_+(\Omega)}\begin{pmatrix}\hat{q}_{\mathrm{in}}(\Omega)\\ \hat{y}_{\mathrm{in}}(\Omega)\end{pmatrix}\\ &+ e^{i\frac{\theta(\Omega)-\theta(-\Omega)}{2}}\boldsymbol{\mathcal{R}}_{\frac{\theta(\Omega)+\theta(-\Omega)}{2}}\sqrt{\Gamma(\Omega)}\begin{pmatrix}i\zeta(\Omega)\\ 1\end{pmatrix}\hat{X}(\Omega)\end{aligned} \tag{3.95}$$

Finaly it is convinient to define a set rotated and phase shifted input and output quadratures:

$$\begin{pmatrix}\hat{q}'_{\mathrm{out}}(\Omega)\\ \hat{y}'_{\mathrm{out}}(\Omega)\end{pmatrix} = e^{-i\phi_-(\Omega)}\boldsymbol{\mathcal{R}}^{\mathrm{T}}_{\phi_+(\Omega)}\begin{pmatrix}\hat{q}_{\mathrm{out}}(\Omega)\\ \hat{y}_{\mathrm{out}}(\Omega)\end{pmatrix} \tag{3.96}$$

$$\begin{pmatrix}\hat{q}'_{\mathrm{in}}(\Omega)\\ \hat{y}'_{\mathrm{in}}(\Omega)\end{pmatrix} = e^{i\phi_-(\Omega)}\boldsymbol{\mathcal{R}}_{\phi_+(\Omega)}\begin{pmatrix}\hat{q}_{\mathrm{in}}(\Omega)\\ \hat{y}_{\mathrm{in}}(\Omega)\end{pmatrix} \tag{3.97}$$

Note that the input and output transformation have opposite phases and rotate the quadratures in opposite directions.

So the input-output relation becomes:

$$\begin{pmatrix}\hat{q}'_{\mathrm{out}}(\Omega)\\ \hat{y}'_{\mathrm{out}}(\Omega))\end{pmatrix} = \begin{pmatrix}\hat{q}'_{\mathrm{in}}(\Omega)\\ \hat{y}'_{\mathrm{in}}(\Omega)\end{pmatrix} + \sqrt{\Gamma(\Omega)}\begin{pmatrix}i\zeta(\Omega)\\ 1\end{pmatrix}\hat{X}(\Omega) \tag{3.98}$$

We see that the position of the oscillator is generally imprinted on both the phase and amplitude quadrature. In particular the position is maximally read out, at a rate $\Gamma(\Omega)$ in the phase quadrature $\hat{y}'_{\text{out}}(\Omega)$, while the sideband asymmetry leads to an non-optimal readout of $\hat{X}$ in the amplitude quadrature $\hat{q}'_{\text{out}}(\Omega)$.

3.4.3 Solving the Mechanical EOM

To obtain the actual solution for the mechanical EOM Eq. (3.83b), i.e. one which only depends on the input noise operators, we must insert the solution for the intra-cavity field Eq. (3.91). However since the intra-cavity field depends on $\hat{X}$, inserting the light operators will lead to a renormalization of the oscillator susceptibility. If we write the cavity transfer matrix Eq. (B.9) as:

$$\boldsymbol{T}_0 = \frac{\sqrt{\Gamma(\Omega)}}{\kappa\sqrt{\Gamma_0}}\begin{bmatrix} 1 & i\zeta(\Omega) \\ -i\zeta(\Omega) & 1 \end{bmatrix} e^{i\frac{\theta(\Omega)-\theta(-\Omega)}{2}}\boldsymbol{\mathcal{R}}_{\frac{\theta(\Omega)+\theta(-\Omega)}{2}} \tag{3.99}$$

Then the intra-cavity field quadratures can be written:

$$\begin{aligned}\begin{pmatrix} \hat{q}(\Omega) \\ \hat{y}(\Omega) \end{pmatrix} = \left(\sqrt{\frac{\kappa\Gamma_0}{2}}\right)^{-1}\sqrt{\Gamma(\Omega)}\begin{bmatrix} 1 & i\zeta(\Omega) \\ -i\zeta(\Omega) & 1 \end{bmatrix}\begin{pmatrix} \hat{q}'_{\text{in}}(\Omega) \\ \hat{y}'_{\text{in}}(\Omega)) \end{pmatrix} \\ + \sqrt{\frac{\kappa\Gamma_0}{2}}\boldsymbol{T}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{X}(\Omega)\end{aligned} \tag{3.100}$$

So the mechanical EOM can then be written:

$$\underbrace{\left[\chi_{m,0}^{-1} - \frac{\kappa\Gamma_0}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\boldsymbol{T}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]}_{\chi_m^{-1}(\Omega)}\hat{X}(\Omega) = \left[\sqrt{\Gamma(\Omega)}\begin{pmatrix} 1 & i\zeta(\Omega) \end{pmatrix}\begin{pmatrix} \hat{q}'_{\text{in}}(\Omega) \\ \hat{y}'_{\text{in}}(\Omega) \end{pmatrix} + \hat{f}(\Omega)\right] \tag{3.101}$$

And the solution is then readily obtained, simply by inverting the susceptibility:

$$\hat{X}(\Omega) = \chi_m(\Omega)\left[\sqrt{\Gamma(\Omega)}\begin{pmatrix} 1 & i\zeta(\Omega) \end{pmatrix}\begin{pmatrix} \hat{q}'_{\text{in}}(\Omega) \\ \hat{y}'_{\text{in}}(\Omega) \end{pmatrix} + \hat{f}(\Omega)\right] \tag{3.102}$$

The renormalized mechanical susceptibility has the form:

$$\chi_m^{-1}(\Omega) = \chi_{m,0}^{-1}(\Omega) + \Sigma(\Omega), \tag{3.103}$$

where the self energy is given by:

$$\Sigma(\Omega) = \frac{\kappa\Gamma_0}{4i}\left(\chi_c(\Omega) - \chi_c^*(-\Omega)\right) \Rightarrow \tag{3.104}$$

$$\begin{aligned}\Sigma(\Omega) = \frac{\kappa\Gamma}{4}\left[\frac{\Omega+\Delta}{\kappa^2+(\Omega+\Delta)^2} - \frac{\Omega-\Delta}{\kappa^2+(\Omega-\Delta)^2}\right] \\ + i\frac{\kappa\Gamma}{4}\left[\frac{\kappa}{\kappa^2+(\Omega-\Delta)^2} - \frac{\kappa}{\kappa^2+(\Omega+\Delta)^2}\right]\end{aligned} \tag{3.105}$$

Again, this renormalization arises due to the coherent feedback loop described earlier.

3.4.4 Dynamical Back-Action: The Optical Spring Effect and Dynamical Broadening

Since $\Sigma(\Omega)$ generally is complex, it will lead to a modification of the resonance frequency and the linewidth of the oscillator. To see this we write the susceptibility explicitly:

$$\chi_m^{-1}(\Omega) = \frac{1}{\Omega_m}\left(\Omega_m^2 + \Omega_m \mathrm{Re}\left[\Sigma(\Omega)\right] - \Omega^2 - 2i\Omega\left(\gamma - \frac{\Omega_m}{2i\Omega}\mathrm{Im}\left[\Sigma(\Omega)\right]\right)\right) \tag{3.106}$$

$$\Rightarrow \chi_m^{-1}(\Omega) = \frac{1}{\Omega_m}\left(\Omega'^2 - \Omega^2 - 2i\Omega\gamma'\right) \tag{3.107}$$

where we define a new resonance and linewidth

$$\Omega'_m = \sqrt{\Omega_m^2 + \Omega_m \mathrm{Re}\left[\Sigma(\Omega)\right]} = \Omega_m\sqrt{1 + \frac{\mathrm{Re}\left[\Sigma(\Omega)\right]}{\Omega_m}} \tag{3.108}$$

$$\gamma' = \gamma - \frac{\Omega_m}{2i\Omega}\mathrm{Im}\left[\Sigma(\Omega)\right] \tag{3.109}$$

The change in the resonance is called the optical spring effect, since the spring constant k satisfies $k \propto \Omega_m^2$, while the change in the linewidth is referred to as dynamical broadening. If the readout is weak, $\Gamma_0 \ll \kappa$, then we can expand the renormalized resonance frequency and get:

$$\Omega'_m \approx \Omega_m + \frac{1}{2}\mathrm{Re}\left[\Sigma(\Omega)\right] \tag{3.110}$$

So in the weak read-out regime, the renormalization amount to a shift in the resonance frequency $\delta\Omega_m(\Omega)$ and a broadening/narrowing of the oscillator $\delta\gamma(\Omega)$:

$$\Omega'_m = \Omega_m + \delta\Omega_m(\Omega) \tag{3.111}$$

$$\gamma' = \gamma + \delta\gamma(\Omega) \tag{3.112}$$

with the optical spring shift and the dynamical broadening resulting from the dynamical back-action given by:

$$\delta\Omega_m(\Omega) = \mathrm{Re}\left[\Sigma(\Omega)\right] = \frac{\kappa\Gamma}{4}\left[\frac{\Omega+\Delta}{\kappa^2 + (\Omega+\Delta)^2} - \frac{\Omega-\Delta}{\kappa^2 + (\Omega-\Delta)^2}\right] \tag{3.113}$$

$$\delta\gamma(\Omega) = -\frac{\Omega_m}{2i\Omega}\mathrm{Im}\left[\Sigma(\Omega)\right] = \frac{\Omega_m}{2\Omega}\frac{\kappa\Gamma}{4}\left[\frac{\kappa}{\kappa^2 + (\Omega+\Delta)^2} - \frac{\kappa}{\kappa^2 + (\Omega-\Delta)^2}\right] \tag{3.114}$$

This dynamical BA is related to the unequal response at the two sidebands (see Fig. 3.2) generated by the optomechanical interaction. We will return to this point in more detail once we discuss two-tone driving.

3.4.5 Input-Output relation for Optomechanics

The output light $\hat{a}_{\mathrm{out}}(t)$ from e.g. a cavity is related to the input light $\hat{a}_{\mathrm{in}}(t)$ and the intra-cavity light $\hat{a}(t)$, through the input-output relation[5]:

$$\hat{a}_{\mathrm{out}}(t) = -\hat{a}_{\mathrm{in}}(t) + \sqrt{2\kappa}\,\hat{a}(t) \tag{3.115}$$

We can now write an input-output relation purely in terms of the input fields, simply by plugging the solution for $\hat{X}$, as given in Eq. (3.102), into the input-output relation for the light Eq. (3.98):

$$\begin{pmatrix}\hat{q}'_{\mathrm{out}}(\Omega)\\ \hat{y}'_{\mathrm{out}}(\Omega)\end{pmatrix} = \begin{pmatrix}\hat{q}'_{\mathrm{in}}(\Omega)\\ \hat{y}'_{\mathrm{in}}(\Omega)\end{pmatrix} + \chi_m(\Omega)\sqrt{\Gamma(\Omega)}\,\boldsymbol{M}_\zeta\left[\sqrt{\Gamma(\Omega)}\begin{pmatrix}\hat{q}'_{\mathrm{in}}(\Omega)\\ \hat{y}'_{\mathrm{in}}(\Omega)\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\hat{f}(\Omega)\right] \tag{3.116}$$

where we have defined:

$$\boldsymbol{M}_\zeta = \begin{bmatrix} i\zeta(\Omega) & -\zeta^2(\Omega)\\ 1 & i\zeta(\Omega)\end{bmatrix} \tag{3.117}$$

We can now see that the output light contains three different noise terms. First there is the shot noise:

$$\vec{N}_{\text{shot}} = \begin{pmatrix} \hat{q}'_{\text{in}}(\Omega) \\ \hat{y}'_{\text{in}}(\Omega) \end{pmatrix}, \tag{3.118}$$

which is due to the vacuum fluctuations of the drive laser, and the optical reservoir. Next there is the QBA

$$\vec{N}_{\text{QBA}} = \chi_m(\Omega)\Gamma(\Omega)\boldsymbol{M}_\zeta \begin{pmatrix} \hat{q}'_{\text{in}}(\Omega) \\ \hat{y}'_{\text{in}}(\Omega) \end{pmatrix} \tag{3.119}$$

Like the shot noise the QBA is due to vacuum fluctuations of the optical field driving the mechanical oscillator, however, it is specifically due to the probing of the oscillator by the light, as can be seen by the dependence on $\Gamma(\Omega)$. Finally there is the thermal noise due to phonons:

$$\vec{N}_{\text{th}} = \chi_m(\Omega)\sqrt{\Gamma(\Omega)}\,\boldsymbol{M}_\zeta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \hat{f}(\Omega) \tag{3.120}$$

If the mechanical oscillator is driven by additional forces, such as a classical force $f_{\text{cl}}(t)$, they will simply be added to the thermal noise:

$$\begin{aligned}\begin{pmatrix} \hat{q}'_{\text{out}}(\Omega) \\ \hat{y}'_{\text{out}}(\Omega) \end{pmatrix} = \begin{pmatrix} \hat{q}'_{\text{in}}(\Omega) \\ \hat{y}'_{\text{in}}(\Omega) \end{pmatrix} + \chi_m(\Omega)\sqrt{\Gamma(\Omega)}\,\boldsymbol{M}_\zeta \Bigg[\sqrt{\Gamma(\Omega)} \begin{pmatrix} \hat{q}'_{\text{in}}(\Omega) \\ \hat{y}'_{\text{in}}(\Omega) \end{pmatrix} \\ + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left(\hat{f}(\Omega) + f_{\text{cl}}(\Omega)\right)\Bigg]\end{aligned} \tag{3.121}$$

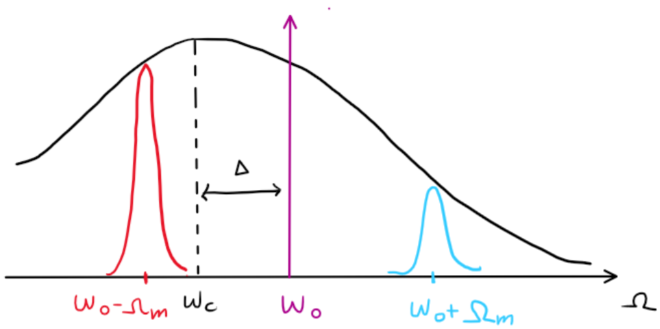

Figure 3.2: Sketch of the heterodyne spectrum for an optomechanical system driven by a laser with frequency ω_o. The mechanical motion is imprinted sidebands (red and blue Lorentzian) at frequencies $\omega_o \pm \Omega_m$. If the cavity susceptibility (black Lorentzian) and the drive tone are detuned by an amount Δ the response from the sidebands will be unequal and lead to dynamical BA

CHAPTER 4

Cavity Optomechanics with a Two-tone Drive

> "This is getting out of hand, now there are two of them!"
>
> NUTE GUNRAY
>
> *- Star Wars: The Phantom Menace,* George Lucas

In this chapter we up the ante and consider an optomechanical cavity driven by two coherent tones. We start by introducing the two-tone drive Hamiltonian and then discuss some subtleties in linearization procedure, arising from the additional dynamics introduced by the second drive tone. At the end of the chapter we arrive at the Heisenberg-Langevin EOM for the two-tone driven problem.

4.1 The Two-Tone Optomechanical Hamiltonian

We start this section by discussing how to write the drive term in the Hamiltonian for a system driven by two strong coherent tones

4.1.1 The Drive Hamiltonian

We now consider an optomechanical system driven by two laser tones at frequencies $\omega_\pm$. The Drive Hamiltonian is simply the sum of the drive Hamiltonians for each tone, of the the same form as the drive Hamiltonian Eq. (3.20) we used previously:

$$\hat{H}_d = i\hbar\sqrt{2\kappa}\left(\alpha_{\text{in},+}e^{-i\omega_+ t} + \alpha_{\text{in},-}e^{-i\omega_- t}\right)\hat{a}^\dagger + \text{H.c.} \tag{4.1}$$

The two amplitudes of the coherent drive fields are generally complex:

$$\alpha_{\text{in},\pm} = |\alpha_{\text{in},\pm}|e^{i\phi_\pm} \tag{4.2}$$

As we have two frequencies, it is more natural to rewrite the problem in terms of an average drive frequency ω_o and a relative frequency $\tilde{\Omega}$:

$$\omega_o = \frac{\omega_+ + \omega_-}{2} \tag{4.3}$$
$$\tilde{\Omega} = \frac{\omega_+ - \omega_-}{2} \tag{4.4}$$

This is easily achieved by factoring out the mean frequency:

$$\hat{H}_d = i\hbar\sqrt{2\kappa}\, e^{-i\frac{\omega_{d+}+\omega_{d-}}{2}t} \left(\alpha_{\text{in},+}e^{-i\frac{\omega_{d+}-\omega_{d-}}{2}t} \right. \tag{4.5}$$
$$\left. +\alpha_{\text{in},-}e^{i\frac{\omega_{d+}-\omega_{d-}}{2}t}\right) \hat{a}^\dagger + \text{H.c.}$$
$$= i\hbar\sqrt{2\kappa}\, e^{-i\omega_o t} \left(\alpha_{\text{in},+}e^{-i\tilde{\Omega}t} + \alpha_{\text{in},-}e^{i\tilde{\Omega}t}\right) \hat{a}^\dagger + \text{H.c.} \tag{4.6}$$

If we then transform to an interaction picture using the unitary:

$$\hat{U} = e^{-i\omega_o \hat{a}^\dagger \hat{a} t}, \tag{4.7}$$

we simply have:

$$\hat{H}_d = i\hbar\sqrt{2\kappa} \left(\alpha_{\text{in},+}e^{-i\tilde{\Omega}t} + \alpha_{\text{in},-}e^{i\tilde{\Omega}t}\right) \hat{a}^\dagger + \text{H.c.} \tag{4.8}$$

Moreover, since the complex amplitudes generally have different phases, we can use the same trick and write:

$$\hat{H}_d = i\hbar\sqrt{2\kappa} \left(|\alpha_{\text{in},+}|e^{i\phi_+}e^{-i\tilde{\Omega}t} + |\alpha_{\text{in},-}|e^{i\phi_-}e^{i\tilde{\Omega}t}\right) \hat{a}^\dagger + \text{H.c.} \tag{4.9}$$
$$= e^{i\frac{\Phi_- + \Phi_+}{2}} i\hbar\sqrt{2\kappa} \left(|\alpha_{\text{in},+}|e^{-i\frac{\phi_- - \phi_+}{2}}e^{-i\tilde{\Omega}t} \right. \tag{4.10}$$
$$\left. +|\alpha_{\text{in},-}|e^{i\frac{\phi_- - \phi_+}{2}}e^{i\tilde{\Omega}t}\right) \hat{a}^\dagger + \text{H.c.},$$

and then we can likewise define a mean phase and a relative phase:

$$\bar{\phi} = \frac{\phi_- + \phi_+}{2} \tag{4.11}$$
$$\phi = \frac{\phi_- - \phi_+}{2} \tag{4.12}$$

So the drive Hamiltonian becomes:

$$\hat{H}_d = i\hbar\sqrt{2\kappa} \left(|\alpha_{\text{in}+}|e^{-i\phi}e^{-i\tilde{\Omega}t} + |\alpha_{\text{in}-}|e^{i\phi}e^{i\tilde{\Omega}t}\right) \hat{a}^\dagger(t)e^{i\bar{\phi}} + \text{H.c.}$$

We can now eliminate the mean phase by performing a gauge transformation of the intra-cavity photon field:

$$\hat{a} \longrightarrow \hat{a}e^{i\bar{\phi}} \tag{4.13a}$$
$$\hat{a}^\dagger \longrightarrow \hat{a}^\dagger e^{-i\bar{\phi}} \tag{4.13b}$$

So finally, by transforming to an interaction picture rotating with the mean drive frequency ω_o, and by using a gauge transformation to eliminate the mean phase, we are left with a drive Hamiltonian on the form:

$$\hat{H}_d = i\hbar\sqrt{2\kappa} \left(|\alpha_{\text{in}+}|e^{-i\phi}e^{-i\tilde{\Omega}t} + |\alpha_{\text{in}-}|e^{i\phi}e^{i\tilde{\Omega}t}\right) \hat{a}^\dagger(t) + \text{H.c.} \tag{4.14}$$

In terms of the intracavity quadratures this is:

$$\hat{H}_d = \hbar\sqrt{2\kappa}\,\sin(\tilde{\Omega}t + \phi)\left[\sqrt{2}|\alpha_{\text{in}+}| - \sqrt{2}|\alpha_{\text{in}-}|\right]\hat{q}, \tag{4.15}$$
$$-\hbar\sqrt{2\kappa}\,\cos(\tilde{\Omega}t + \phi)\left[\sqrt{2}|\alpha_{\text{in}+}| + \sqrt{2}|\alpha_{\text{in}-}|\right]\hat{y}$$

from which we see that for arbitrary drive amplitudes, both quadratures are driven by the laser light.

An important special case of this, is for equal drive amplitudes $|\alpha_{\text{in}+}| = |\alpha_{\text{in}-}| = |\alpha_{\text{in}}|$:

$$\hat{H}_d = -i\hbar\sqrt{2\kappa}\,2|\alpha_{\text{in}}-\cos(\tilde{\Omega}t+\phi)\left(\hat{a}-\hat{a}^\dagger\right), \tag{4.16}$$

which looks just like a single tone drive Eq. (3.20) with a cosine modulation, and double the maximal amplitude. In terms of quadratures it reduce to:

$$\hat{H}_d = \hbar\sqrt{2\kappa}\,2|\alpha_{\text{in}}-\sqrt{2}\cos(\tilde{\Omega}t+\phi)\,\hat{y}(t), \tag{4.17}$$

so for equal drive amplitudes, only the phase quadrature $\hat{y}$ of the intra-cavity field is driven.

4.1.2 Linearization for Equal Drive Amplitudes

Let us for a moment consider just a cavity with no mechanical degree of freedom. If we drive such a cavity two with equal amplitude drive tones, we would expect that after some transient period the sinusoidal driving of intra-cavity field would lead to a steady state where the intra-cavity field also oscillates sinusoidally, be it with some retardation due to the nontrivial response function of the cavity. If we now add a mechanical degree of freedom, there will once again be a transient period until the mechanical oscillator, which is driven by the intra-cavity field, reaches a steady state. However since the intra-cavity field oscillates, our intuition tells us the same will be the case for the mechanics[1], albeit it doesn't have to be at the same frequency nor in phase with the light. In our case, it could potentially mean that the classical steady state values to linearize about would be time dependent, which would lead to several parameters, such as the detuning between the driving light and the cavity and picking up a time dependence. We wish to examine further if this is the case. To do so we choose to focus on the simple case of equal drive amplitudes.

The Hamiltonian for the mechanical system and intra-cavity field is the same as in the single-tone case. If we now transform to the frame rotating a ω_o, i.e. using Eq. (3.20), and drop the vacuum terms, the Hamiltonian for equal drive amplitudes can be written:

$$\begin{aligned}\hat{H} = \hbar\left(\omega_c(\hat{X})-\omega_o\right)\hat{a}^\dagger(t)\hat{a}(t) + \hbar\Omega_m\hat{b}^\dagger(t)\hat{b}(t) \\ - i\hbar\sqrt{2\kappa}\,2|\alpha_{\text{in}}-\cos(\tilde{\Omega}t+\phi)\left(\hat{a}-\hat{a}^\dagger\right)\end{aligned} \tag{4.18}$$

If we now linearize about the classical solution using the same ansatz Eq. (3.20) as we did for single tone, we find that the classical EOM for the intracavity field $\bar{\alpha}(t)$, the mechanical position $\bar{X}(t)$ and the mechanical momentum $\bar{P}(t)$ are:

$$\begin{aligned}\dot{\bar{\alpha}}(t) &= \sqrt{2\kappa}\,2\alpha_{\text{in}}\cos(\tilde{\Omega}t+\phi) + \left(i\Delta(t)-\kappa\right)\bar{\alpha}(t) \\ \frac{1}{\Omega_m}\left[\ddot{\bar{X}}(t) + 2\gamma\dot{\bar{X}} + \Omega_m^2\bar{X}(t)\right] &= g_0(t)|\bar{\alpha}(t)|^2\end{aligned}$$

where the detuning and the optomechanical coupling are given by:

(1) Recall that in general, when a oscillatory system is driven by a periodic force, it will eventually reach a steady state determined by the drive and the response function. Typical examples include a RLC circuit driven by an AC current, an electric dipole driven by a monochromatic electric field, a single spin in a RF magnetic field and so on.

$$g_0(t) = -x_{\text{xpf}} \left.\frac{\partial \omega_c}{\partial X}\right|_{\bar{X}(t)} \tag{4.19}$$

$$\Delta(t) = \omega_o - \omega_c(\bar{X}(t)) + g_0(t) \tag{4.20}$$

$$\tag{4.21}$$

Formal integration of the classical intracavity field EOM yields the steady state solution (see Appendix C for the details):

$$\bar{\alpha}(t) = \sqrt{2\kappa}\, 2\alpha_{\text{in}} \int_{-\infty}^{t} \mathrm{d}t' \left[\cos(\tilde{\Omega}t' + \phi) e^{-\kappa(t-t') + i\int_{t'}^{t} \mathrm{d}t'' \Delta(t'')}\right]$$

Let us now assume that the frequency of the cavity mode does not vary appreciably in time, such that the detuning and the optomechanical coupling are approximately constant

$$\Delta \approx \omega_o - \omega_c(\bar{X}(t_0)) + g_0 \tag{4.22}$$

$$g_0 \approx -x_{\text{xpf}} \left.\frac{\partial \omega_c}{\partial x}\right|_{\bar{x}(t_0)} \tag{4.23}$$

The solution for the intra-caivty field then reduces to:

$$\bar{\alpha}(t) = \sqrt{2\kappa}\, 2\alpha_{\text{in}} \int_{-\infty}^{t} \mathrm{d}t' e^{-(\kappa - i\Delta)(t-t')} \cos(\tilde{\Omega}t' + \phi) \tag{4.24}$$

The integral can now be evaluated, and results in an intra-cavity field which can be written on the form:

$$\bar{\alpha}(t) = \alpha_+ e^{-i\tilde{\Omega}t} + \alpha_- e^{i\tilde{\Omega}t}, \tag{4.25}$$

where the positive and negative frequency component of the intracavity field are given by:

$$\alpha_\pm = \sqrt{2\kappa}\, 2|\alpha_{\text{in}}| e^{i\phi_\pm} \chi_c(\pm\tilde{\Omega}) \tag{4.26}$$

If we now consider the mechanical EOM, we can write it:

$$\frac{1}{\Omega_m}\left[\ddot{\bar{X}}(t) + 2\gamma\dot{\bar{X}} + \Omega_m^2 \bar{X}(t)\right] = F_{\text{rad}}(t),$$

where the radiation preasure force now is time dependent:

$$F_{\text{rad}}(t) = g_0 |\bar{\alpha}(t)|^2$$

Using the solution for the intracavity field we find that the radiation pressure force has a DC term and and a term oscillating at $2\tilde{\Omega}$:

$$F_{\text{rad}}(t) = \underbrace{g_0\left(|\alpha_+|^2 + |\alpha_-|^2\right)}_{F_{rad,DC}} + \underbrace{g_0\left(\alpha_+\alpha_-^* e^{-2i\tilde{\Omega}t} + \alpha_+^*\alpha_- e^{2i\tilde{\Omega}t}\right)}_{F_{rad,AC}}$$

The DC term is just a constant force, and only leads to a constant shift of the equilibrium position of the oscillator in steady state, we also saw a term like this in the single-tone driving case. The AC term is new, and potentially problematic. It will in general lead to oscillations of the "equilibrium" of the oscillator, and potentially invalidate the assumption that Δ and g_0 are constant, and furthermore make the linearization problematic. For the physics we aim to examine, we assume that the mechanical resonance Ω_m and the relative drive frequency $\tilde{\Omega}$ are relatively close. In this case the AC force is non-resonant and can simply be neglected. In reality, the single mechanical mode we consider will be one of many possible modes.

If it just so happens that the spacing between modes is on the order of $\tilde{\Omega}$, then the AC term in the radiation pressure force may be resonant with a neighboring mechanical mode. To mitigate other mechanical modes being activated, one can apply a compensating feedback force $f_{\text{comp}}(t)$:

$$\frac{1}{\Omega_m}\left[\ddot{\bar{X}}(t) + 2\gamma\dot{\bar{X}} + \Omega_m^2\bar{X}(t)\right] = F_{\text{rad}}(t) + f_{\text{comp}}(t), \tag{4.27}$$

where the compensating force is exactly opposite the AC part of the radiation pressure force, which is a deterministic and thus in theory known force:

$$f_{\text{comp}}(t) = -F_{rad,AC} \tag{4.28}$$

By neglecting the non-resonant contribution from the AC term, there is no longer any explicit time dependence in the mechanical EOM, and thus there will be a stationery steady state. This then ensures that Δ and g_0 indeed are constant, and thus we are consistent. If we proceed as in the single-tone case and make the steady state value $\bar{X}_{\text{ss}}$ the reference point for position measurements, then everything is just as in the single tone case, except the classical part $\bar{\alpha}(t)$ of the intra-cavity field now depends on time, even in steady state.

4.1.3 Linearized Hamiltonian For general Drive Ampltitudes

We now turn to the general two tone driving problem. In a frame rotating at the mean driving frequency, the quantum part of the Linearized Hamiltonian has the same form as for single-tone driving:

$$\hat{H} = -\Delta\hbar\delta\hat{a}^\dagger(t)\delta\hat{a}(t) + \hbar\Omega_m\left(\delta\hat{X}^2 + \delta\hat{P}^2\right) - \hbar g_0\left[i\bar{\alpha}(t)\delta\hat{a}^\dagger + \text{H.C.}\right]\delta\hat{X} \tag{4.29}$$

where the classical steady state intra-cavity field still has the form:

$$\bar{\alpha}(t) = \alpha_+ e^{-i\tilde{\Omega}t} + \alpha_- e^{i\tilde{\Omega}t} \tag{4.30}$$

with the complex amplitudes given by:

$$\alpha_\pm = \alpha_{\pm,\text{in}}\chi_c(\pm\tilde{\Omega}), \tag{4.31}$$

which is valid previously discussed limit where the classical mechanical motion doesn't influence the classical steady state of the intra-cavity field. Importantly we can freely control the amplitudes $|\alpha_\pm|$ and phases $\Phi_\pm$ of the intra-cavity field is, by adjusting the amplitudes $|\alpha_{\text{in},\pm}|$ and phases $\phi_\pm$ of the two drive tones, although we must remember to account for amplitude and phase the intra-cavity susceptibility $\chi_c(\pm\tilde{\Omega})$.

We can equally well write the Hamiltonian in terms of phonon and photon operators:

$$\hat{H} = -\Delta\hbar\delta\hat{a}^\dagger(t)\delta\hat{a}(t) + \hbar\Omega_m\delta\hat{b}^\dagger(t)\delta\hat{b}(t) - \frac{\hbar g_0}{\sqrt{2}}\left[\bar{\alpha}(t)\delta\hat{a}^\dagger + \bar{\alpha}^*(t)\delta\hat{a}\right]\left(\delta\hat{b} + \delta\hat{b}^\dagger\right) \tag{4.32}$$

As each coherent component of the intra-cavity field has a different phase $\Phi_\pm$, however only the relative phase will affect the dynamics. To that end we define the relative and mean phase:

$$\Phi = \frac{\Phi_- - \Phi_+}{2}, \tag{4.33a}$$

$$\bar{\Phi} = \frac{\Phi_- + \Phi_+}{2}, \tag{4.33b}$$

and factor out the mean phase:

$$\hat{H} = -\Delta\hbar\delta\hat{a}^\dagger(t)\delta\hat{a}(t) + \hbar\Omega_m\delta\hat{b}^\dagger(t)\delta\hat{b}(t) - \frac{\hbar g_0}{\sqrt{2}}\left[\left(|\alpha_+|e^{-i\Phi}e^{-i\tilde{\Omega}t} + |\alpha_-|e^{i\Phi}e^{i\tilde{\Omega}t}\right)e^{i\bar{\Phi}}\delta\hat{a}^\dagger + \text{H.C.}\right]\left(\delta\hat{b} + \delta\hat{b}^\dagger\right) \tag{4.34}$$

If we then perform the gauge transformation:

$$\delta\hat{a} \longrightarrow \delta\hat{a}e^{i\bar{\Phi}},$$

to eliminate the mean phase, then we finally achieve Hamiltonian we will consider:

$$\hat{H} = -\hbar\Delta\delta\hat{a}^\dagger(t)\delta\hat{a}(t) + \hbar\Omega_m\delta\hat{b}^\dagger(t)\delta\hat{b}(t) - \frac{\hbar g_0}{\sqrt{2}}\left[\left(|\alpha_+|e^{-i\Phi}e^{-i\tilde{\Omega}t} + |\alpha_-|e^{i\Phi}e^{i\tilde{\Omega}t}\right)\delta\hat{a}^\dagger + \text{H.C.}\right]\left(\delta\hat{b} + \delta\hat{b}^\dagger\right) \tag{4.35}$$

If we define the classical quadrature-like variables:

$$q_{\text{cl}} = \frac{|\alpha_-| + |\alpha_+|}{\sqrt{2}} \tag{4.36a}$$

$$y_{\text{cl}} = \frac{|\alpha_-| - |\alpha_+|}{\sqrt{2}} \tag{4.36b}$$

and two corresponding read-out rates from the relation(2):

$$\sqrt{\Gamma_k} = \sqrt{\frac{4g_0^2}{\kappa}}k, \quad k \in \{y_{\text{cl}}, q_{\text{cl}}\},$$

the Hamiltonian can be written succinctly in terms of the quadrature operators:

$$\hat{H} = -\hbar\Delta\left(\delta\hat{q}^2 + \delta\hat{y}^2\right) + \hbar\Omega_m\left(\delta\hat{X}^2 + \delta\hat{P}^2\right) \tag{4.37}$$

$$- \sqrt{\kappa}\left[\sqrt{\Gamma_q}\cos(\tilde{\Omega}t + \Phi)\hat{q} + \sqrt{\Gamma_y}\sin(\tilde{\Omega}t + \Phi)\hat{y}\right]\hat{X} \tag{4.38}$$

From the quadrature Hamiltonian we can make the observation that no matter the choice of $|\alpha_\pm|$, there will always be a coupling between the intra-cavity amplitude quadrature $\hat{q}$ and the mechanical position. However, the intra-cavity phase quadrature $\hat{y}$ only couples to the mechanical position when $|\alpha_+| \neq |\alpha_-|$.

4.2 Equations of Motion for Two-Tone Optomechanics

4.2.1 Heisenberg Langevin Equation for creation and annihilation operators

Starting from the Hamiltonian Eq. (4.35) the corresponding Heisenberg-Langevin equations for the intra-cavity photon annihilation operator $\hat{a}$ and the phonon annihilation operator $\hat{b}$ are:

$$\dot{\hat{a}}(t) = i\Delta - \kappa\delta\hat{a}(t) + \sqrt{2\kappa}\,\delta\hat{a}_{\text{in}}(t) + i\frac{g_0}{\sqrt{2}}\left[|\alpha_-|e^{i\Phi}e^{i\tilde{\Omega}t} + |\alpha_+|e^{-i\Phi}e^{-i\tilde{\Omega}t}\right]\left(\hat{b}(t) + \hat{b}^\dagger(t)\right) \tag{4.39a}$$

$$\dot{\hat{b}}(t) = -\,i\Omega_m\hat{b}(t) - \gamma\left(\hat{b}(t) - \hat{b}^\dagger(t)\right) + i\frac{1}{\sqrt{2}}\hat{f}(t) \tag{4.39b}$$

$$+ i\frac{g_0}{\sqrt{2}}\left[\left(|\alpha_-|e^{-i\Phi}e^{-i\tilde{\Omega}t} + |\alpha_+|e^{i\Phi}e^{i\tilde{\Omega}t}\right)\hat{a}(t) + \text{H.C.}\right] \tag{4.39c}$$

(2) We define the read-out rates from their square roots to ensure that $\sqrt{\Gamma_y} \propto |\alpha_+| - |\alpha_-|$ has the correct sign.

An important feature of these coupled differential equations is that $\hat{a}$ does not couple to $\hat{a}^\dagger$, while $\hat{b}$ does couple to $\hat{b}^\dagger$. As we shall see, this allows us to easily obtain a formal solution for $\hat{a}$ in terms of the mechanical degrees of freedom, while the mechanical equations require a more refined approach. The difference in the damping terms when comparing the photon and phonon EOM arises from using the RWA on the photon reservoir, but not the phonon reservoir.

4.2.2 Heisenberg Langevin Equation for quadratures

The EOM for the quadrature operators can now be obtained from the Hamiltonian Eq. (4.37) or from the creation and annihilation operator EOM Eq. (4.37). In any case the relevant EOM are:

$$\dot{\hat{q}}(t) = \sqrt{2\kappa}\,\hat{q}_{\text{in}}(t) - \Delta\hat{y}(t) - \kappa\hat{q}(t) - \sqrt{\kappa\Gamma_y}\sin(\tilde{\Omega}t + \Phi)\hat{X}(t) \tag{4.40}$$

$$\dot{\hat{y}}(t) = \sqrt{2\kappa}\,\hat{y}_{\text{in}} + \Delta\hat{q} - \kappa\hat{y} + \sqrt{\kappa\Gamma_q}\cos(\tilde{\Omega}t + \Phi)\hat{X}(t) \tag{4.41}$$

$$\frac{1}{\Omega_m}\left[\delta\ddot{\hat{X}}(t) + 2\gamma\delta\dot{\hat{X}}(t) + \Omega_m^2\delta\hat{X}(t)\right] = \hat{f}(t) + \tag{4.42}$$

$$\sqrt{\kappa\Gamma_q}\cos(\tilde{\Omega}t + \Phi)\hat{q} + \sqrt{\kappa\Gamma_y}\sin(\tilde{\Omega}t + \Phi)\,\hat{y} + \hat{f}(t)$$

Both the amplitude and phase quadrature of the light are driven by the mechanical oscillator, however they are driven $\frac{\pi}{2}$ out of phase and with different strengths, set by $\sqrt{\Gamma_q}$ and $\sqrt{\Gamma_y}$. Compared to the single-tone EOM Eq. (3.81), we see the indication of a much more intricate set of feedback loops, not only because there is a direct driving of $\hat{X}$ by both optical quadratures and vice verse, but also to the time dependence and relative phase between the two readout-rates[3] . We will return to the nature of these feedback mechanisms in Chapter 5.

Another interesting feature is the form of the optical force $\hat{f}_{\text{opt}}(t)$ driving the mechanical oscillator:

$$\hat{f}_{\text{opt}}(t) = \sqrt{\kappa\Gamma_q}\cos(\tilde{\Omega}t + \Phi)\hat{q} + \sqrt{\kappa\Gamma_y}\sin(\tilde{\Omega}t + \Phi)\,\hat{y},$$

is not simply a rotation of $\hat{q}$ and $\hat{y}$.

(3) If we wanted to we could have defined two time dependent read-out rates $\sqrt{\Gamma_q(t)} = \sqrt{\frac{4g_0^2}{\kappa}}\,q_{\text{cl}}\cos(\tilde{\Omega}t + \Phi)$ and $\sqrt{\Gamma_y(t)} = \sqrt{\frac{4g_0^2}{\kappa}}\,y_{\text{cl}}\sin(\tilde{\Omega}t + \Phi)$

CHAPTER 5

Engineering an effective oscillator in the narrow-sideband regime

“Y’know,’ he said, ‘it’s very hard to talk quantum using a language originally designed to tell other monkeys where the ripe fruit is.”

LU-TZE

- *Night Watch,* Terry Pratchett

In this chapter we use a time-domain approach, to examine how the two-tone-driven system described in the preceding chapter can be engineered to be have like a single-tone-driven system with negative mass and down-converted frequency. To this end, we focus on the narrow sidebands regime and try to derive a simpler set of equations, a toy model that captures the essence of the scheme. From such a set of simpler solutions one may, in future work, better understand how the complexity of the more general problem arise, such a the coupling of feedback loops arising from dynamical BA.

5.1 The General Mechanical and Optical Equations of Motion

5.1.1 The Intra-cavity Field Solution

Let us first consider the EOM for the intra-cavity photon field Eq. (4.39a). Since this is a first order differential equation, and $\hat{a}(t)$ doesn't explicitly couple to $\hat{a}^\dagger(t)$, we can write a formal solution by formal integration:

$$\hat{a}(t) = \hat{a}(t_0)e^{(i\Delta-\kappa)(t-t_0)} + \int_{t_0}^{t} \mathrm{d}t' e^{(i\Delta-\kappa)(t-t')} \Big[\sqrt{2\kappa}\,\hat{a}_{\mathrm{in}}(t') + i\frac{g_0}{\sqrt{2}}\Big[|\alpha_-|e^{i(\tilde{\Omega}t+\Phi)} + |\alpha_+|e^{-i(\tilde{\Omega}t+\Phi)}\Big]\Big(\hat{b}(t)+\hat{b}^\dagger(t)\Big)\Big] \tag{5.1}$$

As usual we will consider the system when it has reached steady state; when all transients have died out. This can be achieved saying the experiment started long ago which corresponds to taking $t_0 \to -\infty$:

$$\hat{a}(t) = \sqrt{2\kappa} \int_{-\infty}^{t} \mathrm{d}t' e^{(i\Delta-\kappa)(t-t')} \hat{a}_{\mathrm{in}}(t') \tag{5.2}$$

$$+ i\frac{g_0}{\sqrt{2}} \int_{-\infty}^{t} \mathrm{d}t' e^{(i\Delta-\kappa)(t-t')} \Big[|\alpha_-| e^{i(\tilde{\Omega}t'+\Phi)} \hat{b}(t') + |\alpha_-| e^{i(\tilde{\Omega}t'+\Phi)} \hat{b}^\dagger(t')$$
$$+ |\alpha_+| e^{-i(\tilde{\Omega}t'+\Phi)} \hat{b}(t') + \hat{b}^\dagger(t') |\alpha_+| e^{-i(\tilde{\Omega}t'+\Phi)} \Big] \tag{5.3}$$

This is only formally a solution, as it still depends on the mechanical operators, which themselves depend on the intra-cavity light.

5.1.2 Input-Output Relation and Mechanical sidebands

Before we move on with the solution of the mechanical and optical equations of motion, let us see what we can learn from the solution for the intra-cavity field we just found. The input-output relation Eq. (4.39a) for the cavity now becomes:

$$\hat{a}_{\mathrm{out}}(t) = \int_{-\infty}^{t} \mathrm{d}t' \left[2\kappa e^{(i\Delta-\kappa)(t-t')} - \delta(t-t') \right] \hat{a}_{\mathrm{in}}(t') + \tag{5.4}$$
$$i\frac{g_0}{\sqrt{2}} \int_{-\infty}^{t} \mathrm{d}t' e^{(i\Delta-\kappa)(t-t')} \Big[|\alpha_-| e^{i(\tilde{\Omega}t'+\Phi)} \hat{b}(t') + |\alpha_-| e^{i(\tilde{\Omega}t'+\Phi)} \hat{b}^\dagger(t')$$
$$+ |\alpha_+| e^{-i(\tilde{\Omega}t'+\Phi)} \hat{b}(t') + \hat{b}^\dagger(t') |\alpha_+| e^{-i(\tilde{\Omega}t'+\Phi)} \Big]$$

To better understand how the mechanical motion is imprinted on the light we make the following consideration: A free mechanical oscillator would simply oscillate at frequency Ω_m, however the interaction with the intra-cavity light will generally lead to an optical spring effect which will renormalize[(1)] the frequency to Ω_{eff}. Based on this we make the following Ansatz:

$$\hat{b}(t) = \hat{\tilde{b}}(t) e^{-i\Omega_{\mathrm{eff}} t}, \tag{5.5}$$

where the essence of the Ansatz is that $\hat{\tilde{b}}(t)$ is a slowly varying operator compared to $e^{-i\Omega_{\mathrm{eff}} t}$. We will show that this is indeed the case once we solve the mechanical EOM. It is also convenient to define the detuning between this effective mechanical frequency and half the relative drive frequency:

$$\Lambda_{\mathrm{eff}} = \Omega_{\mathrm{eff}} - \tilde{\Omega}, \tag{5.6}$$

which allows us to express the input-output relation Eq. (5.4) in terms of this slowly varying operator as follows:

$$\hat{a}_{\mathrm{out}}(t) = \int_{-\infty}^{t} \mathrm{d}t' \left[2\kappa e^{(i\Delta-\kappa)(t-t')} - 2\delta(t-t') \right] \hat{a}_{\mathrm{in}}(t') \tag{5.7}$$
$$+ i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}} \int_{-\infty}^{t} \mathrm{d}t' e^{(i\Delta-\kappa)(t-t')} \Big[|\alpha_-| e^{i\Phi} \hat{\tilde{b}}(t') e^{-i\Lambda_{\mathrm{eff}} t'} + |\alpha_-| e^{i\Phi} \hat{\tilde{b}}^\dagger(t') e^{i(\Lambda_{\mathrm{eff}}+2\tilde{\Omega})t'}$$
$$+ |\alpha_+| e^{-i\Phi} \hat{\tilde{b}}(t') e^{-i(\Lambda_{\mathrm{eff}}+2\tilde{\Omega})t'} + |\alpha_+| e^{-i\Phi} \hat{\tilde{b}}^\dagger(t') e^{i\Lambda_{\mathrm{eff}} t'} \Big]$$

(1) Generally the story is complicated by the fact that the renormalization may be frequency/time dependent and thus we have a frequency dependent "resonance" $\Omega_{\mathrm{eff}}(\Omega)$. In this chapter we will end up examining a regime where the renormalization is frequency independent and thus we need not worry about it. Moreover, for the renormalized system to be considered an oscillator, the parameters such as the resonance frequency must be constant, so in a sense the assumption that this is the case is necessary for us.

The interpretation of this expression is slightly clearer in frequency space, as the convolution integrals just become products. We note that the cavity susceptibility[(2)] is given by the Fourier transform of the cavity response function :

$$\chi_c(\Omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau e^{(i(\Omega+\Delta)-\kappa)\tau}\theta(\tau), \tag{5.8}$$

And that we also need to use:

$$\int_{0}^{\infty} \mathrm{d}\tau e^{i\Omega\tau}\delta(\tau) = \frac{1}{2} \tag{5.9}$$

Using the above relations we find the input-output relation is a simple algebraic expression in Fourier space:

$$\begin{aligned}\hat{a}_{\mathrm{out}}(\Omega) = e^{i2\theta_c(\Omega)}\hat{a}_{\mathrm{in}}(\Omega) + i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}\chi_c(\Omega)\Big[&e^{i\Phi}|\alpha_-|\hat{\tilde{b}}(\Omega-\Lambda_{\mathrm{eff}}) + e^{i\Phi}|\alpha_-|\hat{\tilde{b}}^\dagger(-\Omega-\Lambda_{\mathrm{eff}}-2\tilde{\Omega}) \\ &+ e^{-i\Phi}|\alpha_+|\hat{\tilde{b}}(\Omega-\Lambda_{\mathrm{eff}}-2\tilde{\Omega}) + e^{-i\Phi}|\alpha_+|\hat{\tilde{b}}^\dagger(-\Omega-\Lambda_{\mathrm{eff}})\Big]\end{aligned} \tag{5.10}$$

The term proportional to $\hat{a}_{\mathrm{in}}(\Omega)$ arises due to the interference of input light which is reflected from the cavity and the input light which was processed by the cavity. We note that is simply results in a cavity induced phase shift of the input noise, which follows from the rewriting (see Appendix D)

$$\underbrace{2\kappa\chi_c(\Omega)}_{\text{from cavity}} - \underbrace{1}_{\text{reflected}} = e^{i2\theta_c(\Omega)}$$

The mechanical signal in Eq. (5.10) is expressed as four sidebands[(3)]: two inner sidebands at frequencies $\pm\Lambda_{\mathrm{eff}}$ and two outer sidebands at frequencies $\pm\left(\Lambda_{\mathrm{eff}}+2\tilde{\Omega}\right)$ (see Fig. 5.1). Note that what is the upper and lower sideband of the inner sidebands depends on the sign of Λ_{eff}, so:

$$\begin{aligned}\Lambda_{\mathrm{eff}} > 0 \longrightarrow \hat{a}_{\mathrm{out}}(\Omega) &= e^{i2\theta_c(\Omega)}\hat{a}_{\mathrm{in}}(\Omega) + \\ &i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}\chi_c(\Omega)\left[e^{i\Phi}|\alpha_-|\underbrace{\hat{\tilde{b}}^\dagger(-\Omega-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})}_{\text{lower outer sideband}} + e^{-i\Phi}|\alpha_+|\underbrace{\hat{\tilde{b}}^\dagger(-\Omega-\Lambda_{\mathrm{eff}})}_{\text{lower inner sideband}}\right] \\ &+ i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}\chi_c(\Omega)\left[e^{i\Phi}|\alpha_-|\underbrace{\hat{\tilde{b}}(\Omega-\Lambda_{\mathrm{eff}})}_{\text{upper innner sideband}} + e^{-i\Phi}|\alpha_+|\underbrace{\hat{\tilde{b}}(\Omega-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})}_{\text{upper outer sideband}}\right]\end{aligned} \tag{5.11a}$$

$$\begin{aligned}\Lambda_{\mathrm{eff}} < 0 \longrightarrow \hat{a}_{\mathrm{out}}(\Omega) &= e^{i2\theta_c(\Omega)}\hat{a}_{\mathrm{in}}(\Omega) + \\ &i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}\chi_c(\Omega)\left[e^{i\Phi}|\alpha_-|\underbrace{\hat{\tilde{b}}^\dagger(-\Omega-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})}_{\text{lower outer sideband}} + e^{i\Phi}|\alpha_-|\underbrace{\hat{\tilde{b}}(\Omega-\Lambda_{\mathrm{eff}})}_{\text{lower innner sideband}}\right] \\ &+ i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}\chi_c(\Omega)\left[e^{-i\Phi}|\alpha_+|\underbrace{\hat{\tilde{b}}^\dagger(-\Omega-\Lambda_{\mathrm{eff}})}_{\text{upper inner sideband}} + e^{-i\Phi}|\alpha_+|\underbrace{\hat{\tilde{b}}(\Omega-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})}_{\text{upper outer sideband}}\right]\end{aligned} \tag{5.11b}$$

(2) Here $\tau = t - t'$ appears due to the $t - t'$ dependence of the cavity response function and rewriting the integrals using $\int_{-\infty}^{t}\mathrm{d}t' = \int_{-\infty}^{\infty}\mathrm{d}t'\theta(t-t')$.

(3) Generally these "sidebands" may be broadened and moved around frequency space by dynamical BA, and so it is not guaranteed that the sidebands are narrow and well separated enough to identify as individual sidebands in the actual spectrum, however the mechanical dynamics in that limit are no longer oscillator like, and thus beyond our interest.

We see that for $\Lambda_{\text{eff}} > 0$ the inner sidebands have the same structure as for a single-tone driven optomechanical cavity driven at frequency ω_o with mechanical frequency Λ_{eff}: the upper sidebands is due to $\hat{b}$ while the lower sidebands is due to $\hat{b}^{\dagger}$. For $\Lambda_{\text{eff}} < 0$ the spectrum is radically different, the role of $\hat{b}$ and $\hat{b}^{\dagger}$ are interchanged, and we instead have a mechanical oscillator with a negative frequency Λ_{eff}. We will see exactly these conclusions hold true once we actually solve the mechanical EOM.

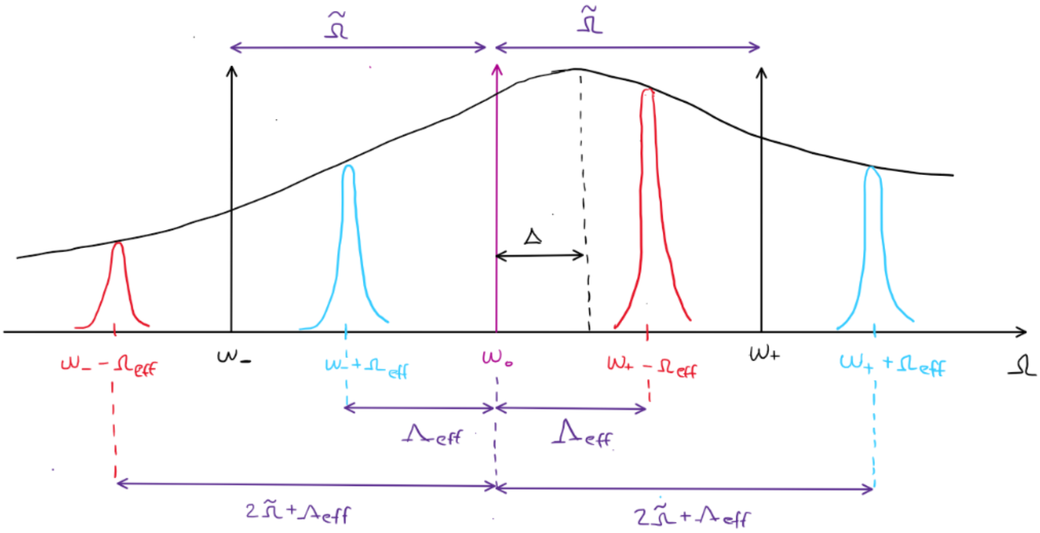

Figure 5.1: Sketch of the heterodyne spectrum from a two-tone optomechanical system, in the narrow sideband regime.

5.1.3 Eliminating the Intracavity Field

Let us now turn our attention to the mechanical EOM Eq. (4.39b). Following the approach we used to analyze the mechanical signal in the light, we use the ansatz Eq. (5.5) and wish to find an EOM for the slowly varying operator $\hat{\hat{b}}(t)$:

$$\dot{\hat{b}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\hat{b}}(t)e^{-i\Omega_{\text{eff}}t}\right) \Leftrightarrow, \tag{5.12}$$

$$\dot{\hat{\hat{b}}}(t) = i\Omega_{\text{eff}}\hat{\hat{b}} + e^{i\Omega_{\text{eff}}t}\dot{\hat{b}}(t) \tag{5.13}$$

It then follows from the mechanical EOM Eq. (4.39b) that the slowly varying operator $\hat{\hat{b}}$ is the solution to:

$$\begin{aligned}\dot{\hat{\hat{b}}}(t) = &-i\Delta_{\text{eff}}\hat{\hat{b}}(t) - \gamma\left(\hat{\hat{b}}(t) - \hat{\hat{b}}^{\dagger}(t)e^{i2\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)}\right) + i\frac{1}{\sqrt{2}}e^{i\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t}\hat{f}(t)+ \\ &+ i\frac{g_0}{\sqrt{2}}\left[\left(e^{i\Phi}|\alpha_-|e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t} + e^{-i\Phi}|\alpha_+|e^{i\Lambda_{\text{eff}}t}\right)\hat{a}^{\dagger}(t)\right] \\ &+ i\frac{g_0}{\sqrt{2}}\left(e^{-i\Phi}|\alpha_-|e^{i\Lambda_{\text{eff}}t} + e^{i\Phi}|\alpha_+|e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t}\right)\hat{a}(t),\end{aligned} \tag{5.14}$$

where we have defined the detuning between the renormalized mechanical resonance frequency Ω_{eff} and the bare mechanical resonance Ω_m:

$$\Delta_{\text{eff}} = \Omega_m - \Omega_{\text{eff}} \tag{5.15}$$

For weak read-out this detuning will simply be minus the optical spring shift induced by the dynamical BA[(4)], via the terms depending on $\hat{a}$ and $\hat{a}^\dagger$, we will see this later. If we insert the solution Eq. (5.2) for the intra-cavity field we find that the Mechanical EOM takes the somewhat lengthy form:

$$
\begin{aligned}
\dot{\hat{\tilde{b}}}(t) = {} & -i\Delta_{\text{eff}}\hat{\tilde{b}}(t) - \gamma\left(\hat{\tilde{b}}(t) - \hat{\tilde{b}}^\dagger(t)e^{i2\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)}\right) + i\frac{1}{\sqrt{2}}e^{i\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t}\hat{f}(t) \\
& + \frac{g_0^2}{2}\int_{-\infty}^{t}\mathrm{d}t'\,\Big[e^{-\left(i\left(\Delta-\Lambda_{\text{eff}}-2\tilde{\Omega}\right)+\kappa\right)(t-t')}\left(|\alpha_-|^2 + |\alpha_+||\alpha_-|e^{i2\Phi}e^{i2\tilde{\Omega}t'}\right) \\
& + e^{-(i(\Delta-\Lambda_{\text{eff}})+\kappa)(t-t')}\left(|\alpha_-||\alpha_+|e^{-i2\Phi}e^{-i2\tilde{\Omega}t'} + |\alpha_+|^2\right) \\
& - e^{(i(\Delta+\Lambda_{\text{eff}})-\kappa)(t-t')}\left(|\alpha_-|^2 + |\alpha_+||\alpha_-|e^{-i2\Phi}e^{-i2\tilde{\Omega}t'}\right) \\
& - e^{\left(i\left(\Delta+\Lambda_{\text{eff}}+2\tilde{\Omega}\right)-\kappa\right)(t-t')}\left(|\alpha_-||\alpha_+|e^{i2\Phi}e^{i2\tilde{\Omega}t'} + |\alpha_+|^2\right)\Big]\hat{\tilde{b}}(t') \\
& + \frac{g_0^2}{2}\int_{-\infty}^{t}\mathrm{d}t'\,\Big[e^{-(i(\Delta-\Lambda_{\text{eff}})+\kappa)(t-t')}\left(|\alpha_-||\alpha_+|e^{-i2\Phi}e^{i2\Lambda_{\text{eff}}t'} + |\alpha_+|^2e^{i2\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t'}\right) \\
& + e^{-\left(i\left(\Delta-\Lambda_{\text{eff}}-2\tilde{\Omega}\right)+\kappa\right)(t-t')}\left(|\alpha_-|^2e^{i2\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t'} + |\alpha_+||\alpha_-|e^{i2\Phi}e^{i2\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t'}\right) \\
& - e^{(i(\Delta+\Lambda_{\text{eff}})-\kappa)(t-t')}\left(|\alpha_-|^2e^{i2\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t'} + |\alpha_+||\alpha_-|e^{-i2\Phi}e^{i2\Lambda_{\text{eff}}t'}\right) \\
& - e^{\left(i\left(\Delta+\Lambda_{\text{eff}}+2\tilde{\Omega}\right)-\kappa\right)(t-t')}\left(|\alpha_-||\alpha_+|e^{i2\Phi}e^{i2\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t'} + |\alpha_+|^2e^{i2\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t'}\right)\Big]\hat{\tilde{b}}^\dagger(t') \\
& + \hat{a}_{\text{nom}}(t) + \hat{a}_{\text{extra}}(t),
\end{aligned}
\tag{5.16}
$$

where we have defined the nominal quantum back-action:

$$
\begin{aligned}
\hat{a}_{\text{nom}}(t) = {} & i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}|\alpha_-|e^{-i\Phi}\int_{-\infty}^{t}\mathrm{d}t'\,e^{(i(\Delta+\Lambda_{\text{eff}})-\kappa)(t-t')}\hat{a}_{\text{in}}(t')e^{i\Lambda_{\text{eff}}t'} \\
& + i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{-i\Phi}|\alpha_+|\int_{-\infty}^{t}\mathrm{d}t'\,e^{-(i(\Delta-\Lambda_{\text{eff}})+\kappa)(t-t')}\hat{a}_{\text{in}}^\dagger(t')e^{i\Lambda_{\text{eff}}t'}
\end{aligned}
\tag{5.17}
$$

By nominal, we mean that this is the QBA we should expect for the effective single-tone driven oscillator, with resonance frequency Λ_{eff}, corresponding to the coupling to the inner sideband frequencies. We have also define thed extraneous QBA:

$$
\hat{a}_{\text{extra}}(t) = i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{i\Phi}|\alpha_+|\int_{-\infty}^{t}\mathrm{d}t'\,e^{\left(i\left(\Delta+\Lambda_{\text{eff}}+2\tilde{\Omega}\right)-\kappa\right)(t-t')}\hat{a}_{\text{in}}(t')e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t'} \tag{5.18}
$$

$$
+ i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{i\Phi}|\alpha_-|\int_{-\infty}^{t}\mathrm{d}t'\,e^{-\left(i\left(\Delta-\Lambda_{\text{eff}}-2\tilde{\Omega}\right)+\kappa\right)(t-t')}\hat{a}_{\text{in}}^\dagger(t')e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t'}, \tag{5.19}
$$

which extraneous in the sense that it is QBA acting on the effective oscillator the outer sideband frequencies that we wish to eliminate in engineering the effective oscillator.

The EOM Eq. (5.16) is not particularly practical for solving, we can extract some important physical insights: We first and foremost see that there are several contributions from dynamical BA, namely the second to fourth line. These terms show the presence of several dynamical BA loops. Firstly there are terms proportional to $|\alpha_\pm|^2\hat{\tilde{b}}(t')$. Such terms would also arise in single-tone optomechanics; they are exactly the usual dynamical BA associated with the individual drive tones. However there are several terms proportional to $|\alpha_\pm||\alpha_\mp|e^{\pm 2i\Phi}\hat{\tilde{b}}(t')$; these terms

(4) Generally the renormalized frequency isn't simply $\Omega_{\text{eff}} \neq \Omega_m + \delta\Omega_m^{(\text{opt})}$, where $\delta\Omega_m^{(\text{opt})}$ is the optical spring shift, but nothing stops us from defining a detuning Δ_{eff}, we just have to be careful how we interpret it.

are unique to two-tone driving, and in particular they are are due to dynamical BA from both drive tones combining in a non-trivial way. Moreover, the coupling to $\hat{\hat{b}}^\dagger$ complicates the problem significantly, as we can not simply solve the EOM for $\hat{\hat{b}}$ must include the Hermitian conjugate equation, the reason being that different sidebands overlap leading to yet more BA. In general the non-trivial dynamical BA arises from overlapping sidebands, and coupling to other sidebands. Based on this we expect simpler dynamics in the limit where all sidebands are well separated. In addition to the dynamical BA terms which depend on $\hat{b}$ and $\hat{\hat{b}}^\dagger$, there are also QBA terms, we will return to those in the next section.

5.2 Obtaining a Simple Description

5.2.1 Defining the Limits of Interest

We are now ready to pursue a simpler limit, to better understand the admittedly complicated EOM which generally arise due to the intricate dynamical BA. As we previously discussed there are two types of dynamical BA which complicates the dynamics: dynamical BA which involves combinations of the feedback loops arising from both the drives and BA arising from the overlap of sidebands. To allow for a simple description we thus want the sidebands to be well-separated, and we also want the sidebands to be narrow such that their dynamics only concern a local range of frequencies. But how do we characterize such a limit? Let us first think about the problem for resonant driving $\Delta = 0$. In this case the distance between the sidebands of a given tone, i.e. the separation of the outer and inner sidebands from one tone, is simply $2\tilde{\Omega}$. However the separation of the two inner sidebands is 2Λ. We will now introduce a number of assumptions on the parameters that ensures that the theory derived here is meaningful: the mechanical linewidth must be small compared to the differences in sidebands resonances. In particular we will assume that the mechanical oscillator has a high Q factor:

$$\gamma \ll \Omega_m \quad \text{(high Q)} \tag{5.20}$$

First and for most, we must be in the limit where the dynamical BA indeed only leads to dynamical broadening and the optical spring effect. As we shall see this requires that the driving be weak compared to the cavity linewidth, as this ensures that the renormalization is frequency independent:

$$\Gamma_\pm \ll \kappa \quad \text{(weak read-out)} \tag{5.21}$$

Next we need to ensure that the dynamical BA does not broaden the sidebands enough nor shift the resonances enough to overlap the sidebands. The magnitude of the renormalizations is set by the read-out rate, and so we must require that it is small compared to the sideband separation:

$$\Gamma_\pm \ll 2\Lambda, 2\tilde{\Omega} \quad \text{(seperated sidebands)} \tag{5.22}$$

The weak read-out and narrow-sideband conditions are in fact related to the same assumption: We assume that the intra-cavity field equilibrates much faster than the evolution of the slowly varying mechanical oscillator, i.e. we assume an adiabatic read-out. By far the largest limitation here is assuming that $\Gamma_\pm \ll 2\Lambda$, which essentially states that the read.out must be slow compared to the effective oscillator, however it is the assumptions which eventually allows us to completely eliminate the coupling to $\hat{\hat{b}}^\dagger$. For the bare sidebands to be narrow enough that the cavity can't resolve their frequency dependence we must have:

$$\kappa \gg \gamma \quad \text{(narrow sidebands)}$$

Together with the weak read-out assumption this generally ensures that the cavity susceptibility is constant over any given sideband, however the cavity can still in principle vary with frequency over several sidebands. This is in fact a crucial point: we have not made any assumptions on κ compared to the sideband resonances(5), and thus we are not limited to neither the sideband resolved nor sidebands unresolved regimes; our theory works in both regimes.

5.2.2 Mechanical EOM in the Narrow Sidebands regime

Let us now turn to the practical consequences of these assumptions in the mechanical EOM Eq. (5.16). First and foremost the large sideband separation allows us to employ a RWA and neglect mechanical terms oscillation like $e^{\pm i2\tilde{\Omega}t'}$, $e^{\pm i4\tilde{\Omega}t'}$ and $e^{\pm i2\Lambda_{\text{eff}}t}$ as all of these are fast compared to $e^{\pm i\Lambda_{\text{eff}}t}$. This eliminates not only all dynamical BA terms which depends on $|\alpha_{\pm}||\alpha_{\mp}|$, but crucially also the coupling to $\hat{\tilde{b}}^{\dagger}$:

$$\begin{aligned}\dot{\hat{\tilde{b}}}(t) \approx &-(i\Delta_{\text{eff}}+\gamma)\,\hat{\tilde{b}}(t) + i\frac{1}{\sqrt{2}}e^{i\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t}\hat{f}(t) \\ &-\frac{g_0^2|\alpha_-|^2}{2}\int_{-\infty}^{t}\mathrm{d}t'\,e^{(i(\Delta+\Lambda_{\text{eff}})-\kappa)(t-t')}\hat{\tilde{b}}(t') \\ &+\frac{g_0^2|\alpha_+|^2}{2}\int_{-\infty}^{t}\mathrm{d}t'\,e^{-(i(\Delta-\Lambda_{\text{eff}})+\kappa)(t-t')}\hat{\tilde{b}}(t') \\ &-\frac{g_0^2|\alpha_+|^2}{2}\int_{-\infty}^{t}\mathrm{d}t'\,e^{\left(i\left(\Delta+\Lambda_{\text{eff}}+2\tilde{\Omega}\right)-\kappa\right)(t-t')}\hat{\tilde{b}}(t')+ \\ &+\frac{g_0^2|\alpha_-|^2}{2}\int_{-\infty}^{t}\mathrm{d}t'\,e^{-\left(i\left(\Delta-\Lambda_{\text{eff}}-2\tilde{\Omega}\right)+\kappa\right)(t-t')}\hat{\tilde{b}}(t') \\ &+\hat{a}_{\text{nom}}(t)+\hat{a}_{\text{extra}}(t)\end{aligned} \tag{5.23}$$

By employing the narrow sideband approximation we can neglect the frequency dependence of the cavity susceptibility near the four sidebands. To see how this is done consider for example the dynamical BA contribution via the sideband at Λ_{eff} as seen in thes second line of Eq. (5.16). By transforming to Fourier space, evaluating the cavity susceptibility at the sideband resonance, and then transforming back we get:

$$\frac{g_0^2|\alpha_-|^2}{2}\int_{-\infty}^{t}\mathrm{d}t'\,e^{(i(\Delta+\Lambda_{\text{eff}})-\kappa)(t-t')}\hat{\tilde{b}}(t') = \frac{g_0^2|\alpha_-|^2}{2}\int_{-\infty}^{\infty}\mathrm{d}\Omega\, e^{-i\Omega t}\chi_c(\Omega+\Lambda_{\text{eff}})\hat{\tilde{b}}(\Omega) \tag{5.24}$$

$$\approx \frac{g_0^2|\alpha_-|^2}{2}\chi_c(\Lambda_{\text{eff}})\hat{\tilde{b}}(t) \tag{5.25}$$

For the remaining three sidebands the narrow sideband approximations of the mechanical terms are:

(5) The sideband resonances appear in many expression as χ_c is evaluated at them. See e.g. the definition of the four sideband operators in Eq. (5.16)

$$-\frac{g_0^2|\alpha_+|^2}{2}\int_{-\infty}^{t}\mathrm{d}t' e^{-(i(\Delta-\Lambda_{\rm eff})+\kappa)(t-t')}\hat{\tilde{b}}(t') \approx -\frac{g_0^2|\alpha_+|^2}{2}\chi_c^*(-\Lambda_{\rm eff})\hat{\tilde{b}}(t') \tag{5.26}$$

$$\frac{g_0^2|\alpha_+|^2}{2}\int_{-\infty}^{t}\mathrm{d}t' e^{\left(i\left(\Delta+\Lambda_{\rm eff}+2\tilde{\Omega}\right)-\kappa\right)(t-t')}\hat{\tilde{b}}(t') \approx \tag{5.27}$$
$$\frac{g_0^2|\alpha_+|^2}{2}\chi_c(\Lambda_{\rm eff}+2\tilde{\Omega})\hat{\tilde{b}}(t')$$

$$-\frac{g_0^2|\alpha_-|^2}{2}\int_{-\infty}^{t}\mathrm{d}t' e^{-\left(i\left(\Delta-\Lambda_{\rm eff}-2\tilde{\Omega}\right)+\kappa\right)(t-t')}\hat{\tilde{b}}(t') \approx \tag{5.28}$$
$$-\frac{g_0^2|\alpha_-|^2}{2}\chi_c^*(-\Lambda_{\rm eff}-2\tilde{\Omega})\hat{\tilde{b}}(t')$$

We can likewise approximate the the nominal QBA Eq. (5.17) and the extraneous QBA Eq. (5.18):

$$\int_{-\infty}^{t}\mathrm{d}t' e^{(i(\Delta+\Lambda_{\rm eff})-\kappa)(t-t')}\hat{a}_{\rm in}(t')e^{i\Lambda_{\rm eff}t'} \approx \chi_c(\Lambda_{\rm eff})\hat{a}_{\rm in}(t)e^{i\Lambda_{\rm eff}t} \tag{5.29}$$

$$\int_{-\infty}^{t}\mathrm{d}t' e^{-(i(\Delta-\Lambda_{\rm eff})+\kappa)(t-t')}\hat{a}_{\rm in}^\dagger(t')e^{i\Lambda_{\rm eff}t'} \approx \chi_c^*(-\Lambda_{\rm eff})\hat{a}_{\rm in}^\dagger(t')e^{i\Lambda_{\rm eff}t} \tag{5.30}$$

$$\int_{-\infty}^{t}\mathrm{d}t' e^{\left(i\left(\Delta+\Lambda_{\rm eff}+2\tilde{\Omega}\right)-\kappa\right)(t-t')}\hat{a}_{\rm in}(t')e^{i\left(\Lambda_{\rm eff}+2\tilde{\Omega}\right)t'} \approx \tag{5.31}$$
$$\chi_c(\Lambda_{\rm eff}+2\tilde{\Omega})\hat{a}_{\rm in}(t)e^{i\left(\Lambda_{\rm eff}+2\tilde{\Omega}\right)t}$$

$$\int_{-\infty}^{t}\mathrm{d}t' e^{-\left(i\left(\Delta-\Lambda_{\rm eff}-2\tilde{\Omega}\right)+\kappa\right)(t-t')}\hat{a}_{\rm in}^\dagger(t')e^{i\left(\Lambda_{\rm eff}+2\tilde{\Omega}\right)t'} \approx \tag{5.32}$$
$$\chi_c^*(-\Lambda_{\rm eff}-2\tilde{\Omega})\hat{a}_{\rm in}^\dagger(t)e^{i\left(\Lambda_{\rm eff}+2\tilde{\Omega}\right)t}$$

So the nominal noise simplifies to:

$$\begin{aligned}\hat{a}_{\rm nom}(t) = {} & i\frac{\kappa}{2}\sqrt{\Gamma_+}\,e^{-i\Phi}\chi_c(\Lambda_{\rm eff})\hat{a}_{\rm in}(t)e^{i\Lambda_{\rm eff}t} \\ & + i\frac{\kappa}{2}\sqrt{\Gamma_-}\,e^{-i\Phi}|\alpha_+|\chi_c^*(-\Lambda_{\rm eff})\hat{a}_{\rm in}^\dagger(t)e^{i\Lambda_{\rm eff}t},\end{aligned} \tag{5.33}$$

and likewise the extraneous noise can now be written:

$$\begin{aligned}\hat{a}_{\rm extra}(t) = {} & i\frac{\kappa}{2}\sqrt{\Gamma_+}\,e^{i\Phi}\chi_c(\Lambda_{\rm eff}+2\tilde{\Omega})\hat{a}_{\rm in}(t)e^{i\left(\Lambda_{\rm eff}+2\tilde{\Omega}\right)t} \\ & + i\frac{\kappa}{2}\sqrt{\Gamma_-}\,e^{i\Phi}\chi_c^*(-\Lambda_{\rm eff}-2\tilde{\Omega})\hat{a}_{\rm in}^\dagger(t)e^{i\left(\Lambda_{\rm eff}+2\tilde{\Omega}\right)t},\end{aligned} \tag{5.34}$$

where we now have introduced a readout rate for each drive-tone, defined by:

$$\Gamma_\pm = \frac{4g_0^2|\alpha_\pm|^2}{\kappa} \tag{5.35}$$

We then have a simple first order differential equation for the slowly oscillating mechanical operator:

$$\dot{\hat{\tilde{b}}}(t) \approx -\left(i\Delta_{\rm eff}+\gamma+i\Sigma\right)\hat{\tilde{b}}(t) + i\frac{1}{\sqrt{2}}e^{i\left(\Lambda_{\rm eff}+\tilde{\Omega}\right)t}\hat{f}(t) + \hat{a}_{\rm nom}(t) + \hat{a}_{\rm extra}(t), \tag{5.36}$$

where we have defined the self energy resulting from the dynamical BA

$$\begin{aligned}\Sigma = {} & \frac{\kappa\Gamma_-}{8i}\left[\chi_c(\Lambda_{\rm eff}) - \chi_c^*(-\Lambda_{\rm eff}-2\tilde{\Omega})\right] \\ & + \frac{\kappa\Gamma_+}{8i}\left[\chi_c(\Lambda_{\rm eff}+2\tilde{\Omega}) - \chi_c^*(-\Lambda_{\rm eff})\right]\end{aligned} \tag{5.37}$$

Before we solve the EOM let us first examine the self energy further.

5.2.3 The Mechanical Self Energy

The first thing we should note is that the self energy is merely the sum of the self energy associated with the dynamical BA due to the upper and lower drive tone:

$$\Sigma = \Sigma_+ + \Sigma_-, \tag{5.38}$$
$$\Sigma_+ = \frac{\kappa\Gamma_-}{8i}\left[\chi_c(\Lambda_{\text{eff}}) - \chi_c^*(-\Lambda_{\text{eff}} - 2\tilde{\Omega})\right], \tag{5.39}$$
$$\Sigma_- = \frac{\kappa\Gamma_+}{8i}\left[\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega}) - \chi_c^*(-\Lambda_{\text{eff}})\right], \tag{5.40}$$

as we can see by comparing with the single tone self energy(6) Eq. (3.104). Consequently the shift in the resonance $\delta\Omega_m^{(\text{opt})}$ (optical spring effect) and the dynamical broadening $\delta\gamma^{(\text{opt})}$is just the sum of the contributions from each drive tone. The optical spring shift is given by:

$$\delta\Omega_m^{(\text{opt})} = \text{Re}\left[\Sigma\right] \tag{5.41}$$
$$\approx \frac{\Gamma_-}{8}\left[(\Lambda + \Delta)\,\kappa|\chi_c(\Lambda)|^2 - \left(\Lambda + 2\tilde{\Omega} - \Delta\right)\kappa|\chi_c(-\Lambda - 2\tilde{\Omega})|^2\right] \tag{5.42}$$
$$+ \frac{\Gamma_+}{8}\left[\left(\Lambda + 2\tilde{\Omega} + \Delta\right)\kappa|\chi_c(\Lambda + 2\tilde{\Omega})|^2 - (\Lambda - \Delta)\,\kappa|\chi_c(-\Lambda)|^2\right]$$

While the dynamical broadening is:

$$\delta\gamma^{(\text{opt})} = -\text{Im}\left[\Sigma\right] \tag{5.43}$$
$$\approx \frac{\Gamma_-}{8}\left[\kappa^2|\chi_c(\Lambda)|^2 - \kappa^2|\chi_c(-\Lambda - 2\tilde{\Omega})|^2\right] \tag{5.44}$$
$$+ \frac{\Gamma_+}{8}\left[\kappa^2|\chi_c(\Lambda + 2\tilde{\Omega})|^2 - \kappa^2|\chi_c(-\Lambda)|^2\right]$$

Here we use the weak read-out assumption to justify that we evaluate the self energy at the bare resonance Ω_m, or more precisely we evaluate it at the bare detuning $\Lambda = \Omega_m - \tilde{\Omega}$.

We can now make the following observations. First and foremost we see that the magnitude of resonance shift $\delta\Omega_m^{(\text{opt})}$and the dynamical broadening $\delta\gamma^{(\text{opt})}$ are both set by the read-out rates $\Gamma_\pm$, as we expected. Another rather important observation is that we can have dynamical broadening or anti-broadening depending on the relative strengths of the sidebands, which depends on the detuning and the read-out rates. If the bare oscillator has a narrow intrinsic linwidth γ, and the dynamical anti-broadening $\delta\gamma^{(\text{opt})}$ is larger than the instrinscic linwidth, i.e. $\gamma < |\delta\gamma^{(\text{opt})}|$ the total linewidth will be negative and thus we have a completely anti-damped oscillator; at such a negative-linewidth instability our theory breaks down.

Another interesting observation is the fact that there is dynamical BA even for $\Delta = 0$, in contrast to the single-tone case:

$$\delta\Omega_m^{(\text{opt})}\Big|_{\Delta=0} = \frac{\Gamma_- - \Gamma_+}{8}\left[\Lambda\kappa|\chi_c(\Lambda)|^2 - \left(\Lambda + 2\tilde{\Omega}\right)\kappa|\chi_c(\Lambda + 2\tilde{\Omega})|^2\right] \tag{5.45}$$
$$\delta\gamma^{(\text{opt})}\Big|_{\Delta=0} = \frac{\Gamma_- - \Gamma_+}{8}\left[\kappa^2|\chi_c(\Lambda)|^2 - \kappa^2|\chi_c(\Lambda + 2\tilde{\Omega})|^2\right] \tag{5.46}$$

(6) The factor 2 difference in the prefactor of the single tone self energy Eq. (3.104) and the self energy Σ_+ from the upper drive tone and the self energy Σ_- , is simply a matter of convention. The choice of convention is related whether we define the self energy for the phonon annihilation operator or the self energy for the phonon quadrature.

Note that for zero detuning[7] the susceptibility at the inner sidebands is strictly larger than the susceptibility at the outer sidebands, i.e. $|\chi_c(\Lambda)|^2 > |\chi_c(\Lambda + 2\tilde{\Omega})|^2$, as the resonance[8] of the cavity susceptibility is at $\Omega = 0$. Thus we see that for $\Delta = 0$, we have anti-broadening for $\Gamma_+ > \Gamma_-$, as the upper drive tone causes net anti-broadening. This does of course not necessarily mean that the system is unstable, as long as the net linewidth is positive i.e. $\gamma^{(\text{opt})} = \delta\gamma^{(\text{opt})} + \gamma > 0$. Likewise we have broadening for $\Gamma_+ < \Gamma_-$.

5.2.4 Solving the Mehcanical EOM in the Narrow Sideband Regime

We will now proceed to solving the mechanical EOM Eq. (5.36). We start by noting that the renomalized frequency and linewidth are:

$$\Omega_{\text{eff}} = \Omega_m + \delta\Omega_m^{(\text{opt})}, \tag{5.47}$$

$$\gamma^{(\text{opt})} = \gamma + \delta\gamma^{(\text{opt})}, \tag{5.48}$$

and so by using the definition of $\Lambda_{\text{eff}}$from Eq. (5.15) we see that:

$$\Delta_{\text{eff}} = -\delta\Omega_m^{(\text{opt})}, \tag{5.49}$$

and thus the EOM simplifies to:

$$\dot{\hat{\tilde{b}}}(t) \approx -\gamma^{(\text{opt})}\hat{\tilde{b}}(t) + i\frac{1}{\sqrt{2}}e^{i(\Lambda_{\text{eff}}+\tilde{\Omega})t}\hat{f}(t) + \hat{a}_{\text{nom}}(t) + \hat{a}_{\text{extra}}(t) \tag{5.50}$$

It should be no supprise that there is no oscillation as this was done per construction by choosing a frame rotating at the oscillation frequency Ω_{eff} of the renormalized oscillator. By direct integration we obtain the steady state solution:

$$\hat{\tilde{b}}(t) = \int_{-\infty}^{t} \mathrm{d}t' e^{-\gamma^{(\text{opt})}(t-t')}\left[i\frac{1}{\sqrt{2}}e^{i(\Lambda_{\text{eff}}+\tilde{\Omega})t'}\hat{f}(t') + \hat{a}_{\text{nom}}(t') + \hat{a}_{\text{extra}}(t')\right] \tag{5.51}$$

5.3 Input-Output Relation and Sideband Operators

We now have the solution for the mechanical degree of freedom, and so we now continue the discussion of the output light from the cavity. By virtue of the regime we are working in, the four sidebands can be described as separate degrees of freedom. We define a set of four input and output operators, two sidebands operators corresponding to the outermost sidebands, and two corresponding to the innermost sidebands:

$$\hat{a}^{(\text{outer})}_{\text{in/out},\pm}(t) = \hat{a}_{\text{in/out}}(t)e^{\pm i(\Lambda_{\text{eff}}+2\tilde{\Omega})} \tag{5.52}$$

$$\hat{a}^{(\text{inner})}_{\text{in/out},\pm}(t) = \hat{a}_{\text{in/out}}(t)e^{\mp i\Lambda_{\text{eff}}} \tag{5.53}$$

We use '+' to label the upper and '-' to label the lower sideband. Note that four the inner sidebands we have chosen notation which is sensible for $\Lambda_{\text{eff}} < 0$, which

(7) For equal read-out rates but non-zero detunning the dynamical BA remains non-zero, however there are no particular simplifications of the expressions for the optical spring shift $\delta\Omega_m^{(\text{opt})}$ and the dynamical broadening $\delta\gamma^{(\text{opt})}$ in this limit, so we will not learn anything new from writing the expressions for $\Gamma_+ = \Gamma_- = \Gamma$. The reason we see now simplification is that we already eliminated the most complicated behavior arising from different drive strengths when we neglected the fast oscillating terms, as these where the ones depending on $|\alpha_\pm||\alpha_\mp|$

(8) Remember that we are in a frame rotating at ω_o, but for $\Delta = 0$ this is the same as being in a frame rotating a the cavity resonance ω_c

is the case we are most interested in[(9)]. The sidebands operators are independent degrees of freedom in the sense that they are separate optical modes and satisfy the canonical algebra:

$$\left[\hat{a}^{(\text{inner/outer})}_{\text{in/out},n}(t), \hat{a}^{(\text{inner/outer})}_{\text{in/out},m}(t')\right] = 0, \quad n, m \in \{+, -\}$$
$$\left[\hat{a}^{(\text{inner/outer})}_{\text{in,/n/out},n}(t), \hat{a}^{(\text{inner/outer})}_{\text{in/out},m}(t')\right] = \delta_{n,m}\delta(t - t')$$

Using these sideband operators and the input-output relation Eq. (5.15), we can explicitly see how the mechanical motion is imprinted in four separate components of the light[(10)]:

$$\hat{a}^{(\text{inner})}_{\text{out},-}(t) = e^{i2\theta_c(\Lambda_{\text{eff}})}\hat{a}^{(\text{inner})}_{\text{in},-}(t) + i\sqrt{\Gamma_-}\frac{\kappa\chi_c(\Lambda_{\text{eff}})}{2}e^{i\Phi}\hat{\tilde{b}}(t) \tag{5.54a}$$

$$\hat{a}^{(\text{inner})}_{\text{out},+}(t) = e^{i2\theta_c(-\Lambda_{\text{eff}})}\hat{a}^{(\text{inner})}_{\text{in},+}(t) + i\sqrt{\Gamma_+}\frac{\kappa\chi_c(-\Lambda_{\text{eff}})}{2}e^{-i\Phi}\hat{\tilde{b}}^\dagger(t) \tag{5.54b}$$

$$\hat{a}^{(\text{outer})}_{\text{out},+}(t) = e^{i2\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})}\hat{a}^{(\text{outer})}_{\text{in},+}(t) + i\sqrt{\Gamma_+}\frac{\kappa\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega})}{2}e^{-i\Phi}\hat{\tilde{b}} \tag{5.54c}$$

$$\hat{a}^{(\text{outer})}_{\text{out},-}(t) = e^{i2\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})}\hat{a}^{(\text{outer})}_{\text{in},-}(t) + i\sqrt{\Gamma_-}\frac{\kappa\chi_c(-\Lambda_{\text{eff}} - 2\tilde{\Omega})}{2}e^{i\Phi}\hat{\tilde{b}}^\dagger(t) \tag{5.54d}$$

We can likewise express the nominal QBA Eq. (5.33) and Eq. (5.34) QBA terms from the equation for $\hat{\tilde{b}}$:

$$\begin{aligned}\hat{a}_{\text{nom}}(t) = {} & ie^{-i\Phi}\frac{\kappa}{2}\sqrt{\Gamma_-}\,\chi_c^*(-\Lambda_{\text{eff}})\hat{a}^{(\text{inner})}_{\text{in}-}(t) \\ & + ie^{-i\Phi}\frac{\kappa}{2}\chi_c(\Lambda_{\text{eff}})\sqrt{\Gamma_+}\,\hat{a}^{(\text{inner})\dagger}_{\text{in}+}(t)\end{aligned} \tag{5.55}$$

$$\begin{aligned}\hat{a}_{\text{extra}}(t) = {} & ie^{i\Phi}\frac{\kappa}{2}\sqrt{\Gamma_-}\,\chi_c^*(-\Lambda_{\text{eff}} - 2\tilde{\Omega})\hat{a}^{(\text{outer})\dagger}_{\text{in}-}(t) \\ & + ie^{i\Phi}\frac{\kappa}{2}\sqrt{\Gamma_+}\,\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega})\hat{a}^{(\text{outer})}_{\text{in}+}(t)\end{aligned} \tag{5.56}$$

We can now note that the extraneous noise is reminiscent of the non-Hermitian quadratures Eq. (2.21) we discussed in relation to homodyne measurement in Eq. (5.34), but with the important distinction that the two sideband operators not are equally weighted. If we consider the commutator:

$$\begin{aligned}&\left[\hat{a}_{\text{extra}}(t), \hat{a}^\dagger_{\text{extra}}(t')\right] = \\ &-\frac{\kappa^2}{4}\left(\Gamma_+|\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega})|^2 - \Gamma_-|\chi_c(-\Lambda_{\text{eff}} - 2\tilde{\Omega})|^2\right)\delta(t - t'),\end{aligned} \tag{5.57}$$

we see that is does not vanish, so $\hat{a}_{\text{extra}}(t)$ is indeed not a non-Hermitian quadrature measurable by a Homodyne measurement. For the commutator to vanish, it turns out that the strength of the outer sidebands must be matched according to:

$$|\alpha_+||\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega})| - |\alpha_-||\chi_c(-\Lambda_{\text{eff}} - 2\tilde{\Omega})| = 0 \Leftrightarrow \tag{5.58}$$

$$\left[\hat{a}_{\text{extra}}(t), \hat{a}^\dagger_{\text{extra}}(t)\right] = 0 \tag{5.59}$$

We will put this knowledge to use once we start discussing noise removal in the next section.

(9) We could define the operators as $\hat{a}^{(\text{inner})}_{\text{in/out},+}(t) = \hat{a}_{\text{in/out}}(t)e^{\pm i|\Lambda_{\text{eff}}|t}$, however this becomes cumbersome to work with in practice.

(10) Yes this could be writtenmore compactly as e.g. $\hat{a}^{(\text{inner})}_{\text{out},\pm}(t) = e^{i2\theta_c(\mp\Lambda_{\text{eff}})}\hat{a}^{(\text{inner})}_{\text{in},\pm}(t) + i\sqrt{\Gamma_\pm}\frac{\kappa\chi_c(\mp\Lambda_{\text{eff}})}{2}e^{i\Phi}\hat{\tilde{b}}_{\text{eff}}(t)$, but we write the expression explicitely to stress that there are four distinc sidebands.

5.4 Extraneous Noise Suppression and Read-out

Let us now consider actually measuring the light from the cavity. We wish to separate the light form the inner and the outer sidebands, so that we can measure the light from the outer sidebands. In particular we wish to use the measurement of the outer sideband light to track the extraneous QBA so that we can eliminate or at least suppress it, via feedback on the mechanical oscillator, while the light from the inner sidebands will be used to either probe the effective mechanical oscillator or as input to other systems (see Fig. 5.2). We can filter the high frequency outer sideband light from the lower frequency inner sideband light by using a filter cavity. For simplicity we will assume that the cavity susceptibility is simply a box function in Fourier space, so the frequencies are perfectly filtered by the cavity.

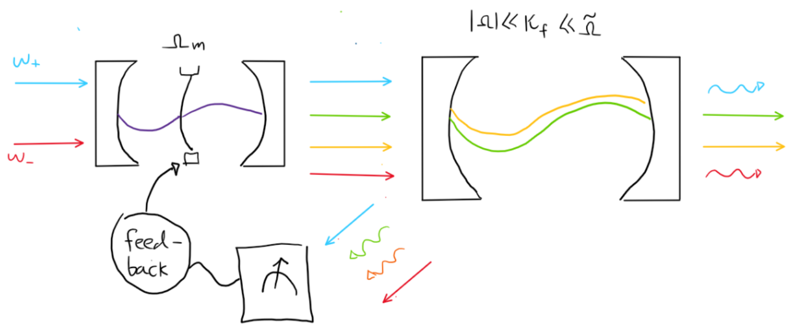

Figure 5.2: Sketch of the extraneous noise suppression scheme. First the candidate negative mass oscillator, in this case depicted as a membrane-in-the-middle system, is driven by two tones at frequency $\omega_\pm$. The optomechanical interaction then generates four sidebands (the four arrow exiting the cavity) Using a filter cavity with linewidth $\|\Omega\| \ll \kappa_f \ll \tilde{\Omega}$, we separate out the outer sidebands. We then measure the outer sidebands by homodyning them, and use this to construct a classical feedback force to suppress the extraneous noise corresponding to the outer sideband. The inner sidebands now contain the signal of an effective negative mass oscillator and can either be measured or used for e.g. a cascaded experiment.

5.4.1 Non-Hermitian Quadratures for Homodyne Detection

We will consider the use of Homodyne detection for both the inner and outer sidebands. This is primarily motivated by the practical consideration that Homodyne detection is a standard tool in most quantum optics labs. As we use two homodyne detections we have two new phases at our disposal. We will now use these extra phase degrees of freedom and redefine the phases of most quantities, such that they are simpler. By simpler we in particular have in mind the many instances where the complex phase of χ_c appears and where we would like to eliminate the cavity phase. We do so to allow us to identify whether or note the extraneous noise may be measured using a homodyne detection.

The expressions are much easier to understand if we can factor out the cavity susceptibility phases. We detail how this is done in Appendix E where we also define what all the primed and double primed operators are, but the end result is that the mechanical operator now reads

$$\hat{\hat{b}}'(t) = \int_{-\infty}^{t} dt' e^{-\gamma^{(\text{opt})}(t-t')} \Big[i e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \frac{1}{\sqrt{2}} e^{i\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t'} \hat{f}(t') \\ + \hat{a}'_{\text{nom}}(t') + \hat{a}'_{\text{extra}}(t') \Big] \tag{5.60}$$

where the change in phase reference simplifies the nominal QBA Eq. (5.55)so that it now contains no phase factors:

$$\hat{a}'_{\text{nom}}(t') = i\frac{\kappa}{2}\left(\sqrt{\Gamma_-}\,|\chi_c(\Lambda_{\text{eff}})|\hat{a}'_{\text{in},-}(t) + \sqrt{\Gamma_+}\,|\chi_c^*(-\Lambda_{\text{eff}})|\hat{a}'^{\dagger}_{\text{in},+}(t)\right) \tag{5.61}$$

This simplification happens at the price of introducing additional phase factors in the extraneous noise Eq. (5.56) which now reads:

$$\begin{aligned}\hat{a}'_{\text{extra}}(t') = {} & i\frac{\kappa}{2}e^{2i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \\ & \times\left(\sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|\hat{a}''_{\text{in},+}(t) + \sqrt{\Gamma_-}\,|\chi_c^*(-\Lambda_{\text{eff}}-2\tilde{\Omega})|\hat{a}''^{\dagger}_{\text{in},-}(t)\right)\end{aligned} \tag{5.62}$$

note that we have used different phase references for the input noise in the extraneous noise and nominal noise operators. We do so to simplify the expressions as much as possible. The choice to do so is justified by the fact that we use a different local oscillator for the detection of the inner and outer sidebands. By homodyning the two inner sidebands with an LO with phase $\theta_{\text{LO}}^{(\text{i})}$, we can measure observables corresponding to the non-Hermitian quadrature (see Eq. (E.25) for the input-output relation leading to this)

$$\hat{Q}^{(\text{inner})}_{\theta_{\text{LO}}^{(\text{i})}} = \frac{1}{\sqrt{2}}e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]}\left(e^{-i\theta_{\text{LO}}^{(\text{i})}}\hat{a}^{(\text{inner})\prime}_{\text{out},+}(t) + e^{i\theta_{\text{LO}}^{(\text{i})}}\hat{a}^{(\text{inner})\prime\dagger}_{\text{out},-}(t)\right) \tag{5.63}$$

$$\begin{aligned}= {} & \frac{1}{\sqrt{2}}\left[\left(e^{-i\theta_{\text{LO}}^{(\text{i})}}\hat{a}^{(\text{inner})\prime}_{\text{in},+}(t) + e^{i\theta_{\text{LO}}^{(\text{i})}}\hat{a}^{(\text{inner})\prime\dagger}_{\text{in},-}(t)\right)\right. \\ & \left. +i\left(e^{-i\theta_{\text{LO}}^{(\text{i})}}\sqrt{\Gamma_+}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}})|}{2} - e^{i\theta_{\text{LO}}^{(\text{i})}}\sqrt{\Gamma_-}\frac{\kappa|\chi_c(\Lambda_{\text{eff}})|}{2}\right)\hat{b}'^{\dagger}(t)\right]\end{aligned} \tag{5.64}$$

And likewise by homodyning the outer sidebands with an LO with phase $\theta_{\text{LO}}^{(\text{o})}$ we can measure observables corresponding to the non-Hermitian quadrature:

$$\begin{aligned}\hat{Q}^{(\text{outer})}_{\theta_{\text{LO}}^{(\text{o})}} = {} & \frac{1}{\sqrt{2}}e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]}\left(e^{-i\theta_{\text{LO}}^{(\text{o})}}\hat{a}^{(\text{outer})\prime\prime}_{\text{out},+}(t) + e^{i\theta_{\text{LO}}^{(\text{o})}}\hat{a}''^{(\text{outer})\dagger}_{\text{out},-}(t)\right) \\ = {} & \frac{1}{\sqrt{2}}\left(\left[e^{-i\theta_{\text{LO}}^{(\text{o})}}\hat{a}''_{\text{in}}(t) + e^{i\theta_{\text{LO}}^{(\text{o})}}\hat{a}''^{\dagger}_{\text{in}}(t)\right]e^{i(\Lambda_{\text{eff}}+2\tilde{\Omega})t}\right. \\ & +i\left[e^{-i\theta_{\text{LO}}^{(\text{o})}}\sqrt{\Gamma_+}\frac{\kappa|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{2} - e^{i\theta_{\text{LO}}^{(\text{o})}}\sqrt{\Gamma_-}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}{2}\right] \\ & \left.\times e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}e^{-2i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{b}'(t)\right)\end{aligned} \tag{5.65}$$

Importantly the output light from the outer quadratures generally also contain a signal from the oscillator, so when we measure this light we will not only detect the input light responsible for the extraneous QBA, which we wish to use for noise cancellation, but also some mechanical contribution.

5.4.2 Feedback with Outer Sideband Matching

We will now consider a particularly simple scheme where the all extraneous quantum noise can be measured using homodyne detection and eliminated completely. It however relies on fixing the value of the drive power of both drive tones according to a certain matching condition. Later we will generalize to arbitrary drive powers ,i.e. without the matching condition.

Consider the extraneous noise as given in Eq. (5.62). Due to the unequal weighting of the upper and lower sideband operators, this is generally not measurable by a homodyne detection, unless the strength of the outer sidebands are matched, as we noted in Eq. (5.58). Let us now pursue the idea of adjusting the drive power such that the sidebands become matched according to the relation:

$$\Gamma_- = \Gamma_+ \frac{|\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega})|^2}{|\chi_c^*(-\Lambda_{\text{eff}} - 2\tilde{\Omega})|^2} \tag{5.66}$$

The extraneous noise then simplifies to a Homodyne non-Hermitian quadrature:

$$\begin{aligned}\hat{a}'_{\text{extra}}(t') = {} & i\frac{\kappa}{\sqrt{2}} e^{2i\Phi} e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \\ & \times \sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega})| \left(\frac{\hat{a}''_{\text{in},+}(t) + \hat{a}''^{\dagger}_{\text{in},-}(t)}{\sqrt{2}} \right)\end{aligned}$$

By setting the LO phase to $\theta_{\text{LO}}^{(o)} = 0$ in Eq. (5.65) we indeed measure a signal corresponding the desired non-Hermitian quadrature. Incidentally, for $\theta_{\text{LO}}^{(o)} = 0$ and with outer sideband matching, the measured signal Eq. (5.65) contains no information about the mechanical oscillator. We now use this measured signal to construct a feedback force which will completely suppress the extraneous noise:

$$\hat{a}_{\text{fb}}(t) = -i\frac{\kappa}{\sqrt{2}} e^{2i\Phi} e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{5.67}$$

$$\times \sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega})| \hat{Q}^{(\text{outer})}_{\theta_{\text{LO}}^{(o)}=0}(t)$$

$$= -\hat{a}'_{\text{extra}}(t) \tag{5.68}$$

The scheme for engineering the effective oscillator proceeds as follows. We separate the inner and outer sidebands signals using a filter cavity, and continuously measure the outer sideband light. Using the measurement results we then perform continuous feedback on the mechanical system by applying an appropriate classical force $f_{\text{fb}}(t)$. The feedback operator Eq. (5.67) tells us how the classical feedback force can be determined from stochastic measurement current obtained fro the homodyne detection. Including the feedback, the equation of motion, so the solution to the oscillator reads:

$$\hat{b}'(t) = \int_{-\infty}^{t} \mathrm{d}t' e^{-\gamma^{(\text{opt})}(t-t')} \left[i\frac{1}{\sqrt{2}} e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} e^{i\Phi} e^{i(\Lambda_{\text{eff}}+\tilde{\Omega})t'} \hat{f}(t') \right. \tag{5.69}$$

$$\left. + \hat{a}'_{\text{nom}}(t') + \hat{a}'_{\text{extra}}(t) + \hat{a}_{\text{fb}}(t) \right] =$$

$$\int_{-\infty}^{t} \mathrm{d}t' e^{-\gamma^{(\text{opt})}(t-t')} \left[i\frac{1}{\sqrt{2}} e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} e^{i\Phi} e^{i(\Lambda_{\text{eff}}+\tilde{\Omega})t'} \hat{f}(t') + \hat{a}'_{\text{nom}}(t') \right], \tag{5.70}$$

We now consider the information contained in the inner sidebands of the output light. For easy comparison with the single-tone case, we consider the two-photon homodyne quadratures that one would measure by invoking an LO with $\omega_{\text{LO}} = \omega_o$. Due to our clever choice of phases, we easily see that setting the LO phase to $\theta_{\text{LO}}^{(i)} = \frac{\pi}{2}$ in the non-Hermitian measurement quadrature Eq. (5.65), will give the maximum readout of the mechanical oscillator, in particular we find:

$$\hat{Q}^{(\text{inner})}_{\theta_{\text{LO}}=\frac{\pi}{2}} = \left(\frac{\hat{a}^{(\text{inner})\prime}_{\text{in},+}(t) - \hat{a}^{(\text{inner})\prime\dagger}_{\text{in},-}(t)}{\sqrt{2}\,i} \right) \tag{5.71}$$

$$+ \sqrt{\Gamma_+}\frac{\kappa}{2} \left(\frac{|\chi_c(\Lambda_{\text{eff}} + 2\tilde{\Omega})|}{|\chi_c(-\Lambda_{\text{eff}} - 2\tilde{\Omega})|} |\chi_c(\Lambda_{\text{eff}})| + |\chi_c(-\Lambda_{\text{eff}})| \right) \hat{b}'^{\dagger}(t)$$

If we define an effective read-out rate:

$$\Gamma_{\text{eff}} = \frac{\kappa^2}{4}\left(\frac{|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}|\chi_c(\Lambda_{\text{eff}})| + |\chi_c(-\Lambda_{\text{eff}})|\right)^2 \tag{5.72}$$

The non-Hermitian quadrature measured by Homodyning the inner sidebands is simply:

$$\hat{Q}^{(\text{inner})}_{\theta_{\text{LO}}=\frac{\pi}{2}} = \left(\frac{\hat{a}^{(\text{inner})\prime}_{\text{in},+}(t) - \hat{a}^{(\text{inner})\prime\dagger}_{\text{in},-}(t)}{\sqrt{2}i}\right) + \sqrt{\Gamma_{\text{eff}}}\,\hat{b}'^{\dagger}(t) \tag{5.73}$$

Going back to the input-output relation Eq. (5.7), we could define a new mechanical operator:

$$\hat{b}'_{\text{eff}}(t) = e^{-i\Lambda_{\text{eff}}t}\hat{b}'(t) \tag{5.74}$$

If we write this mechanical oscillator explicitly, we now see that the effective mechanical oscillator which is read-out when Homodyning the inner sidebands, oscillates at frequency Λ_{eff}:

$$\hat{b}'_{\text{eff}}(t) = \int_{-\infty}^{\infty}\mathrm{d}t'\,\chi_b(t-t')\left[\hat{f}_{\text{eff}}(t') + \hat{a}'_{\text{nom,eff}}(t')\right], \tag{5.75}$$

Let us go trough the newly defined operators and function. Using the outer sideband matching condition Eq. (5.65), the nominal QBA Eq. (5.61) can be written:

$$\begin{aligned}
\hat{a}'_{\text{nom,eff}}(t') &= \sqrt{\Gamma_+}\frac{\kappa}{2}\left(\frac{|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}|\chi_c(\Lambda_{\text{eff}})|\hat{a}'_{\text{in}}(t) + |\chi_c(-\Lambda_{\text{eff}})|\hat{a}'^{\dagger}_{\text{in}}(t)\right) &(5.76)\\
&= \sqrt{\Gamma_{\text{eff}}}\left(\alpha_{\text{nom}}\hat{a}'_{\text{in}}(t) + \beta_{\text{nom}}\hat{a}'^{\dagger}_{\text{in}}(t)\right) &(5.77)
\end{aligned}$$

where we have defined two coefficient which determine the form of the nominal QBA:

$$\alpha_{\text{nom}} = \frac{|\chi_c(\Lambda_{\text{eff}})|}{|\chi_c(\Lambda_{\text{eff}})| + \frac{|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}{|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}|\chi_c(-\Lambda_{\text{eff}})|} \tag{5.78}$$

$$\beta_{\text{nom}} = \frac{|\chi_c(-\Lambda_{\text{eff}})|}{\frac{|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}|\chi_c(\Lambda_{\text{eff}})| + |\chi_c(-\Lambda_{\text{eff}})|} \tag{5.79}$$

We notice that the effective nominal QBA isn't a balanced homodyne (two-photon) quadrature. The effective thermal noise is given by:

$$\hat{f}_{\text{eff}}(t') = \frac{1}{\sqrt{2}}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i(\tilde{\Omega}t'+\Phi)}f(t') \tag{5.80}$$

Notably, the effective oscillator couples only to certain frequency components of the thermal noise, and with a certain phase. If any other forces act on the mechanical oscillator, such as a classical force, these would enter into the equation of motion in the same way. Thus if one wishes to use the effective oscillator as a sensing probe, it needs to be taken into consideration that the force is folded in frequency space around frequency $\tilde{\Omega}$.

We have defined the mechanical response function for the phonon operator as:

$$\chi_b(t-t') = ie^{-\left(i\Lambda_{\text{eff}}+\gamma^{(\text{opt})}\right)(t-t')}\Theta(t-t') \tag{5.81}$$

The susceptibility for the phonon operator in Fourier space is:

$$\chi_b(\Omega) = \int_{-\infty}^{\infty}\mathrm{d}\tau\,\chi_b(\tau)e^{i\Omega\tau} = \frac{1}{\Lambda_{\text{eff}} - \Omega - i\gamma^{(\text{opt})}} \tag{5.82}$$

The susceptibility $\chi_{\mathrm{m,eff}}(\Omega)$ for the mechanical oscillator, i.e. the susceptibility for the Hermitian quadratures:

$$\hat{X}_{\mathrm{eff}}(\Omega) = \frac{\hat{b}'_{\mathrm{eff}}(\Omega) + \hat{b}'^{\dagger}_{\mathrm{eff}}(-\Omega)}{\sqrt{2}}, \tag{5.83a}$$

$$\hat{P}_{\mathrm{eff}}(t) = \frac{\hat{b}'_{\mathrm{eff}}(\Omega) - \hat{b}'^{\dagger}_{\mathrm{eff}}(-\Omega)}{\sqrt{2}\,i}, \tag{5.83b}$$

which can be defined from the mechanical operator Eq. (5.61), is the Hermitian part of the annihilation operator susceptibility(11):

$$\chi_{\mathrm{m,eff}}(\Omega) = \frac{\chi_{\mathrm{b}}(\Omega) + \chi^{*}_{\mathrm{b}}(-\Omega)}{2} \tag{5.84}$$

$$= \frac{\Lambda_{\mathrm{eff}}}{\Lambda^2_{\mathrm{eff}} + \gamma^2_{(\mathrm{opt})} - \Omega^2 - 2i\Omega\gamma_{(\mathrm{opt})}} \tag{5.85}$$

We note that the resonance is not only modified by the optical spring effect but also by the linewidth so the resonance is:

$$\Omega_{\mathrm{res}} = \sqrt{\Lambda^2_{\mathrm{eff}} + \gamma^2_{(\mathrm{opt})}}, \tag{5.86}$$

this is a direct consequence of having damping in both the position and momentum due to the RWA we made earlier. For $\Lambda_{\mathrm{eff}} \ll \Omega_{\mathrm{eff}}, \tilde{\Omega}$, we see that the resonance frequency has been down-converted, potentially by several orders of magnitude!

In contrast to the resonance frequency, the evolution frequency is:

$$\Omega_{\mathrm{evo}} = \Lambda_{\mathrm{eff}},$$

From the definition Eq. (5.6) of Λ_{eff} , we see that we can control the sign of the evolution frequency by adjusting the relative drive frequency $\tilde{\Omega}$. This in turn now allow us to control the sign of the effective mechanical susceptibility or equivalently the effective mass, and thus paves the way for coherent QBA cancellation. As for the dynamical BA, the renormalized quantities are:

$$\Lambda_{\mathrm{eff}} = \Lambda + \delta\Omega_m^{(\mathrm{opt})}$$
$$\gamma^{(\mathrm{opt})} = \gamma + \delta\gamma^{(\mathrm{opt})}$$

When the outer sidebands are balanced, the dynamical broadening and anti-broadening Eq. (5.43) from each of them cancel, leaving only the inner sideband contributions:

$$\delta\gamma^{(\mathrm{opt})} = \frac{\Gamma_+}{8}\left(\frac{|\chi_c(\Lambda + 2\tilde{\Omega})|^2}{|\chi_c(-\Lambda - 2\tilde{\Omega})|^2}|\chi_c(\Lambda_{\mathrm{eff}})|^2 - |\chi_c(-\Lambda_{\mathrm{eff}})|^2\right),$$

Note that for general detuning, balancing the outer sidebands necessarily means that the inner sidebands are unbalanced, as we can see by the prefactor on the upper sidebands term. The optical spring effect Eq. (5.41) oddly enough retains a contribution from the outer sidebands:

$$\delta\Omega_m^{(\mathrm{opt})} = \frac{\Gamma_+}{8}\left[(\Lambda + \Delta)\,\kappa\frac{|\chi_c(\Lambda + 2\tilde{\Omega})|^2}{|\chi_c(-\Lambda - 2\tilde{\Omega})|^2}|\chi_c(\Lambda)|^2\right] + \Delta\frac{\Gamma_+}{4}\kappa|\chi_c(\Lambda + 2\tilde{\Omega})|^2 \tag{5.87}$$

(11) i.e the quadrature solution can be written. $\begin{pmatrix}\hat{X}_{\mathrm{eff}}(\Omega) & \hat{P}_{\mathrm{eff}}(\Omega)\end{pmatrix}^{\mathrm{T}} = \chi_{m,\mathrm{eff}}(\Omega)\,[\cdots]$

5.4.3 Feedback For Arbitrary Read-Out Rates

Let us now examine to what extent it is possible to remove the extraneous noise, if we want to keep $\Gamma_\pm$ unconstrained. In this case we want to identify which of the quadratures contributes the most to the extraneous noise and eliminate that particular quadrature, at the price of keeping the smallest component of the extraneous noise. To that end we write the extraneous noise in terms of the canonical Hermitian quadratures:

$$\hat{a}'_{\text{extra}}(t') = \hat{a}'_{\text{extra},q}(t') + \hat{a}'_{\text{extra},y}(t') = \tag{5.88}$$

$$i\frac{\kappa}{2}e^{2i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\times \tag{5.89}$$
$$\left[\left(\sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})| + \sqrt{\Gamma_-}\,|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|\right)\hat{q}''_{\text{in}}(t)\right.$$
$$\left.+i\left(\sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})| - \sqrt{\Gamma_-}\,|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|\right)\hat{y}''_{\text{in}}(t)\right]e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t}$$

Once again, the phase choices we made earlier makes it clear that the predominant contribution to the extraneous noise is the amplitude quadrature $\hat{q}''_{\text{in}}(t)$. Using an LO with phase $\theta^{(o)}_{\text{LO}} = 0$, we can measure a signal from the outer sidebands corresponding to non-Hermitian quadrature:

$$\hat{Q}^{(\text{outer})}_{\theta^{(o)}_{\text{LO}}=0} = \hat{q}''_{\text{in}}(t)e^{i(\Lambda_{\text{eff}}+2\tilde{\Omega})t} \tag{5.90}$$
$$+\, e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}e^{-2i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}$$
$$\times\, i\frac{\kappa}{2}\left[\sqrt{\Gamma_+}|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})| - \sqrt{\Gamma_-}|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|\right]\hat{b}'(t)$$

We now define a feedback operator:

$$\hat{a}_{\text{fb}}(t) = -ie^{2i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{5.91}$$
$$\times\,\frac{\kappa}{2}\left(\Gamma_+\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})| + \sqrt{\Gamma_-}\,|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|\right)\hat{Q}^{(\text{outer})}_{\theta^{(o)}_{\text{LO}}=0} =$$
$$-\,i\frac{\kappa}{2}e^{2i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{5.92}$$
$$\times\left(\sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})| + \sqrt{\Gamma_-}\,|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|\right)\hat{q}''_{\text{in}}(t)e^{i(\Lambda_{\text{eff}}+2\tilde{\Omega})t}$$
$$+\left[\frac{\kappa^2\Gamma_+}{8}|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|^2 - \frac{\kappa^2\Gamma_-}{8}|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|^2\right]\hat{b}'(t)$$

This feedback force will first and foremost completely suppress the extraneous amplitude noise $\hat{a}'_{\text{extra},q}(t')$, but secondly it will also renormalize the mechanical oscillator due to the oscillator signal contained in the force, leading to a self energy correction:

$$\Sigma_{\text{fb}} = -i\left[\frac{\kappa^2\Gamma_+}{8}|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|^2 - \frac{\kappa^2\Gamma_-}{8}|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|^2\right] \tag{5.93}$$

So we can write the feedback force as:

$$\hat{a}_{\text{fb}}(t) = -\hat{a}'_{\text{extra},q}(t') + \Sigma_{\text{fb}}\hat{b}'(t)$$

If we now consider the equation of motion for the mechanical operator when we include this feedback force we have:

$$\dot{\hat{b}}'_{\text{eff}}(t) = -\left(\gamma^{(\text{opt})} + i\Sigma_{\text{fb}}\right)\hat{b}'_{\text{eff}}(t) + \hat{a}'_{\text{nom}}(t) + \hat{a}'_{\text{extra},y}(t') + \tag{5.94}$$
$$i\frac{1}{\sqrt{2}}e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t}\hat{f}(t)$$

The renormalization due to the feedback causes additional broadening, which is exactly the same as the dynamical broadening from the outer sidebands:

$$\begin{aligned}\delta\gamma^{(\mathrm{fb})} &= \frac{\kappa^2\Gamma_+}{8}|\chi_c(\Lambda_{\mathrm{eff}}+2\tilde{\Omega})|^2 - \frac{\kappa^2\Gamma_-}{8}|\chi_c(-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})|^2 \\ &= \delta\gamma^{(\mathrm{opt})}\end{aligned}$$

So to reiterate, the feedback doubles the dynamical broadening induced by the dynamical BA. Unlike for the matched outer sidebands scheme, we also have the residual extraneous noise from the phase quadrature:

$$\hat{a}'_{\mathrm{extra},y}(t) = -e^{2i\Phi}e^{-i[\theta_c(\Lambda_{\mathrm{eff}})-\theta_c(-\Lambda_{\mathrm{eff}})]/2}e^{i[\theta_c(\Lambda_{\mathrm{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})]/2} \tag{5.95}$$

$$\times\frac{\kappa}{2}\left[|\alpha_+|\,|\chi_c(\Lambda_{\mathrm{eff}}+2\tilde{\Omega})| - |\alpha_-|\,|\chi_c(-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})|\right]\hat{y}''_{\mathrm{in}}(t)e^{i\left(\Lambda_{\mathrm{eff}}+2\tilde{\Omega}\right)t} \tag{5.96}$$

To understand the ramifications of this remaining noise we examine its power spectral density. To determine this we find the symmetrized correlation function:

$$\mathcal{S}^{(\mathrm{extra})}_{yy}\delta(t-t') = \left\langle\frac{\hat{a}'_{\mathrm{extra},y}(t)\hat{a}'^{\dagger}_{\mathrm{extra},y}(t') + \hat{a}'^{\dagger}_{\mathrm{extra},y}(t')\hat{a}'_{\mathrm{extra},y}(t)}{2}\right\rangle \tag{5.97}$$

The phases cancel and we are left with:

$$\mathcal{S}^{(\mathrm{extra})}_{yy}\delta(t-t') = \tag{5.98}$$

$$\left[\frac{\kappa}{2}\left(\sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\mathrm{eff}}+2\tilde{\Omega})| - \sqrt{\Gamma_-}\,|\chi_c(-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})|\right)\right]^2\underbrace{\left\langle\frac{\hat{y}_{\mathrm{in}}(t)\hat{y}_{\mathrm{in}}(t') + \hat{y}_{\mathrm{in}}(t')\hat{y}_{\mathrm{in}}(t)}{2}\right\rangle}_{\frac{1}{2}\delta(t-t')}$$

$$\Rightarrow \mathcal{S}^{(\mathrm{extra})}_{yy} = \frac{\kappa^2}{4}\left(\sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\mathrm{eff}}+2\tilde{\Omega})| - \sqrt{\Gamma_-}\,|\chi_c(-\Lambda_{\mathrm{eff}}-2\tilde{\Omega})|\right)^2 \tag{5.99}$$

The residual extraneous noise is uncorrelated with all other noise sources, i.e. the thermal noise and the shot noise and the nominal QBA. this allows us to absorb the residual extraneous noise into the thermal noise as an effective heating. The thermal force term:

$$\hat{f}'(t) = i\frac{1}{\sqrt{2}}e^{i\Phi}e^{-i[\theta_c(\Lambda_{\mathrm{eff}})-\theta_c(-\Lambda_{\mathrm{eff}})]/2}e^{i\left(\Lambda_{\mathrm{eff}}+\tilde{\Omega}\right)t}\hat{f}(t),$$

has the correlation function:

$$S_{ff} = 2\gamma\left(\bar{n}_m + \frac{1}{2}\right) \tag{5.100}$$

We can then define an effective thermal spectral function:

$$S^{(\mathrm{eff})}_{ff} \equiv 2\gamma\left(\bar{n}_{\mathrm{eff}} + \frac{1}{2}\right) \tag{5.101}$$

where the effective phonon occupation number is:

$$n_{eff} = \bar{n}_m + \frac{\mathcal{S}^{(\mathrm{extra})}_{yy}}{2\gamma}, \tag{5.102}$$

Besides this effective heating of the mechanical oscillator and the doubling of the linewidth we can use the same approach as in the previous section to obtain a mechanical oscillator where we can control the sign of the susceptibility.

CHAPTER 6

Conclusion & What's Next

> "There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable.
>
> There is another theory which states that this has already happened."
>
> THE NARRATOR
>
> - *The Restaurant at the End of the Universe,* Douglas Adams

6.1 Conclusion

In this thesis we have investigated the dynamics of a two-tone driven optomechanical system with the aim of engineering an effective oscillator with negative effective frequency – a key ingredient in quantum backaction evasion schemes.. In the limit where the four sidebands generated by the two drive tones are sufficiently narrow compared to the cavity susceptibility and sufficiently well-separated compared to their widths we found a simple theory describing the mechanics, intra-cavity field and the output light. Importantly this theory accounts for the dynamical back-action arising due to detuning between the relative frequency of the two drive tones and the cavity resonance frequency. We have showed that by adjusting the power of the two drive tones, such that the strength of the outer sidebands is equal, the extraneous noise generated by these sidebands can be measured with homodyne detection and thus be completely suppressed using continuous feedback or subtraction of the measured signal in post-processing. We also went beyond this matching condition and showed that for general amplitudes of the two drives, the extraneous noise on the mechanical oscillator from the outer sidebands cannot be measured fully with a homodyne detection. This more complex noise structure is partially due to the aforementioned detuning but also due to different drive powers, and is a new effect not described in the original work by Zeuthen et. al. However we also showed that by measuring the dominant noise quadrature, the residual extraneous noise simply amounts to an effective thermal heating of the mechanical oscillator. In both the matched and unmatched schemes we have shown that the mechanical signal imprinted on the inner sidebands can be used as an effective mechanical oscillator with a tunable resonance frequency and mass sign, where the effective resonance is down-converted compared to the original mechanical frequency. Hence, the scheme of Zeuthen et al. has been successfully extended to the experimentally relevant scenario of an optomechanical system with finite sideband resolution and detuning.

6.2 Future perspectives

Our investigation in this thesis provides an excellent starting point for further analysis in several directions. Within our simple theory we have yet to estimate the performance of concrete protocols e.g. generating entanglement of two mechanical oscillators with disparate energy spectra, or quantum enhanced sensing of a classical force [9]. A number of corrections to the simple theory presented here should be investigated. For instance, we assumed that the measurement and feedback for the extraneous noise removal scheme were perfect, in the sense that we assumed no optical losses and didn't account for the spectral response of the filtering cavity except to leading order. Moreover, it is of interest to include the coupling between $\hat{b}^\dagger$ and $\hat{b}$ and examine how this affects the effective oscillator scheme. For example it would be useful to gain a better understanding of the general structure of the extraneous noise, and in particular investigate whether it can be measured, perhaps by utilizing synodyne measurements instead of homodyne.

APPENDIX A

Conventions

In this appendix we provide an overview of the convention used e.g. for Fourier transforms in this thesis.

A.1 Fourier transform convention

For fourier transforms we use the convention:

$$f(t) = \int_{-\infty}^{\infty} \mathrm{d}\Omega f(\Omega) e^{-i\Omega t} \tag{A.1}$$

$$f(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t f(t) e^{i\Omega t} \tag{A.2}$$

A.2 Convention for Hermitian Conjugate in Fourier space

By considering the Hermitian conjugate of an annihilation operator in the time domain:

$$(\hat{a}(t))^\dagger = \left[\int_{-\infty}^{\infty} \mathrm{d}\Omega \hat{a}(\Omega) e^{-i\Omega t} \right]^\dagger = \int_{-\infty}^{\infty} \mathrm{d}\Omega \underbrace{(\hat{a}(\Omega))^\dagger}_{\hat{a}^\dagger(\Omega)} e^{i\Omega t} \tag{A.3}$$

$$\int_{-\infty}^{\infty} \mathrm{d}\Omega \underbrace{(\hat{a}(\Omega))^\dagger}_{\hat{a}^\dagger(\Omega)} e^{i\Omega t} = \tag{A.4}$$

$$\int_{-\infty}^{\infty} \mathrm{d}\Omega \underbrace{(\hat{a}(-\Omega))^\dagger}_{\hat{a}^\dagger(-\Omega)} e^{-i\Omega t}, \tag{A.5}$$

we define the relation between the time domain and Fourier space Hermitian conjugate as:

$$\hat{a}^\dagger(t) = \mathcal{F}_\Omega^{-1} \left[\hat{a}^\dagger(-\Omega) \right] \tag{A.6}$$

which is follows from our requirement that:

$$\hat{a}^\dagger(\Omega) \equiv (\hat{a}(\Omega))^\dagger \tag{A.7}$$

For the quadratures this mean we have the relations

$$\hat{q}(t) = \frac{\hat{a}(t) + \hat{a}^\dagger(t)}{\sqrt{2}} \Rightarrow \hat{q}(\Omega) = \frac{\hat{a}(\Omega) + \hat{a}^\dagger(-\Omega)}{\sqrt{2}} \tag{A.8}$$

$$\hat{y}(t) = \frac{\hat{a}(t) - \hat{a}^\dagger(t)}{\sqrt{2}i} \Rightarrow \hat{y}(\Omega) = \frac{\hat{a}(\Omega) - \hat{a}^\dagger(-\Omega)}{\sqrt{2}i} \tag{A.9}$$

With the inverse relation given by:

$$\hat{a}(\Omega) = \frac{\hat{q}(\Omega)) + i\hat{y}(\Omega)}{\sqrt{2}}, \quad \hat{a}^\dagger(\Omega) = \frac{\hat{q}(-\Omega) - i\hat{y}(-\Omega)}{\sqrt{2}} \tag{A.10}$$

For the quadratures which are Hermitian in the time domain, we see that taking the Hermitian conjugate in Fourier space gives:

$$\hat{q}^\dagger(\Omega) = (\hat{q}(\Omega))^\dagger = \tag{A.11}$$

$$\left[\frac{\hat{a}(\Omega) + \hat{a}^\dagger(-\Omega)}{\sqrt{2}}\right]^\dagger = \tag{A.12}$$

$$\frac{\hat{a}^\dagger(\Omega) + \hat{a}(-\Omega)}{\sqrt{2}} \tag{A.13}$$

So we find:

$$(\hat{q}(\Omega))^\dagger = \hat{q}(-\Omega) \Leftrightarrow \tag{A.14}$$

$$\hat{q}(\Omega) = \hat{q}^\dagger(-\Omega) \tag{A.15}$$

In general for any Hermitian operator:

$$\hat{Q}(t) = \hat{Q}^\dagger(t) = \left(\hat{Q}(t)\right)^\dagger \Leftrightarrow \tag{A.16}$$

$$\hat{Q}(\Omega) = \hat{Q}^\dagger(-\Omega) = \left(\hat{Q}(-\Omega)\right)^\dagger \tag{A.17}$$

Where we note that in Fourier space the Hermitian conjugate operator $\hat{Q}^\dagger$ is *not* the same as the Hermitian conjugate of the operator $\hat{Q}$:

$$\hat{Q}^\dagger(t) = \hat{Q}(t) \Rightarrow \tag{A.18}$$

$$\hat{Q}^\dagger(\Omega) \neq \hat{Q}(\Omega) \tag{A.19}$$

A.3 Rotations

We define a rotation as (i.e. anti-clockwise rotations) by the matrix:

$$\mathcal{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

APPENDIX B

Transfer Matrix for a Cavity

B.1 Quadrature transfer Matrix for a Driven Cavity

where κ is the HWHM of the cavity susceptibility. Even though the solution is simple for the annihilation operator, we want to solve the problem for the quadrature operators $\hat{q}$ and $\hat{y}$. Our goal will be to find a transfer matrix $\boldsymbol{T}$ which maps the input light vector $\begin{pmatrix}\hat{q}_{\text{in}}(\Omega) & \hat{y}_{\text{in}}(\Omega)\end{pmatrix}^{\mathrm{T}}$ to the output light $\begin{pmatrix}\hat{q}_{\text{out}}(\Omega) & \hat{y}_{\text{out}}(\Omega)\end{pmatrix}^{\mathrm{T}}$:

$$\begin{pmatrix}\hat{q}_{\text{out}}(\Omega)\\ \hat{y}_{\text{out}}(\Omega))\end{pmatrix} = \boldsymbol{T}\begin{pmatrix}\hat{q}_{\text{in}}(\Omega)\\ \hat{y}_{\text{in}}(\Omega))\end{pmatrix} \tag{B.1}$$

First we consider the equation of motion for the intra-cavity field, which can be write neatly as:

$$\mathbf{T}_0^{-1}\begin{pmatrix}\hat{q}(\Omega)\\ \hat{y}(\Omega)\end{pmatrix} = \begin{pmatrix}\hat{q}_{\text{in}}(\Omega)\\ \hat{y}_{\text{in}}(\Omega))\end{pmatrix} \Rightarrow \tag{B.2}$$

$$\begin{pmatrix}\hat{q}(\Omega)\\ \hat{y}(\Omega)\end{pmatrix} = \sqrt{2\kappa}\,\boldsymbol{T}_0\begin{pmatrix}\hat{q}_{\text{in}}(\Omega)\\ \hat{y}_{\text{in}}(\Omega))\end{pmatrix} \tag{B.3}$$

Where we have defined the inverse of the cavity transfer matrix:

$$\mathbf{T}_0^{-1} = \begin{bmatrix}(\kappa - i\Omega) & \Delta\\ -\Delta & (\kappa - i\Omega)\end{bmatrix} \Leftrightarrow \tag{B.4}$$

$$\boldsymbol{T}_0 = \frac{1}{2}\begin{bmatrix}\chi_c(\Omega) + \chi_c^*(-\Omega) & i\left(\chi_c(\Omega) - \chi_c^*(-\Omega)\right)\\ -i\left(\chi_c(\Omega) - \chi_c^*(-\Omega)\right) & \chi_c(\Omega) + \chi_c^*(-\Omega)\end{bmatrix} \tag{B.5}$$

We can bring the cavity transfermatrix on a more illuminating form by writing the cavity susceptibility χ_c on polar form according to Eq. (3.88) and defining:

$$\phi_\pm = \frac{\theta(\Omega) \pm \theta(-\Omega)}{2} \tag{B.6}$$

If we factor out the relative phase ϕ_- and and rewrite the complex exponentials in terms of sines and cosines we have

$$\boldsymbol{T}_0 = \frac{1}{2}e^{i\phi_-(\Omega)}\boldsymbol{\mathcal{R}}_{\phi_+(\Omega)}\left[\mathbb{1} - i\boldsymbol{\mathcal{R}}_{-\frac{\pi}{2}}\right]|\chi_c(\Omega)| \tag{B.7}$$

$$+\frac{1}{2}e^{i\phi_-(\Omega)}\boldsymbol{\mathcal{R}}_{\phi_+(\Omega)}\left[\mathbb{1} + i\boldsymbol{\mathcal{R}}_{-\frac{\pi}{2}}\right]|\chi_c(-\Omega)| \tag{B.8}$$

where the $\boldsymbol{\mathcal{R}}$-matrices are two-dimensional rotation matrices defined by:

$$\mathcal{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

We note that since rotations in 2D commute:

$$[\mathcal{R}_\theta, \mathcal{R}_\phi] = 0,$$

the transfer matrix is rotationally invariant. By changing the order of the rotations we find:

$$\boldsymbol{T}_0 = e^{i\frac{\theta(\Omega)-\theta(-\Omega)}{2}} \mathcal{R}_{\frac{\theta(\Omega)+\theta(-\Omega)}{2}} \begin{bmatrix} \frac{|\chi_c(\Omega)|+|\chi_c(-\Omega)|}{2} & i\frac{|\chi_c(\Omega)|-|\chi_c(-\Omega)|}{2} \\ -i\frac{|\chi_c(\Omega)|-|\chi_c(-\Omega)|}{2} & \frac{|\chi_c(\Omega)|+|\chi_c(-\Omega)|}{2} \end{bmatrix} \tag{B.9}$$

This form of the cavity transfer matrix will be useful for discussing how the mechanical motion is imprinted on different quadratures, and will later allow us to define a more mathematically convenient set of light variables.

B.2 Transfer Matrix Form of Input-Output Relation

The input-output relation for the cavity is:

$$\begin{pmatrix} \hat{q}_{\text{out}}(\Omega) \\ \hat{y}_{\text{out}}(\Omega)) \end{pmatrix} = -\begin{pmatrix} \hat{q}_{\text{in}}(\Omega) \\ \hat{y}_{\text{in}}(\Omega)) \end{pmatrix} + \sqrt{2\kappa}\begin{pmatrix} \hat{q}(\Omega) \\ \hat{y}(\Omega)) \end{pmatrix} \tag{B.10}$$

Using the solution Eq. (B.3)for the intracavity field we can write the input-output relation:

$$\begin{pmatrix} \hat{q}_{\text{out}}(\Omega) \\ \hat{y}_{\text{out}}(\Omega)) \end{pmatrix} = [-\mathbb{1} + 2\kappa \boldsymbol{T}_0] \begin{pmatrix} \hat{q}_{\text{in}}(\Omega) \\ \hat{y}_{\text{in}}(\Omega)) \end{pmatrix} \tag{B.11}$$

So the transfer matrix $\boldsymbol{T}$, in the absence of a mechanical DOF is given by:

$$\boldsymbol{T} = 2\kappa \boldsymbol{T}_0 - \mathbb{1} \tag{B.12}$$

We can in fact rewrite this purely in terms of a phase and a rotation by making the following observation: The modulus of the cavity susceptibility can be rewritten in terms of the argument:

$$|\chi_c(\Omega)| = \frac{1}{\kappa}\cos(\theta(\Omega)) \tag{B.13}$$

We can then rewrite the matrix containing $|\chi_c(\pm\Omega)|$ in the expression for $\boldsymbol{T}_0$ in Eq. (B.9) as

$$\begin{bmatrix} \frac{|\chi_c(\Omega)|+|\chi_c(-\Omega)|}{2} & i\frac{|\chi_c(\Omega)|-|\chi_c(-\Omega)|}{2} \\ -i\frac{|\chi_c(\Omega)|-|\chi_c(-\Omega)|}{2} & \frac{|\chi_c(\Omega)|+|\chi_c(-\Omega)|}{2} \end{bmatrix} = \frac{1}{2\kappa}\left[e^{i\phi_-(\Omega)}\mathcal{R}_{\phi_+(\Omega)} + e^{-i\phi_-(\Omega)}\mathcal{R}^{\mathrm{T}}_{\phi_+(\Omega)}\right] \tag{B.14}$$

Which reduces the transfermatrix to:

$$\boldsymbol{T} = e^{i2\phi_-(\Omega)}\mathcal{R}_{2\phi_+(\Omega)} \tag{B.15}$$

We thus see that the cavity simply rotates the input quadratures by $2\phi_+$ and change the global phase by $2\phi_-$.

APPENDIX C

Formal integration of the classical intra-cavity field EOM for two-tone driving

The EOM in question is

$$\dot{\bar{\alpha}}(t) = -\left(i\Delta(t) - \kappa\right)\bar{\alpha}(t) + \sqrt{2\kappa}\,2\alpha_{\text{in}}\cos(\tilde{\Omega}t) \tag{C.1}$$

First we note that by multiplying with the integration factor $e^{\kappa - i\int_{t_0}^{t}\mathrm{d}t'(i\Delta(t'))}$ we can rewrite the EOM using the product rule for differentiation:

$$\dot{\bar{\alpha}}(t) - \left(i\Delta(t) - \kappa\right)\bar{\alpha}(t) = \sqrt{2\kappa}\,2\alpha_{\text{in}}\cos(\tilde{\Omega}t) \Rightarrow \tag{C.2}$$

$$\dot{\bar{\alpha}}(t)e^{\kappa - i\int_{t_0}^{t}\mathrm{d}t'(i\Delta(t'))} - \left(i\Delta(t) - \kappa\right)\bar{\alpha}(t)e^{\kappa - i\int_{t_0}^{t}\mathrm{d}t'(i\Delta(t'))} \tag{C.3}$$
$$= \sqrt{2\kappa}\,2\alpha_{\text{in}}\cos(\tilde{\Omega}t)e^{\kappa t - i\int_{t_0}^{t}\mathrm{d}t'(i\Delta(t'))}$$

Which gives us the EOM:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\bar{\alpha}(t)e^{\kappa t - i\int_{t_0}^{t}\mathrm{d}t'(i\Delta(t'))}\right] = \sqrt{2\kappa}\,2\alpha_{\text{in}}\cos(\tilde{\Omega}t)e^{\kappa t' - i\int_{t_0}^{t}\mathrm{d}t'(i\Delta(t'))} \tag{C.4}$$

We can now directly integrate the expression, but it pays to by careful:

$$\int_{t_0}^{t}\mathrm{d}t'\frac{\mathrm{d}}{\mathrm{d}t'}\left[\bar{\alpha}(t')e^{\kappa t' - i\int_{t_0}^{t'}\mathrm{d}t''(\Delta(t''))}\right] = \tag{C.5}$$
$$\int_{t_0}^{t}\mathrm{d}t'\left[\sqrt{2\kappa}\,2\alpha_{\text{in}}\cos(\tilde{\Omega}t')e^{\kappa t' - i\int_{t_0}^{t'}\mathrm{d}t''(\Delta(t''))}\right] \Rightarrow$$

$$\bar{\alpha}(t)e^{\kappa t - i\int_{t_0}^{t}\mathrm{d}t'(i\Delta(t'))} - \bar{\alpha}(t_0)e^{\kappa t_0} = \tag{C.6}$$
$$\sqrt{2\kappa}\,2\alpha_{\text{in}}\int_{t_0}^{t}\mathrm{d}t'\left[\cos(\tilde{\Omega}t')e^{\kappa t' - i\int_{t_0}^{t'}\mathrm{d}t''(\Delta(t''))}\right] \Rightarrow$$

$$\bar{\alpha}(t) = \bar{\alpha}(t_0)e^{-\kappa(t-t_0) + i\int_{t_0}^{t}\mathrm{d}t'(\Delta(t'))} +$$
$$\sqrt{2\kappa}\,2\alpha_{\text{in}}\int_{t_0}^{t}\mathrm{d}t'\left[\cos(\tilde{\Omega}t')e^{-\kappa(t-t') + i\int_{t_0}^{t}\mathrm{d}t''(i\Delta(t'')) - i\int_{t_0}^{t'}\mathrm{d}t''(\Delta(t''))}\right] \tag{C.7}$$

We can rewrite the exponential in the drive term using:

$$\int_{t_0}^{t} \mathrm{d}t'' \left(\Delta(t'')\right) - \int_{t_0}^{t'} \mathrm{d}t'' \left(\Delta(t'')\right) = \int_{t_0}^{t} \mathrm{d}t'' \left(\Delta(t'')\right) + \int_{t'}^{t_0} \mathrm{d}t'' \left(\Delta(t'')\right) = \int_{t'}^{t} \mathrm{d}t'' \left(\Delta(t'')\right)$$

So the solution reduces to:

$$\bar{\alpha}(t) = \bar{\alpha}(t_0) e^{-\kappa(t-t_0)+i\int_{t_0}^{t} \mathrm{d}t' (\Delta(t'))} + \sqrt{2\kappa}\, 2\alpha_{\mathrm{in}} \int_{t_0}^{t} \mathrm{d}t' \left[\cos(\tilde{\Omega}t') e^{-\kappa(t-t')+i\int_{t'}^{t} \mathrm{d}t'' (\Delta(t''))}\right]$$

Which in steady state, i.e. for $t_0 \to -\infty$ becomes:

$$\bar{\alpha}(t) = \sqrt{2\kappa}\, 2\alpha_{\mathrm{in}} \int_{-\infty}^{t} \mathrm{d}t' \left[\cos(\tilde{\Omega}t') e^{-\kappa(t-t')+i\int_{t'}^{t} \mathrm{d}t'' (\Delta(t''))}\right]$$

APPENDIX D

Rewriting the Interference Term in the Input-Output relation

The interference term can be rewritten as follows:

$$2\kappa\chi_c(\Omega) - 1 = -\frac{2\kappa}{i(\Omega+\Delta)-\kappa} - 1 = \tag{D.1}$$

$$\frac{1+i\left(\frac{\Omega+\Delta}{\kappa}\right)}{1-i\left(\frac{\Omega+\Delta}{\kappa}\right)} \tag{D.2}$$

Recall that the cavity susceptibility and its phase are given by:

$$\chi_c(\Omega) = \frac{1}{\kappa - i(\Omega+\Delta)} = \frac{\kappa + i(\Omega+\Delta)}{\kappa^2 + (\Omega+\Delta)^2}$$

$$\theta_c(\Omega) \equiv \text{Arg}\left[\chi_c(\Omega)\right] = \arctan\left(\frac{\Omega+\Delta}{\kappa}\right)$$

If we now define a complex number:

$$z = 1 + i\left(\frac{\Omega+\Delta}{\kappa}\right) \Rightarrow \tag{D.3}$$

$$\text{Arg}\left[z\right] = \text{Arg}\left[\chi_c(\Omega)\right], \tag{D.4}$$

we can use the simpel relation:

$$\frac{1+i\left(\frac{\Omega+\Delta}{\kappa}\right)}{1-i\left(\frac{\Omega+\Delta}{\kappa}\right)} = \frac{z}{z^*} = e^{2i\text{Arg}[z]}. \tag{D.5}$$

So we find that the interference term just leads to a frequency dependent phase-lag

$$2\kappa\chi_c(\Omega) - 1 = e^{i2\theta_c(\Omega)} \tag{D.6}$$

Appendix

E

Phase gymnastics for the Nominal and Extraneous QBA

In this appendix we seek to rewrite the nominal and extraneous QBA in a simplet form by redefining our phase references in a clever maner. Let us start by rewriting the input-output relations for the side-band operators by factoring out the mean phase of the caivty susceptibility

$$e^{-i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{out},-}(t) \approx e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \times \left[e^{i\theta_c(\Lambda_{\text{eff}})}\hat{a}^{(\text{inner})}_{\text{in},-}(t) + i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(\Lambda_{\text{eff}})|}{2}e^{i\Phi}\hat{\tilde{b}}(t)\right] \tag{E.1}$$

$$e^{-i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{out},+}(t) \approx e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \times \left[e^{i\theta_c(-\Lambda_{\text{eff}})}\hat{a}^{(\text{inner})}_{\text{in},+}(t) + i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}})|}{2}e^{-i\Phi}\hat{\tilde{b}}^\dagger(t)\right] \tag{E.2}$$

$$e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})}_{\text{out},+}(t) \approx e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \times \left[e^{i\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})}\hat{a}^{(\text{outer})}_{\text{in},+}(t) + i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{2}e^{-i\Phi}\hat{\tilde{b}}\right] \tag{E.3}$$

$$e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})}_{\text{out},-}(t) \approx e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \times \left[e^{i\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})}\hat{a}^{(\text{outer})}_{\text{in},-}(t) + i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}{2}e^{i\Phi}\hat{\tilde{b}}^\dagger(t)\right] \tag{E.4}$$

Let us try to do the same rewriting to the remaining phase factors in front of the input operators:

$$e^{-i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{out},-}(t) \approx e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \tag{E.5}$$

$$\times\left[e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\left(e^{i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{in},-}(t)\right)+i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(\Lambda_{\text{eff}})|}{2}e^{i\Phi}\hat{\tilde{b}}(t)\right],$$

$$e^{-i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{out},+}(t) \approx e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \tag{E.6}$$

$$\times\left[e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\left(e^{i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{in}+}(t)\right)+i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}})|}{2}e^{-i\Phi}\hat{\tilde{b}}^\dagger(t)\right],$$

$$e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})}_{\text{out},+}(t) \approx e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{E.7}$$

$$\times\left[e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\left(e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})}_{\text{in},+}(t)\right)\right.$$

$$\left.+i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{2}e^{-i\Phi}\hat{\tilde{b}}\right] \tag{E.8}$$

$$e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})}_{\text{out},-}(t) \approx e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{E.9}$$

$$\times\left[e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\left(e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})}_{\text{in},-}(t)\right)\right. \tag{E.10}$$

$$\left.+i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}{2}e^{i\Phi}\hat{\tilde{b}}^\dagger(t)\right] \tag{E.11}$$

Let us define a set of phase shifted output operators

$$\hat{a}^{(\text{inner})'}_{\text{out},\pm}(t) = e^{-i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{out},\pm}(t), \tag{E.12}$$

$$\hat{a}^{(\text{outer})''}_{\text{out},\pm}(t) = e^{-i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{outer})}_{\text{out},\pm}(t) \tag{E.13}$$

and input operators:

$$\hat{a}^{(\text{inner})'}_{\text{in},\pm}(t) = e^{i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{in},\pm}(t) \tag{E.14}$$

$$\hat{a}^{(\text{outer})''}_{\text{in},\pm}(t) = e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})}_{\text{in},\pm}(t) \tag{E.15}$$

So the input-output relation reads:

$$\hat{a}^{(\text{inner})'}_{\text{out},-}(t) \approx e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\times \tag{E.16}$$

$$\left[e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})'}_{\text{in},-}(t)+i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(\Lambda_{\text{eff}})|}{2}e^{i\Phi}\hat{\tilde{b}}(t)\right]$$

$$\hat{a}^{(\text{inner})'}_{\text{out},+}(t) \approx e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\times \tag{E.17}$$

$$\left[e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})'}_{\text{in}+}(t)+i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}})|}{2}e^{-i\Phi}\hat{\tilde{b}}^\dagger(t)\right]$$

$$\hat{a}^{(\text{outer})''}_{\text{out},+}(t) \approx e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{E.18}$$

$$\times\left[e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})''}_{\text{in},+}(t)+i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{2}e^{-i\Phi}\hat{\tilde{b}}\right]$$

$$\hat{a}^{(\text{outer})''}_{\text{out},-}(t) \approx e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{E.19}$$

$$\times\left[e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})''}_{\text{in},-}(t)+i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}{2}e^{i\Phi}\hat{\tilde{b}}^\dagger(t)\right] \tag{E.20}$$

We now try to factor out the phase factor in front of the input noise terms:

$$e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})'}_{\text{out},-}(t) \approx e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \tag{E.21}$$
$$\times\left[\hat{a}^{(\text{inner})'}_{\text{in},-}(t) + i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(\Lambda_{\text{eff}})|}{2}e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{\tilde{b}}(t)\right]$$
$$e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})'}_{\text{out},+}(t) \approx e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \tag{E.22}$$
$$\times\left[\hat{a}^{(\text{inner})'}_{\text{in}+}(t) + i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}})|}{2}e^{-i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{\tilde{b}}^\dagger(t)\right]$$
$$e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})''}_{\text{out},+}(t) \approx e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{E.23}$$
$$\times\left[\hat{a}^{(\text{outer})''}_{\text{in},+}(t) + i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{2}e^{-i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{\tilde{b}}\right]$$
$$e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})''}_{\text{out},-}(t) \approx e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{E.24}$$
$$\times\left[e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})''}_{\text{in},-}(t)\right.$$
$$\left.+i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}{2}e^{i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{\tilde{b}}^\dagger(t)\right]$$

Let us now try to absorb the phase $e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}$ into $\hat{\tilde{b}}$:

$$e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]}\hat{a}^{(\text{inner})-}_{\text{out},-}(t) \approx \left[\hat{a}^{(\text{inner})\prime\prime}_{\text{in},-}(t) + i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(\Lambda_{\text{eff}})|}{2}\hat{\tilde{b}}'(t)\right] \tag{E.25a}$$
$$e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]}\hat{a}^{(\text{inner})'}_{\text{out},+}(t) \approx \left[\hat{a}^{(\text{inner})\prime\prime}_{\text{in}+}(t) + i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}})|}{2}\hat{\tilde{b}}'^\dagger(t)\right] \tag{E.25b}$$
$$e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]}\hat{a}^{(\text{outer})\prime\prime}_{\text{out},+}(t) \approx \left[\hat{a}^{(\text{outer})''}_{\text{in},+}(t)+\right. \tag{E.25c}$$
$$\left. i\sqrt{\Gamma_+}\frac{\kappa|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|}{2}e^{-i2\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{\tilde{b}}'(t)\right] \tag{E.25d}$$
$$e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]}\hat{a}^{(\text{outer})\prime\prime}_{\text{out},-}(t) \approx \left[e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}^{(\text{outer})\prime\prime}_{\text{in},-}(t)\right. \tag{E.25e}$$
$$\left.+i\sqrt{\Gamma_-}\frac{\kappa|\chi_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})|}{2}e^{i2\Phi}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{\tilde{b}}'^\dagger(t)\right] \tag{E.25f}$$

We must now check that we can infact absorb this phase in the phonon operator. To do so we first note that th nominal QBA can be rewritten as follows by factoring out the relative phase of the cavity susceptibility at the inner sidebands:

$$\hat{a}^{(\text{inner})\prime}_{\text{out},\pm}(t) = e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{-i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^{(\text{inner})}_{\text{out},\pm}(t)$$

$$\hat{a}_{\text{nom}}(t) = i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{-i\Phi}\left(|\alpha_-|\,|\chi_c(\Lambda_{\text{eff}})|e^{i\theta_c(\Lambda_{\text{eff}})}\hat{a}_{\text{in}}(t) + |\alpha_+|\,|\chi_c^*(-\Lambda_{\text{eff}})|e^{-i\theta_c(-\Lambda_{\text{eff}})}\hat{a}^\dagger_{\text{in}}(t)\right)e^{i\Lambda_{\text{eff}}t} \tag{E.26}$$
$$= i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{-i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\left[|\alpha_-|\,|\chi_c(\Lambda_{\text{eff}})|e^{i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}_{\text{in}}(t)\right. \tag{E.27}$$
$$\left.+|\alpha_+|\,|\chi_c^*(-\Lambda_{\text{eff}})|e^{-i[\theta_c(\Lambda_{\text{eff}})+\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}^\dagger_{\text{in}}(t)\right]e^{i\Lambda_{\text{eff}}t}$$
$$= i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{-i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \tag{E.28}$$
$$\times\left(|\alpha_-|\,|\chi_c(\Lambda_{\text{eff}})|\hat{a}'_{\text{in}}(t) + |\alpha_+|\,|\chi_c^*(-\Lambda_{\text{eff}})|\hat{a}'^\dagger_{\text{in}}(t)\right)e^{i\Lambda_{\text{eff}}t} \tag{E.29}$$

We can do the same for the extraneous noise:

$$\hat{a}_{\text{extra}}(t) = i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{i\Phi}\Big[|\alpha_+|\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|e^{i\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})}\hat{a}_{\text{in}}(t) \tag{E.30}$$

$$+|\alpha_-|\,|\chi_c^*(-\Lambda_{\text{eff}}-2\tilde{\Omega})|e^{-i\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})}\hat{a}_{\text{in}}^\dagger(t)\Big]\,e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t}$$

$$= i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{E.31}$$

$$\times\Big[|\alpha_+|\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}_{\text{in}}(t) \tag{E.32}$$

$$+|\alpha_-|\,|\chi_c^*(-\Lambda_{\text{eff}}-2\tilde{\Omega})|e^{-i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})+\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}\hat{a}_{\text{in}}^\dagger(t)\Big]\,e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t}$$

$$= i\sqrt{2\kappa}\frac{g_0}{\sqrt{2}}e^{i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2} \tag{E.33}$$

$$\times\Big(|\alpha_+|\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|\hat{a}_{\text{in}}''(t)+|\alpha_-|\,|\chi_c^*(-\Lambda_{\text{eff}}-2\tilde{\Omega})|\hat{a}_{\text{in}}''^\dagger(t)\Big)\,e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t}$$

We now absorb the phase-lag from the relative cavity phase from the inner sidebands into the mechanical operator and define:

$$\hat{\dot{b}}'(t) \equiv e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{\dot{b}}(t) \Leftrightarrow$$
$$\hat{\dot{b}}(t) \equiv e^{-i\Phi}e^{i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{b}'(t)$$

This is possible as we can rewrite the EOM:

$$e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\dot{\hat{\dot{b}}}(t) = e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2} \tag{E.34}$$

$$\times\Big[-\left(i\delta\Omega_m+\gamma+i\Sigma\right)\hat{\dot{b}}(t)+i\frac{1}{\sqrt{2}}e^{i\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t}\hat{f}(t)+\hat{a}_{\text{nom}}(t)+\hat{a}_{\text{extra}}(t)\Big]\Rightarrow$$

$$\dot{\hat{b}}'(t) = -\left(i\delta\Omega_m+\gamma+i\Sigma\right)\hat{b}'(t)+i\frac{1}{\sqrt{2}}e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t}\hat{f}(t) \tag{E.35}$$

$$+e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}_{\text{nom}}(t)+e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}_{\text{extra}}(t)\Rightarrow$$

$$\dot{\hat{b}}'(t) = -\left(i\delta\Omega_m+\gamma+i\Sigma\right)\hat{b}'(t)+i\frac{1}{\sqrt{2}}e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i\left(\Lambda_{\text{eff}}+\tilde{\Omega}\right)t}\hat{f}(t) \tag{E.36}$$

$$+\hat{a}_{\text{nom}}'(t)+\hat{a}_{\text{extra}}'(t),$$

where the nominal QBA now is written in term of $\hat{a}_{\text{in}}'$, but no longer has any explicit phases:

$$\hat{a}_{\text{nom}}'(t) \equiv e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}_{\text{nom}}(t) =$$
$$i\frac{\kappa}{2}\left(\sqrt{\Gamma_-}\,|\chi_c(\Lambda_{\text{eff}})|\hat{a}_{\text{in}}'(t)+\sqrt{\Gamma_+}\,|\chi_c^*(-\Lambda_{\text{eff}})|\hat{a}_{\text{in}}'^\dagger(t)\right)e^{i\Lambda_{\text{eff}}t}$$

As we used the phases which make the nominal QBA nice and tidy, the extraneous noise now involve phases from both the inner and outer sidebands:

$$\hat{a}_{\text{extra}}'(t) \equiv e^{i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}\hat{a}_{\text{extra}}(t) = \tag{E.37}$$

$$i\frac{\kappa}{2}e^{2i\Phi}e^{-i[\theta_c(\Lambda_{\text{eff}})-\theta_c(-\Lambda_{\text{eff}})]/2}e^{i[\theta_c(\Lambda_{\text{eff}}+2\tilde{\Omega})-\theta_c(-\Lambda_{\text{eff}}-2\tilde{\Omega})]/2}$$

$$\times\left(\sqrt{\Gamma_+}\,|\chi_c(\Lambda_{\text{eff}}+2\tilde{\Omega})|\hat{a}_{\text{in}}''(t)+\sqrt{\Gamma_-}\,|\chi_c^*(-\Lambda_{\text{eff}}-2\tilde{\Omega})|\hat{a}_{\text{in}}''^\dagger(t)\right)e^{i\left(\Lambda_{\text{eff}}+2\tilde{\Omega}\right)t} \tag{E.38}$$

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