Non-relativistic realisation of $\mathcal{N} = 4$ super Yang–Mills theory

First you guess. Don't laugh, this is the most important step.

London 1941
R.Feynman

Thesis for the title of
M.Sc. in Physics

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… to my father

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Bakirlis Nikos

Abstract

In this thesis, we mainly review and reproduce the results from the papers [2], [3], [65] that $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$ with gauge group $SU(N)$ is described by the near BPS-limit of a lower-dimensional non-relativistic field theory with $SU(1,1) \times U(1)$ algebra. We show that interactions of this theory, match the one-loop dilatation operator of the $SU(1,1)$ sector, thus demonstrating the consistency of our procedure , at quantum level.

In addition, we review the calculation of the one-loop dilatation operator using the standard procedure of first quantising the the $\mathcal{N} = 4$ super Yang-Mills theory at one-loop, and then constructing the dilation operator in the near-BPS. We do this using purely algebraic arguments and symmetries of the Feynman diagrams.

The construction of this lower-dimensional non-relativistic field theory gives the possibility of expanding beyond pure bosonic theories, to richer field theories that include more exotic fields like fermions, supersymmetry, gauge field. The aim is to provide a means to approach a better understanding of strongly coupled finite-N dynamics of gauge theories.

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1
Introduction

The journey of understanding how the universe works, has (at least for the 21st century) led us to two deep theories that try to describe the very large structure and very small structure of the universe: *General Relativity* and *Quantum Mechanics/Quantum Field Theory*. However, these enormous successful and beautiful mathematical/physical theories are in their fundamental core very incompatible with each other. This is mainly due to the non-locality of quantum theory, which is in contrast with the locality of GR. Some of the main results which arise from the non-locality of QM when trying to contract a quantum theory of gravity (QG) are: gravitational singularities, the black-hole information paradox, problem of time etc. see [50].

Unfortunately, despite decades of research on quantum gravity, we still haven't develop such a concrete self-consistent theory, although a major candidate with a consistent quantization of gravity exists and it is the *Superstring Theory*; which is also a serious candidate for a *theory of everything*(TOE) since not only incorporates quantum gravity in one mainframe but also naturally incorporates gauge theories of the *Standard Model* (SM).

Supersymmetry has also been applied to QFT models giving rise to very beautiful structures. One very important application is the study of $\mathcal{N} = 4$ super-Yang-Mills (SYM), a theory which in a way is the most promising ”nice-behaved” conformal supersymmetric quantum field theory of all, since it has the remarkable feature to be conformally invariant even at the quantum level and two- and three- point correlations functions are completely determined by the scaling dimensions and the structure constants of the involved operators.

When studying D-branes and string/string dualities, Juan Martín Maldacena conjectured that type IIB string theory on $AdS_5 \times S^5$ should be equivalent to $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ and coupling constant g. This correspondence is supported by the match between the mathematical structure of the underlying symmetries of both theories, i.e. $PSU(2, 2|4)$.

Our end-goal for this thesis (and the motivation behind all our calculations), is to be able to study the emergence of gravity/black holes through the quantum theory living at the boundary of the AdS/CFT correspondence. In principle, we would like to use the AdS/CFT-correspondence since by solving the gauge theory side we would get (through the correspondence) the full dynamics of the string/gravity side thus revealing emergent black holes from a quantum theory; sadly in practice such a task is impossible. So we have to approach it by different means. One way is to take the planar limit $N \to \infty$ while keeping the ’t Hooft coupling $\lambda = g^2 N$ fixed. This approach has the limitation that we cannot reach our desired strong-gravity (i.e. black holes) limit since we work in fixed-geometry with potential fluctuations of the scale $1/N$, i.e. very small perturbative corrections to gravity(geometry). On the other hand, at weak coupling , finite N contributions are way simpler to compute but the dual string theory is now strongly coupled (we may say that ii has a pure algebraic structure in this limit) thus loses its geometrical form, at least in the semiclassical sense.

In this thesis, we will explore an alternative idea in which we will start from a non-relativistic limit of AdS/CFT, where both strong dynamics of gravity and semiclassical geometry are present and the CFT side is simple enough to be studied in its strongly coupled finite-N regime. We approach the desired non-relativistic stringy dynamics by considering near-BPS limits of $\mathcal{N} = 4$ SYM. In particular, we will demonstrate that $\mathcal{N} = 4$ SYM Kaluza-Klein decompactified on S^3 close to a particular BPS bound is described by a lower-dimensional non-relativistic field theory generalised[1] Hamiltonian H.

The BPS bounds used are of the type

$$E \geq S_1 + \sum_{i=1}^{3} \omega_i Q_i \tag{1.1}$$

where E is the energy, S_1 of the angular momenta and Q_i, $i = 1, 2, 3$ are the three R-charges of $\mathcal{N} = 4$ SYM on S^3, moreover ω_i are the three constants that characterize the BPS bounds. We mainly focus on the $(\omega_1, \omega_2, \omega_3) = (1, 0, 0)$ in which we obtain a scalar theory (full bosonic) with $U(1) \times SU(1, 1)$ global symmetry, where $U(1)$ corresponds to the conservation of particle numbers as non-relativistic quantum theory dictates and $SU(1, 1)$ is the symmetry of the interactions in our particular choice of BPS bounds. Other choices of ω_i lead to different global symmetries (possibly including supersymmetry) and to the presence of fermions; more details on the decoupling limits that can be taken are given in [29]. The procedure discussed in this thesis to approach these near-BPS limits was introduced in [2] and further applied in other sectors in [3] and [65].

For more choice of ω_i which include different global symmetries plus supersymmetry (i.e. fermions etc) we refer to [29],[3]. The near BPS-limit now is

$$\lambda \to 0 \quad \text{with} \quad \frac{E - S_1 - \sum_{i=1}^{3} \omega_i Q_i}{\lambda} \quad \text{finite} \quad , \quad \text{N fixed} \tag{1.2}$$

This is a type of limit, which we will study in chapter 5, known as *Spin Matrix theory* (SMT) limit.

Using now sphere reduction on our classical $\mathcal{N} = 4$ SYM on S^3, imposing the near-BPS limit at quadratic order, integrating out non-dynamical modes from the Hamiltonian and then computing the interaction Hamiltonian using the exact near-BPS limit; we get a lower-dimensional non-relativistic field theory classical Hamiltonian H_{lim}. Then we quantize this Hamiltonian to get the near-BPS quantum-mechanical Hamiltonian H_q. This quantisation procedure results in self-energy corrections that can be calculated from a standard normal-ordering procedure. This near-BPS theory is now a full SMT and all this procedure is completely dual to the equivalent derivation of the H_q using first path-integral quantisation on the original classical $\mathcal{N} =$ SYM and doing a loop expansion to derive the dilatation operator D (we will

[1]See Appendix A.

demonstrate how to do this also in the next chapters). Then from D we follow the recipe from SMT [25] and take the near-BPS limit in which only one-loop contributions of D survives , to derive the H_q which is exactly equal, as we will demonstrate, in both procedures. We conjecture that this commutative of procedures is probably due to the fact that the highly non-trivial QFT computations which lead to the particular form for the dilatation D are captured by the normal ordering contributions used in our H_{lim}.

2
Field theory

In this chapter we will briefly discuss the basic notions (needed for our work) of QFT, SCFT (super-conformal field theory), superconformal algebra and its representation theory. The main goal is to understand the full Lagrangian of the $\mathcal{N} = 4$ SYM theory and its symmetries. Then we will describe some fundamental features of gravity, in particular related to the holographic principle.

signature	$\eta^{\mu\nu}$	η^{mn}	spacetime sym.	internal sym
physical	$(3,1)$	$(6,0)$	$\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{su}(4)$
Euclidian	$(4,0)$	$(5,1)$	$\mathfrak{sp}(1)\times\mathfrak{sp}(1)$	$\mathfrak{sl}(2,\mathbb{H})$
Minkowski, non-compact	$(3,1)$	$(4,2)$	$\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{su}(2,2)$
maximally non-comapct	$(2,2)$	$(3,3)$	$\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(4,\mathbb{R})$
complex	4	6	$\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{sl}(4,\mathbb{C})$

(2.1)

Figure 2.1: Possible signatures of spacetime, internal space and symmetry algebras.

This will be concretely realized with the AdS/CFT correspondence, which relates a gravitational theory in $(d+1)$-dimensions with a CFT on its d-dimensional boundary, the most known example involving $\mathcal{N}=4$ SYM and type IIB string theory on $AdS_5 \times S^5$.

After analysing in more detail how the duality relates conformal dimensions, masses and correlation functions between the two sides, we will conclude the Chapter with a discussion on the superalgebra of $\mathcal{N}=4$ SYM and its role in Integrability.

We also introduce the main notations that we will follow in this thesis, along with some important conventions.

2.1 The $\mathcal{N}=4$ Super Yang-Mills Theory

In this chapter, we will discuss various aspects of $\mathcal{N}=4$ Super Yang-Mills Theory, which are important for the thesis. We will discuss the foundations of the field theory, superconformal algebra, and its representation theory.

We will follow the "letters" formalism (see [32]), which is very convenient for our study and it is the standard notation used widely in the active research field literature. In this formalism, we collect the various fields, into a unique symbol, which we denote as $\mathcal{W}$[1]

$$\mathcal{W}_A = (\mathcal{D}_\mu, \Psi_{\alpha a}, \dot{\Psi}^a_{\dot{\alpha}}, \Phi_m) \tag{2.2}$$

where $\mathcal{D}$ is the covariant derivative, constructed by the gauge field $\mathcal{A}$, the spinors Ψ and the six scalars Φ. We replace the gauge fields $\mathcal{A}$ in our notation with the covariant derivative in order to have uniform gauge transformation property (on $\mathcal{W}$). We also follow the standard conventions for all the indexes[2].

In order to write down a real-valued Lagrangian, the signatures of spacetime and internal space must be correlated. We have collected all these possible choices in the table (2.1). We will not concern too much about the signature and we will either work with maximally non-compact or complex version of the algebra. We define the covariant derivative as:

$$\mathcal{D}_\mu = \partial_\mu - ig\mathcal{A}_\mu, \quad \mathcal{D}_\mu\mathcal{W} := [\mathcal{D}_\mu, \mathcal{W}] = \partial_\mu\mathcal{W} - ig\mathcal{A}_\mu\mathcal{W} + ig\mathcal{W}\mathcal{A}_\mu \tag{2.3}$$

where g is a dimensionless coupling constant which, in the classical theory,

[1] Note that the covariant derivative $\mathcal{D}$ is not a field. Instead of the gauge field $\mathcal{A}$, we shall place it here, so that all fields in $\mathcal{W}$ have uniform gauge transformations.

[2] We collect the notation for spinorial fields in the Appendix

can be absorbed into the fields, but it will play a crucial role in the quantum theory. Also, throughout this thesis, we will work with either $SU(N)$ or $U(N)$ gauge group and represent all the adjoint fields $\mathcal{W}$ by (traceless) hermitian $N \times N$ matrices

The transformation of the letters under unitary actions on the supergroup is given by,

$$\mathcal{W} \to U\mathcal{W}U^{-1} \tag{2.4}$$

As usual, we construct the field strength $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ as the change (i.e. the covariant derivative) between a vector and its parallel transport around the boundary defined by its arguments, and together with the second *Bianchi identity*[3]:

$$\mathcal{F}_{\mu\nu} = ig^{-1}\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig\left[\mathcal{A}_\mu, \mathcal{A}_\nu\right], \quad \mathcal{D}_{[\rho}\mathcal{F}_{\mu\nu]} = 0 \tag{2.5}$$

We are now ready to write down the Lagrangian of $\mathcal{N} = 4$ SYM, it is:

$$\begin{aligned}\mathcal{L}_{\text{SYM}}[\mathcal{W}] = &\frac{1}{4}\ \text{Tr}\ \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\ \text{Tr}\ \mathcal{D}^\mu\Phi^n\mathcal{D}_\mu\Phi_n \\ &- \frac{1}{4}g^2\ \text{Tr}\ [\Phi^m, \Phi^n][\Phi_m, \Phi_n] + \ \text{Tr}\ \dot{\Psi}^a_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}_\mu\mathcal{D}^\mu\Psi_{\beta a} \\ &- \frac{1}{2}ig\ \text{Tr}\ \Psi_{\alpha a}\sigma^{ab}_m\epsilon^{\alpha\beta}[\Phi^m, \Psi_{\beta b}] - \frac{1}{2}ig\ \text{Tr}\ \dot{\Psi}^a_{\dot{\alpha}}\sigma^m_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}[\Phi_m, \dot{\Psi}^b_{\dot{\beta}}]\end{aligned} \tag{2.6}$$

where σ are the chiral projections of the gamma matrices in four and six dimensions, and satisfy the usual algebra:

$$\sigma^{\{\mu}\sigma^{\nu\}} = \eta^{\mu\nu}, \quad \sigma^{\{m}\sigma^{n\}} = \eta^{mn}$$

also the ϵ symbols are the totally antisymmetric tensors of $\mathfrak{su}(2)$ and $\mathfrak{su}(4)$.

The equations of motion which follow from the variation of this action are:

$$\begin{cases} \mathcal{D}_\nu\mathcal{F}^{\mu\nu} = ig\left[\Phi_n, \mathcal{D}^\mu\Phi^n\right] - ig\sigma^{\dot{\alpha}\beta}_\mu\left\{\dot{\Psi}^a_{\dot{\alpha}}, \Psi_{\beta a}\right\} \\ \mathcal{D}_\nu\mathcal{D}^\nu\Phi^m = -g^2\left[\Phi_n, \left[\Phi^n, \Phi^m\right]\right] + \frac{1}{2}ig\sigma^{m,ab}\varepsilon^{\alpha\beta}\left\{\Psi_{\alpha a}, \Psi_{\beta b}\right\} + \frac{1}{2}ig\sigma^m_{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}\left\{\dot{\Psi}^a_{\dot{\alpha}}, \dot{\Psi}^b_{\dot{\beta}}\right\} \\ \sigma^{\dot{\alpha}\beta}_\mu\mathcal{D}^\mu\Psi_{\beta a} = ig\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^m_{ab}\left[\Phi_m, \dot{\Psi}^b_{\dot{\beta}}\right] \\ \sigma^{\alpha\dot{\beta}}_\mu\mathcal{D}^\mu\dot{\Psi}^a_{\dot{\beta}} = ig\varepsilon^{\alpha\beta}\sigma^{ab}_m\left[\Phi^m, \Psi_{\beta b}\right] \end{cases}$$

[3] Which reflects the notions of the boundary of a boundary term ,since $\mathcal{F} = d\mathcal{A}+\mathcal{A}\wedge\mathcal{A} \implies d\mathcal{F} = 0$

Our theory, as it can be shown from the Lagrangian and the EOM[4] , is invariant under the full $\mathcal{N}=4$ super Poincare algebra, consisting of the usual Lorentz symmetries and internal rotations: $\mathfrak{L}, \dot{\mathfrak{L}}, \mathfrak{R}$ of $\mathfrak{su}(2)\times\mathfrak{su}(2)\times\mathfrak{su}(4)$ as well as the supertranslations $\mathfrak{Q}, \dot{\mathfrak{Q}}, \mathfrak{P}$.

We parametrize the supertranslations using the fermionic and bosonic shifts $\epsilon_a^\alpha, \dot{\epsilon}^{\dot{\alpha}a}$ and e^μ i.e. :

$$\delta_{\epsilon,\dot{\epsilon},e} = \epsilon_a^\alpha \mathfrak{Q}_\alpha^a + \dot{\epsilon}^{\dot{\alpha}a}\dot{\mathfrak{Q}}_{\dot{\alpha}a} + e^\mu \mathfrak{P}_\mu \tag{2.7}$$

where

$$\begin{aligned}
\delta_{\epsilon,\dot{\epsilon},e}\mathcal{D}_\mu =& ig\epsilon_a^\alpha \varepsilon_{\alpha\beta}\sigma_\mu^{\beta\dot{\gamma}}\dot{\Psi}_{\dot{\gamma}}^a + ig\dot{\epsilon}^{a\dot{\alpha}}\varepsilon_{\dot{\alpha}\dot{\beta}}\sigma_\mu^{\dot{\beta}\gamma}\Psi_{\gamma a} + ige^\nu \mathcal{F}_{\mu\nu}\\
\delta_{\epsilon,\dot{\epsilon},e}\Phi_m =& \epsilon_a^\alpha \sigma_m^{ab}\Psi_{\alpha b} + \dot{\epsilon}^{a\dot{\alpha}}\sigma_{m,ab}\dot{\Psi}_{\dot{\alpha}}^b + e^\mu \mathcal{D}_\mu \Phi_m\\
\delta_{\epsilon,\dot{\epsilon},e}\Psi_{\alpha a} =& -\frac{1}{2}\sigma^\mu_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\gamma}}\sigma^\nu_{\dot{\gamma}\delta}\epsilon_a^\delta \mathcal{F}_{\mu\nu} + \frac{1}{2}ig\sigma^m_{ab}\sigma_n^{bc}\varepsilon_{\alpha\beta}\epsilon_c^\beta\left[\Phi_m, \Phi^n\right]\\
&+\sigma^n_{ab}\sigma^\mu_{\alpha\dot{\beta}}\dot{\epsilon}^{b\dot{\beta}}\mathcal{D}_\mu\Phi_n + e^\mu\mathcal{D}_\mu\Psi_{\alpha a}\\
\delta_{\epsilon,\dot{\epsilon},e}\dot{\Psi}^a_{\dot{\alpha}} =& -\frac{1}{2}\sigma^\mu_{\dot{\alpha}\beta}\varepsilon^{\beta\gamma}\sigma^\nu_{\gamma\delta}\dot{\epsilon}^{a\delta}\mathcal{F}_{\mu\nu} + \frac{1}{2}ig\sigma_m^{ab}\sigma^n_{bc}\varepsilon_{\dot{\alpha}\dot{\beta}}\dot{\epsilon}^c\dot{\beta}\left[\Phi^m, \Phi_n\right]\\
&+\sigma_n^{ab}\sigma^\mu_{\dot{\alpha}\beta}\epsilon_b^\beta\mathcal{D}_\mu\Phi^n + e^\mu\mathcal{D}_\mu\dot{\Psi}^a_{\dot{\alpha}}
\end{aligned} \tag{2.8}$$

The algebra of supertranslations resulting from the above variations is:

$$\begin{aligned}
&\left\{\mathfrak{Q}_\alpha, \mathfrak{Q}^b_\beta\right\} = -2ig\epsilon_{\alpha\beta}\sigma_m^{ab}\Phi^m, && \left[\mathfrak{P}_\mu, \mathfrak{Q}^a_\alpha\right] = -ig\varepsilon_{\alpha\beta}\sigma_\mu^{\beta\dot{\gamma}}\dot{\Psi}^a_{\dot{\gamma}}\\
&\left\{\dot{\mathfrak{Q}}_{\dot{\alpha}a}, \dot{\mathfrak{Q}}_{\dot{\beta}b}\right\} = -2ig\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^m_{ab}\Phi_m, && \left[\mathfrak{P}_\mu, \dot{\mathfrak{Q}}_{\dot{\alpha}a}\right] = -ig\varepsilon_{\dot{\alpha}\dot{\beta}}\sigma_\mu^{\dot{\beta}\gamma}\Psi_{\gamma a}\\
&\left\{\mathfrak{Q}^a_\alpha, \dot{\mathfrak{Q}}_{b\dot{\beta}}\right\} = 2\delta^a_b\sigma^\mu_{\alpha\dot{\beta}}\mathfrak{P}_\mu, && \left[\mathfrak{P}_\mu, \mathfrak{P}_\nu\right] = -ig\mathcal{F}_{\mu\nu}
\end{aligned} \tag{2.9}$$

It is very useful to introduce a more dense and unified notation by replacing all the vector indices by a pair of spinors, so we get

$$\begin{cases}
\mathcal{D}_\mu \sim \sigma_\mu^{\dot{\alpha}\beta}\mathcal{D}_{\dot{\alpha}\beta}\\
\mathcal{F}_{\mu\nu} \sim \sigma_\mu^{\alpha\dot{\gamma}}\varepsilon_{\dot{\gamma}\dot{\delta}}\sigma_\nu^{\delta\beta}\mathcal{F}_{\alpha\beta} + \sigma_\mu^{\dot{\alpha}\gamma}\varepsilon_{\gamma\delta}\sigma_\nu^{\delta\dot{\beta}}\dot{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}\\
\Phi_m \sim \sigma_m^{ba}\Phi_{ab}
\end{cases}$$

In this notation Φ_{ab} is antisymmetric while $\mathcal{F}_{\alpha\beta}$ and $\dot{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}$ are both symmetric.

Now the set of fields is given by:

$$\mathcal{W} = \left(\mathcal{D}_{\dot{\alpha}\beta}, \Phi_{ab}, \Psi_{\alpha b}, \dot{\Psi}^b_{\dot{\alpha}}, \mathcal{F}_{\alpha\beta}, \dot{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}\right)$$

which now are all bi-spinors.

The $\mathcal{N}=4$ gauge theory is a pure theory, which means that it only consists

[4]Equations of motion i.e. EOM

of the superspace gauge field, as such $\mathcal{N} = 4$ SYM must be a massless theory with additional (super)conformal symmetry to its traditional Poincaré symmetry.

2.2 The Quantum Theory

There are many ways to quantize a theory; in the present work, we conveniently work with the path integral quantisation. The path integral measures the expectation value of some operator functional $\mathcal{O}[\mathcal{W}]$ by summing over all field configurations weighted by the exponential of the action. We work in the Euclidean signature[5]

$$\langle \mathcal{O}[\mathcal{W}] \rangle := \int D\mathcal{W} \mathcal{O}[\mathcal{W}] \exp(-S[\mathcal{W}])$$

The Yang-Mills action S is the spacetime integral of the gauge theory Lagrangian

$$S[\mathcal{W}] = \frac{2}{g_{\text{YM}}^2} \int d^4x \mathcal{L}_{\text{YM}}[\mathcal{W}, g = 1]$$

where we have used the common definition of the Yang-Mills coupling constant g_{YM}. It will be more convenient to work with a different coupling constant:

$$g^2 := \frac{g_{\text{YM}}^2 N}{8\pi^2}$$

In this thesis, we will consider objects which are the local operators $\mathcal{O}(x)$ and their correlators $< \mathcal{O}(x_1)\mathcal{O}(x_2)... >$ as of major interest.

In particular the *two-point functions*: $< \mathcal{O}(x_1)\mathcal{O}(x_2) >$ are very important object that we will work very frequently. They describe the creation/annihilation of a particle propagating thought spacetime, by that operator.

2.3 Gauge Theory

In this section we will introduce some basic notation, in order to deal with the matrix representation of $(\mathcal{W}_{\mathcal{A}})^{\mathfrak{a}}_{\mathfrak{b}}$. For convince, we also introduce the notation $\check{\mathcal{W}}^{\mathcal{A}}$ such that[6]

[5] In order to avoid geometric abnormalities

[6] Most of the time we will not write out the matrix indices and we will just write $\check{\mathcal{W}}^{\mathcal{A}} := \frac{\delta}{\delta \mathcal{W}_{]\mathcal{A}}}$

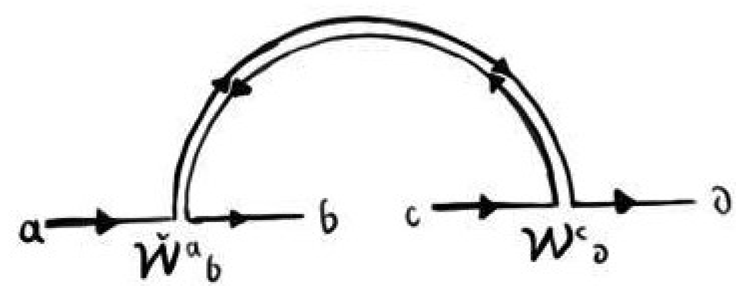


Figure 2.2: $\mathcal{W}$ and $\check{\mathcal{W}}$ contractions. We use the double line notation for adjoint fields in a gauge theory.

$$\left(\check{\mathcal{W}}^{\mathcal{A}}\right)^{\mathfrak{a}}_{\ \mathfrak{b}} := \frac{\delta}{\delta\left(\mathcal{W}_{\mathcal{A}}\right)^{\mathfrak{b}}_{\ \mathfrak{a}}}, \quad \left(\check{\mathcal{W}}^{\mathcal{A}}\right)^{\mathfrak{a}}_{\ \mathfrak{b}}\left(\mathcal{W}_{\mathcal{B}}\right)^{\mathfrak{c}}_{\ \mathfrak{d}} = \delta^{\mathcal{A}}_{\mathcal{B}}\delta^{\mathfrak{a}}_{\mathfrak{d}}\delta^{\mathfrak{c}}_{\mathfrak{b}} \tag{2.10}$$

We furthermore introduce normal ordering :... : which suppresses all possible contractions between fields and variations by moving all variations to the right, for example

$$: \ldots \left(\check{\mathcal{W}}^{\mathcal{A}}\right)^{\mathfrak{a}}_{\ \mathfrak{b}} \ldots \left(\mathcal{W}_{\mathcal{B}}\right)^{\mathfrak{c}}_{\ \mathfrak{d}} \ldots : \ := \ \ldots \left(\mathcal{W}_{\mathcal{B}}\right)^{\mathfrak{c}}_{\ \mathfrak{d}} \ldots \left(\check{\mathcal{W}}^{\mathcal{A}}\right)^{\mathfrak{a}}\ \mathfrak{b}$$

In this notations, it is very convenient to introduce the variation introduced by the matrix gauge transformation of the form $\mathcal{W} \mapsto U\mathcal{W}U^{-1}$ which is generated by

$$\delta_\epsilon \mathcal{W} = i[\epsilon, \mathcal{W}] \quad \text{i.e. } \delta_\epsilon = \ \mathrm{Tr}\ \epsilon\mathfrak{j} \quad \text{with} \quad \mathfrak{j} = \ i : [\mathcal{W}_{\mathcal{A}}, \check{\mathcal{W}}^{\mathcal{A}}] : \tag{2.11}$$

Going away from the matrix representation, we can more general work for the $SU(N)$ gauge group with generators $\mathfrak{t}_{\mathfrak{m}}$, we use the standard notation for the fields:

$$\mathcal{W}_{\mathcal{A}} = \mathcal{W}^{\mathfrak{m}}_{\mathcal{A}}\ \mathfrak{t}_{\mathfrak{m}} \tag{2.12}$$

where

$$\mathrm{Tr}\,\mathfrak{t}_{\mathfrak{m}}\mathfrak{t}_{\mathfrak{n}} = \mathfrak{g}_{mn}, \quad [\mathfrak{t}_{\mathfrak{m}}, \mathfrak{t}_{\mathfrak{n}}] = i\mathfrak{f}^{\mathfrak{p}}_{\mathfrak{mn}}\mathfrak{t}_{\mathfrak{p}} \tag{2.13}$$

in which the structure constants $\mathfrak{f}^{\mathfrak{p}}{}_{\mathfrak{m}\mathfrak{n}}$ and the generators are normalised adequately.

This leads to the more general definition for the variations:

$$\tilde{\mathcal{W}}^{\mathcal{A}} := \mathfrak{t}_{\mathfrak{m}} \mathfrak{g}^{\mathfrak{m}\mathfrak{n}} \frac{\delta}{\delta \mathcal{W}^{\mathfrak{n}}_{\mathcal{A}}}, \quad \frac{\delta}{\delta \mathcal{W}^{\mathfrak{m}}_{\mathcal{A}}} \mathcal{W}^{\mathfrak{n}}_{\mathcal{B}} = \delta^{4}_{\mathcal{B}} \delta^{\mathfrak{n}}_{\mathfrak{m}} \tag{2.14}$$

2.4 The AdS/CFT correspondence

2.4.1 Why in quantum gravity different?

Fundamental physics has been governed by reductionism since its beginning. And the discovery of quantum mechanics taught us that we cannot refine our ”microscopes” without using larger energies/momenta. When you study higher and higher energies, sometimes you hit some critical energy points where the physics changes drastically, and you can try to adjust or create new models to describe the new mechanisms in this energy scale. For example, when we first probed the QCD around a GeV, we found a plethora of new strongly interactive particles which we could understand their mechanisms. So we create new theories, like primitive-String theory, EFT-models, and new ways to use local quantum fields to describe the theory.

On the other hand, Gravity differs tremendously for all the QFT-examples. If you try to probe to the energy scales corresponding to Plank length, you would just get black holes due to the attractive nature of gravity, but we can make black holes without passing through a regime of physics that we do not understand (Gedanken experiments), and pushing more energy will just result in a larger and larger black hole, so our reductionist method stops working.

In hindsight, we have many hints of how to proceed. The most simple hint comes from black hole thermodynamics in which the entropy of a black hole is proportional to its area A:

$$S_{BH} = \frac{A}{4\ell_{pl}^2} \tag{2.15}$$

where ℓ_{pl}^2 is the Plank length. Since you can throw any type of information into a black hole and the entropy increases regardless, the BH entropy must be some fundamental feature of the universe. This hints to us that our notion of spacetime(i.e. General Relativity) is just an approximation of a real (maybe algebraic?) quantity that generalizes our today’s notion of spacetime and in which information is stored in the boundary. But we know also that gravitational energy is not well-defined locally, and can be made well defined only if we look/measure the system from the infinity(its boundary!). So in well-defined gravitational theories, the Hamiltonian should live at its boundary(at the infinity). And the AdS/CFT conjecture is one of the most promising answers to this search.

2.4.2 Understanding the big picture

The big and very simplistic picture of AdS/CFT is that, *any complete theory of quantum gravity(QG) in an asymptotically AdS spacetime defines a CFT.* One can think of the AdS background as a "gravity in a box"

The AdS/CFT essentially implies that the Hilbert spaces are isomorphic:

$$\mathcal{H}_{CFT} \cong \mathcal{H}_{AdS-QG} \tag{2.16}$$

and all global symmetries can be matched between the two sides.

2.4.3 The statement

The AdS/CFT predicts the exact equivalence of $\mathcal{N} = 4$ super Yang-Mills theory (often abbreviated as SYM) with a IIB supersymmetric string theory propagating on $AdS_5 \times S^5$ background. One of the most fascinating tests of this statement is the complete matching between their global symmetry (super-)groups.

Even though agreement of the symmetry groups is far from being a sufficient reason to prove a full duality between the two sides, we can notice amazing hints from studying them, after all, there are no coincidences in mathematics.[7] The $\mathcal{N} = 4$ superconformal symmetry (see [37],[12],[8],[9],[10],[11],[5]) on the SYM side and the isometries of the $AdS_5 \times S^5$ superspace on the superstrings side, are both given by the same supergroup $P\tilde{S}U(2,2|4)$ or its algebra $\mathfrak{psu}(2,2|4)$.

The matching of symmetries hints to us that our two (to be dual) theories have similar properties and dualities between structural constraints e.g. correlation functions. Furthermore, the existence of supersymmetry implies that (probably) exist quantities that are protected from quantum corrections which will occur as we quantize the theory – e.g. on $\mathcal{N} = 4$ SYM, the absence

[7] That is most of the activity of Integrability of AdS/CFT, to make tests involving non-protected(by symmetry) dynamical quantities

of the beta-function and the exactness of correlators for certain BPS-operators as the AdS/CFT predicts [37].

We will split this section into two subsections:

- One involving the standard text-book study of AdS/CFT, the D3-branes and their two dual faces, the field-map operator map, and some brief discussion about the correlation functions.
- And one involving the study of some relevant aspects of Lie superalgebra $\mathfrak{psu}(2,2|4)$. For an introduction into Integrability of AdS/CFT we refer to [36].

2.5 The AdS/CFT correspondence

We can make various formulations of AdS/CFT, but in this thesis, we will work with the original formulation on $\mathcal{N}=4$ SYM on $3+1$-dimensions and type IIB superstring theory on $AdS_5 \times S^5$. Then, the strong form of the conjecture states that:

Theorem: Strong form of the AdS/CFT conjecture:

The $\mathcal{N}=4$ SYM with $SU(N)$ gauge group and g_{YM} as coupling constant is **dynamically** equivalent to a type IIB superstring theory with string length $\sqrt{\alpha'}$ and g_s string coupling constant on $AdS_5 \times S^5$ background with radius of curvature L and N units of $F_{(5)}$ flux on S^5.

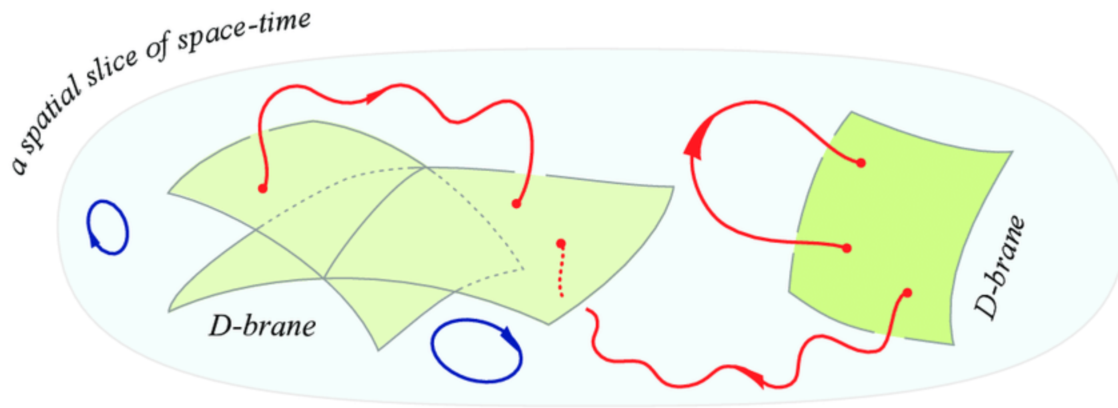
The free parameters of the field theory side are mapped to the free parameters on the sting theory side by:

$$g_{\text{YM}}^2 = 2\pi g_s \quad \text{and} \quad 2g_{\text{YM}}^2 N = L^4/(\alpha')^2 \tag{2.17}$$

Essentially the correspondence states, that the two theories, the $\mathcal{N}=4$ $SU(N)$-SYM CFT side and the string theory side are dual theories, in the sense that they describe the exact same physics from two different perspectives, and by extension, every object of one theory can be mapped to an object on the other side. Moreover, the AdS/CFT correspondence is a realization of the *holographic principle* since, the information is projected to the boundary conformal theory of the five-dimensional theory, from a Kaluza-Klein-reduction of the string theory around the S^5 sphere.

Forms of AdS/CFT correspondence		
	$\mathcal{N}=4$ SYM	IIB on $AdS_5 \times S^5$
Strongest form	any N and λ	Quantum string theory, $g_s \neq 0, \alpha'/L^2 \neq 0$
Strong form	$N \to \infty, \lambda$ fixed but arbitrary	Classical string theory, $g_s \to 0, \alpha'/L^2 \neq 0$
Weak form	$N \to \infty, \lambda$ large	Classical supergravity, $g_s \to 0, \alpha'/L^2 \to 0$

The original statement of the correspondence stated above is unfortunately too strong in order to be used for calculations. Thus we have to soften a bit the statement by adjusting a bit its two sides and working on the effective theories and/or limited cases of our parameters e.g. $N \to \infty$, λ fixed and quantum corrections on the string side turned to zero i.e. $g_s \to 0$, α'/L^2 non zero, i.e. non interacting big classical string. For a full discussion we refer to [36],[8]. Here we will only present the table below:



2.5.1 The duality of $D3$-branes

In this section we will study a particular, very education example of AdS/CFT, and in particular on the weak form of the correspondence. We will study the two perpectives of $D3$ branes: *open string* and *closed string* perspective. For an amazing treatment on string theory and Dp-branes and more, we refer to [12], [13], [15], [16], [17], [18], [19].

Open string perspective

We can visualize the D-branes as the higher dimension points where open strings can end. Since the open strings vibrate, this can only occur when we only have a very small perturbation i.e. when the $g_s \to 0$, moreover if we neglect the excitations of the string i.e. low energies, the dynamics of the open string is described by a supersymmetric gauge theory living on the worldvolume of the D-branes. The excitations of the open strings on the $D3$-branes are the A^μ gauge fields, while the transverse excitations of the string are the scalar fields from the worldvolume point of view. By stacking N D-branes we get the gauge group $U(N)$. Working a bit with our free parameters, the stacking of N D-branes produces an effective coupling constant on the gauge filed theory, given by g_sN which must be very small, since we have weak perturbations, i.e. $g_sN \to 0$.

Now let's look at the math:

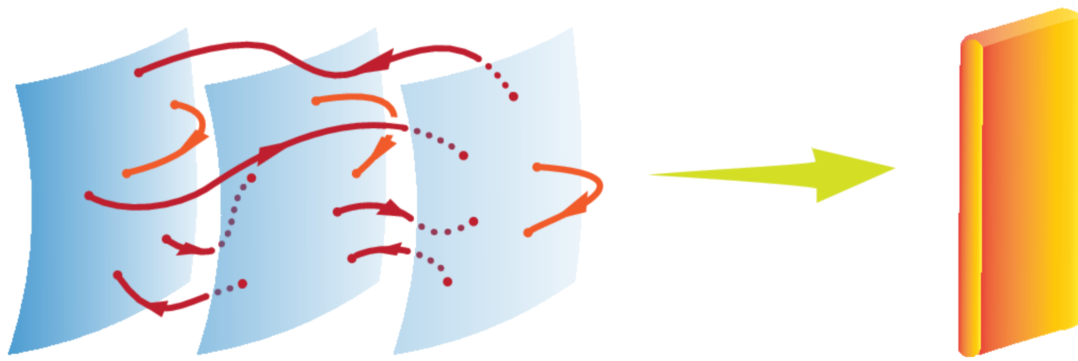


Figure 2.3: The stacking of many $D3$-branes on the open string perspective

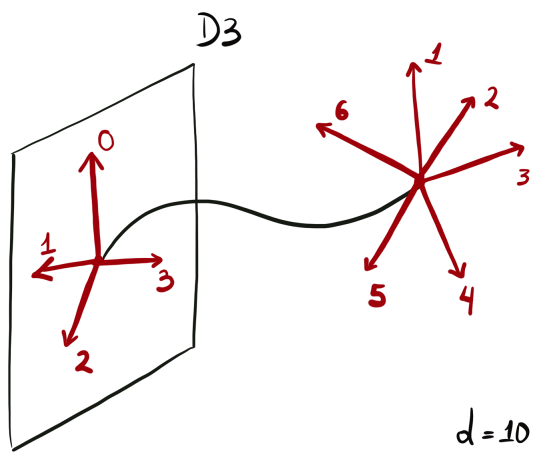


Figure 2.4: A picture of an open string and its degrees of freedom in each space

As we discussed, in our open string sector, perturbative string theory[8] only makes sense in the $g_s N \ll 1$ limit.

The scheme of the perturbation theory is: open strings(i.e. excitations of the $(3+1)$-dimensions-hyperplane) which start and end at the $D3$-branes and closed strings(i.e. excitations of the $(9+1)$-dimensions flat spacetime). Also, as we discussed previously, we only take into account massless excitations, since every other excitation has energies of the order $\alpha^{'-1/2}$.

The *massless open strings* can be grouped into a four-dimensional $\mathcal{N}=4$ supermultiplet, consisting of the gauge field A^μ and six bosonic scalar field ϕ^i along with their fermionic superpartners ψ. More precisely, the massless open strings give rise to a gauge field A^μ who lives longitudinal to the $D3$-branes and to six-bosonic fields ϕ^i who live in the transversal direction. The interactions between the N different D3-branes are achieved by massless open strings like in figure 2.4.

The *massless closed strings* are essentially the ten-dimensional $\mathcal{N}=1$ supergravity multiplet who lives on the whole flat-spacetime.

The complete theory of the open-string sector is written :

$$\mathcal{S} = \mathcal{S}_{\text{open}} + \mathcal{S}_{\text{closed}} + \mathcal{S}_{\text{int}} \tag{2.18}$$

where,$\mathcal{S}_{\text{open}}$, $\mathcal{S}_{\text{closed}}$ are the effective action for the corresponding strings, and S_{int} is the effective action for the interactions between open and closed strings. The closed strings action is the action of the ten-dimensional supergravity plus some higher derivative terms containing the *Kalb-Ramond field* B_{MN} and the *Kalb-Ramond field strength* $H = dB$:

$$\begin{aligned} \mathcal{S}_{\text{closed}} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-g}e^{-2\phi}(R + 4\partial_M\phi\partial^M\phi\Big) + \cdots \\ &\sim -\frac{1}{2}\int \mathrm{d}^{10}x\partial_M h\partial^M h + \mathcal{O}(\kappa) \end{aligned} \tag{2.19}$$

where R is the Ricci scalar, $2\kappa^2 = (2\pi)^7\alpha^{'4}g_s^2$, g_{MN} the metric and ϕ the dilaton. In the second line, we have expand our metric around small perturbations of the flat metric i.e. $g = \eta + \kappa h$, where h is a fluctuations metric and κ a normalisation factor. We have also omitted the display of the Ramond-Ramond(RR)form fields as well as fermionic fields, since we will only

[8]We study type IIB superstring theory in flat $(9+1)$-dimensional Minkowski spacetime where we also embed N coincident $D3$-branes. Our particular configuration of $D3$-branes i.e. the imposing of the boundary conditions breaks half for the 32 supercharges of the type IIB superstring theory

work in the bosonic sector.

To derive the open strings action and the interactions, we have to use the infamous Dirac-Born-Infeld action[9] for a single $D3$-brane [15],[16]:

$$\boxed{\mathcal{S}_{\text{DBI}} = -\frac{1}{(2\pi)^3\alpha'^2 g_{\text{s}}} \int \mathrm{d}^4x e^{-\phi} \sqrt{-\det\left(\varphi * [g]_{ab} + \varphi * [B]_{ab} + 2\pi\alpha' F\right)} \pm \mathcal{S}_{CS}} \tag{2.20}$$

where the $\varphi*$ denotes the pull-back of the NS-NS sector bulk fields g_{MN} and B_{MN}:

$$\varphi * [g]_{ab} = \frac{\partial X^M}{\partial \xi^a}\frac{\partial X^N}{\partial \xi^b} g_{MN} \tag{2.21}$$

and $\mathcal{S}_{CS}$ are additional actions from non trivial couplings to the R-R forms. The R-R forms $C_{(p)}$ define the topological charges for the Dp-branes in the natural way i.e.

$$S_p = \frac{\tau_p}{g_s}\int_{\Sigma_{p+1}} \varphi * [C_{(p+1)}] \tag{2.22}$$

and give rise to *Chern-Simons terms*:

$$\mathcal{S}_{\text{CS}} = \mu_p \int \sum_q \varphi * \left[C_{(q+1)}\right] \wedge e^{\varphi*[B]+2\pi\alpha' F} \tag{2.23}$$

We will not study this terms in this thesis, for more informations we reference the reader to [15],[17].

Setting the Kalb-Ramond field and the Chern-Simons terms to zero and setting $x^i = 2\pi\alpha'\phi^i$ then the *pullback* (see [20],[21]) of the metric to the worldvolume is now given by the expression:

$$\varphi * [g]_{\mu\nu} = g_{\mu\nu} + (2\pi\alpha')\left(g_{i+3}\nu\partial_\mu\phi^i + g_{\mu j+3}\partial_\nu\phi^j\right) + (2\pi\alpha')^2 g_{i+3j+3}\partial_\mu\phi^i\partial_\nu\phi^j \tag{2.24}$$

Expanding the $e^{-\phi} \simeq 1 + \kappa\phi$, $\det(1+M) = 1 - \frac{1}{2}\,\mathrm{Tr}\,(M^2)$ and $g = \eta + \kappa h$, we find to leading order in α',

[9] You can intuitively understand the DBI action as the action which minimizes the hyper-volume of the Dp-brane, in the abscense of Kalb-Ramond and gauge field fields. So we can think the DBI action as generalisation of the worldsheet action of strings to higher dimensions, this is precisely the motivation behind the existence of Dp-branes. However, unlike fundamental strings, Dp-branes are non-pertubative objects.

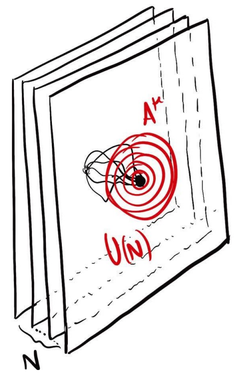


Figure 2.5: The emergence of a $U(N)$ gauge field living on the $D3$-branes by stacking of N branes in a flats-spacetime, form the open string perspective

$$\begin{aligned}\mathcal{S}_{\text{open}} &= -\frac{1}{2\pi g_{\text{s}}}\int \mathrm{d}^4x\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i+\mathcal{O}\left(\alpha'\right)\right)\\ \mathcal{S}_{\text{int}} &= -\frac{1}{8\pi g_{\text{s}}}\int \mathrm{d}^4x\phi F_{\mu\nu}F^{\mu\nu}+\cdots\end{aligned}$$

To generalise the discussion to N coincident $D3$-branes[10], we promote the scalars and gauge field to $U(N)$ group objects:

$$\phi^i=\phi^{ia}t_a \qquad A_\mu = A^a_\mu t_a \tag{2.25}$$

and we trace over the gauge group in order to ensure gauge invariance. We also promote the derivatives to covariant derivatives, as usual, and we add the scalar potential V to the open string action :

$$V=\frac{1}{2\pi g_{\text{s}}}\sum_{i,j}\operatorname{Tr}\left[\phi^i,\phi^j\right]^2 \tag{2.26}$$

As we discussed in the previous section, we can take the limit: $a'\to 0$. In this limit, the open string action is the $\mathcal{N}=4$ SYM bosonic action with $g^2_{YM}=2\pi g_s$. The interactions vanish since the dilaton ϕ scales as $\kappa\sim a'^2\to 0$ and so the open and closed strings decouple. Lastly, the closed strings in this limit are just the ten-dimensional Minkowski spacetime supergravity, which is exactly the picture we have discuss in the introductory discussion.

A pictorial representation of the stacking of N $D3$-branes can been seen in the figures 2.3 and 2.4.

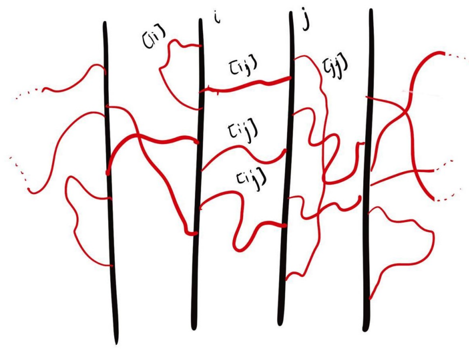


Figure 2.6: The open string sector, Chan-Paton factors

Closed string perspective

We can also look at the D-branes as solitonic solutions of the low-energy limit of superstring theory(i.e. supergravity), so from this perspective the $D3$-branes are sources of gravitational field which curve its surrounding spacetime. In order to have a weak curvature, and essentially supergravity, we must consider large characteristic length scale L.

We now work in the strongly coupled limit: $g_s N \to \infty$. In this regime, we will have to work with the closed strings. The N stacking of $D3$-branes can be viewed as massive charged objects which source type IIB supergravity and by extension type IIB superstring theory.

Our problem now, is to solve the corresponding supergravity problem. We seek SUGRA solutions of the N $D3$-branes which preserve the isometries of $\mathbb{R}^{9,1}$ i.e. the $SO(3,1) \times SO(6)$ and of course, preserve the 1/2-BPS, which means that half of the supercharges will annihilate the solution itself.

To achieve this we make the following ansatz:

$$\boxed{\begin{aligned} ds^2 &= H(r)^{-1/2}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + H(r)^{1/2}\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \\ e^{2\phi(r)} &= g_{\mathrm{s}}^2 \\ C_{(4)} &= \left(1 - H(r)^{-1}\right)\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \cdots \end{aligned}} \tag{2.27}$$

where $\mu, \nu = 0, ..., 3$ and $i, j = 1, ..., 9$ and $r^2 = \sum_{i=4}^{9} x_i^2$. We will not be

[10]For oriented strings.

interested in the higher-form potentials, and then from now on we will neglect such terms.

We insert now our ansatz into the equations of motion for the type IIB SUGRA:

$$\mathcal{S}_{\mathrm{IIB}} = \frac{1}{2\tilde{\kappa}_{10}^2}\left[\int \mathrm{d}^{10}X\sqrt{-g}\left(e^{-2\phi}\left(R + 4\partial_M\phi\partial^M\phi - \frac{1}{2}\left|H_{(3)}\right|^2\right)\right.\right.$$
$$\left. - \frac{1}{2}\left|F_{(1)}\right|^2 - \frac{1}{2}\left|\tilde{F}_{(3)}\right|^2 - \frac{1}{4}\left|\tilde{F}_{(5)}\right|^2\right)$$
$$\left. - \frac{1}{2}\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)}\right]$$

where we use the notation:

$$\int \mathrm{d}^{10}X\sqrt{-g}\left|F_{(p)}\right|^2 = \frac{1}{p!}\int \mathrm{d}^{10}X\sqrt{-g}g_{M_1N_1}\cdots g_{M_pN_p}F^{M_1\cdots M_p}F^{N_1\cdots N_p}$$

and also:

$$^*\tilde{F}_{(5)} = \tilde{F}_{(5)} \qquad \text{Self Duality} \tag{2.28}$$

Firstly, we find that the function $H(r)$ has to be harmonic, i.e. to satisfy the condition $\Box_p H(r) = 0$ which has solution :

$$H(r) = 1 + \frac{L^4}{r^4} \tag{2.29}$$

where $L^4 = 4\pi g_s N\alpha'^2$ which can be calculated using the fact that the flux of the $F_{(5)}$ through the sphere S^5 has to be quantized, since the flux measures the number of coincident Dp-branes.

For large $r \gg L$ the $H(r) \simeq 1$ and the metric reduces to ten-dimensional flat spacetime.

For small $r \ll L$, we call this region *near-horizon region* or *throat* and then $H(r) \simeq \frac{L^4}{r^4}$ and the metric takes the form:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{r^2}{L^2}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{L^2}{r^2}\delta_{ij}\ \mathrm{d}x^i\ \mathrm{d}x^j \\ &= \frac{L^2}{z^2}\left(\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}z^2\right) + L^2\ \mathrm{d}s^2_{S^5} \qquad \cong AdS_5 \times S^5 \end{aligned} \tag{2.30}$$

where $z = L^2/r$ and we have used the regular spherical coordinates instead of the flat ones:

$$\delta_{ij}dx^i dx^j = dr^2 + r^2 ds^2_{S^5} \tag{2.31}$$

From this analysis, we see that in this regime ($\alpha' \to 0$), we have two different kind of closed strings:[11] closed strings living on the standard flat ten-dimensional spacetime i.e. IIB sugra modes on 10-dim flat space and closed strings propagating in the near-horizon region with $AdS_5 \times S^5$ geometry i.e. IIB sugra excitations on $AdS_5 \times S^5$.

Combining of the two perspective

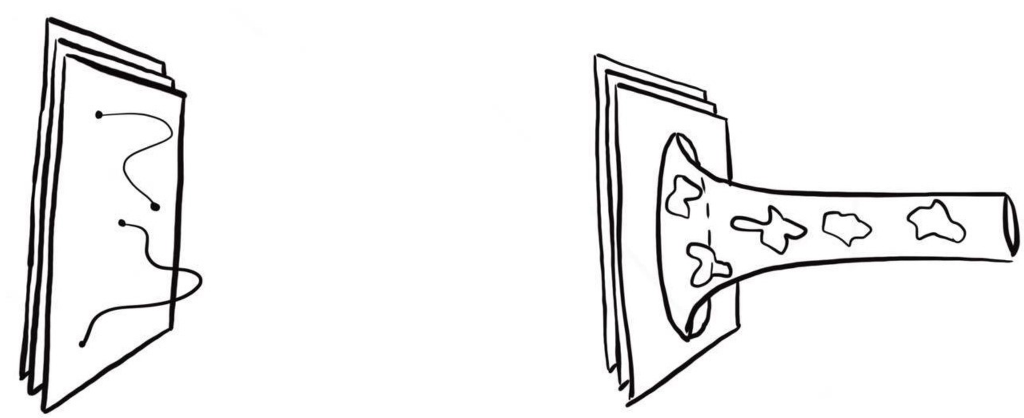


Figure 2.7: D-branes: The open(closed) string perspective left(right)

The two regime analysed in our extended previous discussion should correspond to the same physics, and type IIB sugra on $\mathbb{R}^{9,1}$ is present on both sides. This hints that the two theories should also be identified and this is exactly one way to motivate the *Maldacena conjecture* that $\mathcal{N} = 4$ SYM in four-dimensions is equivalent to type IIB sugra on $AdS_5 \times S^5$, even though their fundamental degrees of freedom are quite different.

2.6 The field-map operator

Since the AdS/CFT conjecture implies a connection between the two different theories, there should be a map between the two regimes which relates operators from the one side to the other. This is exactly what we will explore in this section. This map between this two theories, is called a *dictionary* and its existence arises from the fact that both sides have matching symmetries, which allows the field theory operators in some particular representation of the

[11] Decoupled strings

$PSU(2,2|4)$ to be mapped to $AdS_5 \times S^5$ string theory states on the other side of the correspondence.

Given that we have established that the global symmetry groups on both sides of the AdS/CFT correspondence coincide, it remains to show that the actual representations of the supergroup $SU(2,2 \mid 4)$ also coincide on both sides. Suffice it to recall here the special significance of the short multiplet representations, namely 1/2 BPS representations with a span of spin $2, 1/4$ BPS representations with a span of spin 3 and 1/8 BPS representations with a span of spin 7/2. Non-BPS representations in general have a span of spin 4 .

A very crucial role is played by the *single color trace operators* because out of them, all higher trace operators may be constructed using the OPE (Operator Product Expansion)[22],[23]. Thus one should expect single trace operators on the SYM side to correspond to single particle states (or canonical fields) on the AdS side [1]. Multiple trace states should then be interpreted as bound states of these one particle states. Multiple trace BPS operators have the property that their dimension on the AdS side is simply the sum of the dimensions of the BPS constituents. Such bound states occur in the spectrum at the lower edge of the continuum threshold and are therefore called threshold bound states. A good example to keep in mind when thinking of threshold bound states in ordinary quantum field theory is another case of BPS objects : magnetic monopoles in the Bogomol'nyi-Prasad-Sommerfield limit (BPS) (or exactly in the Coulomb phase of $\mathcal{N}=4$ SYM). A collection of N magnetic monopoles, were like charges, forms a static solution of the BPS equations and therefore form a threshold bound state.

We describe all type IIB massless sugra and massive string DOFs by field φ living on $AdS_5 \times S^5$. We also introduce coordinates $z^\mu \in AdS_5$ and $y^u \in S^5$, and decompose the metric as

$$ds^2 = g^{AdS}_{\mu\nu} dz^\mu dz^\nu + g^S_{uv} dy^u dy^v \tag{2.32}$$

Then the fields become functions of z, y associated with the various $D = 10$ DOFs. We decompose the fields φ as Kaluza-Klein towers on S^5 by expanding into a complete set of spherical harmonics $Y^l(\Omega_5)$ of S^5:

$$\varphi(z,y) = \sum_{l=0}^{\infty} \varphi^l(z) Y^l(y) \tag{2.33}$$

For scalars for example, Y^l are labelled by the rank $l = \Delta$ of the totally symmetric traceless representation $[l, 0, 0]$ of $SO(6)$(or equivalent the $[0, l, 0]$ of $SU(4)$)[12]. Just as fields on a circle receive a mass contribution from the momentum mode on the circle, so also fields compactified on S^5 receive a mass contribution. From the eigenvalues of the Laplacian on S^5 for various spin, we calculate the following relations between mass and scaling dimensions $l = \Delta$ and with the radii L of the S^5 set to one:

$$\begin{aligned} \text{scalars} \quad & m^2 = \Delta(\Delta - 4) \\ \text{spin } 1/2, 3/2 \quad & |m| = \Delta - 2 \\ p- \text{ form} \quad & m^2 = (\Delta - p)(\Delta + p - 4) \\ \text{spin } 2 \quad & m^2 = \Delta(\Delta - 4) \end{aligned} \tag{2.34}$$

We summarise the complete mapping between the presentations of $SU(2, 2|4)$ in the table below:

Type IIB string theory	$\mathcal{N} = 4$ conformal super-Yang-Mills
Supergravity Excitations 1/2 BPS, spin ≤ 2	Chiral primary + descendants $\mathcal{O}_2 = \mathrm{tr}\, X^{\{i} X^{j\}} +$ desc.
Supergravity Kaluza-Klein 1/2 BPS, spin ≤ 2	Chiral primary + Descendants $\mathcal{O}_\Delta = \mathrm{tr}\, X^{\{i_1} \cdots X^{i_\Delta\}} +$ desc.
Type IIB massive string modes non-chiral, long multiplets	Non-Chiral operators, dimensions $\sim \lambda^{1/4}$ e.g. Konishi tr $X^i X^i$
Multiparticle states	products of operators at distinct points $\mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n)$
Bound states	product of operators at same point $\mathcal{O}_{\Delta_1}(x) \cdots \mathcal{O}_{\Delta_n}(x)$

(2.35)

The mapping of the descendant states is also very interesting and can be found in the work of [23].

[12] The spherical harmonics satisfy the equation $\Box_{S^5} Y^l = -\frac{1}{L^2} l(l+4) Y^l$ which is the corresponding object on S^5 as the equation for the Hydrogen atom where one derives the standard spherical harmonics for S^3.

2.7 Correlation functions

In our previous sections, we have shown that there is a dictionary (a duality) between field theory operators $\mathcal{O}$ and gravity fields φ in the same representation of the symmetry group of isometry, respectively.

In this context, the boundary value of the gravity fields φ_∂ acts as a source for the field operator $\mathcal{O}$. This suggests a duality between the generating functionals on both side of the correspondence.

In this section, we will present a more detailed version of the AdS/CFT correspondence by mapping the correlators on both sides of the conjecture.

2.7.1 Mapping SYM and AdS correlators

To ensure regularity we work on Euclidean AdS_5, with metric:

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu) \tag{2.36}$$

Often we will graphically represent this space as a disc, whose boundary is a circle, as we will see graphically in the next paragraph on Witten diagrams, [7],[37]. We notice that the metric diverges at the boundary, because of the overall scale factor, but using Weyl rescaling we can avoid this blown up in the metric; but with a small caveat, such a rescaling in not unique. A unique well-defined limit to the ∂AdS_5 can only exist if the boundary theory is scale invariant. For finite $z > 0$ values the geometry will still be Poincare invariant but it doesn't have to be scale invariant.

On the one side, $\mathcal{N} = 4$ SYM is scale invariant and thus is a perfect candidate as a boundary theory. Its dynamical observables are local gauge invariant polynomial operators which live on the boundary and are characterized by their dimension, Lorentz group $SO(1,3)$ and $SU(4)_R$ quantum numbers.

On the other side, the AdS side, as we discussed , we decompose all 10-dimensional fields onto KK-towers on S^5, so all fields are labelled by their scaling dimension Δ. Going away from the bulk interaction region, the bulk fields are free asymptotically. The free fields then satisfy the Klein-Gordon(KG) equation $(\Box + m_\Delta^2)\varphi_\Delta^0 = 0$ with $m_\Delta^2 = \Delta(\Delta - 4)$ for scalars. Thus we get[13]

[13] with $z_0 \to 0$

$$\varphi^0_\Delta\left(z_0,\vec{z}\right)=\begin{cases} z_0^{\Delta} & \text{normalizable} \\ z_0^{4-\Delta} & \text{non-normalizable}\end{cases} \tag{2.37}$$

In [52] it is argued that the normalizable modes determine the VEV of operators of associated dimensions and quantum numbers. On the other hand, the non-normalizable solutions, do not correspond to bulk excitations because they are not properly square normalizable. They represent the coupling of external sources to the string theory (or its low energy effective theory i.e. sugra). The non-normalizable solutions define *associated boundary fields* $\bar{\varphi}_\Delta$ as:

$$\bar{\varphi}_\Delta(\vec{z})\equiv\lim_{z_0\to 0}\varphi_\Delta\left(z_0,\vec{z}\right)z_0^{4-\Delta} \tag{2.38}$$

We introduce now a generating functional $W[\bar{\varphi}_\Delta]$ for all the correlators of single trace operators $\mathcal{O}_\Delta$ on the SYM side with source fields $\bar{\varphi}_\Delta$,

$$Z[\bar{\varphi}_\Delta]=\exp\left\{-W\left[\bar{\varphi}_\Delta\right]\right\}\equiv\left\langle\exp\left\{\int_{\partial H}\bar{\varphi}_\Delta\mathcal{O}_\Delta\right\}_{\mathbb{E}}\right\rangle_{\mathrm{CFT}} \tag{2.39}$$

where H is the AdS disc whose boundary is a circle $\partial H=\mathbb{R}^4$, as we discussed previously.

On the AdS side, the action $S[\varphi_\Delta]$ summarizes the dynamics of the type IIB string theory on the $AdS_5\times S^5$. In the sugra approximation, $S[\varphi_\Delta]$ is just the type IIB sugra action on $AdS_5\times S^5$. Going away from the sugra approximation, the $S[\varphi_\Delta]$ will now also include stringy corrections due to massive string effects.

The mapping between correlators is given by

$$\boxed{\Gamma[\bar{\varphi}_\Delta]=\mathrm{extr}\left\{S[\varphi_\Delta]\right\}} \tag{2.40}$$

where by extr we mean the extremum, which on the RHS is taken over all fields φ_Δ that satisfy the asymptotic behaviour (2.33) for the boundary fields that are the sources of $\mathcal{O}_\Delta$ of SYM of the LHS.[14]

[14] The map between generating functionals is the starting point for the holographic calculations of correlation functions of composite gauge invariant operators, where for every $\mathcal{O}_i$ on the field side we obtain a gravity field on the boundary side $\bar{\varphi}^i$ by the relation:

2.7.2 1/N expansion and Witten diagrams

The actions that we are interested , have an overall coupling constant factor. For example, the part of the type IIB sugra for the dilaton field Φ and the axion C is given by

$$S[G,\Phi,C] = \frac{1}{2\kappa_5^2}\int_H \sqrt{G}\left[-R_G + \Lambda + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{2}e^{2\Phi}\partial_\mu C\partial^\mu C\right] \tag{2.42}$$

where $\kappa = 4\pi^2/N^2$, where N comes from the gauge group e.g. $SU(N)$. For large N, the theory simplifies considerably since the coupling constant κ becomes very small and we can perform semi-classical expansion of the correlators generated by this action. The result, is a set of rules, very similar to Feynman rules, which we call *Witten diagrams.* We represent them by a disc, whose interior corresponds to the interior of the AdS while the boundary circle corresponds to the ∂AdS. The corresponding Feynman rules are as follows,

- Each external source to $\bar{\varphi}_\Delta(x_I)$ is located at the boundary of the WItten diagram at a point x_I.
- Propagators depart from external sources, either to another boundary point or to an interior interaction point via a *boundary-to-bulk propagator.*
- The structure of the interior points is governed by the interaction vertices of the action S, just as in Feynman diagrams. We can derive these interactions terms by performing a KK-reduction of type IIB sugra on S^5.
- Two interior interactions points may be connected by a *bulk-to-bulk propagators*, again following the rules of ordinary Feynman rules.

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle_{\mathrm{CFT,c}} = -\left.\frac{\delta^n W}{\delta\bar{\varphi}^1(x_1)\,\delta\bar{\varphi}^2(x_2)\ldots\delta\bar{\varphi}^n(x_n)}\right|_{\bar{\varphi}^i=0} \tag{2.41}$$

A simple formula for obtaining gauge invariant operators $\mathcal{O}$ from the gravity side is:

- Determine bulk field φ which is dual to $\mathcal{O}_\Delta$ and compute the sugra action on KK-reduced S^5.
- Solve sugra EOM for φ on the boundary (i.e. impose the asymptotical condition discusser previously)
- Insert the $\bar{\varphi}$ into the sugra action, with the appropriate boundary conditions(bcs)
- Use the last formula to take variational derivatives with respect to the source, to obtain the correlation functions.

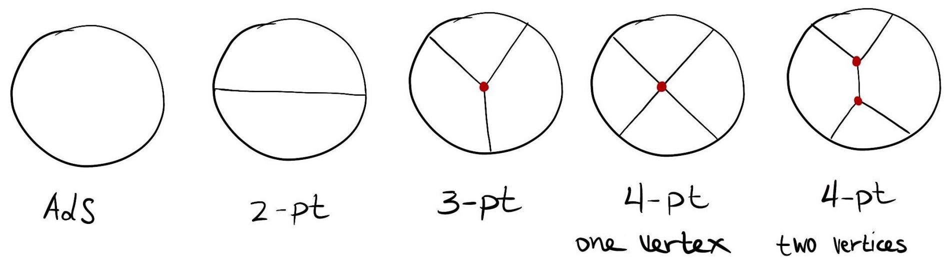


Figure 2.8: Witten diagrams

Tree level 2pt, 3pt and 4pt functions contributions are given in figure (2.6). There are two ways of making progress. One is using the components formulation of sugra and the other using superspace [45],[38],[39].

3

The Superconformal Algebra

In this chapter we will discuss various mathematical definitions and results from the study of superspaces and superconfromal algebras, mainly focused on the $\mathfrak{u}(2,2|4)$-algebra. We will also study two representations of the $PSU(2,2|4)$-supergroup. For more in depth analysis we reference to [46],[53],[54],[38], [45].

3.1 The $\mathfrak{psu}(2,2|4)$ superalgebra and Integrability

3.1.1 Multilinear Algebra

The ground rule is that all objects should be mod 2 graded and that in all classical formulas, whenever the order in which two odd quantities appear is changed, a minus sign must be introduced. For example, a *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space:

$$V = V_0 \otimes V_1 \tag{3.1}$$

An element $u \in V_0$ is called even, an element $v \in V_1$ is called odd. If V is finite dimensional, we define its *dimension* to be the pair of integers $m_0|m_1$, where the $m_i = \dim(V_i)$.

We can also define the *tensor product* of two super spaces; lets say V and W is the tensor product of their underlying vector spaces, with a $\mathbb{Z}/2\mathbb{Z}$-grading, i.e.

$$(V \otimes W)_k = \otimes_{i+j=k} V_i \otimes W_j \tag{3.2}$$

Definition 3.1.1 (Super Algebra)**.** A *super algebra* over k is a super vector space A, given with a morphism, called the *product* $A \otimes A \to A$.

The super algebra A is *associative* if $(xy)z = x(yz)$ and the *unit* is an even element 1 (where by 1 we mean a morphism $\underline{1} \to A$). In our usual discussion, by *"super algebra"* we will mean an associated super algebra with a unit.

We also define the *commutativity* property which will be a little different that usual because of the sign rule:

$$xy = (-1)^{p(x)p(y)} yx \tag{3.3}$$

where $p(x)$ is the parity of the x element.

Besides, we need the notion of free modulus which is defined as:

Definition 3.1.2 (Free Module)**.** A *free module* is a module that is free as an ungraded module, with a homogeneous basis.

Fix a commutative superalgebra A. The standard free module $A^{p|q}$ is the module freely generated by even elements $e_1, ..., e_p$ and odd elements $e_{p+1}, ..., e_{p+q}$. A morphism $T : A^{p|q} \to A^{r|s}$ can be represented by a matrix of size $(r+s) \times (p+q)$ with blocks of even and odd entities as follows:

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{\hphantom{\text{even}}}^{p} & \overbrace{\hphantom{\text{even}}}^{q} \end{array} \\ \begin{array}{c} r\{ \\ \\ s\{ \end{array} & \begin{pmatrix} \text{even} & \text{odd} \\ & \\ \text{odd} & \text{even} \end{pmatrix} \end{array}$$

We will represent an element x of $A^{p|q}$ by the column vector x^i s.t. $x = e_i x^i$. We shall define also the entries of the matrix of T by $T(e_j) = e_i t^i_j$.

We can also define the trace as follows, let $T : A^{p|q} \to A^{r|s}$ be a morphism :

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{}^{p} & \overbrace{}^{q} \end{array} \\ \begin{array}{c} p\{ \\ \\ q\{ \end{array} & \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ & \\ \mathrm{C} & \mathrm{D} \end{pmatrix} \end{array}$$

the *supertrace* of T, is defined by: $\mathrm{Tr}(T) := \mathrm{str}(T) =$ sum of diagonal entries of A - sum of diagonal entries of D.

3.2 The $\mathfrak{u}(2,2|4)$ superalgebra

The $\mathcal{N} = 4$ SYM has the superconformal symmetry defined by the supergroup $PSU(2,2|4)$ as we previously discussed. It is represented as a subgroup of the slightly enlarged supergroup $U(2,2|4) \cong \mathfrak{gl}(4|4)$.Here we replace the "curly" notation for the generators, with the most standard notation:
$\mathfrak{P} = P, \mathfrak{Q} = Q, \mathfrak{S} = S$ etc..

We decompose the $U(2,2|4)$ into:

$$T^{-} \oplus T^{0} \oplus T^{+} \tag{3.4}$$

where with T^0 represents the generators of the compact group $U(2,2) \otimes U(4) \otimes U(1)$, and $T^- \oplus T^+$ represents non-compact ones s.t.

$$[T^0, T^{\pm}] = T^{\pm} \qquad [T^{\pm}, T^{\pm}\} = T^{\pm} \tag{3.5}$$

Where the $[,\}$ is the graded bosonic/fermionic (anti-)commutator. We introduce two set of bosonic oscillators $(a_\alpha, a^{\dagger\alpha})$ and $(b_{\dot\alpha}, b^{\dagger\dot\alpha})$ and a set of fermionic ones $(c_a, c^{\dagger a})$ in order to realise the generators. We have,

$$\left[a^{\alpha}, a^{\dagger}_{\beta}\right] = \delta^{\alpha}_{\beta}, \quad \left[b^{\dot\alpha}, b^{\dagger}_{\dot\beta}\right] = \delta^{\dot\alpha}_{\dot\beta}, \quad \left\{c^{a}, c^{\dagger}_{b}\right\} = \delta^{a}_{b} \tag{3.6}$$

The compact subgroup T^0 consists of the generators [1] [2] [3]

$$\begin{aligned} L^{\alpha}_{\beta} &= a^{\dagger}_{\beta} a^{\alpha} - \tfrac{1}{2}\delta^{\alpha}_{\beta} a^{\dagger}_{\gamma} a^{\gamma} \\ L^{\dot\alpha}_{\dot\beta} &= b^{\dagger}_{\dot\beta} b^{\dot\alpha} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta} b^{\dagger}_{\dot\gamma} b^{\dot\gamma} \\ R^{a}_{b} &= c^{\dagger}_{b} c^{a} - \tfrac{1}{4}\delta^{a}_{b} c^{\dagger}_{c} c^{c} \end{aligned} \tag{3.7}$$

and the three $U(1)$ generators:

[1] $U(m|n)$:
[2] $SU(m,p|n+q)$:
[3] Both footnotes are calculations derived in the excellent work from [44]

$$
\begin{aligned}
&D = 1 + \tfrac{1}{2}a^\dagger_\gamma a^\gamma + \tfrac{1}{2}b^\dagger_{\dot\gamma} b^{\dot\gamma} = \tfrac{1}{2}a^\dagger_\gamma a^\gamma + \tfrac{1}{2}b^{\dot\gamma} b^\dagger_{\dot\gamma} \\
&C = 1 - \tfrac{1}{2}a^\dagger_\gamma a^\gamma + \tfrac{1}{2}b^\dagger_{\dot\gamma} b^{\dot\gamma} - \tfrac{1}{2}c^\dagger_c c^c = -\tfrac{1}{2}a^\dagger_\gamma a^\gamma + \tfrac{1}{2}b^{\dot\gamma} b^\dagger_{\dot\gamma} - \tfrac{1}{2}c^\dagger_c c^c \\
&B = -1 + \tfrac{1}{2}a^\dagger_\gamma a^\gamma - \tfrac{1}{2}b^\dagger_{\dot\gamma} b^{\dot\gamma} = \tfrac{1}{2}a^\dagger_\gamma a^\gamma - \tfrac{1}{2}b^{\dot\gamma} b^\dagger_{\dot\gamma}
\end{aligned}
\tag{3.8}
$$

The non-compact T^+ generators:

$$
Q^a_\alpha = a^\dagger_\alpha c^a, \quad \dot{Q}_{\dot\alpha a} = b^\dagger_{\dot\alpha} c^\dagger_a, \quad P_{\dot\alpha\beta} = b^\dagger_{\dot\alpha} a^\dagger_\beta, \tag{3.9}
$$

while those in T^- by

$$
S^\alpha_a = c^\dagger_a a^\alpha \quad \dot{S}^{\dot\alpha a} = b^{\dot\alpha} c^a, \quad K^{\alpha\dot\beta} = a^\alpha b^{\dot\beta} \tag{3.10}
$$

Then the generators in (3.7) form the subalgebra $\mathrm{SU}(2) \otimes \mathrm{SU}(2) \otimes \mathrm{SU}(4)$ of $\mathrm{U}(2,2 \mid 4)$

$$
\left[L^\alpha_\beta, L^\gamma_\delta\right] = -\delta^\gamma_\beta L^\alpha_\delta + \delta^\alpha_\delta L^\gamma_\beta, \quad \left[L^{\dot\alpha}_{\dot\beta}, L^{\dot\gamma}_{\dot\delta}\right] = -\delta^{\dot\gamma}_{\dot\beta} L^{\dot\alpha}_{\dot\delta} + \delta^{\dot\alpha}_{\dot\delta} L^\gamma_\beta
$$

$$
\left[R^a_b, R^c_d\right] = -\delta^c_b R^a_d + \delta^a_d R^c_d \tag{3.11}
$$

The algebra $[T^\pm, T^\pm\} = T^\pm$ is nilpotent i.e. $[T^\pm\,[T^\pm, T^\pm\}\} = 0$, and is now given by:

$$
\left\{\dot{Q}_{\dot\alpha b}, Q^a_\beta\right\} = \delta^a_b P_{\dot\alpha\beta}, \quad \left\{\dot{S}^{a\dot\beta}, S^\alpha_b\right\} = \delta^a_b K^{\beta\dot\alpha} \tag{3.12}
$$

while the algebra $[T^+, T^-] = T^0$ is given by:

$$
\begin{cases}
\left[K^{\alpha\dot\beta}, P_{\dot\gamma\delta}\right] = \delta^{\dot\beta}_{\dot\gamma} L^\alpha_\delta + \delta^\alpha_\delta \dot{L}^{\dot\beta}_{\dot\gamma} + \delta^{\dot\beta}_{\dot\gamma}\delta^\alpha_\delta D \\
\left\{S^\alpha_b, Q^a_\beta\right\} = \delta^a_b L^\alpha_\beta + \delta^\alpha_\beta R^a_b + \tfrac{1}{2}\delta^a_b\delta^\alpha_\beta (D - C) \\
\left\{\dot{S}^{a\dot\beta}, \dot{Q}_{\dot\alpha b}\right\} = \delta^a_b \dot{L}^{\dot\beta}_{\dot\alpha} - \delta^{\dot\beta}_{\dot\alpha} R^a_b + \tfrac{1}{2}\delta^a_b\delta^{\dot\beta}_{\dot\alpha}(D + C)
\end{cases}
\tag{3.13}
$$

Lastly, the algebra $[T^+, T^-]$, which does not close into T^0, is given by

$$
\begin{cases}
\left[S^\alpha_b, P_{\dot\alpha\beta}\right] = \delta^\alpha_\beta \dot{Q}_{\dot\alpha b}, \quad \left[K^{\alpha\dot\beta}, Q_{\dot\alpha b}\right] = \delta^{\dot\beta}_{\dot\alpha} S^\alpha_b \\
\left[\dot{S}^{a\dot\beta}, P_{\dot\alpha\beta}\right] = \delta^{\dot\beta}_{\dot\alpha} Q^a_\beta, \quad \left[K^{\alpha\dot\beta}, Q^a_\beta\right] = \delta^\alpha_\beta \dot{S}^{a\dot\beta}
\end{cases}
\tag{3.14}
$$

Combining all the above, we get the Lie-superalgebra of the complex supergroup $U(2,2|4)$. Notice also, that D is the dilaton, the hypercharge B never appears in the superalgebra, and C is the central charge, since all generators commute with it.

The quadratic Casimir has the form

field	$\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ h.w.	$\mathrm{SU}(4)$ h · w.
$\mathcal{D}^k\mathcal{F}$	$[k+2,k]$	$[0,0,0]$
$\mathcal{D}^k\Psi$	$[k+1,k]$	$[1,0,0]$
$\mathcal{D}^k\Phi$	$[k,k]$	$[0,1,0]$
$\mathcal{D}^k\dot{\Psi}$	$[k,k+1]$	$[0,0,1]$
$\mathcal{D}^k\dot{\mathcal{F}}$	$[k,k+2]$	$[0,0,0]$

(3.18)

Figure 3.1: The $\mathcal{N}=4$ SYM field strength multiplet.

$$L^\alpha_\beta L^\beta_\alpha - R^a_b R^b_a + L^{\dot\alpha}_{\dot\beta} L^{\dot\beta}_{\dot\alpha} + D^2 - \left\{P_{\dot\alpha\beta}, K^{\beta\dot\alpha}\right\} - [Q^a_\alpha, S^\alpha_b] - \left[\dot{Q}_{\dot\alpha a}, \dot{S}^{a\dot\alpha}\right] - 2BC \quad (3.15)$$

Putting now the generators in tensor product form

$$\psi^\dagger = \left(a^\dagger_\beta, c^\dagger_b, b^{\dot\beta}\right), \quad \psi = \begin{pmatrix} a^\alpha \\ c^a \\ b^\dagger_{\dot\alpha} \end{pmatrix} \quad (3.16)$$

we obtain the following convenient matrix representation of

$$\psi^\dagger\otimes\psi = \left[\begin{array}{c|c|c} L^\alpha_\beta & S^\alpha_b & K^{\alpha\dot\beta} \\ \hline Q^a_\beta & R^a_b & \dot{S}^{a\dot\beta} \\ \hline P_{\dot\alpha\beta} & \dot{Q}_{\dot\alpha b} & \dot{L}^{\dot\beta}_{\dot\alpha} \end{array}\right] + \left[\begin{array}{c|c|c} \frac{1}{2}\delta^\alpha_\beta(D+B) & 0 & 0 \\ \hline 0 & -\frac{1}{2}(C+B) & 0 \\ \hline 0 & 0 & \frac{1}{2}\delta^{\dot\beta}_{\dot\alpha}(D-B) \end{array}\right]. \quad (3.17)$$

This tensor form, motivates the usefulness of the Matrix representation of the $PSU(2,2|4)$ as we will discuss later on.

3.3 Oscillator representation of $PSU(2,2|4)$

In QFT we represent the superconformal transformations , as unitary linear transformations of Hilbert space. In particular, the superconformal transformations that act on $\mathcal{N}=4$ SYM field strength multiplet are given in the table below, which shows the Dynkin labels of the heighest weight for various letters.

We are ,mainly, interested in the unitary rep. of the $PSU(2,2|4)$ of the $\mathcal{N}=4$ SYM theory. Since $PSU(2,2|4)$ is non-compact, the unitary rep. is infinite-dimensional. As it is the standard mathematical procedure , a unitary operator U of $U(2,2|4)$ is given by the **exp** super-map:

$$U = \exp\left(i\bar{\psi}M\psi\right) \tag{3.19}$$

with $M, \gamma \in \mathrm{Mat}(8 \times 8)$ s.t.

$$M = \left(\begin{array}{c|c|c} V & \theta & X \\ \hline \theta^\dagger & W & \epsilon \\ \hline -X^\dagger & -\epsilon^\dagger & Z \end{array}\right), \quad \gamma = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & -1 \end{array}\right) \tag{3.20}$$

where V, W, Z Hermitian matrices, X complex matrix, θ is $2 \otimes 4$ and ϵ is $4 \otimes 2$ with Grassmannian elements. The minus in the matrix M represents the non-compacteness of $U(2,2|4)$.

The general Fock space for the $\mathfrak{u}(2,2|4)$ is given by

$$\prod_{\alpha=1}^{2}\prod_{n_{a_\alpha}=1}^{\infty}\left(a_\alpha^\dagger\right)^{n_{a_\alpha}}\prod_{\dot{\alpha}=1}^{2}\prod_{n_{b_{\dot{\alpha}}}}^{\infty}\left(b_{\dot{\alpha}}^\dagger\right)^{n_{b_{\dot{\alpha}}}}\prod_{a=1}^{4}\prod_{n_{c_a}=1}^{\infty}\left(c_a^\dagger\right)^{n_{c_a}}|0\rangle \tag{3.21}$$

To go to our desired subsector of $\mathfrak{psu}(2,2|4)$, we impose the constrain:

$$C = 1 - \frac{1}{2}\sum_{\alpha}^{2} n_{a_\alpha} + \frac{1}{2}\sum_{\dot{\alpha}}^{2} n_{b_{\dot{\alpha}}} - \frac{1}{2}\sum_{a}^{4} n_{c_a} = 0 \tag{3.22}$$

If we choose the ground states,

$$a^\alpha|0\rangle = 0, \quad b^{\dot{\alpha}}|0\rangle = 0 \quad c^a|0\rangle = 0 \tag{3.23}$$

we notice that the vacuum can not belong to $\mathfrak{psu}(2,2|4)$ subsector, since $C = 1$, hence we define a physical vacuum Z with $C = 0$ by

$$Z = c_3^\dagger c_4^\dagger|0\rangle \tag{3.24}$$

For convenience we rename the whole fermionic oscillators $c^a, c_a^\dagger, a = 1,2,3,4$ as

$$\begin{aligned} \left(c^1, c^2\right) \equiv c^{\bar{a}}, \quad \left(c^3, c^4\right) = \left(d_3^\dagger, d_4^\dagger\right) \equiv d_{\dot{a}}^\dagger \\ \left(c_1^\dagger, c_2^\dagger\right) \equiv c_{\bar{a}}^\dagger, \quad \left(c_3^\dagger, c_4^\dagger\right) = \left(d^3, d^4\right) \equiv d^{\dot{a}} \end{aligned} \tag{3.25}$$

Z now satisfies

$$a^\alpha|0\rangle = 0, \quad b^{\dot{\alpha}}|0\rangle = 0, \quad c^{\bar{a}}|0\rangle = 0, \quad d^{\dot{a}}|0\rangle = 0$$

The physical Fock space is now

$$\prod_{\alpha=1}^{2}\left(a_\alpha^\dagger\right)^{n_{a_\alpha}}\prod_{\dot\alpha=1}^{2}\left(b_{\dot\alpha}^\dagger\right)^{n_{b_{\dot\alpha}}}\prod_{\bar a=1}^{2}\left(c_{\bar a}^\dagger\right)^{n_{c_{\bar a}}}\prod_{\dot a=3}^{4}\left(d_{\dot a}^\dagger\right)^{n_{d_{\dot a}}} Z$$

Lastly, the constraint equation for the central charge $C = 0$ becomes

$$C=\sum_{\alpha=1}^{2} n_{a_\alpha}-\sum_{\dot\alpha=1}^{2} n_{b_{\dot\alpha}}+\sum_{\bar a=1}^{2} n_{c_{\bar a}}-\sum_{\dot a=3}^{4} n_{d_{\dot a}}=0$$

Acting on Z with the generators we get the table below

Field	States
$\mathcal{F}$	$a_\alpha^\dagger a_\beta^\dagger d_3^\dagger d_4^\dagger Z$
Ψ	$a_\alpha^\dagger d_{\dot a}^\dagger Z, a_\alpha^\dagger c_{\bar a}^\dagger d_3^\dagger d_4^\dagger Z$
Φ	$Z, c_{\bar a}^\dagger d_{\dot a}^\dagger Z, c_1^\dagger c_2^\dagger d_3^\dagger d_4^\dagger Z$
$\dot\Psi$	$b_{\dot\alpha}^\dagger c_{\bar a}^\dagger Z, b_{\dot\alpha}^\dagger d_{\dot a}^\dagger c_1^\dagger c_2^\dagger Z$
$\dot{\mathcal{F}}$	$b_{\dot\alpha}^\dagger b_{\dot\beta}^\dagger c_1^\dagger c_2^\dagger Z$

(3.26)

As we can notice, they exactly correspond with the fundamental fields of $\mathcal{N}=4$ SUSY field strength multiplet we have shown before. Acting on those states P and R create the $SU(2)\otimes SU(2)$ excited states with Dynkin label $[k,k]$ respectively. Using these we get the covariant derivative $\mathcal{D}$. The remaining generators annihilate Z. More specifically, the fermionic which are

$$Q^{\bar a}_{\beta}=a^\dagger_\beta c^{\bar\alpha} \quad \dot{Q}_{\dot\alpha\dot b}=b^\dagger_{\dot\alpha d^{\dot b}} \tag{3.27}$$

They are half of the 16 total supercharges. Thus our states on the table in (3.26) form a *half-multiplet*, which is the smallest BPS multiplet.

3.4 The matrix representation of $PSU(2,2|4)$

We will now discuss a matrix rep. of $U(2,2|4)$ which it will deduced from our previous discussion of the oscillator-unitary rep. To start, we build the supermatrix

with the index convention:

$$\left(\begin{array}{c|c|c} t^\gamma_\delta & t^\gamma_d & t^\gamma_{\dot\delta} \\ \hline t^c_\delta & t^c_d & t^c{}_{\dot\delta} \\ \hline t^{\dot\gamma}_\delta & t^{\dot\gamma}_d & t^{\dot\gamma}_{\dot\delta} \end{array}\right) \tag{3.28}$$

we can write the generators as:

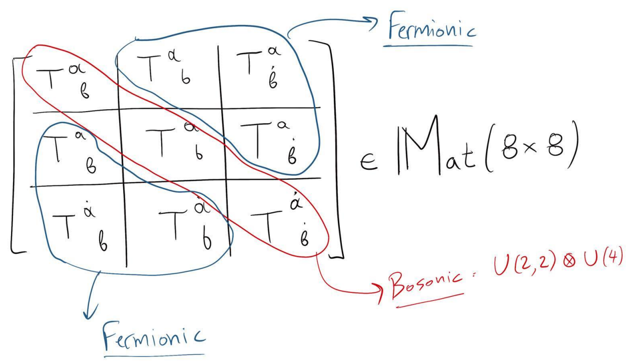


Figure 3.2: the supermatrix

$$T^{\alpha}_{\beta} = \left(\begin{array}{c|c|c} \delta^{\alpha}_{\delta}\delta^{\gamma}_{\beta} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad T^{a}_{\beta} = \left(\begin{array}{c|c|c} 0 & \delta^{a}_{d}\delta^{\gamma}_{\beta} & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad T^{\dot{\alpha}}_{\beta} = \left(\begin{array}{c|c|c} 0 & 0 & \delta^{\gamma}_{\beta}\delta^{\dot{\alpha}}_{\dot{\delta}} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \tag{3.29}$$

$$T^{\alpha}_{b} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \delta^{\alpha}_{\delta}\delta^{c}_{b} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad T^{a}_{b} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \delta^{a}_{d}\delta^{c}_{b} & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad T^{\dot{\alpha}}_{b} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & \delta^{c}_{b}\delta^{\dot{\alpha}}_{\dot{\delta}} \\ \hline 0 & 0 & 0 \end{array}\right) \tag{3.30}$$

$$T^{\alpha}_{\dot{\beta}} = -\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \delta^{\alpha}_{\delta}\delta^{\dot{\gamma}}_{\dot{\beta}} & 0 & 0 \end{array}\right), \quad T^{a}_{\dot{\beta}} = -\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & \delta^{\dot{\gamma}}_{\dot{\beta}}\delta^{a}_{d} & 0 \end{array}\right), \quad T^{\dot{\alpha}}_{\dot{\beta}} = -\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & \delta^{\dot{\alpha}}_{\dot{\delta}}\delta^{\dot{\gamma}}_{\dot{\beta}} \end{array}\right) \tag{3.31}$$

We have the Lie algebra $\mathfrak{u}(2,2) \otimes \mathfrak{u}(4)$ which is formed by the Bosonic generators:

$$\left[T^{\alpha}_{\beta}, T^{\gamma}_{\delta}\right] = -\delta^{\gamma}_{\beta}T^{\alpha}_{\delta} + \delta^{\alpha}_{\delta}T^{\gamma}_{\beta}, \quad \left[T^{\dot{\alpha}}_{\dot{\beta}}, T^{\dot{\gamma}}_{\dot{\delta}}\right] = \delta^{\dot{\gamma}}_{\dot{\beta}}T^{\dot{\alpha}}_{\dot{\delta}} - \delta^{\dot{\alpha}}_{\dot{\delta}}T^{\gamma}_{\beta}$$
$$[T^{a}_{b}, T^{c}_{d}] = -\delta^{c}_{b}T^{a}_{d} + \delta^{a}_{d}T^{c}_{d}$$
$$\left[T^{\alpha}_{\dot{\beta}}, T^{\dot{\gamma}}_{\delta}\right] = \delta^{\dot{\gamma}}_{\dot{\beta}}T^{\alpha}_{\delta} - \delta^{\alpha}_{\delta}T^{\dot{r}}_{\dot{\beta}}$$

With the anti-commuting fermionic generators we get the following algebra:

$$\left\{T^{\alpha}_{b}, T^{a}_{\beta}\right\} = \delta^{\alpha}_{\beta}T^{a}_{b} + \delta^{a}_{b}T^{\alpha}_{\beta}, \quad \left\{T^{a}_{\dot\beta}, T^{\dot\alpha}_{b}\right\} = -\delta^{a}_{b}T^{\dot\alpha}_{\dot\beta} + \delta^{\dot\alpha}_{\dot\beta}T^{a}_{b}$$
$$\left\{T^{\alpha}_{b}, T^{a}_{\dot\beta}\right\} = \delta^{a}_{b}T^{\alpha}_{\dot\beta}, \quad \left\{T^{a}_{\beta}, T^{\dot\alpha}_{b}\right\} = \delta^{a}_{b}T^{\dot\alpha}_{\beta}$$

Commuting these fermionic generators with bosonic generators yields

$$\begin{aligned}
&[T^{\alpha}_{b}, T^{\gamma}_{\delta}] = \delta^{\alpha}_{\delta}T^{\gamma}_{b}, && [T^{\alpha}_{b}, T^{c}_{d}] = -\delta^{c}_{b}T^{\alpha}_{d}, && \left[T^{\alpha}_{b}, T^{\dot\gamma}_{\dot\delta}\right] = 0\\
&\left[T^{a}_{\beta}, T^{\gamma}_{\delta}\right] = -\delta^{\gamma}_{\beta}T^{a}_{\delta}, && \left[T^{a}_{\beta}, T^{c}_{d}\right] = \delta^{a}_{d}T^{c}_{\beta}, && \left[T^{a}_{\beta}, T^{\dot\gamma}_{\dot\delta}\right] = 0\\
&\left[T^{a}_{\dot\beta}, T^{\gamma}_{\delta}\right] = 0, && \left[T^{a}_{\dot\beta}, T^{c}_{d}\right] = \delta^{a}_{d}T^{c}_{\dot\beta}, && \left[T^{a}_{\dot\beta}, T^{\dot\gamma}_{\dot\delta}\right] = \delta^{\dot\gamma}_{\dot\beta}T^{a}_{\dot\delta},\\
&\left[T^{\dot\alpha}_{b}, T^{\gamma}_{\delta}\right] = 0, && [T^{\dot\alpha}_{b}, T^{c}_{d}] = -\delta^{c}_{b}T^{\dot\alpha}_{d}, && \left[T^{\dot\alpha}_{b}, T^{\dot\gamma}_{\dot\delta}\right] = -\delta^{\dot\alpha}_{\dot\delta}T^{\dot\gamma}_{b}
\end{aligned}$$

while commuting them with bosonic generators

$$\begin{aligned}
&\left[T^{\alpha}_{b}, T^{\gamma}_{\dot\delta}\right] = 0, && \left[T^{\alpha}_{b}, T^{\dot\gamma}_{\delta}\right] = \delta^{\alpha}_{\delta}T^{\dot\gamma}_{b}\\
&\left[T^{a}_{\beta}, T^{\gamma}_{\dot\delta}\right] = -\delta^{\gamma}_{\beta}T^{a}_{\dot\delta}, && \left[T^{a}_{\beta}, T^{\dot\gamma}_{\delta}\right] = 0\\
&\left[T^{a}_{\dot\beta}, T^{\gamma}_{\dot\delta}\right] = 0, && \left[T^{a}_{\dot\beta}, T^{\dot\gamma}_{\delta}\right] = \delta^{\dot\gamma}_{\dot\beta}T^{a}_{\delta}\\
&\left[T^{\dot\alpha}_{b}, T^{\gamma}_{\dot\delta}\right] = -\delta^{\dot\alpha}_{\dot\delta}T^{\gamma}_{\gamma}, && \left[T^{\dot\alpha}_{b}, T^{\dot\gamma}_{\delta}\right] = 0
\end{aligned}$$

All the rest (that we not show here) are zero. As we discussed again in the previous section, in the diagonal blocks we have the generators of the subgroup $SU(2) \otimes SU(2) \otimes SU(4)$ given by

$$\mathcal{L}^{\alpha}_{\beta} = T^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}T^{\gamma}_{\gamma}, \quad \dot{\mathcal{L}}^{\dot\alpha}_{\dot\beta} = T^{\dot\alpha}_{\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}T^{\dot\gamma}_{\dot\gamma}, \quad \mathcal{R}^{a}_{b} = T^{a}_{b} - \frac{1}{4}\delta^{a}_{b}T^{c}_{c} \tag{3.32}$$

and the three $U(1)$:

$$D = \frac{1}{2}\left(\begin{array}{c|c|c} \delta^{\gamma}_{\delta} & 0 & 0\\ \hline 0 & 0 & 0\\ \hline 0 & 0 & -\delta^{\dot\gamma}_{\dot\delta}\end{array}\right) = \frac{1}{2}\left(T^{\alpha}_{\alpha} + T^{\dot\alpha}_{\dot\alpha}\right)$$
$$B = \frac{1}{2}\left(\begin{array}{c|c|c} \delta^{\gamma}_{\delta} & 0 & 0\\ \hline 0 & 0 & 0\\ \hline 0 & 0 & \delta^{\dot\gamma}_{\dot\delta}\end{array}\right) = \frac{1}{2}\left(T^{\alpha}_{\alpha} - T^{\dot\alpha}_{\dot\alpha}\right)$$
$$C = -\frac{1}{2}\left(\begin{array}{c|c|c} \delta^{\gamma}_{\delta} & 0 & 0\\ \hline 0 & \delta^{c}_{d} & 0\\ \hline 0 & 0 & \delta^{\dot\gamma}_{\dot\delta}\end{array}\right) = -\frac{1}{2}\left(T^{\alpha}_{\alpha} - T^{\dot\alpha}_{\dot\alpha} + T^{a}_{a}\right), \tag{3.33}$$

Using now our generators (3.32) – (3.33) we get:

$$\left[\mathcal{L}^{\alpha}_{\beta}, \mathcal{L}^{\gamma}_{\delta}\right] = -\delta^{\gamma}_{\beta}\mathcal{L}^{\alpha}_{\delta} + \delta^{\alpha}_{\delta}\mathcal{L}^{\gamma}_{\beta}, \quad \left[\dot{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \dot{\mathcal{L}}^{\dot\gamma}_{\dot\delta}\right] = \delta^{\dot\gamma}_{\dot\beta}\dot{\mathcal{L}}^{\dot\alpha}_{\dot\delta} - \delta^{\dot\alpha}_{\dot\delta}\dot{\mathcal{L}}^{\dot\gamma}_{\dot\beta}$$
$$[\mathcal{R}^{a}_{b}, \mathcal{R}^{c}_{d}] = -\delta^{c}_{b}\mathcal{R}^{a}_{d} + \delta^{a}_{d}\mathcal{R}^{c}_{d} \tag{3.34}$$

$$\begin{cases} \left[T^{\alpha}_{\dot\beta}, T^{\dot\gamma}_{\delta}\right] = \delta^{\dot\gamma}_{\dot\beta}\mathcal{L}^{\alpha}_{\delta} + \delta^{\alpha}_{\delta}\dot{\mathcal{L}}^{\dot\gamma}_{\dot\beta} + \delta^{\dot\gamma}_{\dot\beta}\delta^{\alpha}_{\delta} D \\ \left\{T^{\alpha}_{b}, T^{a}_{\beta}\right\} = \delta^{\alpha}_{\beta}\mathcal{R}^{a}_{b} + \delta^{a}_{b}\mathcal{L}^{\alpha}_{\beta} + \frac{1}{2}\delta^{\alpha}_{\beta}\delta^{a}_{b}(D - C) \\ \left\{T^{a}_{\dot\beta}, T^{\dot\alpha}_{b}\right\} = \delta^{a}_{b}\dot{\mathcal{L}}^{\dot\alpha}_{\dot\beta} - \delta^{\dot\alpha}_{\dot\beta}\mathcal{R}^{a}_{b} + \frac{1}{2}\delta^{a}_{b}\delta^{\dot\alpha}_{\dot\beta}(D + C) \end{cases} \tag{3.35}$$

To get our required $\mathfrak{psu}(2,2|4)$ we just have to set the $U(1)$ central charge $C = 0$ and hypercharge $B = 0$.

We can see clearly the indentification between the two representations, unitary and matrix rep. by the equalities:

$$\left\{ \begin{array}{ccc} \epsilon^{\dot\alpha\dot\delta}\epsilon_{\dot\beta\dot\gamma}\dot{\mathcal{L}}^{\dot\gamma}_{\dot\delta} = \dot{L}^{\dot\alpha}_{\dot\beta} = -\dot{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, & \epsilon^{\dot\beta\delta}T^{\alpha}_{\dot\delta} = K^{\alpha\dot\beta}, & \epsilon_{\dot\alpha\dot\gamma}T^{\dot\gamma}_{\beta} = P_{\dot\alpha\beta} \\ \epsilon^{\dot\beta\dot\delta}T^{a}_{\dot\delta} = \dot{S}^{a\dot\beta}, & & \epsilon_{\dot\alpha\dot\gamma}T^{\dot\gamma}_{b} = \dot{Q}_{\dot\alpha b} \\ \mathcal{L}^{\alpha}_{\beta} = L^{\alpha}_{\beta}, \mathcal{R}^{a}_{b} = R^{a}_{b}, & \mathcal{S}^{\alpha}_{b} = T^{\alpha}_{b}, \quad T^{a}_{\beta} = Q^{a}_{\beta}, & \mathcal{D} = D \end{array} \right\} \tag{3.36}$$

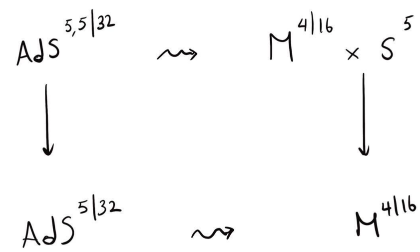


Figure 3.3: Here we represent the basic geometric set up for the AdS/CFT in superspace .The squiggly lines represent the passing to the boundary. Each of these superspaces is a coset space of the supergroup $PSU(2,2|4)$, with the notation indicating the $(even|odd)$ dimensions

3.5 The superconformal algebra and the Dilatation operator $\mathfrak{D}$

Lets now return to our original notation. Our main focus in this section will be to understand the algebraic picture for the $\mathfrak{psu}(2,2|4)$ since it is the algebra of generators of the $\mathcal{N} = 4$ SYM, as we discussed previously. All our generators are:

$$\boxed{\mathfrak{J} \in \{\mathfrak{L}, \dot{\mathfrak{L}}, \mathfrak{R}, \mathfrak{P}, \mathfrak{K}, \mathfrak{D}, \mathfrak{B}, \mathfrak{C} \mid \mathfrak{Q}, \dot{\mathfrak{Q}}, \mathfrak{S}, \dot{\mathfrak{S}}\}} \tag{3.37}$$

Which are the $\mathfrak{su}(2), \mathfrak{su}(2), \mathfrak{su}(4)$:
Rotations:

$$\mathfrak{L}, \dot{\mathfrak{L}}, \mathfrak{R}$$

Super-translations:

$$\mathfrak{Q}, \dot{\mathfrak{Q}}, \mathfrak{P}$$

Super-boosts:

$$\mathfrak{S}, \dot{\mathfrak{S}}, \mathfrak{K}$$

and the $\mathfrak{u}(1)$ charges:

Dilatation generator :

$$\mathfrak{D}$$

Hypercharge :

$$\mathfrak{B}$$

Central Charge :

$$\mathfrak{C}$$

In the irreducible super-conformal-algebra $\mathfrak{psu}(2, 2 \mid 4)$, the generators $\mathfrak{B}, \mathfrak{C}$ are absent because:

The $\mathfrak{u}(1)$ hypercharge $\mathfrak{B}$ of $\mathfrak{pu}(2, 2 \mid 4) = u(1) \ltimes \mathfrak{psu}(2, 2 \mid 4)$ is an external automorphism which consistently assigns a charge to all the generators of $\mathfrak{psu}(2, 2 \mid 4)$.

The $\mathfrak{u}(1)$ central charge $\mathfrak{C}$ of $\mathfrak{su}(2, 2 \mid 4) = \mathrm{psu}(2, 2 \mid 4) \ltimes \mathfrak{u}(1)$ must vanish to be able to reduce to $\mathrm{psu}(2, 2 \mid 4)$.

The Lorentz algebra $\mathfrak{so}(3, 1) = \mathfrak{su}(2) \times \mathfrak{su}(2)$ is formed by $\mathfrak{L}, \dot{\mathfrak{L}}$. Together with $\mathfrak{P}, \mathfrak{K}, \mathfrak{D}$ one gets the conformal algebra $\mathfrak{so}(4, 2) = \mathfrak{su}(2, 2)$.

It is very important to note that after the quantization the only symmetries that remain exact are the Lorentz and the internal symmetries. All the other symmetries receive quantum corrections of the powers of our theories coupling constant g. In particular, the dilatation operator receives loop corrections. That is why is it very wise to define an operator which measures the *Classical Dimension Dilatation Operator*, we denote this operation as $\mathfrak{D}_0$. We also define the change of the scaling dimension by quantum effects by $\delta\mathfrak{D} := \mathfrak{D} - \mathfrak{D}_0$ and we call it the *Anomalous Dilatation Operator*. We can also identify the Anomalous Dilatation Operator with the Hamiltonian $\mathcal{H}$ with eigenvalues E:

$$\mathcal{H} = -\frac{1}{g^2}\delta\,\mathfrak{D}$$

For a bosonic, semi-simple Lie algebra the Dynking diagram is unique, but the addition of fermionic roots breaks that uniqueness because of the freedom to distribute the simple fermionic roots. We will follow the particular choice made in the paper [32], which gives the Dynkin diagram of figure 3.3, with generators associated with positive and negative roots and elements of the Cartan (super)-subalgebra:

$$\left\{\begin{array}{l} \mathfrak{J}^+ \in \left\{\mathfrak{K}^{\alpha\dot{\beta}}, \mathfrak{S}^{\alpha}_{b}, \dot{\mathfrak{S}}^{a\dot{\beta}}, \mathfrak{L}^{\alpha}_{\beta}(\alpha < \beta), \dot{\mathfrak{L}}_{\beta}(\dot{\alpha} < \dot{\beta}), \mathfrak{R}^{a}_{b}(a < b)\right\} \\ \mathfrak{J}^0 \in \left\{\mathfrak{L}^{\alpha}_{\beta}(\alpha = \beta), \dot{\mathfrak{L}}^{\dot{\alpha}}_{\dot{\beta}}(\dot{\alpha} = \dot{\beta}), \mathfrak{R}^{a}_{b}(a = b), \mathfrak{D}, \mathfrak{B}, \mathfrak{C}\right\} \\ \mathfrak{J}^- \in \left\{\mathfrak{P}_{\dot{\alpha}\beta}, \mathfrak{Q}^{a}_{\beta}, \mathfrak{Q}_{\dot{\alpha}b}, \mathfrak{L}^{\alpha}_{\beta}(\alpha > \beta), \dot{\mathfrak{L}}^{\dot{\alpha}}_{\beta}(\dot{\alpha} > \dot{\beta}), \mathfrak{R}^{a}_{b}(a > b)\right\} \end{array}\right\} \tag{3.38}$$

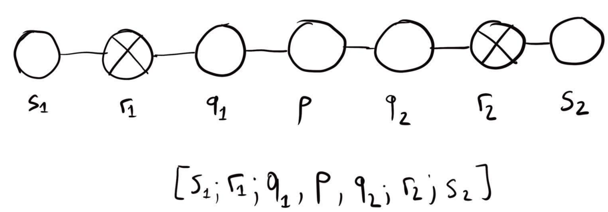


Figure 3.4: Our choice of Dynkin diagram of $\mathfrak{psu}(2,2|4)$

All elements of the Cartan subalgebra, spanned by $\{\mathfrak{J}^0\}$, commute with each other. Therefore one can find simultaneous eigenstates with respect to all of its elements, the eigenvalues are the quantum numbers n_i or what we will call *labels* of the state.

The Dynkin labels corresponding to our choice of fermionic roots, figure 3.3, are[4]

[4]Note that s_1, s_2 equal twice the spin, $q_1, q_2 \in \mathbb{Z}$, and $r_1, r_2, r \in \mathbb{R}$

$$w = [s_1; r_1, q_1, p, q_2, r_2; s_2] \quad \cong \quad \overbrace{\mathfrak{su}(2)}^{\text{2*spin label: } [s_1]} \otimes \overbrace{\mathfrak{su}(2)}^{\text{2*spin label: } [s_2]} \otimes \overbrace{\mathfrak{su}(4)}^{[q_1,p,q_2] \text{ Dynkin labels of } \mathfrak{su}(4)} \otimes \underbrace{\mathfrak{u}(1)}_{E\in\mathbb{R}} \tag{3.39}$$

which are defined as combinations of eigenvalues of our Cartan generators $\mathfrak{J}$:

$$\begin{array}{ll} n_1 := s_1 = L_2^2 - L_1^1, & n_7 := s_2 = \dot{L}_2^2 - \dot{L}_1^1 \\ n_2 := r_1 = \frac{1}{2}D - \frac{1}{2}C - L_1^1 + R_1^1, & n_6 := r_2 = \frac{1}{2}D + \frac{1}{2}C - \dot{L}_1^1 - R_4^4 \\ n_3 := q_1 = R_2^2 - R_1^1, & n_5 := q_2 = R_4^4 - R_3^3 \\ n_4 := p = R_3^3 - R_2^2, & \text{The R-charge: } \; r = -D + L_1^1 + \dot{L}_1^1 \end{array} \tag{3.40}$$

4

The Dilatation Operator

The dilatation operator is a means to investigate scaling dimensions in a conformal field theory.

4.1 Scaling Dimensions

4.1.1 The 2-pt function

As we have discussed before, in CFTs, the correlation operators obey certain relations due to the conformal symmetry. We will mainly work in the framework of CFT theories without assuming the further SUSY invariance since it would restrict the scenario and require to work using superspace techniques[1]. The two-point and three-point function of scalar primary operators $\mathcal{O}_i$ are[2][3]

[1]But the results would be similar.
[2]Due to symmetries
[3]Conformal transformations restrict the form of these 2-pt and 3-pt functions to these particular forms

$$\boxed{\langle \mathcal{O}_1 (x_1)\, \mathcal{O}_2 (x_2) \rangle = \frac{M_{12}}{|x_{12}|^{2D}}} \tag{4.1}$$

were r_i is the distance $x_i - x_j$.

$$\boxed{\langle \mathcal{O}_1 (x_1)\, \mathcal{O}_2 (x_2)\, \mathcal{O}_3 (x_3) \rangle = \frac{C_{123}}{|x_{12}|^{D_1+D_2-D_3}\, |x_{23}|^{D_2+D_3-D_1}\, |x_{31}|^{D_3+D_1-D_2}}} \tag{4.2}$$

In particular, it is possible to use the invariance of the theory to normalize M_{12} to unity. The C_{123} are called OPE coefficients or structure constants; together with the scaling dimensions D of the fields, they uniquely specify the CFT.

Of course, for non-scalar primary operators, we should introduce the correct spacetime/spinor indices. Although we work on flat Euclidean spacetime $\mathbb{R}^4$, from the point of view of conformal symmetry, spacetime is not flat; rather it is, the coset space of the conformal group by the Poincare group and dilatations. So we can not just compare the tangent space at two different points. We must introduce a *connection.* For example, for a vector, we may use the connection:

$$J_{12}^{\mu\nu} = -\frac{1}{2} \sigma^{\mu}_{\dot{\alpha}\beta} J_{12}^{\dot{\alpha}\delta} J_{12}^{\beta\dot{\gamma}} \sigma^{\nu}_{\dot{\gamma}\delta} = \eta^{\mu\nu} - 2\frac{x_{12}^{\mu} x_{12}^{\nu}}{|x_{12}|^2} \tag{4.3}$$

thus we get,

$$\langle \mathcal{O}_1^{\mu} (x_1)\, \mathcal{O}_2^{\nu} (x_2) \rangle = \frac{M_{12} J_{12}^{\mu\nu}}{|x_{12}|^{2D}} \tag{4.4}$$

For further analysis on 4+ point-functions and on descendant operators and their mixing we ref to the book [37].

Moving on from this small review, we return to our main purpose, which is to calculate the scaling dimension i.e. the dilatation operator, in a CFT. We introduce the dependence of $D = D(g)$ in the 2-pt function such that:

$$\langle \mathcal{O} (x_1)\, \mathcal{O} (x_2) \rangle = \frac{M(g)}{|x_{12}|^{2D(g)}} \tag{4.5}$$

Our aim, is not perform a pertubative analysis in powers of g, as it is the standard procedure in quantum physics. In order to be consistent with dimensional analysis, we must introduce an arbitrary scale μ and rescale $\mathcal{O}$ by $\mu^{\delta D(g)}$ to a fixed mass dimension $D_0 = D(0)$, then we expand:

$$\mu^{-2\delta D(g)} \langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle = \frac{M_0}{|x_{12}|^{2D_0}} + g^2 \frac{M_2 + M_0 D_2 \log |\mu x_{12}|^{-2}}{|x_{12}|^{2D_0}} + \dots \quad (4.6)$$

For the operator:

$$\mathcal{O}_{mn} = \text{ Tr } \Phi_m \Phi_n \quad (4.7)$$

the tree-level 2-pt function is evaluated using $SU(N)$-gauge group properties in the evaluation of the diagrams

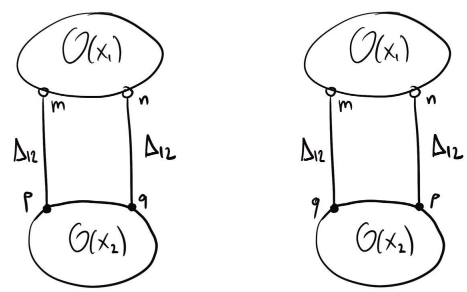


Figure 4.1: Tree-level contributions to the 2-pt function of $\mathcal{O}_{mn} = \text{ Tr } \Phi_m \Phi_n$

$$\begin{aligned} \langle \mathcal{O}_{mn}(x_1)\, \mathcal{O}_{pq}(x_2) \rangle &= \frac{\eta_{mp}\eta_{nq}\mathfrak{g}^{\mathfrak{mp}}\mathfrak{g}^{\mathfrak{nq}} \text{ Tr } \mathfrak{t}_\mathfrak{m}\mathfrak{t}_\mathfrak{n} \text{ Tr } \mathfrak{t}_\mathfrak{p}\mathfrak{t}_\mathfrak{q}}{N^2 |x_{12}|^4} + \frac{\eta_{mq}\eta_{np}\mathfrak{g}^{\mathfrak{mq}}\mathfrak{g}^{\mathfrak{np}} \text{ Tr } \mathfrak{t}_\mathfrak{m}\mathfrak{t}_\mathfrak{n} \text{ Tr } \mathfrak{t}_\mathfrak{p}\mathfrak{t}_\mathfrak{q}}{N^2 |x_{12}|^4} + \mathcal{O}(g) \\ &= \frac{2\left(1 - N^{-2}\right) \eta_{m\{p}\eta_{q\}n}}{|x_{12}|^4} + \mathcal{O}(g). \end{aligned} \quad (4.8)$$

Thus, at 0-th order, the classical dimension, as can be read by the above expansion is $D_0 = 2$.

Going now to the first order in perturbation theory, we fall in the usual shenanigan of quantum field theory, our results diverge and must regularize/renormalise the theory. In particular, we will work with dimensional regularisation in which we will have a $(4 - 2\epsilon)$-dimensional spacetime.

Our working action will be the modified/re-scaled/dimensional reduced

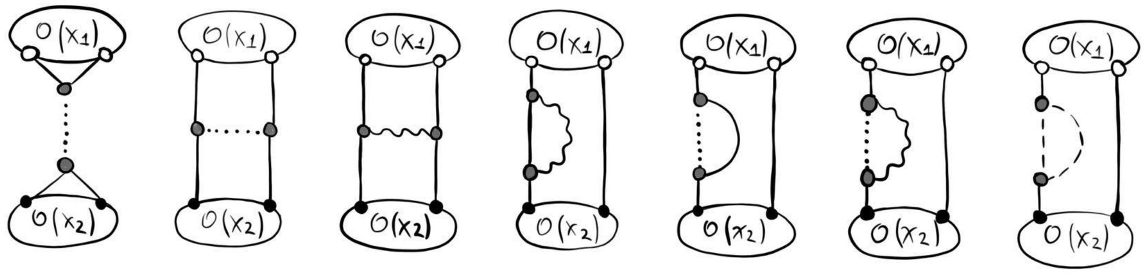


Figure 4.2: One-loop contributions to the 2-pt function .Sold, wiggly, dashed lines represent scalars, gluons, fermions respectively. The dotted lines represent ghosts.

S_{YM}:

$$S_{\mathrm{DR}}[\mathcal{W}] = N \int \frac{d^{4-2\epsilon}x}{(2\pi)^{2-\epsilon}} \mathcal{L}_{\mathrm{YM}}\left[\mathcal{W}, g\mu^{\epsilon}\right] \tag{4.9}$$

our propagator is

$$\Delta(x,y) = \frac{e^{\epsilon\Gamma(1-\epsilon}}{|x-y|^{(2-2\epsilon}} \tag{4.10}$$

Using the one-loop diagrams in figure (4.2) we get, [4]

[4]To perform the calculations we use the integrals:

$$\begin{aligned}
I_{x_1x_2} &= \frac{1}{2}\Delta\left(x_1 - x_2\right) \\
Y_{x_1x_2x_3} &= \mu^{2\epsilon}\int \frac{d^{4-2\epsilon}z}{(2\pi)^{2-\epsilon}} I_{x_1z}I_{x_2z}I_{x_3z} \\
X_{x_1x_2x_3x_4} &= \mu^{2\epsilon}\int \frac{d^{4-2\epsilon}z}{(2\pi)^{2-\epsilon}} I_{x_1z}I_{x_2z}I_{x_3z}I_{x_4z} \\
\tilde{H}_{x_1x_2,x_3x_4} &= \frac{1}{2}\mu^{2\epsilon}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}\right)^2 \int \frac{d^{4-2\epsilon}z_1 d^{4-2\epsilon}z_2}{(2\pi)^{4-2\epsilon}} I_{x_1z_1}I_{x_2z_1}I_{z_1z_2}I_{z_2x_3}I_{z_2x_4}
\end{aligned} \tag{4.11}$$

where the shape of the letter, very cleverly, represent the connections in terms of scalar propagators. They evaluate as:

$$Y_{112} = \frac{\xi I_{12}}{\epsilon(1-2\epsilon)}, \quad X_{1122} = \frac{2(1-3\epsilon)\kappa\xi I_{12}^2}{\epsilon(1-2\epsilon)^2}, \quad \tilde{H}_{12,12} = -\frac{2(1-3\epsilon)(\kappa-1)\xi I_{12}^2}{\epsilon^2(1-2\epsilon)} \tag{4.12}$$

$$\langle \mathcal{O}_{mn}(x_1)\, \mathcal{O}_{pq}(x_2)\rangle = 2\left(1-N^{-2}\right)\eta_{m\{p}\eta_{q\}n}\Delta_{12}^2 + \left(1-N^{-2}\right)g^2\left(\frac{1}{2}\eta_{m\{p}\eta_{q\}}\tilde{H}_{12,12} - \frac{1}{4}\eta_{mn}\eta_{pq}X_{1122}\right) + \mathcal{O}\left(g^3\right). \tag{4.14}$$

Which can be split into irreducible representations of $\mathfrak{su}(4)$, i.e into the singlet $[0,0,0]$ and the symmetric-traceless $[0,2,0]$

$$\mathcal{K} = \eta^{mn}\mathcal{O}_{mn} \qquad \text{"Konishi operator"} \tag{4.15}$$

$$\mathcal{Q} = \mathcal{O}_{(mn)} = \mathcal{O} - \frac{1}{6+2\epsilon}\eta_{mn}\eta^{pq}\mathcal{O}_{pq} \tag{4.16}$$

which correspond to the classical weights

$$w_{\mathcal{Q}} = (2;0,0;0,2,0;0,2) \qquad w_{\mathcal{K}} = (2;0,0;0,0,0;0,2) \tag{4.17}$$

We expect that the 2-pt function of the symmetric-traceless part should vanish, since the operator $\mathcal{Q}$ is part of the half-BPS multiplet and it is also part of the current multiplet of superconformal symmetry, were it is not protected, superconformal symmetry would be broken, see [55].

As such we expect at the limit $\epsilon \to 0$ the $\langle \mathcal{Q}_{mn}(x_1)\, \mathcal{Q}_{pq}(x_2)\rangle = 0$ as we can also verify doing the calculations.

On the other hand, forthe Konishi operator $\mathcal{K}$, we are forced to renormalise the operator in order to avoid its divergent behaviour. First of all, the 2-pt of the Konishi is given by

$$\begin{aligned}\langle \mathcal{K}(x_1)\,\mathcal{K}(x_2)\rangle &= 4\left(1-N^{-2}\right)(3+\epsilon)\left(\Delta_{12}^2 + \frac{1}{4}g^2\tilde{H}_{12,12} - \frac{1}{4}g^2(3+\epsilon)X_{1122}\right) + \mathcal{O}\left(g^3\right) \\ &= 4\left(1-N^{-2}\right)(3+\epsilon)\Delta_{12}^2\left(1-g^2\gamma\xi/\epsilon\right) + \mathcal{O}\left(g^3\right)\end{aligned} \tag{4.18}$$

where we have an $1/\epsilon$ pole which we have to get rid of. Here renormalisation is done by

$$Z\mathcal{K} = (1 + 1/2g^2\gamma/\epsilon\xi_0\,, \qquad \xi_0 = 2^{-\epsilon}\Gamma(1-\epsilon) \tag{4.19}$$

In a correlator of $Z\mathcal{K}$'s we replace $\xi \to \xi - \xi_0$ and evaluate the regular term at $\epsilon \to 0$:

Where ξ, κ are:

$$\xi = \frac{\Gamma(1-\epsilon)}{\left|\frac{1}{2}\mu^2 x_{12}^2\right|^{-\epsilon}}, \quad \kappa = \frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)^2\Gamma(1-3\epsilon)}{\Gamma(1-2\epsilon)^2\Gamma(1+2\epsilon)} = 1 + 6\zeta(3)\epsilon^3 + \mathcal{O}\left(\epsilon^4\right) \tag{4.13}$$

$$
\begin{aligned}
-g^2\gamma \lim_{\epsilon\to 0}\frac{\xi-\xi_0}{\epsilon} &= -g^2\gamma \lim_{\epsilon\to 0}\frac{\left(|\mu x_{12}|^{-2}\right)^{-\epsilon}-1}{\epsilon}\xi_0 \\
&= -g^2\gamma \lim_{\epsilon\to 0}\frac{\partial\left(|\mu x_{12}|^{-2}\right)^{-\epsilon}}{\partial\epsilon} = g^2\gamma\log|\mu x_{12}|^{-2}
\end{aligned} \tag{4.20}
$$

So are final results is

$$
\langle Z\mathcal{K}(x_1)\, Z\mathcal{K}(x_2)\rangle = \frac{12\left(1-N^{-2}\right)}{|x_{12}|^4}\left(1+6g^2\log|\mu x_{12}|^{-2}\right)+\mathcal{O}\left(3^3\right) \tag{4.21}
$$

where we can read that $D_2 = 6$ and thus

$$
\boxed{D = 2+6g^2+\mathcal{O} = 2+\frac{3g_{\text{YM}}^2}{4\pi^2}+\mathcal{O}(g_{\text{YM}}^3)} \tag{4.22}
$$

4.1.2 Matrix Quantum Mechanics

There are plenty other ways of calculating the dilatation operator, a particular nice one, is with the use of *matrix quantum mechanics*.

Here the dilatation generator, which scales the $r \mapsto cr$, maps[5] to the Hamiltonian. Also rotations (i.e. $SU(4)$ elements) map to rotations on the sphere naturally, whereas translations and boosts act on both $\mathbb{R}$ and S^3. It is therefore natural to KK-decompose field on a time slice S^3 in terms of spherical harmonics. This decomposition turns the qft into a quantum mechanical system of infinitely many matrices. For our theory of interest, i.e. $\mathcal{N}=4$ SYM this yields our desired field spectrum from the first chapter. Also this matrix quantum mechanics model is equivalent to $\mathcal{N}=4$ SYM[6]. The Hamiltonian can be derived by performing a Legendre transformation of the Lagrangian, but unfortunately it is not quite in the desired form, which leads to a very involved and tedious diagonalisation due to the infinite number of matrices.

[5] We work in radial coordinates of $\mathbb{R}^4$ where

$$
(r,\theta,\phi,\psi)\mapsto(t=\log r,\theta,\phi,\psi)
$$

[6] As we will see in the chapter on *Spin Matrix Theory* or *SMT* form sort.

4.2 Subsectors

In an idea scenario, it would very desirable to derive the dilatation operator for all the $\mathcal{N} = 4$ SYM. Unfortunately, such a task is very hard and we need to invent a way to restrict our calculations to *subsectors* of fields in such a way that the $\mathfrak{D}(g)$ closes on the subsector. We expect that inside a subsector, the number of fields as well as the symmetry algebra, to be reduced. This reduction will simplify the calculations enough in order to be able to calculate the dilatation generator within this subsector and thus deduce the anomalous dimensions. All possible closed subsectors are: the Half-BPS subsector, the short subsector, the BPS-subsector, the combined subsectors and the excitations subsectors. For the full derivation of these subsectors we refer to [32]. For an example of calculating the dilation generator from a perturbative expansion of the 2-pt function on the $\mathfrak{su}(2)$ quarter-BPS at one-loop level, we refer to [32].

5
Spin Matrix Theory

5.1 Motivation for SMT

The underlying motivation of Spin Matrix Theory (SMT) [25] was the need for a connection between the gauge and the string theory side of the AdS/CFT outside the $N = \infty$ planar limit. For example, in the study of black holes in AdS/CFT, where one needs to go beyond infinite N and must include non-perturbative effects, known as finite-N effects. Another motivation comes from the study of emergent D-branes in AdS/CFT where there, one is forced to study finite-N effects, e.g. Giant Gravitons [56] [57].

In our present paper, the SMT will be the underlying non relativistic theory that governs the near-BPS limit that we work on.

5.2 Definition of SMT

Spin Matrix Theory is a (non-relativistic) quantum mechanical theory. Its main feature is its separable complex Hilbert space $(\mathcal{H}, +, \cdot, \langle\cdot|\cdot\rangle)$, which for simplicity we will called it just $\mathcal{H}$, and the Hamiltonian on $\mathcal{H}$. It is built on a representation R_s of a semi-simple Lie super-group G_s which is called *spin group* (not to be confused with the usual spin/pin group of Spinor Representation Theory) and on the adjoint matrix representation R_m of the $U(N)$ group. [1]

5.3 Hilbert Space of SMT

Because in this thesis we only care about the bosons, we will introduce SMT for only the purely bosonic case, for fermions see [25].

- Step 1 Motivated by spin-chains models, we begin the construct of the SMT by defining the raising operators $(a_s^\dagger)^i{}_j$ with $s \in R_s$ and $i, j \in R_m$ where $i, j = 1, ..., N$ labels the fundamental/antifundamental of $U(N)$ accordingly.

- Step 2 The we define the *vacuum state* $|0\rangle$ as the state which is annihilated by the lowering operators, and we demand the all raising/lowering operators to commute with each other:

$$(a^s)^j{}_i\,|0\rangle = 0\ , \quad [(a^s)^j{}_i, (a^\dagger_{s'})^k{}_l] = \delta^s_{s'}\delta^k_i\delta^j_l \tag{5.1}$$

- Step 3 We can construct a Hilbert space $\mathcal{H}'$ corresponding to the symmetric[2] sum of all the possible symmetric(direct) products of the representations R_m and R_s:

$$\mathcal{H}' = \sum_{L=1}^{\infty} \mathrm{sym}[(R_s \otimes R_m)^L] \tag{5.2}$$

The corresponding base for $\mathcal{H}'$ is

$$(a^\dagger_{s_1})^{i_1}{}_{j_1}(a^\dagger_{s_2})^{i_2}{}_{j_2}...(a^\dagger_{s_L})^{i_L}{}_{j_L}\,|0\rangle\ , \quad L = 1, 2, 3, ... \tag{5.3}$$

- Step 4 Then our SMT Hilbert space $\mathcal{H}$ is just a linear subspace of $\mathcal{H}'$ such that $\mathcal{H}$ is the space of all singlet states in R_m, where the singlet condition is

[1] We can also generalize to different groups like $SU(N), SO(N), Osp(N)$, etc

[2] Because of the boson statistics

$$\Phi^i{}_j \left|\phi\right\rangle = 0 \text{ with } \Phi^i_j \equiv \sum_{s\in R_s} \sum_{k=1}^{N} \left[\left(a_s^\dagger\right)^i_k \left(a^s\right)^k j - \left(a_s^\dagger\right)^k j \left(a^s\right)^i k \right] \tag{5.4}$$

The Hilbert space $\mathcal{H}$ is spanned by:

$$\sum_{i_1,i_2,\ldots,i_L=1}^{N} \left(a_{s_1}^\dagger\right)^{i_1}_{i_{\sigma(1)}} \left(a_{s_2}^\dagger\right)^{i_2}_{i_{\sigma(2)}} \cdots \left(a_{s_L}^\dagger\right)^{i_L}_{i_{\sigma(L)}} \left|0\right\rangle, \quad L = 1, 2, \ldots \tag{5.5}$$

where $\sigma \in S(L)$ is an element of the permutation group $S(L)$ of L elements.

We can equivalently span $\mathcal{H}$ using a slightly different notation

$$\mathrm{Tr}\left(a_{s_1}^\dagger a_{s_2}^\dagger \cdots a_{s_l}^\dagger\right) \mathrm{Tr}\left(a_{s_{l+1}}^\dagger \cdots\right) \cdots \mathrm{Tr}\left(a_{s_{k+1}}^\dagger \cdots a_{s_L}^\dagger\right) \left|0\right\rangle, \quad L = 1, 2, \ldots \tag{5.6}$$

Where the individual cycles of the permutation elements correspond to single traces.

Before we move to the Hamiltonian of SMT, we make a last comment about how to general approach the fermionic case:

To introduce fermionic excitations you should split up the spin representation into a Bosonic and a Fermionic part $R_s = B_s \oplus F_s$. Then the $s \in B_s$ behaves just like discussed above, but the $s \in F_s$ must be treated with anticommutators instead, look e.g. [25].

This split up into fermions and bosons occurs in Lie supergroups like $SU(p, q|r)$ with $p + q \neq 0$ and $r \neq 0$ and it a very important case in AdS/CFT.

5.4 The Hamiltonian of SMT

It is time to consider interaction terms in the Spin Matrix Theory. We are mainly interested in the "two annihilations, two creation" type of interactions where we require the Hamiltonian to be *normal ordered*, and we also demand that the interactions should commute with all the generators of the spin group G_s. Additional the spin and matrix parts must factorize, to prevent measurements of the one affecting the measurements of the other. This leads to a Hamiltonian of the form

$$\boxed{H_{\text{int}} = \frac{1}{N} U^{s'r'}_{sr} \sum_{\sigma\in S(4)} T_\sigma \left(a_{s'}^\dagger\right)^{i_{\sigma(1)}} \left(a_{i_3}^\dagger\right)^{i_{\sigma(2)}} \left(a^{s_j}\right)^{i_{\sigma(3)}} i_1 \left(a^r\right)^{i_o(4)} i_2} \tag{5.7}$$

Where T_σ are coefficients defined below and $\sigma \in S(4)$ are permutations element of the $S(4)$ symmetric group. The U is a Hermitian linear operator to be defined below, and the sum is over s, r, s', r' and i_1, i_2, i_3, i_4.

It is clear to see that this Hamiltonian preserves the singlet condition and therefore belongs to the Hilbert Space of SMT.

The specific choice for the T is due to the behaviour of SMT near zero-temperature critical points of $\mathcal{N} = 4$ SYM. For more details see [25]

$$\sum_{\sigma \in S(4)} T_\sigma \sigma = (14) + (23) - (12) - (34) \tag{5.8}$$

The U linear operator is defined as

$$U : R_s \otimes R_s \to R_s \otimes R_s \tag{5.9}$$

and it is the spin part of the Hamiltonian which maps the direct(representation of) spin product of two states to a new (representation of) spin product of two states. Expanding this direct product into irreducible representations $V_{\mathcal{J}}$(where we label them with $\mathcal{J}$) we get:

$$R_s \otimes R_s = \sum_{\mathcal{J}} V_{\mathcal{J}} \tag{5.10}$$

We can prove that the U has a general form which is proportional to a constant multiplied by a *projection operator* $P_{\mathcal{J}}$. This can be achieved by imposing our defining condition that the H_{int} should commute with all the generators of the G_s spin group. Hence,

$$U_{sr}^{s'r'} = \sum_{\mathcal{J}} C_{\mathcal{J}} \left(P_{\mathcal{J}}\right)_{sr}^{s'r'} \tag{5.11}$$

Having defined now how interaction can be included in SMT, we can now present the form of the most general form of SMT Hamiltonian is

$$\boxed{H = L + gH_{int} - \sum_p \mu_p K_p} \tag{5.12}$$

As the reader can see, there are still two things we need to defined.

First the **Length operator** L which is essentially the kinetic part of the Hamiltonian H. It has the form,

$$L = \sum_s \mathrm{Tr}\left(a_s^\dagger a^s\right) \tag{5.13}$$

and it is a "diagonal"[3] operator, which when acted on a state it reads its *length* and commutes with all the generators of G_s.

Second, we also include *chemical potential* factor, which is essentially constructed by the Cartan generators of G_s (denoted by K_p), we also name the μ_p as chemical potentials.

Eventually the partition function for SMT is

$$Z(\beta, \mu_p) = \mathrm{Tr}\left(e^{-\beta H}\right) = \mathrm{Tr}\left(e^{-\beta\left(L + gH_{\mathrm{int}} - \sum_p \mu_p K_p\right)}\right) \tag{5.14}$$

where the trace is over the Hilbert space $\mathcal{H}$

For this thesis we are mostly interested for the $G_s = SU(1,1)$ case.

5.5 SMT from $\mathcal{N} = 4$ SYM near critical points

It is quite a beautiful demonstration to show how one gets the SMT from $\mathcal{N} = 4$ SYM near zero-temperature critical points in the grand canonical ensemble. In this chapter we will do a small representation of the derivation, for the whole original discussion see [25][29].

5.5.1 The $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ partition function

The global symmetry of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with gauge group $U(N)$ is the Lie supergroup $\widetilde{\mathrm{PSU}}(2,2|4)$ or its algebra $\mathfrak{psu}(2,2|4)$.

- The bosonic subgroup group $SU(2,2)$ give us the Cartan generators operators D, S_1, S_2
- The R-symmetry bosonic subgroup $SU(4) \cong SO(6)$ has the generators R_1, R_2, R_3.

The grand canonical partition function has the standard form,

$$Z(\beta, \vec{\Omega}) = \mathrm{Tr}\left(e^{-\beta D + \beta \vec{\Omega} \cdot \vec{J}}\right) \tag{5.15}$$

[3] Diagonal is the matrix representation of the operator

where $T = 1/\beta$, chemical potentials $\vec{\Omega} = (\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3)$ and 't Hooft coupling constant $\lambda = g_{\text{YM}}^2 N$. We also introduce the notation $\vec{J} = (S_1, S_2, R_1, R_2, R_3)$.

5.5.2 The Grand-Canonical Ensemble (GCE)

○

In the last subsection, we introduce the notion of the GCE of $\mathcal{N} = 4$ SYM which has the form (4.15). Our goal from now on is to show that the SMT essentially describes $\mathcal{N} = 4$ SYM near zero-temperature critical points in the GCE.

Definition 5.5.1. The *zero-temperature critical points* of the GCE are the points that one can obtain by expanding the submanifold of phase transitions points to zero temperature. Thus, for a given critical point $(T, \vec{\Omega}) = (0, \vec{\Omega}^{(c)})$, there are confinement/deconfinement transition points that lie arbitrary close to it.

In this thesis we are mainly interested in the critical point $(T, \vec{\Omega}^{(c)}) = (0, 1, 0, 1, 0, 0)$ with spin group $G_s = SU(1, 1)$, Dynkin label: ◯ ,in the $[-1]$-Representation R_s of G_s.

Another very interesting critical point of $\mathcal{N} = 4$ SYM is the:

- $(T, \vec{\Omega}^{(c)}) = (0, 2/3, 0, 1, 2/3, 2/3)$ with spin group $G_s = SU(1|1)$, Dynkin label: $\otimes$, in the $[1]$-Representation R_s of G_s.
- $(T, \vec{\Omega}^{(c)}) = (0, 1, 0, 1, 1/2, 1/2)$ with spin group $G_s = SU(1, 1|1)$, Dynkin label: $\otimes$—$\otimes$, in the $[0, 1]$-Representation R_s of G_s.
- $(T, \vec{\Omega}^{(c)}) = (0, 1, 1, 1, 1, 1)$ with spin group $G_s = SU(1, 2|3)$, Dynkin label: ○—$\otimes$—○—○—$\otimes$, in the $[0, 0, 0, 1, 0]$-Representation R_s of G_s.

We can now take the near zero temperature limit of (4.14) and by requiring that the $\beta(\Omega - \Omega^{(c)})$ to finite in this limit we get

$$\beta D - \beta\vec{\Omega} \cdot \vec{J} = \beta\delta D + \beta(D_0 - \vec{\Omega}^{(c)} \cdot \vec{J}) - \beta(\vec{\Omega} - \vec{\Omega}^{(c)}) \cdot \vec{J} \tag{5.16}$$

We now analyse the different limit cases:

- The $\lambda = 0$ limit : We define $\Delta = D_0 - \vec{\Omega^{(c)}} \cdot \vec{J}$ for the states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. Then we get either $\Delta = 0$ or $\Delta \geq 1/2$ and so only states with zero Δ contributes to the partition function after the limit.[4] Quite remarkably at $\Delta = 0$ the states span the *Hilbert space* $\mathcal{H}$ of SMT [REF] with $G_s \leq PSU(2,2|4)$ and $R_s \subseteq \mathcal{A}$.

We can also notice that the term $-\beta\vec{\Omega} \cdot \vec{J}$ is a linear combination of the Cartan generators of K_p and length operator with $\Delta = 0$.

There only one term in [5.16] that needs to be discussed which can only be analysed only if we work in the $\lambda \neq 0$ regime , the $\beta\delta D$.

- The $\lambda \neq 0$ and fixed with $\beta \to \infty$ limit : We have $\delta D = 0$ on those states. To get an interactive theory start we require that $\beta\lambda$ to finite in this limit. We can then introduce a new $\tilde{\beta}$ s.t. $\beta\lambda \to \tilde{\beta}\lambda$, then $\beta\delta D \to \tilde{\beta}\tilde{\lambda}D_2$ since all the higher loops vanish. For the $\Delta = 0$ states we have $D_2 = H_{\text{int}}$.[5]

So to sum all the above discussion , by approaching one of the critical points in the limit:

$$(T,\vec{\Omega}) \to \left(0,\vec{\Omega}^{(c)}\right) \text{ and } \lambda \to 0 \quad \text{with} \quad \beta\left(\vec{\Omega}-\vec{\Omega}^{(c)}\right) \text{ and } \beta\lambda \text{ finite} \tag{5.17}$$

one gets as we foreshadow, the partition function of SMT:

$$Z\left(\tilde{\beta},\mu_p\right) = \text{Tr}\left(e^{-\tilde{\beta}\left(L+gH_{\text{int}}-\sum_p \mu_p K_p\right)}\right) \tag{5.18}$$

the trace is over the SMT Hilbert space in the subsector $\Delta = 0$ of $\mathcal{N} = 4$ SYM, the coefficients in the interaction term H_{int} are given by $C_j = \frac{1}{8\pi^2}h(j)$ as was discussed previously and in the main paper [25]

5.5.3 The Micro-Canonical Ensemble (MCE)

It is also equivalent successful to work in the MCE in order to get the same limits. To do this we take the *low energy limit*

$$D - \vec{\Omega^{(c)}} \cdot \vec{J} \to 0 \tag{5.19}$$

This essentially in the trick that force us to work on $\Delta = 0$ since anything above the energy gap ($\Delta \geq 1/2$) decouples. For the remaining states we have

$$D - \vec{\Omega}^{(c)} \cdot \vec{J} = \delta D = \lambda D_2 + \mathcal{O}(\lambda^{3/2}) \tag{5.20}$$

[4] Since $D \geq \Omega^{(c)} \cdot J$ which is a requirement for any critical point in our main theory

[5] We can see the $\Delta = 0$ states as states in the Hilbert space $\mathcal{H}$ of the SMT corresponding to the R_s of G_s.

and in order to get a non trivial energy spectrum we should work on the limit

$$D - \vec{\Omega}^{(c)} \cdot \vec{J} \to 0 \text{ and } \lambda \to 0 \quad \text{with} \quad \frac{D - \vec{\Omega}(c) \cdot \vec{J}}{\lambda} \text{ finite} \tag{5.21}$$

which gives again the SMT. The two limits (the MCE and the GCE) are equivalent limits and one can jump from one to another freely.

5.6 Decoupling limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

We list 14 possible decoupling limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with gauge group $SU(N)$ are they were calculated in the paper [31]. Note that we use the notation from SMT in which $n = (n_1, n_2, n_3, n_4, n_5) = (\mu_1, \mu_2, \omega_1, \omega_2, \omega_3)$ where μ_1, μ_2 are the chemical potentials (note the we previously use the notation ω for the chemical potentials, but due to the fact the in the next chapters our notation will different for various reasons, we change the notations accordingly).

# derivatives	0	0	0	0	1	1	1	1	2	2	2	2
# scalars	0	1	2	3	0	1	2	3	0	1	2	3
0 fermions		+	+			+			+			
1 fermion	+	+	+		+	+			+			
2 fermions				+			+			+		
5 fermions												+

(5.22)

- The bosonic $U(1)$ limit. Given by $n = (0, 0, 1, 0, 0)$.
- The fermionic $U(1)$ limit. Given by $n = (3/5, -3/5, 3/5, 3/5, 3/5)$.
- The $SU(2)$ limit. Given by $n = (0, 0, 1, 1, 0)$.
- The $SU(1|1)$ limit. Given by $n = (2/3, 0, 1, 2/3, 2/3)$.
- The $SU(1|2)$ limit. Given by $n = (1/2, 0, 1, 1, 1/2)$.
- The $SU(2|3)$ limit. Given by $n = (0, 0, 1, 1, 1)$.
- The $SU(1, 1|1)$ limit. Given by $n = (1, 0, 1, 1/2, 1/2)$.
- The bosonic $SU(1, 1)$ limit. Given by $n = (1, 0, 1, 0, 0)$.
- The fermionic $SU(1, 1)$ limit. Given by $n = (1, 0, 2/3, 2/3, 2/3)$..

- The $SU(1,1|2)$ limit. Given by $n = (1,0,1,1,0)$.
- The $SU(2|3)$ limit. Given by $n = (0,0,1,1,1)$.
- The $SU(1,2)$ limit. Given by $n = (1,1,0,0,0)$.
- The $SU(1,2|1)$ limit. Given by $n = (1,1,1/2,1/2,0)$.
- The $SU(1,2|2)$ limit. Given by $n = (1,1,1,0,0)$.
- The $SU(1,2|3)$ limit. Given by $n = (1,1,1,1,1)$.

These above, are all the possible decoupling limits, other choices of n give decoupling limits which result in equivalent theories to one of the theories listed above, e.g. $n = (a,-a,b,b,b)$ with $0 < a, b < 1$ and $2a + 3b = 3$ will always result in a $U(1)$-fermionic theory.

6

The 1–loop order

The purpose of this chapter, is to present a sketch for the calculation of the complete one-loop dilatation operator of $\mathcal{N} = 4$ Super Yang-Mills theory. We follow the original papers [30] [31] [32].

6.1 The general form of the dilatation generator

The (bosonic) Hamiltonian has the form

$$\begin{aligned}\mathcal{H} = &- N^{-1} (C_\mathrm{a})^{\mathcal{AB}}_{\mathcal{CD}} : \mathrm{Tr}\left[\mathcal{W}_\mathcal{A}, \check{\mathcal{W}}^\mathcal{C}\right] \left[\mathcal{W}_\mathcal{B}, \check{\mathcal{W}}^\mathcal{D}\right] : \\ &- N^{-1} (C_\mathrm{b})^{\mathcal{AB}}_{\mathcal{CD}} : \mathrm{Tr}\left[\mathcal{W}_\mathcal{A}, \mathcal{W}_\mathcal{B}\right] \left[\check{\mathcal{W}}^\mathcal{C}, \check{\mathcal{W}}^\mathcal{D}\right] \\ &+ N^{-1} (C_\mathrm{c})^{\mathcal{A}}_{\mathcal{B}}\, \mathfrak{g}^{\mathfrak{mn}} : \mathrm{Tr}\left[\mathcal{W}_\mathcal{A}, \mathfrak{t}_\mathfrak{m}\right] \left[\mathfrak{t}_\mathfrak{n}, \check{\mathcal{W}}^\mathcal{B}\right] :\end{aligned} \tag{6.1}$$

which is it the algebraic representation of the Feynman diagrams on figure (6.1). Using gauge invariance and the Jacobi identity we get the compact form of $\mathcal{H}$

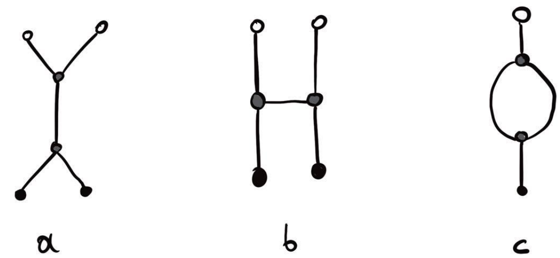


Figure 6.1: The one-loop contribution diagrams.

$$\mathcal{H} = -N^{-1} C^{\mathcal{AB}}_{cD} : \mathrm{Tr}\left[\mathcal{W}_{\mathcal{A}}, \tilde{\mathcal{W}}^{\lrcorner}\right]\left[\mathcal{W}_{\mathcal{B}}, \check{\mathcal{W}}^{\mathcal{D}}\right] : \tag{6.2}$$

with coefficients

$$C^{\mathcal{AB}}_{\mathcal{CD}} = -\left((C_{\mathrm{a}})^{\mathcal{AB}}_{\mathcal{CD}} + (C_{\mathrm{b}})^{\mathcal{AB}}_{\mathcal{CD}} - (C_{\mathrm{b}})^{\mathcal{AB}}_{\mathcal{DC}} + \frac{1}{2}\delta^{\mathcal{A}}_{\mathcal{C}} \left(C_{\mathrm{c}}\right)^{\mathcal{B}}_{\mathcal{D}} + \frac{1}{2}\left(C_{\mathrm{c}}\right)^{\mathcal{A}}_{C} \delta^{\mathcal{B}}_{\mathcal{D}} \right) \tag{6.3}$$

which remain to be determined. To compute $C^{\mathcal{AB}}_{\mathcal{CD}}$ we will use the fact that it must be invariant under the classical superconformal algebra. The $C^{\mathcal{AB}}_{\mathcal{CD}}$ can be essentially described as the endomorphism *interwining map* s.t. $C : \mathcal{V}_F \otimes \mathcal{V}_F \rightarrow \mathcal{V}_F \otimes \mathcal{V}_F$.[1] This puts tight constrains on the coefficients, where its independent components can be obtained by investigating the irreducible modules in $\mathcal{V}_F \otimes \mathcal{V}_F$[2]

Following the procedure/arguments from the Spin Matrix Theory chapter,

[1]Where the $\mathcal{V}_j$ modules have primary weights

$$w_0 = (2; 0, 0; 0, 2, 0; 0, 2)$$
$$w_1 = (2; 0, 0; 1, 0, 1; 0, 2)$$
$$w_j = (j; j-2, j-2; 0, 0, 0; 0, 2)$$

[2]Notice the similarity between SMT and this discussion. We do exactly the same procedure to get the $C^{\mathcal{AB}}_{\mathcal{CD}} \rightleftarrows U^{s'r'}_{sr}$

we get that

$$C^{\mathcal{AB}}_{\mathcal{CD}} = \sum_{j=0} (\mathcal{P}_j)^{\mathcal{AB}}_{\mathcal{CD}} \tag{6.4}$$

where $\mathcal{P}_j$ is the projector, that projects two fields $\mathcal{W}_{\mathcal{A}}, \mathcal{W}_{\mathcal{B}}$ to the $\mathcal{V}_j$ module.

In our new notation we have that a new more compact form for the Hamiltonian

$$\mathcal{H} = \sum_{j=0}^{\infty} C_j \, (\mathcal{P}_j)^{AB}_{CD} \left\{ \begin{array}{c} \mathcal{CD} \\ \mathcal{AB} \end{array} \right\} \tag{6.5}$$

where the $\{\}$ planar interactions notations is:

$$\text{"planar interactions"}: \left\{ \begin{array}{c} \mathcal{A}_1 \ldots \mathcal{A}_{E_\mathrm{i}} \\ \mathcal{B}_1 \ldots \mathcal{B}_{E_\mathrm{o}} \end{array} \right\} := N^{1-E_\mathrm{i}} \operatorname{Tr} \mathcal{W}_{\mathcal{B}_1} \ldots \mathcal{W}_{\mathcal{B}_{\mathrm{E}_0}} \tilde{\mathcal{W}}^{\mathcal{A}_{E_\mathrm{i}}} \ldots \tilde{\mathcal{W}}^{\mathcal{A}_1} \tag{6.6}$$

Essentially, it searches for the sequence of fields $\mathcal{W}_{\mathcal{A}_1} ... \mathcal{W}_{\mathcal{A}_{E_i}}$ and replaces it with the sequence $\mathcal{W}_{\mathcal{B}_1} ... \mathcal{W}_{\mathcal{B}_{E_0}}$. To be more precise, on a single-trace state: $|\mathcal{C}_1 ... \mathcal{C}_L\rangle :=$ Tr $[\mathcal{W}_{\mathcal{C}_1} ... \mathcal{W}_{\mathcal{C}_L}]$, if we act with it, we get:

$$\sum_{p=1}^{L} (-1)^{(\mathcal{C}_1 \ldots \mathcal{C}_{p-1})(\mathcal{B}_1 \ldots \mathcal{B}_{E_\mathrm{o}} \mathcal{A}_1 \ldots \mathcal{A}_{E_\mathrm{i}})} \delta^{\mathcal{A}_1}_{\mathcal{C}_p} \cdots \delta^{\mathcal{A}_{E_\mathrm{i}}}_{\mathcal{C}_{p+E_\mathrm{i}-1}} |\mathcal{C}_1 \ldots \mathcal{C}_{p-1} \mathcal{B}_1 \ldots \mathcal{B}_{E_\mathrm{o}} \mathcal{C}_{p+E_\mathrm{i}} \ldots \mathcal{C}_L\rangle \tag{6.7}$$

where E_0 is the number of fields , E_i are the number of variations, $E = (E_0, E_i)$ is the number of the external legs of the Feynman diagrams for the *interactions*. An example is:

$$\left\{ \begin{array}{c} \mathcal{AB} \\ \mathcal{BA} \end{array} \right\} |12345\rangle = |21345\rangle \pm |13245\rangle \pm |12435\rangle \pm |12354\rangle \pm |52341\rangle \tag{6.8}$$

Returning now to the Hamiltonian, we want to present another notation that will be convinient for our discussions later, and it is related to the fact that $\mathcal{H}$ acts on spin chains(with length L) and transforms two adjacent nodes s.t.

$$\mathcal{H} = \sum_{p=1}^{L} \mathcal{H}_{p,p+1}, \quad \text{where} \quad \mathcal{H}_{p,p+1} = \sum_{j=0}^{\infty} C_j \mathcal{P}_{p,p+1,j} \tag{6.9}$$

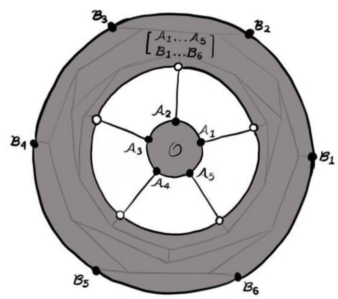


Figure 6.2: An example of action for a wrapping interactions, we do not concern ourselves with this kind of interactions, for more were refer to [32].

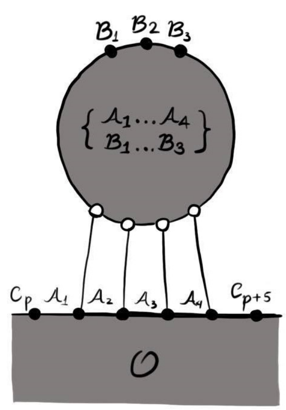


Figure 6.3: Insertions of planar interactions. White dotes corresponds to variations and black dots to fields, inside the blob there are some irrelevant planar diagrams that connect the dots.

As we can see we can read of all the C_j from this Hamiltonian and therefore we can generalize the non-planar $\mathcal{H}$ to the Hamiltonian density $\mathcal{H}_{12}$ uniquely. We also use the total spin operator $\mathcal{J}_{12}$ with eigenvalue equation

$$\mathcal{J}_{12}\mathcal{V}_j = j\mathcal{V}_j \tag{6.10}$$

So our final short notation for the $\mathcal{H}_{12}$ is

$$\boxed{\mathcal{H}_{12} = C(\mathcal{J}_{12}) \quad \text{where} \quad C(\mathcal{J}_{12}) = \sum_{j=0}^{\infty} C_j \mathcal{P}_{12,j}} \tag{6.11}$$

6.1.1 The bosonic $\mathfrak{su}(1,1)$ subsector

It remains to calculate the coefficients C_j, to achieve this we restrict our analysis to the $\mathfrak{su}(1,1)$ subsector.[3]

The calculations for the Hamiltonian density was fully derived in [31][30][32] and it is:

$$\boxed{\mathcal{H}_{12}|m, m-k\rangle = \sum_{k'=0}^{m} \left(\delta_{k=k'}(h(k) + h(m-k)) - \frac{\delta_{k\neq k'}}{|k-k'|} \right) |k', m-k'\rangle} \tag{6.12}$$

this is equivalent to

$$\mathcal{H}_{12} = 2h(\mathcal{J}_{12}) \tag{6.13}$$

where $h(m)$ is the *Harmonic series*:

$$h(m) := \sum_{k=1}^{m} \frac{1}{k} = \Psi(m+1) - \Psi(1) \quad \Psi(x) = \Gamma'(x)/\Gamma(x) \text{ is the digamma function} \tag{6.14}$$

This result is exactly the result we will attempt to derive in the next chapter using a very different by probably simpler approach; and it is the main result for this thesis.

[3] To be exact, we work on the $(2,2)$ closed subsector, [32].

7

Non–relativistic corner of $\mathcal{N} = 4$ SYM

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Classical}\\ \mathcal{N} = 4\end{array}} \arrow[r] \arrow[d] & \boxed{\begin{array}{c}\text{Dilatation}\\ \text{Operator}\end{array}} \arrow[d] \\
\boxed{H_c} \arrow[r] & \boxed{H_q}
\end{tikzcd}
$$

As we highlighted in the introduction discussion, our starting goal is to find a way to calculate using the AdS/CFT framework, the dynamics of strong gravity/black holes from the quantum theory living on the boundary. As we argued in the introduction chapter, we will achieve this by using a particular near-BPS limit which reduces our theory into a regular non-relativistic quantum mechanics model such that we preserve the desired strong gravity dynamics and semi-classical geometry without sacrificing the simplicity of direct quantitative study of the strongly coupled finite-N regime, on the QFT side.

We sketch all the procedure as follows:

- We first impose near-BPS bound SMT-type bound:

$$\lambda \to 0 \quad \text{with} \quad \frac{E - S_1 - \sum_{i=1}^{3} \omega_i Q_i}{\lambda} \quad \text{finite} \quad , \quad \text{N fixed} \tag{7.1}$$

  where E is the energy,S_1 of the angular momenta and Q_i, $i = 1, 2, 3$ are the three R-charges of $\mathcal{N} = 4$ SYM on S^3, moreover, ω_i are the three constants that characterize the BPS bounds. We use $(\omega_1, \omega_2, \omega_3) = (1, 0, 0)$ in which we obtain a scalar theory (full bosonic) with $U(1) \times SU(1, 1)$ global symmetry, where $U(1)$ corresponds to the conservation of particle numbers as nonrelativistic quantum theory dictates and $SU(1, 1)$ is the symmetry of the interactions in our particular choice of BPS bounds. Other choices of ω_i were shown below in the chapter of SMT.

- We do a sphere reduction of our classical $\mathcal{N} = 4$ SYM on S^3 by expanding the scalar fields into spherical harmonics, then we see that most of the massive modes on S^3 decouple. Also, we note that we work in the Coulomb gauge $\nabla_i A^i = 0$ which helps integrate out spurious DOF but keeps track of interactions between scalars fields that are mediated by the longitudinal and temporal gluons.

- We find the propagating modes from the near-BPS limit at quadratic order.

- We integrate out non-dynamical modes from the Hamiltonian.

- We can compute the interacting Hamiltonian using the exact near-BPS bounds.

- We combine all our results to write down the full classical effective Hamiltonian H_{lim}.

- Then we quantize the Hamiltonian using arguments from Appendix A. This quantization procedure results in self-energy corrections that can be calculated from a standard normal-ordering procedure.

Before we start the above prescription, we have to introduce and work out some basic mathematical facts about the spherical expansion on the manifold S^3.

7.1 The Harmonic Expansion on S^3

We will follow the procedure first developed in the paper by Salam and Strathdee [47] where the expansion to harmonic functions on the coset space G/H is developed. In our particular case we have our main group $S^3 = SO(4)/SO(3)$, where we identify the $G = SO(4)$ and the stability group $H = SO(3)$ which is essentially the local Lorentz group on S^3. Also the decomposition of $SO(4) \cong SU(2) \times \widetilde{SU}(2)$ allows us to interpret each $SU(2)$ term as spin, with generators J_i and $\tilde{J}_i$ accordingly, $i = 1, 2, 3$. Then the generators of the H are the direct sum of the spins i.e.

$$L_i = J_i + \tilde{J}_i \tag{7.2}$$

We naturally denote the basis of the $(J, \tilde{J})$ representation by $|Jm\rangle|\tilde{J}\tilde{m}\rangle$. Then the basis for the spin L representation of H is,

$$|Ln; J\tilde{J}\rangle\rangle = \sum_{m\tilde{m}} C^{Ln}_{Jm\tilde{J}\tilde{m}} |Jm\rangle|\tilde{J}\tilde{m}\rangle \tag{7.3}$$

where the Clebsch-Gordan coefficient of $SU(2)$ and the triangular $|J - \tilde{J}| \leq L \leq J + \tilde{J}$ inequality must be satisfied.

We can represent the element of G/H by using the exponential map

$$\exp : T_e(G/H) \to G/H \tag{7.4}$$

For more formal details look e.g. [47] For our particular case we are interested in the $S^3 = SO(4)/SO(3)$ so using spherical polar coordinates the corresponding elements $\exp(S^3)$ are,

$$\Upsilon(\Omega) = e^{-i\psi L_1} e^{-i\varphi L_3} e^{-i\theta K_1} \tag{7.5}$$

where $K_i = J_i - \tilde{J}_i$.

The spherical harmonics on S^3 with spin L are,

$$\mathcal{Y}^{Ln}_{Jm,\tilde{J}\tilde{m}}(\Omega) = N^L_{J\tilde{J}} \left\langle \left\langle Ln; J\tilde{J} \left| \Upsilon^{-1}(\Omega) \right| Jm \right\rangle \mid \tilde{J}\tilde{m} \right\rangle \quad , \quad N^L_{J\tilde{J}} = \sqrt{\frac{(2J+1)(2\tilde{J}+1)}{2L+1}} \tag{7.6}$$

where the normalisation factor is fixed to satisfy the orthonormality condition,

$$\int d\Omega \sum_n \left(\mathcal{Y}^{Ln}_{Jm,\tilde{J}\tilde{m}}\right)^* \mathcal{Y}^{Ln}_{J'm',\tilde{J}'\tilde{m}'} = \delta_{JJ'}\delta_{\tilde{J}\tilde{J}'}\delta_{mm'}\delta_{\tilde{m}\tilde{m}'} \tag{7.7}$$

$$\sum_{\alpha\beta} C^{c\gamma}_{a\alpha b\beta} C^{c'\gamma'}_{a\alpha b\beta} = \delta_{cc'}\delta_{\gamma\gamma'}$$

The complex conjugate of $\mathcal{Y}^{Ln}_{Jm,\tilde{J}\tilde{m}}$ is

$$\left(\mathcal{Y}^{Ln}_{Jm,\tilde{J}\tilde{m}}\right)^* = (-1)^{-J+\tilde{J}-L+m-\tilde{m}+n} \mathcal{Y}^{L-n}_{J-m,\tilde{J}-\tilde{m}} \tag{7.8}$$

It is important to note that the Haar measurement of the group G is identified with our choice of normalisation for the integral $\int d\Omega = 1$ since the integrand is invariant under the action of the stability group H.

$$\nabla_i \mathcal{Y}^{Ln}_{Jm,\tilde{j}\tilde{m}}(\Omega) = N^L_{J\bar{J}} \left\langle \left\langle Ln; J\tilde{J} \left|(-iK_i)\,\Upsilon^{-1}(\Omega)\right| Jm \right\rangle \mid \tilde{J}\tilde{m} \right\rangle$$

$$\nabla^2 \mathcal{Y}^{Ln}_{Jm,j_{\tilde{m}}}(\Omega) = -(2J(J+1) + 2\tilde{J}(\tilde{J}+1) - L(L+1))\mathcal{Y}^{Ln}_{Jm,\tilde{J}\tilde{m}}(\Omega)$$

$$\begin{aligned}&\int d\Omega \textstyle\sum_{n_1 n_2 n_3} \left(\mathcal{Y}^{L_1 n_1}_{J_1 m_1, J_1 \bar{m}_1}\right)^* \mathcal{Y}^{L_2 n_2}_{J_2 m_2, J_2 \bar{m}_2} \mathcal{Y}^{L_3 n_3}_{J_3 m_3, \tilde{J}_3 \bar{m}_3} C^{L_1 n_1}_{L_2 n_2} L_3 n_3 \\ &= \sqrt{(2L_1+1)(2J_2+1)\left(2\tilde{J}_2+1\right)(2J_3+1)\left(2\tilde{J}_3+1\right)} \begin{Bmatrix} J_1 & \tilde{J}_1 & L_1 \\ J_2 & \tilde{J}_2 & L_2 \\ J_3 & \tilde{J}_3 & L_3 \end{Bmatrix} C^{J_1 m_1}_{J_2 m_2, J_3 \tilde{J} m_3} C^{\tilde{J}_1 \tilde{m}_1}_{\tilde{J}_2 \tilde{m}_2, \tilde{J}_3 \bar{m}_3}\end{aligned} \tag{7.9}$$

7.1.1 Tensors on S^3

- The scalars on S^3

The scalars on S^3 correspond to $L = 0$ and are defined by:

$$Y_{JM} \equiv \mathcal{Y}^{L=0,n=0}_{Jm,J\tilde{m}} \tag{7.10}$$

- The vector harmonics on S^3

The vector harmonics on S^3 correspond to $L = 1$ and are defined by:

$$\begin{aligned}|1; J\tilde{J}\rangle\rangle &= \tfrac{1}{\sqrt{2}}(-|1,1; J\tilde{J}\rangle\rangle + |1,-1; J\tilde{J}\rangle\rangle) \\ |2; J\tilde{J}\rangle\rangle &= \tfrac{i}{\sqrt{2}}(|1,1; J\tilde{J}\rangle\rangle + |1,-1; J\tilde{J}\rangle\rangle) \\ |3; J\tilde{J}\rangle\rangle &= |1,0; J\tilde{J}\rangle\rangle\end{aligned} \tag{7.11}$$

where $\rho = 1, 2, 3$ are a value signed to each of the states $\left|\rho\,; J\tilde{J}\right\rangle$ in order to numerate each particular case implied by the particular triangular inequality for $(J, \tilde{J}) \to (J+1, J), (J, J+1), (J, J)$. From our above notation we defined the vector harmonics on the three-sphere as

$$\mathcal{Y}^i_{Jm,\tilde{J}\tilde{m}} = N^1_{J\tilde{J}} \left\langle \left\langle i; J\tilde{J} \left| \Upsilon^{-1}(\Omega) \right| Jm \right\rangle \mid \tilde{J}\tilde{m} \right\rangle \quad (i = 1, 2, 3) \tag{7.12}$$

We use the following natural notations for the vectors:

$$\begin{aligned} Y^{\rho=1}_{JMi} &= i\mathcal{Y}^i_{J+1m,J\tilde{m}} \\ Y^{\rho=-1}_{JMi} &= -i\mathcal{Y}^i_{Jm,J+1\tilde{m}} \\ Y^{\rho=0}_{JMi} &= \mathcal{Y}^i_{Jm,J\tilde{m}} \end{aligned} \tag{7.13}$$

For the discussion on spinors i.e. $L = 1/2$ we refer to [42]

7.2 Sphere Reduction of the $\mathcal{N} = 4$ SYM on S^3

We are ready to expand our theory in terms of infinitely many KK modes[1], according to the [47] prescription. This will result in a very nice matrix quantum mechanics with uncountably many matrices. The quantization procedure will be the climax point of this thesis. We will now prove the main point of the subject, i.e. the equivalence between quantizing the $\mathcal{N} = 4$ SYM at one-loop near the BPS-limit and taking the same limit on S^3 and then quantizing the resulting effective theory.

We start by taking the $\mathcal{N} = 4$ SYM Langrangian with all the fermionic fields turned off and also introduce the combination

$$\Phi_a = \phi_{2a-1} + i\phi_{2a} \quad , \quad a = 1, 2, 3 \tag{7.14}$$

for the real scalar fields that belong to the **6** of $SU(4) \cong SO(6)$, so we get

$$L = \int_{S^3} \text{tr} \left[-\frac{1}{4}F^2_{\mu\nu} - |D_\mu \Phi_a|^2 - |\Phi_a|^2 - \frac{g^2}{2} \sum_{a,b} (|[\Phi_a, \Phi_b]|^2 + |[\Phi_a, \bar{\Phi}_b]|^2) \right] \tag{7.15}$$

Before we start the individual limits we must first discuss the gauge field and our gauge fixing. In all four limits, the gauge field DOF will decouple

[1] Kaluza-Klein modes.

on-shell, but it still contributes to the dynamics as an off-shell longitudinal gluon and by integrating it out it gives rise to an effective interaction of the surviving mode at order g^2. We will work, as we highlighted before, in the Coulomb gauge:

$$\nabla_i A^i = 0 \ , \quad \int d\Omega A_0 = 0 \tag{7.16}$$

We integrate out all the auxiliary DOF (in our case are the longitudinal and the temporal components of the gauge field) by solving the constraints. We consider the action with a general source j^μ as follows:

$$S_A = \int_{\mathbb{R}\times S^3} \sqrt{-g} \ \mathrm{tr} \ \left(-\frac{1}{4} F_{\mu\nu}^2 - A \cdot j \right) \tag{7.17}$$

We can calculate, from their definitions, the canonical momenta Π_μ to be

$$\Pi_0 = 0 \ , \quad \Pi_i = F_{0i} \tag{7.18}$$

We calculate the Hamiltonian from the action to be

$$H_A = \int_{\mathbb{R}\times S^3} \sqrt{-g} \ \mathrm{tr} \left(\frac{1}{2} \Pi_i^2 + \frac{1}{4} F_{ij}^2 - A_0 \left(\nabla_i \Pi^i + j_0 \right) + A^i j_i + \eta \nabla_i A^i \right) \tag{7.19}$$

where we have introduce the Legendre multipliers η in order to enforce the gauge conditions. We thus obtain the constraint equations:

$$\nabla_i \Pi^i + j_0 = 0 \ , \quad \nabla_i A^i = 0 \tag{7.20}$$

Since the field A_0 has no dynamics, the non-trivial spatial dependence is only encoded into the momentum Π_i, thus we can treat the A_0 as a Lagrange multiplier which enforces Gauss's law. This, in total, gives us two second-class constraints which eliminate the remaining spurious DOF.

We can now decompose the fields into spherical harmonics according to the discussion of the previous section:

$$\Phi_a = \sum_{J,M} \Phi_a^{JM} \mathcal{Y}_{JM} \tag{7.21}$$

$$A_i = \sum_{J,M,\rho} A_{(\rho)}^{JM} \mathcal{Y}_{JM\rho,i} \tag{7.22}$$

with $M \equiv (m, \tilde{m})$ running from $-J$ to J for the scalar spherical harmonics and for $-Q$ to Q and $-\tilde{Q}$ to $\tilde{Q}$ for the vector, where

$$Q = J + \frac{(1+\rho)\rho}{2} \quad , \quad \tilde{Q} = \tilde{J} - \frac{(1-\rho)\rho}{2} \tag{7.23}$$

Note also that ρ takes ± 1 values because of the Coulomb gauge condition.

In order to directly solve the constrains and obtain an algebraic condition, we decompose the fields in modes and we use the identity

$$Y_{J_1 M_1}(\Omega) Y_{J_2 M_2}(\Omega) = \sum_{J_1 M_1 J_2 M_2} \mathcal{C}^{J_3 M_3}_{J_1 M_1 J_2 M_2} Y_{J_3 M_3}(\Omega) \tag{7.24}$$

Our gauge constraints are now given by their final, very useful for our analysis, form

$$2i\sqrt{J(J+1)}\Pi^{JM}_{(0)} + j_0^{\dagger JM} = 0 \quad , \quad A^{JM}_{(0)} = 0 \tag{7.25}$$

Returning now to the calculation of our total Hamiltonian, if we insert the spherical decomposition of all the fields, we get the decomposed expression

$$\begin{aligned} H = \ \mathrm{tr} \sum_{J,M} \frac{1}{2}(|\Pi^{JM}_{(\rho)}|^2 + \omega^2_{A,J}|A^{JM}_{(\rho)}|^2) \\ + |\Pi^{JM}_a|^2 + \omega_J^2 |\Phi^{JM}_a|^2 + \frac{1}{8J(J+1)}|j_0^{JM}|^2 \\ - 4g \sum_{J_i\ M_i} \sqrt{J_1(J_1+1)} \mathcal{D}^{J_2 M_2}_{J_1 M_1, JM\rho} A^{JM}_\rho [\Phi^{J_1 M_2}_a, \bar{\Phi}^{J_2 M_2}_a] \\ + \frac{g^2}{2} |\sum_{J_i, M_i} \mathcal{C}^{J_2 M_2}_{J_1 M_1, JM} [\Phi^{J_1 M_1}_a, \bar{\Phi}^{J_2 M_2}_a]|^2 \end{aligned} \tag{7.26}$$

with

$$j_0^{JM} = ig \sum_{J_i M_i} \mathcal{C}^{J_2 M_2}_{J_1 M_1, JM} ([\bar{\Phi}^{J_2 M_2}_a, \bar{\Pi}^{J_1 M_1}_a] + [\Phi^{J_2 M_2}_a, \Pi^{J_1 M_1}_a]) \tag{7.27}$$

We compute all the relevant currents corresponding to the symmetries of our action using the canonical energy momentum tensor:

$$T_{\mu\nu} \equiv T^{(\Phi)}_{\mu\nu} + T^{(A)}_{\mu\nu} + \frac{g_{\mu\nu}}{\sqrt{-g}}\mathcal{L} \tag{7.28}$$

where $\mathcal{L}$ is the Lagrangian density and also we have

$$\begin{aligned} T^{(\Phi)}_{\mu\nu} &= (\partial_\mu \Phi_a)^\dagger \partial_\nu \Phi^a + (\partial_\mu \Phi_a)^\dagger \partial_\nu \Phi^a \\ T^{(A)}_{\mu\nu} &= F^\sigma_\mu F_{\nu\sigma} \end{aligned} \tag{7.29}$$

We can now compute the rotation generators S_i from

$$S_i \equiv S_i^{(\Phi)} + S_i^{(A)} = \int_{S^3} d\Omega\, T_i^0 \tag{7.30}$$

So for our particular case, we can calculate the whole rotation generator S_1 is,

$$S_1 = i \sum_{JM} \Delta m \text{ tr } (\Phi_a^{JM}\Pi_a^{JM} - \bar{\Phi}_a^{JM}\bar{\Pi}_a^{JM} + \frac{1}{2}(A_\rho^{JM}\Pi_\rho^{JM} - \bar{A}_\rho^{JM}\bar{\Pi}_\rho^{JM})) \tag{7.31}$$

The Q_a charges associated to the Cartan subalgebra of the global R-symmetry are given by (for $a = 1$):

$$Q_1 = i \sum_{JM} \text{ tr } (\Phi_1^{JM}\Pi_1^{JM} - \bar{\Phi}_1^{JM}\bar{\Pi}_1^{JM}) \tag{7.32}$$

Our goal is to obtain the interactive Hamiltonian H_{int} in the decoupling limit, i.e. the near-BPS limit, as we discuss above:

$$\lambda \to 0 \text{ with } \frac{E - Q_1 - S_1}{\lambda} \text{ finite, N fixed.} \tag{7.33}$$

To achieve this we first demand that the Hamiltonian $H - S_1 - Q_1$ is of order g when $g \to 0$. This limit essentially determines the propagating degrees of freedom we interested in. This yields:

$$\begin{aligned} H - S_1 - Q_1|_{g=0} = \text{tr} \textstyle\sum_{JM} \Big(\tfrac{1}{2} \Big(\Big|\Pi_{(\rho)}^{JM} - i\Delta m \bar{A}_{(\rho)}^{JM}\Big|^2 \\ + \left(\omega_{A,J}^2 - \Delta m^2\right) \Big|A_{(\rho)}^{JM}\Big|^2 \Big) + \left|\Pi_a^{JM} + i\left(\delta_1^a - \Delta m\right)\bar{\Phi}_a^{JM}\right|^2 \\ + \Big(\omega_J^2 - (\delta_1^a - \Delta m)^2\Big) \left|\Phi_a^{JM}\right|^2 \Big) \end{aligned} \tag{7.34}$$

Using now this result we can derive various constraints which will be very useful, because the dynamics of the theory close to the near-BPS bound can now be realised by solving these constrains.

First since $|\Delta m| \leq 2J + 1$: we find that

$$A_{(\rho)}^{JM} = \mathcal{O}(g), \quad \Pi_{(\rho)}^{JM} - i\Delta m \bar{A}_{(\rho)}^{JM} = \mathcal{O}(g) \tag{7.35}$$

Second, for the scalar Φ_1: we find for $J = -m = \tilde{m}$:

$$\begin{gathered} \Pi_1^{J,-J,J} + i\omega_J \bar{\Phi}_1^{J,-J,J} = \mathcal{O}(g) \\ \text{and for all other } m, \tilde{m} \\ \Phi_1^{JM} = \Pi_1^{JM} = \mathcal{O}(g) \end{gathered} \tag{7.36}$$

All the other scalars: are for all possible values of the $m, \tilde{m}$:

$$\Phi_2^{JM} = \Phi_3^{JM} = \Pi_2^{JM} = \Pi_3^{JM} = \mathcal{O}(g) \tag{7.37}$$

All of the above constraints are essentially relations that help eliminate dynamical DOF from our theory by removing our freedom to choose initial conditions. Thus we expect that the corresponding fields should commute with the Hamiltonian i.e. they should be compatible with the Hamiltonian evolution. This requirement gives us the final set of constraints

$$A_{(\rho)}^{JM} = -\frac{j_{(\rho)}^{JM}}{\omega_{A,J}^2 - (\Delta m)^2} = \sum_{J_i, M_i} \frac{4g\sqrt{J_1\left(J_1+1\right)}}{\omega_{A,J}^2 - \Delta m^2} \times \mathcal{D}_{J_1 M_1, JM\rho}^{J_2 M_2} \left[\Phi_1^{J_1 M_2}, \bar{\Phi}_1^{J_2 M_2}\right] \tag{7.38}$$

$$\Phi_{a=2,3}^{JM} = 0 \quad , \quad \Pi_{a=2,3}^{JM} = 0 \tag{7.39}$$

$$\Pi_q^{J,-J,J} + i\omega_J \tilde{\Phi}_1^{J,-J,J} = 0 \tag{7.40}$$

$$\Phi_1^{JM} = 0 \quad , \quad \Pi_1^{JM} = 0 \quad \text{except when } J = -m = \tilde{m} \tag{7.41}$$

We can start to see from 7.40 a decouple between particle/anti-particles which is what makes our theory non-relativistic. More concretely, the equation 7.40 relates the momentum with the complex conjugate of the field, which implies that at the quantum level the bosonic fields will annihilate a particle and the hermitian conjugate will create it. This is the $U(1)$ global symmetry responsible for the conservation of particle number, and it is standard behavior for non-relativistic low momentum limits of QFTs.

Free Hamiltonian:

Before we start getting involved with interactions, we must first analyze the free-part of the Hamiltonian. For convenience, we introduce the notation where $s = J/2$ in such a way to sum only over integer numbers

$$\bar{\Phi}_s \equiv \sqrt{2(1+s)}\Phi_1^{\frac{s}{2},-\frac{s}{2},\frac{s}{2}} \tag{7.42}$$

with the canonical Dirac-brackets satisfying the relation,

$$\left\{\Phi_s, \bar{\Phi}_{s'}\right\} = i\delta_{ss'} \tag{7.43}$$

For more details about the Dirac-bracket see Appendix A on Generalised Hamiltonian Procedure. This leads us to the quadratic form for the effective Hamiltonian H_0, where we note that the only surviving contribution to the kinematics comes from Φ_1 (i.e. Φ_s in the new notation) which is further restricted by the constrain on angular momentum $\tilde{m} = -m = J$.

$$H_0 = \mathrm{tr} \sum_{s \geq 0} (s+1) \left|\Phi_s\right|^2 \tag{7.44}$$

A last discussion before moving to the H_{int}. Consider the

$$L_0 = \sum_{n=0}^{\infty} (n + 1/2) \;\; \mathrm{tr} \; |\Phi_n|^2 \tag{7.45}$$

and

$$L_+ = \bar{L}_- = \sum_{n=0}^{\infty} (n+1) \;\; \mathrm{tr} \; \bar{\Phi}_{n+1} \Phi_n \tag{7.46}$$

Using the Dirac-bracket commutators relation for the Φ_n we get

$$\{L_0, L_\pm\} = \pm L_\pm \quad , \quad \{L_+, L_-\} = -2iL_0 \tag{7.47}$$

which is exactly the algebra of the group $SU(1,1)$. So we clearly see the emergence of the $SU(1,1)$ global symmetry of all our theory.

Interacting Hamiltonian

We can now obtain the desirable interacting Hamiltonian H_{int} in the decoupling near-BPS limit:

$$H_{\mathrm{int}} = \lim_{g^2 N \to 0} \frac{H - S_1 - Q_1}{g^2 N} \tag{7.48}$$

Applying now the constraint 7.38, we find that the contributions of the gauge field constrain are

$$\sum_{J,M} \mathrm{tr} \left(\frac{1}{8J(J+1)} \left| j_0^{JM} \right|^2 - \sum_{\rho = \pm 1} \frac{1}{2\left(\omega_{A,J}^2 - (\Delta m)^2\right)} \left| j_{(\rho)}^{JM} \right|^2 \right) \tag{7.49}$$

From the scalar sector, we read of the terms:[2]

[2] we use the notation $\mathcal{J}$ form the Appendix B for a more clear presentation of the result since we only work with the surviving scalar modes; see constrains, we then convert back to s notation afterward.

$$\sum_{J,M}\operatorname{tr}\left\{\frac{g^2}{2}\mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1,JM}\mathcal{C}^{\mathcal{J}_3}_{\mathcal{J}_4,JM}\left[\Phi_1^{\mathcal{J}_1},\bar{\Phi}_1^{\mathcal{J}_2}\right]\left[\Phi_1^{\mathcal{J}_3},\bar{\Phi}_1^{\mathcal{J}_4}\right]-4g\sqrt{J_1\left(J_1+1\right)}\mathcal{D}^{\mathcal{J}_2}_{\mathcal{J}_1,JM\rho}A^{JM}_{(\rho)}\left[\Phi_1^{\mathcal{J}_1},\bar{\Phi}_1^{\mathcal{J}_2}\right]\right\} \tag{7.50}$$

We can from here, read the expressions for the currents:

$$\bar{j}_0^{JM}=2g(1+J_1+J_2)\mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1,JM}[\Phi_1^{\mathcal{J}_1},\bar{\Phi}_1^{\mathcal{J}_2}] \tag{7.51}$$

$$\bar{j}_{(\rho)}^{JM}=-4g\sqrt{J_1(J_1+1)}\mathcal{D}^{\mathcal{J}_2}_{\mathcal{J}_1,JM\rho}[\Phi_1^{\mathcal{J}_1},\bar{\Phi}_1^{\mathcal{J}_2}] \tag{7.52}$$

Combining everything and returning to the $s=J/2$ notation we get

$$H_{\text{int}}=\frac{1}{4N}\operatorname{tr}\sum_{s_1,s_2\geq 0}\sum_{l\geq 0}V_l^{s_1,s_2}\left[\Phi_{s_1},\Phi_{s_1+l}\right]\left[\bar{\Phi}_{s_2+l},\Phi_{s_2}\right] \tag{7.53}$$

$$\begin{aligned}V_l^{s_1,s_2}\equiv\textstyle\sum_{JM}\Big(&\tfrac{(2+2s_1+l)(2+2s_2+l)}{8J(J+1)}\mathcal{C}^{s_1+l}_{s_1,JM}\mathcal{C}^{s_2+l}_{s_2,JM}\\ -\textstyle\sum_{\rho=\pm1}&\tfrac{2\sqrt{s_1(s_1+2)}\sqrt{s_2(s_2+2)}}{\omega^2_{A,J}-(m-\tilde{m})^2}\mathcal{D}^{s_1+l}_{s_1,JM\rho}\overline{\mathcal{D}}^{s_2+l}_{s_2,JM\rho}\\ &+\tfrac{1}{2}\mathcal{C}^{s_1+l}_{s_1,JM}\mathcal{C}^{s_2+l}_{s_2,JM}\Big)\end{aligned} \tag{7.54}$$

Where we have to use the fact that all the non-trivial contributions come only from the $m=-\tilde{m}$. The most important "trick" to proceed with the calculations is to notice that upon shifting $J\to J-1$ when $\rho=1$, all terms in the sum cancel except for a nontrivial remainder from the lower boundary of summation. See the crossing relation from Appendix B. It is easier to split the calculation into two cases, $J_1=J_2$ and $J_1>J_2$ where we can account for the converse by adding a factor of 2.

For the $J_1=J_2$ case, the $s\to s-1$ trick lead to a vanishing contribution from the corresponding $H^{s_1=s_2}_{\text{int}}$, as a consequence of the Gauss law on the S^3 which implies $q_0=0$ and the fact that $H^{s_1=s_2}_{\text{int}}$ can be calculated to be linear in q_0 in this sector.

So for the $s_1>s_2$ case (and thus for the whole calculation) we calculate $V^{s_1,s_2}_{l>0}$ to be

$$V^{s_1,s_2}_{l>0}=\frac{2}{l} \tag{7.55}$$

Which is a potential term which behaves exactly like the classical EM-theory thus the $l=0$ contributions vanish on all physical states.

Thus the full interacting Hamiltonian is given by

$$H_{\text{int}}=\frac{g_0}{2N}\sum_{s>0}\frac{1}{s}\left|q_s\right|^2 \tag{7.56}$$

where the $SU(N)$ charge density in Fourier space

$$q_s \equiv \sum_{n\geq 0} [\bar{\Phi}_n, \Phi_{n+s}] \tag{7.57}$$

Indeed, the $s = 0$ mode vanished at classical level due to Gauss' law and at quantum level because it it the $SU(N)$ singlet constraint.

The total Hamiltonian

The total Hamiltonian must have the form $H = H_0 + g_0 H_{int}$. With the help of the $SU(N)$ charge density in Fourier space

$$q_s \equiv \sum_{n\geq 0} [\bar{\Phi}_n, \Phi_{n+s}] \tag{7.58}$$

we deduce the result

$$\boxed{H = \mathrm{tr}\left(\sum_{\beta\geq 0} (s+1) \left|\Phi_s\right|^2 + \frac{g_0}{2N} \sum_{s>0} \frac{1}{s} \left|q_s\right|^2\right)} \tag{7.59}$$

This is a beautiful non-relativistic $\mathcal{N} = 4$ SYM field theory near the $SU(1,1)$ BPS limit. As we discuss, it is non-relativistic because 7.40 relates canonical momenta and complex conjugate field.

We are now ready to proceed to the quantization of this theory, which is remarkably non trivial and with some surprises. We will also prove that 7.59 is equivalent to $SU(1,1)$ SMT.

7.3 Quantisation

To start the quantization , we follow the standard procedure of promoting the "Dirac-Poisson"-Bracket [3] to commutators i.e.

$$\{.\ ,.\}_{\text{Dirac}} \to i\,[.\ ,.] \tag{7.60}$$

and the ladder operators,

$$a_s \equiv \Phi_s \quad \text{and} \quad a_s^\dagger \equiv \bar{\Phi}_s \quad \text{where} \quad [(a_r)^i_j, (a_s^\dagger)^k_l] = \delta^i_l \delta^k_j \delta_{rs} \tag{7.61}$$

This leads to,

$$H_{QM} = \text{tr}\left(\sum_{s\geq 0}(s+1)a_s^\dagger a_s + \frac{g_0}{2N}\sum_{s>0}\frac{1}{s}q_s^\dagger q_s\right) \tag{7.62}$$

We can check now that this is the desired Hamiltonian, starting by the one-loop dilatation operator which is originally derived in [35] which reads,

$$\begin{aligned} H_{\text{int}} &= \tfrac{1}{4N}\textstyle\sum_{m=0}^{\infty}\sum_{k,k'=0}^{m}\text{tr}\left(:\left[a_{k'}^\dagger, a_k\right]\left[a_{m-k'}^\dagger, a_{m-k}\right]:\right) \\ &\times\left(\delta_{k=k'}(h(k)+h(m-k)) - \delta_{k\neq k'}\tfrac{1}{|k-k'|}\right) \end{aligned} \tag{7.63}$$

In which the second line is the so-called one-loop dilatation operator in the bosonic $SU(1,1)$-sector. Also the $h(n) = \sum_{m=1}^{n}\frac{1}{m}$ are the so-called harmonic numbers. To prove the connection between SMT/near-BPS limits we see that the 7.63 i.e. the interacting part of our Hamiltonian, has the exact same Dilatation operator D Eq.6.12 as the $SU(1,1)$-SMT as we demonstrated in previous chapters.

Using the normal ordering relation in the equation 7.63 we get

$$\begin{aligned} &\textstyle\sum_{l>0}\frac{1}{l}\,\text{tr}\left(:q_l^\dagger q_l:\right) = \sum_{l>0}\frac{1}{l}\,\text{tr}\left(q_l^\dagger q_l\right) \\ &-2N\textstyle\sum_{n=0}^{\infty}h(n)\,\text{tr}\left(a_n^\dagger a_n\right) + 2\sum_{n=0}^{\infty}h(n)\,\text{tr}\left(a_n^\dagger\right)\text{tr}\left(a_n\right) \end{aligned} \tag{7.64}$$

we easily get back our quantized Hamiltonian H_{QM}. This essentially completes our calculations, and we have proved that there is a way to reach the quantized Hamiltonian using a non-relativistic QM theory.

[3] Called Dirac-brackets in our generalised Hamiltonian Procedure

7.4 Local Formulation

Our goal for this section is to build a QFT description for our near-BPS limit $SU(1,1)$-bosonic. Our main task is to reproduce the interaction Hamiltonian H_{int}, which is given in momentum space, in terms of local field theory containing our complex scalar field. As we are going to prove, our bosonic complex field will share some features with $\beta - \gamma$ ghost fields and thus behave like a ghost field.

The $\mathfrak{su}(1,1)$ algebra of the bosonic part is non-compact and thus we expect the $SU(1,1)$ representations that we have to be infinite-dimensional; for this reason, we can find a *local representation* of the states that we have with respect to their $SU(1,1)$ representations. The $SU(1,1)$ has three generators, $L_0, L_\pm$ with algebra

$$\{L_0, L_\pm\} = \pm i L_\pm \quad , \quad \{L_-, L_+\} = -2iL_0 \tag{7.65}$$

where

$$\begin{aligned} L_0 &= \operatorname{tr} \textstyle\sum_{m\geq 0} \left(m + \frac{1}{2}\right) |\Phi_m|^2 \\ L_+ = (L_-)^* &= \operatorname{tr} \textstyle\sum_{m\geq 0} (m+1) \Phi^\dagger_{m+1} \Phi_m \end{aligned} \tag{7.66}$$

All the generators commute with the interaction part of the Hamiltonian H on the singlet constrain surface $q_0 = 0$. The existence of such a singlet conditions implies that the $SU(N)$ remains gauged. This implies that we can introduce an (appropriate non-dynamical) auxiliary field A_s in order to conveniently reproduce that interactions. We can interpret such a step as the position space version of the mediation given by the non-dynamical gauge field in our sphere reduction procedure.

We can write our Hamiltonian solution using our appropriate chosen auxiliary field A_s as

$$H = \operatorname{tr} \sum_{s\geq 0} \left((s+1)\bar{\Phi}_s \Phi_s + s\bar{A}_s A_s + \sqrt{\frac{g_0}{2N}} \left(A_s \bar{q}_s + \bar{A}_s q_s \right) \right) \tag{7.67}$$

We can see the non-dynamical nature of A_s from its EOM

$$sA_s + \sqrt{\frac{g_0}{2N}} q_s = 0 \tag{7.68}$$

Note that for $s = 0$ we get the $SU(N)$ singlet constraint $q_0 = 0$. We can also directly obtain this solution starting from the action of a $(1+1)$-dimensional field theory on a circle of radii one, parametrized by the spatial coordinate x with periodic identification $x \sim x + 2\pi$.

$$S = \int dtdx \text{ tr } \left(i\Phi^\dagger(\partial_0 + \partial_x)\Phi + iA^\dagger\partial_x A + \tilde{g}(A^\dagger q + Aq^\dagger)\right) \tag{7.69}$$

using $A(t,x) = \sum_{s=0}^{\infty} A_s(t)e^{isx}$,our previous form for $\Phi(t,x)$, the EOM for the auxiliary field A for $s \geq 0$ and then performing an Legendre transformetion of this action, we get the Hamiltonian (7.49) as we desired.

To end the section we comment a bit on the $(1+1)$-dimensional quantum field theory (7.56). As we can see from the action S the kinetic term of the theory is both linear in time and space derivatives. This is quite different from both the standard Schroedinger operator and the Klein-Gordon operator. Instead, our kinetic term corresponds to an ultra-relativistic dispersion relation between energy and momentum $E = P$ but only with the constraint $P > 0$, see Carrollian theories [58]. This makes our theory non-relativistic, as we were anticipating.

As a closing remark, we must comment on the promise that we gave, that we will prove that our bosonic fields behave like $\beta - \gamma$-ghosts [59]. This can be seen from the fact that the scalar field must be complex to avoid our kinetic term to be a total derivative, thus we can introduce real-scalar fields β, γ s.t.

$$\Phi = \beta + i\gamma \tag{7.70}$$

which makes our kinetic term to be

$$\mathcal{L}_0 = -2\beta(\partial_0 + \partial_x)\gamma \tag{7.71}$$

Which proves that our bosonic part of the action can be seen as a $\beta - \gamma$ CFT theory. We will investigate further this intriguing emergence of lower-dimensional locality in future work.

Conclusion & Outlook

In this dissertation, we have shown a new way of developing non-relativistic field theories from near-BPS bounds of $\mathcal{N} = 4$ SYM. Our procedure applies to any BPS bound of $\mathcal{N} = 4$ SYM but we, in particular, worked on a specific near-BPS limit and we got a new non-relativistic theory with a global $U(1)$ symmetry (conservation of particles number) as well as $SU(1,1)$ interactions consisting of dynamical complex chiral scalar field interacting with a boson gauge field without dynamical DOF.

Furthermore, we can study the new non-relativistic models that we obtained to study any coupling (strong or weak) and thus get potential new insights about the holographic dual. For example, in the planar limit, we have non-relativistic string theory duals. In our particular limit, the bosonic $SU(1,1)$ case leads to a $U(1)$-Galilean geometry which is a $\mathbb{R}$ times a cigar-geometry. Exploring this further will probably teach us great insights about some features of Holography [64].

Going away from the planar limit will also be a very important exploration; there we expect the possibility to observe the emergence of dual D-branes or M-brane configurations in the form of Giant Gravitons, [61].

The study of all the other BPS bounds is of course another natural extension of all the work in this field. This was done for most of the sectors in [3], [65] where additional features of these limits were pointed out: it is possible to build all the interacting Hamiltonians starting from fundamental blocks and studying their highest weight representation.

Lastly, we hope that by using a thermodynamical treatment, we will be able to retrieve some greater insights into the precise mechanics of the confinement/deconfinement transition. In particular, we are excited to extend our study to the $SU(1,2|3)$ symmetry (which contains supersymmetric, asymptotical, AdS_5 black holes, see [62]) with temperatures above the Hawking-Page transition; where they should exhibit maximal chaos [63]!

In conclusion, we expect our new novel methods/approach to shed some light on our never-ending search for understanding how to approach and understand strongly coupled gauge theories and finally leave the weak coupling regime.

Appendix

A

Generalized Hamiltonian Procedure

For a more complete approach with a lot more developments for the quantization of gauge systems, we refer to [49], which we follow closely in this discussion.

A.1 Introduction

In Classical Lagrangian mechanics, if the system has holonomic constraints $f(q_1, q_2, ..., q_n, t) = 0$ with $n = 1, ..., N$, then we have to add Lagrange multipliers to the Lagrangian to account for them. The extra terms vanish when the constraints are satisfied, thereby forcing the path of stationary action to be on the constraint surface. In this case, going to the Hamiltonian formalism introduces a constraint on phase space in Hamiltonian mechanics.

Our new generalized Hamiltonian procedure, first proposed by Dirac [60] starts with a Lagrangian and the usual canonical momentums:

$$L = L(q, p) \quad , \quad p_n = \frac{\partial L}{\partial \dot{q}_n} \tag{A.1}$$

Some of those definitions may not be invertible and instead give a constraint in phase space (as above). Constraints derived in this way or imposed from the beginning of the problem are called *primary constraints*.[1]

The M-constraints, labelled ϕ_m, must weakly vanish,[2]

$$\phi_m(x, y) \approx 0 \quad , \quad m = 1, 2, ..., M \tag{A.2}$$

The Euler-Lagrange equations now can only fix $N - M$ functions of the acceleration and give M equations between q and $\dot{q}$. The conditions A.2 are called *primary constrains*. From the A.2 we see that the inverse transformation from the p's to the $\dot{q}$'s is multivalued. From the E-L equations, we get

$$\ddot{q}^{\,n'} \frac{\partial^2 L}{\partial \dot{q}^{n'} \partial \dot{q}^n} = \frac{\partial L}{\partial q^n} - \dot{q}^{n'} \frac{\partial^2 L}{\partial q^{n'} \partial \dot{q}^n} \tag{A.3}$$

we can see that, the accelerations at a given time are uniquely determined by the positions and the velocities at that time iff

$$\det \left[\frac{\partial^2}{\partial \dot{q}^n \partial \dot{q}^{n'}} \right] \neq 0 \tag{A.4}$$

We are interested thought, on the case where the determinant is zero, i.e. we cannot invert $\frac{\partial^2}{\partial \dot{q}^n \partial \dot{q}^{n'}}$. The A.2 now defines a $2N - M'$, $M' < M$ submanifold[3] smoothly embedded in phase space. The inverse image of a point in A.2 forms a manifold of dimensions M', and in order to render the transformation single-valued, we need to introduce extra parameters which indicate the location of $\dot{q}$ on the inverse manifold. These parameters are exactly the parameters that appear like Lagrangian Multipliers for our new Hamiltonian $H* = H + c^m \phi_m \approx H$.

Theorem 1. If a smooth phase space function g vanishes on the surface $\phi_m = 0$, then $g = g^m \phi_m$ for some function g^m

A.2 Canonical Hamiltonian

Introducing, as usual, the canonical Hamiltonian

[1] For example, if not all the canonical momenta (e.g. $N - M$) are not independent functions of $\dot{q}$, then will have M independent relations.

[2] Two functions on phase space, f and g, are *weakly equal* if they are equal when the constraints are satisfied, but not throughout the phase space, denoted $f \approx g \Leftrightarrow f - g = c^m(q, p)\phi_m$. If f and g are equal independently of the constraints being satisfied, they are called *strongly equal*, written $f = g$. It is important to note that, in order to get the right answer, no weak equations may be used before evaluating derivatives or Poisson brackets.

[3] We allowed the possibility that not all M constraints are independent.

$$H = \dot{q}^n p_n - L \tag{A.5}$$

But now, in our more generalized case, the Hamiltonian is well defined only on the submanifold defined by the primary constraints and can be extended arbitrary off that manifold, thus the formalism should be unchanged by the transformation

$$H \to H + c^m(q,p)\phi_m \tag{A.6}$$

These can be proven by:

$$\begin{aligned} \delta H &= \dot{q}^n \delta p_n + \delta \dot{q}^n p_n - \delta \dot{q}^n \frac{\partial L}{\partial \dot{q}^n} - \delta q^n \frac{\partial L}{\partial q^n} \\ &= \dot{q}^n \delta p_n - \delta q^n \frac{\partial L}{\partial q^n} \end{aligned} \tag{A.7}$$

which we can write it as

$$\left(\frac{\partial H}{\partial q^n} + \frac{\partial L}{\partial q^n}\right)\delta q^n + \left(\frac{\partial H}{\partial p_n} - \dot{q}^n\right)\delta p_n = 0$$

so

$$\dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}$$

$$-\left.\frac{\partial L}{\partial q^n}\right|_{\dot{q}} = \left.\frac{\partial H}{\partial q^n}\right|_p + u^m \frac{\partial \phi_m}{\partial q^n}$$

where we have used the theorem below

Theorem 2. if $\lambda_n \delta q^n + \mu^n \delta p_n = 0$ for some arbitrary variations tangent to the constraint surface, then for some u^m we have

$$\lambda_n = u^m \frac{\partial \phi_m}{\partial q^n} \tag{A.8}$$

$$\mu_n = u^m \frac{\partial \phi_m}{\partial p_n} \tag{A.9}$$

where the equalities are on the surface defined by primary constrains A.2

We can define now, the "Legendre Transformation" from the $(q, \dot{q})$-space to the surface $\phi_m = 0$ of (q,p,u)-space by

$$\begin{cases} q^n = q^n \\ p_n = \dfrac{\partial L}{\partial \dot{q}^n}(q,\dot{q}) \\ u^m = u^m \cdot (q,\dot{q}) \end{cases} \tag{A.10}$$

So the price we pay , in order to restored the invertibility of the Legendre transformation is the introduction of extra variables.

Moving on, our new EoM are

$$\begin{aligned} &\dot{q}^n = \tfrac{\partial H}{\partial p_n} + u^m \tfrac{\partial \phi_m}{\partial p_n} \\ &\dot{p}_n = -\tfrac{\partial H}{\partial q^n} - u^m \tfrac{\partial \phi_m}{\partial q^n} \\ &\phi_m(q,p) = 0 \end{aligned} \tag{A.11}$$

These EoM can also be derived by least action principle using the new variation principle for our new action

$$\delta \int_{t_1}^{t_2} (\dot{q}^n p_n - H - u^m \phi_m) = 0 \tag{A.12}$$

where our new variable u^m is been introduced to enforce the primary constraints into the action and ensure the invertibility of the Legendre transformation. As usual, we can derive the EoM by the integral A.12 using the Poisson brackets. For a random function $f = f(q,p)$ we have

$$\dot{f} = \{f, H\}_{\text{P.B.}} + u^m \{f, \phi_m\}_{\text{P.B.}} \tag{A.13}$$

where for two arbitrary function $f = f(q,p), g = g(q,p)$ we have

$$\{f, g\}_{\text{P.B.}} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \tag{A.14}$$

A.3 Secondary constrains

If we impose $f = \phi_m$ for a particular value of m in the EoM we get

$$\dot{\phi}_m = \{\phi, H\}_{\text{P.B.}} + u^{m'} \{\phi_m, \phi_{m'}\}_{\text{P.B.}} = 0 \tag{A.15}$$

if the relation between $p, \dot{q}$ is independent of the primary constrains , this equation is called *secondary constraint*, which are on-shell constrains. If we have K secondary constrains, we are then left with the secondary constrains

$$\phi_k = 0 \ , \quad k = M+1, ..., M+K \tag{A.16}$$

and all our constrains are

$$\phi_j = 0 \ , \quad k = 1, ..., M+K = J \tag{A.17}$$

Also from these splitting of the constrains we are forced to restrict the Lagrange Multipliers

$$u^m = U^m + V^m \tag{A.18}$$

where U^m is a solution of the $\dot{\phi}_j \approx 0$ and V^m is the solution of the homogeneous system

$$V^m[\phi_j, \phi_m] \approx 0 \tag{A.19}$$

The most general V^m is a linear combination of A-linear independent solution V_a^m, $a = 1, ..., A$, thus our complete solution is

$$u^m \approx U^m + v^a V_a^m \tag{A.20}$$

where v^a are arbitrary coefficients from the linear combination of V_a^m.

A.4 The Total Hamiltonian

We can now rewrite B.13 using our previous result

$$\dot{f} \approx \{f, H' + v^a \phi_a\}_{\text{P.B.}} \quad , \quad H' = H + U^m \phi_m \quad , \quad \phi_a = V_a^m \phi_m \tag{A.21}$$

We can now define our *total Hamiltonian* as

$$H_T = H' + v^a \phi_a \tag{A.22}$$

such that we satisfy the very elegant property

$$\dot{f} \approx \{f, H_T\}_{\text{P.B.}} \tag{A.23}$$

So in a way, we manage to find the right Hamiltonian such that the EoM are now in the regular, from classical mechanics, form; avoiding all the annoying correcting terms.

A.5 First-class constrains: generators of the gauge transformations

The presence of v^a in the H_T is an indication that not all p, q are observables i.e. exists a redundancy. This redundancy is exactly what we call the gauge freedom and so we see that the existence of these v^a implies some gauge freedom which can be proven to have the form, for example[4]

$$\delta f = \delta v^a \{f, \phi_a\}_{\text{P.B.}} \tag{A.24}$$

for some arbitrary function f and $\delta v^a = (v^a - \tilde{v}^a)\delta t$. So we can see that the *first class primary constrains generate gauge transformations.*

We the sake of convenience we introduce the notion of the *extended Hamiltonian*; which is derived by distinguishing between first- and second-class constraints. We denote the first-class constrains by the letter γ — and, subsequently by G — and the second-class ones' by χ, also we denote the set of all constraints by $\{\phi_j\}$ as before. Thus we can derive the form for the extended Hamiltonian as

$$H_E = H' + u^a \gamma_a \quad \text{where the index runs over a complete set of fist-class constraints.} \tag{A.25}$$

A.6 Second-class constrains: The Dirac Bracket

Finally now we can talk about the second-class constrains; which are present when the matrix $C_{jj'} = \{\phi_j, \phi_{j'}\}_{\text{P.B}}$ is not zero on the constraint surface.

The theorem below is very useful for deriving the splitting intro first and second class constrains

Theorem 5. If $\det C_{jj'} \approx 0$, then there exists at least one first-class constraint among the ϕ_j's.

The final result, invented by Dirac, is the so called *Dirac bracket* and it is the generalisation of the Poisson brackets for an arbitrary set of second-class constrains:

$$\{f, g\}_{\text{D.B.}} = \{f, g\}_{\text{P.B}} - \{f, \chi_\alpha\}_{\text{P.B}}\, C^{\alpha\beta}\, \{\chi_\beta, g\}_{\text{P.B}} \tag{A.26}$$

[4]This transformation is not the only one that does not change the physical state, more generally we have:

Theorem 3. The $\{\phi_a, \phi_{a'}\}_{\text{P.B.}}$ of any two first-class primary constrains generates a gauge transformation.

Theorem 4. The $\{\phi_a, H'\}_{\text{P.B.}}$ of any two first-class primary constrains ϕ_a with the first class Hamiltonian H' generates a gauge transformation.

where P.B is the notations for the Poisson brackets and $_{\text{D.B}}$ is the notation for the Dirac brackets $C_{jj'} = \{\phi_j, \phi_{j'}\}_{\text{P.B}}$.

It can be proven that the new Dirac brackets follow all the properties it must follow, i.e.: bilinearity, antisymmetry, Jacobi identity, and Leibnitz rule; so we can see that the function space under the D.B. is a *Lie Algebra* and our new/generalized Hamiltonian mechanics can be fruitfully studied from the point of view of Lie algebras!

In general, the original Poisson bracket is discarded after having served its purpose of distinguishing between first-class and second-class constraints. All the equations of our theory must now be formulated in terms of the new Dirac bracket, and the second-class constraints merely become identities expressing some canonical variables in terms of others. In some simple cases, the second-class constraints can be used to eliminate some canonical variables from formalism. However, in more complicated situations, the elimination of some DOF in favor of others may be involved.

B

Properties and symmetries of the Clebsch-Gordan coefficients

B.1 Spherical Harmonics

Here we just present some very useful identities which the reader can easily prove using the act of the covariant derivative on the spherical harmonics,

$$\nabla_i Y^{\pm 1}_{JMi} = 0 \tag{B.1}$$

$$\begin{aligned} \epsilon_{ijk}\nabla_j Y^{\rho}_{JMk} &= -2\rho(J+1)Y^{\rho}_{JMi} \\ \nabla_i Y_{JM} &= -2i\sqrt{J(J+1)}Y^{0}_{JMi} \end{aligned} \tag{B.2}$$

The corresponding eigenvalues are for the Laplacian are

$$\begin{aligned} \nabla^2 Y_{JM} &= -4J(J+1)Y_{JM} \\ \nabla^2 Y^{\pm 1}_{JMi} &= -(4J(J+2)+2)Y^{\pm 1}_{JMi} \\ \nabla^2 Y^{0}_{JMi} &= -(4J(J+1)-2)Y^{0}_{JMi} \\ \nabla^2 Y^{\kappa}_{JM\alpha} &= -\left(2J(2J+3)+\tfrac{3}{4}\right)Y^{\kappa}_{JM\alpha} \end{aligned}$$

Using now the integral of the product of three spherical harmonics, we get the, very important for, Clebsch-Gordan generalised coefficients:

$$\begin{array}{l} \mathcal{C}^{J_1M_1}_{J_2M_2J_3M_3} \equiv \int d\Omega \left(Y_{J_1M_1}\right)^* Y_{J_2M_2}Y_{J_3M_3} \\ \mathcal{C}_{J_1M_1J_2M_2J_3M_3} \equiv \int d\Omega Y_{J_1M_1}Y_{J_2M_2}Y_{J_3M_3} \\ \mathcal{D}^{JM}_{J_1M_1\rho_1J_2M_2\rho_2} \equiv \int d\Omega \left(Y_{JM}\right)^* Y^{\rho_1}_{J_1M_1i}Y^{\rho_2}_{J_2M_2i} \end{array} \tag{B.3}$$

B.2 Definition of Clebsch-Gordan

In this section we give some basic definitions and properties for the CG-coefficients that we will use through out calculations.

$$\mathcal{C}^{JM}_{J_1M_1,J_2M_2} = \sqrt{\frac{(2J_1+1)(2J_2+1)}{2J+1}} \times C^{Jm}_{J_1m_1,J_2m_2}C^{J\tilde{m}}_{J_1\tilde{m}_1,J_2\tilde{m}_2} \tag{B.4}$$

$$\mathcal{D}^{JM}_{J_1M_1,J_2M_2\rho} = (-1)^{-1/2+J+J_1+J_2}(2J_1+1) \tag{B.5}$$

$$\times\sqrt{\frac{(2J_1+1)(2J_2+1)}{2J+1}} \begin{Bmatrix} J_1 & J_1 & 1 \\ J_2-\frac{\rho-1}{2} & bJ_2+\frac{\rho+1}{2} & J \end{Bmatrix} \mathcal{C}^{Jm}_{J_1M_1,J_2-\frac{\rho-1}{2},m2}\mathcal{C}^{J\tilde{m}}_{J_1\tilde{m}_1,J_2+\frac{\rho+1}{2}\tilde{m}_2}$$

$$\begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & 1 \\ J & J & 0 \end{Bmatrix} = \frac{(-1)^{\tilde{Q}_1+Q_2+J+1}}{\sqrt{3(2J+1)}} \times \begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & J \end{Bmatrix} \tag{B.6}$$

where the $9-j$ symbols are also:

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{Bmatrix}$$

$$= [(2c+1)(2f+1)(2g+1)(2h+1)]^{-\frac{1}{2}}(2j+1)^{-1} \sum_{\alpha\beta\gamma\delta\epsilon\varphi\eta\mu\nu} C^{c\gamma}_{a\alpha b\beta}C^{f\varphi}_{d\delta e\epsilon}C^{j\nu}_{c\gamma}f_{\varphi}C^{g\eta}_{a\alpha d\delta}C^{h\mu}_{b\beta e\epsilon}C^{j\nu}_{g\eta h\mu} \tag{B.7}$$

where we also have defined

$$Q \equiv J + \frac{\rho(\rho+1)}{2} \quad , \quad \tilde{Q} \equiv J + \frac{\rho(\rho-1)}{2} \tag{B.8}$$

B.3 Crossing relations

In order to make our calculations simpler, we introduce the notation $\mathcal{J} = (J, -J, J)$. For our particular calculation of the Hamiltonian, we are interested in the expression (for $\rho = \pm 1$):

$$\mathcal{D}^{\mathcal{J}_1}_{\mathcal{J}_2;Jm\tilde{m},\rho=1} = -\,i(-1)^{J-J_1+J_2}\sqrt{\frac{(2J_1+1)\,(J+\Delta J+1)(J-\Delta J+1)}{J_2(J+1)\,(J_2+1)\,(2J_2+1)}} \times \frac{(2J_1)\,!\,(2J_2+1)\,!}{2\,(J+1+J_1+J_2)\,!\,(J_1+J_2-J-1)\,!} \tag{B.9}$$

and

$$\mathcal{D}^{\mathcal{J}_1}_{\mathcal{J}_2;Jm\tilde{m},\rho=-1} = -\mathcal{D}^{\mathcal{J}_1}_{\mathcal{J}_2;Jm\tilde{m},\rho=1} \tag{B.10}$$

From the Hamiltonian that we wish to calculate we are motivated to define some expressions which will greatly reduce the complexity of our calculations

$$\mathcal{A}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;Jm\tilde{m}} = \left(1+\frac{(J_1+J_2+1)\,(J_3+J_4+1)}{J(J+1)}\right)\mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1;Jm\tilde{m}}\mathcal{C}^{\mathcal{J}_3}_{\mathcal{J}_4;Jm\tilde{m}} \tag{B.11}$$

$$\mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;Jm\tilde{m}\rho} = \frac{16}{\omega^2_{A,J}-(\Delta m)^2}\sqrt{J_1\,(J_1+1)\,J_4\,(J_4+1)}\mathcal{D}^{\mathcal{J}_2}_{\mathcal{J}_1;Jm\tilde{m}\rho}\overline{\mathcal{D}}^{\mathcal{J}_3}_{\mathcal{J}_4;Jm\tilde{m}\rho} \tag{B.12}$$

where we can see that this two expression are related as below ($J \geq 1$):

$$\mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;Jm\tilde{m}\rho=-1} + \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;J-1m\tilde{m}\rho=1} = \mathcal{A}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;Jm\tilde{m}} \tag{B.13}$$

where we calculate in particular that

$$\begin{aligned}\mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;Jm\tilde{m},\rho=1} &= \frac{(2+J+J_1+J_2)\,(2+J+J_3+J_4)}{(J+1)(2J+3)}\mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1,J+1m\tilde{m}}\mathcal{C}^{\mathcal{J}_3}_{\mathcal{J}_4,J+1m\tilde{m}}\\ \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;Jm\tilde{m},\rho=-1} &= \frac{(J_1+J_2-J)\,(J_3+J_4-J)}{(J+1)(2J+1)}\mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1,Jm\tilde{m}}\mathcal{C}^{\mathcal{J}_3}_{\mathcal{J}_4,Jm\tilde{m}}\end{aligned} \tag{B.14}$$

Our goal is to calculate the expression (from the H_{int}):

$$\sum_{JM}\left(\left(1+\frac{(J_1+J_2+1)\,(J_3+J_4+1)}{J(J+1)}\right)\mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1;Jm\tilde{m}}\mathcal{C}^{\mathcal{J}_3}_{\mathcal{J}_4;Jm\tilde{m}}\right. \tag{B.15}$$

$$\left.-\sum_{\rho=\pm1}\frac{16}{\omega^2_{A,J}-(\Delta m)^2}\sqrt{J_1\,(J_1+1)\,J_4\,(J_4+1)}\mathcal{D}^{\mathcal{J}_2}_{\mathcal{J}_1;Jm\tilde{m}\rho}\overline{\mathcal{D}}^{\mathcal{J}_3}_{\mathcal{J}_4;Jm\tilde{m}\rho}\right) \tag{B.16}$$

which is equal to (we use $\Delta J = J_2 - J_1 = J_3 - J_4$)

$$\sum_{J\geq J_{\min}(\rho)}\mathcal{A}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;J,-\Delta J,\Delta J} - \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;-\Delta J,\Delta J,\Delta J,\rho=1} - \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;-\Delta J,\Delta J,\Delta J,\rho=-1} \tag{B.17}$$

Combing our results we get that $J_{\min} = |\Delta J|$ and by shifting the summation with the trick $J \to J-1$ we cancel all terms in the sum with $J > |\Delta J|$. Thus the equation B.17 becomes

$$\sum_{J\geq|\Delta J|} \mathcal{A}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;J,-\Delta J,\Delta J} - \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;J,-\Delta J,\Delta J,\rho=1} - \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;J,-\Delta J,\Delta J,\rho=-1} \tag{B.18}$$

which can be calculated to be using eq. B.13 to be:

$$\sum_{J\geq|\Delta J|} \mathcal{A}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;J,-\Delta J,\Delta J} - \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;J,-\Delta J,\Delta J,\rho=1} - \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;J,J,-\Delta J,\rho=-1} = \mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;-|\Delta J|-1,-\Delta J,\Delta J,\rho=1} \tag{B.19}$$

Which is a result that we can calculate explicitly:

$$\mathcal{B}^{\mathcal{J}_2,\mathcal{J}_3}_{\mathcal{J}_1,\mathcal{J}_4;|\Delta J|-1,-\Delta J,\Delta J,\rho=1} = \frac{(1+|\Delta J|+J_1+J_2)\,(1+|\Delta J|+J_3+J_4)}{|\Delta J|(2|\Delta J|+1)} \mathcal{C}^{\mathcal{J}_2}_{J_1,|\Delta J|,-\Delta J,\Delta J} \mathcal{C}^{\mathcal{J}_3}_{\mathcal{J}_4,|\Delta J|,-\Delta J,\Delta J} \tag{B.20}$$

Which can be used to calculate our final result.

C

Spinors in $D = 4, 6, 10$-dimensions

Some very useful formulas for when dealing with spinors in various dimensions.

C.1 Four dimensions spinors

In four dimensions, there are only two types of spinor indices belonging to the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ factors of $\mathfrak{so}(4)$, usually denoted by small greek letters, and differentiated by a dot e.g. $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$.

There are only two invariant objects, ϵ and σ:

$$\sigma_{\{m}\sigma_{n\}} = \eta_{mn} \tag{C.1}$$

Some identities of ϵ:

$$\epsilon\epsilon = -1 \tag{C.2}$$

$$\epsilon^T = -\epsilon \tag{C.3}$$

$$[\epsilon, \sigma] = 0 \tag{C.4}$$

The completeness relation which is very useful:

$$\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} - \delta^{\delta}_{\alpha}\delta^{\gamma}_{\beta} = 2\delta^{\gamma}_{[\alpha}\delta^{\delta}_{\beta]} \tag{C.5}$$

And the Fierz identities:

$$\sigma^{\dot{\alpha}\beta}_{\mu}\sigma^{\mu}_{\dot{\gamma}\delta} = 2\delta^{\dot{\alpha}}_{\dot{\gamma}}\delta^{\beta}_{\delta}, \quad \sigma^{\dot{\alpha}\beta}_{\mu}\sigma^{\mu,\dot{\gamma}\delta} = 2\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}, \quad \sigma_{\mu,\dot{\alpha}\beta}\sigma^{\mu}_{\dot{\gamma}\delta} = 2\varepsilon_{\dot{\alpha}\dot{\gamma}}\varepsilon_{\beta\delta} \tag{C.6}$$

C.2 Six dimensions spinors

In six dimensions, because of the symmetry groups, we only have one type of spinor index, which runs from one to four. There are only two totally antisymmetric tensors ϵ and ϵ and only two sigma symbols σ and σ. Te sigma's are antisymmetric

We raise/lower indices by using the ϵ symbols:

$$\sigma^{m,ab} = \frac{1}{2}\varepsilon^{abcd}\sigma^{m}_{cd}, \quad \sigma_{m,ab} = \frac{1}{2}\varepsilon_{abcd}\sigma^{cd}_{m}$$

They satisfy the Clifford algebra

$$\sigma_{\{m}\sigma_{n\}} = \eta_{mn}$$

Fierz identities for the $\sigma's$

$$\sigma^{ab}_{m}\sigma^{m}_{cd} = 2\delta^{a}_{d}\delta^{b}_{c} - 2\delta^{a}_{c}\delta^{b}_{d}, \quad \sigma^{ab}_{m}\sigma^{m,cd} = -2\varepsilon^{abcd}, \quad \sigma_{m,ab}\sigma^{m}_{cd} = -2\varepsilon_{abcd}$$

C.3 Ten dimensions spinors

In ten-dimensions, we denote spinor indices by $A, B, \ldots = 1, \ldots, 16$. There are two sigma symbols Σ^{M}_{AB} and Σ^{AB}_{M} and we can suppress spinor indices. The sigma symbols are symmetric

$$\Sigma^{\top}_{M} = \Sigma_{M}$$

and satisfy

$$\Sigma_{\{M}\Sigma_{N\}} = \eta_{MN}$$

For the construction of supersymmetric gauge theory, there is one particularly useful identity

$$\Sigma_{M,AB}\Sigma^{M}_{CD} + \Sigma_{M,AC}\Sigma^{M}_{DB} + \Sigma_{M,AD}\Sigma^{M}_{BC} = 0$$

In order to obtain our desired theory:$\mathcal{N} = 4$ SYM from the ten-dimensional supersymmetric gauge theory, we have to reduce the starting ten-dimensional spacetime to a four spacetime and six internal dimensions space.

We will assume that a spinor Ψ^A in ten dimensions decomposes into $\Psi_{\alpha a} + \Psi^a_{\dot{\alpha}}$ in $4+6$ dimensions. Then the sigma symbols in ten dimensions split likewise:

$$\begin{aligned} \Sigma_\mu^{AB} &= \sigma_{\mu,\alpha\dot{\beta}}\delta_a^b + \sigma_{\mu,\dot{\alpha}\beta}\delta_a^b \\ \Sigma_{\mu,AB} &= \sigma_\mu^{\alpha\dot{\beta}}\delta_b^a + \sigma_\mu^{\dot{\alpha}\beta}\delta_a^b \\ \Sigma_m^{AB} &= -\sigma_{m,ab}\varepsilon_{\alpha\beta} - \sigma_m^{ab}\varepsilon_{\dot{\alpha}\dot{\beta}} \\ \Sigma_{m,AB} &= \sigma_m^{ab}\varepsilon^{\alpha\beta} + \sigma_{m,ab}\varepsilon^{\dot{\alpha}\dot{\beta}} \end{aligned} \tag{C.7}$$

D
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