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Master's Thesis

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Integrable Matrix Product State Overlaps from Twisted Yangian Representations

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Abstract

This thesis aims to examine overlaps between Matrix Product States and Bethe states that are relevant to defect versions of $\mathcal{N} = 4$ Super Yang-Mills. The relation between conformal operators in the $so(6)$ and $su(2)$ sectors with Bethe states of the corresponding spin chains in the planar limit and at one-loop order is explained through Feynman diagram calculations and the relevance of spin chain overlaps to the dCFT is established. The Heisenberg model is introduced and solved using the algebraic Bethe ansatz. The Yangian of $gl(N)$ is defined, its connection to the algebraic Bethe ansatz is noted and it is used to derive the Bethe equations for a spin chain in an arbitrary $gl(N)$ representation. The connection between the boundary Yang-Baxter equation and integrable matrix product states is explained and the relation between solutions to that relation and twisted Yangian representations is established. For the (SU(3),SO(3)) and (SO(6),SO(5)) symmetric pairs, twisted Yangian representations are used to reduce ratios of overlaps between the matrix product states and Bethe states to transfer matrix eigenvalues. The transfer matrix eigenvalues are computed.

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Introduction and Outline

Holography is a property which is generally expected of theories of quantum gravity. The Holographic principle states that the information theory of a gravitational theory defined in a volume (called the bulk) can be encoded on the boundary of that volume. In string theory, holography is realized in the form of gauge/gravity dualities, where the dynamics of a string theory in the bulk can be reduced to those of a gauge theory on the boundary.
A concrete example of such a duality is the AdS_5/CFT_4 correspondence, first conjectured in [16]. The string theory on one side of the duality is a type IIB superstring theory on an $AdS_5 \times S_5$ background. Its holographic dual is a $\mathcal{N} = 4$ Super Yang-Mills QFT, with an $SU(N)$ gauge group, defined on the boundary of the anti-de Sitter space. Though this conjecture is believed to hold in general, it is usually studied in the planar limit of the CFT, where $g_{YM} \to 0$ and $N \to \infty$, such that the parameter $\lambda = g_{YM}^2 N$ is constant. The correspondence has the form of a strong/weak duality i.e. if λ is chosen to be large (strong coupling), the dual theory exhibits weak string coupling and low curvature. This is an example of the utility of the correspondence, as it allows for perturbative calculations to be carried out on the AdS-side, that would not be possible on the strongly-coupled CFT.
One of the several remarkable properties of $\mathcal{N} = 4$ SYM is its planar integrablity, namely that, in the planar limit, the full spectrum of anomalous dimensions and the corresponding good conformal operators can be determined. This was first found for the scalar sector of the theory at 1-loop order by mapping the corrections to the dilatation operator to the Hamiltonian of integrable SO(6) spin chain [18]. The extension of this result to the entire theory and at all loop orders similarly involves mapping the dilatation operator to a (super)spin chain, the integrability of which implies the integrability of $\mathcal{N} = 4$ SYM [3],[4]. The most simple spin chain, namely the su(2) spin chain or Heisenberg model, has been known to be exactly solvable since 1931, through the what is now called the coordinate Bethe ansatz method [5] Other techniques that were later developed to solve the Heisenberg model, such as the algebraic Bethe ansatz, have been successfully generalized to solve more complicated spin-chain models.
Some interesting variations of the original AdS/CFT correspondence rise by introducing certain D-brane setups on the string theory side. This breaks part of the symmetry of the original theory and its holographic dual has the form of a defect conformal field theory (dCFT), where integrable structures can also appear. In particular, in some defect variants of $\mathcal{N} = 4$ SYM, the tree-level vacuum expectation value of some conformal operators, which can be non trivial due to the partial breaking of the symmetry, has been mapped to the overlap between spin-chain eigenstates and certain integrable matrix product states (MPS)[15] . In some cases these overlaps have been calculated by utilizing the relation between the MPSs and boundary integrability. The aim of this thesis is to present and explain these calculations.

To that end, the first chapter introduces $\mathcal{N} = 4$ SYM with an emphasis on its conformal symmetry and the constraints it imposes on two-point scalar functions at tree level. Then, some 1-loop order calculations are performed in the planar limit, leading to the relation between the dilatation operator of the $SO(6)$ and $SU(2)$ sectors with the respective spin chains. Furthermore, we will see an example of how tree-level one-point functions in the dCFT can be calculated through overlaps between MPSs and spin chains, which motivates their calculation. In the second chapter, we will introduce the Heisenberg model and explain its original solution by Bethe. However, the main focus of that chapter is the algebraic Bethe ansatz (ABA) approach of solving the system. In particular, we will see how the fundamental commutation relations (FCR) of the Lax operators guarantee the commutativity of the transfer matrix, which renders the system integrable. In chapter 3 we will introduce the Yangian algebra and investigate the connection between its representations and the fundamental objects of the ABA approach. Then, as an example of the utility of the Yangian in integrability, we use it to derive the Bethe equations and transfer matrix eigenvalues for a spin chain in an arbitrary gl(N) representation. In chapter 4 we will see how integrable MPSs can be generated from boundary integrability and, in particular, from solutions of the boundary Yang-Baxter (BYB) relation. We will then see how the BYB equation is related to representations of twisted Yangian algebras and how the latter can be used to extract relations between MPSs. The last two chapters are devoted to calculating overlaps between specific MPSs and Bethe states of their corresponding spin chains. This is achieved by initially using twisted Yangian representation theory to extract the exact relation between the MPSs and some simpler states. Then, the computation of the overlaps reduces to some transfer matrix eigenvalues, which we will also see how to obtain.

Contents

1 $\mathcal{N} = 4$ Super Yang-Mills

1.1 The Lagrangian and the Gauge group

The theory consists of six scalar fields ϕ_i, four 4-dimensional Majorana fermions ψ and the 4-dimensional gauge field A_μ, subject to the action

$$S_{SYM} = \frac{2}{g^2} \int d^4x Tr \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi_i D^\mu \phi_i + \frac{i}{2} \bar{\psi} \Gamma^\mu D_\mu \psi + \frac{1}{2} \bar{\psi} \Gamma^i [\phi_i, \psi] + \frac{1}{4} [\phi_i, \phi_j][\phi_i, \phi_j] \right], \tag{1.1}$$

where Γ_i are ten-dimensional gamma matrices. The covariant derivative is defined as

$$D_\mu = \partial_\mu - i[A_\mu, \cdot] \tag{1.2}$$

and the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1.3}$$

The gauge group we consider is $U(N)$. The scalar and fermion fields transform covariantly under gauge group transformations, which means that they transform in the adjoint representation of $U(N)$. For an infinitesimal transformation parametrized by ε, this implies

$$\chi \to \chi + [\varepsilon, \chi] \tag{1.4}$$

for χ being either a scalar or fermion. The gauge connection instead transforms as

$$A_\mu \to A_\mu + \partial_\mu \varepsilon + [\varepsilon, A_\mu] \tag{1.5}$$

Using this, it can be shown that $F_{\mu\nu}$ and $D_\mu \chi$ transform covariantly. Gauge invariant operators are constructed as traces of products of covariant fields. Taking, for example, two scalar fields ϕ_1,ϕ_2, it is straightforward to show that $Tr[\phi_1\phi_2]$ is invariant under the infinitesimal transformation. We have

$$Tr[\phi_1\phi_2] \to Tr[\phi_1\phi_2] + Tr[\phi_1[\varepsilon, \phi_2] + \phi_2[\varepsilon, \phi_1]] + \mathcal{O}(\varepsilon^2) \tag{1.6}$$

and using the cyclicity of the trace we can see that

$$Tr[\phi_1[\varepsilon, \phi_2]] = -Tr[\phi_2[\varepsilon, \phi_1]], \tag{1.7}$$

and the $\mathcal{O}(\varepsilon)$ term vanishes. Additional operators can be constructed by taking products of such traces. However, this thesis focuses on the t'Hooft limit, where $g \to 0$ and $N \to \infty$ with $\lambda = g^2 N$ constant and in that case it suffices to look at single trace operators. Also, in that limit, $N \to \infty$ suppresses the contribution of non-planar Feynman diagrams to correlation functions, so that only the planar ones need to be considered.

1.2 Conformal Symmetry

Apart from the gauge symmetry, $\mathcal{N} = 4$ SYM theory enjoys extended spacetime symmetry in the form of the projective superconformal group PSU(2,2| 4). We will focus on its $SO(4, 2)$ subgroup, which generates conformal transformations.

The conformal algebra and CFTs

The conformal algebra is an extension of the Poincare algebra. In addition to the generators of Lorentz transformations $J_{\mu\nu}$ and translations P_μ, it includes the dilatation D and the special conformal transformation (SCT) K_μ and reads

$$[J_{\mu\nu}, P_\rho] = i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right) \tag{1.8}$$
$$[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}\right) \tag{1.9}$$
$$[J_{\mu\nu}, K_\rho] = i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right) \tag{1.10}$$
$$[J_{\mu\nu}, D] = 0 \tag{1.11}$$
$$[D, P_\mu] = iP_\mu \tag{1.12}$$
$$[D, K_\mu] = -iK_\mu \tag{1.13}$$
$$[D, K_\mu] = -2i\left(\eta_{\mu\nu}D - J_{\mu\nu}\right) \tag{1.14}$$
$$[K_\mu, K_\nu] = 0 \tag{1.15}$$

The general infinitesimal coordinate transformation generated by this algebra is

$$\delta x^\mu = x'^\mu - x^\mu = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + b_\nu(g^{\mu\nu}x^2 - 2x^\mu x^\nu), \tag{1.16}$$

where $a^\mu + \omega^\mu_\nu x^\nu$ is the familiar Poincare transformation, while λx^μ and $b_\nu(g^{\mu\nu}x^2 - 2x^\mu x^\nu)$ are generated by D and K_μ respectively. The finite transformation of the dilatation is a simple rescaling of the coordinates

$$x^\mu \to \lambda x^\mu \tag{1.17}$$

while K_μ generates the rather complicated finite transformation

$$x \to \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2x^2} \tag{1.18}$$

where $b \cdot x = g_{\mu\nu}b^\mu x^\nu$

The operators of a Conformal Field Theory are representations of this algebra. In $d = 4$ dimensions, the conformal algebra includes $so(1,1) \oplus so(3,1)$ as a subalgebra. This allows us to characterise each representation as (Δ, j_L, j_R), where Δ is the Conformal dimension and j_L, j_R are the $su(2)$ weights in the $su(2)_L \oplus su(2)_R$ decomposition of the Lorentz algebra. In particular, we postulate that for an operator with fixed conformal dimension the algebra acts at the origin as

$$[J_{\mu\nu}, \mathcal{O}(0)] = -J_{\mu\nu}\mathcal{O}(0), \tag{1.19}$$

where $J_{\mu\nu}$ is now some representation of the Lorentz group and

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0). \tag{1.20}$$

The action at an arbitrary point is determined using the differential operator representation of the algebra

$$P_\mu = -i\partial_\mu \tag{1.21}$$
$$J_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{1.22}$$
$$D = -ix^\mu\partial_\mu \tag{1.23}$$
$$K^\mu = -i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu) \tag{1.24}$$

and using P_μ to translate from the origin to x. As an example, we can calculate the infinitesimal action of the dilatation operator. We have

$$[D, \mathcal{O}(x)] = [D, e^{-ix^\mu P_\mu} \mathcal{O}(0) e^{ix^\mu P_\mu}] \tag{1.25}$$
$$= e^{-ix^\mu P_\mu} [e^{ix^\mu P_\mu} D e^{-ix^\mu P_\mu}, \mathcal{O}] e^{ix^\mu P_\mu} \tag{1.26}$$
$$= e^{-ix^\mu P_\mu} [D - ix^\mu P_\mu, \mathcal{O}] e^{ix^\mu P_\mu} \tag{1.27}$$
$$= -i(\Delta + x^\mu \partial_\mu)\mathcal{O}(x). \tag{1.28}$$

In the penultimate equality, we use the Baker-Hausdorff formula.

In what follows we will focus on scalar operators, i.e. representations of the form $(\Delta, 0, 0)$, that are anihilated by the SCT generator

$$[K_\mu, \mathcal{O}_\Delta(0)] = 0 \tag{1.29}$$

Such operators play a central role in conformal field theory, since all operators of a CFT can be generated from them. To see that, note that the conformal dimension of $\mathcal{O}(0)$ is determined by the action of D on it. Using the Jacobi identity, we have

$$[D, [P_\mu, \mathcal{O}(0)]] = -[\mathcal{O}(0), [D, P_\mu]] - [P_\mu, [\mathcal{O}(0), D]] \tag{1.30}$$
$$= i[\mathcal{O}(0), P_\mu] - i\Delta[P_\mu, \mathcal{O}(0)] \tag{1.31}$$
$$= -i(\Delta + 1)[P_\mu, \mathcal{O}(0)] \tag{1.32}$$

thus P_μ raises the conformal dimension by 1. Acting on conformal primaries with P_μ generates "towers" of descendant operators, which include all the operators relevant to the theory. In that sense, a CFT is completely determined by its primary operators.

Two-point correlations in a CFT

The extended symmetry of a CFT imposes heavy restrictions on its correlation functions. In particular, one-point correlations are trivial and any other correlation can be calculated using the conformal data (Δ, c_{ijk}), where c_{ijk} are some structure constants that determine the three-point correlations.

Of particular importance to us are the two-point functions, which are completely determined by the conformal dimension of the fields involved. One can show this for two scalar primary operators $\mathcal{O}_I(x)$, $\mathcal{O}_J(x)$, using the fact that under a finite conformal transformation $x \to x'$, their two point correlation transforms as

$$\langle \mathcal{O}_I(x_I)\mathcal{O}_J(x_J)\rangle \longrightarrow \Big|\frac{\partial x'}{\partial x}\Big|_{x=x_I}^{-\Delta_I/d} \Big|\frac{\partial x'}{\partial x}\Big|_{x=x_J}^{-\Delta_J/d} \langle \mathcal{O}_I(x_I)\mathcal{O}_J(x_J)\rangle, \tag{1.33}$$

where d is the dimensionality of spacetime, in our case 4. The first constraint comes from spacetime homogeneity and isotropy, which restrict the two-point correlation to

$$\langle \mathcal{O}_I(x_I)\mathcal{O}_J(x_J)\rangle = f_{IJ}(|x_I - x_J|) \tag{1.34}$$

Next, note that under a rescaling of the coordinates $x^\mu \to \lambda x^\mu$, an operator with fixed conformal dimension transforms as

$$\mathcal{O}(x) = \lambda^{-\Delta}\mathcal{O}(\lambda x) \tag{1.35}$$

which means that our correlation transforms as

$$\langle \mathcal{O}_I(x_I)\mathcal{O}_J(y_J)\rangle \longrightarrow \lambda^{-\Delta_I-\Delta_J}\langle \mathcal{O}_I(\lambda x_I)\mathcal{O}_J(\lambda x_J)\rangle. \tag{1.36}$$

Then, dilatation invariance requires that

$$f_{IJ}(|x_I - x_J|) = \lambda^{-\Delta_I-\Delta_J} f_{IJ}(\lambda|x_I - x_J|), \tag{1.37}$$

which is satisfied by

$$\langle \mathcal{O}_I(x_I)\mathcal{O}_J(x_J)\rangle = \frac{M_{IJ}}{|x_I - x_J|^{\Delta_I+\Delta_J}} \tag{1.38}$$

The final constraint comes from invariance to special conformal transformations, for which we need to calculate the Jacobian $|\frac{\partial x'}{\partial x}|$. A nice way to do that is to start by noting that we can decompose the SCT into a translation and two inversion transformations. In particular, the SCT is equivalent to an inversion

$$x^\mu \longrightarrow \frac{x^\mu}{x^2}. \tag{1.39}$$

followed by a translation

$$\frac{x^\mu}{x^2} \longrightarrow \frac{x^\mu}{x^2} + b^\mu \tag{1.40}$$

followed by another inversion

$$\frac{x^\mu}{x^2} + b^\mu \longrightarrow \frac{x + b^\mu + b^\mu x^2}{x^2}\frac{x^2}{1 + 2b\cdot x + b^2x^2} = \frac{x + b^\mu + b^\mu X^2}{1 + 2b\cdot x + b^2x^2} \tag{1.41}$$

The Jacobian of the SCT is thus

$$\mathcal{J}_{SCT} = \mathcal{J}_{inv}(\frac{x^\mu}{x^2} + b^\mu)\mathcal{J}_{tr}(\frac{x^\mu}{x^2})\mathcal{J}_{inv}(x^\mu) \tag{1.42}$$

The Jacobian of the translation is trivial, and for the inversion we have

$$\frac{\partial}{\partial x^\nu}(\frac{x^\mu}{x^2}) = \frac{x^2\delta^\mu_\nu - 2x^\mu x_\nu}{x^4}, \tag{1.43}$$

from which it follows that

$$\mathcal{J}_{SCT} = \gamma^{-4}, \tag{1.44}$$

where we have introduced $\gamma = 1 + 2b\cdot x + b^2x^2$ to simplify notation. It can also be shown that under an SCT

$$(x_I - x_J)^2 \rightarrow \frac{(x_I - x_J)^2}{\gamma_I\gamma_J}. \tag{1.45}$$

Combining (1.33), (1.44) and (1.45) we get that, under a SCT, eq. (1.38) transforms as

$$\gamma_x^{\Delta_I}\gamma_y^{\Delta_J}\langle \mathcal{O}_I(x_I)\mathcal{O}_J(x_J)\rangle = (\gamma_x\gamma_y)^{(\Delta_I+\Delta_J)/2}\frac{M_{IJ}}{|x_I - x_J|^{\Delta_I+\Delta_J}} \tag{1.46}$$

or

$$\langle \mathcal{O}_I(x_I)\mathcal{O}_J(x_J)\rangle = \frac{(\gamma_I\gamma_J)^{(\Delta_I+\Delta_J)/2}}{\gamma_x^{\Delta_I}\gamma_y^{\Delta_J}}\frac{M_{IJ}}{|x-y|^{\Delta_I+\Delta_J}} \tag{1.47}$$

Conformal invariace then requires

$$\frac{(\gamma_I\gamma_J)^{(\Delta_I+\Delta_J)/2}}{\gamma_I^{\Delta_I}\gamma_J^{\Delta_J}} = 1, \tag{1.48}$$

which is identically true only for $\Delta_I = \Delta_J$.

Summing up, we have concluded that the two-point correlation between two scalar operators is restricted, by conformal invariance, to be

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle = \frac{M_{IJ}}{|x-y|^{\Delta_I+\Delta_J}} \tag{1.49}$$

with $M_{IJ} = 0$ for $I \neq J$.

1.3 Scalar two-point functions in $\mathcal{N} = 4$ SYM

In what follows we restrict ourselves to operators made up of the scalar fields of $\mathcal{N} = 4$ SYM. These operators make up the so-called SO(6) sector of the theory. This sector is closed at first loop order, which means that, at 1st order in perturbation theory, correlations between operators within and outside the sector vanish. We will also consider its subsector which consists of the two complex fields

$$X = \phi_1 + i\phi_4 \tag{1.50}$$

$$Y = \phi_2 + i\phi_5 \tag{1.51}$$

and their conjugates. It is called the SU(2) sector and it is closed at all loop orders.

Tree level correlation

From the action (1.1), one see that the free scalar propagator is

$$\langle [\phi_i]_{ab}(x)[\phi_j]_{b'a'}(y)\rangle = \delta_{ij}\delta_{aa'}\delta_{bb'}\frac{g^2}{8\pi^2}\frac{1}{|x-y|^2}, \tag{1.52}$$

where we have made the U(N) indices, also called the colour indices, explicit. The conformal dimension of ϕ_i is 1, so tracing over the colour indices results in an expression consistent with (1.49).

Now, consider an operator in the SO(6) sector,

$$\mathcal{O}_I(x) = Tr[\phi_{i_1}(x)\ldots\phi_{i_L}(x)], \tag{1.53}$$

where $I=\{i_1,\dots,i_L\}$ is an ordered set of indices which defines the operator. This operator obviously has conformal dimension L, at tree level. The tree level two-point function of two such operators

$$\langle \mathcal{O}_I(x)\bar{\mathcal{O}}_J(y)\rangle = \left(\frac{g^2}{8\pi^2}\right)^L \langle [\phi_{i_1}]_{ab}(x)[\phi_{i_2}]_{bc}(x)\cdots[\phi_{i_L}]_{fa}(x)[\phi_{j_L}]_{a'f'}(y)\cdots[\phi_{j_2}]_{c'b'}(y)[\phi_{j_1}]_{b'a'}(y)\rangle \tag{1.54}$$

is the sum of all possible contractions of two fields, according to Wick's theorem. Terms where all contractions have the form

$$\langle \phi_{i_k}(x)\phi_{j_{k+m}}(y)\rangle, \tag{1.55}$$

for some integer $1\leq m\leq L$, correspond to planar Feynman diagrams. For example, m=0 gives the term

$$\langle \phi_{i_1}(x)\phi_{j_1}(y)\rangle\langle \phi_{i_2}(x)\phi_{j_2}(y)\rangle\cdots\langle \phi_{i_L}(x)\phi_{j_L}(y)\rangle \tag{1.56}$$

and m=2

$$\langle \phi_{i_1}(x)\phi_{j_3}(y)\rangle\langle \phi_{i_2}(x)\phi_{j_5}(y)\rangle\cdots\langle \phi_{i_L}(x)\phi_{j_2}(y)\rangle \tag{1.57}$$

In the t'Hoof limit only planar diagrams contribute to the correlation. To see that, we can compare (1.56) to the non-planar contraction

$$\langle \phi_{i_1}(x)\phi_{i_2}(y)\rangle\langle \phi_{i_2}(x)\phi_{i_1}(y)\rangle\langle \phi_{i_3}(x)\phi_{j_3}(y)\rangle\cdots\langle \phi_{j_L}(x)\phi_{j_L}(y)\rangle \tag{1.58}$$

Using the propagator (1.52) in the planar case, the sum over repeated colour indices gives

$$\delta_{aa'}\delta_{aa'}\delta_{bb'}\delta_{bb'}\cdots\delta_{ff'}\delta_{ff'} = N^L. \tag{1.59}$$

and in the non planar case

$$\delta_{aa'}(\delta_{ab'}\delta_{bc'}\delta_{ba'}\delta_{cb'})\delta_{cc'}\delta_{dd'}\delta_{dd'}\cdots\delta_{ff'} = N^{L-2}, \tag{1.60}$$

where the parenthesis is added to emphasize what has changed compared to the planar case. In general, a diagram of genus gn [1] comes with a factor of N^{2-2gn}. Thus in the t'Hooft limit, where $N\to\infty$, non planar diagrams are suppressed.

The last step is to note that (1.55) includes a factor $\delta_{i_k,j_{k+m}}$. Thus, a planar contraction is non zero only if $i_k=j_{k+m}$ for all $k=1,\dots,L$ which implies that the set I needs to be a cyclic permutation of J. We can now conclude that the tree level two-point function is

$$\langle \mathcal{O}_I(x)\bar{\mathcal{O}}_J(y)\rangle = c_I N^L\left(\frac{g^2}{8\pi^2}\right)^L \frac{\delta_{IJ}}{|x-y|^{2L}}, \tag{1.61}$$

where c_I is the number of cyclic permutations that leave I invariant. This result is consistent with (1.49).

[1] The genus of the diagram is the genus of the surface on which the diagram can be drawn with no propagators intersecting. The cylinder, on which (1.56) can be drawn, has genus 0 and (1.58) requires a surface of genus 1.

Renormalization and operator mixing

Do deal with divergences in loop diagrams we employ dimensional regularization

$$S = \frac{2}{g^2}\int \mathcal{L} \to S_\varepsilon = \frac{2}{(g\mu^\varepsilon)^2}\int d^{4-2\varepsilon}x\mathcal{L}, \tag{1.62}$$

where μ has mass dimension $[\mu] = 1$. The scalar propagator is then modified to

$$\langle [\phi_i]_{ab}(x)[\phi_j]_{cd}(y)\rangle = \delta_{ij}\delta_{ad}\delta_{bc}K_\varepsilon(x,y), \tag{1.63}$$

where

$$K_\varepsilon(x,y) = \frac{\Gamma(1-\varepsilon)}{8\pi^{2-\varepsilon}(x-y)^{2(1-\varepsilon)}} \tag{1.64}$$

Calculating the two-point function at first order in perturbation theory yields

$$\langle \mathcal{O}_I^{bare}(x)\bar{\mathcal{O}}_J^{bare}(y)\rangle_\varepsilon = \sqrt{c_I c_J}N^\Delta(K_\varepsilon(x,y))^\Delta(\tilde{\delta}_{IJ} + \frac{\lambda}{16\pi^2}M_{IJ}(\varepsilon)[\mu|x-y|]^{2\varepsilon}) + \mathcal{O}(g^4) \tag{1.65}$$

where $\tilde{\delta}_{IJ} = 1$ if J is a cyclic permutation of I and is zero otherwise. The $M_{IJ}(\varepsilon)$ matrix turns out to have a pole at $\varepsilon = 0$, so we write it as

$$M_{IJ}(\varepsilon) = -\frac{1}{\varepsilon}D_{IJ} + M_{IJ}^{fin} + \mathcal{O}(\epsilon) \tag{1.66}$$

Usually, removing the divergence at $\varepsilon \to 0$ involves adding counterterms to the Lagrangian, such that the divergences of the original Lagrangian and the counterterms cancel each other. In $\mathcal{N} = 4$ SYM this turns out to not be necessary. Instead, the two-point correlations are finite after defining appropriate operators

$$\mathcal{O}_I^{ren} = \frac{\mathcal{Z}_{IJ}}{\sqrt{c_I}}(g,\varepsilon)\mathcal{O}_J^{bare}. \tag{1.67}$$

Here, $\mathcal{O}_J^{bare}$ are single-trace operators and $\mathcal{Z}_{IJ}$ is a matrix that rescales and mixes them. At first loop order, we choose

$$\mathcal{Z}_{IJ}(g,\varepsilon) = \delta_{IJ} + \frac{\lambda^2}{2\epsilon}D_{IJ} - \frac{\lambda^2}{2}M_{IJ}^{fin}, \tag{1.68}$$

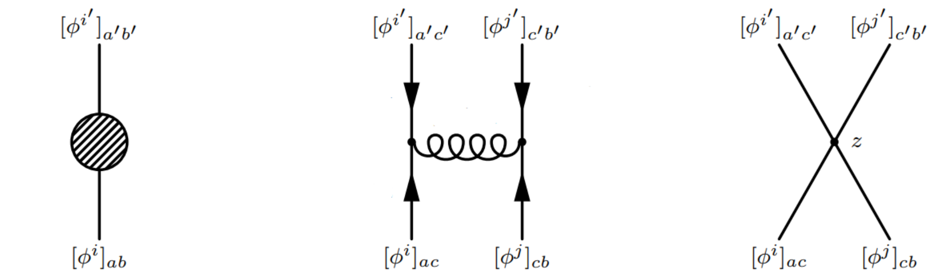
in which case the renormalized correlation function is

$$\langle \mathcal{O}_I^{ren}(x)\bar{\mathcal{O}}_J^{ren}(y)\rangle = \lim_{\varepsilon\to 0} N^\Delta K_\varepsilon(x,y)^\Delta(\delta_{IJ} - \lambda^2 D_{IJ}\log(\mu^2|x-y|^2) + \mathcal{O}(g^4) \tag{1.69}$$

We see that, at the quantum level, not all operators have the property (1.49). Our next goal is to determine the operators that do, which is achieved by diagonalizing D_{IJ}.

Determining the dilatation matrix

The non-vanishing Feynman diagrams at 1-loop order are shown in the figure below. Let us now briefly see how the term that corresponds to diagram (c) is computed. The computation of the other two follows similar steps, but is more complicated.



(a) Self energy **(b)** Gluon exchange **(c)** Scalar vertex

The term that we need to calculate is

$$A = \frac{ig^2}{2} \int d^{4-2\varepsilon} z \langle [\phi_i \phi_j]_{ab}(x) (Tr[(\phi_k \phi_l \phi_k \phi_l)](z) - Tr[(\phi_k \phi_k \phi_l \phi_l)](z) [\phi_{j'} \phi_{i'}]_{b'a'}(y)) \rangle \tag{1.70}$$

For each of the two terms in the correlation there are four ways to contract the incoming and outgoing fields with the ones coming from the vertex. For the first term, all contractions create a $\delta_{ij'}\delta_{ji'}$, so it reverses the incoming and outgoing flavors. In the second term, there are two contractions that lead to $\delta_{ii'}\delta_{jj'}$ and the other two give $\delta_{ij}\delta_{i'j'}$. We thus get

$$\langle [\phi_i \phi_j]_{ab}(x) Tr([\phi_k, \phi_l][\phi_k, \phi_l])(z) [\phi_{j'} \phi_{i'}]_{b'a'}(y)) \rangle = \tag{1.71}$$

$$N^2 \delta_{aa'} \delta_{bb'} (4\delta_{ij'}\delta_{ji'} - 2\delta_{ii'}\delta_{jj'} - 2\delta_{ij}\delta_{i'j'}) K_\varepsilon^2(x,z) K_\varepsilon^2(z,y) \tag{1.72}$$

The integral over z is a common loop integral and yields

$$A = \frac{g\mu^\varepsilon \Gamma(1-\varepsilon)^4 \delta_{aa'}\delta_{bb'}}{2^{11}\pi^{6-3\varepsilon}} \frac{G(2-2\varepsilon, 2-2\varepsilon)}{|x-y|^{2(2-\varepsilon)}} (2\delta_{ij'}\delta_{ji'} - \delta_{ii'}\delta_{jj'} - \delta_{ij}\delta_{i'j'}) \tag{1.73}$$

where

$$G(x,y) = \frac{\Gamma(x+y-\varepsilon-2)\Gamma(2-\varepsilon-x)\Gamma(2-\varepsilon-y)}{\Gamma(x)\Gamma(y)\Gamma(4-x-y-2\epsilon)} \tag{1.74}$$

Expanding everything in powers of ε leads to

$$A = \lambda^2 N K_\varepsilon^2(x,y) \delta_{aa'} \delta_{vv'} \left(\frac{1}{\varepsilon} + 2 + \gamma_E + log(\pi |x-y|^2) \mathcal{O}(\varepsilon) \right) \tag{1.75}$$

The remaining two diagrams give rise to similar terms. It is useful to note that the self energy diagram (a) does not change the flavour of the incoming field, so the corresponding term contains a $\delta_{ii'}$. Similarly, a gluon exchange between two scalars cannot alter their flavour, so in diagram (b) the indices are paired as $\delta_{ii'}\delta_{jj'}$.

The 1-loop order contribution to the two-point function of two composite operators is obtained by performing every possible contraction such that one contraction is made using one of these three diagrams and the rest are tree-level. For a diagram to be planar, the gluon exchange and scalar vertex can only contract adjacent fields. In order to keep track of the flavor indices, we introduce two operators acting on the Kronecker deltas. The permutation operator

$$\mathbb{P}_{n,n+1} \ldots \delta_{j_n, j_m} \delta_{i_{n+1}, j_{m+1}} = \ldots \delta_{i_{n+1}, j_m} \delta_{i_n, j_{m+1}} \ldots, \tag{1.76}$$

which exchanges two adjacent fields on one of the operators, and the trace operator

$$\mathbb{K}_{n,n+1}\ldots\delta_{j_n,j_m}\delta_{i_{n+1},j_{m+1}} = \ldots\delta_{i_n,j_{n+1}}\delta_{i_m,j_{m+1}}\ldots \tag{1.77}$$

which contracts two neighboring operators. For example, in the term that corresponds to contracting the n and n+1 sites using the scalar vertex, the index structure can be written as

$$2\delta_{i_n,j_{n+1}}\delta_{j_n,i_{n+1}} - \delta_{i_n i_{n+1}}\delta_{i_n,j_{n+1}} - \delta_{i_n j_n}\delta_{i_{n+1}j_{n+1e}} = (2\mathbb{P}_{n,n+1} - \mathbb{K}_{n,n+1} - 1)\delta_{i_n j_n}\delta_{i_{n+1}j_{n+1}} \tag{1.78}$$

Including all the corrections, the dilatation matrix in (1.69) turns out to be

$$D_{IJ} = \sum_{n=1}^{L}(2 - 2\mathbb{P}_{n,n+1} + \mathbb{K}_{n,n+1})(\delta_{i_1,j_1}\ldots\delta_{i_L,j_L} + \text{cyclic permutations}), \tag{1.79}$$

where we identify $L+1=1$. This result is interesting because this matrix has the same form as the Hamiltonian of a quantum mechanical model, the SO(6) spin chain. We will now see some of the consequences of a similar correspondence from in the SU(2) sector of the field theory to the SU(2) spin chain.

The SU(2) sector and the SU(2) spin chain

If we further restrict ourselves to the SU(2) sector, the process is similar. In this case, we only have the two fields X and Y, which we choose to denote as

$$X = \phi_\uparrow \tag{1.80}$$
$$Y = \phi_\downarrow \tag{1.81}$$

Single-trace composite operators are then of the form

$$\mathcal{O}(x) = Tr[\phi_{s_1}\ldots\phi_{s_L}] \tag{1.82}$$

with $s_k \in \{\uparrow,\downarrow\}$. The difference in the dilatation matrix is that the contractions that would require introducing the $\mathbb{K}_{n,n+1}$ operators vanish and we obtain the simpler matrix

$$D^{su(2)}_{SS'} = 2\sum_{n=1}^{L}(1 - \mathbb{P}_{n,n+1})(\delta_{s_1,s'_1}\ldots\delta_{s_L,s'_L} + \text{cyclic permutations}) \tag{1.83}$$

This matrix has the same form as the Hamiltonian of a quantum mechanical spin chain called the Heisenberg model. Thus, the problem of diagonalizing the dilatation matrix reduces to determining the eigenstates of the Heisenberg hamiltonian. In particular, as we will see shortly, a basis of the Hilbert space of that model is $\{|s_1, s_2, \ldots, s_L\rangle\}$, hence any eigenstate of the hamiltonian can be written in the form

$$|\Psi\rangle = \sum_{\{\tau\}} C(\tau)\,|s_{\tau_1}\ldots s_{\tau_L}\rangle\,, \tag{1.84}$$

where the sum is over all L-fold combinations of $\uparrow$ and $\downarrow$. A good conformal operator can be constructed as

$$\mathcal{O} = \frac{A}{\sqrt{\langle\Psi|\Psi\rangle}} Tr\left[\prod_{l=1}^{L}(\langle\uparrow_l|\otimes X + \langle\downarrow_l|\otimes Y)\,|\Psi\rangle\right], \tag{1.85}$$

where the prefactor ensures proper normalization. The right hand side of this equation is simply a linear combination of single trace operators, with the coefficients determined by the state $|\Psi\rangle$.

1.4 The defect version

Another theory is created by positioning a defect of codimension one at $x_3 = 0$. On either side of the defect, one still has $\mathcal{N} = 4$ SYM with gauge group $U(N-k)$ for $x_3 < 0$ and $U(N)$ for $x_3 > 0$, but the latter is partially broken by some fields having a non-zero vacuum expectation value (vev). The presence of the defect breaks part of the conformal symmetry of the original theory. In particular, spacetime homogeniety along the x_3 direction is broken and thus the theory is no longer invariant under translations generated by P_3. Also, the defect breaks part of the isotropy and the $SO(3,1)$ rotations generated by $M_{\mu,3}$ are not symmetries. The remaining Poincare symmetry is then generated by $M_{\hat{\mu},\hat{\nu}}$ and $P_{\hat{\mu}}$ for $\mu, \hat{\nu} \hat{=} 0, 1, 2$. Similarly, K_3-symmetry is broken and the remaining symmetry is with respect to the three-dimensional conformal group $SO(3,2)$.

This breaking of conformal symmetry partially eases the constraints on the correlation functions. In particular, the one point correlation of a good conformal operator is no longer trivial, but still restricted by the remaining symmetry to be

$$\langle \mathcal{O}(x) \rangle \propto \frac{1}{x_3^{\Delta}} \tag{1.86}$$

At tree-level, this correlation is determined by the vev of the fields involved. We consider the case where ϕ_i^{cl} for $i = 1, 2, 3$ are the only non-vanishing vevs. The classical solutions are subject to the equations of motion which, in this case, are

$$\nabla^2 \phi_i^{cl} = [\phi_j^{cl}, [\phi_j^{cl}, \phi_i^{cl}]]. \tag{1.87}$$

and it is straightforward to check that

$$\begin{aligned} \phi_i^{cl} &= \frac{1}{x_3} \begin{pmatrix} (t_i)_{k\times k} & 0_{k\times(N-k)} \\ 0_{(N-k)\times k} & 0_{(N-k)\times(N-k)} \end{pmatrix}, \text{ for } i = 1,2,3 \\ \phi_i^{cl} &= 0\,, \text{ for } i = 4,5,6 \end{aligned} \tag{1.88}$$

solve this equation if the matrices t_i form a k-dimensional representation of su_2, that is if

$$[t_i, t_j] = i\epsilon_{ijk} t_k. \tag{1.89}$$

Going to the t'Hooft limit, we can restrict ourselves to linear combinations of single-trace operators

$$\mathcal{O} = \Psi^{i_1 \dots i_L} Tr(\phi_{i_1} \dots \phi_{i_L}), \tag{1.90}$$

for which at tree-level

$$\langle \mathcal{O} \rangle^{tree} = (-1)^L \Psi^{i_1 \dots i_L} \frac{Tr(t_{i_1} \dots t_{i_L})}{x_3^L} \tag{1.91}$$

The results of the previous section can be used to rewrite the last expression by introducing the so-called matrix product state (MPS)

$$|MPS_k\rangle = Tr\left[\prod_{n=1}^{L} t_1 \otimes |1\rangle_n + t_2 \otimes |2\rangle_n + t_3 \otimes |3\rangle\right] = \sum_{i_1,\dots,i_L=1}^{3} Tr(t_{i_1} \dots t_{i_L})\,|i_1 \dots i_L\rangle\,, \tag{1.92}$$

where $|i\rangle$, $i = 1, 2, 3$ are states in an SO(6) representation. Then, we can use the correspondence between the SO(6) sector and the SO(6) spin chain to write

$$\langle \mathcal{O} \rangle^{tree} \propto \frac{\langle MPS|\Psi\rangle}{x_3^L}, \tag{1.93}$$

where $|\Psi\rangle$ is an eigenstate of the SO(6) Hamiltonian. The prefactor we have omitted is related to normalization. This form of the tree-level one-point function motivates the calculation of the $\langle MPS|\Psi\rangle$ overlaps, which is investigated in the last chapters of this thesis for two different solutions of (1.89).

2 The Heisenberg model

2.1 Description of the model and its original solution

The term *spin chain* refers to a family of quantum mechanical models, where the Hilbert space is of the form

$$\mathcal{H} = \bigotimes_{n=1}^{L} h_n \tag{2.1}$$

and each local space h_n carries a representation of some algebra. In one of the most simple cases, that algebra is su_2 and the local spaces carry its spin-$\frac{1}{2}$ representation. That is, $h_n = \mathbb{C}^2$ and we have spin-$\frac{1}{2}$ operators S_i^n that act trivially on every individual h_n, except for the n^{th} one:

$$S_n^i = I \otimes I \otimes \cdots \otimes \frac{\sigma_i}{2} \otimes \cdots \otimes I \tag{2.2}$$

where σ_i are the pauli matrices. We can also define the i^{th} component of the total spin as

$$S^i = \sum_{n=1}^{L} S_n^i \tag{2.3}$$

The Hamiltonian of the system couples the sites in nearest-neighbor pairs

$$H = \sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1} - \frac{1}{4} \tag{2.4}$$

with the periodic boundary condition $S_n = S_{n+L} \Leftrightarrow S_{L+1} = S_L$. Let us now introduce the *permutation* matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.5}$$

which acts in $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$P(a \otimes b) = b \otimes a. \tag{2.6}$$

Using the permutation, the Hamiltonian can be written as

$$H = \frac{1}{2} \sum_n P_{n,n+1} - \frac{L}{2} \tag{2.7}$$

where $P_{n,n+1}$ acts in $v \in \mathcal{H}$ in the obvious way

$$P_{n,n+1}(v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} \otimes \cdots \otimes v_L) = v_1 \otimes \cdots \otimes v_{n+1} \otimes v_n \otimes \cdots \otimes v_L \tag{2.8}$$

It is easy to see that $[H, S^i] = 0$, which means that this Hamiltonian exhibits su_2 symmetry. This implies that that the total number of $\uparrow$ or $\downarrow$ spins in a state is conserved. Thus, if we call the number of $\downarrow$'s M, and express the Hamiltonian in the following eigenbasis of S^3

$$\{\left|\uparrow\uparrow\cdots\uparrow\right\rangle, \left|\downarrow\uparrow\cdots\uparrow\right\rangle, \left|\uparrow\downarrow\uparrow\cdots\uparrow\right\rangle, \cdots, \left|\downarrow\downarrow\cdots\downarrow\right\rangle\} \tag{2.9}$$

it will be block-diagonal with each $\binom{L}{M}\times\binom{L}{M}$ block corresponding to states of the same M and energy E_M. It turns out that to each block we can associate a set of M complex numbers $\{u_i\}$, such that:

$$E_M = 2\sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}} \tag{2.10}$$

iff $\{u_i\}$ is a set of *finite* and *distinct* solutions to the set of equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j\neq k} \frac{u_k - u_j + i}{u_k - u_j - i} \tag{2.11}$$

We will now briefly see the original method for obtaining this system of equations, energies and the corresponding eigenstates.

The coordinate Bethe ansatz

All eigenstates of the Hamiltonian can be constructed starting from the (pseudo)vacuum state

$$|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle \tag{2.12}$$

by the action of lowering operators S^-. An arbitrary in the Hilbert space $\mathcal{H}$ with M spin-down excitations can be written as

$$|\Psi\rangle = \sum \Psi(l_1,\ldots,l_M) S^-_{l_1}\cdots S^-_{l_M}|0\rangle\,, \tag{2.13}$$

where the sum is over $1 \leq l_1 < l_2 \leq \cdots \leq l_m \leq L$. The assumption made by Bethe is that in an eigenstate, the coefficients are of the form

$$\Psi(l_1,\ldots,l_M) = \sum_{\{\tau\}} A(\tau) e^{ip_{\tau_1}l_1} + \cdots + e^{ip_{\tau_M}l_M}, \tag{2.14}$$

where the sum is over permutations of $\{\tau_i\}$. The $\{p_i\}$ are sometimes interpreted as the lattice momenta of the spin $\downarrow$ pseudoparticles (called magnons) that propagate around the chain. The function $A(\tau)$ is determined by requiring that (2.13) is an eigenstate of the Hamiltonian, which leads to the constraints

$$e^{ip_kL} = \prod_{j\neq k} S(p_k,p_j)\,,\; S(p_k,p_j) = -\frac{e^{ip_k+ip_j} - 2e^{ip_k} + 1}{e^{ip_k+ip_j} - 2e^{ip_j} + 1} \tag{2.15}$$

for the momenta and

$$A(\tau) = sign(\tau)\prod_{j<k}\left(e^{ip_k+ip_j} - 2e^{ip_k} + 1\right), \tag{2.16}$$

where $sign(\tau)$ is the signature of the permutation. The eigenvalues are then

$$E_M = \sum_{k=1}^{M} 8sin^2(\frac{p_k}{2}). \tag{2.17}$$

The equations (2.15) and the energy eigenvalues are equivalent to (2.8) and (2.9), after we identify

$$u_k = \frac{1}{2}cot(\frac{p_k}{2}) \tag{2.18}$$

Note that due to the prescence of $sign(\tau)$ in $A(\tau)$, the wavefunctions $\Psi(\{l_i\})$ are antisymmetric with respect to the exchange of two momenta, therefore they vanish if two momenta are the same. This is why we only look for distinct roots of the Bethe equations.[2]

2.2 The algebraic Bethe ansatz

2.2.1 Preliminary definitions and the Hamiltonian

Lax operators, the Monodromy and the Transfer Matrix

In this approach we need to first consider and auxiliary space, denoted as V, which is also isomorphic to $\mathbb{C}^2$. Then, we can define *Lax operators* acting in $h_n \otimes V$, which for the $sl(2)$ spin chain in the fundamental representation are

$$L_{n,a}(u) = uI_n \otimes I_a + i\sum_{\alpha} S_n^{\alpha} \otimes \sigma^{\alpha}, \tag{2.19}$$

where I_n, I_a are the identity operators in the respective spaces, σ^{α} are the pauli matries acting in V,and u is a complex parameter called the *rapidity*. Noting that we can rewrite the permutation as,

$$P = \frac{1}{2}(I \otimes I + \sum_{\alpha} \sigma^{\alpha} \otimes \sigma^{\alpha}) \tag{2.20}$$

it is not hard to rewrite this Lax operator as

$$L_{n,a}(u) = (u - \frac{i}{2})I_{n,a} + P_{n,a} \tag{2.21}$$

It is useful to think of it as a matrix in the auxiliary space with element acting in h_n:

$$L_{n,a}(u) = \begin{pmatrix} u + iS_n^3 & iS_n^- \\ iS_n^+ & u + iS_n^3 \end{pmatrix} \tag{2.22}$$

The Lax operator defines a transport between two sites, in the sense that it acts on a vector $\psi_n = \begin{pmatrix} \psi_{\substack{1\\n}} \\ \psi_{\substack{2\\n}} \end{pmatrix}$ with entries in $\mathcal{H}$ as

$$L_n \psi_n = \psi_{n+1} \tag{2.23}$$

In words, acting on a vector in the h_n space with the appropriate Lax operator generates a vector in the neighboring subspace h_{n+1}. It is then quite obvious that an ordered product of Lax operators of the form

$$L_{n_2,a}(u)L_{n_2-1,a}(u)\ldots L_{n_1,a}(u) \tag{2.24}$$

[2] According to [12], there is no concrete proof that the roots must be distinct, although it is generally accepted that they should be. The CBA wavefunction is proportional to the ABA state and the latter does not vanish for coinciding rapidities. The CBA wavefunction can then be made non-zero after some sort of renormalisation.

transports from site n_1 to $n_2 + 1$. The *monodromy* is defined as the transport from site 1 to $L + 1$, i.e.

$$T_a(u) = L_{L,a}(u) \dots L_{1,a}(u). \tag{2.25}$$

By definition, it is a polynomial in u of order L.

$$T_a(u) = u^L + iu^{L-1} \sum_{\alpha} S^{\alpha} \otimes \sigma^{\alpha} + \cdots \tag{2.26}$$

As with the Lax operators, we can also treat it as a matrix in the auxiliary space V

$$T_a = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{2.27}$$

with components acting in the entire Hilbert space $\mathcal{H}$.
We can now define the *Transfer Matrix* as the trace of the monodromy over the auxiliary space i.e.

$$F(u) = tr_V(T(u)) = A(u) + D(u) \tag{2.28}$$

When we expand it in powes of u, using (2.25), the $O(u^{L-1})$ term vanishes (due to the Pauli matrices being traceless) and the $O(u^L)$ term is trivial:

$$F(u) = 2u^L + \sum_{l=0}^{L-2} Q_l u^l, \tag{2.29}$$

where Q_l are some u-independent operators acting in $\mathcal{H}$.

The Fundamental Commutation Relation (FCR)

At this point we need to introduce two auxiliary spaces V_1, V_2, so that the product of two Lax operators can act in $h_n \otimes V_1 \otimes V_2$. It turns out that $L_{n,a_1}(u)L_{n,a_2}(v)$ and $L_{n,a_2}(v)L_{n,a_1}(u)$ are similar operators with the intertwiner acting trivially in h_n. In particular, we have

$$R_{a_1,a_2}(u-v)L_{n,a_1}(u)L_{n,a_2}(v) = L_{n,a_2}(v)L_{n,a_1}(u)R_{a_1,a_2}(u-v), \tag{2.30}$$

where

$$R_{a_1,a_2}(u) = uI_{a_1,a_2} + iP_{a_1,a_2}. \tag{2.31}$$

It's worth noticing that, in this model, this so-called *R-Matrix* (acting in $V_1 \otimes V_2$) is identical to the Lax operators (acting in $h_n \otimes V_{1,2}$), after a rapidity shift. We can show that the FCR also hold for any transport operator of the form in (2.24). It suffices to show that it holds for the two-site transport $L_{n+1,a}(u)L_{n,a}$. Dropping the rapidity arguments to compactify notation, we have

$$\begin{aligned} &R_{a_1,a_2}L_{n+1,a_1}L_{n,a_1}L_{n+1,a_2}L_{n,a_2} = \\ &R_{a_1,a_2}L_{n+1,a_1}L_{n+1,a_2}L_{n,a_1}L_{n,a_2} = \\ &L_{n+1,a_2}L_{n+1,a_1}L_{n,a_2}L_{n,a_1}R_{a_1,a_2} = \\ &L_{n+1,a_2}L_{n,a_2}L_{n+1,a_1}L_{n,a_1}R_{a_1,a_2}, \end{aligned} \tag{2.32}$$

where we used the commutativity of Lax operators that act in different pairs of spaces and the original FCR. This implies that the FCR holds, in particular, for the monodromy:

$$R_{a_1,a_2}(u-v)T_{a_1}(u)T_{a_2}(v) = T_{n,a_2}(v)T_{a_1}(u)R_{a_1,a_2}(u-v) \tag{2.33}$$

By tracing over the auxiliary spaces in the last relation, which "removes" the R-matrices, we can obtain

$$[F(u), F(v)] = 0. \tag{2.34}$$

Recovering the Hamiltonian

An immediate and important consequence of this commutation relation is that the Q_l in (2.29) constitute a family of $L-1$ commuting operators. This set can be completed with a component of the total spin , say S^3, so that it consists of L commuting operators. We can now prove that the Hamiltonian (2.7) can be constructed out of these operators.

We start by noting that at the special point $u = \frac{i}{2}$, the monodromy reduces to a string of permutation operators

$$T_a(i/2) = i^L P_{L,a}P_{L-1,a}\dots P_{1,a}. \tag{2.35}$$

Using the properties of the permutation

$$P_{n,a_1}P_{n,a_2} = P_{a_1,a_2}P_{n,a_1} = P_{n,a_2}P_{a_2,a_1} \tag{2.36}$$

$$P_{a_1,a_2} = P_{a_2,a_1} \tag{2.37}$$

this can be written as

$$T_a(i/2) = i^L P_{1,2}P_{2,3}\dots P_{L,a} = i^L U P_{L,a}, \tag{2.38}$$

where the matrix U is defined in the obvious way. Now, using

$$\frac{d}{du}L_{n,a}(u) = I_{n,a} \tag{2.39}$$

we can see that

$$\frac{d}{du}T_a(u)\Big|_{u=\frac{i}{2}} = i^{L-1}\sum_n P_{L,a}\dots \hat{P}_{n,a}\dots P_{1,a} \tag{2.40}$$

where the hat denotes that the operator is missing from the sum. Using (2.37)-(2.38), this can be rewritten as

$$\frac{d}{du}T_a(u)|_{u=\frac{i}{2}} = i^{L-1}\sum_n P_{1,2}\dots P_{n-1,n+1}\dots P_{L-1,L}P_{L,a} \tag{2.41}$$

Using the property of the permutation $Tr_a(P_{a,n}) = I_n$, we can obtain

$$\frac{d}{du}F(u)|_{u=\frac{i}{2}} = i^{L-1}\sum_n P_{1,2}\dots P_{n-1,n+1}\dots P_{L-1,L} \tag{2.42}$$

Finally, noting that

$$F(i/2) = i^L U \tag{2.43}$$

we can multiply by U^{-1} to remove most permutation operators and obtain

$$\frac{d}{du}F(u)F(u)^{-1}|_{u=\frac{i}{2}} = \frac{d}{du}lnF(u)|_{u=\frac{i}{2}} = -i\sum_n P_{n,n+1}. \tag{2.44}$$

We can thus rewrite the Hamiltonian (2.7) as

$$H = \frac{i}{2}\frac{d}{du}lnF(u)|_{u=\frac{i}{2}} - \frac{L}{2}. \tag{2.45}$$

This means that the problem of diagonalizing the Hamiltonian reduces to the simultaneous diagonalization of the commuting operators in the expansion of the transfer matrix, which renders the system integrable.

2.2.2 Derivation of the Bethe Ansatz Equations and some comments

The derivation

From the FCR (2.33), we can extract relations between the components of the transfer matrix. The ones relevant to the derivation are the following.

$$[B(u), B(v)] = 0 \tag{2.46}$$
$$A(u)B(v) = f(u-v)B(v)A(u) + g(u-v)B(u)A(v) \tag{2.47}$$
$$D(u)B(v) = h(u-v)B(v)D(u) + k(u-v)B(u)D(v) \tag{2.48}$$

where

$$f(u) = \frac{u-i}{u}\,,\; g(u) = \frac{i}{u} \tag{2.49}$$
$$h(u) = \frac{u+i}{u}\,,\; k(u) = \frac{i}{u} \tag{2.50}$$

We now define the *pseudovacuum* $\Omega \in \mathcal{H}$ by requiring that it is annihilated by $C(u)$

$$C(u)\Omega = 0 \tag{2.51}$$

To find such a state, we can notice that a Lax operator acts on the state $\omega_n = e_+ \otimes v \in h_n \otimes V$ as

$$L_n(u)\omega_n = \begin{pmatrix} u+\frac{i}{2} & * \\ 0 & u-\frac{i}{w} \end{pmatrix}\omega_n \tag{2.52}$$

Thus, for $\Omega = (\otimes_n e_+) \otimes V$, we have

$$T_a(u)\Omega = \begin{pmatrix} \alpha^L(u) & * \\ 0 & \delta^L(u) \end{pmatrix}\Omega\,, \tag{2.53}$$

where

$$\alpha(u) = u + \frac{i}{2} \tag{2.54}$$
$$\delta(u) = u - \frac{i}{2}. \tag{2.55}$$

Clearly, we have

$$C(u) = 0 \tag{2.56}$$

and

$$A(u)\Omega = \alpha^L(u)\Omega \tag{2.57}$$
$$D(u)\Omega = \delta^L(u)\Omega \tag{2.58}$$

Thus, Ω is an eigenvector of $A(u), D(u)$ and, consequently, of the transfer matrix. Other eigenvectors of $F(u)$, called *Bethe states*, are of the form

$$\Phi(\{u_i\}) = B(u_1)\cdots B(u_l)\Omega \tag{2.59}$$

By requiring that $\Phi(\{u_i\})$ are indeed eigenstates, we now derive the Bethe equations for the sl_2 model.

Using the commutation relation (2.47) to move $A(u)$ to the right, so that it acts on Ω, we obtain

$$\begin{aligned} A(u)B(u_1)\ldots B(u_l)\Omega = \prod_{k+1}^{l} f(u-u_k)\alpha^N(u)B(u_1)\ldots B(u_l)\Omega + \\ + \sum_{k=1}^{l} M_k(u,\{u\})B(u_1)\ldots B(u_{k-1})B(u_{k=1})\ldots B(u_l)\Omega \end{aligned} \tag{2.60}$$

The first term here comes from only using the first term in (2.47), while the second term comes from using combinations of both terms. One of the M_k, which corresponds to using the second term of (2.47) on the first permutation and the other on the rest, is simple to calculate

$$M_1(u,\{u\}) = g(u-u_1)\prod_{k=2}^{l} f(u_1-u_k)\alpha^N(u_1). \tag{2.61}$$

We can avoid the complicated computation of the other factors by arguing that, due to the commutativity of the $B(u_i)$, they are of the same form of the one we computed, i.e.

$$M_j(u,\{u\}) = g(u-u_i)\prod_{k\neq j}^{l} f(u_j-u_k)\alpha^N(u_j) \tag{2.62}$$

Similarly, using (2.48), one can obtain

$$\begin{aligned} D(u)B(u_1)\ldots B(u_l)\Omega = \prod_{k+1}^{l} f(u-u_k)\alpha^N(u)B(u_1)\ldots B(u_l)\Omega + \\ + \sum_{k=1}^{l} N_k(u,\{u\})B(u_1)\ldots B(u_{k-1})B(u_{k+1})\ldots B(u_l)\Omega \end{aligned} \tag{2.63}$$

with

$$N_j(u,\{u\}) = k(u-u_j)\prod_{k\neq j}^{l} h(u_j-u_k)\delta^N(u_j) \tag{2.64}$$

Now, we can see that

$$F(u)\Phi(\{u\}) = u(u,\{u\})\Phi(\{u\}) \tag{2.65}$$

requires that the "extra" terms cancel each other. This is possible due to

$$g(u-u_j) = -k(u-u_j) \tag{2.66}$$

and leads to the condition

$$\prod_{k\neq j}^{l} f(u_j-u_k)\alpha^N(u_j) = \prod_{k\neq j}^{l} h(u_j-u_k)\delta^N(u_j) \tag{2.67}$$

Substituting the explicit forms of the functions involved, we obtain the Bethe equations.

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{j\neq k}^{l} \frac{u_k-u_j+i}{u_k-u_j-i} \tag{2.68}$$

Bethe states are Highest Weight States

Taking the $v\to\infty$ limit in the FCR and using the expansion of $T_a(u)$ (2.26), we obtain:

$$\left[T_a(u), \frac{1}{2}\sigma^\alpha + S^\alpha\right] = 0, \tag{2.69}$$

which implies the $sl(2)$ invariance of T_a in $\mathcal{H}\otimes V$. Acting with this commutator on the pseudovacuum, we obtain

$$S^+\Omega = 0\,,\ S^3\Omega = \frac{L}{2}\Omega. \tag{2.70}$$

Using this and the commutators

$$[S^3, B(u)] = B(u) \tag{2.71}$$

$$[S^+, B(u)] = A(u) - D(u) \tag{2.72}$$

one can show that

$$S^3\Phi(\{u_i\}) = \left(\frac{L}{2} - M\right)\Phi(\{u_i\}) \tag{2.73}$$

$$S^+\Phi(\{u_i\}) = 0. \tag{2.74}$$

This implies that the Bethe ansatz only generates highest weight states of the total spin. The rest of the eigenstates can be generated by acting on the Bethe states with the lowering operator S^-. Since highest weight states must, by definition, have a positive S^3 eigenvalue, (2.74) implies that we must restrict the number of excitations to $M \leq \frac{L}{2}$, or

$$\Phi(\{u_i\}) = \prod_{i=1}^{M} B(u_i)\Omega = 0\,, \text{ for } M > \frac{L}{2} \tag{2.75}$$

Singular Solutions and the completeness of the Bethe-state basis

It is quite obvious that the Bethe equations, as formulated in (2.68), are singular for $i = \pm i/2$. However, such roots can still rise from equivalent formulations, for example as roots of Baxter polynomials. This raises the question of whether solutions that contain these singular roots are are valid sets of rapidities. Specifically, one can examine whether (2.59) does indeed generate eigenstates of the Hamiltonian for singular solutions.

One way to do that , presented in [21], is to regularize the singular roots as

$$\frac{i}{2} \to \frac{i}{2} + \epsilon + c\epsilon^L \tag{2.76}$$

$$-\frac{i}{2} \to -\frac{i}{2} + \epsilon \tag{2.77}$$

and determine the parameter c by requiring that the corresponding Bethe state is an eigenstate of the transfer matrix. This leads to two solutions for the constant that must be simultaneously satisfied, leading to the following consistency condition for the non-singular roots

$$\left[-\prod_{k=3}^{M} \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right) \right]^L = 1, \tag{2.78}$$

where we have assumed that the set of roots is ordered as $\{\frac{i}{2}, -\frac{i}{2}, u_3, \ldots, u_M\}$. Note that the restrictions imposed on Bethe roots in N=4 SYM applications, namely that they must be paired and that L is even, are enough for this constraint to be always satisfied.

The above result is related to questions regarding the completeness of the Bethe equations, i.e. whether Bethe states and their descendants constitute a full basis of the Hilbert space. In particular, it is believed that the number of solutions to the B.A.E. is generally greater than what is required to form the basis. The number of states needed can be calculated by recalling that the Hilbert space of the Heisenberg model is essentially the vector space of a representation of su(2), which is in particular L-fold tensor product of spin-$\frac{1}{2}$ representations. By the Clebsch-Gordan theorem, this representation can be decomposed into a direct sum of irreducible spin-s representations,

$$\mathbf{\frac{1}{2}} \otimes \cdots \otimes \mathbf{\frac{1}{2}} = \bigoplus_{s=0}^{\frac{L}{2}} n_s \mathbf{s}, \tag{2.79}$$

where

$$n_s = \binom{L}{\frac{L}{2} - s} - \binom{L}{\frac{L}{2} - s - 1}, \tag{2.80}$$

is the number of representations with spin-s. Each Bethe state corresponds to the highest weight of one of these irreps, so the number of solutions to the Bethe equations for a given $M = \frac{L}{2} - s$ should be

$$\mathscr{N}(L, M) \stackrel{?}{=} \binom{L}{M} - \binom{L}{M-1} \tag{2.81}$$

Numerical calculations presented in [12] have verified that, for small values of L, the equality holds if out of the singular solutions we only take into account those that satisfy the constraint (2.78).

3 The Yangian of gl(N) and the gl(N) spin chain

3.1 The Yangian $Y(N)$

Definition and the RTT relation

The Yangian $Y(N)$ is the complex, unital, associative algebra generated by $\{t_{ij}^{(n)}|1 \leq i,j \leq N\,,\, n=0,1,\cdots\}$ subject to the relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}, \tag{3.1}$$

where $[\cdot,\cdot]$ denotes the commutator and $t_{ij}^{(0)} = \delta_{ij}$. This abstract definition can be brought to a more tangible form by introducing a complex parameter λ and collecting the generators into

$$t_{ij}(\lambda) = \sum_{r=0}^{\infty} \frac{(-i)^r}{\lambda^r} t_{ij}^{(r)} \tag{3.2}$$

We can then introduce an auxiliary space $V_a = \mathbb{C}^N$ to gather them in a matrix in $End(\mathbb{C}^N) \otimes Y(N)[[\lambda^{-1}]]$

$$T_a(\lambda) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(\lambda) = \sum_{r=0}^{\infty} \frac{(-i)^r}{\lambda^r} T^{(r)} \tag{3.3}$$

We can now show that these matrices satisfy a familiar relation

$$R_{ab}(\lambda_a - \lambda_b) T_a(\lambda_a) T_b(\lambda_b) = T_b(\lambda_b) T_a(\lambda_a) R_{ab}(\lambda_a - \lambda_b) \tag{3.4}$$

where R_{ab} is the gl(N) R-matrix

$$R_{ab}(\lambda) = \mathbb{I}_N \otimes \mathbb{I}_N + \frac{i\mathbb{P}_{ab}}{\lambda}. \tag{3.5}$$

The permutation in $V_a \otimes V_b \cong \mathbb{C}^N \otimes \mathbb{C}^N$ is defined as

$$P_{ab} = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}, \tag{3.6}$$

where e_{ij} is the matrix with entry 1 at the (i,j) position and 0 elsewhere i.e. $[e_{ij}]_{ab} = \delta_{ia}\delta_{jb}$.

In order to see that this so-called RTT relation is equivalent to the defining relations, we first need to notice that

$$[t_{ij}(\lambda_a), t_{kl}(\lambda_b)] = \frac{1}{\lambda_a - \lambda_b} (t_{kj}(\lambda_a) t_{il}(\lambda_b) - t_{kj}(\lambda_b) t_{il}(\lambda_a)) \tag{3.7}$$

is equivalent to (3.1). This can be seen by multiplying with $\lambda_a - \lambda_b$ and comparing the terms of same order in λ_a and λ_b. The RTT relation reads

$$(1 - \frac{P}{\lambda_a - \lambda_b}) \sum_{i,j,k,l} t_{ij}(\lambda_a) t_{kl}(\lambda_b)(e_{ij} \otimes e_{kl}) = \sum_{i,j,k,l} t_{kl}(\lambda_b) t_{ij}(\lambda_a)(e_{ij} \otimes e_{kl})(1 - \frac{P}{\lambda_a - \lambda_b}) \tag{3.8}$$

which we can rewrite as

$$\begin{aligned}&\sum_{i,j,k,l}[t_{ij}(\lambda_a),t_{kl}(\lambda_b)](e_{ij}\otimes e_{kl})=\\ \frac{1}{\lambda_a-\lambda_b}&\sum_{i,j,k,l}t_{ij}(\lambda_a)t_{kl}(\lambda_b)P(e_{ij}\otimes e_{kl})-\frac{1}{\lambda_a-\lambda_b}\sum_{i,j,k,l}t_{kl}(\lambda_b)t_{ij}(\lambda_a)(e_{ij}\otimes e_{kl})P\end{aligned}\tag{3.9}$$

Using the definition of P and the property $e_{ab}e_{cd} = e_{ad}\delta_{bc}$, it is easy to see that

$$\begin{aligned}P(e_{ij}\otimes e_{kl})&=e_{kj}\otimes e_{il}\\ (e_{ij}\otimes e_{kl})P&=e_{il}\otimes e_{kj}\end{aligned}\tag{3.10}$$

and we thus have

$$\begin{aligned}&\sum_{i,j,k,l}[t_{ij}(\lambda_a),t_{kl}(\lambda_b)](e_{ij}\otimes e_{kl})=\\ \frac{1}{\lambda_a-\lambda_b}&\sum_{i,j,k,l}t_{ij}(\lambda_a)t_{kl}(\lambda_b)(e_{kj}\otimes e_{il})-\frac{1}{\lambda_a-\lambda_b}\sum_{i,j,k,l}t_{kl}(\lambda_b)t_{ij}(\lambda_a)(e_{il}\otimes e_{kj})\end{aligned}\tag{3.11}$$

By renaming the indices on the right hand side, we can obtain

$$\sum_{i,j,k,l}[t_{ij}(\lambda_a),t_{kl}(\lambda_b)](e_{ij}\otimes e_{kl})=\frac{1}{\lambda_a-\lambda_b}\sum_{i,j,k,l}(t_{kj}(\lambda_a)t_{il}(\lambda_b)-t_{kj}(\lambda_b)t_{il}(\lambda_a))e_{ij}\otimes e_{kl})\tag{3.12}$$

which is equivalent to (3.7), thus proving the RTT relation.

The Yangian, in addition to the usual addition and multiplication of an algebra, is equipped with a *coproduct*, which maps the Yangian onto two copies of itself

$$\Delta:\ Y(N)\mapsto Y(N)\otimes Y(N)\tag{3.13}$$

and is defined by its action on the generators:

$$\Delta(t_{ij}(u))=\sum_k t_{ik}(u)\otimes t_{kj}(u)\tag{3.14}$$

One can show that the $\tilde{t}(u)_{ij} = \Delta(t_{ij}(u)) \in Y(N)\otimes Y(N)$ also satisfy the RTT relation. It can then be inductively deduced that $\Delta^{(n)}(t_{ij}(u))$ also fulfil it, after properly defining $\Delta^{(n)}$.

The Heisenberg model as a $Y(2)$ representation

In order to find a connection between between the Yangian and integrable models, let us see how the Heisenberg model emerges from selecting the appropriate $Y(2)$ representation. The su_2 R-matrix itself provides a representation of Y(2) i.e. we can choose

$$(t_{ij}(\lambda))^{kl}=R^{kl}_{ij}(\lambda-\frac{i}{2}),\tag{3.15}$$

where in the LHS i,j identify the generator and k,l the component of the matrix representing the generator. We can verify that this is a valid representation by noting that $T_a(\lambda)$ is represented by $R_{ac}(\lambda)$, where a denotes the auxiliary space already introduced and c denotes the representation's vector space. Then, the RTT relation becomes

$$R_{ab}(\tilde{\lambda}_a - \tilde{\lambda}_b)R_{ac}(\tilde{\lambda}_a)R_{bc}(\tilde{\lambda}_b) = R_{bc}(\tilde{\lambda}_b)R_{ac}(\tilde{\lambda}_a)R_{ab}(\tilde{\lambda}_a - \tilde{\lambda}_b) \tag{3.16}$$

$$\tilde{\lambda} = \lambda - \frac{i}{2} \tag{3.17}$$

This is the Yang-Baxter equation, which the R-matrix fulfils by definition.

Now recall that, in the gl_2 case, R_{ac} acts in $\mathbb{C}^2 \otimes \mathbb{C}^2$, while T_a is a matrix in $\mathbb{C}^2 \otimes Y(2)$. It follows that the $Y(2)$ representation that we chose is carried by $\mathbb{C}^2$. This is the local Hilbert space of the Heisenberg model. Furhtermore, the RTT relation can now be seen to be identical to the FCR (2.30) fulfiled by the Lax operators, since in that model the Lax operators are identical to the R-matrix up to a rapidity shift (see 2.21).[3]Thus, we could define the Lax operator of the Heisenberg model as this representation of the T-matrix and the local Hilbert space as the corresponding vector space.

The global Hilbert space $\mathcal{H}$ would then need to carry a representation of $\otimes^L Y(2)$. The monodromy, which acts in $\mathcal{H}$, can be defined using the co-product of the Yangian. First, because the co-product turns out not to be co-associative, one needs to specify how it acts on multiple copies of the algebra. We recursively define

$$\Delta^{(n)} = (\Delta \otimes id^{n-2})\Delta^{(n-1)}\,,\; n > 2 \tag{3.18}$$

$$\Delta^{(2)} = \Delta, \tag{3.19}$$

so that it always acts on the left. Then

$$\Delta^L(t_{ij}(u)) = \sum_{a_i=1}^{2} t_{ia_1}(u) \otimes t_{a_1a_2}(u) \otimes \cdots \otimes t_{a_Lj}(u) \tag{3.20}$$

Note that the index structure in this relation is reminiscent of that in matrix multiplication. One can then see that, after picking the representation (3.15) and identifying $t_{ij}(u)$ as the components of the Lax operator, (3.20) yields the same result as the product of Lax operators which define the monodromy. Thus, we can identify the monodromy as the matrix with components

$$\mathcal{T}_{ij} = \Delta^{(L)}(t_{ij}), \tag{3.21}$$

The property of the coproduct discussed after its definition implies that the monodromy fulfils the FCR relation, as expected. This implies that the transfer matrix defined as

$$F(u) = \mathscr{T}_{11} + \mathscr{T}_{22} \tag{3.22}$$

has the commutation property (2.34) which in turn implies that the charges in the expansion (2.29) commute.

3.2 Properties and representations of Y(N)

Here we present, mostly without proof, the properties of Y(N) that will be needed at some later point. The derivations can be found, for instance, in [19].

[3]In models where the Hilbert space carries a higher representation of $Y(2)$, the Lax operator is distinct from the fundamental R-matrix.

The quantum determinant and the centre of Y(N)

The quantum determinant of $T(\lambda)$ is defined as a powerseries in $\lambda^{(-1)}$ with coefficients in $Y(N)$ as follows

$$\text{qdet}(T(\lambda)) = \sum_{\sigma\in\mathscr{P}_N} sign(\sigma)T_{1,\sigma(1)}(\lambda+iN+i)\cdots T_{N,\sigma(N)}(\lambda) \tag{3.23}$$

, where $\mathscr{P}_N$ is the N-object permutation group. An important feature of qdet$T(\lambda)$ is that its coefficients generate the centre of $Y(N)$. One of its properties, which we will later use, is the following:

Let A_m be the antisymmetrizer in $(\mathbb{C}^N)^{\otimes m}$, defined by its action on the canonical basis as

$$A_m(e_{i_1}\otimes\cdots\otimes e_{i_m}) = \frac{1}{m!}\sum_{s=\mathscr{P}_N} sign(\sigma)e_{i_1}\otimes\cdots\otimes e_{i_m}, \tag{3.24}$$

The following identity holds

$$A_mT_1(\lambda)\cdots T_m(\lambda+im-i)A_m = T_m(\lambda+im-i)\cdots T_1(\lambda)A_m = A_mT_1(\lambda)\cdots T_m(\lambda+im-i), \tag{3.25}$$

which, for $m=N$, can be used to show

$$\text{qdet}\,T(\lambda)A_N = T_N(\lambda+iN-i)\cdots T_1(\lambda)A_N. \tag{3.26}$$

The latter, which can also be treated as the definition of the quantum determinant, is the equation we will need.

Evaluation representations of Y(N)

The *evaluation map* is a Hopf algebra homomorphism from $Y(N)$ to $\mathcal{U}(gl_n)$, the universal enveloping algebra of gl_N:

$$ev: t_{ij}(\lambda)\longmapsto \delta_{ij}+\frac{iE_{ij}}{\lambda}, \tag{3.27}$$

where E_{ij} are the generators of gl_N.[4] This mapping allows one to create representations of $Y(N)$ that are directly related to gl_N representations.

More specifically, $M(\boldsymbol{\alpha})$ be a highest weight representation of gl_N with highest weight $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_N)$ and highest weight vector v. Recall that the latter implies

$$E_{kj}\cdot v = 0\,,\ 1\le k<j\le N \tag{3.28}$$

$$E_{kk}\cdot v = \alpha_k v\,,\ 1\le k\le N, \tag{3.29}$$

where $\alpha_{k+1}\le\alpha_k$, which is the requirement for the representation to be finite dimensional. The evaluation map allows one to define an action of the elements of the Yangian on a gl_N-module, thus defining a so-called *evaluation representation* ($M_\lambda(\boldsymbol{\alpha})$) of the Yangian. In particular, in accordance to (3.27), the generators of

[4]Throughout this thesis we will treat this mapping as a simple relation between $Y(N)$ and gl_N generators. The algebra $\mathcal{U}(gl_n)$ enters because, formally, a homomorphism needs to map between algebras of the same type. Both $Y(N)$ and $\mathcal{U}(gl_n)$ have a Hopf algebra sructure, while Lie algebras do not.

$Y(N)$ act on the highest weight vector as

$$T_{kj}(\lambda) \cdot v = 0\,,\ 1 \leq k < j \leq N \tag{3.30}$$

$$t_{kk}(\lambda) \cdot v = \left(1 + \frac{i\alpha_k}{\lambda}\right) v\,,\ 1 \leq k \leq N. \tag{3.31}$$

It is important to notice that the evaluation representation of each Yangian generator has a simple pole at $\lambda = 0$. In the derivation that follows, we will use elements of Y(N) with the purpose of choosing a specific evaluation representation at the very end. In that context we say that the above equations imply that the components of $\lambda t_{ij}(\lambda)$ are analytical (in the sense that their representation is analytical), despite (3.2) perhaps suggesting otherwise.

Two automorphisms

$$\text{Inversion: } T(\lambda) \longmapsto T^{(-1)}(-\lambda) \tag{3.32}$$

$$\text{Shift: } T(\lambda) \longmapsto T(\lambda + \alpha)\,,\ a \in \mathbb{C}. \tag{3.33}$$

Embedding of $\mathcal{U}(gl_n)$ into Y(N)
Using an equivalent form of the defining relations of Y(N) , one can easily see that

$$[t^{(1)}_{ij}, t^{(1)}_{kl}] = \delta_{kj} t^{(1)}_{il} - \delta_{il} t^{(1)}_{kj} \tag{3.34}$$

Thus, $t^{(1)}_{ij}$ generate a gl_N algebra.

3.3 The gl_N spin chain

As an example of the usage of Yangians in integrability, we present the part of the derivation found in [1] that regards closed spin chains. The starting point is to utilize the Yangian to define the transfer matrix as an element in $Y(N)^{\otimes L}$. This abstract definition has the advantage of being representation independent, which allows us to take steps that hold for any evaluation representation of $Y(N)$. The final result is a general form of the Bethe Equations and transfer matrix eigenvalues for any spin chain in a highest weight representation of gl_N.

Note that in this section we use a different notation for the generators of Y(N). The standard notation is the one used in previous sections but we will denote them as $l^{(s)}_{ij}$ instead, in order to to distinguish between $Y(N)$ and $Y(N)^{\otimes L}$. The notation in (3.2) and (3.3) is changed accordingly.

The Monodromy and the Transfer Matrix
Using the "extension" of the coproduct (3.18), we can define the monodromy as

$$T_a(\lambda) = \Delta^{(L)}(L(\lambda)) \tag{3.35}$$

and the transfer matrix as usual

$$t(\lambda) = tr_a(T). \tag{3.36}$$

As mentioned earlier, the generators of the Yangian and their coproduct both satisfy the RTT relation. We can thus iteratively deduce that the monodromy also satisfies the same relation, i.e.

$$R_{ab}(u - v) T_a(u) T_b(v) = T_b(v) T_a(u) R_{ab}(u - v) \tag{3.37}$$

Similarly to the algebraic Bethe ansatz, this implies the commutativity of the transfer matrix at different rapidities

$$[t(\lambda), t(\mu)] = 0. \tag{3.38}$$

The transfer matrix also exhibits gl_N symmetry, which can be seen by recalling that $\mathcal{L}^{(1)}(\lambda)$ generates a local gl_N transformation. Then, a global transformation is generated by

$$t_{ij}^{(1)} = l_{ij}^{(1)} \otimes^{L-1} 1 + 1 \otimes l_{ij}^{(1)} \otimes^{(L-2)} 1 + \cdots + \otimes^{(L-1)} 1 + l_{ij}^{(1)} \in \otimes^L Y(N) \tag{3.39}$$

Now, taking the trace over V_a in the RTT relation gives

$$(\lambda_a - \lambda_b)[t(\lambda_a), T(\lambda_b)] = -i[T(\lambda_a), T(\lambda_b)]. \tag{3.40}$$

Keeping the λ_b-free term, we obtain

$$[t(\lambda_a), T^{(1)}] = 0, \tag{3.41}$$

which proves the gl_N symmetry of the transfer matrix.

Representations of $Y(N)^{\otimes L}$ are built out of evaluation representations of $Y(N)$. First, note that the shift automorphism (3.33) implies that if $M_\lambda(\boldsymbol{\alpha})$ is a representation, $M_{\lambda+\theta}(\boldsymbol{\alpha})$ defines another representation. Then, for highest weights $\boldsymbol{\alpha}^n = (\alpha_1^n, \ldots, \alpha_N^n)$, $1 \leq n \leq L$

$$M_{\lambda+\theta_1}(\boldsymbol{\alpha}^1) \otimes \cdots \otimes M_{\lambda+\theta_L}(\boldsymbol{\alpha}^L) \tag{3.42}$$

provides a finite-dimensional representation of $\mathcal{T}(\lambda)$ with highest weight vector

$$v^+ = v^1 \otimes \cdots \otimes v^L, \tag{3.43}$$

which means that

$$T_{kj}(\lambda)v^+ = 0\,,\ 1 \leq k < j \leq N \tag{3.44}$$

$$T_{kk}(\lambda)v^+ = \prod_{n=1}^{L} \left(1 + \frac{i\,\alpha_k^n}{\lambda + \theta_n}\right) v^+\,,\ 1 \leq k \leq N \tag{3.45}$$

In principle, this discussion of representations allows every local space to carry a different $Y(N)$ or, equivalently, gl_N representation. However, in the context of spin chains, we are interested in all local representations being identical, i.e. all $\boldsymbol{\alpha}^n$ being the same. A consequence of that is that the global representation is guaranteed to be irreducible.

Derivation of the Bethe equations

In the following, we will use a different normalisation for the local and global matrices

$$\hat{L}_{a,n}(\lambda) = (\lambda + \theta_n) L_{a,n}(\lambda) \tag{3.46}$$

$$\hat{T}(\lambda) = \prod_{n=1}^{L} (\lambda + \theta_n) T(\lambda), \tag{3.47}$$

which guarantees that the monodromy is analytical. We then need to rewrite the highest weights as

$$\hat{T}_{kk}v^{+} = P_k(\lambda)v^{+}\,,\ 1 \leq k \leq N, \tag{3.48}$$

where

$$P_k(\lambda) = \prod_{n=1}^{L}(\lambda + \theta_n + i\alpha_k^n) \tag{3.49}$$

are the so-called Drinfeld polynomials. It is quite obvious that the highest weight vector is an eigenvector of the transfer matrix with eigenvalue

$$\Lambda^0(\lambda) = \sum_{k=1}^{N} P_k(\lambda). \tag{3.50}$$

Now, we make an ansatz by assuming that every other eigenvalue of the transfer matrix can be written as

$$\Lambda(\lambda) = \sum_{k=1}^{N} P_k(\lambda)D_k(\lambda), \tag{3.51}$$

where $D_k(\lambda)$ are some dressing functions. The form of these functions is determined by the asymptotic behaviour, as well as the analyticity of the eigenvalues. From (3.47) we can see that for $\lambda \to \infty$, the transfer matrix $\hat{t}(\lambda)$ is dominated by λ^L. We require that the eigenvalues have the same asymptotic behaviour which implies $D_k(\lambda) \to 1$, given that the Drinfeld polynomials have degree L. Then, the most simple ansatz that ensures that the eigenvalues can be analytical is

$$D_k(\lambda) = \prod_{n=1}^{M^{(k-1)}} \frac{\lambda + u_n^{(k-1)}}{\lambda - \lambda_n^{(k-1)} + \frac{i(k-1)}{2}} \prod_{n=1}^{M^{(k)}} \frac{\lambda + v_n^{(k)}}{\lambda - \lambda_n^{(k-1)} + \frac{ik}{2}}. \tag{3.52}$$

Here, $M^{(k)}$, with $M^{(0)} = M^{(N)} = 0$ are related to the action of the Cartan generators of sl_N on the eigenvectors.[5] The next step is to determine the parameters $u_n^{(k)}, v_n^{(k)}$ in terms of $\lambda_n^{(k)}$.

To constrain the dressing functions, consider the antisymmetrizer defined in (3.24), now acting on auxiliary spaces $a_1, \ldots, a_N$. Then, (3.26) can be rewritten as

$$\hat{T}_{a_N}(\lambda + iN - i)\cdots\hat{T}_{a_1}(\lambda) = \text{qdet}\hat{T}(\lambda)\mathcal{A}_N + \hat{T}_{a_N}(\lambda + iN - i)\cdots\hat{T}_{a_1}(\lambda)(1 - A_N). \tag{3.53}$$

By tracing over all auxiliary spaces, we get

$$\hat{t}(\lambda + iN - i)\,\hat{t}(\lambda + iN - 2i)\cdots\hat{t}(\lambda) = \text{qdet}\hat{T} + \hat{t}_f(\lambda), \tag{3.54}$$

where we used that $\mathcal{A}_N$ is a 1-dimensional projector and, as such, has trace 1. The

$$t_f(\lambda) = Tr_{a_1\otimes\cdots\otimes a_N}\left[\hat{T}_{a_N}(\lambda + iN - i)\cdots\hat{T}_{a_1}(\lambda)(1 - A_N)\right]$$

[5] Recall that that in the Heisenberg model, M is related to the eigenvalue of S^3.

is the so-called fused transfer matrix, which is not relevant to what follows. By definition, the quantum determinant acts on the pseudovacuum as

$$\text{qdet}\hat{T}(\lambda)\cdot v^+ = \sum_{s\in\mathscr{P}_N} sgn(\sigma)\hat{T}_{1,\sigma(1)}(\lambda+iN-i)\cdots\hat{T}_{N,\sigma(N)}(\lambda)\cdot v^+ \tag{3.55}$$

Using the properties of the highest weight, it can be seen that the only non-vanishing term is the one that comes from the permutation where $\sigma(i)=i$, i.e.

$$\text{qdet}\hat{T}(\lambda)\cdot v^+ = \prod_{k=1}^{N} P_k(\lambda+iN-ik)v^+ \tag{3.56}$$

Since the quantum determinant is a central element, it is represented by a matrix proportional to the identity, in any representation. We thus have

$$\text{qdet}\hat{T}(\lambda) = \prod_{k=1}^{N} P_k(\lambda+iN-ik), \tag{3.57}$$

where the identity matrix on the RHS is implied. Now, acting with (3.54) on an eigenstate would give

$$\Lambda(\lambda+iN-i)\cdots\Lambda(\lambda) = \prod_{k=1}^{N} P_k(\lambda+iN-ik) + \Lambda_f(\lambda). \tag{3.58}$$

The terms proportional to $\prod_{k=1}^{N} P_k(\lambda+iN-ik)$ on both sides of the equation must have the same coefficient, which implies

$$D_1(\lambda+iN-i)\cdots D_N(\lambda) = 1 \tag{3.59}$$

This is the constraint we have been looking for and it implies

$$D_k(\lambda) = \prod_{n=1}^{M^{(k-1)}} \frac{\lambda-\lambda_n^{(k-1)}+\frac{i(k+1)}{2}}{\lambda-\lambda_n^{(k-1)}+\frac{i(k-1)}{2}} \prod_{n=1}^{M^{(k)}} \frac{\lambda-\lambda_n^{(k)}+\frac{i(k-2)}{2}}{\lambda-\lambda_n^{(k)}+\frac{ik}{2}} \tag{3.60}$$

The final step is to invoke the analyticity of the monodromy. Recall that we chose our normalization in (3.47) such that the entries of $\hat{T}(\lambda)$ are analytical. Then, the eigenvalues of $\hat{t}(\lambda)=\sum_{i=1}^{N}\hat{T}_{ii}(\lambda)$ are also analytical. This is a consequence of (3.38), which implies that the matrix which diagonalizes the transfer matrix must be λ-independent. Thus, diagonalizing the matrix does not introduce any singularities.

The Bethe equations are derived from the requirement $\Lambda(\lambda)$ is analytical, which is equivalent to imposing that the residue at $\lambda=\lambda_n^{(k)}-\frac{ik}{2}$ vanishes for all n and k.[6] We can focus on

$$D_k(\lambda)P_k(\lambda)+D_{k+1}(\lambda)P_{k+1}(\lambda), \tag{3.61}$$

[6] $\Lambda(\lambda)$ has a simple pole at each of these points. This means that the principal part of the Laurent series of $\Lambda(\lambda)$ around these points only contains the $\mathcal{O}(\lambda^{-1})$ term. The residue of each pole is the coefficient of that term and it vanishing implies that the principal part of the series vanishes and $\Lambda(\lambda)$ is analytical.

as these are the only two terms where $\lambda^{(k)}$ appears. The vanishing of the residue requires

$$\lim_{\lambda\to\lambda_n^{(k)}-\frac{ik}{2}}\left(\lambda-\lambda_n^{(k)}+\frac{ik}{2}\right)\left(D_k(\lambda)P_k(\lambda)+D_{k+1}(\lambda)P_{k+1}(\lambda)\right)=0 \tag{3.62}$$

which straightforwardly leads to

$$\prod_{m=1}^{M^{(k-1)}} e_{-1}\left(\lambda_n^{(k)}-\lambda_m^{(k-1)}\right)\prod_{m\neq n}^{M(k)}\frac{\lambda_n^{(k)}-\lambda_m^{(k)}-i}{\lambda_n^{(k)}-\lambda_m^{(k)}}P_k\left(\lambda_n^{(k)}-\frac{ik}{2}\right)=$$
$$\prod_{m=1}^{M^{(k)}} e_{+1}\left(\lambda_n^{(k)}-\lambda_m^{(k+1)}\right)\prod_{m\neq n}^{M(k)}\frac{\lambda_n^{(k)}-\lambda_m^{(k)}+i}{\lambda_n^{(k)}-\lambda_m^{(k)}}P_k\left(\lambda_n^{(k)}-\frac{ik}{2}\right),$$

where we have introduced

$$e_x(\lambda)=\frac{\lambda+\frac{ix}{2}}{\lambda-\frac{ix}{2}} \tag{3.63}$$

to simplify notation. Noting that $e_{-1}(\lambda)=\frac{1}{e_{+1}(\lambda)}$, we can rewrite the last relation as

$$\prod_{m=1}^{M^{(k-1)}} e_{-1}\left(\lambda_n^{(k)}-\lambda_m^{(k-1)}\right)\prod_{m=1,m\neq n}^{M^{(k)}} e_2\left(\lambda_n^{(k)}-\lambda_m^{(k)}\right)\prod_{m=1}^{M^{(k+1)}} e_2\left(\lambda_n^{(k)}-\lambda_m^{(k+1)}\right)=\frac{P_k\left(\lambda_n^{(k)}-\frac{ik}{2}\right)}{P_{k+1}\left(\lambda_n^{(k)}-\frac{ik}{2}\right)} \tag{3.64}$$

The set of these equations for $1\leq k\leq N-1$ and $1\leq n\leq M^{(k)}$ are the Bethe equations the $gl(N)$ spin chain whose Hilbert space carries the representation defined by $\boldsymbol{\alpha}$. As a byproduct of the derivation, we also have an expression for the eigenvalues of the gl_N transfer matrix in (3.51).

4 Integrable Matrix Product States and Twisted Yangians

4.1 Integrable MPSs and boundary integrability

The R-Matrix

An R-matrix $R_{12}(u) \in End(V_1 \otimes V_2)$, where $V_1 \cong V_2 \cong \mathbb{C}^N$ is generally a solution of the Yang-Baxter equation (3.16). A solution is symmetric with respect to some Lie group $\mathcal{G}$ if for any G_1 and G_2 acting in the defining representation of $\mathcal{G}$

$$(G_1 \otimes G_2)R_{12}(u) = R_{12}(u)(G_1 \otimes G_2) \tag{4.1}$$

For $\mathcal{G} = SU(N)$, we have already encountered the R-matrix

$$R_{12}^{su_N}(u) = I_1 \otimes I_2 - \frac{1}{u}\mathcal{P}_{12}, \tag{4.2}$$

where

$$\mathcal{P} = \sum_{i,j}^{N} e_{ij} \otimes e_{ji} \tag{4.3}$$

is the permutation operator in $V_1 \otimes V_2$.

For $\mathcal{G} = SO(N)$, a solution that is often used is

$$R_{12}^{so_6}(u) = I_1 \otimes I_2 - \frac{1}{u}\mathcal{P}_{12} + \frac{\mathcal{K}_{12}}{u-\kappa} \tag{4.4}$$

where $\kappa = \frac{N}{2} - 1$ and K is the trace operator, with matrix elements $K_{ab}^{cd} = \delta_{ab}\delta^{cd}$. Note that we can write

$$\mathcal{K} = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij} = \mathcal{P}^T \tag{4.5}$$

Where $\cdot^T$ denotes partial transposition, that is, transposition acting either on V_1 or V_2. It is not hard to see that $\mathcal{P}^{T_1} = P\mathcal{P}^{T_2}$, so one does not need to distinguish between the two cases. Similarly,

$$R_{12}^{T_1}(u) = R_{12}^{T_2}(u) = R_{12}^{T}(u) \tag{4.6}$$

for both R-matrices that we have introduced. Using (4.5) it is straightforward to see that in the SO(N) case

$$(R_{12}^{so_N}(u))^T = R_{12}^{so_N}(\kappa - u) \tag{4.7}$$

which is the so-called *crossing relation* of the SO(N) R-matrix. In some calculations we will make use of the *unitarity* property

$$R_{12}(u)R_{12}(-u) \propto I \tag{4.8}$$

which holds for both the SU(N) and SO(N) R-matrix.

The spin chain

Similar to the algebraic ansatz, we can use an R-matrix to construct a spin chain by treating one of the vector spaces a a local Hilbert space $h_n = \mathbb{C}^N$ and the other the auxiliary space V_0. Then, the monodromy acting in

$V_0 \otimes \mathcal{H}$, where $\mathcal{H} = h_n^{\otimes L}$, is defined as

$$T = R_{0L}(u)R_{0L-1}(u)\cdots R_{01}(u) \tag{4.9}$$

and the transfer matrix by tracing over the auxiliary space

$$t(u) = Tr_0\left[T(u)\right]. \tag{4.10}$$

A family of commuting charges is generated from the transfer matrix as

$$Q_j = \left(\frac{d}{du}\right)^{(j-1)} log\, t(u)\Big|_{u=0}, \; j = 2, 3, \ldots L \tag{4.11}$$

The Q_j operator can be seen to act on j neighboring sites [22], thus a two-site Hamiltonian can be defined using Q_2[7]. Is is also known that for odd j, the Q_j are odd under space reflections, while the even charges are even, namely

$$\Pi Q_j \Pi = (-1)^j Q_j, \tag{4.12}$$

where Π is the spatial reflection operator, which acts on Bethe states as

$$\Pi \left|\{u_i\}\right\rangle = \left|\{-u_i\}\right\rangle. \tag{4.13}$$

Integrable Matrix Product States

Consider a second aulxiliary space V_A (distinct from the auxiliary space of the spin chain) and a set of N d-dimensional matrices $\omega_i \in End(V_A)$. A Matrix Product State (MPS) is a state in $\mathcal{H}$ which is of the form

$$|\Psi\rangle = \sum_{j_1,\ldots,j_L=1}^{N} tr_A[\omega_{j_L}\cdots\omega_{j_2}\omega_{j_1}] \left|j_L,\ldots,j_2,j_1\right\rangle, \tag{4.14}$$

where $\left|j_L,\ldots,j_2,j_1\right\rangle$ are the basis vectors of the Hilbert space. An MPS can be invariant under a subgroup of the symmetry group of the spin chain $\mathcal{G}' \subset \mathcal{G}$, forming a so-called *symmetric pair* $(\mathcal{G}, \mathcal{G}')$. This means that for every $G \in \mathcal{G}'$

$$\left[\otimes_{j-1}^{L} G_j \left|\Psi\right\rangle\right] = |\Psi\rangle. \tag{4.15}$$

This can be achieved by the ω_i matrices being group invariant., which is the case if the auxiliary space V_A carries a representation of $\mathcal{G}'$, denoted Λ such that for every j

$$\Lambda(G^{-1})\omega_j\Lambda(G) = \sum_k g_{jk}\omega_k, \tag{4.16}$$

where g_{jk} are the components of G in the fundamental representation.

A MPS is said to be integrable if it is annihilated by the odd charges associated with the spin chain:

$$Q_{2j+1}\left|\Psi\right\rangle = 0 \tag{4.17}$$

[7] Compare with (2.45)

This is equivalent to

$$t(u)\,|\Psi\rangle = \bar{t}(u)\,|\Psi\rangle\,, \tag{4.18}$$

where $\bar{t}(u) = \Pi t(u)\Pi$ is the reflected transfer matrix, which also implies that the state has to be parity invariant

$$\Pi\,|\Psi\rangle = \pm\,|\Psi\rangle\,. \tag{4.19}$$

As a consequence of this and (4.13), an overlap

$$\langle\{u_i\}|\Psi\rangle \tag{4.20}$$

is non-vanishing only if the Bethe state is parity invariant. This requires the set of Bethe roots to be parity invariant, i.e. $\{u_i\} = \{-u_i\}$. If the state contains an even number of excitations, this is the case if the roots come in pairs of opposite signs, namely for every root u_i there needs to exists some other root $u_{i'} = -u_i$. For an odd number of excitations, the roots again need to be paired and the "extra" root needs to be zero.

The twisted Boundary Yang-Baxter equation

We define the $K-$matrix acting in $V_0 \otimes V_A$ as

$$K(u) = \sum_{a,b=1}^{N} e_{ab} \otimes \phi_{ab}(u), \tag{4.21}$$

where e_{ab} are the usual elementary matrices acting in V_0 with matrix elements $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$. We are interested in K-matrices that are solutions to the twisted Boundary Yang-Baxter [8] (BYB) equation

$$K_2(v)R_{21}^T(-u-v)K_1(u)R_{12}(u-v) = R_{21}(u-v)K_1(u)R_{12}^T(-u-v)K_2(v), \tag{4.22}$$

where each side of this relation is a matrix acting in $V_1 \otimes V_2 \otimes V_A$ and

$$K_1(u) = \sum_{a,b=1}^{N} e_{ab} \otimes I \otimes \phi_{ab}(u) \tag{4.23}$$

$$K_2(u) = \sum_{a,b=1}^{N} I \otimes e_{ab} \otimes \phi_{ab}(u) \tag{4.24}$$

It turns out that, given a solution of that relation such that the two site block factorizes at some special point $u = u_f$.

$$\psi_{ij}(u_f) \propto \omega_i\omega_j \tag{4.25}$$

the MPS defined as

$$|MPS\rangle = \sum_{a_i,b_i=1}^{N} Tr[\psi_{a_1,b_1}(u_f)\cdots\psi_{a_{\frac{L}{2}},b_{\frac{L}{2}}}(u_f)]\,|a_1,b_1,\ldots,a_{\frac{L}{2}},b_{\frac{L}{2}}\rangle \tag{4.26}$$

is integrable [23]. Unless stated otherwise, we will assume that $u_f = 0$.

[8] This relation also appears in the context of open spin chains where it is used to determine the behaviour of the system at the boundaries.

4.2 The Twisted Yangian

Definition
Consider the Yangian Y(N) generated by $t_{ij}^{(k)}$, with the indices shifted so that $i, j \in \{-N, \dots, N\}$ for N even and $i, j \in \{-N, \dots, 0, \dots, N\}$, for N odd. Using the T-matrix defined in (3.3), we can now define the following matrix in terms of the generators of Y(3)

$$S(u) = T(u)G(u)T^t(-u), \tag{4.27}$$

where $G(u)$ fulfils the equation

$$G_2(v)R_{21}^t(-u-v)G_1(u)R_{12}(u-v) = R_{12}(u-v)G_1(u)R_{21}^t(-u-v)G_2(v). \tag{4.28}$$

In these relations, we use the transposition

$$A^t = CA^TC \tag{4.29}$$

where C is the *charge conjugation* matrix with elements $[C]_{i,j} = \delta_{i,-j}$. This operation is essentially transposition with respect to the secondary diagonal of the matrix and is commonly used in the context of twisted Yangians. The elements of the S-matrix are then

$$s_{ij}(u) = \sum_{\alpha,\beta}[G(u)]_{\alpha,\beta}t_{i\alpha}(u)\, t_{-j,-\beta}(-u). \tag{4.30}$$

The twisted Yangian $Y^{tw}(N)$ is the algebra generated by $s_{ij}(u)$, which makes it a subalgebra of $Y(N)$.

The quaternary relation
We will now show that, by definition, the S-matrix satisfies the so-called *quaternary* relation

$$S_2(v)R_{21}^t(-u-v)S_1(u)R_{12}(u-v) = R_{12}(u-v)S_1(u)R_{21}^t(-u-v)S_2(v). \tag{4.31}$$

During this derivation, we drop the indices in the R-matrix to simplify notation.[9] We will make use of the RTT relation and some more relations that are equivalent to it. By acting with partial transposition on V_1 or V_2 in the RTT relation we obtain the two relations

$$T_1^t(u)R^t(u-v)T_2(v) = T_2(v)R^t(u-v)(v)T_1^t(u) \tag{4.32}$$
$$T_2^t(v)R^t(u-v)T_1(u) = T_1(u)R^t(u-v)(v)T_2^t(v) \tag{4.33}$$

By transposing in both spaces and recalling that $R^{t_1}(u) = R^{t_2}(u)$, which implies that $R^{t_1t_2}(u) = R(u)$, we obtain

$$T_1^{t_1}(u)T_2^{t_2}(v)R(u-v) = T_2^{t_2}(v)T_1^{t_1}(u)R(u-v). \tag{4.34}$$

Finally, we can multiply the RTT relation from both sides with the R-matrix and apply the unitarity condition (4.8), to obtain

$$T_1(u)T_2(v)R(v-u) = R(v-u)T_2(v)T_1(u) \tag{4.35}$$

Before moving on with the derivation, let us spell out some points that, although trivial, might not be obvious

[9] The purpose of these indeces is to distinguish between the R-matrix acting in $V_1 \otimes V_2$ and the one acting in $V_2 \otimes V_1$. In our case, where $V_1 \cong V_2$ it is obvious that $R_{12}(u)$ and $R_{21}(u)$ are the same matrix, so the distinction is redundant.

- $T_1(u)$ and $T_2(v)$ do not commute. This has already been stated in the form of the RTT relation, but it might be helpful to see why it is the case. We have

$$T_1(u) = \sum_{i,j=1}^{N} t_{ij}(u) \otimes e_{ij} \otimes I \in Y(N) \otimes End(V_1) \otimes End(V_2)$$
$$T_2(u) = \sum_{i,j=1}^{N} t_{ij}(u) \otimes I \otimes e_{ij} \in Y(N) \otimes End(V_1) \otimes End(V_2)$$

Then,

$$T_1(u)T_2(u) = \sum_{i,j,k,l} t_{ij}(u)t_{kl}(u) \otimes e_{ij} \otimes e_{kl}, \tag{4.36}$$

so $T_1(u)T_2(u) \neq T_2(u)T_1(u)$, because $Y(N)$ is a non commutative algebra.

- $T_1(u)$ and $G_1(u)$ do not commute. To see that we also need to define $G_1(u)$ as an element in $Y(N) \otimes End(V_1) \otimes End(V_2)$:

$$G_1(u) = E \otimes K(u) \otimes I, \tag{4.37}$$

where E is the multiplicative identity in $Y(N)$. Then,

$$T_1(u)G_1(u) = \sum_{i,j=1}^{N} t_{ij}(u) \otimes e_{ij}K(u) \otimes I \tag{4.38}$$

which is not equal to $G_1(u)T_1(u)$ because of the ordinary matrix multiplication in $End(V_1)$.

- $T_1(u)$ and $G_2(u)$ commute, since

$$T_1(u)K_2(u) = \sum_{i,j=1}^{N} t_{ij}(u) \otimes e_{ij} \otimes G(u) = G_2(u)T_1(u) \tag{4.39}$$

Similarly, '$T_2(u)G_1(u) = G_1(u)T_2(u)$ and $G_1(u)G_2(u) = G_1(u)G_2(u)$.

Using all of these exchange properties on the two sides of (4.31), we obtain

$$\begin{aligned} LHS &= T_2(v)G_2(v)T_2^t(-u)R^t(u-v)T_1(u)G_1(u)T_1^t(-u)R(u-v) \\ &= T_2(v)G_2(v)T_1(u)R^t(u-v)T_2^t(-u)G_1(u)T_1^t(-u)R(u-v) \\ &= T_2(v)T_1(u)G_2(v)R^t(u-v)G_1(u)T_2^t(-u)T_1^t(-u)R(u-v) \\ &= T_2(v)T_1(u)G_2(v)R^t(u-v)G_1(u)R(u-v)T_1^t(-u)T_2^t(-v) \end{aligned}$$

and

$$\begin{aligned} RHS &= R(u-v)T_1(u)G_1(u)T_1^t(-u)R_{21}^t(-u-v)T_2(v)G_2(v)T_2^t(-v) \\ &= R(u-v)T_1(u)G_1(u)T_2(v)R_{21}^t(-u-v)T_1^t(-u)G_2(v)T_2^t(-v) \\ &= T_2(v)T_1(u)R(u-v)G_1(u)R_{21}^t(-u-v)G_2(v)T_1^t(-u)T_2^t(-v) \end{aligned}$$

It then immediately follows from (4.28) that $LHS = RHS$, which concludes the derivation of the quaternary relation.

The reason for introducing the twisted Yangians is that the quaternary relation, which is equivalent to their defining relations, is very similar to the twisted boundary Yang-Baxter relation introduced previously. The difference between the two relations is the convention used for partial transposition. Fortunately, there exist ways of transforming between solutions of (4.31) and (4.22). They are explained in appendix A. Given that, one can see a clear connection between twisted Yangians and integrable MPSs: Solutions of the BYB relation can constitute twisted Yangian representation, after the appropriate conversion. Conversely, one can use twisted Yangian representation theory to generate solutions of the BYB relation.

We can construct representations of $Y^{tw}(N)$ based on evaluation representations of $Y(N)$ in a somewhat straightforward way. Recall that the evaluation homomorphism (3.27) allows us to define a $Y(N)$ representation carried by a gl_N module. Restricting the evaluation representation to $Y^{tw}(N) \subset Y(N)$ defines a representation of the twisted Yangian, also carried by the gl_N module. We will use representations of this type in the subsequent sections. Before that, we discuss a result that will also be used later.

4.3 Dressing MPSs

Clearly, (4.28) and (4.31) are identical, thus the "twisting" matrix also forms a $Y^{tw}(N)$ representation. To emphasize that, let us denote $S_0(u) = G(u)$. Then, (4.27) implies that dressing S_0 with a representation of the untwisted $Y(N)$ also gives a $Y^{tw}(N)$ representation $S^D(u)$. The fact that these two S-matrices are related by a dressing allows us to find a relation between the corresponding MPSs.

More specifically, in the next section we will make use of the integrable state that corresponds to the twisting matrix of a specific twisted Yangian. According to (A.5), the K-matrix that corresponds to $S_0(u)$ and generates this state is

$$K_0(u) = S_0(u)C. \tag{4.40}$$

A dressed S-matrix is

$$S^D(u) = T(u)S_0(u)T^t(-u). \tag{4.41}$$

In order to create a MPS out of it, we need to map it to a K-matrix

$$K^D(u) = S^D(u)C = T(u)S_0(u)T^T(u)C. \tag{4.42}$$

Substituting (4.40), we obtain

$$K^D(u) = T(u)K_0(u)T^T(-u) \tag{4.43}$$

where we used that $T^t(-u) = CT^T(-u)C$. We will now use this relation between K^D and K_0 to show that their corresponding MPSs are related via the action of a transfer matrix.

Let V_Λ be the vector space carrying a gl_N representation and denote by $\mathcal{L}^{(\Lambda)}_{i,j}(u)$ the evaluation representation of the generators of Y(N), carried by V_Λ. The $\mathcal{L}^{(\Lambda)}_{i,j}(u)$ are the components of the T-matrix with which we dress $K_0(u)$. We can then define a Lax operator acting in $V_\Lambda \otimes h_k$, where h_k denotes a local Hilbert space

$$L^{(k,\Lambda)}(u) = \mathcal{L}^{(\Lambda)}_{i,j}(u) \otimes e_{i,j} \tag{4.44}$$

where $e_{i,j}$ are the unit matrices acting in V_k and summation is implied. The corresponding transfer matrix acting in a product of L physical spaces is

$$t^{(\Lambda)}(u) = Tr_\Lambda \left[L^{(L,\Lambda)}(u) L^{(L-1,\Lambda)}(u) \cdots L^{(1,\Lambda)}(u) \right] \tag{4.45}$$

The difference between this transfer matrix and the one in (4.10) is that the auxiliary space carries a higher representation of su_N. Let us denote as ϕ_{ij} the two-site block of $K_0(u)$, acting in some auxiliary space V_Ω. Then, (4.43) implies

$$\phi^D_{ab}(u) = \mathcal{L}^{(\Lambda)}_{ac}(-u)\mathcal{L}^{(\Lambda)}_{bd}(u)\phi_{cd}(u), \tag{4.46}$$

where $\phi^D_{ab}(u)$ acts in $V_\Omega \otimes V_\Lambda$. The MPSs corresponding to the two solutions are related by

$$|\Phi^D\rangle = t^{(\Lambda)}(0)\,|\Phi_0\rangle\,. \tag{4.47}$$

<u>Proof:</u> We can focus on the two-site state. The generalisation to an L-site state is simple, using that each two-site block acts in a different pair of physical spaces. The two-site MPSs are

$$|\Phi_0\rangle = \sum_{a,b} Tr_\Omega \left[\phi^{(0)}_{ab}(0) \right] |a, b\rangle \tag{4.48}$$

$$|\Phi^D\rangle = \sum_{a,b}\sum_{c,d} Tr_{\Lambda\otimes\Omega} \left[\mathcal{L}^{(\Lambda)}_{ac}(0)\mathcal{L}^{(\Lambda)}_{bd}(0)\phi_{cd}(0) \right] |a, b\rangle \tag{4.49}$$

Using that $Tr_{\Lambda\otimes\Omega}[\cdot] = Tr_\Omega[\cdot]Tr_\Lambda[\cdot]$, the latter can be written as

$$|\Phi^D\rangle = \sum_{a,b}\sum_{c,d} Tr_\Lambda \left[\mathcal{L}^{(\Lambda)}_{ac}(0)\mathcal{L}^{(\Lambda)}_{bd}(0) \right] Tr_\Omega\left[\phi_{cd}(0)\right] |a, b\rangle \tag{4.50}$$

It is now straightforward to show that (4.50) is the same as (4.47):

$$t^{(\Lambda)}(0)\,|\Phi\rangle = Tr_\Lambda \left[L^{(2,\Lambda)}(0) L^{(1,\Lambda)}(0) \right] \sum_{c,d} Tr_\Omega\left[\phi_{cd}(0)\right] |c, d\rangle \tag{4.51}$$

$$= \sum_{c,d}\sum_{a,b}\sum_{e,f} Tr_\Lambda \left[\mathcal{L}_{ae}(0)\mathcal{L}_{bf} \otimes e_{ae} \otimes e_{bf}(0) \right] Tr_\Omega\left[\phi_{cd}(0)\right] |c, d\rangle \tag{4.52}$$

$$= \sum_{c,d}\sum_{a,b}\sum_{e,f} Tr_\Lambda \left[\mathcal{L}_{ae}(0)\mathcal{L}_{bf}(0) \right] Tr_\Omega\left[\phi_{cd}(0)\right] (e_{ae}\,|c\rangle \otimes e_{bf}\,|d\rangle) \tag{4.53}$$

$$= \sum_{c,d}\sum_{a,b}\sum_{e,f} Tr_\Lambda \left[\mathcal{L}_{ae}(0)\mathcal{L}_{bf}(0) \right] Tr_\Omega\left[\phi_{cd}(0)\right] \delta_{ce}\delta_{fd}\,|a, b\rangle \tag{4.54}$$

$$= \sum_{a,b}\sum_{c,d} Tr_\Lambda \left[\mathcal{L}_{ac}(0)\mathcal{L}_{bd}(0) \right] Tr_\Omega\left[\phi_{cd}(0)\right] |a, b\rangle \tag{4.55}$$

In the fourth equality, we used $e_{ab}\,|c\rangle = \delta_{bc}\,|a\rangle$.

5 The (SU(3),SO(3)) symmetric pair

In this chapter we will work with the integrable state

$$|MPS_k\rangle = \sum_{a_i,b_i=1}^{3} Tr[S_{a_1}S_{b_1}\dots S_{a_{\frac{L}{2}}}S_{b_{\frac{L}{2}}}]\,|a_1,b_1\dots,a_{\frac{L}{2}}b_{\frac{L}{2}}\rangle\,, \tag{5.1}$$

where S_a , $a = 1, 2, 3$ form the $k = 2s + 1$ dimensional spin-s representation of su_2 and the Hilbert space is that of an su_3 spin chain. To argue for the so_3 symmetry of this MPS, we can show that (4.16) is fulfilled for some representation of so_3. Since $so_3 \cong su_2$, we can use the adjoint representation of su_2 to transform the building block S_i. Under the action of the su_2 generator S_k, the building block S_i transforms as

$$S_i \to [S_k, S_i] = i\varepsilon_{k,i,l}S_l. \tag{5.2}$$

Noting that in the fundamental representation of so_3 the components of the k^{th} generator are $[g_k]_{il} = \varepsilon_{kil}$, (5.2) is equivalent to (4.16).

The two-site block from which this state is obtained is

$$\tilde{\chi}^{(s)}_{ab}(u) = -u^2\chi^{(s)}_{ab}(u) \tag{5.3}$$

where

$$\chi^{(s)}_{ab} = \delta_{ab} + u^{-1}[S_a, S_b] - u^{-2}S_aS_b, \tag{5.4}$$

The K-matrix that corresponds to $\chi^{(s)}_{ab}(u)$, denoted $K_\chi(u)$ solves the twisted BYB relation [23] and $\tilde{\chi}^{(s)}_{ab}(u)$ factorizes at $u = 0$ and produces $|MPS_k\rangle$ through (4.26).

Our first goal is to relate this MPS with the so-called δ-state

$$|\Psi_\delta\rangle = \bigotimes_{j=1}^{\frac{L}{2}}(|11\rangle + |22\rangle + |33\rangle). \tag{5.5}$$

As we will later see, this state can be constructed from the twisting matrix of the twisted Yangian $Y^+(3)$.

5.1 Representations of $Y^+(3)$

The algebra $Y^+(3)$ [20] is a subalgebra of $Y(3)$ generated by

$$S(u) = T(u)T^t(u), \tag{5.6}$$

which corresponds to

$$s_{ij}(u) = \sum_{a=-1}^{1} t_{i,a}(u)t_{-j,-a}(-u). \tag{5.7}$$

Following the discussion of section (4.3), these s_{ij} generate a twisted Yangian with $G(u) = I$. In addition to the quaternary relation (4.31), the S-matrix of this algebra also satisfies the symmetry relation

$$S^t(-u) = S(u) + \frac{1}{2u}(S(u) + S(-u)), \tag{5.8}$$

as a consequence of $G(u) = G^t(u)$. As explained previously, $Y^+(3)$ representations are based on gl_3 representations.

gl_3 Highest weight representations

The algebra is generated by E_{ij}, with $i, j \in \{-1, 0, 1\}$, subject to the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} \tag{5.9}$$

Given a highest weight representation of $gl(3)$ with highest weights $\lambda_1, \lambda_2, \lambda_3$, we denote the corresponding module as $L(\lambda_1, \lambda_2, \lambda_3)$. The highest weight vector is an element of that vector space, such that

$$E_{ij} |\lambda_1, \lambda_2, \lambda_3\rangle = 0, \text{ for } i < j \tag{5.10}$$

$$E_{ii} |\lambda_1, \lambda_2, \lambda_3\rangle = \lambda_{i+2} |\lambda_1, \lambda_2, \lambda_3\rangle \tag{5.11}$$

The module is finite dimensional iff $\lambda_1 - \lambda_2 \in \mathbb{N}$ and $\lambda_2 - \lambda_3 \in \mathbb{N}$

$Y(3)$ Evaluation representations

Recall that the evaluation homomorphism maps the generators of $Y(3)$ to the generators of gl_3:

$$t_{ij}(u) \longrightarrow \delta_{ij} + u^{-1} E_{ij} \tag{5.12}$$

Thanks to this mapping, the elements of Y(3) can act on the gl_3-module $L(\lambda_1, \lambda_2, \lambda_3)$. The gl_3 module carries a $Y(3)$ highest weight representation and the two representations share the highest weight vector. That is,

$$t_{ij}(u) |\lambda_1, \lambda_2, \lambda_3\rangle = 0, \text{ for } i < j \tag{5.13}$$

$$t_{ii}(u) |\lambda_1, \lambda_2, \lambda_3\rangle = \lambda_i(u) |\lambda_1, \lambda_2, \lambda_3\rangle \tag{5.14}$$

From the evaluation map, it is obvious that the $Y(3)$ highest weights are

$$\lambda_i(u) = 1 + u^{-1} \lambda_i. \tag{5.15}$$

The matrix representing $t_{ij}(u)$ will be denoted as $\mathcal{L}_{ij}^{(\lambda_1, \lambda_2 \lambda_3)}(u)$.

$Y^+(3)$ Representations

A highest weight module of $Y^+(3)$, denoted as V, is generated by a vector $v \in V$, such that

$$s_{ij}(u) \cdot v = 0, \text{ for } i < j \tag{5.16}$$

$$s_{ij}(u) \cdot v = \mu_i(u) v, \text{ for } i = 0, 1 \tag{5.17}$$

Since $Y^+(3)$ is a subalgebra of $Y(3)$, $L(\lambda_1, \lambda_2, \lambda_3)$ can also be treated as a $Y^+(3)$ module. In particular, it is a highest weight module of $Y^+(3)$ [20], the highest weights of can be calculated using (5.7) and (5.15). They are

$$\mu_1(u) = (1 + \lambda_2 u^{-1})(1 - \lambda_1 u^{-1}) \tag{5.18}$$

$$\mu_0(u) = (1 + \lambda_1 u^{-1})(1 - \lambda_1 u^{-1}). \tag{5.19}$$

It is interesting to note another way of constructing $Y^+(3)$ representations. Using the definition of the $Y(3)$ co-product, it is straightforward to show that

$$\Delta(s_{i,j}) = \sum_{a,b} t_{i,a}(u)t_{-j,-b}(-u) \otimes s_{a,b}(u) \in Y(3) \otimes Y^+(3), \tag{5.20}$$

which means that $Y^+(3)$ is a left co-ideal subalgebra of $Y(3)$. Given a $Y(3)$ module L and a $Y^+(3)$ module V, this property allows us to define an action of $Y^+(3)$ on $L \otimes V$ as

$$s \cdot (w \otimes v) = \Delta(s)(w \otimes v) \tag{5.21}$$

where $s \in Y^+(3)$, $w \in L$ and $v \in V$. Thus, $L \otimes V$ can carry a $Y^+(3)$ representation with highest weights

$$\mu_i^{L\otimes V}(u) = \mu_i^L(u)\mu_i^V(u) \tag{5.22}$$

so_3 Highest weight representations
Let us also introduce so_3 representations, which are relevant to our discussion due to the two-site block being expressed in terms of $su_2 \cong so_3$ generators. We denote the generators as $F_{i,j}$ with $i, j \in \{-1, 0, 1\}$ and defining relations

$$[F_{ij}, F_{kl}] = \delta_{jk}F_{il} - \delta_{il}F_{kj} + \delta_{j,-l}F_{k,-i} - \delta_{i,-k}F_{j,-l} \tag{5.23}$$

$$F_{-i,-j} = -F_{i,j} \tag{5.24}$$

For the highest weight w of an so_3 representation, we have

$$F_{ij} \cdot w = 0, i < j \tag{5.25}$$

$$F_{11} \cdot w = \lambda w \tag{5.26}$$

It is straightforward to use (5.9) to check that if we define

$$F_{i,j} = E_{i,j} - E_{-j,-i}, \tag{5.27}$$

the defining relations (5.23-24) are fulfilled. This relation constitutes an embedding of so_3 into gl_3 and defines an action of so_3 in a gl_3 module. Note that an embedding is generally not unique. That is, there might be other ways to define F_{ij} in terms of the E_{ij}, such that (5.23-24) is fulfilled. With our choice of embedding, for $|\alpha_1, \alpha_2, \alpha_3\rangle \in L(\lambda_1, \lambda_2, \lambda_3)$ (which is not necessarily the highest weight vector), such that

$$E_{ii} |\alpha_1, \alpha_2, \alpha_3\rangle = \alpha_i |\alpha_1, \alpha_2, \alpha_3\rangle, \tag{5.28}$$

we have

$$-F_{11} |\alpha_1, \alpha_2, \alpha_3\rangle = (\alpha_1 - \alpha_3) |\alpha_1, \alpha_2, \alpha_3\rangle. \tag{5.29}$$

The Cartan subalgebra of so_3 is generated by F_{11} and the F_{01}, F_{10} act as ladder operators, which motivates the identification

$$S_z = -F_{11} \tag{5.30}$$

$$S_+ = F_{01} \tag{5.31}$$

$$S_- = F_{10} \tag{5.32}$$

5.2 Relating $|MPS_k\rangle$ to $|\Psi_\delta\rangle$

The strategy

The starting point is to transform K_χ and the matrix that produces the δ-state, denoted K_δ, into S-matrices that form $Y^+(3)$ representations. The K-matrices are solutions of (4.22) and the S-matrices need to fulfil (4.31), thus the transformation can be achieved as described in appendix A. Then, we aim to find a relation between the S-matrix that corresponds to K_χ and a dressing of the S-matrix that corresponds to K_δ, denoted S_δ. To create dressed representations, S_δ needs to be the twist matrix that defines $Y^+(3)$. This can be achieved by rewriting the δ-state a different basis of the Hilbert space,

$$|+\rangle = \frac{1}{\sqrt{2}}(i\,|1\rangle + |2\rangle) \tag{5.33}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(-i\,|1\rangle + |2\rangle) \tag{5.34}$$

$$|0\rangle = |3\rangle\,, \tag{5.35}$$

obtained from the standard one by the transformation matrix

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ -i & 1 & 0 \end{pmatrix}. \tag{5.36}$$

In this basis the δ-state is

$$|\Psi_\delta\rangle = (\bigotimes_{j=1}^{\frac{L}{2}}(|+-\rangle + |-+\rangle + |00\rangle) = \sum_{i,j\in\{+,-,0\}} \delta_{i_1,-j_1}\dots\delta_{i_{\frac{L}{2}},-j_{\frac{L}{2}}}\,|i_1,j_1,\dots i_{\frac{L}{2}},j_{\frac{L}{2}}\rangle \tag{5.37}$$

and it is clear that the two-site block should satisfy the initial condition $\chi^{(\delta)}_{i,j}(0) = \delta_{i,-j}$. The most simple choice of K-matrix is the charge conjugation matrix

$$K_\delta = C \tag{5.38}$$

A corresponding $Y^+(3)$ representation is obtained as $S_\delta = K_\delta C$, or

$$S_\delta = I, \tag{5.39}$$

which is the matrix we aimed to create.

From K_χ to an S-matrix

The basis transformation defined above provides with a natural way of producing a $Y^+(3)$ representation from K_χ, by defining

$$S(u) = PK_\chi(u)P^\dagger, \tag{5.40}$$

This P- matrix fulfils the constraints (A.9-A.10) and it is straightforward to check that $S(u)$ fulfils the symmetry relation (5.8).[10] Therefore, the components of this $S(u)$, denoted $\psi^{(s)}_{ij}(u)$, form a $Y^+(3)$ representation We could, however, have created a representation by using any other matrix that fulfils the constraints. An argument

[10] This is not a result of the choice of P. Any choice of matrix would lead to an $S(u)$ that fulfils the symmetry relation, due to the two-sit e block fulfilling the relation $\phi_{ab}(-u) = \phi_{ab}(u) + \frac{1}{2u}(\phi_{ab}(u) - \phi_{ab}(-u))$

for this specific choice can be made after calculating the new two-site block, which turns out to be

$$\psi^{(s)}_{1,1}(u) = 1 - u^{-1}S_z - \frac{1}{2}u^{-2}(s(s+1) - S_z(S_z+1)) \tag{5.41}$$
$$\psi^{(s)}_{1,0}(u) = iu^{-1}S_- - u^{-2}iS_-S_z \tag{5.42}$$
$$\psi^{(s)}_{1,-1}(u) = u^{-2}S_-^2 \tag{5.43}$$
$$\psi^{(s)}_{0,1}(u) = -iu^{-1}S_+ + iu^{-2}S_zS_+ \tag{5.44}$$
$$\psi^{(s)}_{0,0}(u) = 1 - u^{-2}S_z^2 \tag{5.45}$$
$$\psi^{(s)}_{0,-1}(u) = -iu^{-1}S_- - u^{-2}iS_zS_- \tag{5.46}$$
$$\psi^{(s)}_{-1,1}(u) = u^{-2}S_+^2 \tag{5.47}$$
$$\psi^{(s)}_{-1,0}(u) = iu^{-1}S_+ + u^{-2}iS_+S_z \tag{5.48}$$
$$\psi^{(s)}_{-1,-1}(u) = 1 + u^{-1}S_z - \frac{1}{2}u^{-2}(s(s+1) - S_z(S_z+1)) \tag{5.49}$$

These $\psi^{(s)}_{i,j}(u)$ act on the highest weight state of a spin-s so_3 module as would be expected of the $Y^+(3)$ generators to act on the highest weight of a $Y^+(3)$ module. Namely, $\psi^{(s)}_{i,i}(u)$ give the highest weights (they are the equivalent of the Cartan generators of a Lie algebra) while $\psi_{i,j}$ with $i < j$ annihilate the highest weight state. This observation should justify the choice of P in (5.40): Out of the various $Y^+(3)$ representations that are created by all possible transformation matrices, this is the one which is consistent with the definition of a highest weight representation on a gl_3 module. In other words, a different transformation might lead to $\psi_{ij}(u)$ that do not fulfil (5.16-17).

Embedding V(s) in a gl_3 module

The $Y^+(3)$ highest weights on the so_3 module can be calculated from (5.45) and (5.41):

$$\mu_1(u) = (1 - u^{-1}s) \tag{5.50}$$
$$\mu_0(u) = (1 - u^{-1}s)(1 + u^{-1}s) \tag{5.51}$$

Comparing them with (5.18) and (5.19), they can be immediately identified with the highest weights of $Y^+(3)$ on the gl_3 module $L(s,s,0)$. Since $V(s)$ and $L(s,s,0)$ carry representations with the same $Y^+(3)$ highest weight, we could assume that they are they are isomorphic vector spaces. However, the dimension of V is generally lower than that of L. This implies that V can instead be embedded within L, i.e. $V(s) \subseteq L(s,s,0)$. Note that $L(s,s,0)$ is finite dimensional only if s is integer or, equivalently, if $k = 2s+1$ is odd. In what follows, we restrict ourselves to this case.[11] We then have

$$L(s,s,0) = V(s) \oplus W, \tag{5.52}$$

where W consists of the vectors in $L(s,s,0)$ that are not in $V(s)$. The action of the twisted Yangian on $y \in L(s,s,0)$ can now be decomposed as

$$s_{ij}(u) \cdot y = \begin{pmatrix} \psi^{(s)}_{i,j}(u) & X \\ 0 & Y \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \tag{5.53}$$

[11] For even k, $V(s)$ can be embedded in a $Y^+(3)$ module of the form $V(1/2) \otimes L(a,b,c)$.

where $v \in V(s)$ and $w \in W$. The reason for the (2,1) block being zero is that $V(s)$ is a $Y^+(3)$-module and, as such, must be closed under the action of $s_{i,j}(u)$, which means

$$y \in V(s) \Rightarrow s_{i,j}(u) \cdot y \in V(s). \tag{5.54}$$

Assuming that there is something non vanishing in that position, we get

$$\begin{pmatrix} \psi_{i,j}^{(s)}(u) & X \\ Z & Y \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_{i,j}^{(s)}(u) \cdot v \\ Z \cdot v \end{pmatrix}, \tag{5.55}$$

which is in $V(s)$ only if

$$Z = 0. \tag{5.56}$$

It can also be argued that we are not interested in the form of the X block: Recall that, in the end, the MPSs are obtained by tracing over the auxiliary space. Thus, X does not enter the relations between the states, and can be ignored.

Calculations for odd $k = 2s + 1$

For $s = 1$, both $V(1)$ and $L(1, 1, 0)$ have dimension 3 and have the same highest weight, so we can immediately identify $V(1) = L(1, 1, 0)$

The first non-trivial case is for $s = 2$. The dimension of $L(2, 2, 0)$ is 6, while the dimension of $V(2)$ is 5. We thus need to identify $V(s)$ with a subspace of $L(2, 2, 0)$. In other words, we need to find a 2-dimensional subspace of $L(2, 2, 0)$ which has a 1-dimensional intersection with $V(2)$. The $L(2, 2, 0)$ module is spanned by the following vectors.

$$|2, 0, 0\rangle \tag{5.57}$$
$$|2, 2, 1\rangle = E_{3,2}\,|2, 2, 0\rangle \tag{5.58}$$
$$|2, 0, 0\rangle = E_{3,2}^2\,|2, 2, 0\rangle \tag{5.59}$$
$$|1, 2, 1\rangle = E_{2,3}E_{3,2}\,|2, 2, 0\rangle \tag{5.60}$$
$$|1, 1, 2\rangle = E_{2,1}E_{3,2}^2\,|2, 2, 0\rangle \tag{5.61}$$
$$|2, 0, 0\rangle = E_{2,1}^2E_{3,2}^2\,|2, 2, 0\rangle \tag{5.62}$$

Since $V(s)$ is a $Y^+(3)$ module, its basis vectors should be distinguished by their $Y^+(3)$ highest weights. From the explicit form of $\psi_{i,i}$ it is obvious that states with the same so_3 weight also have the same $Y^+(3)$ weight. Using the embedding of so_3 into gl_3 (5.27) and (5.30), we can calculate the S_z eigenvalue for each of the basis vectors, shown in the table.

S_z		
2	$\|2, 2, 0\rangle$	
1	$\|2, 1, 1\rangle$	
0	$\|2, 0, 2\rangle$	$\|1, 2, 3\rangle$
-1	$\|1, 1, 2\rangle$	
-2	$\|0, 2, 2\rangle$	

One can now see that $V(s) = \text{span}\{|2,2,0\rangle, |2,1,1\rangle, |1,2,1\rangle + a|2,0,2\rangle, |1,1,2\rangle, |0,2,2\rangle\}$. In order to determine a, we require that acting on the highest weight with the $Y^+(3)$ lowering generator $s_{1,-1}$ gives a vector in $V(s)$.

$$s_{1,-1}(u)|2,2,0\rangle \tag{5.63}$$
$$= [t_{1,-1}(u)t_{1,1}(-u) + t_{1,0}(u)t_{1,0}(-u) + t_{1,1}(u)t_{1,-1}(-u)]|2,2,0\rangle \tag{5.64}$$
$$= [u^{-1}E_{1,-1}(1 - u^{-1}E_{11}) - u^{-1}E_{10}^2 + (1 + u^{-1}E_{11})(-u^{-1})E_{1,-1}]|2,2,0\rangle \tag{5.65}$$
$$= u^{-1}|1,2,1\rangle - u^{-2}|2,0,2\rangle + u^{-1}|1,2,1\rangle + u^{-2}|1,2,1\rangle \tag{5.66}$$
$$= u^{-2}(|1,2,1\rangle - |2,0,2\rangle) \tag{5.67}$$

Here, we used the evaluation map, the properties of the highest weight states and $E_{1,-1}|2,2,0\rangle = -|1,2,1\rangle$. The latter can be seen from $[E_{0,-1}, E_{1,0}] = -E_{1,-1}$. We can now conclude that

$$V(s) = \text{span}\{|2,2,0\rangle, |2,1,1\rangle, u^{-2}(|1,2,1\rangle - |2,0,2\rangle), |1,1,2\rangle, |0,2,2\rangle\} \tag{5.68}$$

and

$$L(2,2,0) = V(s) \oplus W, \tag{5.69}$$

where $W = span\{|1,2,1\rangle + |2,0,2\rangle\}$. In order to calculate the Y block in (5.35), we need to calculate the action of the diagonal generators on W, that is $s_{00}(u)(|1,2,1\rangle + |2,0,2\rangle)$ and $s_{11}(u)(|1,2,1\rangle + |2,0,2\rangle)$. We have

$$s_{00}(u) = t_{0,-1}(u)t_{0,1}(-u) + t_{0,0}(u)t_{0,0}(u) + t_{0,1}(u)t_{0,-1}(u) \tag{5.70}$$

and using the evaluation representation

$$s_{0,0}(u)(|1,2,1\rangle + |2,0,2\rangle) = \tag{5.71}$$
$$(-u^{-2}E_{0,-1}E_{0,1} + (1 - u^{-2}E_{0,0}^2) - u^{-2}E_{0,1}E_{0,-1})(|1,2,1\rangle + |2,0,2\rangle) \tag{5.72}$$

Noticing that both $E_{0,1}$ and $E_{0,-1}$ annihilate $|1,2,1\rangle$, it is easy to see that

$$s_{0,0}(u)|1,2,1\rangle = (1 - 4u^{-2})|1,2,1\rangle. \tag{5.73}$$

To calculate $s_{00}|2,0,2\rangle$, first note that $[E_{0,-1}, E_{0,1}] = 0$, which implies

$$s_{00}(u)|2,0,2\rangle = (-2u^{-2}E_{0,-1}E_{0,1} + (1 - u^{-2}E_{0,0}^2))|2,0,2\rangle \tag{5.74}$$

The second term can be immediately seen to be

$$(1 - u^{-2}E_{0,0}^2)|2,0,2\rangle = |2,0,2\rangle \tag{5.75}$$

One can then use the commutation relations of gl_3 to show that

$$E_{0,-1}E_{0,1}|2,0,2\rangle = E_{0,-1}E_{0,1}E_{1,0}E_{1,0}|2,2,0\rangle = 2|1,2,1\rangle. \tag{5.76}$$

Putting everything together, we obtain

$$s_{0,0}(u)(|2,0,2\rangle + |1,2,1\rangle) = |2,0,2\rangle - 4u^{-2}|1,2,1\rangle + (1-4u^{-2})|1,2,1\rangle \tag{5.77}$$
$$= (1-4u^{-2})(|2,0,2\rangle + |1,2,1\rangle) + 4u^{-2}(|2,0,2\rangle - |1,2,1\rangle) \tag{5.78}$$

Similarly, one can show that

$$s_{1,1}(u)(|2,0,2\rangle + |1,2,1\rangle) = \tag{5.79}$$
$$(1-4u^{-2})(|2,0,2\rangle + |1,2,1\rangle) - u^{-2}(|2,0,2\rangle - |1,2,1\rangle) \tag{5.80}$$

We see that the action of the $Y^+(3)$ on W leads to a component in W and a component in $V(2)$ (not in W). These components correspond to the action of Y and X respectively. As discussed earlier, X will eventually get traced out. Focusing on the action of Y, we see that it is identical to the action of $Y^+(3)$ on the 1-dimensional gl_3 module $L(2,2,2)$. We thus have

$$L(2,2,0) \sim V(2) \oplus L(2,2,2) \tag{5.81}$$

It should be stressed that this relation is not an equation (or isomorphism), since W and $L(2,2,2)$ are clearly not the same vector space. However, the MPSs generated from $L(2,2,0)$ and $V(2) \oplus L(2,2,2)$ is the same. In other words, we can replace W with $L(2,2,2)$ without altering the trace of the block matrix in (5.35). The advantage of using the gl_3 module is that it carries a $Y^+(3)$ highest weight representation, while W does not (it is not closed under the action of the algebra). This $Y^+(3)$ representation generates an integrable MPS and the equivalence (5.81) allows us to relate the states generated from the three representations involved. Similar calculations presented in [15] lead to the conjecture

$$L(s,s,0) \sim V(s) \oplus L(s,s,2) \tag{5.82}$$

From twisted Yangian representations MPSs

In what follows, we change the normalisation of the R-matrix to

$$\tilde{R}(u) = uI_{12} + i\mathcal{P}_{12} \tag{5.83}$$

which rescales the Yangian generators as

$$t_{ij}(u) \to u t_{ij}(iu) \tag{5.84}$$
$$s_{ij}(u) \to u^2 s_{ij}(iu) \tag{5.85}$$

We do that so that the two-site block is non-singular at $u = 0$, where it factorizes.

For $v \in L(s,s,0)$, $w_1 \in V(s)$ and $w_2 \in L(s,s,2)$, the result of the previous section means that we can decompose the action of $Y^+(3)$ as

$$\mathcal{L}^{(s,s,0)}_{i,a}(u)\mathcal{L}^{(s,s,0)}_{-j,-a}(-u) \cdot v \sim \begin{pmatrix} \psi^{(s)}_{i,j}(u) & X \\ 0 & \mathcal{L}^{(s,s,2)}_{i,a}(u)\mathcal{L}^{(s,s,2)}_{-j,-a}(-u) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2, \end{pmatrix} \tag{5.86}$$

where summation over a is implied.

In order to build the MPSs, we need to transform the $Y^+(3)$ representations into elements some K-matrix.

Using (A.5) this leads to the following two-site blocks

$$\mathcal{L}^{(s,s,m)}_{i,a}(u)\mathcal{L}^{(s,s,m)}_{-j,-a}(-u) \to \phi^{(s,s,m)}_{i,j}(u) = \mathcal{L}^{(s,s,m)}_{i,a}(u)\mathcal{L}^{(s,s,m)}_{j,-a}(-u) \tag{5.87}$$

$$\psi^{(s)}_{i,j}(u) \to \phi^{(s)}_{i,j}(u) = \psi^{(s)}_{i,-j}(u), \tag{5.88}$$

where m=1 or 2. We can also rewrite

$$\phi^{(s,s,m)}_{i,j}(u) = \mathcal{L}^{(s,s,m)}_{i,a}(u)\mathcal{L}^{(s,s,m)}_{j,b}(-u)\delta_{a,-b} \tag{5.89}$$

to emphasize that this two-site block is a dressing of the two-site block of $K_\delta = C$. Then, according to the result of section 4.3, the state generated from $\phi^{(s,s,m)}_{i,j}(u)$ is

$$\sum_{i_k,j_k} Tr[\phi^{(s,s,m)}_{i_1,j_1}(0),\ldots,\phi^{(s,s,m)}_{i_{\frac{L}{2}},j_{\frac{L}{2}}}(0)]\,|i_1,j_1,\ldots,i_{\frac{L}{2}},j_{\frac{L}{2}}\rangle = \hat{t}^{(s,s,m)}(0)\,|\Psi_\delta\rangle \tag{5.90}$$

where $\hat{t}(u)$ denotes a transfer matrix acting on the sate.

Let us now verify that $\phi^{(s)}_{ij}(u)$ generates the same MPS as $\chi^{(s)}_{ij}(u)$. We have

$$K_\phi(u) = S(u)C = PK_\chi(u)P^{-1}C. \tag{5.91}$$

For the two-site blocks involved, this corresponds to

$$\phi^{(s)}_{ij}(u) = \psi^{(s)}_{i,-j}(u) = \sum_{a,b\in\{1,2,3\}} P_{ia}\chi^{(s)}_{ab}(u)P^{-1}_{b,-j} \tag{5.92}$$

We can now show that $\phi^{(s)}_{ij}(u)$ generates the original MPS, expressed in the new basis (5.33)-(5.35). We start from

$$\sum_{a,b\in\{1,2,3\}} \chi^{(s)}_{ab}(0)\,|a,b\rangle\,, \tag{5.93}$$

which would give us the two-site version of (5.1) after taking the trace. We now simultaneously transform the basis and express the χ block in terms of ψ

$$\sum_{a,b=1}^{3} \chi^{(s)}_{ab}(0)\,|a,b\rangle = \sum_{a,b=1}^{3}\sum_{i,j=-1}^{1}\sum_{k,l=-1}^{1} P^{-1}_{ai}\psi^{(s)}_{ij}(0)P_{jb}P^{-1}_{ak}\,|k\rangle\,P^{-1}_{bl}\,|l\rangle \tag{5.94}$$

$$= \sum_{a=1}^{3}\sum_{i,j=-1}^{1}\sum_{k,l=-1}^{1} P^{-1}_{ai}P^{-1}_{ak}\psi^{(s)}_{ij}(0)\delta_{jl}\,|k,l\rangle \tag{5.95}$$

Making use of $PP^\dagger = I \Rightarrow P^{-1} = (P^*)^T$, we can rewrite

$$\sum_a P^{-1}_{ai}P^{-1}_{ak} = \sum_a P^*_{ia}P^{-1}_{ak} = \delta_{i,-k}. \tag{5.96}$$

The last equality holds due to $PP^T = C \Rightarrow P^* P^{-1} = C$. We thus obtain

$$\sum_{a,b=1}^{3} \chi_{ab}^{(s)}(0) = \sum_{i,j=-1}^{1} \sum_{k,l=-1}^{1} \psi_{i,j}^{(s)}(0) \delta_{jl} \delta_{i,-k} |k, l\rangle \tag{5.97}$$

$$= \sum_{i,j=-1}^{1} \psi_{-i,j}^{(s)}(0) |i, j\rangle \tag{5.98}$$

Therefore, we can rewrite the MPS as

$$|MPS_k\rangle = \sum_{i_k, j_k \in \{-1,0,1\}} Tr \left[\psi_{-i_1,j_1}^{(s)}(0) \dots \psi_{-i_{\frac{L}{2}}, j_{\frac{L}{2}}}^{(s)}(0) \right] |i_1, j_1, \dots i_{\frac{L}{2}}, j_{\frac{L}{2}}\rangle \tag{5.99}$$

Next, from (5.41)-(5.49) and after the rescaling (5.85) we can see that

$$\psi_{i,j}^{(s)}(0) = -\tilde{S}_{-i} \tilde{S}_j, \tag{5.100}$$

where

$$\tilde{S}_{+1} = -iS_+ \tag{5.101}$$

$$\tilde{S}_0 = S_z \tag{5.102}$$

$$\tilde{S}_{-1} = iS_- \tag{5.103}$$

The $\tilde{S}_i$ are a simple redefinition of the Cartan generators of su_2. Then, under the su_2 automorphism

$$\tilde{S}_i \to V \tilde{S}_i V^{-1} = -\tilde{S}_{-i} \tag{5.104}$$

the two-site block transforms as

$$\psi_{ij}^{(s)}(0) \to V \psi_{i,j}^{(s)}(0) V^{-1} = \psi_{-i,-j}^{(s)}(0) \tag{5.105}$$

and we can rewrite

$$\psi_{-i,j}^{(s)}(0) = V^{-1} \psi_{i,-j}^{(s)}(0) V. \tag{5.106}$$

Due to the cyclicity of the trace, this allows us to rewrite (5.99) as

$$|MPS_k\rangle = \sum_{i_k, j_k} Tr \left[\psi_{i_1,-j_1}^{(s)}(0) \dots \psi_{i_{\frac{L}{2}}, -j_{\frac{L}{2}}}^{(s)}(0) \right] |i_1, j_1, \dots i_{\frac{L}{2}}, j_{\frac{L}{2}}\rangle \tag{5.107}$$

$$= \sum_{i_k, j_k} Tr \left[\phi_{i_1,j_1}^{(s)}(0) \dots \phi_{i_{\frac{L}{2}}, j_{\frac{L}{2}}}^{(s)}(0) \right] |i_1, j_1, \dots i_{\frac{L}{2}}, j_{\frac{L}{2}}\rangle \tag{5.108}$$

The relation between the states

The equivalent of (5.86) for the ϕ is

$$\phi_{i,j}^{(s,s,0)}(u) \sim \begin{pmatrix} \phi_{i,j}^{(s)}(u) & \tilde{X} \\ 0 & \phi_{i,j}^{(s,s,2)}(u) \end{pmatrix} \tag{5.109}$$

from which it follows that

$$\phi^{(s,s,0)}_{i_1,j_1}(u)\cdots\phi^{(s,s,0)}_{i_{\frac{L}{2}},j_{\frac{L}{2}}}(u)\sim\begin{pmatrix}\phi^{(s)}_{i_1,j_1}(u)\cdots\phi^{(s)}_{i_{\frac{L}{2}},j_{\frac{L}{2}}}(u) & \tilde{X}' \\ 0 & \phi^{(s,s,2)}_{i_1,j_1}(u)\cdots\phi^{(s,s,2)}_{i_{\frac{L}{2}},j_{\frac{L}{2}}}(u)\end{pmatrix} \tag{5.110}$$

Using the property $Tr_{V(s)\oplus L(s,s,2)}[\cdot]=Tr_{V(s)}[\cdot]+Tr_{L(s,s,2)}[\cdot]$ of the trace over the auxiliary space and (5.108), (5.90), we obtain

$$\hat{t}^{(s,s,0)}(0)\,|\Psi_\delta\rangle=|MPS_k\rangle+\hat{t}^{(s,s,2)}(0)\,|\Psi_\delta\rangle \tag{5.111}$$

or

$$|MPS_k\rangle=(\hat{t}^{(s,s,0)}(0)-\hat{t}^{(s,s,2)}(0))\,|\Psi_\delta\rangle \tag{5.112}$$

This is result we wanted, but we can take a few extra steps to replace the gl_3 representations with equivalent ones that simplify the calculations in the next part. The mapping

$$E_{i,j}\to -E_{j,i} \tag{5.113}$$

is an automorphism of gl_3. After using this mapping to redefine the generators, the vector space $L(s,s,m)$ still carries a representation of the algebra. To determine which representation it is note that, since $E_{ii}\to -E_{ii}$, the weights of each vector will change sign. Also, swapping the order of the indices means that we exchange the raising and lowering generators. This has the effect of reversing the role of the highest and lowest weight vectors. The lowest weight vector of our (s,s,m) representations is $|m,s,s\rangle$ and overall, our original (s,s,m) module is the $(-m,-s,-s)$ module of the transformed algebra. Under this automorphism, the Lax operators defined in (4.44) transform as

$$L^{(s,s,m)}(u)=\mathcal{L}^{(s,s,m)}(u)\otimes e_{ij}=(uI-iE^{(s,s,m)}_{i,j})\otimes e_{i,j}=(uI+iE^{(-m,-s,-s)}_{i,j})\otimes e_{j,i}, \tag{5.114}$$

where $E^{(s,s,m)}_{i,j}$ is the matrix representing $E_{i,j}$ in the specified representation. Another trick we can use is that we can rewrite these matrices as

$$E^{(-m,-s,-s)}_{i,j}=-sI+E^{(s-m,0,0)}_{i,j}. \tag{5.115}$$

After defining

$$L^{(s)}(u)=(u-i\frac{s-1}{2})I+iE^{(s,0,0)}_{i,j}\otimes e_{j,i} \tag{5.116}$$

we can then rewrite the Lax operators as

$$L^{(s,s,0)}(u)=L^{(s)}(u-i\frac{s+1}{2}) \tag{5.117}$$

$$L^{(s,s,2)}(u)=L^{(s-2)}(u-i\frac{s+3}{2}) \tag{5.118}$$

Finally, (5.112) becomes

$$|MPS_s\rangle=\left(\hat{t}^{(s)}\left(-i\frac{s+1}{2}\right)-\hat{t}^{(s)}\left(-i\frac{s+1}{2}\right)\right)|\Psi_\delta\rangle \tag{5.119}$$

5.3 Calculation of overlap ratios

The tableaux sum formula

Having related the matrix product states $|MPS_k\rangle$ to the more simple state $|\Psi_\delta\rangle$ the computation of the overlap $\langle\{u_i\}|MPS_s\rangle$ reduces to computing the eigenvalues of the transfer matrices. In particular, since the Bethe states are by construction eigenstates of the transfer matrix, we have

$$\frac{\langle\{u_i\}|MPS_s\rangle}{\langle\{u_i\}|\Psi_\delta\rangle} = t^{(s)}\left(-i\frac{s+1}{2}\right) - t^{(s)}\left(-i\frac{s+1}{2}\right) \tag{5.120}$$

The transfer matrix eigenvalues can be calculated using the *tableaux sum* formula, which can be found in [7], and gives the eigenvalues of transfer matrices whose auxiliary space is in an SU(N) representation that corresponds to a rectangular Young diagram. Given rectangular a Young diagram with a rows and s columns, the formula reads

$$t_m^{(a)}(u) = \sum_\tau \left[\prod_{k=1\ldots,a}\prod_{l=1,\ldots,m} z^{(\tau_{kl})}\left(u + i\frac{a-2k+1}{2} - i\frac{m-2l+1}{2}\right)\right] \tag{5.121}$$

The summation is over semistandard Young tableaux and τ_{ij} is the number inserted in the (i,j) position in a given tableau. The z-functions are such that the eigenvalue of the fundamental transfer matrix $t_1^1(u)$ is

$$t_1^{(1)}(u) = \sum_{l=1}^{N} z^{(l)}(u) \tag{5.122}$$

We will shortly determine these z-functions.

The tableaux sum formula might seem incompatible with our results, since deals with sl_N representations, while in the context of Yangians we use representations of the gl_N generators. We can get around that using the fact that any gl_N representation can be restricted to an sl_N one. From now on, we will be putting the gl_N highest weights in square brackets and sl_N weights in ordinary brackets, to avoid confusion. Let us choose an embedding of sl_N into g_N such that the Cartan generators are

$$E_{ii}^{sl_N} = E_{ii}^{gl_N} - E_{i+1,i+1}^{gl_N}. \tag{5.123}$$

It can be immediately seen that, using this convention for the embedding, the $[\lambda_1,\ldots,\lambda_N]$ representation of gl_N contains the $(\lambda_1-\lambda_2,\ldots,\lambda_{N-1}-\lambda_N)$ representation of sl_N[12].

The form of the z-functions can now be recovered from eq. (3.51). Recall that this equation gives the eigenvalues of a transfer matrix with the auxiliary space in the fundamental representation and the physical space in some other representation. In the present section we have been dealing with the opposite case, namely the physical space is in the fundamental representation but the auxiliary might not. In the special case of the fundamental transfer matrix, where both spaces are in the fundamental representation, these two cases are identical. Thus, plugging the highest weights of the fundamental representation of gl_N in (3.51) is equivalent to (5122). In that case, we can identify each term in (3.51) as one of the z-functions. In order to be consistent with [7] we need to change the normalization of the transfer matrix by defining

$$z^{(l)}(u) = \frac{1}{Q_0(u+i)} D_l(u) P_l(u), \tag{5.124}$$

[12] To be concrete, we should also define the off-diagonal generators $E_{i,j}^{sl_N}$ in terms of the $E_{i,j}^{gl_N}$ such that they behave as raising and lowering generators. However, we are not going to use them, so that is not necessary here.

where $Q_0(u)$ is one of the Baxter polynomials, here defined as

$$Q_0(u) = u^L \tag{5.125}$$

$$Q_l(u) = \prod_{n=1}^{M^{(l)}} (u - u_n^{(l)}) \text{ , for } 1 \leq l \leq N-1 \tag{5.126}$$

$$Q_N(u) = 1 \tag{5.127}$$

We can use these $Q_0(u)$ to rewrite the Drinfeld polynomials as

$$P_1(u) = \prod_{n=1}^{L} (u+i) = Q_0(u+i) \tag{5.128}$$

$$P_k(u) = u^N = Q_0(u) \text{ , } k \geq 2, \tag{5.129}$$

where we have set all impurities $\theta_n = 0$ and have substituted the highest weight vector $\alpha = (1, 0, \ldots, 0)$. The dressing functions can also be written as

$$D_1(u) = \frac{Q_1(u - \frac{i}{2})}{Q_1(u + \frac{i}{2})} \tag{5.130}$$

$$D_l(u) = \frac{Q_{l-1}(\lambda + \frac{i(l+1)}{2}) Q_l(\lambda + \frac{i(l-2)}{2})}{Q_{l-1}(\lambda + \frac{i(l-1)}{2}) Q_l(\lambda + \frac{i(l-2)}{2})} \text{ , for } 0 \leq l \leq N \tag{5.131}$$

and the z-functions as

$$z^{(l)}(u) = \frac{Q_0(u)}{Q_0(u+i)} \frac{Q_{l-1}(u + i\frac{l+1}{2}) Q_l(u + i\frac{l-2}{2})}{Q_{l-1}(u + i\frac{l-1}{2}) Q_l(u + i\frac{l}{2})} \tag{5.132}$$

These z-functions are identical to the ones presented in [7].

Calculation of the overlap ratios

In our particular case, the $[s, 0, 0]$ module contains $(s, 0)$. The latter corresponds to a Young diagram with a single row of s boxes, thus the eigenvalue $t^{(s)}(u)$ is $t_s^{(1)}$. For such Young diagrams, the tableaux sum formula for gl_3 is simplified and can be found in [7]. In order to be consistent with [15], we change, yet again, the normalization of the eigenvalues and redefine the Baxter polynomials as

$$Q_0(u) = \prod_{i=1}^{N_0} (iu - u_i) \tag{5.133}$$

$$Q_+(u) = \prod_{j=1}^{N_+} (iu - v_i) \tag{5.134}$$

that respectively correspond to the first and second simple roots of sl_3. Note that as a consequence of the integrability conditions discussed in section 4, N_0 is restricted to being even and $N_+ = N_0/2$. In what follows we assume that N_+ is also even. The tableau sum formula reads

$$t^{(s)}(u) = Q_0(-iu - \frac{s}{2})Q_+(-iu + \frac{s+3}{2}) \sum_{k=0}^{s} \frac{(u + i\frac{s+1}{2} - ik)^L Q_+(-iu + \frac{s+1}{2} - k)}{Q_0(-iu + \frac{s}{2} - k)Q_0(-iu + \frac{s+2}{2} - k)} \times \sum_{l=0}^{k} \frac{Q_0(-iu + \frac{s+2}{2} - l)}{Q_+(-iu + \frac{s+1}{2} - l)Q_+(-iu + \frac{s+3}{2} - l)} \tag{5.135}$$

and the two eigenvalues that appear in (5.120) are

$$t^{(s)}(-i\frac{s+1}{2}) = Q_0(s + \frac{1}{2})Q_+(1) \sum_{k=0}^{(s)} (ik)^L \frac{Q_+(k)}{Q_0(k + \frac{1}{2})Q_0(k - \frac{1}{2})} \sum_{l=0}^{k} \frac{Q_0(l + \frac{1}{2})}{Q_+(l)Q_+(l-1)}$$

and

$$\begin{aligned} t^{(s-2)}(-i\frac{s+3}{2}) &= Q_0(s + \frac{1}{2})Q_+(1) \sum_{k=1}^{(s-2)} (ik + 2i)^L \frac{Q_+(k+2)}{Q_0(k + \frac{5}{2})Q_0(k + \frac{3}{2})} \sum_{l=0}^{k} \frac{Q_0(l + \frac{3}{2})}{Q_+(l+2)Q_+(l+1)} \\ &= Q_0(s + \frac{1}{2})Q_+(1) \sum_{k=2}^{s} (ik)^L \frac{Q_+(k)}{Q_0(k + \frac{1}{2})Q_0(k - \frac{1}{2})} \sum_{l=2}^{k} \frac{Q_0(l - \frac{1}{2})}{Q_+(u)Q_+(u-1)}, \end{aligned}$$

where in the second equality we shift the parameters and the bounds of the summation. To simplify notation, let us set

$$\begin{aligned} A(k) &= (ik)^L \frac{Q_+(k)}{Q_0(k + \frac{1}{2}Q_0(k - \frac{1}{2})} \\ B(l) &= \frac{Q_0(l - \frac{1}{2})}{Q_+(u)Q_+(u-1)} \end{aligned}$$

We then have

$$\begin{aligned} &\sum_{k=0}^{s} A(k) \sum_{l=0}^{k} B(l) - \sum_{k=2}^{s} A(k) \sum_{l=2}^{k} B(l) = \\ &\sum_{k=0}^{1} A(k) \sum_{l=0}^{k} B(l) + \sum_{k=2}^{s} A(k) \left(\sum_{l=0}^{1} B(l) + \sum_{l=2}^{k} B(l) \right) - \sum_{k=2}^{s} A(k) \sum_{l=2}^{k} B(l) \\ &\sum_{k=0}^{1} A(k) \sum_{l=0}^{k} B(l) + \sum_{k=2}^{s} A(k) \sum_{l=0}^{1} B(l) = \\ &A(0)B(0) + \sum_{k=1}^{s} A(k) \sum_{l=0}^{1} B(l) = \\ &\sum_{k=1}^{s} A(k) \sum_{l=0}^{1} B(l), \end{aligned}$$

where in the last equality we used that $A(0) = 0$. The two sums are now decoupled, so we can independently investigate

$$\sum_{l=0}^{1} B(l) = \left(\frac{Q_0(-\frac{1}{2})}{Q_+(0)Q_+(-1)} + \frac{Q_0(\frac{1}{2})}{Q_+(1)Q_+(0)} \right) = \frac{Q_0(\frac{1}{2})}{Q_+(1)Q_+(0)} \left(\frac{Q_+(1)Q_0(-\frac{1}{2})}{Q_+(-1)Q_0(\frac{1}{2})} + 1 \right) \tag{5.136}$$

To simplify this expression, note that due to the pairing of the Bethe roots and N_0 being even, the polynomial $Q_0(u)$ is even:

$$Q_0(u) = \prod_{i=1}^{N_0} (iu - u_i) = \prod_{i=1}^{N_0/2} (iu - u_i)(iu + u_i) = Q_0(-u). \tag{5.137}$$

Since we have assumed that N_+ is also even, the same holds for $Q_+(u)$ and it immediately follows that

$$\frac{Q_+(1)Q_0(-\frac{1}{2})}{Q_+(-1)Q_0(\frac{1}{2})} = 1 \tag{5.138}$$

and

$$\sum_{l=0}^{1} B(l) = 2\frac{Q_0(\frac{1}{2})}{Q_+(1)Q_+(0)}. \tag{5.139}$$

Putting everything together, we obtain

$$t^{(s)}(-i\frac{s+1}{2}) - t^{(s-2)}(-i\frac{s+3}{2}) = 2\frac{Q_0(s+\frac{1}{2})Q_0(\frac{1}{2})}{Q_+(0)} \sum_{k=1}^{s} (ik)^L \frac{Q_+(k)}{Q_0(k+\frac{1}{2}Q_0(k-\frac{1}{2})} \tag{5.140}$$

6 The (SO(6),SO(5)) symmetric pair

The MPSs that we are interested here are related to the matrices G_i introduced in Appendix B. An $SO(5)$-invariant MPS in the Hilbert space of an $SO(6)$ spin-chain can be built out of the two-site block [23]

$$\phi_{ab}(u) = 2(u+1)G_aG_b - 2u(u-1)[G_a, G_b] - u(4u^2 + C)\delta_{ab} \ , \ a, b = 1, \ldots, 5 \tag{6.1}$$

$$\phi_{66}(u) = u(4u(u+2) - C), \tag{6.2}$$

where $C = \sum_{i=1}^{5} G_i^2 = n(n+4)$. Our goal is now to express these MPSs in terms of the 0-state

$$|\Psi_0\rangle = \otimes_{i=1}^{L} |6\rangle = \otimes_{i=1}^{\frac{L}{2}} |66\rangle \tag{6.3}$$

6.1 The extended Yangian and the twisted extended Yangian

Convention for the R-matrix

The two-site block (6.1-2) has been derived as a solution to the BYB relation (4.22) using the R-matrix (4.4). In the context of extended Yangians, it is standard to use (A.13) instead. The form of the R-matrix is the same, but the trace operator is now defined as

$$\mathcal{K} = \sum_{i,j=-N/2}^{N/2} e_{ij} \otimes e_{-i,-j}. \tag{6.4}$$

Using our modified transposition $\cdot^t$, we can relate this matrix to the permutation as $\mathcal{K}_{12} = \mathcal{P}_{12}^{t_1} = \mathcal{P}_{12}^{t_2} = \mathcal{P}_{12}^{t}$. Thus, the crossing relation for the R-matrix now becomes

$$R_{12}^{t}(u) = R_{12}(\kappa - u) \tag{6.5}$$

The extended Yangian $X(so_6)$

The extended Yangian $X(so_6)$ is another unital, associative algebra generated by $t_{ij}^{so_6}(u)$ with $1 \leq i, j \leq N$. The relations that these generators are subject to have the form of the RTT relation, with the $SO(N)$ R-matrix in the place of the $SU(N)$ one [2]. The one that is relevant to us is clearly $X(so_6)$, which can be mapped to the Yangian of gl_4 via the homomorphism

$$T^{so_6}(u) \rightarrow (I - P)T_1^{gl_4}(u)T_2^{gl_4}(u-1), \tag{6.6}$$

where P is the permutation in $\mathbb{C}^4 \otimes \mathbb{C}^4$. This mapping should be understood as follows. The (I-P) projects $T_1^{gl_4}(u)T_2^{gl_4}(u-1)$ on the anti-symmetric subspace of $\mathbb{C}^4 \otimes \mathbb{C}^4$. The latter is isomorphic to $\mathbb{C}^6$, after we construct a basis for $\mathbb{C}^6$ out of the canonical basis of $\mathbb{C}^4 \otimes \mathbb{C}^4$ as

$$\begin{aligned} v_{-3} &= e_1 \otimes e_2 - e_2 \otimes e_1 \\ v_{-2} &= e_3 \otimes e_1 - e_1 \otimes e_3 \\ v_{-1} &= e_1 \otimes e_4 - e_4 \otimes e_1 \\ v_1 &= e_2 \otimes e_3 - e_3 \otimes e_2 \\ v_2 &= e_2 \otimes e_4 - e_4 \otimes e_1 \\ v_3 &= e_2 \otimes e_3 - e_3 \otimes e_2 \end{aligned} \tag{6.7}$$

We then need to restrict the action of $T_1^{gl_4}(u)T_2^{gl_4}(u-1)$ to $\mathbb{C}^6$, which is achieved by calculating its matrix elements in the above basis. We thus have

$$[T^{so_6}(u)]_{ij} = v_i^\dagger \left(T_1^{gl_4}(u)T_2^{gl_4}(u-1)\right) v_j. \tag{6.8}$$

The resulting expressions can then be simplified using the RTT relation for $Y(4)$, which leads to the expressions of $X(so_6)$ generators

$$\begin{gathered} t_{ij}^{so_6}(u) = t_{f_1(i),f_1(j)}^{gl_4}(u)t_{f_2(i),f_2(j)}^{gl_4}(u-1) - t_{f_2(i),f_1(j)}^{gl_4}(u)t_{f_1(i),f_2(j)}^{gl_4}(u-1) \\ f_1(-3)=-2,\ f_1(-2)=1,\ f_1(-1)=-2,\ f_1(1)=-1,\ f_1(2)=-1,\ f_1(3)=1, \\ f_2(-3)=-1,\ f_2(-2)=-2,\ f_2(-1)=2,\ f_2(1)=1,\ f_2(2)=2,\ f_2(3)=2, \end{gathered} \tag{6.9}$$

These relations allow us to use the evaluation homomorphism for $Y(4)$ to treat a gl_4-module as a $X(so_6)$-module.

The extended twisted Yangian $X(so_6, so_5)$

Similar to the twisted Yangian, we can define a twisted extended Yangian $X^{tw}(so_6)$ as the subalgebra of $X(so_6)$, generated by

$$S(u) = T^{so_6}(u)S_0(u)(T^{so_6})^t(-u), \tag{6.10}$$

where $S_0(u)$ satisfies the relation (4.28), with the SO(N) R-matrix. We choose

$$S_0(u) = \begin{pmatrix} \frac{u}{u+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{u}{u+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{u+1} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{u+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{u}{u+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{u}{u+1} \end{pmatrix} \tag{6.11}$$

In addition to the reflection equation, this matrix fulfils the symmetry equation

$$S_0^t = S_0(-u) - \frac{2u}{(u+1)(u-1)} I \tag{6.12}$$

and the corresponding twisted extended Yangian is denoted $X(so_6, so_5)$ [10][11]. The process of deriving the quaternary relation from the RTT relation, presented in the previous section, is unaffected by the different choice of R-matrix, thus it holds for this $S(u)$. In addition to that, it fulfils the symmetry relation

$$S^t(u) = S(-u) + \frac{1}{2u}(S(u) - S(-u)) - \frac{1}{2u-2} Tr[S(u)]I, \tag{6.13}$$

as a consequence of the symmetry relation for $S_0(u)$.

The connection between this algebra and our goal becomes clear after we find the K-matrix that corresponds to $S_0(u)$. As explained in Appendix A, we can transform between a K-matrix and an $X(so_6, so_5)$ representation as

$$S_0(u) = P^{-1} K_0(u) P \tag{6.14}$$

with P fulfilling (A.18-19). Using the matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 1 & -i & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.15}$$

we obtain

$$K_0(u) = \begin{pmatrix} \frac{u}{u+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{u}{u+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{u}{u+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{u}{u+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{u}{u+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{u+1} - 2 \end{pmatrix}. \tag{6.16}$$

Noticing that

$$[K_0(0)]_{ab} = 0\,,\ (a,b) \neq (6,6) \tag{6.17}$$

$$[K_0(0)]_{66} = -1 \tag{6.18}$$

we can see that this matrix generates $|\Psi_0\rangle$, up to a phase.

The definition of a $X(so_6, so_5)$ highest weight representation is slightly different from the one for $Y^+(N)$. We call a vector space V an $X(so_6, so_5)$ highest weight module if it is generated from $v \in V$, such that[13]

$$s_{i_j}(u) \cdot v = 0, \text{ for } i < j \text{ and } (i,j) \neq (1,-1) \tag{6.19}$$

$$s_{ii}(u) \cdot v = \mu_i(u) v \tag{6.20}$$

$$s_{1,-1}(u) \cdot v = \mu^{(+)}(u) v \tag{6.21}$$

$$s_{-1,1}(u) \cdot v = \mu^{(-)}(u) v \tag{6.22}$$

The main difference between this definition and the definition of $Y^+(N)$ highest weight representations is that $s_{1,-1}(u)$ and $s_{-1,1}(u)$ behave like Cartan generators, in that they do not alter the weights of the state. This can be attributed to the fact that they are fixed points of the $\cdot^t$ transposition, since

$$s_{i,-i}(u) \xrightarrow{t} \sum_{a,b=-1}^{1} \delta_{a,-i} s_{b,a}(u) \delta_{b,i} = s_{i,-i}(u) \tag{6.23}$$

Then, through the symmetry relation (6.13), they are expressed in terms of the $s_{ii}(u)$. As consequence of this, the $\mu^{(\pm)}(u)$ are not indepenent, as they can be expressed in terms of the $\mu_i(u)$.

6.2 Identification of V with a dressing of $S_0(u)$

The strategy for relating the $|MPS_n\rangle$ generated from (6.1) and $|\Psi_0\rangle$ is similar to the one employed in the previous chapter. Namely, look for a relation between the $X(so_6, so_5)$ representation that corresponds to $|MPS_n\rangle$ and a dressing of the representation that corresponds to $|\Psi_0\rangle$. This is achieved by comparing the highest weights

[13]This is not a standard definition, but was conjectured in [15].

of the corresponding $X(so_6, so_5)$-modules.

Calculation of the $X(so_6, so_5)$ highest weights
Using the matrix P, we create an S-matrix out of the K-matrix which generates $|MPS_n\rangle$

$$S(u) = \frac{1}{4}u^{-3}(1-u^{-1})PK(u)P^{-1} \tag{6.24}$$

The components of the S-matrix are better expressed in terms the matrices $\tilde{G}_i$ defined in (B.17)-(B.19). Since we are interested in computing the highest weights, we only need some of the components of the S-matrix, in particular

$$\psi_{3,3}(u) = g_1(u)\tilde{G}_1\tilde{G}_{-1} + g_2(u)[\tilde{G}_1, \tilde{G}_-1] + f(u) \tag{6.25}$$
$$\psi_{2,2}(u) = g_1(u)\tilde{G}_2\tilde{G}_{-2} + g_2(u)[\tilde{G}_2, \tilde{G}_-2] + f(u) \tag{6.26}$$
$$\psi_{1,1}(u) = \frac{1}{2}(g_1(u)\tilde{G}_0^2 + f(u) + h(u)) \tag{6.27}$$
$$\psi_{1,-1}(u) = \frac{1}{2}(g_1(u)\tilde{G}_0^2 + f(u) - h(u)) \tag{6.28}$$
$$\psi_{-1,1}(u) = \frac{1}{2}(g_1(u)\tilde{G}_0^2 + f(u) - h(u)), \tag{6.29}$$

where

$$g_1(u) = -\frac{1}{2}u^{-2}(1-u^{-2}) \tag{6.30}$$
$$g_2(u) = \frac{1}{2}u^{-1}(1-u^{-2}) \tag{6.31}$$
$$f(u) = (1-u^{-1})(1+\frac{C}{4}u^{-2}) \tag{6.32}$$
$$h(u) = -(1-u^{-1})(1+2u^{-1}-\frac{C}{4}u^{-2}). \tag{6.33}$$

As discussed in appendix B, the

$$F_{i,j} = \frac{1}{4}[\tilde{G}_i, \tilde{G}_{-j}] \tag{6.34}$$

furnish an so_5 representation with lowest weights $(-\frac{n}{2}, -\frac{n}{2})$. Using (B.21) one can see that G_{-1} and G_{-2} lower the so_5 weights. For our lowest weight vector $\tilde{v}$, we thus have

$$\tilde{G}_{-1}\cdot\tilde{v} = \tilde{G}_{-2}\cdot\tilde{v} = 0. \tag{6.35}$$

It is then easy to see that

$$s_{3,3}(u)\cdot\tilde{v} = (f(u) + 4g_2(u)F_{1,1})\cdot\tilde{v} = (f(u) - 2ng_2(u))\tilde{v} \tag{6.36}$$
$$s_{2,2}(u)\cdot\tilde{v} = (f(u) + 4g_2(u)F_{2,2})\cdot\tilde{v} = (f(u) - 2ng_2(u))\tilde{v}. \tag{6.37}$$

To compute the remaining weights we use $C = \sum_{i=1}^{5} G_i^2 = n(n+1)$ to write

$$\tilde{G}_0^2\cdot\tilde{v} = C\tilde{v} - (\tilde{G}_1\tilde{G}_{-1} + \tilde{G}_{-1}\tilde{G}_1 + \tilde{G}_2\tilde{G}_{-2} + \tilde{G}_{-2}\tilde{G}_2) \tag{6.38}$$

Then, noticing that

$$\tilde{G}_i\tilde{G}_{-i} + \tilde{G}_{-i}\tilde{G}_i = [\tilde{G}_{-i}, \tilde{G}_i] + 2\tilde{G}_i\tilde{G}_{-i} = -4F_{ii} + 2\tilde{G}_i\tilde{G}_{-i} \tag{6.39}$$

we obtain

$$s_{1,1}(u)\cdot\tilde{v} = \frac{1}{2}(n^2 g_1(u) + f(u) + h(u))\tilde{v} \tag{6.40}$$

$$s_{1,-1}(u)\cdot\tilde{v} = s_{-1,1}(u)\cdot\tilde{v} = \frac{1}{2}(n^2 g_1(u) + f(u) - h(u))\tilde{v} \tag{6.41}$$

We have therefore calculated the highest weights of our $X(so_6, so_5)$ module, which we can factorize and write as

$$\mu_3(u) = \mu_2(u) = (1-u^{-1})(1-\frac{n}{2}u^{-1})^2 \tag{6.42}$$

$$\mu_1(u) = -u^{-1}(1-u^{-1})(1-\frac{n}{2}u^{-1})^2 \tag{6.43}$$

$$\mu^{(+)}(u) = \mu^{(-)}(u) = (1-u^{-2})(1-\frac{n^2}{4}u^{-2}) \tag{6.44}$$

Determining the corect dressing

We now look for a gl_4 module $L(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, such that the highest weights of the dressed representation

$$S^D(u) = T^{so_6}(u)S_0(u)(T^{so_6})^t(-u) \tag{6.45}$$

match the ones we just obtained. Unlike the $(SU(3), SO(3))$ case, where the twisted Yangian module was a subspace of L, we can now find a single gl_4 module such that $V(n) \cong L(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$.

Since the $X(so_6, so_5)$ weights of V are related to the so_5 weights, we can start by fixing the so_5 highest weights of the gl_4 module. Similar to (5.27), an embedding of so_5 in gl_4 is given by

$$F_{i,j} = E_{i,j} - E_{-j,-i} \tag{6.46}$$

If we then choose the Cartan subalgebra to be generated by

$$H_1 = \frac{1}{2}(F_{11} + F_{22}) \tag{6.47}$$

$$H_2 = \frac{1}{2}(F_{11} - F_{22}), \tag{6.48}$$

we obtain that the so_5 highest weights on $L(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ are $\left(-\frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{2}, -\frac{\lambda_1-\lambda_2+\lambda_3-\lambda_4}{2}\right)$. By requiring that these weights are equal to $(-\frac{n}{2}, -\frac{n}{2})$, we get two equations that lead to the constraints

$$\begin{aligned} \lambda_2 &= \lambda_3 \\ \lambda_4 - \lambda_1 &= n. \end{aligned} \tag{6.49}$$

We therefore restrict our search to representations of the form $[n+c_1, c_2, c_2, c_1]$. We can further restrict our options by comparing the dimension of the gl_4 module to that of the $X(so_6, so_5)$ module. The general formula for the dimension of a gl_N representation is

$$\dim[\lambda_1 \ldots, \lambda_N] = \prod_{i<j}^{N} \frac{\lambda_i - \lambda_j - i + j}{j - i}, \tag{6.50}$$

which in our case reads

$$\begin{aligned} &\dim[n + c_1, c_2, c_2, c_1] = \\ &\qquad \frac{1}{12}(n+3)(c_2 - c_1 + 1)(c_2 - c_1 + 2)(-c_2 + c_1 + n + 1)(-c_2 + c_1 + n + 2) \end{aligned} \tag{6.51}$$

Equating this to the dimension of the G-matrices (B.13), gives four solutions for the c_1 and c_2. Two of them contain square roots, which we ignore as they do not correspond to finite-dimensional representations. The remaining solutions give the representations $[n+c, n+c, n+c, c]$ and $[n+c, c, c, c]$. We will now show that the correct choice is the first of the two, by explicitly calculating its $X(so_6, so_5)$ highest weights.

In what follows, $v = |n+c, n+c, n+c, c\rangle$ denotes the highest weight of the gl_4 module. The definition of the twisted extended Yangian (6.10) reads

$$s^D_{3,3}(u) \cdot v = \sum_{a,b} t^{so_6}_{3,a}(u)[S_0(u)]_{ab} t^{so_6}_{-3,-b}(-u) \cdot v \tag{6.52}$$

Since $t^{so_6}_{ij}$ with $i < j$ annihilate the highest weight, the only term that survives is

$$\frac{u}{u+1} t^{so_6}_{3,3}(u) t^{so_6}_{3,3}(-u) \cdot v = \frac{u}{u+1}\Big(t^{gl_4}_{1,1}(u) t^{gl_4}_{22}(u-1) - t^{gl_4}_{2,1}(u) t^{gl_4}_{12}(u-1)\Big) \times \tag{6.53}$$

$$\Big(t^{gl_4}_{-2,-2}(-u) t^{gl_4}_{-1,-1}(-u-1) - t^{gl_4}_{-2,-1}(-u) t^{gl_4}_{-1,-2})(-u-1)\Big) \cdot v \tag{6.54}$$

Similarly, due to $t^{gl_6}_{ij} \cdot v = 0$ for $i < j$ and $t^{gl_6}_{ii} \cdot v = \lambda_i v$, only one term remains

$$s^D_{3,3}(u) \cdot v = \frac{u}{u+1} t^{gl_4}_{11}(u) t^{gl_4}_{22}(u-1) t^{gl_4}_{-2,-2}(-u) t^{gl_4}_{-1,-1}(-u-1) \cdot v \tag{6.55}$$

Using (5.12), and simplifying the expression, one can eventually obtain

$$\mu^D_3(u) = \frac{(1 - (\frac{n}{2}+1)^2 u^{-2})(1 - \frac{n}{2}u^{-1})^2}{(1+u^{-1})^2(1-u^{-1})} \tag{6.56}$$

The same process yields the remaining highest weights

$$\mu^D_2(u) = \mu^D_3(u) \tag{6.57}$$

$$\mu^D_1(u) = -\frac{(u^2 - (n+c)^2)(u+c-1)(u-n-c+1)}{u^2(u+1)^2(u-1)} \tag{6.58}$$

$$\mu^{(+)}_D(u) = \frac{(u^2 - (n+c)^2)(u^2 - (n+c-1)^2)}{u^2(u^2-1)} \tag{6.59}$$

$$\mu^{(-)}_D(u) = \frac{(u^2 - (n+c)^2)(u^2 - (c-1)^2)}{u^2(u^2-1)} \tag{6.60}$$

We can now fix the constant c by requiring that $\mu^{(+)}_D(u) = \mu^{(-)}_D(u)$, according to (6.44). We find $c = 1 - \frac{n}{2}$. Finally, we can notice that the highest weights of the dressed representation are proportional to those of the

$X(so_6, so_5)$-module:

$$\mu_i^D(u) = \frac{(1-(\frac{n}{2}+1)^2u^{-2})}{(1-u^{-2})^2}\mu_i(u) \tag{6.61}$$

We can thus conclude that

$$S(u) = \frac{(1-u^{-2})^2}{(1-(\frac{n}{2}+1)^2u^{-2})}T^{so_6}(u)S_0(u)(T^{so_6}(-u))^t \tag{6.62}$$

6.3 The transfer matrix eigenvalues

The result of the previous section is that we can treat V(n) as

$$V \cong L(1+\frac{n}{2}, 1+\frac{n}{2}, 1+\frac{n}{2}, 1-\frac{n}{2}) \tag{6.63}$$

This implies that the MPS is related to $|\Psi_0\rangle$ via the action of a transfer matrix at some special point, which we leave unspecified. The overlaps of the form $\langle\{u_i\}|MPS_n\rangle$ are therefore related to the eigenvalues of the transfer matrix at that special point. In order to simplify the calculation of the transfer matrix eigenvalues, we now employ the same tricks we used in the previous chapter to replace this gl_4 module with an equivalent one, which corresponds to a single-row Young tableaux. By rewriting all matrices in the representation as

$$\tilde{E}_{ij}^{rep} = E_{ij}^{rep} + (1+\frac{n}{2})I \tag{6.64}$$

we replace our original highest weights with

$$L(0,0,0,-n). \tag{6.65}$$

Then, under the gl_4 automorphism $E_{ij} \to -E_{j,i}$, this module is mapped to

$$L(n,0,0,0). \tag{6.66}$$

Using the embedding (5.123), $[n,0,0,0]$ contains the sl_4 representation $(n,0,0)$. We can therefore use the tableaux sum formula to calculate the eigenvalues of this transfer matrix. Before doing that we need to recall that the formula treats su_N spin chains, while we are working with the Hilbert space which carries the fundamental representation of so_6. This is not a problem, since so_6 is isomorphic to su_4. The fundamental representation of so_6, which we have been using, corresponds to the antisymmetric representation of su_4. We thus need to fix the z-functions such that (5.122) gives the eigenvalues of a transfer matrix with the auxiliary space in the fundamental of su_N and the Hilbert space in the antisymmetric. Then, the tableaux sum will give us the eigenvalues for auxiliary spaces in higher representations of su_4 while keeping the physical representation the same. We can, again, obtain the appropriate z-functions from (5.31) by plugging in a gl_4 representation in which the antisymmetric of su_4, with Dynkin labels $(0,1,0)$, can be embedded. Choosing (5.123) as the embedding of the algebras, we see that this sl_4 module can be embedded in any gl_4 module of the form $[a,a,a+1,a+1]$. This introduces an ambiguity that was swept under the rug in the previous section, in that there are several gl_4 modules where a given sl_4 module can be embedded. We now argue that the different choices of module correspond to simply changing the point at which we need to evaluate the transfer matrix eigenvalue.

Let us compare the gl_N representation $[\lambda_1, \dots, \lambda_N]$ with one where all the weights are shifted by the same constant $[\lambda_1+\theta, \dots, \lambda_N+\theta]$. The objects related to the shifted representation will be denoted with tildes. The highest weights enter the description of the gl_N spin chain through the Drinfeld polynomials. By investigating

(3.49) it can be immediately seen that the two cases are related by

$$\tilde{P}_k(u) = P_k(u + i\theta). \tag{6.67}$$

These polynomials enter the RHS of the Bethe equations (3.64), while the LHS is independent of the choice of representation. Now, consider the Bethe equations in the shifted representation. It is easy to check that by redefining $\tilde{u}_n^{(k)} \to \tilde{u}_n^{(k)} + \theta$, for all n and k, the Bethe equations take the form of those of the non-shifted case. Thus, the roots in the two cases are related by the shift

$$u_n^{(k)} = \tilde{u}_n^{(k)} + i\theta \tag{6.68}$$

Since the roots enter the dressing functions (3.60), we have

$$\tilde{D}_k(u) = D_k(u + i\theta) \tag{6.69}$$

Due to $z^{(k)}(u) \propto D_k(u)P_k(u)$, the z-functions are also related by a shift in the argument

$$\tilde{z}^{(k)}(u) = z^{(k)}(u + i\theta). \tag{6.70}$$

Therefore, shifting all weights in a representation has the effect of shifting the rapidity in the transfer matrix eigenvalues.

To be consistent with [15], we choose the gl_4 representation $[\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}]$ and define the z-functions as

$$z^{(l)}(u) = \frac{1}{(u + \frac{i}{2})^L} D_l(u - i)P_l(u - i). \tag{6.71}$$

For that representation, the Drinfeld polynomials are

$$P_1(u) = P_2(u) = \prod_{n=1}^{L}(u + i\frac{3}{2}) = (u + i\frac{3}{2})^L \tag{6.72}$$

$$P_3(u) = P_4(u) = (u + \frac{1}{2}i)^L \tag{6.73}$$

Introducing the Baxter polynomials

$$Q_-(u) = \prod_{i=1}^{N_-}(iu - u_i^{(1)}) \tag{6.74}$$

$$Q_0(u) = \prod_{i=1}^{N_0}(iu - u_i^{(2)}) \tag{6.75}$$

$$Q_+(u) = \prod_{i=1}^{N_+}(iu - u_i^{(3)}) \tag{6.76}$$

we can write the dressing functions as

$$D_1(u) = \frac{Q_-(-iu-\frac{1}{2})}{Q_-(-iu+\frac{1}{2})} \tag{6.77}$$

$$D_2(u) = \frac{Q_-(-iu+\frac{3}{2})}{Q_-(-iu+\frac{1}{2})}\frac{Q_0(-iu)}{Q_0(-iu+1)} \tag{6.78}$$

$$D_3(u) = \frac{Q_0(-iu+2)}{Q_0(-iu+1)}\frac{Q_+(-iu+\frac{1}{2})}{Q_+(-iu+\frac{3}{2})} \tag{6.79}$$

$$D_4(u) = \frac{Q_+(-iu+\frac{5}{2})}{Q_+(-iu+\frac{3}{2})}, \tag{6.80}$$

and the z-functions as

$$z^{(1)}(u) = \frac{(u+\frac{i}{2})^L}{(u-\frac{i}{2})^L}\frac{Q_-(-iu-\frac{3}{2})}{Q_-(-iu-\frac{1}{2})} \tag{6.81}$$

$$z^{(2)}(u) = \frac{(u+\frac{i}{2})^L}{(u-\frac{i}{2})^L}\frac{Q_0(-iu-1)Q_-(-iu+\frac{1}{2})}{Q_0(-iu)Q_-(-iu-\frac{1}{2})} \tag{6.82}$$

$$z^{(3)}(u) = \frac{Q_0(-iu+1)Q_+(-iu-\frac{1}{2})}{Q_0(-iu)Q_+(-iu+\frac{1}{2})} \tag{6.83}$$

$$z^{(4)}(u) = \frac{Q_+(-iu+\frac{3}{2})}{Q_+(-iu+\frac{1}{2})} \tag{6.84}$$

To calculate the required eigenvalue, we start from the tableaux sum formula (5.121) which in the case of a single-row tableaux reads

$$t_n^{(1)}(u) = \sum_{\tau}\prod_{l=1,\dots,n} z^{(\tau_l)}\left(\tilde{u}+il)\right), \tag{6.85}$$

where we have introduced $\tilde{u} = u - i\frac{n+1}{2}$. For su_4, such a tableau is of the form

$$\boxed{1}\boxed{\cdots}\boxed{1}\boxed{2}\boxed{\cdots}\boxed{2}\boxed{3}\boxed{\cdots}\boxed{3}\boxed{4}\boxed{\cdots}\boxed{4} \tag{6.86}$$

Denoting by k_i the number of boxes that contain $1 \le i \le 4$, we have $k_1+k_2+k_3+k_4 = n$. We can then rewrite the sum over all possible tableaux as

$$\sum_{\tau_i} = \sum_{k_1,k_2,k_3} = \sum_{k_1}^{n}\sum_{k_2=0}^{n-k_1}\sum_{k_3=0}^{n-k_1-k_2}. \tag{6.87}$$

and the formula as

$$\begin{aligned} t_n^{(1)} = \sum_{k_1,k_2,k_3}\prod_{l_1=1}^{k_1} z^{(1)}(\tilde{u}+il_1)\prod_{l_2=k_1+1}^{k_1+k_2} z^{(2)}(\tilde{u}+il_2)\times \\ \prod_{l_3=k_1+k_2+1}^{k_1+k_2+k_3} z^{(3)}(\tilde{u}+il_3)\prod_{l_4=k_1+k_2+k_3+1}^{n} z^{(4)}(\tilde{u}+il_4) \end{aligned} \tag{6.88}$$

For the first product, we have

$$\prod_{l_1=1}^{k_1} z^{(1)}(\tilde{u}+il_1) = \frac{\prod_{l_1=1}^{k_1}(\tilde{u}+\frac{i}{2}+il_1)\prod_{l_1=1}^{k_1}Q_-(-i\tilde{u}-l_1-\frac{3}{2})}{\prod_{l_1'=1}^{k_1}(\tilde{u}-\frac{i}{2}+il_1')\prod_{l_1'=1}^{k_1}Q_-(-i\tilde{u}-l_1'-\frac{1}{2})} \tag{6.89}$$

The factors in the numerator and denominator for which $l_1' - l_1 = 1$ cancel each other and the remaining ones correspond to the maximum and minimum values of l_1. Reinstating the original parameter u, we get

$$\prod_{l_1=1}^{k_1} z^{(1)}\left(u - i\frac{n-2l_1+1}{2}\right) = \frac{(u-i\frac{n-2k_1}{2})^L}{(u-i\frac{n}{2})^L}\frac{Q_-(-iu-\frac{n+2}{2})}{Q_-(-iu-\frac{n-2k_1+2}{2})}$$

The remaining products can be similarly seen to be

$$\prod_{l_2=1+k_1}^{k_1+k_2} z^{(2)}\left(u - i\frac{n-2l_2+1}{2}\right) = \frac{(u-i\frac{n-2k_1-2k_2}{2})^L}{(u-i\frac{n-2k_1}{2})^L}\frac{Q_0(-iu-\frac{n-2k_1+1}{2})}{Q_0(-iu-\frac{n-2k_1-2k_2+1}{2})}\frac{Q_-(-iu-\frac{n-2k_1-2k_2}{2})}{Q_-(-iu-\frac{n-2k_1}{2})}$$

$$\prod_{l_3=1+k_1+k_2}^{k_1+k_2+k_3} z^{(3)}\left(u - i\frac{n-2l_3+1}{2}\right) = \frac{Q_0(-iu-\frac{n-2k_2-2k_2-2k_3-1}{2})}{Q_0(-iu-\frac{n-2k_2-2k_2-1}{2})}\frac{Q_+(-iu-\frac{n-2k_1-2k_2}{2})}{Q_+(-iu-\frac{n-2k_1-2k_2-2k_3}{2})}$$

$$\prod_{l_4=1+k_1+k_2+k_3}^{n} z^{(4)}\left(u - i\frac{n-2l_4+1}{2}\right) = \frac{Q_+(-iu+\frac{n+2}{2})}{Q_+(-iu-\frac{n-2k_1-2k_2-2k_3-2}{2})}$$

Finally, introducing the variables

$$\begin{aligned} p &= k_1 \\ q &= k_1+k_2-\frac{n}{2} \\ r &= k_1+k_2+k_3-\frac{n}{2} \end{aligned}$$

we can write the transfer matrix eigenvalue as

$$t_n^{(1)} = \frac{Q_-(-iu-\frac{n}{2}-1)Q_+(-iu+\frac{n}{2}+1)}{(u-i\frac{n}{2})^L}\sum_{q=-n/2}^{n/2}(u++iq)^L\frac{Q_-(-iu+q)Q_+(-iu+q)}{Q_0(-iu+q-\frac{1}{2})Q_0(-iu+q+\frac{1}{2})}$$
$$\sum_{p=-n/2}^{q}\frac{Q_0(-iu+p-\frac{1}{2})}{Q_-(-iu+p-1)Q_-(-iu+p)}\sum_{r=q}^{n/2}\frac{Q_0(-iu+r+\frac{1}{2})}{Q_+(-iu+r)Q_+(-iu+r+1)} \tag{6.90}$$

7 Conclusion and Outlook

Summary

In the first section of this thesis we briefly introduced $\mathcal{N} = 4$ SYM and derived the constraints imposed on scalar two-point functions of the theory by its conformal symmetry. Then, through some 1-loop order calculations in the planar limit, we established the correspondence between the corrections to the dilatation operator of the $SO(6)$ and $SU(2)$ sectors and the corresponding spin chain models. This was then used to explain the relevance of MPS overlaps to certain defect versions of the theory.

In section 2 we solved the Heisenberg model through the Algebraic Bethe Ansatz. Interestingly the main focus in that approach is not the Hamiltonian but the transfer matrix, the eigenstates of which we determined and called Bethe states. We also saw that the transfer matrix generates a family of commuting operators, one of which is the Hamiltonian. Therefore, by diagonalizing the transfer matrix, we also determine the energy eigenstates and eigenvalues. A crucial part of the process was the RTT relation, which is fulfilled by the Lax operators and the monodromy. The latter implies the commutativity of the transfer matrix for different rapidities, which renders the system integrable. This illustrates the importance of the RTT relation as a requirement for the integrability of the model.

In section 3 we introduced the Yangian of gl_N and noticed its intrinsic connection with integrability, as a result of its generators satisfying the RTT relation. In particular, this implies that if one defines the Lax operator as a representation of that algebra, the corresponding transfer matrix will generate commuting charges and the model will be solvable. After introducing the evaluation representations of $Y(N)$, this allowed us to construct an integrable gl_N spin chain as a representation of $\otimes^L Y(N)$. We then derived the Bethe equations of this model through the Analytical Bethe Ansatz approach. As a byproduct, we also obtained a formula for the eigenvalues of the corresponding transfer matrices, which were used at a later point.

The introduction of $Y(N)$ turned out to be useful towards achieving or goal of calculating overlaps relevant to the dCFT. After introducing matrix product states and their integrability conditions, we saw that integrable MPSs can be generated from solutions of the twisted BYB relation. This equation was then shown to be fulfilled by representations of a family of subalgebras of $Y(N)$, the twisted Yangians. We then noticed that a family of representations of $Y^{tw}(N)$ can be obtained by dressing the twisting matrix with $Y(N)$ representations and derived that the MPSs that correspond to the undressed and dressed matrices are related by the action of a transfer matrix. This relation provided us with a strategy for calculating overlaps between MPSs and Bethe states: Identifying the $Y^{tw}(N)$ representation that corresponds to a given MPS with a dressed representation enables the calculation of the overlaps in question.

In what followed, we applied this strategy to two MPSs that appear in the study of defect versions of $\mathcal{N} = 4$ SYM. First, we worked with a MPS generated by spin representations of su_2 and belongs in a (SU(3),SO(3)) symmetric pair. For odd-dimensional representations, we were able to derive relations between this MPS and the δ-state, which is generated from the twisting matrix of $Y^+(3)$. This was achieved by calculating the highest weights of the MPS-module V and identifying them as the $Y^+(3)$ weights on a gl_3 module L. This indicated that V can be embedded within L. Then, the orthogonal complement on V with respect to L was investigated and seen to be equivalent to another gl_3 module, which allowed us to express the MPS as the action of two transfer matrices on the δ-state. As a result, the calculation of the overlaps was reduced to the calculation of some transfer matrix eigenvalues, which we performed using the tableaux sum formula. The input to this formula is related to the eigenvalues of the fundamental transfer matrix and we were able to extract from elements of section 3.

In the final section we followed similar steps for the MPS of the (SO(6),SO(5)) symmetric pair. The main difference was the replacement of the Yangian with an extended Yangian, which was required due to the $SO(6)$ symmetry of the underlying spin chain. Fortunately, an algebra homomorphism allowed us to reduce $X(so_6)$ representations to evaluation representations $Y(4)$, which we had already used. Another difference compared to the previous case, was that the $X(so_6, so_5)$ module corresponding to our MPS could be identified with a single gl_4 module, instead of being embedded in one. This simplified the process, as there was no complement to be examined and each overlap was related to a single transfer matrix eigenvalue, which we again calculated through the tableaux sum formula.

Outlook: The $(SO(6), SO(3) \otimes SO(3))$ case

One more family of MPSs which is relevant to a defect version of $\mathcal{N} = 4$ SYM belongs in the Hilbert space of an $SO(6)$ spin chain and is generated from the two-site block

$$\psi_{ab}^{(s)}(u) = (1+u)S_a S_b - u(u+1)[S_a, S_b] - \frac{1}{2}u(u^2+u+s(s+1))\delta_{ab}$$
$$\psi_{Aa}^{(s)}(u) = \psi_{aA}^{(s)}(u) = 0$$
$$\psi_{AB}^{(s)}(u) = \frac{1}{2}u(u^2+u-s(s+1))\delta_{AB},$$

where $a, b \in \{1,2,3\}$ and $A, B \in \{4,5,6\}$. Similar to the cases discussed in the main text, it would be useful to the calculation of overlaps ratios to relate these states to the δ-state

$$|\Psi_\delta\rangle = \otimes^{\frac{L}{2}}(|11\rangle + |22\rangle + |33\rangle - |66\rangle - |66\rangle - |66\rangle).$$

The K-matrix that generates this δ-state is related to the extended twisted Yangian $X(so_6, so_3 \oplus so_3)$. The transformation matrix which creates an $X(so_6, so_3 \oplus so_3)$ highest weight representation from the K-matrix of the MPS through () has been identified. It then should be possible, through a process similar to section 6.2, to achieve the relation in question through twisted extended Yangian representations. The difficulty in this case lies in the embedding of the $X(so_6, so_3 \oplus so_3)$ module in a gl_4 module, which appears to be significantly more complicated than the previous two cases. Some progress has been made by studying the branching rules of su_4 highest weight representations, which will hopefully help to overcome this obstacle. If that is achieved, the calculation of the overlaps will once again reduce to the computation of some transfer matrix eigenvalues. This would constitute a formal derivation of some overlap formulas that have already been discovered numerically.

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A Different versions of the BYB relation

A.1 The SU(N) case

The su_N R-matrix is

$$R_{12}(u) = \mathcal{I}_{12} - \frac{\mathcal{P}_{12}}{u}. \tag{A.1}$$

We have encountered the following two versions of the BYB relation:

$$K_2(v)R^T_{21}(u+v)K_1(u)R_{12}(u-v) = R_{21}(u-v)K_1(u)R^T_{12}(-u-v)K_2(v) \tag{A.2}$$

and what we called the quaternary relation

$$K'_2(v)R^t_{21}(u+v)K'_1(u)R_{12}(u-v) = R_{21}(u-v)K'_1(u)R^t_{12}(-u-v)K'_2(v) \tag{A.3}$$

There exist two ways to transform a solution of the first into a solution of the second. The first is to multiply by the charge conjugation matrix either from the left or right, i.e.

$$K'(u) = CK(u) \tag{A.4}$$
$$\text{or } K'(u) = K(u)C \tag{A.5}$$

In general, these two multiplications lead to different matrices, both of which are solutions of (a.3). The second is to find a matrix P such that

$$P_1P_2R(u)P_1^{-1}P_2^{-1} = R(u) \tag{A.6}$$
$$P_1P_2R^T_{12}(u)P_1^{-1}P_2^{-1} = R^t(u) \tag{A.7}$$

and define

$$K'(u) = PK(u)P^{-1} \tag{A.8}$$

The conditions on the P-matrix (A.5)-(A.6) are fulfilled if

$$P \in SU(N) \tag{A.9}$$
$$PP^T = C. \tag{A.10}$$

In particular, (A.9) guarantees (A.6) due to the $SU(N)$ symmetry of the R-matrix and (A.10) implies (A.7). Since (A.3) is identical to the defining relations of $Y^{tw}(N)$, any solution of that relation could form a representation of that algebra. Therefore, (A.4-5) and (A.8) can be used to create $Y^{tw}(N)$ representations out of solutions of the ordinary twisted BYB relation (4.22). It should be noted that, in addition to this, one should verify that the symmetry relation of the specific twisted Yangian is fulfilled. Conversely, inverting these transformations generates solutions of (4.22) out of $Y(N)^+$ representations. Verifying these relations is identical to the proofs for the SO(N) case shown below.

A.2 The SO(N) case

This case is a little more complicated, as it turns out that the two transformations above lead to solutions of different equations. These two equations include to two different versions for the $SO(N)$ R-matrix.

Conventions for the R-matrix

$$R_{12}(u) = \mathcal{I}_{12} - \frac{\mathcal{P}_{12}}{u} + \frac{\mathcal{K}_{12}}{u-\kappa}, \tag{A.11}$$

$$\text{where } \mathcal{K}_{12} = \mathcal{P}_{12}^T = \sum_{i,j} e_{i,j} \otimes e_{i,j} \tag{A.12}$$

$$\tilde{R}_{12}(u) = \mathcal{I}_{12} - \frac{\mathcal{P}_{12}}{u} + \frac{\tilde{\mathcal{K}}}{u-\kappa}, \tag{A.13}$$

$$\text{where } \tilde{\mathcal{K}}_{12} = \mathcal{P}_{12}^t = \sum_{i,j} e_{i,j} \otimes e_{-i,-j} \tag{A.14}$$

Versions for the BYB relation

- Version 1: The ordinary twisted BYB relation

$$K_2(v)R_{21}^T(u+v)K_1(u)R_{12}(u-v) = R_{21}(u-v)K_1(u)R_{12}^T(-u-v)K_2(v) \tag{A.15}$$

- Version 2: We have not encountered this equation, but we define it here in order to clarify some points

$$\begin{gathered} K_2'(v)R_{21}^t(u+v)K_1'(u)R_{12}(u-v) = R_{21}(u-v)K_1'(u)R_{12}^t(-u-v)K_2'(v), \\ R^t(u) = C_1R^{T_1}(u)C_1 = C_2R^{T_2}(u)C_2 \end{gathered} \tag{A.16}$$

- Version 3: The quaternary relation

$$\begin{gathered} \tilde{K}_2(v)\tilde{R}_{21}^t(-u-v)\tilde{K}_1(u)\tilde{R}_{12}(u-v) = \tilde{R}_{21}(u-v)\tilde{K}_1(u)\tilde{R}_{12}^t(-u-v)\tilde{K}_2(v), \\ \tilde{R}^t(u) = C_1\tilde{R}^{T_1}(u)C_1 = C_2\tilde{R}^{T_2}(u)C_2 \end{gathered} \tag{A.17}$$

Note that going from 1 to 2 requires the replacement of $\cdot^T$ with $\cdot^t$ while keeping the same convention for the R-matrix. Going from 1 to 3 requires us to simultaneously replace $\cdot^T$ with $\cdot^t$ and switch between the two conventions. In the SU(N) case there is no distinction between the R-matrix conventions, due to the absence of the trace operator, thus 2 and 3 are identical and the two transformations lead to solutions of the same equation.

Relations between the solutions of the different relations

- Relation between 1 and 2: $K'(u) = CK(u)$ or $K'(u) = K(u)C$.

  Proof: We can start by showing the relations $C_1C_2R^T(u)C_1C_2 = R^T(u)$ and $C_1C_2R(u)C_1C_2 = R(u)$. It is enough to show that C_1C_2 is a symmetry of $\mathcal{I}, \mathcal{P}$ and $\mathcal{K}$:

$$\begin{aligned} C_1C_2\mathcal{P}C_1C_2 &= \sum_{i,j} Ce_{ij}C \otimes Ce_{ji}C = \sum_{i,j} e_{-i,-j} \otimes e_{-j,-i} = \mathcal{P} \\ C_1C_2\mathcal{K}C_1C_2 &= \sum_{i,j} Ce_{ij}C \otimes Ce_{ij}C = \sum_{i,j} e_{-i,-j} \otimes e_{-i,-j} = \mathcal{K} \\ (C_1C_2)^2 &= \mathcal{I} \Rightarrow C_1C_2\mathcal{I}C_1C_2 = \mathcal{I} \end{aligned}$$

  The rest of the derivation consists of multiplying the BYB relation by C_1C_2 and inserting $C^2 = I$ where needed to use these two relations.

$$
\begin{aligned}
&K_2'(v)R_{21}^t(u+v)K_1'(u)R_{12}(u-v) = R_{21}(u-v)K_1'(u)R_{12}^t(-u-v)K_2'(v)\\
&C_2K_2(v)C_1R_{21}^T(u+v)C_1C_1K_1(u)R_{12}(u-v) = R_{21}(u-v)C_1K_1(u)C1R_{12}^T(-u-v)C_1C_2K_2(v)\\
&C_2C_1K_2(v)R_{21}^T(u+v)K_1(u)R_{12}(u-v) = R_{21}(u-v)C_1K_1(u)C1R_{12}^T(-u-v)C_1C_2K_2(v)\\
&K_2(v)R_{21}^T(u+v)K_1(u)R_{12}(u-v) = C_2C_1R_{21}(u-v)C_1K_1(u)C1R_{12}^T(-u-v)C_1C_2K_2(v)\\
&K_2(v)R_{21}^T(u+v)K_1(u)R_{12}(u-v) = C_2C_1R_{21}(u-v)C_1C_2K_1(u)C_2C1R_{12}^T(-u-v)C_1C_2K_2(v)\\
&K_2(v)R_{21}^T(u+v)K_1(u)R_{12}(u-v) = R_{21}(u-v)K_1(u)R_{12}^T(-u-v)K_2(v)
\end{aligned}
$$

- Relation between 1 and 3:
  Given a matrix P such that

$$P_1P_2R(u)P_1^{-1}P_2^{-1} = \tilde{R}(u) \tag{A.18}$$

$$\text{and} \quad PP^T = C \tag{A.19}$$

then

$$\tilde{K}(u) = PK(u)P^{-1} \tag{A.20}$$

<u>Proof</u>: The first step is to show that $PP^T = C \Rightarrow \tilde{R}^t(u) = P_1P_2R^T(u)P_1^{-1}P_2^{-1}$:

$$
\begin{aligned}
\tilde{R}^t(u) &= C_1\tilde{R}^{T_1}C_1\\
&= C_1(P_1^{-1})^T P_2R^T(u)P_1^TP_2^{-1}\\
&= P_1P_2R^T(u)P_1^{-1}P_2^{-1}
\end{aligned}
$$

where we used that $PP^T = C \Rightarrow P^T = P^{-1}C$ and $P = C(P^{-1})^T$. Now, the process of showing that the solution transformation is correct is similar to that of the previous part. Start from

$$K_2(v)R_{21}^T(u+v)K_1(u)R_{12}(u-v) = R_{21}(u-v)K_1(u)R_{12}^T(-u-v)K_2(v)$$

and multiply by P_1P_2 from the left and P_1^{-1}, P_2^{-1} from the right to obtain

$$P_2K_2(v)P_1R_{21}^T(u+v)K_1(u)R_{12}(u-v)P_1^{-1}P_2^{-1} = P_2P_1R_{21}(u-v)K_1(u)R_{12}^T(-u-v)P_1^{-1}K_2(v)P_2^{-1}.$$

Now insert some $PP^{-1} = I$ to obtain

$$
\begin{aligned}
&P_2K_2(v)P_2^{-1}P_1P_2R_{21}^T(u+v)P_2^{-1}P_1^{-1}P_1K_1(u)P_1^{-1}P_1P_2R_{12}(u-v)P_1^{-1}P_2^{-1} =\\
&P_2P_1R_{21}(u-v)P_2^{-1}P_1^{-1}P_1K_1(u)P_1^{-1}P_1P_2R_{12}^T(-u-v)P_1^{-1}P_2^{-1}P_2K_2(v)P_2^{-1}
\end{aligned}
$$

and use the relations between $K(u), R(u), R^T(u)$ and $\tilde{K}(u), \tilde{R}(u), \tilde{R}^t(u)$ to see that this is equivalent to

$$\tilde{K}_2(v)\tilde{R}_{21}^t(u+v)\tilde{K}_1(u)\tilde{R}_{12}(u-v) = \tilde{R}_{21}(u-v)\tilde{K}_1(u)\tilde{R}_{12}^t(-u-v)\tilde{K}_2(v),$$

which is what we wanted to show.

Note that due to the crossing symmetry of the $SO(N)$ R-matrix, versions 1 and 3 are equivalent to the untwisted BYB relations

$$K_2(v)R_{21}(u+v-\kappa)K_1(u)R_{12}(u-v) = R_{21}(u-v)K_1(u)R_{12}(-u-v-\kappa)K_2(v) \tag{A.21}$$

$$K_2(v)\tilde{R}_{21}(u+v-\kappa)K_1(u)\tilde{R}_{12}(u-v) = \tilde{R}_{21}(u-v)K_1(u)\tilde{R}_{12}(-u-v-\kappa)K_2(v) \tag{A.22}$$

It might then seem like the constraint $PP^T = C$ on the basis transformation is redundant, since the transposed R-matrices do not show up. However, one should remember that these relations follow from the original ones (A.15),(A.17) only if the crossing relations $R^T(u) = R(u-\kappa)$ and $\tilde{R}^t = \tilde{R}(u-\kappa)$ hold. The condition $R^T(u) = R(u-\kappa) \Leftrightarrow \tilde{R}^t = \tilde{R}(u-\kappa)$ leads to the same constraint for P.

B The G-matrix representations of so_5

The γ-matrix representation
The gamma matrices

$$\gamma_1 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{pmatrix} \tag{B.1}$$

$$\gamma_4 = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \ \gamma_5 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}, \tag{B.2}$$

where σ_i are the Pauli matrices, form a representation of the SO(5) Clifford algebra, i.e.

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} I_{4\times 4}. \tag{B.3}$$

Now, consider the matrices

$$\gamma_{ij} = \frac{1}{4i}[\gamma_i, \gamma_j]. \tag{B.4}$$

Using the identity

$$\big[[A, B], C\big] = 2A\{B, C\} - 2\{C, A\}B + C\{A, B\} - \{A, B\}C \tag{B.5}$$

one can obtain

$$[\gamma_{ij}, \gamma_k] = \frac{1}{i}(\delta_{jk}\gamma_i - \delta_{ki}\gamma_j), \tag{B.6}$$

from which it follows that the γ_{ij} generate a representation of the so_5 Lie algebra

$$\begin{aligned} [L_{ij}, L_{kl}] &= i(\delta_{jk}L_{il} + \delta_{il}L_{jk} - \delta_{ik}L_{jl} - \delta_{jl}L_{ik}) \\ L_{ij} &= -L_{ji} \end{aligned} \tag{B.7}$$

The highest weights of this representation are determined by diagonalizing the Cartan subalgebra. Since

$$[L_{12}, L_{34}] = 0 \tag{B.8}$$

and so_5 is rank 2, we can choose the Cartan subalgebra to be spanned by

$$h = \{L_{12}, L_{34}\}. \tag{B.9}$$

One can then find a common eigenvector of γ_{12}, γ_{34}, such that

$$\gamma_{12} \cdot v = \frac{1}{2} v \tag{B.10}$$

$$\gamma_{34} \cdot v = \frac{1}{2} v, \tag{B.11}$$

which implies that the γ_{ij} form the representation with highest weights $(\frac{1}{2}, \frac{1}{2})$.

Higher dimensional representations
A higher-dimensional representation of the Lie algebra can be constructed from the γ_i as

$$G_i = (\gamma_i \otimes I \cdots \otimes I + I \otimes \gamma_i \otimes \cdots \otimes I)_{sym} \oplus 0_{N-d_G}, \tag{B.12}$$

where the tensor product needs to be symmetrised for the representation to be irreducible. The dimension of these matrices is

$$d_G = \frac{1}{6}(n+1)(n+2)(n+3) \tag{B.13}$$

From (B.6) it follows that the matrices

$$G_{ij} = \frac{1}{4i}[G_i, G_j] \tag{B.14}$$

also fulfil the same relation, namely

$$[G_{ij}, G_k] = \delta_{jk}G_i - \delta_{ki}G_j. \tag{B.15}$$

and therefore form a representation of (B.7). To determine the highest weights, note that the n-th tensor power of the vector v introduced previously is an eigenvector of G_{12} and G_{34}. In particular

$$\begin{aligned} G_{12} \cdot (v \otimes \cdots \otimes v) &= (\frac{1}{2} + \cdots + \frac{1}{2})v = \frac{n}{2}v \\ G_{34} \cdot (v \otimes \cdots \otimes v) &= (\frac{1}{2} + \cdots + \frac{1}{2})v = \frac{n}{2}v, \end{aligned} \tag{B.16}$$

thus the G_{ij} generate the so_5 representation with highest weights $(\frac{n}{2}, \frac{n}{2})$.

The complex matrices $\tilde{G}_i$
Let us introduce the matrices

$$\tilde{G}_{\pm 1} = \frac{1}{\sqrt{2}}(G_1 \pm iG_2) \tag{B.17}$$

$$\tilde{G}_{\pm 2} = \frac{1}{\sqrt{2}}(G_3 \pm iG_4) \tag{B.18}$$

$$\tilde{G}_0 = G_5. \tag{B.19}$$

and define

$$F_{ij} = \frac{1}{4}[\tilde{G}_i, \tilde{G}_{-j}]. \tag{B.20}$$

The equivalent of (B.15) is now

$$[F_{ij}, \tilde{G}_k] = \delta_{jk}\tilde{G}_i - \delta_{-i,k}\tilde{G}_{-j}, \tag{B.21}$$

which one can use to show that

$$\begin{aligned} [F_{ij}, F_{kl}] &= \delta_{jk}F_{il} - \delta_{il}F_{kj} + \delta_{j,-l}F_{k,-i} - \delta_{i,-k}F_{-j,l} \\ F_{i,j} &= -F_{-j,-i}. \end{aligned} \tag{B.22}$$

Therefore, the F_{ij} defined here form a representation of the so_5 algebra using the convention in the main text (5.23). By substituting (B.17)-(B.19) in (B.20) one can see that

$$F_{11} = -G_{12} \tag{B.23}$$
$$F_{22} = -G_{34}. \tag{B.24}$$

It then follows from (B.16) that

$$F_{11} \cdot v = -\frac{n}{2} v \tag{B.25}$$
$$F_{22} \cdot v = -\frac{n}{2} v, \tag{B.26}$$

thus the F_{ij} form the $(-\frac{n}{2}, -\frac{n}{2})$ representation of this so_5 algebra. The interpretation of the negative highest weights is that v needs to be treated as a lowest weight state, namely a state that is annihilated by all lowering generators.