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A thesis presented to the Faculty of Science in partial fulfillment of the requirements for the degree

Doctor of Philosophy in Physics

The On-Shell Highway To Waveforms

Andrea Cristofoli

Supervisor: Poul Henrik Damgaard

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ABSTRACT

This thesis explores several dynamical aspects of the classical two-body problem using scattering amplitude methods. We first discuss the conservative two-body dynamics in General Relativity in the post-Minkowskian regime, valid for weak gravitational fields and unbound velocities. In doing so, we present novel relations between on-shell scattering amplitudes and classical quantities such as the post-Minkowskian Hamiltonian and scattering angle. We then focus on modern amplitude methods, presenting the string inspired CHY formalism. As an application, we derive covariant expressions for tree-level amplitudes with two massive scalars and an arbitrary number of gravitons, providing the needed input to evaluate classical observables from unitarity methods. We then study classical wave physics, showing how to derive from amplitudes gravitational shock wave solutions to Einstein field equations. We conclude by presenting a generalization of a formalism developed by David Kosower, Ben Maybee and Donal O'Connell, which describes black hole dynamics from on-shell data. With the addition of coherent states in the original framework, we study observables with massless particles as incoming states, such as the bending of light and the Thomson scattering. We also derive

waveforms generated in binary scattering, providing an elegant relation between the Newman-Penrose formalism and helicity amplitudes.

RESUMÉ

I denne afhandling undersøges flere dynamiske aspekter af det klassiske problem med to legemer ved hjælp af spredningsamplitude-metoder. Vi diskuterer først den konservative dynamik for to legemer i den generelle relativitetsteori i det post-Minkowskiske regime, som gælder for svage gravitationsfelter og ubundne hastigheder. I den forbindelse præsenterer vi nye relationer mellem on-shell spredningsamplituder og klassiske størrelser som f.eks. den post-Minkowskiske Hamiltonian og spredningsvinklen. Derefter fokuserer vi påmoderne amplitudemetoder og præsenterer den stringinspirerede CHY-formalisme. Som en anvendelse udleder vi kovariante udtryk for amplituder påtræniveau med to massive scalarer og et vilkårligt antal gravitoner, hvilket giver det nødvendige input til at evaluere klassiske observabler fra unitaritetsmetoder. Derefter fokuserer vi påklassisk bølgefysik og viser, hvordan man ud fra amplituder kan udlede gravitationelle chokbølgeløsninger til Einsteins feltligninger. Vi afslutter med at præsentere en generalisering af en formalisme udviklet af David Kosower, Ben Maybee og Donal O'Connell, som beskriver dynamikken i sorte huller ud fra data om skallen. Med tilføjelsen af kohærente tilstande i den oprindelige ramme studerer vi observabler med masseløse partikler som indkommende tilstande, såsom bøjning af lys og Thomson-spredning. Vi udleder ogsåbølgeformer, der genereres i binær spredning, hvilket giver en elegant relation mellem Newman-Penrose-formalismen og helicity-amplituder.

PREFACE

This thesis contains results originating from a three-year research project in general relativity and theoretical physics. The main topics are classical gravitational physics and scattering amplitudes, divided into three parts:

- The-two body problem in General Relativity
- Scattering amplitude methods
- Classical wave physics

The main text consists of reprints of preprints and published journal articles.

REPRINTS

The thesis is based on the following publications and preprints:

- A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, "Post-Minkowskian Hamiltonians in general relativity", Phys. Rev. D 100 (2019), 084040 [1906.01579]
- A. Cristofoli, "Post-Minkowskian Hamiltonians in modified theories of gravity", Phys. Lett. B 800 (2020) 135095 [1906.05209]
- N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, "Post-Minkowskian Scattering Angle in Einstein Gravity", JHEP 08 (2020) [1910.09366]
- A. Cristofoli, P. H. Damgaard, P. Di Vecchia and C. Heissenberg, "Second-order Post-Minkowskian scattering in arbitrary dimensions", JHEP 122 (2020) [2003.10274]
- N. Bjerrum-Bohr, A. Cristofoli, P. H. Damgaard and H. Gomez, "Scalar-Graviton Amplitudes", JHEP 11 (2019) [1908.09755]

- A. Cristofoli, “Gravitational shock waves and scattering amplitudes”, JHEP 11 (2020) [2006.08283]

- A. Cristofoli, R. Gonzo, D. Kosower and D. O’Connell, ”Waveforms from amplitudes”, [2107.10193]

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CONTENTS

Part I

INTRODUCTION

1

MOTIVATION

The start of the gravitational-wave observational era has spurred theorists to explore new approaches to computing classical observables for the general relativistic two-body problem. Remarkably, in recent years, state of the art results have been obtained using quantum scattering amplitudes. One of the main objectives of this thesis is to extend this new paradigm, increasing the range of observables that we can classically describe in terms of scattering amplitudes. This would simultaneously improve our ability to develop more accurate gravitational wave templates and illuminate underlying structures in classical field theory.

2

THESIS OUTLINE

The main results of the thesis are presented into three parts:

- Part 2: The two-body problem in General Relativity
- Part 3: Scattering amplitude techniques
- Part 4: Classical wave physics

Part 2 contains the following chapters:

- Chapter 4 is an invitation to the classical two-body problem in General Relativity. It explores the relations between classical and quantum scattering, providing a summary of the main ideas addressed in the subsequent chapters.
- Chapter 5 is a reprint of A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, "Post-Minkowskian Hamiltonians in general relativity", Phys. Rev. D 100 (2019), 084040 [1906.01579], containing modifications of the original publication.

- Chapter 6 is a reprint of A. Cristofoli, ”Post-Minkowskian Hamiltonians in modified theories of gravity”, Phys. Lett. B 800 (2020) 135095 [1906.05209], containing minor modifications of the original publication.

- Chapter 7 is a reprint of N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, “Post-Minkowskian Scattering Angle in Einstein Gravity”, JHEP 08 (2020) [1910.09366], containing modifications of the original publication.

- Chapter 8 is a reprint of A. Cristofoli, P. H. Damgaard, P. Di Vecchia and C. Heissenberg, “Second-order Post-Minkowskian scattering in arbitrary dimensions”, JHEP 122 (2020) [2003.10274], containing minor modifications of the original publication.

Part 3 contains the following chapter:

- Chapter 9 is a reprint of N. Bjerrum-Bohr, A. Cristofoli, P. H. Damgaard and H. Gomez, “Scalar-Graviton Amplitudes”, JHEP 11 (2019) [1908.09755].

Part 4 contains the following chapters:

- Chapter 10 is a reprint of A. Cristofoli, “Gravitational shock waves and scattering amplitudes”, JHEP 11 (2020) [2006.08283]. It contains minor modifications of the original publication.

- Chapter 11 is a reprint of A. Cristofoli, R. Gonzo, D. Kosower and D. O’Connell, ”Waveforms from amplitudes”, [arXiv:2107.10193].

The thesis ends with the conclusions in Chapter 12 and an outlook on future work in Chapter 13.

3

INTRODUCTION

> "While the science is advancing thus in one direction by the improvement of physical views, it may advance in another direction also by the invention of mathematical methods."
>
> — W.R. Hamilton, 1834

Few problems in theoretical physics have seen more study and reformulations as the two-body problem. From its first description by Newton [1], till the geodesic reinterpretation in General Relativity [2], up to the two-body scattering in quantum mechanics [3]: its development, from classical to quantum physics, constantly has shed light on new mathematical tools, from Lagrangian to Hamiltonian mechanics, as well as providing insights to new mathematical and physical theories [4, 5]. Nowadays, among its different formulations, the classical gravitational two-body problem has gained a renewed interest due to the fingerprints left in gravitational waves. These signals, detected for the first time by the LIGO/Virgo collaboration in 2015 [6], describes the gravitational radiation emitted by the inspiral, merger and subsequent ringdown of a binary system. Unfortunately, despite its Newtonian analogue, the general relativistic two-body problem admits no exact solution available with analytical tools. To overcome this obstacle, theorists have combined different types of analytical approximations, and numerical simulations [7,8], with the aim of generating accurate gravitational wave templates of the radiation emitted by coalescing black holes. Among the analytical schemes used, the post-Minkowskian approximation to General Relativity [9], valid during the inspiral for relativistic systems and weak gravitational fields, is the one that has seen the most radical revolution in recent years. This paradigm change was marked by a remarkable state of the art computation [10] using scattering amplitudes, the primary tool to describe interactions of quantum particles in quantum field theory. The achievement has been outstanding: following earlier works [11,12], a state of the art result for macroscopic black holes has been derived from the classical limit of the most microscopic structures in quantum field theory, scattering amplitudes [13]. This milestone has marked the beginning of a new field at the crossroads between classical gravitational physics and quantum field theory, with potential implications for improving gravitational wave templates and our understanding of classical physics [14]. In this thesis, we will apply and extend this new paradigm. In the first part, we will provide novel relations between classical observables for spinless binary black holes and on-shell scattering amplitudes computed in quantum field theory [15–18]. We will see how the conservative regime of the two-body problem in General Relativity can be efficiently described in terms of amplitudes, using the Lippmann-Schwinger equation beyond its well known non-relativistic approximation. In the second part of the thesis, we will explore modern scattering amplitudes techniques such as the string inspired CHY formalism, which provides the needed inputs for the computation of post-Minkowskian observables from unitarity techniques [19]. In the third part, we will focus on classical wave physics. We will first show how to derive gravitational shock waves solutions to Einstein field equations using scattering amplitudes [20]. We will then present a generalization of a formalism developed by David Kosower, Ben Maybee and Donal O'Connell, which describes black hole dynamics from on-shell data [21]. With the addition of coherent states in the original framework, we will study observables with massless particles as incoming states, such as the bending of light and the Thomson scattering. We will also derive waveforms generated in binary scattering, providing an elegant relation between the Newman-Penrose formalism and helicity amplitudes [22]. We will then end the thesis with a general outlook on the results in the thesis and the next challenges ahead.

Part II

THE TWO-BODY PROBLEM IN GENERAL RELATIVITY

4

THE S-MATRIX APPROACH TO GENERAL RELATIVITY

With the publication of the Principia in 1687 [1], Newton provided for the first time a physical theory that accounted for the dynamics of gravitational binary systems. Arguing from this theory, he showed that the orbit of any binary acted upon by a gravitational force is always a conic section. Despite the knowledge of the exact orbits, this problem has been revisited and generalized several times over the past centuries, leading to the development of new mathematical methods such as Lagrangian and Hamiltonian mechanics, the Hamilton–Jacobi equations [4], the notion of integrability, and the discovery of the angle-action variables [5]. As we will see, scattering amplitudes and quantum field theory are nowadays providing a new understanding of the two body problem in General Relativity, similar to what Lagrangian mechanics did for Newtonian gravity. To see how this is possible, we first need to understand the intricacy of the general relativistic two body problem which is a far more intricate problem than its Newtonian counterpart. We start following [23,24], and by modeling a general relativistic binary system, with masses $m_{a=1,2}$, using two worldlines $x_a^\mu(\sigma_a)$, parametrized by $\sigma_a = \frac{\tau_a}{m_a}$, being τ_a their proper time. The geodesic equation for each worldline will then follow from Hamilton's equations with a Hamiltonian given by

$$\mathcal{H}(x_a, p_a) = \sum_{a=1,2} \frac{1}{2} g^{\alpha\beta}(x_a) p_{a,\alpha} p_{a,\beta} \; . \tag{1}$$

These are

$$\frac{dx_a^\mu}{d\sigma_a}(\sigma_a) = g^{\mu\nu}\left(x_a(\sigma_a)\right) p_{a\nu}(\sigma_a) \quad , \quad \frac{dp_{a,\mu}}{d\sigma_a}(\sigma_a) = -\frac{1}{2}\partial_\mu g^{\alpha\beta}\left(x_a(\sigma_a)\right) p_{a,\alpha}(\sigma_a) p_{a,\beta}(\sigma_a) \; . \tag{2}$$

The gravitational field $g_{\mu\nu}(x)$ generated by their motion satisfies Einstein equations

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = 8\pi G_N T_{\mu\nu}(x), \tag{3}$$

where the energy momentum tensor of the binary system is given by

$$T^{\mu\nu}(x) = \sum_{a=1,2} \int_{-\infty}^{+\infty} d\sigma_a \, p_a^\mu(\sigma_a) p_a^\nu(\sigma_a) \frac{\delta^4\left(x - x_a\left(\sigma_a\right)\right)}{\sqrt{-\det[g_{\alpha\beta}(x_a(\sigma_a))]}} \; . \tag{4}$$

For well defined initial data, we say that $[x_1(\sigma_1), x_2(\sigma_2), g_{\mu\nu}(x)]$ is a solution to the two problem in General Relativity if (2) and (3) are satisfied simultaneously. To get a glimpse of the intricacy of a similar endeavor, we can consider a scattering-like encounter by solving perturbatively the dynamics in the gravitational coupling G_N order by order. We focus on the impulse experienced during the scattering

$$\Delta p_{a,\mu} = -\frac{1}{2} \int_{-\infty}^{+\infty} d\sigma_a \partial_\mu g^{\alpha\beta}(x_a(\sigma_a)) p_{a,\alpha}(\sigma_a) p_{a,\beta}(\sigma_a) \; . \tag{5}$$

Assuming the knowledge of both worldlines, we start by solving Einstein equations in the controvariant metric $g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x)$ at leading order in G_N. In de Donder gauge

$$\Box h^{\mu\nu}(x) = -16\pi G_N \sum_{a=1,2} \int_{-\infty}^{+\infty} d\sigma_a \, \mathcal{P}^{\mu\nu;\alpha\beta} \, p_{a,\alpha}(\sigma_a) p_{a,\beta}(\sigma_a) \delta^4 \left(x - x_a\left(\sigma_a\right)\right) \, , \tag{6}$$

where we have introduced the projector $\mathcal{P}^{\mu\nu;\alpha\beta} = \left(\eta^{\mu\alpha}\eta^{\nu\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)$. Since radiative effects start to appears only at G_N^3 order [25, 26], we can restrict the problem to the conservative sector by choosing a time-symmetric Green function. We obtain

$$h^{\mu\nu}(x) = 4G_N \int d^4y \, \mathcal{P}^{\mu\nu;\alpha\beta}(x-y) T_{\alpha\beta}(y) + O\left(G_N^2\right) \, , \tag{7}$$

where we have introduced

$$\mathcal{P}^{\mu\nu;\alpha\beta}(x-y) = 4\pi \left(\eta^{\mu\alpha}\eta^{\nu\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right) \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2} \, . \tag{8}$$

We can now substitute (7) in (5). At leading order in G_N, the impulse on the $a = 1$ component of the binary is given by two contributions,

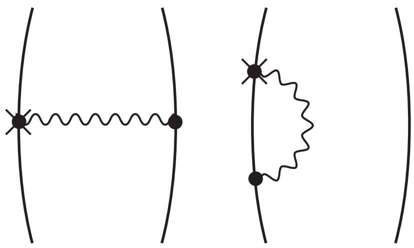
$$\Delta p_{1\mu} = \Delta p_{1\mu}^{tree} + \Delta p_{1\mu}^{self} + O\left(G_N^2\right) \, , \tag{9}$$

where we have defined

$$\Delta p_{1,\mu}^{tree} \equiv 2G_N \int d\sigma_1 d\sigma_2 p_{1,\alpha}(\sigma_1) p_{1,\beta}(\sigma_1) \, \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'} \left(x_1\left(\sigma_1\right) - x_2\left(\sigma_2\right)\right) p_{2,\alpha'}(\sigma_2) p_{2,\beta'}(\sigma_2) \, , \tag{10}$$

$$\Delta p^{self}_{1,\mu} \equiv 2G_N \int d\sigma_1 d\sigma_1' p_{1,\alpha}(\sigma_1) p_{1,\beta}(\sigma_1) \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'} \left(x_1\left(\sigma_1\right) - x_1\left(\sigma_1'\right)\right) p_{1,\alpha'}(\sigma_{1'}) p_{1,\beta'}(\sigma_{1'}) \,. \tag{11}$$

Interestingly, we can rewrite both contributions as Feynman-like diagrams [23, 24] . The first term corresponds to a tree-level topology where we have added a cross to denote a derivative in the Green function. The second one corresponds instead to a self-interaction for one of the constituents of the binary:[1]



The analogy can be extend also to higher order contributions to the impulse, where more convoluted topologies will appear containing also self-interactions of the gravitational field. However, explicit calculations are viable only once the momenta $p_{a,\mu}(\sigma_a)$ are known, which for the moment we haven't specified yet. Considering we are interested in a scattering problem, we parametrize both worldlines in a perturbative expansion around a free motion

$$\begin{aligned} x^\mu_a\left(\sigma_a\right) &= y^\mu_{0,a} + \sigma_a p^\mu_{0,a} + G_N x^\mu_a\left(\sigma_a\right) + \dots \,, \\ p_{a,\mu}\left(\sigma_a\right) &= {}_0p_{a,\mu}\left(\sigma_a\right) + G_N p_{a,\mu}\left(\sigma_a\right) + \dots \,, \end{aligned} \tag{12}$$

1 Reproduced from [24].

where $y^{\mu}_{0,a}$ is a constant four-vector and the momentum ${}_0p_{a,\mu}$ is given by the on-shell quantity in flat space satisfying ${}_0p^{\mu}_{a}\,{}_0p^{\nu}_{a}\eta_{\mu\nu} = -m_a^2$. For ease of notation, in what follows we remove the subscript 0 in the incoming momentum. Using (12), we can now compute both contributions to the impulse. The first term is given by

$$\Delta p^{tree}_{1,\mu} = 8\pi G_N p_{1,\alpha} p_{1,\beta}\, \mathcal{P}^{\alpha\beta;\alpha'\beta'} p_{2,\alpha'} p_{2,\beta'} \int d\sigma_1 d\sigma_2 \int \frac{d^4k}{(2\pi)^4} \frac{ik^{\mu} e^{ik\cdot b}}{k^2} e^{ik\cdot p_1\sigma_1} e^{-ik\cdot p_2\sigma_2}\,, \tag{13}$$

where the covariant impact parameter b^{μ} is defined as $b^{\mu} = y^{\mu}_{0,1} - y^{\mu}_{0,2}$. We obtain

$$\Delta p_{1,\mu} = -2G_N \frac{2(p_1\cdot p_2)^2 - m_1^2 m_2^2}{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \frac{b_{\mu}}{b^2}\,, \tag{14}$$

where we have introduced $b = \sqrt{-b^{\mu}b_{\mu}}$. As for the remaining contribution to the impulse, this is vanishing since proportional to a scaleless integral. Thus, (14) is the value for the impulse at leading order in G_N. From it, we can extract other observables of physical interests, such as the scattering angle in the centre of mass frame. From geometrical considerations, it is simple to show that the scattering angle is given by

$$\sin(\theta) = \frac{\Delta p_{\mu} b^{\mu}}{p_{cm}\, b}\,, \tag{15}$$

where p_{cm} is the modulus of the relative momentum in the center of mass frame. Truncating at leading order in G_N, we obtain the following result

$$\theta_{1PM} = \frac{2G_N}{L} \frac{2\left(p_1\cdot p_2\right)^2 - m_1^2 m_2^2}{\sqrt{\left(p_1\cdot p_2\right)^2 - p_1^2 p_2^2}}\,, \tag{16}$$

where we have introduced the total angular momentum $L = p_{cm}b$. This result is fully relativistic and we say it describes a first post-Minkowskian order calculation (i.e. at linear order in G_N and to all orders in the velocities). We can easily understand that to get a complete knowledge of this observable, all post-Minkowksian contributions need to be calculated. This would require a systematic expansion to higher orders, followed by a delicate interplay of conservative and radiative effects, which for the moment we have ignored. As such, it should not be surprising that the state of the art result was given only by the next to leading order calculation since a few years ago. The calculation at second post-Minkowskian order, published by Westpfhal in 1985 [27], shows many complicated contributions related to the nonlinearity of the field equations and the iteration of the equations of motion. However, the final result is very simple

$$\theta_{2PM} = \frac{G_N^2}{L^2} \frac{3\pi(m_1 + m_2)(5(p_1 \cdot p_2)^2 - m_1^2 m_2^2)}{4E}, \qquad (17)$$

where we have introduced E as the total energy of the system. This is quite reminiscent of complicated Feynman diagram calculations boiling down to shockingly simple results in quantum field theory [28]. As we will see in the next section, this observation, together with the aforementioned diagrammatic expansion, is not a coincidence but the hint that a deeper reformulation of the two-body problem in General Relativity is possible.

THE QUANTUM TWO-BODY PROBLEM IN GENERAL RELATIVITY

This section aims to highlight the close analogy between the theory of scattering in the classical and quantum theory. We have already seen that a scattering encounter in General Relativity is a highly complicated problem, so the reader might wonder what can be gained by looking at its even more intricate quantum counterpart. Moreover, we don't even have a consistent theory of quantum gravity, so how could we even describe a quantum analogue for this two-body problem? Some clarifications are in order. Throughout the years, it has been proven that the quantization of gravity around a flat background makes a well defined effective field theory for energy scales well below the Planck scale [29]. Within this approach, it is possible to compute classical phenomena such as the perihelion-motion of Mercury [30], as well as to compute quantum corrections to the Newton potential. It is precisely in this sense that quantum field theory reveals an unexpected simplicity than the classical theory itself, especially when we are dealing with the weak field limit of General Relativity. To see how this remarkable change of paradigm occurs, we first start by introducing few crucial notions in quantum scattering theory that will allow us to relate the quantization of gravity in the weak-field limit with the calculation of classical observables. Let us start by considering the scattering of two point particles adopting a first quantization scheme. In the center of mass frame, the problem is equivalent to the scattering of a single particle off a given potential.

Thus, in analogy with the classical scattering, the worldlines $x_a^\mu(\sigma_a)$ are replaced by a state $|\Psi(t)\rangle \in L^2(\mathbb{R}^3)$ satisfying the evolution equation

$$i\hbar \frac{\partial |\Psi(t)\rangle}{\partial t} = \hat{H} |\Psi(t)\rangle \qquad \forall t \in \mathbb{R} \,, \tag{18}$$

for a proper self-adjoint Hamiltonian operator $\hat{H}$. In the center of mass frame, the Hamiltonian for a conservative and fully relativistic system is given by

$$\hat{H} = \hat{H}_0 + \hat{V} \quad , \quad \hat{H}_0 = \sqrt{\hat{p}^2 + m_1^2} + \sqrt{\hat{p}^2 + m_2^2} \,, \tag{19}$$

where $\hat{V}$ denotes a generic potential operator. Since (19) is time independent, we can provide the most general solution to (18) as

$$|\Psi(t)\rangle = e^{\frac{i}{\hbar}(t-t_0)\hat{H}} |\Psi\rangle \ . \tag{20}$$

The initial state is then given by

$$|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} \, \varphi(p) \, e^{ip\cdot r} \, |p\rangle \ , \tag{21}$$

which for a gaussian wavepacket $\varphi(p)$ it describes a classical system of two point-particles separated by a distance r. This allows us to introduce the so called S-matrix

$$S = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} e^{it_f \hat{H}_0} e^{-i(t_f - t_i)\hat{H}} e^{-it_i \hat{H}_0} \ , \tag{22}$$

an operator of fundamental importance in scattering theory, relating asymptotic states in the infinite past and future by means of a unitary operator. Having introduced the S-matrix, we proceed by introducing two other operators of importance in our goal to relate quantum field theory to classical physics: the free Green operator $\hat{G}_0(z)$ and the interacting one $\hat{G}(z)$ defined as

$$\hat{G}_0(z) \equiv (z - \hat{H}_0)^{-1} \quad , \quad \hat{G}(z) \equiv (z - \hat{H})^{-1} \qquad z \in \mathbb{C} \, . \tag{23}$$

Both are complex analytic functions with a simple pole for the eigenvalues of the Hamiltonian, as it can be easily seen assuming the knowledge of the eigenstates. Considering for simplicity only a discrete spectrum, we can label a complete set of states as $|n\rangle$. Thus

$$\hat{G}(z) = \sum_n \frac{|n\rangle \langle n|}{z - E_n} \quad , \quad \forall z \in \mathbb{C} \setminus \{E_n\} \, . \tag{24}$$

Using the relation $A^{-1} = B^{-1} + B^{-1}(B - A)A^{-1}$, we can relate both Green operators by

$$\hat{G}(z) = \hat{G}_0(z) + \hat{G}_0(z)\hat{V}\hat{G}(z) \, . \tag{25}$$

It is useful to express the same relation in terms of the so called off-shell scattering operator $\hat{\mathbb{M}}(z)$

$$\hat{\mathbb{M}}(z) \equiv \hat{V} + \hat{V}\hat{G}(z)\hat{V} \, . \tag{26}$$

Using this definition, we can rewrite (25) by means of algebraic manipulations obtaining the so called Lippmann-Schwinger equation for the scattering operator

$$\hat{\mathbb{M}}(z) = \hat{V} + \hat{V}\hat{G}_0(z)\hat{\mathbb{M}}(z) \, . \tag{27}$$

This equation is of fundamental importance in our approach. As we are going to see, it allows us to compute a classical Hamiltonian once the matrix elements of $\hat{\mathbb{M}}(z)$ are known. To show this relation, we evaluate the mean value of the Lippmann-Schwinger equation (27) on the initial state (21), describing a binary system of two point particles

$$\langle\Psi|\,\hat{\mathbb{M}}(z)\,|\Psi\rangle = \langle\Psi|\,\hat{V}\,|\Psi\rangle + \langle\Psi|\,\hat{V}\hat{G}_0(z)\hat{\mathbb{M}}(z)\,|\Psi\rangle \ . \tag{28}$$

Considering the classical limit as in [21], and by using a completeness relation we obtain

$$\langle p'|\,\hat{\mathbb{M}}(z)\,|p\rangle = \widetilde{V}(q,p) + \int \frac{d^3k}{(2\pi\hbar)^3}\,\frac{\langle p'|\,\hat{\mathbb{M}}(z)\,|k\rangle\,\widetilde{V}(k,p)}{E_p - E_k + i\varepsilon}\ , \tag{29}$$

where we have defined the classical potential in momentum space as $\widetilde{V}(q,p)$ where q stands for $\vec{q} = \vec{p} - \vec{p}\,'$. What is missing now is the knowledge of the matrix elements of $\hat{\mathbb{M}}$. To see how it is related to the S matrix, it is enough to compute the matrix element of (22) between two single particle states $|p\rangle$ and $|p'\rangle$

$$\langle p'|\,S\,|p\rangle = \lim_{\substack{t_i\to-\infty\\ t_f\to\infty}} \langle p'|\,e^{it_f\hat{H}_0}e^{-i(t_f-t_i)\hat{H}}e^{-it_i\hat{H}_0}\,|p\rangle \ . \tag{30}$$

Since the order we perform both limits is irrelevant, we can set $t_f = -t_i$ to obtain

$$\langle p'|\,S\,|p\rangle = \lim_{t\to\infty} \langle p'|\,e^{it\hat{H}_0}e^{-i2t\hat{H}}e^{it\hat{H}_0}\,|p\rangle \ . \tag{31}$$

Using the properties of uniformly convergent integrals, we rewrite (31) as

$$\langle p'|\,S\,|p\rangle = \langle p'|p\rangle + \langle p'|\lim_{\varepsilon\to0^+}\int_0^{+\infty} dt\, e^{-\varepsilon t}\,\frac{d}{dt}\left[e^{it\hat{H}_0}e^{-i2t\hat{H}}e^{it\hat{H}_0}\right]|p\rangle \ . \tag{32}$$

$$= \langle p'|p\rangle - i \langle p'| \lim_{\varepsilon\to 0^+} \int_0^{+\infty} dt \left[\hat{V} e^{it(E_{p'}+E_p+i\varepsilon-2\hat{H})} + e^{it(E_{p'}+E_p+i\varepsilon-2\hat{H})} \hat{V} \right] |p\rangle \ , \quad (33)$$

where $E_p = \sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2}$ stands for the eigenvalue of the free Hamiltonian on a state $|p\rangle$. The remaining integration can be easily performed by showing that the elements of the S-matrix depends on a specific value of the Green operator $\hat{G}(z)$, precisely

$$\langle p'| S |p\rangle = \langle p'|p\rangle + \frac{1}{2} \langle p'| \lim_{\varepsilon\to 0^+} \left[\hat{V}\hat{G}\left(\frac{E_p+E_{p'}}{2} + i\varepsilon\right) + \hat{G}\left(\frac{E_p+E_{p'}}{2} + i\varepsilon\right)\hat{V} \right] |p\rangle \ . \quad (34)$$

At this point, we can rewrite the product of operators in the square bracket using (25) and

$$\hat{G}(z)\hat{V} = \hat{\mathbb{M}}(z)\hat{G}_0(z) \quad , \quad \hat{\mathbb{M}}(z)\hat{G}_0(z) = \hat{G}(z)\hat{V} \ . \quad (35)$$

Thanks to this representation, we have the following crucial relation between matrix elements of the S-matrix and $\hat{\mathbb{M}}$

$$\langle p'| S |p\rangle = \langle p'|p\rangle + \lim_{\varepsilon\to 0^+} \left(\frac{1}{E_{p'} - E_p + i\varepsilon} + \frac{1}{E_p - E_{p'} + i\varepsilon} \right) \langle p'| \hat{\mathbb{M}}\left(\frac{E_{p'}+E_p}{2} + i\varepsilon\right) |p\rangle \ , \quad (36)$$

$$= \langle p'|p\rangle - 2\pi i \delta(E_{p'} - E_p) \lim_{\varepsilon\to 0^+} \langle p'| \hat{\mathbb{M}}(E_p + i\varepsilon) |p\rangle \ . \quad (37)$$

We call scattering amplitude the matrix element $\langle p'| \hat{\mathbb{M}}(z) |p\rangle$ at $z = E_p + i0^+$. Given its importance we define it as

$$\lim_{\varepsilon \to 0^+} \langle p'| \hat{\mathbb{M}}(E_p + i\varepsilon) |p\rangle \equiv \widetilde{\mathcal{M}}(p, p') \ . \tag{38}$$

where the tilde denotes that these quantities are computed in quantum mechanics with fixed degrees of freedom and not in quantum field theory. Their relation in the center of mass frame is trivial and mainly given by a different normalization factor for the external states[2]

$$\widetilde{\mathcal{M}}(p, p') = \frac{\mathcal{M}(p, p')}{4E_1(p)E_2(p)} \ . \tag{39}$$

At this point, we have all the ingredients to compute the classical Hamiltonian of a relativistic binary system from the knowledge of a scattering amplitude. We can compute these quantities from the Einstein-Hilbert action minimally coupled to two massive scalar fields $\varphi_{a=1,2}$

$$\mathcal{S}(g_{\mu\nu}, \varphi_{a=1,2}) = \int d^4x \sqrt{-\det(g_{\mu\nu})} \left[\frac{R}{16\pi G_N} - \frac{1}{2} \sum_{a=1,2} \left(g^{\mu\nu} \hbar^2 \partial_\mu \varphi_a \, \partial_\nu \varphi_a + m_a^2 \varphi_a^2 \right) \right] . \tag{40}$$

We then expand the metric around a Minkowski background $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ and quantize the fluctuations assuming a weak field expansion for the gravitational field. In doing so, it is possible to define an effective field theory for a massless spin

2 For further details, see the discussion around (383).

2 particle - called graviton - interacting with massive scalars with masses $m_{a=1,2}$. Choosing the center-of-mass frame, we can describe a two-body scattering using the following momenta

$$\begin{aligned} p_1^\mu &= (E_1(p), \vec{p}\,)\,, \qquad & p_3^\mu &= (E_1(p), \vec{p}\,')\,, \\ p_2^\mu &= (E_2(p), -\vec{p}\,)\,, \qquad & p_4^\mu &= (E_2(p), -\vec{p}\,')\,. \end{aligned} \tag{41}$$

where p_1 and p_2 are incoming momenta and p_3 and p_4 outgoing. For the rest of the Chapter, we will continue to use the mostly positive signature convention and define

$$p \equiv |\vec{p}\,| = |\vec{p}\,'|\,, \tag{42}$$

$$E_1(p) \equiv \sqrt{p^2 + m_1^2}\,, \qquad E_2(p) \equiv \sqrt{p^2 + m_2^2}\,, \tag{43}$$

$$E_p \equiv E_1(p) + E_2(p)\,, \qquad \xi(p) \equiv \frac{E_1(p)E_2(p)}{E_p^2}\,. \tag{44}$$

As for the exchanged momentum in the center of mass frame, this is given by

$$q^\mu \equiv p_1^\mu - p_3^\mu \quad , \quad \vec{q} \equiv \vec{p} - \vec{p}\,' \quad , \quad q^\mu q_\mu = q^2 = |\vec{q}\,|^2 \tag{45}$$

Following these conventions, the tree level on-shell scattering amplitude reads

$$\mathcal{M}_{tree}(p, p') = 16\pi G_N \frac{[m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2 - \hbar^2 \bar{q}^2 (p_1 \cdot p_2)]}{\bar{q}^2}\,, \tag{46}$$

with $p_1 \cdot p_2 = -E_1(p)E_2(p) - p^2$. To highlight the $\hbar$ scaling we have expressed the momenta in terms of their wavenumbers using $q = \hbar\bar{q}$. Once this amplitude is known, the matrix element of the previously defined scattering operator are given by a simple rescaling due to the normalization of the external states

$$\widetilde{\mathcal{M}}_{tree}(p,p') = \frac{4\pi G_N}{E_1(p)E_2(p)} \frac{[m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2 - \hbar^2 \bar{q}^2 (p_1 \cdot p_2)]}{\bar{q}^2}, \tag{47}$$

If we now truncate (29) at linear order in the gravitational coupling, we obtain a linear relation between the Fourier transform of the amplitude and what we define as a first post-Minkowskian potential

$$V_{1PM}(r,p) \equiv \int \hat{d}^3\bar{q}\, \widetilde{\mathcal{M}}_{tree}(\bar{q},p) e^{ir\cdot\bar{q}}\ . \tag{48}$$

Using (47) and keeping only long range contributions to the potential, we obtain

$$V_{1PM}(r,p) = \frac{G_N c_1(p^2)}{E_p^2\, \xi(p)\, r}\ , \tag{49}$$

where

$$c_1(p^2) \equiv m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2 \quad , \quad \xi(p) \equiv \frac{E_1(p)E_2(p)}{E_p^2}\ . \tag{50}$$

This result provides an Hamiltonian describing a system of two classical point-particles at linear order in the coupling G_N and to all orders in the ratio v/c, being v a characteristic velocity of the system

$$H_{1PM}(r,p) = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + \frac{G_N c_1(p^2)}{E_p^2\, \xi(p)\, r}\ . \tag{51}$$

The non relativistic limit agrees with the Hamiltonian for a binary system in Newtonian gravity and in the center of mass frame. As we are going to see, using (51) we can easily re-derive observables such as the scattering angle at first post-Minkowskian order, as well as non trivial informations about a bound system. Let us focus on the former. Since the motion is planar, the conjugate variables describing the phase space are $(r, \varphi, p_r, p_\varphi)$. These are related to the square of the momentum in the center of mass frame by

$$p^2 = p_r^2 + \frac{L^2}{r^2} \, , \tag{52}$$

being L the conserved angular momentum of the binary system. At this point, the dynamics of the system is fully encoded in the Hamilton-Jacobi equation of (51)

$$\sum_{a=1,2} \sqrt{\left(\frac{\partial S}{\partial r}\right)^2 + \frac{L^2}{r^2} + m_a^2} + V_{1PM}\left(r, \frac{\partial S}{\partial r}, L\right) + \frac{\partial S}{\partial t} = 0 \, , \tag{53}$$

where S is the principal Hamilton function. We will see in Chapter 7 a remarkable relation which allows us to rewrite (53) as a quadratic differential equation in the derivatives of S, uniquely in terms of the scattering amplitudes. For the moment, we settle by solving this equation perturbatively in G_N for the ansatz $S = -Et + S_r(r)$, where $S_r(r)$ is known as radial action and E the conserved energy of the system. Defining its derivative as

$$p_r(r) \equiv \frac{\partial S_r(r)}{\partial r} \, , \tag{54}$$

we obtain

$$p_r^2(r) = p_0^2 - \frac{L^2}{r^2} + \frac{G_N f_1}{r} + \cdots \, , \tag{55}$$

where, introducing the Mandelstam variable $s = -(p_1 + p_2)^2$, we have

$$p_0^2 = \frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{s} \quad , \quad f_1 = -\frac{2c_1(p_0^2)}{\sqrt{s}} \, . \tag{56}$$

Using (55), it is now straightforward to derive the change in the angular variable φ during a scattering encounter (see for instance [23, 52])

$$\Delta\varphi = \pi + \theta, \tag{57}$$

where the scattering angle is given by

$$\theta = -2 \int_{r_{min}}^{+\infty} dr \, \frac{\partial p_r(r)}{\partial L} - \pi \, . \tag{58}$$

Here r_{min} is the positive root for the condition of turning point at $p_r(r) = 0$ with

$$p_r(r) = \sqrt{p_0^2 - p_0^2 \frac{b^2}{r^2} + \frac{G_N f_1}{r}} \, . \tag{59}$$

where we have introduced the impact parameter $b \equiv L/p_0$. We note now that $p_r(r)$ can be rewritten as

$$p_r(r) = \frac{p_0}{r} \sqrt{r^2 + r\frac{G_N f_1}{p_0^2} - b^2} = \frac{p_0}{r} \sqrt{r - r^+} \sqrt{r - r^-} \, , \tag{60}$$

$$r^{\pm} = -\frac{G_N f_1}{2p_0^2} \pm \sqrt{\frac{G_N^2 f_1^2}{4p_0^4} + b^2} \, . \tag{61}$$

Since $r_{min} = r^+$, the scattering angle becomes

$$\theta = 2 \int_{r^+}^{+\infty} \frac{dr}{r} \frac{b}{\sqrt{(r - r^+)(r - r^-)}} - \pi \; . \tag{62}$$

This integral can be performed analytically and we get

$$\theta = \frac{4b}{\sqrt{-r^+ r^-}} \arccos \sqrt{\frac{r^+}{r^+ - r^-}} - \pi \; . \tag{63}$$

Truncating the result at linear order in G_N we obtain

$$\theta_{1PM} = \frac{2G_N}{L} \frac{2(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \; , \tag{64}$$

in complete agreement with the first post-Minkowskian calculation of the scattering angle from the classical equations of motion (16). As we are going to see in the next Chapter, the power of our formalism is that it can be easily used to derive also the next-to-leading order contribution computed by Westpfahl (17), as well as higher order corrections, directly from scattering amplitudes.

5

POST-MINKOWSKIAN HAMILTONIANS IN GENERAL RELATIVITY

We describe the computation of post-Minkowskian Hamiltonians in General Relativity from scattering amplitudes. Using a relativistic Lippmann-Schwinger equation, we relate perturbative amplitudes of massive scalars coupled to gravity to the post-Minkowskian Hamiltonians of classical General Relativity to any order in Newton's constant. We illustrate this by deriving a Hamiltonian for binary black holes without spin up to 2nd order in the post-Minkowskian expansion and demonstrate explicitly the equivalence with the recently proposed method based on an effective field theory matching.

INTRODUCTION

The detection of gravitational waves by the LIGO/Virgo collaboration has opened up the exciting possibility of testing Einstein's theory of general relativity at a new and unprecedented level, including the regime of strong gravity as probed by black holes just prior to merging. A combination of Numerical Relativity and analytical methods is needed in order to push theory to the level where it can provide best-fit templates from which physical parameters can be extracted. This has spurred interest in new and innovative ideas that can facilitate computations of the required two-body interaction Hamiltonians to high accuracy.

Conventionally, the calculations of effective interaction Hamiltonians have been carried out in the systematic post-Newtonian expansion of General Relativity. The problem can, however, be attacked from an entirely different angle, that of relativistic scattering amplitudes as computed by standard quantum field theory methods in a quantum field theory of gravity coupled to matter [31]. Modern methods of amplitude computations greatly facilitate this program [10–12,32,35–38]. Incoming and outgoing particles in the scattering process are taken to past and future infinity where the metric by definition is flat Minkowskian, and the full metric is treated perturbatively around that Minkowskian background. The classical piece of the scattering amplitude solves the scattering problem of two black holes to the given order in Newton's constant G_N. When expanding to the appropriate post-Newtonian order and defining the interaction potential with the inclusion of the required lower-order Born subtractions as explained in detail in the next section, the amplitude also contains all the information of the bound state problem of two massive objects to the given order in the expansion in Newton's constant. For the bound-state regime one has, on account of the virial theorem, a double expansion in both Newton's constant and momentum. However, a more daring angle of attack is to treat the bound state problem as not expanded in momentum while still expanding to fixed order in Newton's constant. Such an approach has recently been proposed by Cheung, Rothstein and Solon [12], and it has already been pushed one order higher in the expansion in Newton's constant [10] (and compared to the post-Newtonian expansions in [14]). Here the method of effective field theory is used to extract the interaction Hamiltonian: the underlying Einstein-Hilbert action coupled to matter produces the classical part of the scattering amplitude while an effective theory of two massive objects define the interaction Hamiltonian. The correct matching between the two theories is performed by insisting that the two theories produce the same scattering amplitude to the given order in Newton's constant.

The post-Newtonian expansion (see, *e.g.*, refs. [39,40] for recent comprehensive reviews) of General Relativity dates back to the founding days of the theory. Its perturbation theory is ideal for the low-velocity situations of planetary orbits, satellites, and large-distance effects of General Relativity that occur at velocities far below the speed of light. In contrast to this, the computation of observables in General Relativity based on relativistic scattering amplitudes is valid for all velocities and in particular this is the proper framework for high energy scattering where kinetic energies can exceed potential energies by arbitrarily large amounts. This leads naturally to what has become known in the theory of General Relativity as the post-Minkowskian expansion [23, 24, 27, 41–43].

Extracting the interaction energy from the relativistic scattering amplitude, for consistency with the virial theorem in the bound-state problem one would perform a double expansion where velocity v and G_N are both kept to the appropriate order. To any given order in G_N this would imply a truncation of a Taylor-expanded amplitude in powers of momenta. There is no general argument for whether keeping higher powers of only one of the expansion parameters in the regime where they are of comparable magnitude will increase the accuracy. Considering its potential impact, it is nevertheless of much interest to explore the consequences of keeping higher-order terms of momenta even in the bound state regime where they would not ordinarily have been included [10, 12, 14]. We will here show how that post-Minkowskian Hamiltonian also follows directly from the full relativistic amplitude without having to perform the amplitude matching to the effective field theory, thereby explicitly showing equivalence between the two approaches [12, 32].

PERTURBATIVE GRAVITY AS A FIELD THEORY

We start by introducing the Einstein-Hilbert action minimally coupled to massive scalar fields[1] $\varphi_{a=1,2}$ and by working in natural units ($\hbar = c = 1$)

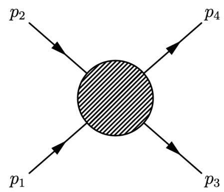
$$\mathcal{S}(g_{\mu,\nu}, \varphi_{a=1,2}) = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G_N} + \frac{1}{2}\sum_{a=1,2}\left(g^{\mu\nu}\partial_\mu\varphi_a\,\partial_\nu\varphi_a - m_a^2\varphi_a^2\right)\right], \quad (65)$$

where R defines the Ricci scalar and $g \equiv \det(g_{\mu\nu})$. Perturbatively, we expand the metric around a Minkowski background: $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$. At large distances we can treat the scattering of two massive objects m_a and m_b as that of two point-like particles with the same masses. This has all been well elucidated in the literature (see, *e.g.*, refs. [44, 45]), although most focus until now seems to have been on considering the quantum mechanical effects. The way classical terms appear from the quantum mechanical loop expansion is subtle [31, 46]; see ref. [21] for a very nice and clear discussion of this issue. Instead of expanding the action (65) in terms of ordinary Feynman rules, it pays to use modern amplitude methods to extract the needed non-analytic pieces in momentum transfer $\vec{q}$ through the appropriate cuts at loop level [11, 35, 36].

1 For a simple comparison with [12], in this Chapter we will use the mostly negative signature convention.

The scattering $m_1 + m_2 \to m_1 + m_2$ mediated by gravitons at an arbitrary loop order is described by

$$\mathcal{M}(p_1, p_2, p_3, p_4) =$$



$$= \sum_{L=0}^{+\infty} \mathcal{M}_{L-loop}(p_1, p_2, p_3, p_4)\,. \tag{66}$$

We choose the center-of-mass frame and parametrize the momenta as follows:

$$\begin{aligned} p_1^\mu &= (E_1(p), \vec{p}\,)\,, \quad p_3^\mu = (E_1(p), \vec{p}\,')\,, \\ p_2^\mu &= (E_2(p), -\vec{p}\,)\,,\; p_4^\mu = (E_2(p), -\vec{p}\,')\,, \end{aligned} \tag{67}$$

and $|\vec{p}\,| = |\vec{p}\,'|$. We also define

$$q^\mu \equiv p_1^\mu - p_3^\mu = (0, \vec{q}\,)\,, \quad \vec{q} \equiv \vec{p} - \vec{p}\,', \tag{68}$$

and the total energy $E_p = E_1(p) + E_2(p)$.

THE LIPPMANN-SCHWINGER EQUATION

It is a classical problem in perturbative scattering theory to relate the scattering amplitude $\mathcal{M}$ to an interaction potential V. This is typically phrased in terms of non-relativistic quantum mechanics, but it is readily generalized to the relativistic case. Crucial in this respect is the fact that we shall consider particle solutions to the relativistic equations only. There will thus be, in the language of old-fashioned (time-ordered) perturbation theory, no back-tracking diagrams corresponding to multiparticle intermediate states. This is trivially so since we neither wish to treat the macroscopic classical objects such as heavy neutron stars as indistinguishable particles with their corresponding antiparticles, nor do we wish to probe the scattering process in any potential annihilation channel. The classical objects that scatter will always be restricted to classical distance scales.

We now briefly outline a systematic procedure for connecting the scattering amplitude in perturbative gravity with post-Minkowskian potentials in classical General Relativity. We start by introducing a bit of notation. First, we assume the existence of a relativistic one-particle Hamiltonian of only particle states describing what in bound-state problems is known as the Salpeter equation,

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}, \qquad \hat{\mathcal{H}}_0 = \sqrt{\hat{k}^2 + m_1^2} + \sqrt{\hat{k}^2 + m_2^2}, \tag{69}$$

where $\hat{V}$ is a so far unspecified potential describing our post-Minkowskian system. We also define, on a proper subset of the complex plane, the following $\mathbb{C}$-valued operators

$$\hat{G}_0(z) \equiv (z - \hat{\mathcal{H}}_0)^{-1}, \qquad \hat{G}(z) \equiv (z - \hat{\mathcal{H}})^{-1}, \tag{70}$$

$$\hat{\mathbb{M}}(z) \equiv \hat{V} + \hat{V}\hat{G}(z)\hat{V}\,. \tag{71}$$

Here $\hat{G}_0(z)$ and $\hat{G}(z)$ are the Green's operator for the free and interacting case, while $\hat{\mathbb{M}}(z)$ is the scattering operator defined in (38). We can relate the two Green's operator by means of the following operator identity

$$A^{-1} = B^{-1} + B^{-1}(B - A)A^{-1} \quad \Rightarrow \quad \hat{G}(z) = \hat{G}_0(z) + \hat{G}_0(z)\hat{V}\hat{G}(z)\,. \tag{72}$$

Multiplying both sides of (71) by $\hat{G}_0(z)$, combined with (72), one has

$$\hat{G}_0(z)\hat{\mathbb{M}}(z) = \hat{G}_0(z)\hat{V} + \hat{G}_0(z)\hat{V}\hat{G}(z)\hat{V} = \hat{G}(z)\hat{V}, \tag{73}$$

$$\hat{\mathbb{M}}(z) = \hat{V} + \hat{V}\hat{G}_0(z)\hat{\mathbb{M}}(z)\,, \tag{74}$$

which is the basis for a perturbative knowledge of $\hat{M}(z)$. We now take the inner product on scattering states $|p\rangle, |p'\rangle$

$$\langle p'|\hat{\mathbb{M}}(z)|p\rangle = \langle p'|\hat{V}|p\rangle + \int \frac{d^3k}{(2\pi)^3} \frac{\langle p'|\hat{V}|k\rangle\langle k|\hat{\mathbb{M}}(z)|p\rangle}{z - E_k}\,, \tag{75}$$

and use the crucial relation (38)

$$\lim_{\epsilon\to 0}\langle p'|\hat{\mathbb{M}}(E_p + i\epsilon)|p\rangle = \widetilde{\mathcal{M}}(p, p')\,, \tag{76}$$

which provides the link to the conventionally defined scattering amplitude $\mathcal{M}$ in quantum field theory, restricted to the particle sector and up to a normalization factor as previously discussed. Substituting (76) into (75) we have a recursive relation between the amplitude and the post-Minkowskian potential

$$\widetilde{\mathcal{M}}(p, p') = \langle p'|\hat{V}|p\rangle + \int \frac{d^3k}{(2\pi)^3} \frac{\langle p'|\hat{V}|k\rangle\widetilde{\mathcal{M}}(p, k)}{E_p - E_k + i\epsilon}\,. \tag{77}$$

Solving this equation iteratively, we can invert it in order to arrive at a relativistic equation for the potential

$$\langle p'|\hat{V}|p\rangle = \widetilde{\mathcal{M}}(p,p') - \int \frac{d^3k}{(2\pi)^3}\frac{\widetilde{\mathcal{M}}(p,k)\widetilde{\mathcal{M}}(k,p')}{E_p - E_k + i\epsilon} + \cdots , \tag{78}$$

or, in position space,

$$V(r,p) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r}\widetilde{V}(q,p), \tag{79}$$

with

$$\widetilde{V}(q,p) \equiv \langle p'|\hat{V}|p\rangle \,. \tag{80}$$

At this stage there has not been any restriction to a non-relativistic limit. The anti-particle sector has been eliminated by hand, as dictated by the physical scattering process. We can thus regard (79) as defining a post-Minkowskian potential.

POST-MINKOWSKIAN HAMILTONIANS

The post-Minkowskian potential to first order

We are now ready to use the above definition of the relativistic interaction potential to describe the post-Minkowskian Hamiltonian to the trivial lowest order for two massive scalars of masses m_1 and m_2 interacting with gravity. With the proper relativistic normalization of external states,

$$\widetilde{\mathcal{M}}_{tree}(p_1, p_2, p_3, p_4) = \frac{4\pi G_N}{\sqrt{E_1(p_1)E_1(p_2)E_2(p_3)E_2(p_4)}} \frac{A(p_1, p_2, p_3, p_4)}{q^2}, \tag{81}$$

with

$$\begin{aligned} A(p_1, p_2, p_3, p_4) = (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4) \\ +(p_1 \cdot p_3)m_2^2 + (p_2 \cdot p_4)m_1^2 - 2m_1^2 m_2^2 \,. \end{aligned} \tag{82}$$

In the center-of-mass frame this reduces to an amplitude which only depends on $\vec{p}$ and $\vec{p}\,'$

$$\widetilde{\mathcal{M}}_{tree}(p, p') = -\frac{4\pi G_N}{E_1(p)E_2(p)} \frac{[2(p_1 \cdot p_2)^2 - m_1^2 m_2^2 - |\vec{q}\,|^2 (p_1 \cdot p_2)]}{|\vec{q}\,|^2}, \tag{83}$$

with $p_1 \cdot p_2 = E_1(p)E_2(p) + |\vec{p}\,|^2$.

In order to facilitate a comparison with [12] we can write the Fourier transform as

$$V_{1PM}(r, p) = \frac{1}{E_p^2 \xi(p)} \frac{G_N c_1(p^2)}{r} + \cdots \,, \tag{84}$$

with

$$c_1(p^2) = m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2 \quad , \quad \xi(p) = \frac{E_1(p)E_2(p)}{E_p^2} \quad . \tag{85}$$

The terms omitted in eq. (84) are either ultra-local or vanishing in the classical limit. This of course agrees with the leading-order potential of ref. [12] while not very easily derived in more traditional approaches.

The post-Minkowskian potential to second order

In order to consider a post-Minkowksian potential at second order in G_N^2, we will need to consider a contribution coming from the iterated tree-level amplitude, as dictated by (78)

$$\widetilde{V}_{2PM}(p,p') = \widetilde{\mathcal{M}}_{1-loop}(p,p') + \widetilde{\mathcal{M}}_{Born}(p,p'), \tag{86}$$

$$\widetilde{\mathcal{M}}_{Born}(p,p') \equiv -\int \frac{d^d k}{(2\pi)^d} \frac{\widetilde{\mathcal{M}}_{tree}(p,k)\widetilde{\mathcal{M}}_{tree}(k,p')}{E_p - E_k + i\epsilon} \,. \tag{87}$$

Infrared divergences are regularized by temporarily switching to $d+1$ space-time dimensions. The classical terms of the one-loop amplitude have been given elsewhere [12,32,33,47]. They can be decomposed in terms of scalar integrals with coefficients that are independent of the exchanged three-momentum $\vec{q}$,

$$\widetilde{\mathcal{M}}_{1-loop}(p,p') = \frac{i16\pi^2 G_N^2}{E_1(p)E_2(p)}\Big(c_{\Box}\mathcal{I}_{\Box} + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\Big)(p,p')\,, \tag{88}$$

where the symbol of each coefficient refers to the topology of the contributions involved while the ellipses denote quantum mechanical contributions that we neglect. In detail, the scalar box and crossed-box integrals are given by

$$\mathcal{I}_{\square} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_1^2 + i\varepsilon)((\ell-p_2)^2 - m_2^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}, \tag{89}$$

$$\mathcal{I}_{\bowtie} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_1^2 + i\varepsilon)((\ell+p_4)^2 - m_2^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}. \tag{90}$$

At leading order in the momentum transfer $\vec{q}$ the coefficients of these integrals are finite at $d = 3$ and given by [35,45]

$$c_{\square}(p^2) = c_{\bowtie}(p^2) = 4\big(m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2\big)^2 . \tag{91}$$

The scalar triangle integrals are given by

$$\mathcal{I}_{\triangleright} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+p_1)^2 - m_1^2 + i\varepsilon)}, \tag{92}$$

$$\mathcal{I}_{\triangleleft} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell-q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell-p_2)^2 - m_2^2 + i\varepsilon)}, \tag{93}$$

with coefficients, at the leading order in $|\vec{q}|$ and around $d = 3$, given by

$$c_{\triangleright}(p^2) = 3m_1^2\big(m_1^2 m_2^2 - 5(p_1 \cdot p_2)^2\big) \quad , \quad c_{\triangleleft}(p^2) = 3m_2^2\big(m_1^2 m_2^2 - 5(p_1 \cdot p_2)^2\big) . \tag{94}$$

These scalar integrals are conveniently evaluated by performing proper contour integrals in ℓ^0 as explained in [32]. Doing so, we see that the box, crossed-box, and triangle contributions are given by [45, 48]

$$\mathcal{I}_{\square} = -\frac{i}{16\pi^2|\vec{q}\,|^2}\left(-\frac{1}{m_1 m_2} + \frac{m_1(m_1-m_2)}{3m_1^2 m_2^2} + \frac{i\pi}{|p|E_p}\right)\left(\frac{2}{3-d} - \log|\vec{q}\,|^2\right) + \cdots\,, \tag{95}$$

$$\mathcal{I}_{\bowtie} = -\frac{i}{16\pi^2|\vec{q}\,|^2}\left(\frac{1}{m_1 m_2} - \frac{m_1(m_1-m_2)}{3m_1^2 m_2^2}\right)\left(\frac{2}{3-d} - \log|\vec{q}\,|^2\right) + \cdots\,, \tag{96}$$

$$\mathcal{I}_{\triangleright} = -\frac{i}{32 m_1}\frac{1}{|\vec{q}\,|} + \cdots\,, \tag{97}$$

$$\mathcal{I}_{\triangleleft} = -\frac{i}{32 m_2}\frac{1}{|\vec{q}\,|} + \cdots\,, \tag{98}$$

at leading order in the $|\vec{q}\,|^2$ expansion and around $d = 3$. We thus arrive at the one-loop amplitude to leading order in $|\vec{q}\,|^2$,

$$\widetilde{\mathcal{M}}_{1-loop}(p,p') = \frac{\pi^2 G_N^2}{E_p^2\xi(p)}\left[\frac{1}{2|\vec{q}\,|}\left(\frac{c_{\triangleright}(p^2)}{m_1} + \frac{c_{\triangleleft}(p^2)}{m_2}\right) + \frac{i}{E_p}\frac{c_{\square}(p^2)}{|\vec{p}\,|}\frac{\left(\frac{2}{3-d} - \log|\vec{q}\,|^2\right)}{\pi|\vec{q}\,|^2}\right]. \tag{99}$$

The imaginary part of this which arises from the box and crossed-box integrals is the infrared divergent Weinberg phase [49]. By restoring the $\hbar$-counting, one sees that it scales as $\hbar^{-1}$, a behavior dubbed super-classical in [21]. We will show below that it cancels in the properly defined potential, a fact already noted in the post-Newtonian expansion [33]. We next evaluate the iterated tree-level contribution given by

$$\widetilde{\mathcal{M}}_{Born}(p,p') = -\frac{16\pi^2 G_N^2}{E_1(p)E_2(p)} \int \frac{d^d k}{(2\pi)^d} \frac{A(\vec{p},\vec{k})}{|\vec{p}-\vec{k}|^2} \frac{A(\vec{k},\vec{p}\,')}{|\vec{p}'-\vec{k}|^2} \frac{\mathcal{G}(p^2,k^2)}{E_1(k)E_2(k)}, \tag{100}$$

where we have introduced the Green function

$$\mathcal{G}(p^2,k^2) = \frac{1}{E_p - E_k + i\epsilon}. \tag{101}$$

The function A is the numerator of the tree-level amplitude (82) with the k-legs satisfying 3-momentum (but not energy) conservation. We notice that $A(\vec{p},\vec{k}\,)$ and $A(\vec{k},\vec{p}\,')$ can be written as

$$A(\vec{p},\vec{k}\,) = \widetilde{A}(p^2,k^2) + B(\vec{p},\vec{k}), \tag{102}$$

$$A(\vec{k},\vec{p}\,') = \widetilde{A}(p^2,k^2) + B(\vec{p}\,',\vec{k}), \tag{103}$$

where $\widetilde{A}$ is $\vec{q}$-independent and function of $|\vec{p}| = p$ and $|\vec{k}| = k$. The classical contribution from the iterated Born amplitude is hence

$$\widetilde{\mathcal{M}}_{Born}(p,p') = -\frac{16\pi^2 G_N^2}{E_1(p)E_2(p)} \int \frac{d^d k}{(2\pi)^d} \frac{\mathcal{G}(p^2,k^2)Q(p^2,k^2)}{|\vec{p}-\vec{k}\,|^2|\vec{p}\,'-\vec{k}\,|^2}, \tag{104}$$

where

$$Q(p^2,k^2) = \frac{\widetilde{A}^2(p^2,k^2)}{E_1(k^2)E_2(k^2)}. \tag{105}$$

We now expand Q around p^2,

$$Q(p^2,k^2) = Q_{k=p} + (k^2 - p^2)\partial_{k^2}Q_{k^2=p^2} + \cdots\,, \tag{106}$$

$$Q_{k^2=p^2} = \frac{\widetilde{A}^2_{k^2=p^2}}{E_1(p^2)E_2(p^2)} \;=\; \frac{c_1^2(p^2)}{E_p^2\xi(p)}, \tag{107}$$

$$\partial_{k^2}Q_{k^2=p^2} = -\frac{1}{E_p^2\xi^2(p)}\left(2c_1(p^2)p_1\cdot p_2 + \frac{c_1^2(p^2)}{2E_p^2\xi(p)}(1-2\xi(p))\right). \tag{108}$$

The Green function $\mathcal{G}$ likewise admits a Laurent expansion in k^2

$$\mathcal{G}(p^2,k^2) = \frac{2E_p\xi(p)}{p^2-k^2} + \frac{3\xi(p)-1}{2E_p\xi(p)} + \cdots\,. \tag{109}$$

Combining terms, the Born subtraction can hence be expressed as

$$\begin{aligned}\widetilde{\mathcal{M}}_{Born}(p,p') &= \frac{32\pi^2G_N^2}{E_p^3\xi(p)}c_1^2(p^2)\int\frac{d^dk}{(2\pi)^d}\frac{1}{|\vec{p}-\vec{k}|^2|\vec{p}'-\vec{k}|^2(k^2-p^2)}\\ &-\frac{16\pi^2G_N^2}{E_p^3\xi^2(p)}\left(\frac{c_1^2(p^2)(1-\xi(p))}{2E_p^2\xi(p)} + 4c_1(p^2)p_1\cdot p_2\right)\int\frac{d^dk}{(2\pi)^d}\frac{1}{|\vec{p}-\vec{k}|^2|\vec{p}\,'-\vec{k}|^2} + \cdots\,.\end{aligned} \tag{110}$$

Evaluating the remaining three-dimensional integrals, we find

$$\begin{aligned}\widetilde{\mathcal{M}}_{Born}(p,p') = \frac{i\pi G_N^2}{E_p^3\xi(p)}\frac{4c_1^2(p^2)}{|\vec{p}|}\frac{(\log|\vec{q}|^2 - \frac{2}{3-d})}{|\vec{q}|^2} + \frac{2\pi^2G_N^2}{E_p^3\xi^2(p)|\vec{q}|}\\ \times\left(\frac{c_1^2(p^2)(\xi(p)-1)}{2E_p^2\xi(p)} - 4c_1(p^2)p_1\cdot p_2\right).\end{aligned} \tag{111}$$

The second-order post-Minkowskian potential in momentum space is thus given by

$$\widetilde{V}_{2PM}(q,p) = \widetilde{\mathcal{M}}_{1-loop}(q,p) + \widetilde{\mathcal{M}}_{Born}(q,p)\,, \tag{112}$$

leading to

$$\begin{aligned}\widetilde{V}_{2PM}(q,p) = \frac{\pi^2 G_N^2}{E_p^2\xi(p)|\vec{q}\,|}\Bigg[\frac{1}{2}\left(\frac{c_\triangleright(p^2)}{m_1}+\frac{c_\triangleleft(p^2)}{m_2}\right)+\frac{2}{E_p\xi(p)} \\ \times\left(\frac{c_1^2(p^2)(\xi(p)-1)}{2E_p^2\xi(p)}-4c_1(p^2)p_1\cdot p_2\right)\Bigg]\,,\end{aligned} \tag{113}$$

or, in coordinate space,

$$V_{2PM}(r,p) = \frac{G_N^2}{r^2}\frac{1}{E_p^2\xi(p)}\left[\frac{1}{4}\left(\frac{c_\triangleright(p^2)}{m_1}+\frac{c_\triangleleft(p^2)}{m_2}\right)+\left(\frac{c_1^2(p^2)(\xi(p)-1)}{2E_p^3\xi^2(p)}-\frac{4c_1p_1\cdot p_2}{E_p\xi(p)}\right)\right]. \tag{114}$$

This agrees with what has been previously obtained in ref. [12] (taking into account that $c_1(p^2)$ here is $E_p^2\xi(p)$ times $c_1(p^2)$ in [12]). As expected on physical grounds, the imaginary part which is composed of super-classical and infrared divergent pieces has cancelled, leaving a finite and well-defined post-Minkowskian potential at $d = 3$. That such cancellation had to occur was expected on physical ground, since the imaginary part clearly cannot affect classical motion. Interestingly, the evaluation of the same potential in $\mathcal{N} = 8$ supergravity has shown no contributions coming from triangle topologies [50].

The post-Minkowskian scattering angle

In [32] a one-loop formula for the gravitational eikonal limit [51] generalized to the scattering of two objects of different masses m_1 and m_2 was used to deduce the classical scattering angle to second post-Minkowskian order directly from the scattering amplitude. An alternative method based on the Hamiltonian [52] has recently been revived in connection with the third post-Minkowskian scattering amplitude calculation [10, 14] and we here briefly summarize the method at second order in G_N. Since the motion lies on a plane, we can introduce the following coordinates on the phase space $(r, \varphi, p_r, p_\varphi)$ so as to express the momentum in the center of mass frame as

$$p^2 = p_r^2 + \frac{L^2}{r^2}, \tag{115}$$

being L the conserved angular momentum of our binary system, with constant energy E

$$\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + V_{1PM}(r, p) + V_{2PM}(r, p) = E\,. \tag{116}$$

This equation can be solved perturbatively in G_N for $p^2 = p^2(E, L, r)$

$$p^2 = p_0^2 + \frac{G_N f_1}{r} + \frac{G_N^2 f_2}{r^2} + \cdots \tag{117}$$

Using $s = (p_1 + p_2)^2$

$$p_0^2 = \frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{s}, \qquad f_1 = -\frac{2c_1(p_0^2)}{\sqrt{s}}, \qquad f_2 = -\frac{1}{2\sqrt{s}}\left(\frac{c_{\triangleright}(p_0^2)}{m_1} + \frac{c_{\triangleleft}(p_0^2)}{m_2}\right). \tag{118}$$

It is straightforward to derive the following expression for the change in the angular variable φ during scattering

$$\Delta\varphi = \pi + \theta, \tag{119}$$

where the scattering angle is given by

$$\theta = -2\int_{r_{min}}^{+\infty} dr \frac{\partial p_r}{\partial L} - \pi\,. \tag{120}$$

Here r_{min} is the positive root for the condition of turning point at $p_r = 0$ with

$$p_r = \sqrt{p_0^2 - \frac{L^2}{r^2} + \frac{G_N f_1}{r} + \frac{G_N^2 f_2}{r^2}}\,. \tag{121}$$

Introducing $b \equiv L/p_0$ we note that p_r can be rewritten as

$$p_r = \frac{p_0}{r}\sqrt{r^2 + r\frac{G_N f_1}{p_0^2} + \frac{G_N^2 f_2}{p_0^2} - b^2} = \frac{p_0}{r}\sqrt{r - r^+}\sqrt{r - r^-}\,, \tag{122}$$

$$r^{\pm} = -\frac{G_N f_1}{2p_0^2} \pm \sqrt{\frac{G_N^2 f_1^2}{4p_0^4} - \frac{G_N^2 f_2}{p_0^2} + b^2}\,. \tag{123}$$

Since $r_{min} = r^+$, the scattering angle becomes

$$\theta = 2\int_{r^+}^{+\infty} \frac{dr}{r} \frac{r_0}{\sqrt{(r - r^+)(r - r^-)}} - \pi\,. \tag{124}$$

The integral so expressed can be performed analytically without the need of regularization. We get

$$\theta = \frac{4b}{\sqrt{-r^+ r^-}} \arccos \sqrt{\frac{r^+}{r^+ - r^-}} - \pi \,. \tag{125}$$

Taylor-expanding the scattering angle to second post-Minkowskian order we arrive at the final result

$$\theta = \frac{G_N f_1}{p_0 L} + \frac{G_N^2 f_2 \pi}{2L^2} + \cdots \tag{126}$$

where

$$\theta_{2PM} = \frac{G_N^2}{L^2} \frac{3\pi (m_1 + m_2)(5(p_1 \cdot p_2)^2 - m_1^2 m_2^2)}{4E} \,, \tag{127}$$

which agrees with the result of [27] at second post-Minkowskian order. In particular, since f_1 and f_2 do not depend on box topologies (324), also the scattering angle (126) receives no contributions from these, a known fact from the eikonal approach in four dimensions. The details of the calculation based on the Hamiltonian is, on the surface, quite different from the eikonal approach. It would be interesting to establish the precise link between the two, first identifying the precise exponentiation formula for the eikonal limit beyond second post-Minkowskian order.

CONCLUSION

Using the conventional approach to determining the interaction potential in perturbative gravity we have demonstrated that it can be extended to the relativistic setting by means of a one-particle Hamiltonian and associated Salpeter equation. We have used the Lippmann-Schwinger equation to derive straightforwardly the needed Born subtractions at arbitrary loop order. The resulting Fourier-transformed post-Minkowskian Hamiltonian

$$\mathcal{H}_{2PM}(r,p) = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + V_{1PM}(r,p) + V_{2PM}(r,p)\,, \tag{128}$$

agrees with the one derived in ref. [12] based on an effective field theory expansion in operators that can contribute to the given order, supplemented with the matching condition that the scattering amplitude as computed in the effective theory agrees with the one computed from the full one-loop expression of the Einstein-Hilbert action (plus scalars).

The resulting post-Minkowskian Salpeter equation is not an effective low-energy theory (momentum is not limited), but rather a small $|\vec{q}\,|/m$ approximation where small momentum is exchanged and only particle states are summed over. It is encouraging that preliminary results indicate that the corresponding two-loop Hamiltonian [10] may improve the computation of two-body dynamics as compared to the conventional post-Newtonian expansion for bound states [14]. The post-Minkowskian Hamiltonian also appears to provide a short-cut towards computing the scattering angle without first demonstrating exponentiation (and potential correction terms) as in the eikonal approach. It would be interesting to demonstrate the equivalence between those two computations in all generality.

6

POST-MINKOWSKIAN HAMILTONIANS IN MODIFIED GRAVITY

The aim of this note is to describe the computation of post-Minkowskian Hamiltonians in modified theories of gravity. Exploiting a recent relation between scattering amplitudes of massive scalars and potentials for relativistic point-particles we derive a contribution to post-Minkowskian Hamiltonians at second order in the Newton's constant coming from $\mathcal{R}^3$ modifications in General Relativity. Using this result we calculate the associated contribution to the scattering angle for binary black holes at second post-Minkowskian order, showing agreement in the non relativistic limit with previous results for the bending angle of a massless particle around a static massive source in $\mathcal{R}^3$ theories.

INTRODUCTION

The first detection of gravitational waves by the LIGO and Virgo collaboration, has opened up the possibility to test Einstein's theory of General Relativity at an unprecedented level, heralding a new era in fundamental physics [53]. A central framework is the Effective One Body approach [7, 8], where information from Numerical Relativity and analytical approaches are combined in order to lead to improved gravitational wave templates. Among these several inputs, it has been recently suggested [23, 24] that also post-Minkowskian (PM) results, valid for weak gravitational fields and unbound velocities, can independently lead to improved modeling of bound binary dynamics. Given the growing results in post-Minkowksian physics [10, 12, 14, 15], we would like to explore how contributions to post-Minkowskian Hamiltonians can be defined in modified theories of gravity. With no loss of generality, we here restrict ourselves on $\mathcal{R}^3$ modifications[1] to General Relativity [54–58]. Recently, these have been studied in the context of scattering amplitudes [59, 60] leading to a post-Newtonian definition of the potential [31, 33]. However, scattering amplitudes contain relativistic information that is lost in the passage to post-Newtonian point-particle potentials. We show how this can be restored defining a post-Minkwoskian potential in cubic theories of gravity, without restricting to the case of non relativistic point-particles. Using this result we derive

1 These arise as further contributions to the Ricci scalar in the Einstein-Hilbert action, where the only non trivial modifications are given by $R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu}$ and $R^{\mu\nu\alpha}{}_{\beta}R^{\beta\gamma}{}_{\nu\sigma}R^{\sigma}{}_{\mu\gamma\alpha}$.

the associated contribution to the fully relativistic scattering angle for binary black holes at second order in the Newton's constant. By then taking the non relativistic limit of one particle and the massless of the other, we are able to reproduce the bending angle recently calculated in [59] for a massless particle around a static massive source.

HIGHER DERIVATIVE CORRECTIONS IN GENERAL RELATIVITY

A non-trivial modification of the one-loop scattering of massive scalars in cubic theories of gravity has been recently studied with amplitudes techniques in [59, 60]. In what follows we focus on the contribution given by $I_1 \equiv R^{\mu\nu}_{\ \ \alpha\beta} R^{\alpha\beta}_{\ \ \rho\sigma} R^{\rho\sigma}_{\ \ \mu\nu}$. As can be seen from [61], this arises as a non trivial modification to the usual Einstein-Hilbert action which for simplicity of discussion we will parametrize by an unknown coefficient α with the dimension of length squared, following [59]. The associated classical information in the scattering of two massive scalars of masses m_1, m_2 has been calculated here [59, 60]. Using a mostly negative signature convention, and working in natural units ($c = \hbar = 1$) we have

$$\widetilde{\mathcal{M}}^{\alpha}(p_1, p_2, p_3, p_4) = \mathcal{D}\Big[\mathcal{I}(m_1)\ c(m_1, m_2) + \mathcal{I}(m_2)\ c(m_2, m_1)\Big](p_1, p_2, p_3, p_4) + ... \tag{129}$$

where, using $s = (p_1 + p_2)^2$ and $t = (p_1 - p_3)^2$, we have defined

$$\mathcal{D} = \frac{i\pi^2 G_N^2 \alpha^2}{\sqrt{E_1(p_1)E_2(p_2)E_3(p_3)E_4(p_4)}}\,, \tag{130}$$

$$\mathcal{I}(m_j) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p_1 - k)^2 (p_2 - k)^2 (k^2 - m_j^2)}\,, \tag{131}$$

$$c(m_i, m_j) = \frac{4t^2 m_i^4}{(4m_i^2 - t)^2} \left[\sum_{k=1}^{3} \beta_k(m_i, m_j) t^{(k-1)} \right], \tag{132}$$

$$\beta_1(m_i, m_j) = 2m_i^2 \left[(m_i^2 + m_j^2 - s)^2 - 4m_i^2 m_j^2 \right], \tag{133}$$

$$\beta_2(m_i, m_j) = -3m_i^4 + 2m_i^2 m_j^2 + (m_j^2 - s)^2\,, \tag{134}$$

$$\beta_3(m_i, m_j) = m_i^2 - m_j^2 + s\,. \tag{135}$$

We choose the center-of-mass frame and parametrize the momenta of the particles as follows

$$\begin{aligned} p_1^\mu &= (E_1(p), \vec{p}\,)\,, \quad p_3^\mu = (E_1(p), \vec{p}\,')\\ p_2^\mu &= (E_2(p), -\vec{p}\,)\,,\ p_4^\mu = (E_2(p), -\vec{p}\,') \end{aligned} \tag{136}$$

$$\vec{q} \equiv \vec{p} - \vec{p}\,'\,, \tag{137}$$

$$|\vec{p}| = |\vec{p}'| \equiv p \quad , \quad |\vec{q}| \equiv q \,. \tag{138}$$

We now proceed to define a post-Minkwoskian potential in the context of this modified theory of gravity using a recent relation between post-Minkowskian amplitudes and Hamiltonians [15]. The simplicity of this computation here lies in the lack of the Born subtraction, as there is no tree level amplitude to iterate that scales in the same way as (129). We can thus define a post-Minkowskian potential to second order in G_N and in the coupling α as

$$V^{I_1}_{2PM}(r,p) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \widetilde{\mathcal{M}}^{\alpha}(q,p)\,. \tag{139}$$

By performing a proper k^0 integration on (131), the scalar triangle integral becomes [32]

$$\mathcal{I}(m_j) = -\frac{i}{32 m_j q} + \ldots \tag{140}$$

where the ellipsis denote quantum contributions.

To leading order in q the associated post-Minkowskian potential is[2]

$$V^{I_1}_{2PM}(r,p) = \frac{\pi^2 G_N^2 \alpha^2}{32 E_1(p) E_2(p)} \int \frac{d^3q}{(2\pi)^3} \left[\frac{c(m_1,m_2)}{m_1} + \frac{c(m_2,m_1)}{m_2} \right] \frac{e^{i\vec{q}\cdot\vec{r}}}{q} \tag{141}$$

$$= \frac{\pi^2 G_N^2 \alpha^2}{128 E_1(p) E_2(p)} \left(\frac{\beta_1(m_1,m_2)}{m_1} + \frac{\beta_1(m_2,m_1)}{m_2} \right) \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} q^3\,. \tag{142}$$

2 The reason we only keep the leading term in q is due by $\hbar$ counting. For a detailed analysis on how to restore the proper classical limit from an amplitude calculation see [21].

$$V_{2PM}^{I_1}(r,p) = \frac{3\alpha^2}{32E_1(p)E_2(p)}\frac{G_N^2}{r^6}\left(\frac{\beta_1(m_1,m_2)}{m_1} + \frac{\beta_1(m_2,m_1)}{m_2}\right). \tag{143}$$

In the non relativistic limit, our post-Minkwoskian potential reduces to

$$V_{2PM}^{I_1}(r,p) = \frac{3\alpha^2}{4}\frac{G_N^2 p^2}{r^6}\frac{(m_1+m_2)^3}{m_1 m_2} + \ldots \tag{144}$$

in agreement with the post-Newtonian calculation in [59]. For the sake of completeness we also report the post-Minkowskian contribution to the potential given by the remaining cubic term $R^{\mu\nu\alpha}{}_{\beta}R^{\beta\gamma}{}_{\nu\sigma}R^{\sigma}{}_{\mu\gamma\alpha}$. This has been recently calculated in [59] as coming from the topological invariant $G_3 = R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu} - 2R^{\mu\nu\alpha}{}_{\beta}R^{\beta\gamma}{}_{\nu\sigma}R^{\sigma}{}_{\mu\gamma\alpha}$. The result has been found equal to

$$V_{2PM}^{G_3}(r,p) = \frac{12\alpha^2 G_N^2}{E_1(p)E_2(p)}\frac{m_1^2 m_2^2(m_1+m_2)}{r^6}. \tag{145}$$

In a natural way, the same procedure for defining a post-Minkwoskian potential can be applied for more general modified theories of gravity.

THE SCATTERING ANGLE

At second post-Minkowskian order in G_N, the Hamiltonian for a binary system of spinless binary black holes, including contributions from cubic gravity, is given by

$$H_{2PM}^{\alpha}(r,p) = \sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2} + V_{2PM}(r,p) + V_{2PM}^{\alpha}(r,p), \tag{146}$$

where $V_{2PM}(r,p)$ has been calculated here [12, 15], being $V^{\alpha}_{2PM}(r,p)$ the sum of (143) and (145). Since the motion lies on a plane, we can introduce the following coordinates on the phase space $(r, \varphi, p_r, p_\varphi)$ so as to express the momentum in the center of mass frame as

$$p^2 = p_r^2 + \frac{p_\varphi^2}{r^2} \quad , \quad p_\varphi = L \, , \tag{147}$$

being L the angular momentum of the system, which is a conserved quantity.

The associated Hamilton-Jacobi equation gives

$$\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + V_{2PM}(r,p) + V^{\alpha}_{2PM}(r,p) = E \, , \tag{148}$$

where E is the constant energy of the system.

By solving now in p^2 we can express the momentum in the center of mass frame as

$$p^2 = p^2(E, L, \alpha, r) \quad , \quad p^2 = p_0^2 + \frac{G_N f_1}{r} + \frac{G_N^2 f_2}{r^2} + \frac{G_N^2 \alpha^2 f_\alpha}{r^6} + \ldots \tag{149}$$

where the ellipsis denotes higher contributions in G_N and

$$p_0^2 = \frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{s} \quad , \quad f_1 = -\frac{2c_1(p_0^2)}{\sqrt{s}} \quad , \quad f_2 = -\frac{1}{2\sqrt{s}} \left(\frac{c_\triangleright(p_0^2)}{m_1} + \frac{c_\triangleleft(p_0^2)}{m_2} \right) , \tag{150}$$

$$f_\alpha = -\frac{3}{16E} \left(\frac{\beta_1(m_1, m_2)}{m_1} + \frac{\beta_1(m_2, m_1)}{m_2} \right) - \frac{24 m_1^2 m_2^2 (m_1 + m_2)}{E} \, . \tag{151}$$

where the coefficients related to f_1 and f_2 has been calculated in the previous Chapter. At this point, by considering the angular variable φ, it is straightforward to derive the following expression for its total change during a scattering

$$\Delta\varphi = \pi + \theta \quad , \quad \frac{\theta}{2} = -\int_{r_{min}}^{+\infty} dr \frac{\partial p_r}{\partial L} - \frac{\pi}{2} \, , \tag{152}$$

where r_{min} is the positive root for $p_r = 0$.

In order to evaluate (152) we proceeds perturbatively by expanding both the integrand and the extreme of integration in G_N, where

$$r_{min} = \frac{L}{p_0} + \ldots \quad , \quad p_r = \sqrt{p_0^2 - \frac{L^2}{r^2}} + \ldots \tag{153}$$

being the leading term of r_{min} equivalent to the impact parameter b.

This expansion give rise to divergent integrals which can be handled only by means of the Hadamard Partie finie (Pf) of the latter as shown by Damour in [52].

Restricting to the contribution to (152) due to $\mathcal{R}^3$ one finds

$$\frac{\theta_{2PM}^{\alpha}}{2} = -\frac{L G_N^2 \alpha^2 f_\alpha}{2} \, \mathbf{Pf} \int_{r_0}^{+\infty} \frac{dr}{r^8} \left(p_0^2 - \frac{L^2}{r^2} \right)^{-\frac{3}{2}} . \tag{154}$$

Changing variables to $u = \frac{1}{r}$ the integral becomes

$$\frac{\theta_{2PM}^{\alpha}}{2} = -\frac{G_N^2 \alpha^2 f_\alpha}{2L^2} \, \mathbf{Pf} \int_0^{u_0} du \frac{u^6}{(u_0^2 - u^2)^{\frac{3}{2}}} \quad , \quad u_0 \equiv \frac{1}{b} \, . \tag{155}$$

The remaining integration is straightforward, leading to

$$\frac{\theta_{2PM}^{\alpha}}{2} = \frac{15\pi G_N^2 \alpha^2 f_\alpha}{32 L^2 b^4} \, , \tag{156}$$

$$\frac{\theta^{\alpha}_{2PM}}{2} = -\frac{45\pi G_N^2 \alpha^2}{512 L^2 b^4 E}\left(\frac{\beta_1(m_1, m_2)}{m_1} + \frac{\beta_1(m_2, m_1)}{m_2} + 128 m_1^2 m_2^2 (m_1 + m_2)\right). \quad (157)$$

Equation (157) has to be considered as an additional contribution to the fully relativistic scattering angle at second order in G_N coming from a cubic theory of gravity. In particular, by taking the non relativistic limit of our result with the additional condition $m_1 = m$ and $m_2 = 0$, we have

$$\theta^{\alpha}_{2PM} = -\frac{45 G_N^2 \alpha^2 \pi m^2}{32 b^6} + \dots \quad (158)$$

which agrees with the non relativistic contribution derived in [59] for the bending angle of a massless particle around a static massive source.[3]. In this case, the G_3 contribution to the potential is found to be absent for the bending angle of a massless particle, but not in the fully relativistic scattering angle of two massive particles as it can be seen from (157).

CONCLUSION

We have derived the post-Minkowskian contribution to relativistic point-particles Hamiltonians in modified theories of gravity. We have restricted ourselves to the

3 The authors in [59] have used a convention for the deflection angle which differs by a minus sign compared to ours.

case of $\mathcal{R}^3$ modifications, although similar changes are expected to appear also for $\mathcal{R}^2$ terms [62–64]. The derived post-Minkowskian contribution, once expanded for small velocities, is in agreement with the recent post-Newtonian computation [59]. The simplicity of the calculation has taken advantage of a recent relation between post-Minkowskian amplitudes and Hamiltonians for relativistic point-particles [15]. Indeed, the computation has required no effective field theory matching as well as no need to known the operator reproducing the $\mathcal{R}^3$ modifications in an effective field theory of scalar fields. We have also derived an additional contribution to the fully relativistic scattering angle of black holes at second order in G_N arising from $\mathcal{R}^3$, showing agreement in the non relativistic limit with a result derived in [59] for the bending angle of a massless particle around a static massive source. It would be interesting to systematically explore similar results in other alternative formulations of General Relativity.

7

POST-MINKOWSKIAN SCATTERING ANGLE IN EINSTEIN GRAVITY

Using the implicit function theorem we demonstrate that solutions to the classical part of the relativistic Lippmann-Schwinger equation are in one-to-one correspondence with those of the energy equation of a relativistic two-body system. A corollary is that the scattering angle can be computed from the amplitude itself, without having to introduce a potential. All results are universal and provide for the case of general relativity a very simple formula for the scattering angle in terms of the classical part of the amplitude, to any order in the post-Minkowskian expansion.

INTRODUCTION

The Post-Minkowskian expansion of general relativity promises to become a new and powerful tool with which to compute observables of two-body gravitational interactions [10, 14, 23, 24, 32]. As a systematic expansion in Newton's constant G_N, the Post-Minkowskian framework is perfectly suited for a standard second-quantized field theory approach to classical gravity [12, 15]. There is now hope that modern field theory techniques may radically change the prospect for how far analytical calculations can be pushed in general relativity. Currently also much work goes into seeing how Post-Minkowskian gravitational interactions of classically spinning objects can be treated by modern quantum field theory techniques [65–72], leading again to a complete revision of how such classical observables can be computed in general relativity.

When the Post-Minkowskian expansion is applied to the two-body bound-state problem it is natural to phrase it in terms of a potential V, either as provided implicitly through the Effective One-Body Hamiltonian [14] or by the large-distance effective Hamiltonian obtained by matching of amplitudes [12]. Up to canonical transformations, this is equivalent to studying the relativistic Salpeter equation [15] based on an Hamiltonian operator

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V} = \sum_{i=1}^{2} \sqrt{\hat{p}^2 + m_i^2} + \hat{V}\,, \tag{159}$$

and then taking the classical limit. Only the positive-energy solutions enter in this Hamiltonian because we remove antiparticles in the scattering process by hand when taking the macroscopic classical limit. The momentum-space potential $\widetilde{V}$ can be easily computed by solving the associated Lippmann-Schwinger equation for the full scattering amplitude[1] [15],

$$\widetilde{\mathcal{M}}(p,p') \;=\; \widetilde{V}(p,p') \;+\; \int \frac{d^3k}{(2\pi)^3}\frac{\widetilde{V}(k,p)\,\widetilde{\mathcal{M}}(k,p')}{E_p - E_k + i\epsilon}\,, \tag{160}$$

inverting it,

$$\widetilde{V}(p,p') \;=\; \widetilde{\mathcal{M}}(p,p') - \int \frac{d^3k}{(2\pi)^3}\frac{\widetilde{\mathcal{M}}(p,k)\,\widetilde{\mathcal{M}}(k,p')}{E_p - E_k + i\epsilon} + \ldots\,, \tag{161}$$

and taking the classical limit. This is the systematics of the *Born subtractions* needed to define a potential from the scattering amplitude. The so-called super-classical terms [21] cancel in the process, rendering the classical limit of the potential well-defined. By performing a suitable Fourier transform, this leads to the conventionally defined position-space potential V in the chosen coordinates.

It should be noted that the effective field theory matching employed in refs. [10, 12] is equivalent to the method of Born subtractions [15]. In four dimensions, the effective field theory matching, after suitable reduction of the four-dimensional amplitude integrals to integrals living in only three dimensions, involves cancel-

1 For ease of notation, we will work in natural units with $c = \hbar = 1$. When needed, we can easily restore the counting in $\hbar$ by following [21].

lations of identical integrals, which hence do not need to be evaluated. The same cancellations among three-dimensions integrals can be achieved also in the Born subtraction method (indeed, the two methods are completely equivalent), but we prefer to evaluate all integrals for clarity.

The position-space potential V seems needed when solving the bound-state problem in general relativity. However, this quantity is not very natural in the field theoretic framework where everything is based on the gauge invariant S-matrix with incoming and outgoing momenta defined at Minkowskian infinity. One would surely prefer as far as possible a formulation in which V would not be needed. This problem is compounded when we consider a coordinate-independent observable such as the classical scattering angle from far infinity to far infinity. Conventionally, we will be led to solve the classical analog of the Salpeter Hamiltonian of eq. (159) and then follow the classical analysis of the scattering problem. While that method is correct, it seems intuitively surprising that it should be necessary to go through the carefully Born subtracted position-space potential V as an intermediate step. Indeed, we know from quantum field theory that all scattering information from far infinity to far infinity is contained in the S-matrix, viz., the scattering amplitude.

This puzzle has become greatly clarified by the observation of Bern et al. [10] that up to two loop order (3PM order in the Post-Minkowskian counting) seemingly miraculous cancellations take place, leaving a perturbatively expanded expression for the two-loop scattering angle expressed entirely in terms of the classical part of the scattering amplitude up to that two-loop order. If this phenomenon is to persist to all orders it means that all the classical pieces of the Born subtractions defined above provide a potential V in precisely such a manner as to compensate, exactly and to all orders in the coupling G_N, the additional terms that arise from solving the expanded classical Salpeter equation. While the apparent conspiracy of two such totally unrelated equations having a one-to-one relation might seem improbable, we shall in this paper elucidate how this indeed will be true. Our tool will be the implicit function theorem that sometimes goes under the name of Dini's Theorem (although a different theorem also carries Dini's name). In the process we will unravel new and compact relations between the classical potential, together with its derivatives, and the classical part of the scattering amplitude.

Having this relationship established, a next burning question is: how do we then compactly express the scattering angle directly in terms of the classical part of the scattering amplitude? To find such an expression, we make use of an idea proposed by Damour in ref. [24], mapping the classical and fully relativistic Salpeter Hamiltonian into an auxiliary Hamiltonian that is formally in the non-relativistic form of a one-particle Hamiltonian for a particle of mass equal to 1/2 in appropriate units and with a potential that is only position-dependent. Considering the quantized analog of this Hamiltonian one immediately proves, in essentially one line, that the solution for the scattering angle indeed only depends on the classical part of the scattering amplitude.[2] But armed with the non-relativistic auxiliary problem we can do far more than that. Indeed, the classical part of the mapped scattering problem must now be WKB-exact and even solved by only its leading-order piece of order $\hbar^0$ in the exponent. This is, consistently, simply the classical Hamilton-Jacobi equation with the phase identified with the generating function S. Much literature exists on the relationship between the scattering angle and the WKB-approximation as well as their relation to the eikonal limit, and we hope our discussion here will clarify some confusion. Our end result is a very simple formula for the scattering angle in terms of the classical part of the scattering amplitude, to all orders in the coupling.

THE LIPPMANN-SCHWINGER EQUATION IN POSITION SPACE

The Lippmann-Schwinger equation is usually expressed as an integral equation involving amplitudes and potentials in momentum space. For the case of non-relativistic systems, its space representation states that the Fourier transform of the classical part of the amplitude is proportional to the potential. However, for the case of fully relativistic systems, this is no longer true [15]. We shall here extend this observation by demonstrating that the position-space representation of the Lippmann-Schwinger equation for fully relativistic systems can be expressed as

2 While this paper was in preparation, the same observation was made in ref. [73].

a differential equation for the potential and the classical part of the amplitude. To show this, we start by considering the fully relativistic Lippmann-Schwinger equation in momentum space

$$\widetilde{\mathcal{M}}(p,p') = \widetilde{V}(p,p') + \int \frac{d^3k}{(2\pi)^3} \frac{\widetilde{V}(k,p)\,\widetilde{\mathcal{M}}(k,p')}{E_p - E_k + i\epsilon}\,. \tag{162}$$

Kinematics will always be that of the center of mass frame. We parametrize the potential in momentum space as

$$\widetilde{V}(k_i,k_j) = \sum_{n=1}^{\infty} \left(\frac{G_N}{2}\right)^n (4\pi)^{\frac{3}{2}} \frac{\Gamma(\frac{3-n}{2})}{\Gamma(\frac{n}{2})} \frac{c_n(k_i,k_j)}{|k_i - k_j|^{3-n}}\,, \quad c_n(k_i,k_j) = c_n\left(\frac{k_i^2 + k_j^2}{2}\right). \tag{163}$$

Eq. (162) allows us to express the momentum-space amplitude as

$$\widetilde{\mathcal{M}}(p,p') = \sum_{n=0}^{\infty} \int_{k_1,k_2,\ldots,k_n} \frac{\widetilde{V}(p,k_1)\,\widetilde{V}(k_1,k_2)\,\cdots\,\widetilde{V}(k_n,p')}{(E_p - E_{k_1})(E_{k_1} - E_{k_2})\cdots(E_{k_{n-1}} - E_{k_n})} = \widetilde{V}(p,p') + \sum_{n=1}^{\infty} S_n(p,p')\,, \tag{164}$$

where the n-th terms of the series has $n+1$ factors of potential $\widetilde{V}$ in the numerator and n energy denominators.

We are only interested in the classical pieces of this equation, which means that we must device a precise mechanism to discard super-classical and quantum terms from the right hand side of eq. (164) based on the $\hbar$-counting [21]. In order to understand this procedure, we start by considering the first non-trivial ($n = 1$) term of eq. (164) and then extend the reasoning to all n. For $n = 1$ we have

$$S_1 = \int \frac{d^3k}{(2\pi)^3} \frac{\widetilde{V}(p,k)\,\widetilde{V}(k,p')}{E_p - E_k}\,. \tag{165}$$

Since we are only interested in classical terms, we can expand the propagator in eq. (165) around $k_i^2 = k_j^2 = k^2$ as

$$\frac{1}{E_{k_i} - E_{k_j}} = \frac{2E_k\xi(k)}{k_i^2 - k_j^2} + \frac{3\,\xi(k) - 1}{2E_k\xi(k)} + \dots\,, \tag{166}$$

$$E_k = E_1(k) + E_2(k) \quad , \quad \xi(k) = \frac{E_1(k)E_2(k)}{E_k^2}\,. \tag{167}$$

Using this expansion, the only classical contributions that could arise from (165) are

$$S_1 = 2E_p\xi(p)I_1 + \left(\frac{3\xi(p) - 1}{2E_p\,\xi(p)}\right)J_1 + \dots\,, \tag{168}$$

where

$$I_1 \equiv \int \frac{d^3k}{(2\pi)^3} \frac{\widetilde{V}(p,k)\,\widetilde{V}(k,p')}{p^2 - k^2}\,, \quad J_1 \equiv \int \frac{d^3k}{(2\pi)^3} \widetilde{V}(p,k)\,\widetilde{V}(k,p')\,. \tag{169}$$

We start by evaluating the classical contributions from I_1, using (163)

$$I_1 = (4\pi)^3 \sum_{n,m=1}^{\infty} \left(\frac{G_N}{2}\right)^{n+m} \frac{\Gamma(\frac{3-n}{2})\,\Gamma(\frac{3-m}{2})}{\Gamma(\frac{n}{2})\,\Gamma(\frac{m}{2})} \int \frac{d^3k}{(2\pi)^3} \frac{c_n(p,k)\,c_m(k,p')}{(p^2 - k^2)\,|k-p|^{3-n}\,|k-p'|^{3-m}}\,. \tag{170}$$

In order to discard super-classical and quantum terms we expand the numerator around $k^2 = p^2$ as

$$c_n(k,p)\,c_m(k,p') = c_n^0\,c_m^0 + \frac{1}{2}(c_n^0\,\partial_{p^2}\,c_m^0 + c_m^0\,\partial_{p^2}\,c_n^0)(k^2-p^2) + \dots\,, \quad c^0 \equiv c_{|_{k^2=p^2}}\,. \tag{171}$$

The $\hbar$-counting thus tells us that the only classical contribution (cl.) from eq. (170) is given by

$$I_1^{cl.} = -(4\pi)^3 \sum_{n,m=1}^{\infty} \left(\frac{G_N}{2}\right)^{n+m} \frac{\Gamma(\frac{3-n}{2})\,\Gamma(\frac{3-m}{2})}{\Gamma(\frac{n}{2})\,\Gamma(\frac{m}{2})} \frac{(c_n^0\partial_{p^2}c_m^0 + c_m^0\partial_{p^2}c_n^0)}{2}\, G_{n,m}^{(2)}(q)\,, \tag{172}$$

where we have introduced $q \equiv p - p'$ and

$$G_{n,m}^{(2)}(q) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|^{3-n}\,|k+q|^{3-m}}\,. \tag{173}$$

It is also convenient to define its Fourier transform

$$g_{n,m}^{(2)}(r) \equiv \int \frac{d^3q}{(2\pi)^3} G_{n,m}^{(2)}(q) e^{iq\cdot r}\,, \tag{174}$$

which is seen to factorize,

$$g_{n,m}^{(2)}(r) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{e^{i(q-k)\cdot r}}{|k|^{3-n}\,|q|^{3-m}} \tag{175}$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{e^{iq\cdot r}}{|q|^{3-m}} \times \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik\cdot r}}{|k|^{3-n}} = g_n(r)\,g_m(r) \tag{176}$$

The function $g_n(r)$ is well known and given by

$$g_n(r) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{3-n}{2})}\left(\frac{2}{r}\right)^n \frac{1}{(4\pi)^{\frac{3}{2}}} . \tag{177}$$

Using this, the position-space representation of eq. (172) becomes

$$\widetilde{I}_1^{cl.} = -\sum_{n,m=1}^{\infty}\left(\frac{G_N}{r}\right)^{n+m}(c_n^0\,\partial_{p^2}\,c_m^0)\,, \quad \widetilde{I}_1^{cl.} \equiv \int \frac{d^3q}{(2\pi)^3}\,e^{iq\cdot r}\,I_1^{cl.} . \tag{178}$$

This can be expressed in an even simpler form by realizing that it can be factorized,

$$\widetilde{I}_1^{cl.} = -\left[\sum_{n=1}^{\infty}\left(\frac{G_N}{r}\right)^n c_n^0\right]\left[\sum_{m=1}^{\infty}\left(\frac{G_N}{r}\right)^m \partial_{p^2}\,c_m^0\right] . \tag{179}$$

This nicely connects with the Fourier transform of the potential in position space,

$$V(r,p) = \sum_{n=1}^{\infty}\left(\frac{G_N}{r}\right)^n c_n(p^2)\,, \tag{180}$$

giving

$$\widetilde{I}_1^{cl.} = -V(r,p)\,\partial_{p^2}V(r,p) . \tag{181}$$

As for the remaining integral, one has

$$J_1 = (4\pi)^3\sum_{n,m=1}^{\infty}\left(\frac{G_N}{2}\right)^{n+m}\frac{\Gamma(\frac{3-n}{2})\,\Gamma(\frac{3-m}{2})}{\Gamma(\frac{n}{2})\,\Gamma(\frac{m}{2})}\,c_n^0\,c_m^0 G_{n,m}^{(2)}(q) . \tag{182}$$

One readily finds that its Fourier transform $\widetilde{J}_1^{cl.}$ simply satisfies $\widetilde{J}_1^{cl.} = V^2$. Defining the real-space representation of the classical part of the amplitude by

$$\widetilde{\mathcal{M}}^{cl.}(r,p) \equiv \int \frac{d^3q}{(2\pi)^3}\, \widetilde{\mathcal{M}}^{cl.}(q,p)\, e^{iq\cdot r}\,, \tag{183}$$

we find that the leading first term to all orders in G_N is given by

$$\widetilde{\mathcal{M}}^{cl.}(r,p) = V(r,p) - 2E_p\xi(p)\, V(r,p)\partial_{p^2}V(r,p) + \left(\frac{3\xi(p)-1}{2E_p\xi(p)}\right) V^2(r,p) + \ldots\,. \tag{184}$$

As for the remaining terms in the series, they can be evaluated in exactly the same fashion by an expansion of the energy denominators and numerators, although the complexity of these analytical expressions grow rapidly and we do not display them here. (Remarkably, the classical part of the series can always be expressed as a linear combination of generalized n-loop massless sunset diagrams with external momentum q; this is shown in the Appendix). We have thus shown that a quite simple differential equation links the classical part of the amplitude to the potential. At higher loop level the order of the differential equation increases, but the structure remains. What is far more interesting is that we can understand the same series from an alternative point of view by applying the implicit function theorem to the relativistic energy equation. Remarkably, this will provide a physical interpretation of the classical part of the amplitude, as here defined.

DINI'S THEOREM AND THE LIPPMANN-SCHWINGER EQUATION

We start by stating the implicit function theorem (Dini's theorem) in a form useful for the present purpose:

Let $F : \mathbb{R}^2 \to \mathbb{R}$ *be a* $\mathcal{C}^\infty$ *function. Consider a point* (x_0, y_0) *such that* $F(x_0, y_0) = 0$ *and* $\partial_x F(x_0, y_0) \neq 0$*. Then there exist a closed neighbourhood of* (x_0, y_0) *and a function* $y = f(x)$ *so that* $F(x, y(x)) = 0$ *for every point in that neighbourhood. The implicit function* $y = f(x)$ *will admit a Taylor expansion in terms of the partial derivatives of* $F(x, y)$ *given by*

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{1}{2}y''(x_0)(x - x_0)^2 + \dots, \tag{185}$$

$$y'(x_0) = -\left.\frac{\partial_x F}{\partial_y F}\right|_{x=x_0\,,\, y=y(x_0)}, \tag{186}$$

$$y''(x_0) = -\left.\frac{\partial^2_{xx}F + 2y'\partial^2_{xy}F + \partial^2_{yy}F y'^2}{\partial_y F}\right|_{x=x_0\,,\, y=y(x_0)}, \tag{187}$$

where the higher order derivatives can be computed from

$$\left(\partial_x + \frac{dy}{dx}\partial_y\right)^n F(x, y(x)) = 0 \quad , \quad \forall n \in \mathbb{N}, \tag{188}$$

by the binomial expansion of operators.

We now apply this theorem to the problem of inverting the relativistic energy equation in terms of three-momenta. This is precisely what arises in the post-Minkowskian two-to-two scattering process where we must solve the classical energy relation of eq. (159),

$$\sum_{i=1}^{2}\sqrt{p^2+m_i^2}+V(r,p) \;=\; E\,,\quad V(r,p) \;=\; \sum_{n=1}^{\infty}\left(\frac{G_N}{r}\right)^n c_n(p^2)\,, \tag{189}$$

where E is the energy of the system, which according to our conventions satisfies $E = E_{p_\infty}$. In order to find a solution to eq.(189) we apply Dini's theorem by choosing p^2 as y and G_N as x respectively.[3] Then,

$$F(p^2, G_N) \;=\; \sum_{i=1}^{2}\sqrt{p^2+m_i^2}+V(r,p)-E\,, \tag{190}$$

$$F(p^2(G_N), G_N) \;=\; 0\,,\quad \partial_{G_N}F(p^2,G_N) \;=\; \partial_{G_N}V(r,p)\neq 0\,. \tag{191}$$

From the theorem we thus know that there exists a p^2 such that

$$p^2 \;=\; p_\infty^2+\sum_{k=1}^{\infty}\frac{G_N^k}{k!}\frac{d^k p^2}{dG_N^k}\bigg|_{G_N=0}\,,\quad p_\infty^2=\frac{(m_1^2+m_2^2-E^2)^2-4m_1^2m_2^2}{4E^2}\,, \tag{192}$$

where the first term is nothing else than the value of p^2 which solves eq.(189) in absence of interactions. The next terms can be found using eqs. (186) and (187), giving

3 We choose G_N for sheer convenience because post-Minkowskian Hamiltonians in the center of mass frame have the same counting in $1/r$ and G_N. In case of higher-derivative gravity this counting is of course broken by new coupling constants [16, 59, 60]. That more general case can be analyzed analogously by simply identifying y with r.

$$\frac{dp^2}{dG_N}\bigg|_{G_N=0} = -\left.\frac{\partial_{G_N} V(r,p)}{\frac{1}{2E_p\xi(p)} + \partial_{p^2} V(r,p)}\right|_{G_N=0} = -2E_p\xi(p)\left[\frac{c_1(p^2)}{r}\right]_{|p=p_\infty}, \tag{193}$$

$$\frac{d^2p^2}{d^2G_N}\bigg|_{G_N=0} = -2E_p\xi(p)\left[\frac{2c_2(p^2)}{r^2} - \frac{4E_p\xi(p)\,c_1(p^2)\partial_{p^2}c_1(p^2)}{r^2} + \frac{c_1^2(p^2)}{r^2}\left(\frac{3\xi(p)-1}{E_p\xi(p)}\right)\right]_{|p=p_\infty}, \tag{194}$$

and so on for higher derivatives.

Apparently, the structure of the k-derivative of p^2 as a function of G_N seems to show no discernible structure, involving the potential and its derivatives. However, almost unbelievably, precisely the same relations also appear in the classical part of the position-space representation of the Lippmann-Schwinger equation that we have just examined above. There they relate the classical part of an n-loop amplitude to the potential and its derivatives. Indeed, by substituting eqs. (193) and (194) into eq. (192), we see that the derivatives of p^2 satisfies a remarkable relation to the classical part of the position-space representation of loop amplitudes:

$$G_N \frac{dp^2}{dG_N}\bigg|_{G_N=0} = -2E\xi(p_\infty)\left[\widetilde{\mathcal{M}}^{cl.}_{tree}(r,p_\infty)\right], \tag{195}$$

$$\frac{G_N^2}{2}\frac{d^2p^2}{d^2G_N}\bigg|_{G_N=0} = -2E\xi(p_\infty)\left[\widetilde{\mathcal{M}}^{cl.}_{1-loop}(r,p_\infty)\right]. \tag{196}$$

By substituting these into eq. (192), we observe that the implicit function we were searching for is precisely the classical part of the Fourier transform of the scattering amplitude,

$$p^2 \;=\; p_\infty^2 - 2E\xi(p_\infty)\left[\widetilde{\mathcal{M}}^{cl.}_{tree}(r,p_\infty) + \widetilde{\mathcal{M}}^{cl.}_{1-loop}(r,p_\infty)\right] + \ldots \,. \tag{197}$$

Indeed, the correspondence between solutions to the classical part of the Lippmann-Schwinger equation and the relativistic energy relation is not a coincidence and can be generalized to any loop order. The validity of eq. (197) is a consequence of Dini's theorem which maps the implicit function p^2 of the relativistic energy equation to the solution of the classical part of the Lippmann-Schwinger equation in position space.

The same relation[4] [10,73],

$$p^2 = p_\infty^2 - 2E\xi(p_\infty)\widetilde{\mathcal{M}}^{cl.}(r,p_\infty)\,, \tag{198}$$

can also be inferred by an intriguing alternative route suggested by Damour [24] and recently generalized to all orders in by Kälin and Porto in [73]. We rephrase it as follows. Consider the energy equation in a fully relativistic system subjected to a post-Minkowskian potential. In the center of mass frame,

4 As discussed in ref. [73], the inclusion of radiative effects introduce a non-linear relation between p^2 and scattering amplitudes. These enter at 4PM order for a non-spinning binary system [10]. Our analysis is valid only in the conservative sector of the two body problem.

$$E = \sum_{i=1}^{2} \sqrt{p^2 + m_i^2} + V(r,p)\,, \quad V(r,p) = \sum_{n=1}^{\infty} \frac{G_N^n c_n(p^2)}{r^n}\,, \tag{199}$$

$$p^2 = p_\infty^2 + \sum_{n=1}^{\infty} \frac{G_N^n f_n(E)}{r^n}\,, \tag{200}$$

where eq. (200) provides the perturbatively expanded solution to the energy condition and the f_n coefficients that can be determined order by order in the coupling constant. A natural quantization of this [15] is the Salpeter Hamiltonian of relativistic particle states (159),

$$\hat{H} = \sum_{i=1}^{2} \sqrt{\hat{p}^2 + m_i^2} + \hat{V}\,, \tag{201}$$

from which we infer the Lippmann-Schwinger equation discussed above. Given the nature of this Hamiltonian, it comes as no surprise that the associated Green function will have an intricate structure involving square roots as we have discussed in the previous section. Damour [24] considers instead the second relation (200) as a formally non-relativistic energy relation for a particle of mass 1/2 in appropriate units. Because there is a map from eq. (199) to eq. (200) it should be equally meaningful to quantize the Hamiltonian in p^2 of eq. (200) as the original Salpeter Hamiltonian (199). This means that we can use a much simpler non-relativistic Hamiltonian to derive relations for the scattering amplitude. Its potential depends only on the radial distance r as we are familiar with in ordinary non-relativistic quantum mechanics. We will thus have all the powerful technology of non-relativistic quantum mechanics (and classical mechanics) at our disposal.

The scattering amplitude will not be normalized as the original one, but this is of no immediate concern since physical observables should not depend on it as long as we rescale units appropriately. Damour's effective Hamiltonian operator is thus

$$\hat{\mathcal{H}} \; = \; \hat{p}^2 + V_{eff}(r) \; , \quad V_{eff}(r) \; \equiv - \sum_{n=1}^{\infty} \frac{G_N^n f_n(E)}{r^n} \; , \tag{202}$$

which is a simple non-relativistic system with a potential given by Newtonian-like contributions of r-dependence only. For such a system, the classical part of the associated Lippmann-Schwinger equation is trivial in $D = 4$. Indeed, all energy denominators in the Born subtractions will be just quadratic in the momenta and since the associated potential has no momentum-dependence, there is no expansion that could lead to classical terms. We thus find that the effective potential $V_{eff}(r)$ to all orders is proportional to the Fourier transform of the classical part of the corresponding amplitude evaluated at p_∞, as before.

Using this relation, we can then easily read the $f_n(E)$ coefficients from a known scattering amplitude. As we will see, these coefficients lead directly to the post-Minkowskian scattering angle in the center of mass frame.

THE SCATTERING ANGLE TO ALL ORDERS

The computation of the scattering angle for non-relativistic quantum mechanical Hamiltonians has a long history. Typically, interest has been mainly on finding approximate (semi-classical) solutions, first through the WKB-approximation, later by considering the eikonal limit (see, e.g., refs. [77–79]). These methods are powerful, but they quickly get complicated and they were, of course, developed as approximate solutions to the full quantum mechanical problem.

Armed with the map of Hamiltonians from (199) to (200) we are in a completely different situation since we can treat (200) as a quantum mechanical Hamiltonian from which we only wish to extract the classical part. Not only is the problem then WKB-exact, it is also WKB-trivial in the sense that we only wish to retain the leading $\hbar^0$-piece of the wave function. This leading term S, as is well known, is a solution of the classical Hamilton-Jacobi equation. At this stage we have therefore come full circle and we are back at analyzing the classical Hamiltonian (200) with the added knowledge that f_n coefficients are simply identified with the Fourier transformed scattering amplitude evaluated at p_∞.

Using this observation, we now provide an all-order expression for the post-Minkowskian scattering angle only in terms of the classical part of the amplitude in position space and the impact parameter b, both gauge invariant quantities.

As is well known that scattering angle is given from Hamilton-Jacobi theory by

$$\frac{\theta}{2} = -\int_{r_m}^{+\infty} dr\, \frac{\partial p_r}{\partial L} - \frac{\pi}{2}, \tag{203}$$

where

$$p_r = \sqrt{p_\infty^2 - \frac{L^2}{r^2} - V_{eff}(r)}, \quad V_{eff}(r) = -\sum_{n=0}^{\infty} \frac{G_N^n f_n(E)}{r^n}, \tag{204}$$

being L the angular momentum of the system and r_m the closest root to the origin of eq. (204) which satisfies

$$1 - \frac{b^2}{r_m^2} - \frac{V_{eff}(r_m)}{p_\infty^2} = 0, \quad b = \frac{L}{p_\infty}, \tag{205}$$

where we have introduced the impact parameter b.

We find it convenient to rewrite the scattering angle as

$$\frac{\theta}{2} = b\int_{r_m}^{+\infty} \frac{dr}{r^2}\left(1 - \frac{b^2}{r^2} - V_{eff}(r)\right)^{-\frac{1}{2}} - \frac{\pi}{2} = b\int_{r_m}^{+\infty} \frac{dr}{r^2}\left(1 - \frac{r_m^2}{r} - W(r)\right)^{-\frac{1}{2}} - \frac{\pi}{2}, \tag{206}$$

where we have defined

$$W(r) \equiv \frac{1}{p_\infty^2}\left[V_{eff}(r) - \frac{r_m^2}{r^2} V_{eff}(r_m)\right], \quad W(r_m) = 0. \tag{207}$$

We next perform a change of variables to highlight the properties of $W(r)$ at r_m,

$$r^2 = u^2 + r_m^2 \quad \Rightarrow \quad \frac{\theta}{2} = b\int_0^{+\infty} \frac{du}{r^2}\left(1 - \frac{r^2 W(r)}{u^2}\right)^{-\frac{1}{2}} - \frac{\pi}{2}. \tag{208}$$

At this point we expand the square root of eq. (208) using the generalized binomial theorem

$$(1+x)^{-\frac{1}{2}} = 1 + \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n+1} x^{n+1}, \tag{209}$$

where

$$\binom{-\frac{1}{2}}{n+1} = \frac{\Gamma(\frac{1}{2})}{\Gamma(n+2)\Gamma(-n-\frac{1}{2})} = \frac{(-1)^{n+1}(2n+1)!!}{2^{n+1}\Gamma(n+2)}. \tag{210}$$

Using eq. (209) the scattering angle becomes

$$\begin{aligned} \frac{\theta}{2} &= \frac{\pi}{2}\left(\frac{b}{r_m}-1\right) + b\sum_{n=0}^{\infty}(-1)^{n+1}\binom{-\frac{1}{2}}{n+1}\int_0^{+\infty}\frac{du}{u^{2(n+1)}}[W^{n+1}(r)r^{2n}] \\ &= \frac{\pi}{2}\left(\frac{b}{r_m}-1\right) + b\sum_{n=0}^{\infty}\frac{(2n+1)!!}{2^{n+1}(n+1)!}\int_0^{+\infty}\frac{du}{u^{2(n+1)}}[W^{n+1}(r)r^{2n}]. \end{aligned} \tag{211}$$

We now use the following properties which holds for $\mathcal{C}^{\infty}$ functions from $\mathbb{R}$ to $\mathbb{R}$ that vanish at infinity and at the origin:

$$\int_0^{+\infty}\frac{du}{u^{2(n+1)}}f(u) = \frac{1}{(2n+1)!!}\int_0^{+\infty}du\left(\frac{1}{u}\frac{d}{du}\right)^{n+1}f(u). \tag{212}$$

Using eq. (212) we obtain

$$\begin{aligned} \frac{\theta}{2} &= \frac{\pi}{2}\left(\frac{b}{r_m}-1\right) + b\sum_{n=0}^{\infty}\frac{1}{2^{n+1}(n+1)!}\int_0^{+\infty}du\left(\frac{1}{u}\frac{d}{du}\right)^{n+1}\left[W^{n+1}(r)r^{2n}\right] \\ &= \frac{\pi}{2}\left(\frac{b}{r_m}-1\right) + b\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\int_0^{+\infty}du\left(\frac{d}{du^2}\right)^{n+1}\left[W^{n+1}(r)r^{2n}\right]. \end{aligned} \tag{213}$$

In order to introduce a systematic expansion we write this as

$$\frac{\theta}{2} = \frac{\pi}{2}\left(\frac{b}{r_m}-1\right) + b\sum_{n=0}^{\infty}\Delta_n(r_m), \tag{214}$$

$$\Delta_n(r_m) \equiv \frac{1}{(n+1)!}\int_0^{+\infty} du \left(\frac{d}{du^2}\right)^{n+1} \left[W^{n+1}(r) r^{2n}\right], \quad r = \sqrt{u^2+r_m^2}. \quad (215)$$

Focusing on eq. (215), we now expand

$$\Delta_n(r_m) = \frac{1}{p_\infty^{2n+2}}\frac{1}{(n+1)!}\sum_{k=0}^{n+1}\binom{n+1}{k}\int_0^{+\infty} du \left(\frac{d}{du^2}\right)^{n+1}\left[V_{eff}^{n+1-k}(r) r^{2n}\right]\left[-\frac{r_m^2 V_{eff}(r_m)}{r^2}\right]^k . \quad (216)$$

Rewriting in terms of b and r_m, and using eq. (205), this leads to

$$\begin{aligned}\Delta_n(r_m) &= \sum_{k=0}^{n+1}\frac{(b^2-r_m^2)^k}{k!}\int_0^{+\infty} du\left(\frac{d}{du^2}\right)^{n+1}\frac{V_{eff}^{n-k+1}(r)\, r^{2(n-k)}}{(n-k+1)!\, p_\infty^{2(n-k+1)}} \quad (217)\\ &= \sum_{k=0}^{n+1}\frac{(b^2-r_m^2)^k}{k!}\left(\frac{d}{dr_m^2}\right)^k\int_0^{+\infty} du\left(\frac{d}{du^2}\right)^{n-k+1}\frac{V_{eff}^{n-k+1}(r)\, r^{2(n-k)}}{(n-k+1)!\, p_\infty^{2(n-k+1)}},\end{aligned}$$

where we have used the fact that derivatives on r_m^2 and u^2 can be interchanged for a function of the radial distance $r = \sqrt{u^2+r_m^2}$, so as to put these outside the integration. This simple trick, allows us to recognize in eq. (217) the following function

$$\theta_m(r_m) \equiv \frac{1}{p_\infty^{2m+2}}\int_0^{+\infty} du\left(\frac{d}{du^2}\right)^{m+1}\frac{V_{eff}^{m+1}(r)\, r^{2m}}{(m+1)!}, \quad (218)$$

using which we can rewrite eq. (217) as

$$\Delta_n(r_m) = \sum_{k=0}^{n+1}\widetilde{\Delta}_{n,k}(r_m), \quad \widetilde{\Delta}_{n,k}(r_m) \equiv \frac{(b^2-r_m^2)^k}{k!}\left(\frac{d}{dr_m^2}\right)^k \theta_{n-k}(r_m). \quad (219)$$

To summarize what we have obtained so far,

$$\frac{\theta}{2} = \frac{\pi}{2}\left(\frac{b}{r_m}-1\right) + b\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\widetilde{\Delta}_{n,k}(r_m)$$

$$= \frac{\pi}{2}\left(\frac{b}{r_m} - 1\right) + b\sum_{n=0}^{\infty}\sum_{k=0}^{n}\widetilde{\Delta}_{n,k}(r_m) + b\sum_{n=0}^{\infty}\widetilde{\Delta}_{n,n+1}(r_m)\,. \tag{220}$$

The last sum can be rewritten in a remarkably simple way

$$\begin{aligned} b\sum_{n=0}^{\infty}\widetilde{\Delta}_{n,n+1}(r_m) &= b\sum_{n=0}^{\infty}\frac{(b^2-r_m^2)^{n+1}}{(n+1)!}\left(\frac{d}{dr_m^2}\right)^{n+1}\theta_{-1}(r_m) \\ &= b\sum_{n=0}^{\infty}\frac{(b^2-r_m^2)^{n}}{n!}\left(\frac{d}{dr_m^2}\right)^{n}\theta_{-1}(r_m) - b\theta_{-1}(r_m) = b\left[\theta_{-1}(b) - \theta_{-1}(r_m)\right], \end{aligned} \tag{221}$$

or simply

$$b\sum_{n=0}^{\infty}\widetilde{\Delta}_{n,n+1}(r_m) = \frac{\pi}{2}\left(1 - \frac{b}{r_m}\right), \tag{222}$$

where we have recognized the Taylor series of $\theta_{-1}(r_m)$ around b. This is equal and opposite to the first contribution of eq. (220), a cancellation which lead to the following expression for the scattering angle

$$\frac{\theta}{2} = b\sum_{n=0}^{\infty}\sum_{k=0}^{n}\widetilde{\Delta}_{n,k}(r_m) = b\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(b^2-r_m^2)^k}{k!}\left(\frac{d}{dr_m^2}\right)^{k}\theta_{n-k}(r_m) = b\sum_{k=0}^{\infty}\theta_k(b)\,. \tag{223}$$

In the last equality we have used the fact that eq. (223) is the sum over n of the Taylor series of $\theta_n(r_m)$ around b. Thus, the main result of this section can be summarized in the following way, which states that the scattering angle can always be expressed in terms of finite integrals without any reference to r_m

$$\theta = \sum_{k=1}^{\infty}\widetilde{\theta}_k(b)\,, \quad \widetilde{\theta}_k(b) \equiv \frac{2b}{k!}\int_0^{+\infty} du\left(\frac{d}{du^2}\right)^k \frac{V_{eff}^k(r)\, r^{2(k-1)}}{p_\infty^{2k}}\,. \tag{224}$$

Since V_{eff} is related to the classical part of the Fourier transform of scattering amplitudes, this concludes the derivation of the scattering angle solely in terms of gauge invariant quantities. The manifest independence of the intermediate parameter r_m (the distance of nearest approach) in our expression for the scattering angle is important. Since r_m in general is determined by a solvable condition relating it to other scattering information it should disappear entirely from the result, as we have shown explicitly. In our approach there is no subtlety involved in the way it drops out of the relation for the scattering angle and there is no need to regularize intermediate expressions on account of it. Independence of r_m is a particularly acute problem in general relativity where this quantity is not even gauge invariant and such it has to disappear from the expression for the gauge invariant scattering angle.

Let us finally explore the simplicity of our expression for the scattering angle as opposed to previous methods. As described above, we can express the fully relativistic scattering angle in terms of an effective position-space potential which for the case of general relativity is given by

$$V_{eff}(r) \; = \; -\sum_{n=1}^{\infty} \frac{G_N^n f_n(E)}{r^n} \,. \tag{225}$$

This is related to the classical part of the scattering amplitude to any loop order as shown. Let us first focus on the angle up to 3PM order in four dimensions. For ease of notation, in what follows we are going to implicitly assume the dependence on energy of the $f_n(E)$ coefficients. We thus consider

$$\theta_{3PM}(b) = \widetilde{\theta}_1(b) + \widetilde{\theta}_2(b) + \widetilde{\theta}_3(b)\,, \tag{226}$$

$$\widetilde{\theta}_1(b) = \frac{2b}{p_\infty^2}\int_0^{+\infty} du\, \frac{d}{db^2} V_{eff}(r)\,,$$

$$\widetilde{\theta}_2(b) = \frac{b}{p_\infty^4}\int_0^{+\infty} du \left(\frac{d}{db^2}\right)^2 r^2 \left[V_{eff}(r)\right]^2, \tag{227}$$

$$\widetilde{\theta}_3(b) = \frac{b}{3p_\infty^6}\int_0^{+\infty} du \left(\frac{d}{db^2}\right)^3 r^4 \left[V_{eff}(r)\right]^3. \tag{228}$$

We start with the first contribution from eq. (226),

$$\widetilde{\theta}_1(b) = \frac{b}{p_\infty^2}\int_0^{+\infty} du\, \frac{\partial_r V_{eff}(\sqrt{u^2+b^2})}{\sqrt{u^2+b^2}}\,. \tag{229}$$

This we recognize as a classic textbook formula, usually presented for the bending angle around static massive sources in the non-relativistic approximation (see, e.g., ref. [80]). Although it is surely of older origin, we will denote it Bohm's formula. The power of our derivation is that this formula describes the motion of fully relativistic particles, with no restriction on masses or range of velocities on account of the exact map. We can also provide a closed formula for this contribution given by a generic effective potential.

$$\widetilde{\theta}_1(b) = \frac{b}{p_\infty^2}\sum_{n=1}^{\infty} n G_N^n f_n \int_0^{+\infty} du \frac{1}{(u^2+b^2)^{\frac{n}{2}+1}}\,. \tag{230}$$

As can be seen, all terms depend on the integral

$$\int_0^{+\infty} du \frac{1}{(u^2+b^2)^{\frac{n}{2}+1}} = \frac{1}{b^{n+1}}\frac{\sqrt{\pi}}{n}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\,, \quad \forall n \in \mathbb{N}\,, \tag{231}$$

and thus

$$\widetilde{\theta}_1(b) = \frac{\sqrt{\pi}}{p_\infty^2}\sum_{n=1}^{\infty}\frac{G_N^n f_n}{b^n}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}. \tag{232}$$

To 3PM order, the other needed contributions are given by

$$\widetilde{\theta}_1(b) = \frac{\sqrt{\pi}}{p_\infty^2}\sum_{n=1}^{3}\frac{G_N^n f_n}{b^n}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} = \frac{G_N f_1}{Lp_\infty} + \frac{G_N^2 f_2 \pi}{2L^2} + \frac{G_N^3 f_3 2p_\infty}{L^3}, \tag{233}$$

which reproduces the linear terms in f_n up to 3PM known in literature. However, to the same order there are also additional contributions which can be regarded as corrections to Bohm's formula beyond leading order as given by eqs. (227)-(228)

$$\widetilde{\theta}_2(b) = \frac{b}{p_\infty^4}\int_0^{+\infty} du \left(\frac{d}{db^2}\right)^2 r^2\left[\frac{2G_N^3 f_1 f_2}{r^3} + \frac{G_N^2 f_1^2}{r^2}\right] = \frac{G_N^3 f_1 f_2}{L^3 p_\infty}. \tag{234}$$

Here is an important observation: The contribution to G_N^2 vanishes in four dimensions. This means that Bohm's formula in eq. (229) is valid, beyond what we could expect, also at 2PM order, a fact which has been previously noticed and from which now we provide a clear understanding. In fact, Bohm's non-relativistic formula holds at 2PM order even if one naively substitutes a static non-relativistic potential for the bending of light, and we now understand why. Furthermore, this formula agrees with the explicit calculations of the eikonal limit of gravity up to 2PM order with arbitrary masses [47, 74–76]. We now also understand why the eikonal exponentiation of classical gravity works out so simply at 2PM order in four dimensions: it is the vanishing of the f_1^2-term for the angle (and the fact that in the eikonal limit the scattering angle enters in terms of the odd function $\sin(\theta)$).

This brings us to another important point. We see from this analysis that the eikonal exponentiation is bound to work for classical gravity to all orders and in any number of dimensions. Not only that, its precise form is already dictated by the formula we provide. In this sense, there would superficially seem to be no need to pursue the computation of the eikonal limit beyond 2PM order. However, given that the actual evaluation of the coefficients f_i require explicit full amplitude calculations it could still be of interest to pursue the eikonal limit to the given order, as an independent check.

Finally, we need to evaluate the remaining term

$$\widetilde{\theta}_3(b) \;=\; -\frac{bG_N^3 f_1^3}{3p_\infty^6}\int_0^{+\infty} du \left(\frac{d}{db^2}\right)^3 r = -\frac{G_N^3 f_1^3}{12L^3 p_\infty^3}\,. \tag{235}$$

Summing these contributions, we obtain the desired scattering angle at 3PM order

$$\theta_{3PM} = \frac{G_N f_1}{Lp_\infty} + \frac{G_N^2 f_2 \pi}{2L^2} + \frac{G_N^3 f_3 2p_\infty}{L^3} + \frac{G_N^3 f_1 f_2}{L^3 p_\infty} - \frac{G_N^3 f_1^3}{12L^3 p_\infty^3} \tag{236}$$

As seen, the computation is quite straightforward, involving only elementary integrals and derivatives. Higher PM contributions can be calculated easily to any desired order as demonstrated in table 1. It is clear that there are interesting patterns in these expressions and it is elementary to express several of the combinations in simple closed form, valid to all orders.

It is perhaps more interesting to note that certain combinations are *missing*. We illustrated this above by pointing out how the f_1^2-contribution vanishes. Equipped with the map between effective potential and coefficient of the amplitude, we can now understand this result in all generality.

In order to analyse the general conditions for such vanishing contribution to the scattering angle we start by reconsidering the previous expression for the post-Minkowskian scattering angle assuming a m post-Minkowskian potential

$$\theta = \sum_{n=1}^{\infty} \widetilde{\theta}_n(b)\,, \quad \widetilde{\theta}_n(b) = \frac{2b}{n!\, p_\infty^{2n}} \int_0^{+\infty} du \left(\frac{d}{db^2}\right)^n r^{2n-2} V_{eff}^n(r)\,, \tag{237}$$

$$V_{eff}(r) = -\sum_{k=1}^{m} \frac{G_N^k f_k}{r^k}\,. \tag{238}$$

We expand the n-power of the potential by using the multinomial theorem

$$V_{eff}^n(r) = (-1)^n \sum_{n_1+n_2+\ldots+n_m=n} \binom{n}{n_1, n_2, \ldots, n_m} \frac{G_N^{\beta_m} f_1^{n_1} f_2^{n_2} \cdots f_m^{n_m}}{r^{\beta_m}}\,, \tag{239}$$

where $\beta_m \equiv n_1 + 2n_2 + 3n_3 + \ldots m n_m$ and $\binom{n}{n_1, n_2, \ldots, n_m} \equiv \frac{n!}{n_1! n_2! \ldots n_m!}$. If we now evaluate eq.(237) using this we have

$$\begin{aligned} \widetilde{\theta}_n(b) &= \frac{2\sqrt{\pi}}{n! p_\infty^{2n}} \sum_{n_1+n_2+\ldots+n_m=n} \frac{f_1^{n_1} f_2^{n_2} \cdots f_m^{n_m}}{b^{\beta_m}} \frac{G_N^{\beta_m}}{\beta_m} \binom{n}{n_1, n_2, \ldots, n_m} \frac{\Gamma(\frac{\beta_m+1}{2})}{\Gamma(\frac{\beta_m}{2})} \prod_{\alpha=0}^{n-1} (1-n+\frac{\beta_m}{2}+\alpha) \\ &= \frac{\sqrt{\pi}}{p_\infty^{2n}} \sum_{n_1+n_2+\ldots+n_m=n} \left(\frac{G_N}{b}\right)^{\beta_m} \left(\prod_{l=1}^{m} \frac{f_l^{n_l}}{n_l!}\right) \frac{\Gamma(\frac{\beta_m+1}{2})}{\Gamma(\frac{\beta_m}{2}+1-n)}\,. \end{aligned} \tag{240}$$

Null contributions in eq.(240) appears for

$$\begin{cases} 2n-2-n_1-2n_2-...-mn_m=0\,, \\ \\ n_1+n_2+...n_m=n\,, \quad \forall n\wedge n_{j=1,2..m}\in\mathbb{N}\,, \end{cases} \tag{241}$$

as well as

$$\begin{cases} 2n-2-n_1-2n_2-...-mn_m-1=0\,, \\ \\ n_1+n_2+...n_m=n\,, \quad \forall n\wedge n_{j=1,2..m}\in\mathbb{N}\,, \end{cases} \tag{242}$$

and so on. All these can be expressed in a compact form as follows

$$\begin{cases} 2n-2-n_1-2n_2-...-mn_m-\alpha=0\,, \\ \\ n_1+n_2+...n_m=n\,, \quad \forall n,\alpha\wedge n_{j=1,2..m}\in\mathbb{N} \quad : \quad 0\leq\alpha\leq n-1\,. \end{cases} \tag{243}$$

This system of equations describes the intersection of two affine hyperplanes in m dimensions, the solutions to which are positive integer points on a parametric $m-2$ affine hyperplane with parameters n and α. Thus, given a m-dimensional post-Minkowskian potential, the vanishing coefficients to the scattering angle are in one-to-one correspondence with the positive integer zeros of the intersection of two affine hyperplanes in m dimensions. As an example, let us evaluate the vanishing contributions to the scattering angle arising from a 3PM potential. The system to be solved is

$$\begin{cases} n_1 + 2n_2 + 3n_3 = 2n - 2 - \alpha\,, \\ n_1 + n_2 + n_3 = n\,, \quad \forall n, \alpha \wedge n_{j=1,2,3} \in \mathbb{N} \;\; : \;\; 0 \leq \alpha \leq n - 1\,, \end{cases} \tag{244}$$

and the solution is given by

$$\begin{cases} n_1 = n_1 \\ n_2 = 2 - 2n_1 + n + \alpha\,, \\ n_3 = n_1 - 2 - \alpha\,, \quad \forall n, \alpha, n_1 \in \mathbb{N} \;\; : \;\; 0 \leq \alpha \leq n - 1\,. \end{cases} \tag{245}$$

We remind the reader that the parameter n labels the $\widetilde{\theta}_n$ contribution to the scattering angle. For $n = 1$ there are no positive integer solution on this hyperplane, while for $n = 2$ we find that there is only one solution given by $n_1 = 2, n_2 = n_3 = 0$, which is nothing else than the vanishing of the $f_1^{n_1} = f_1^2$ term. This procedure is straightforward, it can be easily generalized to any order, and shows that there is an infinite number of such vanishing contributions.

CONCLUSION

We have unravelled an unexpected equivalence between classical solutions to Lippmann-Schwinger equations and solutions to the relativistic energy relation of two-body dynamics. The equivalence ensures that a physical observable such as the scattering angle can be determined directly from the classical part of the amplitude without recourse to the relativistic potential. In detail, we have found that the implicit function theorem applied to the relativistic energy relation is in one-to-one correspondence with the classical part of the solutions to the Lippmann-Schwinger equation of the quantum mechanical scattering problem. The link is a relation between the classical part of the scattering amplitude and the potential (and derivatives thereof). Amazingly, this relation removes all Born subtractions from the problem leaving us with only the classical part of the amplitude when we evaluate the scattering angle.

Using Damour's map to a non-relativistic theory for a particle of mass equal to 1/2, we have derived an explicit formula for the Post-Minkowksian scattering angle to any order in the coupling constants of the potential. This formula is universal and applicable to any classical potential. A distinct advantage of our formula is that it does not require knowledge of the classical turning point r_m, nor does it require regularization with respect to that quantity. When we apply our formula to the problem of Post-Minkowskian general relativity we recover, effortlessly, the perturbative expansions quoted in the literature. We have illustrated the simplicity of our expression for the scattering angle by listing the expression of the scattering angle up to 12PM order.

There are patterns in these expressions for the scattering angle and we have explained why there are certain "vanishing theorems" for particular combinations of terms. The first missing one is the f_1^2-piece of the one-loop scattering angle, which explains the simplicity of the eikonal limit at one-loop order. We have also found the general condition for the vanishing of such contributions to any order.

PM	$\theta_{PM}/\left(\frac{G_N}{p_\infty L}\right)^{PM}$
1	f_1
2	$\frac{1}{2}\pi p_\infty^2 f_2$
3	$2f_3p_\infty^4 + f_1f_2p_\infty^2 - \frac{f_1^3}{12}$
4	$\frac{3}{8}\pi p_\infty^4\left(2f_4p_\infty^2 + f_2^2 + 2f_1f_3\right)$
5	$\frac{8}{3}f_5p_\infty^8 + 4\left(f_2f_3 + f_1f_4\right)p_\infty^6 + f_1\left(f_2^2 + f_1f_3\right)p_\infty^4 - \frac{1}{6}f_1^3f_2p_\infty^2 + \frac{f_1^5}{80}$
6	$\frac{5}{16}\pi p_\infty^6\left(3f_6p_\infty^4 + 3\left(f_3^2 + 2f_2f_4 + 2f_1f_5\right)p_\infty^2 + f_2^3 + 6f_1f_2f_3 + 3f_1^2f_4\right)$
7	$\frac{16}{5}f_7p_\infty^{12} + 8\left(f_3f_4 + f_2f_5 + f_1f_6\right)p_\infty^{10} + 6\left(f_3f_2^2 + 2f_1f_4f_2 + f_1\left(f_3^2 + f_1f_5\right)\right)p_\infty^8$ $+f_1\left(f_2^3 + 3f_1f_3f_2 + f_1^2f_4\right)p_\infty^6 - \frac{1}{8}f_1^3\left(2f_2^2 + f_1f_3\right)p_\infty^4 + \frac{3}{80}f_1^5f_2p_\infty^2 - \frac{f_1^7}{448}$
8	$\frac{35}{128}\pi p_\infty^8\left(4f_8p_\infty^6 + 6\left(f_4^2 + 2\left(f_3f_5 + f_2f_6 + f_1f_7\right)\right)p_\infty^4 + 12\left(f_4f_2^2 + \left(f_3^2 + 2f_1f_5\right)f_2\right.\right.$ $\left.\left.+f_1\left(2f_3f_4 + f_1f_6\right)\right)p_\infty^2 + f_2^4 + 6f_1^2f_3^2 + 12f_1f_2^2f_3 + 12f_1^2f_2f_4 + 4f_1^3f_5\right)$
9	$\frac{128}{35}f_9p_\infty^{16} + \frac{64}{5}\left(f_4f_5 + f_3f_6 + f_2f_7 + f_1f_8\right)p_\infty^{14} + \frac{16}{3}\left(f_3^3 + 6\left(f_2f_4 + f_1f_5\right)f_3 + 3f_2^2f_5\right.$ $\left.+3f_1\left(f_4^2 + 2f_2f_6 + f_1f_7\right)\right)p_\infty^{12} + 8\left(f_3f_2^3 + 3f_1f_4f_2^2 + 3f_1\left(f_3^2 + f_1f_5\right)f_2\right.$ $\left.+f_1^2\left(3f_3f_4 + f_1f_6\right)\right)p_\infty^{10} + f_1\left(f_2^4 + 6f_1f_3f_2^2 + 4f_1^2f_4f_2 + f_1^2\left(2f_3^2 + f_1f_5\right)\right)p_\infty^8$ $-\frac{1}{30}f_1^3\left(10f_2^3 + 15f_1f_3f_2 + 3f_1^2f_4\right)p_\infty^6 + \frac{1}{40}f_1^5\left(3f_2^2 + f_1f_3\right)p_\infty^4 - \frac{1}{112}f_1^7f_2p_\infty^2 + \frac{f_1^9}{2304}$
10	$\frac{63}{256}\pi p_\infty^{10}\left(5f_{10}p_\infty^8 + 10\left(f_5^2 + 2\left(f_4f_6 + f_3f_7 + f_2f_8 + f_1f_9\right)\right)p_\infty^6\right.$ $+30\left(f_6f_2^2 + \left(f_4^2 + 2f_1f_7\right)f_2 + f_3^2f_4 + 2f_3\left(f_2f_5 + f_1f_6\right) + f_1\left(2f_4f_5 + f_1f_8\right)\right)p_\infty^4$ $+10\left(2f_4f_2^3 + 3\left(f_3^2 + 2f_1f_5\right)f_2^2 + 6f_1\left(2f_3f_4 + f_1f_6\right)f_2 + f_1\left(2f_3^3 + 6f_1f_5f_3\right.\right.$ $\left.\left.+f_1\left(3f_4^2 + 2f_1f_7\right)\right)\right)p_\infty^2 + f_2^5 + 30f_1^2f_2f_3^2 + 20f_1f_2^3f_3 + 30f_1^2f_2^2f_4 + 20f_1^3f_3f_4$ $\left.+20f_1^3f_2f_5 + 5f_1^4f_6\right)$
11	$\frac{256}{63}f_{11}p_\infty^{20} + \frac{128}{7}\left(f_5f_6 + f_4f_7 + f_3f_8 + f_2f_9 + f_1f_{10}\right)p_\infty^{18}$ $+32\left(f_7f_2^2 + 2\left(f_4f_5 + f_1f_8\right)f_2 + f_3^2f_5 + f_3\left(f_4^2 + 2f_2f_6 + 2f_1f_7\right)\right.$ $\left.+f_1\left(f_5^2 + 2f_4f_6 + f_1f_9\right)\right)p_\infty^{16} + \frac{80}{3}\left(f_8f_1^3 + 3\left(f_4\left(f_3^2 + f_1f_5\right) + f_1f_3f_6\right)f_1 + f_2^3f_5\right.$ $\left.+3f_2^2\left(f_3f_4 + f_1f_6\right) + f_2\left(f_3^3 + 6f_1f_5f_3 + 3f_1\left(f_4^2 + f_1f_7\right)\right)\right)p_\infty^{14} + 10\left(f_3f_2^4 + 4f_1f_4f_2^3\right.$ $\left.+6f_1\left(f_3^2 + f_1f_5\right)f_2^2 + 4f_1^2\left(3f_3f_4 + f_1f_6\right)f_2 + f_1^2\left(2f_3^3 + 2f_1\left(f_4^2 + 2f_3f_5\right) + f_1^2f_7\right)\right)p_\infty^{12}+$ $f_1\left(f_2^5 + 10f_1f_3f_2^3 + 10f_1^2f_4f_2^2 + 5f_1^2\left(2f_3^2 + f_1f_5\right)f_2 + f_1^3\left(5f_3f_4 + f_1f_6\right)\right)p_\infty^{10}$ $-\frac{1}{12}f_1^3\left(5f_2^4 + 15f_1f_3f_2^2 + 6f_1^2f_4f_2 + f_1^2\left(3f_3^2 + f_1f_5\right)\right)p_\infty^8$ $+\frac{1}{56}f_1^5\left(7f_2^3 + 7f_1f_3f_2 + f_1^2f_4\right)p_\infty^6 - \frac{5}{896}f_1^7\left(4f_2^2 + f_1f_3\right)p_\infty^4 + \frac{5f_1^9f_2p_\infty^2}{2304} - \frac{f_1^{11}}{11264}$
12	$\frac{231}{1024}\pi p_\infty^{12}\left(6f_{12}p_\infty^{10} + 15\left(f_6^2 + 2\left(f_5f_7 + f_4f_8 + f_3f_9 + f_2f_{10} + f_1f_{11}\right)\right)p_\infty^8\right.$ $+20\left(f_4^3 + 6\left(f_3f_5 + f_2f_6 + f_1f_7\right)f_4 + 3\left(f_8f_2^2 + \left(f_5^2 + 2f_3f_7 + 2f_1f_9\right)f_2\right.\right.$ $\left.\left.+f_3^2f_6 + f_1\left(2f_5f_6 + 2f_3f_8 + f_1f_{10}\right)\right)\right)p_\infty^6 + 15\left(f_3^4 + 12\left(f_2f_4 + f_1f_5\right)f_3^2\right.$ $+12\left(f_5f_2^2 + f_1\left(f_4^2 + 2f_2f_6 + f_1f_7\right)\right)f_3 + 2\left(2f_6f_2^3 + 3\left(f_4^2 + 2f_1f_7\right)f_2^2\right.$ $\left.\left.+6f_1\left(2f_4f_5 + f_1f_8\right)f_2 + f_1^2\left(3f_5^2 + 6f_4f_6 + 2f_1f_9\right)\right)\right)p_\infty^4$ $+30\left(f_4f_2^4 + 2\left(f_3^2 + 2f_1f_5\right)f_2^3 + 6f_1\left(2f_3f_4 + f_1f_6\right)f_2^2 + 2f_1\left(2f_3^3 + 6f_1f_5f_3\right.\right.$ $\left.\left.+f_1\left(3f_4^2 + 2f_1f_7\right)\right)f_2 + f_1^2\left(6f_4f_3^2 + 4f_1f_6f_3 + f_1\left(4f_4f_5 + f_1f_8\right)\right)\right)p_\infty^2$ $+f_2^6 + 20f_1^3f_3^3 + 90f_1^2f_2^2f_3^2 + 15f_1^4f_4^2 + 30f_1f_2^4f_3$ $\left.+60f_1^2f_2^3f_4 + 120f_1^3f_2f_3f_4 + 60f_1^3f_2^2f_5 + 30f_1^4f_3f_5 + 30f_1^4f_2f_6 + 6f_1^5f_7\right)$

Table 1: PM corrections to 12th order in G_N.

APPENDIX

CLASSICAL CONTRIBUTIONS FROM THE LIPPMAN-SCHWINGER EQUATION

In this Appendix we elaborate on the point made in the main text regarding the computational topology of the classical contributions from the Lippman-Schwinger equation. As we have seen in eq.(164), the n-th term of the Lippmann-Schwinger equation is given by

$$S_n(p,p') = \int_{k_1,k_2,\ldots,k_n} \frac{\widetilde{V}(p,k_1)\widetilde{V}(k_1,k_2)\cdots\widetilde{V}(k_n,p')}{(E_p - E_{k_1})\cdots(E_{k_{n-1}} - E_{k_n})} \,. \tag{246}$$

Using eq.(163), this can also be rewritten as

$$S_n(p,p') = \sum_{i_1,\ldots,i_{n+1}} \alpha^{i_1\cdots i_{n+1}} \int_{k_1,\ldots,k_n} \frac{1}{|p-k_1|^{2i_1}\cdots|k_n - p'|^{2i_{n+1}}} + \ldots \,, \tag{247}$$

where the ellipsis denotes both quantum and super-classical contributions around $D = 4$ space-time dimensions, while the $\alpha^{i_1\cdots i_{n+1}}$ are the combinations of constants that can be taken out of the integrals by properly expanding the numerators from each of the potential terms. Let us denote the classical part of this series by $S_n^{cl.}$

$$S_n^{cl.}(p,p') = \sum_{i_1,\ldots,i_{n+1}} \alpha^{i_1\ldots i_{n+1}} G^{(n+1)}_{i_i\ldots i_{n+1}}(p,p')\,,\;\; G^{(n+1)}_{i_i\ldots i_{n+1}}(p,p') \equiv \int_{k_1,\ldots,k_n} \frac{1}{|p-k_1|^{2i_1}\cdots|k_n-p'|^{2i_{n+1}}}\,. \tag{248}$$

If we perform a shift in $k_i \to k_i + p$, $\forall k_i$ in the right-hand side of eq.(248) we immediately recognize the definition of a generalized sunset loop-diagram associated with a massless particle with momentum q and arbitrary powers in the denominators,

$$G^{(n+1)}_{i_i\ldots i_{n+1}}(q) \equiv \int_{k_1,\ldots,k_n} \frac{1}{|k_1|^{2i_1}\cdots|k_n+q|^{2i_{n+1}}}\,. \tag{249}$$

These integrals can be easily computed and they do share a nice factorization property in position space. In fact, by taking the Fourier transform in position space we get, using the same notation as for the one-loop case,

$$g^{(n+1)}_{i_i\ldots i_{n+1}}(r) = \int_{q,k_1,\ldots,k_n} \frac{e^{iq\cdot r}}{|k_1|^{2i_1}\cdots|k_n+q|^{2i_{n+1}}} = g_{i_1}(r)\cdots g_{i_{n+1}}(r)\,, \tag{250}$$

thus generalizing to all orders what was already seen at one-loop order in the main text.

8

SECOND ORDER POST-MINKOWSKIAN SCATTERING IN ARBITRARY DIMENSIONS

We extract the long-range gravitational potential between two scalar particles with arbitrary masses from the two-to-two elastic scattering amplitude at 2nd Post-Minkowskian order in arbitrary dimensions. In contrast to the four-dimensional case, in higher dimensions the classical potential receives contributions from box topologies. Moreover, the kinematical relation between momentum and position on the classical trajectory contains a new term which is quadratic in the tree-level amplitude. A precise interplay between this new relation and the formula for the scattering angle ensures that the latter is still linear in the classical part of the scattering amplitude, to this order, matching an earlier calculation in the eikonal approach. We point out that both the eikonal exponentiation and the reality of the potential to 2nd post-Minkowskian order can be seen as a consequence of unitarity. We finally present closed-form expressions for the scattering angle given by leading-order gravitational potentials for dimensions ranging from four to ten.

INTRODUCTION

The study of gravitational collisions has recently received a lot of attention thanks to the amazing experimental breakthroughs in the detection [6, 81–84] of gravitational-waves coming from black-hole or neutron star mergers. Given the expected improvements in detector sensitivity, it will be extremely important in the future to have high-precision theoretical predictions from General Relativity. To this aim the use of quantum field theory amplitudes to extract the post-Minkowskian (PM) expansion of General Relativity has recently gained considerable momentum [10, 12, 14, 15, 23, 24, 32, 73, 85, 86], and progress is now also being made on extensions to spinning objects [65–72, 87, 88]. The underlying physical motivation for this approach lies in the observation that, during the early stages of a merger event, when the two compact objects are still far apart, gravitational interactions are weak and can be conveniently treated in a weak-coupling approximation. The perturbative series that naturally organizes the calculation of scattering amplitudes in quantum field theory then offers a convenient tool to study the dynamics of such systems for weak gravitational fields without the need to consider the limit of small velocities, thanks to the Lorentz invariance of the amplitude. The price one has to pay in order to eventually retrieve predictions for General Relativity is the proper handling of the classical limit. Indeed, going to higher orders in the gravitational coupling in the classical theory entails evaluating Feynman diagrams with more and more loops in the quantum theory and one may wonder as to how the loop expansion may yield precision corrections to classical quantities, an issue that was first clarified in the seminal papers [31, 46] and more recently investigated systematically in [21].

A fundamental and gauge-independent quantity that is most readily computed from quantum field theoretic amplitudes is the scattering angle of two colliding massive objects. Computations of classical gravitational observables using relativistic amplitude techniques have so far been performed with two *a priori* different approaches. One is based on the evaluation of the eikonal phase, while the other proceeds by constructing the Hamiltonian, *i.e.* the effective interaction potential. The deflection angle can then be easily obtained from either of these two quantities.

The eikonal approach began in the late eighties with the work by 't Hooft [74] and independent parallel work of two other groups [89–91], dealing with transplanckian energy collisions of strings in a generic number D of macroscopic dimensions. It was further developed in Refs. [47, 92–98] and extended to the scattering of strings off a stack of D-branes [99, 100] (see Ref. [101] for a review) and recently to supersymmetric theories [102–104].

That approach has its origin in the observation that, in general, a tree diagram in gravity diverges at high energy, implying that unitarity is violated in this regime. A viable way to restore unitarity is to first observe that also the loop diagrams are divergent at high energy and actually their degree of divergence increases with the number of loops. Then, Fourier transforming a suitably normalized amplitude from momentum space to the $(D-2)$-dimensional impact parameter space, one sees that the leading terms for large impact parameter of the various diagrams re-sum into an exponential given by the tree contribution, whose phase is called the leading eikonal. In this way one obtains a quantity that is consistent with unitarity. Sub-leading eikonals can be obtained in a similar way by re-summing diagrams that are subdominant for large impact parameter. Unlike the leading one, they also contain an imaginary part related to inelastic processes, although we do not discuss these new effects in this paper.

Having determined the eikonal, one can then use it to compute the classical deflection angle taking its derivative with respect to the impact parameter. Other physical quantities, as for instance the Shapiro time delay, can also be computed from the eikonal. An interesting aspect of this approach is that, in order to compute the deflection angle to a given order in the coupling, one must still compute, in principle, an infinite number of loops to check the exponentiation.

In contrast, the Hamiltonian approach relies on the calculation of the effective interaction potential between two massive particles from the scattering amplitude, which is achieved as follows. One first imposes that the two-to-two scattering amplitude in General Relativity be equal to that of an effective theory of massive particles interacting via a long-range potential and then reconstructs the potential that ensures this matching condition order by order in Newton's coupling constant G_N. To this purpose one can either employ the relativistic Lippmann–Schwinger equation and the technique of Born subtractions for a first-quantized effective theory [15, 17], or alternatively the Effective Field Theory (EFT) matching procedure [10, 12].

These two methods have proven to be completely equivalent in the cases that have been studied and lead to the same effective potential. Indeed, one would expect the first- and second-quantized effective theories to be equivalent as long as quantum effects such as particle creation are discarded. We shall review the demonstration of equivalence below.

Note that the scattering amplitude contains, in general, not only classical and quantum terms, as identified by their behavior in terms of $\hbar$, but also super-classical terms. With our conventions, classical terms have a finite limit as $\hbar \to 0$ and quantum terms vanish, while super-classical contributions give rise to singular expressions, corresponding to infinitely rapid phase oscillations in the S-matrix. It is therefore crucial that the super-classical terms cancel out in the procedure of extracting the classical potential from the scattering amplitude. We find that this cancellation occurs and in fact also ensures that the potential is real.

In this work we show that indeed both the eikonal exponentiation and the reality of the classical potential are ultimately direct consequences of the unitarity of the quantum theory.

This observation also lies behind the explanation of the following puzzling question: In the Hamiltonian approach one only needs to compute the classical part of the scattering amplitude up to the given order of the expansion in Newton's coupling constant G_N. Classical Hamilton-Jacobi analysis then yields the scattering angle up to that order. Why, then, does the eikonal approach require the computation of the near-forward scattering amplitude to all orders in the coupling G_N in order to derive a fixed-order result for scattering angle? One of the purposes of this paper is to provide an answer to this question. For consistency, it must be that the exponentiation of all higher order terms required in the eikonal limit is automatic. We shall argue that the infinite string of identities needed for the eikonal exponentiation of the classical parts of the near-forward scattering amplitude follow from unitarity. This then resolves the apparent conflict and explains why the two methods for calculating the scattering angle are equivalent.

We consider the scattering problem in a general D-dimensional setting rather than just limiting ourselves to the four-dimensional case. As is known already from non-relativistic quantum mechanics, four space-time dimensions represents a borderline case for scattering in Coulomb-like potentials (such as the leading-order scattering in general relativity) due to the slow fall-off of the potential at infinity and the associated logarithmic phase of the scattered wave. In relativistic quantum field theory this is reflected in the well-known infrared divergences of the scattering amplitude in four dimensions. Once we move beyond four dimensions, even just infinitesimally such as in dimensional regularization, these infrared divergences are regularized.

The need to maintain reparametrization (gauge) invariance at all stages of the amplitude calculations while taming the infrared divergences thus leads us to perform the amplitude calculations beyond $D = 4$ dimensions. Moreover, as we shall demonstrate, it is not correct that the D-dimensional result just trivially mimics the corresponding one in four-dimensional space-time. A new term proportional to $(D-4)$ appears at one-loop order. This could potentially have repercussions at higher loop order if cancelled against infrared sub-divergences, thus threatening to introduce new finite pieces even after taking the $D \to 4$ limit.

To be more specific, we use the relativistic Lippmann–Schwinger equation to derive the long-range effective potential up to 2PM order from the elastic scattering amplitude of two scalar particles with arbitrary masses in a generic D-dimensional space-time.

While in Ref. [47] the box and triangle diagrams were computed for small transferred momentum q, *i.e.* in the classical limit, using a saddle-point evaluation in the space of Schwinger parameters, we here perform the same calculation employing the so-called method of regions [105] in momentum space. This consists in evaluating the asymptotic expansion of the relevant Feynman integrals as $q \to 0$ considering loop momenta k that scale in a definite way with respect to q in this limit.

We identify the soft region, characterized by the scaling relation $k \sim \mathcal{O}(q)$, as the one producing the non-analytic terms that eventually give rise to the long-range potential, namely the ones considered in Ref. [47]. The integrals also receive contributions from the hard region, $k \sim \mathcal{O}(1)$, that are proportional to positive integer powers of q^2 and hence do not contribute to the long-range behavior in position space, although they are needed for the overall consistency of the small-q expansion. Indeed, as is often the case, the hard and soft series separately possess spurious singularities that are just artifacts of the splitting into regions. However, only the singularities originally present in the Feynman integrals survive in the sum of the two series, which provides a nontrivial cross check of the asymptotic expansion thus obtained.

Another region that is often used in order to extract the non-analytic terms in the classical limit is the potential region. Considering a combination of classical limit $q \to 0$ and nonrelativistic limit[1] $v \to 0$, with v the characteristic velocity of the asymptotic states in the center-of-mass frame, one defines the scaling of the loop momenta $k = (k^0, \vec{k})$ in the potential region as $k^0 \sim \mathcal{O}(qv)$ and $\vec{k} \sim \mathcal{O}(q)$. The potential expansion allows one to break down the Feynman integrals into

1 We are grateful to Julio Parra-Martinez and Mikhail Solon for pointing out that the role played by the non-relativistic limit in the definition of the potential region was not properly spelled out in an earlier version of this paper.

$(D-1)$-dimensional integrals in a non-relativistic spirit. In its turn, this opens the possibility to compare General Relativity amplitudes directly to the $(D-1)$-dimensional integrals arising in the effective theory, *i.e.* to perform the matching mentioned above at the level of integrands, disposing with the need to actually evaluate certain integrals. We check that, to leading order in the small-v expansion, the result obtained from the potential region agrees with the non-relativistic limit of the one furnished by the soft region. However, we deem more convenient to apply the method of regions in a covariant fashion directly to the D-dimensional integrals involved in the evaluation of the fully relativistic amplitude, as outlined above, *i.e.* to base our calculation on the soft and hard regions.

An important new feature that appears in our analysis for $D > 4$ is that the 2PM potential receives a nonzero contribution from the sum of the box and crossed box diagrams, which, of course, vanishes if we take $D = 4$. This new contribution comes about because of a nontrivial classical term arising from the sum of box and crossed box diagrams that is not exactly compensated by the Born subtraction of the effective theory. Interestingly, this compensation is exact for any D, and thus no new term appears for $D > 4$ if we limit ourselves to leading order in the non-relativistic expansion, *i.e.* to the leading term of the potential region.

Similarly, when we solve the energy equation for the kinematical relation between momentum and position on the classical trajectory, $p^2(r, G_N)$, in dimensions $D > 4$, we find that new terms that are quadratic in the scattering amplitude appear. To

2PM order, this nonlinearity is precisely canceled by a new term for the classical scattering angle. In this somewhat surprising way, the scattering angle still depends linearly on the amplitude, to this order. The scattering angle we compute here coincides perfectly with the one obtained in Ref. [47] using instead the eikonal method.

The paper is organized as follows. In Sect. 8 we collect the classical and super-classical terms to the one-loop two-to-two amplitude, arising from triangle and box diagrams, which we evaluate with the method of regions. In Sect. 8 we extract the long-range classical potential at 2PM order from the amplitude solving the Lippmann–Schwinger equation by means of Born subtractions and describe the equivalence between this technique and the strategy of EFT matching. Sect. 8 is then devoted to evaluating, given the 2PM potential, the relation $p^2(r, G_N)$ for the classical trajectory, which we then use in Sect. 8 to determine the deflection angle to 2PM order. In Sect. 8 we furnish explicit expressions for the scattering angle given by the 1PM interaction potential for space-time dimensions ranging from four up to ten. The paper contains two appendices. In Appendix 8 we detail our conventions for the normalization of various scattering amplitudes appearing throughout the paper, while in Appendix 8 we present the explicit calculation of the relevant one-loop integrals in the limit $\hbar \to 0$ using the method of the regions.

SCATTERING AMPLITUDES IN D-DIMENSIONAL GENERAL RELATIVITY

In this section we derive the super-classical and classical parts of the one-loop amplitude $\mathcal{M}_{1-\text{loop}}$ in Einstein gravity minimally coupled to two massive scalar fields,

$$S(g_{\mu\nu}, \varphi_{i=1,2}) = \int d^D x \, \frac{\sqrt{-g}\, R}{16\pi G_N} - \frac{1}{2} \int d^D x \, \sqrt{-g} \sum_{i=1,2} \left(g^{\mu\nu} \hbar^2 \partial_\mu \varphi_i \partial_\nu \varphi_i + m_i^2 \varphi_i^2 \right) , \tag{251}$$

for a general space-time dimension D. Focusing on the gravitational interaction of spin-less fields we can compute the large-distance classical scattering of Schwarzschild black holes (or more generically a point-particle) in the perturbative loop expansion. This amplitude has been recently computed in Ref. [47] using a Schwinger parametrization of the various propagators entering the loop and the method of steepest descent in those parameters. One of the surprising results was that the classical piece of $\mathcal{M}_{1-\text{loop}}$ includes, for $D > 4$, a nonvanishing contribution from the sum of box and crossed-box Feynman diagrams. We here employ an alternative method that, in the QCD literature, is known as the method of the regions [105]. It is conveniently used to determine the behavior of a loop integral when one is interested in a kinematical limit involving the external momenta, for instance when one of them is small. Here this method is used to determine an expansion of the loop integrals in powers of $\hbar$, confirming the result of Ref. [47].

Let us consider the scattering of two point-like scalar particles, schematically represented by the diagram in the following figure, whose amplitude is given by a sum over all loop contributions:

We refer to Appendix 8 for more details on our conventions for the normalization of the scattering amplitude.

In the center-of-mass frame we have

$$\begin{aligned} p_1^\mu &= (E_1(p), \vec{p}\,)\,, \qquad & p_3^\mu &= (E_1(p), \vec{p}\,')\,, \\ p_2^\mu &= (E_2(p), -\vec{p}\,)\,, \qquad & p_4^\mu &= (E_2(p), -\vec{p}\,') \end{aligned} \tag{252}$$

and we define

$$p \equiv |\vec{p}\,| = |\vec{p}\,'|\,, \tag{253}$$

$$E_1(p) \equiv \sqrt{p^2 + m_1^2}\,, \qquad E_2(p) \equiv \sqrt{p^2 + m_2^2}\,, \tag{254}$$

$$E_p \equiv E_1(p) + E_2(p)\,, \qquad \xi(p) \equiv \frac{E_1(p)E_2(p)}{E_p^2}\,, \tag{255}$$

$$q^\mu \equiv p_1^\mu - p_3^\mu\,, \qquad \vec{q} \equiv \vec{p} - \vec{p}\,'\,. \tag{256}$$

The previous quantities are related to the Mandelstam variables

$$s = -(p_1 + p_2)^2\,, \qquad t = -(p_1 - p_3)^2 = -q^2 \tag{257}$$

and

$$s = E_p^2, \qquad p^2 = \frac{(E_p^2 - (m_1+m_2)^2)(E_p^2 - (m_1-m_2)^2)}{4E_p^2}. \tag{258}$$

We will use a mostly positive signature metric, so that in particular

$$q^\mu q_\mu = q^2 = |\vec{q}\,|^2 \tag{259}$$

in the center-of-mass frame, and following Ref. [47] we define

$$\kappa_D^2 \equiv 8\pi G_N, \qquad \gamma(p^2) \equiv 2(p_1 \cdot p_2)^2 - \frac{2m_1^2 m_2^2}{D-2}. \tag{260}$$

We first proceed by decomposing the one-loop amplitude in terms of Feynman integrals as follows:

$$\mathcal{M}_{1-\text{loop}} = d_\Box (I_{\Box,s} + I_{\Box,u}) + (d_\triangleleft)_{\mu\nu} I_\triangleleft^{\mu\nu} + d_\triangleleft I_\triangleleft + (d_\triangleright)_{\mu\nu} I_\triangleright^{\mu\nu} + d_\triangleright I_\triangleright + \cdots, \tag{261}$$

where the ellipsis denote quantum contributions. The integrals involved in the above expression are the triangle integrals [2]

$$I_\triangleright = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2 - i\epsilon)\left((q-k)^2 - i\epsilon\right)(k^2 - 2p_1 \cdot k - i\epsilon)}, \tag{262}$$

$$I_\triangleright^{\mu\nu} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^3 k^\mu k^\nu}{(k^2 - i\epsilon)\left((q-k)^2 - i\epsilon\right)(k^2 - 2p_1 \cdot k - i\epsilon)}, \tag{263}$$

2 The dependence on $\hbar$ in the various integrals follows from the fact that, with our conventions, the amplitude in (261) has dimension of $E^3 L^{D-1}$ where E is an energy and L is a length, as detailed in Appendix 8.

together with $I_{\triangleleft}$, $I_{\triangleleft}^{\mu\nu}$ which are given by substituting $p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$ in Eqs. (262) and (263), the box integral

$$I_{\square,s} = \int \frac{d^Dk}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2-i\epsilon)((k-q)^2-i\epsilon)(k^2-2p_1\cdot k-i\epsilon)(k^2+2p_2\cdot k-i\epsilon)} \tag{264}$$

and the crossed box $I_{\square,u}$, obtained by the replacement $p_1 \leftrightarrow -p_3$ from Eq. (264). The associated decomposition coefficients are

$$d_{\square} = 4i\kappa_D^4\gamma^2(p^2)\,, \qquad d_{\triangleright}^{\mu\nu} = \frac{16i\kappa_D^4(D-3)m_1^4}{(D-2)}\frac{\hbar^2 p_2^\mu p_2^\nu}{q^2} \tag{265}$$

and

$$d_{\triangleright} = 4im_1^2\kappa_D^4\left[2m_1^2m_2^2\frac{D^2-4D+2}{(D-2)^2} - 2m_1^2E_p^2 + m_1^4 + \left(m_2^2-E_p^2\right)^2\right], \tag{266}$$

while $d_{\triangleleft}^{\mu\nu}$ and $d_{\triangleleft}$ are obtained by replacing $m_1 \leftrightarrow m_2$ in $d_{\triangleright}^{\mu\nu}$ and $d_{\triangleright}$.

In Appendix 8 we employ the method of expansion by regions to evaluate the classical limit of the one-loop integrals (262), (263) and (264) in arbitrary dimensions D and in a generic reference frame. This limit entails letting $\hbar \to 0$ in such a way that in the center-of-mass frame the three-momentum transfer $\vec{q}$ vanishes, while the transferred wave number $\frac{1}{\hbar}|\vec{q}\,|$, the total energy E_p and the masses m_1, m_2 are kept fixed (see *e.g.* [10,21]). It turns out that this analysis in D dimensions presents some new features as compared with that of Ref. [12], while being in perfect agreement for $D = 4$. The modified expressions for generic $D \geq 4$ will be instrumental in reproducing the correct scattering angle in D dimensions [47].

Quoting first for completeness the tree-level contribution

$$\mathcal{M}_{\text{tree}}(\vec{p},\vec{p}\,') = -2\gamma(p^2)\kappa_D^2 \frac{\hbar^2}{q^2}\,, \tag{267}$$

we are finally able to cast the classical and super-classical terms of the one-loop scattering amplitude in General Relativity and in D dimensions in the following form:

$$\mathcal{M}_{1-\text{loop}}(\vec{p},\vec{p}\,') = \mathcal{M}_{\triangleleft}(\vec{p},\vec{p}\,') + \mathcal{M}_{\triangleright}(\vec{p},\vec{p}\,') + \mathcal{M}_{\Box,s}(\vec{p},\vec{p}\,') + \mathcal{M}_{\Box,u}(\vec{p},\vec{p}\,') + \cdots\,, \tag{268}$$

where

$$\begin{aligned}\mathcal{M}_{\triangleleft}(\vec{p},\vec{p}\,') + \mathcal{M}_{\triangleright}(\vec{p},\vec{p}\,') &= -\frac{2\sqrt{\pi}\kappa_D^4(m_1+m_2)}{(4\pi)^{\frac{D}{2}}} \\ &\times \left(4(p_1\cdot p_2)^2 - \frac{4m_1^2m_2^2}{(D-2)^2} - \frac{(D-3)E_p^2p^2}{(D-2)^2}\right)\frac{\Gamma(\frac{5-D}{2})\Gamma^2(\frac{D-3}{2})}{\Gamma(D-3)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}}\end{aligned} \tag{269}$$

and

$$\begin{aligned}\mathcal{M}_{\Box,s}(\vec{p},\vec{p}\,') + \mathcal{M}_{\Box,u}(\vec{p},\vec{p}\,') = &-\frac{i\pi}{(4\pi)^{\frac{D}{2}}}\frac{2\kappa_D^4\gamma^2(p^2)}{E_p\,p}\frac{\Gamma(\frac{6-D}{2})\Gamma^2(\frac{D-4}{2})}{\Gamma(D-4)}\frac{1}{\hbar}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}} \\ &-\frac{2\sqrt{\pi}\kappa_D^4\gamma^2(p^2)}{(4\pi)^{\frac{D}{2}}}\frac{(m_1+m_2)}{E_p^2\,p^2}\frac{\Gamma(\frac{5-D}{2})\Gamma^2(\frac{D-3}{2})}{\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}}\,.\end{aligned} \tag{270}$$

These results are in agreement with those of Ref. [47] [3].

It should be stressed that the above result for the triangle and box contributions (269), (270) is obtained from the expansion of the corresponding integrals in the

3 Actually the corresponding amplitudes in Ref. [47] are obtained from the ones appearing here multiplying by a factor $-\frac{1}{\hbar}$, since in this paper we use (367), while in Ref. [47] (388) is used instead.

soft region, as detailed in Appendix 8. Such integrals also receive additional contributions from the *hard* region that are, however, proportional to positive integer powers of $\frac{q^2}{\hbar^2}$. We thus discard such terms because they would give rise to strictly local contributions in position space, while we are interested in the long-range behavior of the effective potential. Nevertheless, the interplay between the soft and the hard series is important because it ensures the proper cancellation of spurious divergences that arise for specific dimensions in the above expressions, *e.g.* in $D = 5$, and thus provides a nontrivial consistency check of the asymptotic expansion.

The expression (269) for the triangle topologies could be also alternatively obtained from the leading-order expansion of the associated triangle integrals in the *potential* region, as described in Appendix 8. The potential region also allows for a quick evaluation of the sum of box and crossed box diagrams to leading order in the nonrelativistic limit, $\frac{p}{m_1}, \frac{p}{m_2} \ll 1$.

The result furnished by the leading potential region coincides with the small-velocity limit of (270), which, as we stressed, is based on the soft region. Actually, the first term on the right-hand side of (270), namely the super-classical term, coincides with the corresponding term arising from the leading potential approximation. The second term, instead, agrees with the corresponding classical term in the leading potential expansion only in the nonrelativistic limit, in which $E_p \approx m_1 + m_2$. We refer the reader to Appendix 8 for a detailed discussion of this comparison.

THE POST-MINKOWSKIAN POTENTIAL IN ARBITRARY DIMENSIONS

In this section, we address the calculation of the long-range effective interaction potential to 2PM order in arbitrary dimension, stressing in particular the new elements that appear when away from $D = 4$. Our strategy is based on the method of Born subtractions [15, 17], which is equivalent to the technique of EFT matching [10, 12].

As we have stressed, the two-to-two amplitude presents, to one-loop order, both super-classical, $\mathcal{O}(\hbar^{-1})$, and classical, $\mathcal{O}(\hbar^{0})$, contributions, as identified by their $\hbar$ scaling. The super-classical term arises in particular from the sum of box and crossed box diagrams, which are also the source of the imaginary part of the amplitude and, in $D = 4$, of the infrared divergence. Inverse powers of $\hbar$ are conventionally labelled "IR" in four dimensions since they characterize the terms responsible for infrared divergences there. It should be stressed, however, that the very notion of infrared divergent integrals becomes ambiguous away from four dimensions. Therefore, we shall keep labelling the terms entirely by their scaling (power) with respect to $\hbar$, which is well-defined for any D.

The calculation of the post-Minkowskian potential in the center-of-mass frame will then reveal how the super-classical and imaginary term eventually cancel, providing a well-defined, real and classical expression for the interaction potential, but leave behind nontrivial contributions in generic dimensions $D > 4$. We will also see how this cancellation can be understood as a consequence of the unitarity of the underlying quantum theory.

The Lippmann–Schwinger equation in D dimensions

In order to define a post-Minkowskian potential in momentum space and in an arbitrary number of dimensions $D = d + 1$ we can use a fully relativistic Lippmann-Schwinger equation as in [15]

$$\widetilde{\mathcal{M}}(\vec{p}, \vec{p}\,') = \widetilde{V}^D(\vec{p}, \vec{p}\,') + \int \frac{d^d\vec{k}}{(2\pi\hbar)^d} \frac{\widetilde{V}^D(\vec{p}, \vec{k})\widetilde{\mathcal{M}}(\vec{k}, \vec{p}\,')}{E_p - E_k + i\epsilon} . \tag{271}$$

where in the left-hand side we have defined scattering amplitudes with a proper normalization factor (see Appendix 8, in particular Eq. (383))

$$\widetilde{\mathcal{M}}(\vec{p}, \vec{p}\,') = \frac{\mathcal{M}(\vec{p}, \vec{p}\,')}{4E_1(p)E_2(p)} , \tag{272}$$

while on the right hand side we have denoted by $\widetilde{\mathcal{M}}(\vec{k}, \vec{p}\,')$ their analogue definition off the energy shell with $|\vec{k}| \neq |\vec{p}\,'|$. In what follows our aim is to extract the classical potential to 2PM order for arbitrary $D \geq 4$. We will work in the center-of-mass frame using an isotropic gauge which identifies the phase space (r, p) of a two body Hamiltonian with the Fourier analogue of the exchanged momentum q in the center of mass and with the modulus of the momenta p. The advantage of the latter is the absence of $p \cdot r$ terms in the Hamiltonian and it has shown extremely useful in the computation of post-Minkwoskian Hamiltonians as shown in [12, 15].

We solve perturbatively Eq. (271) for the potential itself

$$\widetilde{V}^D(\vec{p},\vec{p}\,') = \widetilde{\mathcal{M}}(\vec{p},\vec{p}\,') + \sum_{n=1}^{\infty}(-1)^n \int \frac{d^d\vec{k}_1}{(2\pi\hbar)^d}\frac{d^d\vec{k}_2}{(2\pi\hbar)^d}\cdots\frac{d^d\vec{k}_n}{(2\pi\hbar)^d}\frac{\widetilde{\mathcal{M}}(\vec{p},\vec{k}_1)\cdots\widetilde{\mathcal{M}}(\vec{k}_n,\vec{p}\,')}{(E_p-E_{k_1}+i\epsilon)\cdots(E_{k_{n-1}}-E_{k_n}+i\epsilon)} \tag{273}$$

and truncate the series up to order G_N^2

$$\widetilde{V}^D_{\rm 1PM}(\vec{p},\vec{p}\,') + \widetilde{V}^D_{\rm 2PM}(\vec{p},\vec{p}\,') = \widetilde{\mathcal{M}}_{\rm tree}(\vec{p},\vec{p}\,') + \widetilde{\mathcal{M}}_{1-\rm loop}(\vec{p},\vec{p}\,') + \widetilde{\mathcal{M}}_{Born}(\vec{p},\vec{p}\,')\,, \tag{274}$$

where we have denoted the first Born subtraction by

$$\widetilde{\mathcal{M}}_{Born}(\vec{p},\vec{p}\,') \equiv -\int \frac{d^d\vec{k}}{(2\pi\hbar)^d}\frac{\widetilde{\mathcal{M}}_{\rm tree}(\vec{p},\vec{k})\widetilde{\mathcal{M}}_{\rm tree}(\vec{k},\vec{p}\,')}{E_p-E_k+i\epsilon}\,. \tag{275}$$

Although we do not explicitly distinguish between on-shell and off-shell scattering amplitudes in our notation, it should be stressed that the functions $\widetilde{\mathcal{M}}(p,k)$ entering the integrand on the right-hand side of (275) are evaluated for states that do not necessarily respect energy conservation and the sum over states indeed runs over all intermediate $(D-1)$-momenta $\vec{k}$. They are defined by T-matrix elements for asymptotic states with energies unconstrained, *i.e.*, $|\vec{p}\,| \neq |\vec{k}|$. This is analogous to the EFT approach where the potential $\widetilde{V}^D(\vec{p},\vec{k})$ likewise is defined off the energy shell, *i.e.*, with $|\vec{p}\,| \neq |\vec{k}|$. The off-shell extension of the T-matrix and V corresponds to the choice of operator basis in the EFT formalism. For instance, insisting on $(D-1)$-dimensional rotational symmetry, the analog of Wilson coefficients in the expansion of V will not depend on the scalar product $\vec{p}\cdot\vec{k}$ but only on $\vec{p}^2$ and $\vec{k}^2$. After Fourier transforming, this corresponds to the choice of isotropic coordinates. In the center-of-mass frame and using this isotropic parametrization

$$\widetilde{\mathcal{M}}_{\text{tree}}(\vec{k},\vec{k}\,') \equiv G_N \frac{A_1\left(\frac{k^2+k'^2}{2}\right)}{\frac{1}{\hbar^2}|\vec{k}-\vec{k}'|^2}\,, \qquad A_1\left(\frac{k^2+k'^2}{2}\right) = -\frac{4\pi\gamma(\frac{k^2+k'^2}{2})}{E_1(\frac{k^2+k'^2}{2})E_2(\frac{k^2+k'^2}{2})}\,, \tag{276}$$

where $|\vec{k}|$ is not necessarily equal to $|\vec{k}'|$. For a physical on-shell process in which $|\vec{p}| = |\vec{p}\,'|$ this of course reduces to

$$\widetilde{\mathcal{M}}_{\text{tree}}(\vec{p},\vec{p}\,') = G_N \frac{A_1(p^2)}{\frac{1}{\hbar^2}q^2}\,, \qquad A_1(p^2) = -\frac{4\pi\gamma(p^2)}{E_1(p)E_2(p)}\,. \tag{277}$$

At this point we need to evaluate the Born subtraction given by the integral in Eq. (275). We focus on the contributions to (275) arising from the soft region, which are obtained in this case expanding the integrand around $k^2 = p^2$. Indeed, to more directly compare with the discussion of the expansion by regions presented in Appendix 8, we could let $\vec{k} = \vec{p} + \vec{\ell}$ and then expand for $\vec{\ell} \sim \mathcal{O}(\hbar)$, which implies $k^2 = p^2 + \mathcal{O}(\hbar)$. One can also check that performing the expansion with respect to this shifted variable $\vec{\ell}$ eventually leads to the same final answer for the leading and subleading terms. We thus begin by Taylor-expanding the denominator and discard quantum terms. In doing so, we find

$$\begin{aligned}\widetilde{\mathcal{M}}_{Born}(\vec{p},\vec{p}\,') &= -2E_p\xi(p)\int \frac{d^d\vec{k}}{(2\pi\hbar)^d}\frac{\widetilde{\mathcal{M}}_{\text{tree}}(\vec{p},\vec{k})\widetilde{\mathcal{M}}_{\text{tree}}(\vec{k},\vec{p}\,')}{\vec{p}^{\,2}-\vec{k}^{\,2}+i\epsilon} \\ &\quad + \left(\frac{1-3\xi(p)}{2E_p\xi(p)}\right)\int \frac{d^d\vec{k}}{(2\pi\hbar)^d}\,\widetilde{\mathcal{M}}_{\text{tree}}(\vec{p},\vec{k})\widetilde{\mathcal{M}}_{\text{tree}}(\vec{k},\vec{p}\,') + \cdots\,,\end{aligned} \tag{278}$$

where ellipsis denotes quantum contributions which we discard. Using Eq. (276), we find

$$\widetilde{\mathcal{M}}_{Born}(\vec{p},\vec{p}\,') = -2E_p\xi(p)G_N^2 \int \frac{d^dk}{(2\pi\hbar)^d} \frac{\hbar^4 A_1^2\left(\frac{\vec{p}^{\,2}+\vec{k}^{\,2}}{2}\right)}{(\vec{p}^{\,2}-\vec{k}^2+i\epsilon)|\vec{k}-\vec{p}\,|^2|\vec{k}-\vec{p}\,'|^2} + G_N^2\left(\frac{1-3\xi(p)}{2E_p\xi(p)}\right)\int \frac{d^dk}{(2\pi\hbar)^d}\frac{\hbar^4 A_1^2\left(\frac{\vec{p}^{\,2}+\vec{k}^2}{2}\right)}{|\vec{k}-\vec{p}\,|^2|\vec{k}-\vec{p}\,'|^2} + \cdots . \tag{279}$$

We now Taylor-expand also the numerator around $k^2 = p^2$. Using Eq. (276) and reinstating κ_D, we find

$$\widetilde{\mathcal{M}}_{Born}(\vec{p},\vec{p}\,') = -\frac{\gamma^2(p^2)\kappa_D^4}{2E_p^3\xi(p)}\int \frac{d^dk}{(2\pi\hbar)^d}\frac{\hbar^4}{(\vec{p}^{\,2}-\vec{k}^{\,2}+i\epsilon)|\vec{k}-\vec{p}\,|^2|\vec{k}-\vec{p}\,'|^2} + \frac{\kappa_D^4}{4E_p^3\xi^2(p)}\left(\frac{\gamma^2(p^2)(\xi(p)-1)}{2E_p^2\xi(p)} - 4\gamma(p)\,p_1\cdot p_2\right)\int\frac{d^dk}{(2\pi\hbar)^d}\frac{\hbar^4}{|\vec{k}-\vec{p}\,|^2|\vec{k}-\vec{p}\,'|^2} + \cdots , \tag{280}$$

where we have used the following relation, $\frac{\partial}{\partial p^2}\gamma(p^2) = -\frac{2p_1\cdot p_2}{\xi(p)}$.

The first integral in Eq. (280) is given in Eq. (444), while the second can be evaluated with Feynman parameters. The super-classical and classical parts of the Born subtraction to this order can then be written as follows

$$\widetilde{\mathcal{M}}_{Born}(\vec{p},\vec{p}\,') = \frac{i\pi\gamma^2(p^2)\kappa_D^4}{2p\,\xi(p)E_p^3(4\pi)^{\frac{D}{2}}}\frac{\Gamma\left(\frac{6-D}{2}\right)\Gamma^2\left(\frac{D-4}{2}\right)}{\Gamma(D-4)}\frac{1}{\hbar}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}} + \frac{\kappa_D^4\gamma^2(p^2)}{4E_p^3p^2\xi(p)(4\pi)^{\frac{D-1}{2}}}\frac{\Gamma\left(\frac{5-D}{2}\right)\Gamma^2\left(\frac{D-3}{2}\right)}{\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} + \frac{\kappa_D^4}{4E_p^3\xi^2(p)(4\pi)^{\frac{D-1}{2}}}\left(\frac{\gamma^2(p^2)(\xi(p)-1)}{2E_p^2\xi(p)} - 4p_1\cdot p_2\gamma(p^2)\right)\frac{\Gamma^2\left(\frac{D-3}{2}\right)\Gamma\left(\frac{5-D}{2}\right)}{\Gamma(D-3)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} + \cdots \tag{281}$$

where again ellipsis denotes quantum contributions. Remarkably, not only do the box and crossed box diagrams give nonvanishing super-classical and classical contributions for $D \neq 4$, but similar contributions are also contained in the Born subtraction. It turns out, as expected, that the two super-classical contributions exactly cancel each other. The classical terms, however, remain and reproduce for $D = 4$ the result of Ref. [15].

The cancellation of the (super-classical) imaginary part can be interpreted as a consequence of unitarity. Indeed, applying the relation (370) to the two-to-two scattering in the center-of-mass frame, one has

$$\widetilde{\mathcal{M}}(\vec{p},\vec{p}\,') - \overline{\widetilde{\mathcal{M}}(\vec{p}\,',\vec{p}\,)} = -i2\pi \int \frac{d^d\vec{k}}{(2\pi\hbar)^d} \delta(E_p - E_k) \overline{\widetilde{\mathcal{M}}(\vec{k},\vec{p}\,)} \widetilde{\mathcal{M}}(\vec{k},\vec{p}\,')\,. \tag{282}$$

Recalling that the tree-level amplitude is real and that, because of time reversal invariance, the whole invariant amplitude is symmetric under the exchange of $\vec{p}$ and $\vec{p}\,'$, we then have, to 2PM order,

$$\mathrm{Im}\, \widetilde{\mathcal{M}}_{1-\mathrm{loop}}(\vec{p},\vec{p}\,') = -\pi \int \frac{d^d\vec{k}}{(2\pi\hbar)^d} \delta(E_p - E_k)\, \widetilde{\mathcal{M}}_{\mathrm{tree}}(\vec{p},\vec{k}\,) \widetilde{\mathcal{M}}_{\mathrm{tree}}(\vec{k},\vec{p}\,')\,. \tag{283}$$

Comparing the right-hand sides of (274) and (275), this identity guarantees that the imaginary part of $\widetilde{\mathcal{M}}_{1-\mathrm{loop}}$ must cancel against that of the Born subtraction $\widetilde{\mathcal{M}}_{Born}$.

In conclusion, we get the following potential in momentum space up to 2PM:

$$
\begin{aligned}
&\widetilde{V}^D_{1\mathrm{PM}}(\vec{p},\vec{p}\,') + \widetilde{V}^D_{2\mathrm{PM}}(\vec{p},\vec{p}\,') = -\frac{\gamma(p^2)\kappa_D^2\hbar^2}{2\xi(p)E_p^2 q^2} \\
&+\frac{\kappa_D^4(m_1+m_2)}{(4\pi)^{\frac{D-1}{2}}4\xi(p)E_p^2}\left(-4(p_1\cdot p_2)^2+\frac{4m_1^2m_2^2}{(D-2)^2}+\frac{(D-3)E_p^2p^2}{(D-2)^2}\right)\frac{\Gamma(\frac{5-D}{2})\Gamma^2(\frac{D-3}{2})}{\Gamma(D-3)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} \\
&+\frac{\kappa_D^4}{4E_p^3\xi^2(p)(4\pi)^{\frac{D-1}{2}}}\left(\frac{\gamma^2(p^2)(\xi(p)-1)}{2E_p^2\xi(p)}-4p_1\cdot p_2\gamma(p^2)\right)\frac{\Gamma(\frac{5-D}{2})\Gamma^2(\frac{D-3}{2})}{\Gamma(D-3)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} \\
&-\frac{\kappa_D^4\gamma^2(p^2)(m_1+m_2-E_p)}{(4\pi)^{\frac{D-1}{2}}\xi(p)E_p^4p^2}\frac{\Gamma(\frac{5-D}{2})\Gamma^2(\frac{D-3}{2})}{\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} .
\end{aligned}
\tag{284}
$$

Fourier-transforming it to configuration space,

$$
V^D(r,p) = \int \frac{d^d\vec{q}}{(2\pi\hbar)^d}\widetilde{V}^D(\vec{p},\vec{p}\,')e^{\frac{i}{\hbar}\vec{q}\cdot\vec{x}} , \tag{285}
$$

and making use of the identity

$$
\int \frac{d^d\vec{q}}{(2\pi\hbar)^d}\left(\frac{q^2}{\hbar^2}\right)^{\nu} e^{\frac{i}{\hbar}\vec{q}\cdot\vec{x}} = \frac{2^{2\nu}}{\pi^{\frac{d}{2}}}\frac{\Gamma(\nu+\frac{d}{2})}{\Gamma(-\nu)}\frac{1}{r^{2\nu+d}} , \tag{286}
$$

we get the potential in configuration space up to order 2PM

$$
V^D(r,p) = V^D_{1\mathrm{PM}}(r,p) + V^D_{2\mathrm{PM}}(r,p) + \cdots , \tag{287}
$$

$$
V^D_{1\mathrm{PM}}(r,p) = -\frac{\gamma(p^2)G_N}{E_p^2\xi(p)}\frac{\Gamma(\frac{D-3}{2})}{\pi^{\frac{D-3}{2}}}\frac{1}{r^{D-3}} , \tag{288}
$$

$$
\begin{aligned}
&V^D_{2\text{PM}}(r,p) \\
&= \frac{G_N^2(m_1+m_2)}{\pi^{D-3}E_p^2\xi(p)}\left(\frac{4m_1^2m_2^2}{(D-2)^2} + \frac{(D-3)[(p_1\cdot p_2)^2 - m_1^2m_2^2]}{(D-2)^2} - 4(p_1\cdot p_2)^2\right)\frac{\Gamma^2(\frac{D-3}{2})}{r^{2D-6}} \\
&+ \frac{G_N^2}{E_p^3\xi^2(p)}\left(\frac{\gamma^2(p^2)(\xi(p)-1)}{2E_p^2\xi(p)} - 4\gamma(p^2)p_1\cdot p_2\right)\frac{\Gamma^2(\frac{D-3}{2})}{\pi^{D-3}}\frac{1}{r^{2D-6}} \\
&+ \frac{G_N^2\gamma^2(p^2)(E_p-m_1-m_2)}{E_p^4\,p^2\,\xi(p)\pi^{D-3}}\frac{\Gamma^2(\frac{D-3}{2})}{\Gamma(D-4)}\frac{\Gamma(D-3)}{r^{2D-6}}\,.
\end{aligned}
\tag{289}
$$

Let us stress once more that, for $D > 4$, the 2PM potential thus receives a nontrivial contribution from box and crossed-box diagrams that is not exactly compensated by the Born subtraction. The combination of the two is proportional to the difference between the total energy and the sum of the masses as shown in the last line of Eq. (289). As we shall see in the next section, the appearance of this term for $D > 4$ will give rise to a modification in the linear relation between the classical part of the amplitude and the expression for $p^2(r, G_N)$ in the classical trajectory that exists in $D = 4$ dimensions [17,73].

The Effective Field Theory Matching in D dimensions

In the previous section we have shown how the classical effective potential can be obtained from a scattering amplitude by means of the Born subtraction, which involves inverting (271) perturbatively. We have seen in particular how the potential acquires new nontrivial terms at 2PM order in higher dimensions. Let us now briefly explain how this calculation can be performed following the method of EFT amplitude-matching introduced in [12].

We consider two theories: a fundamental one, which we also call the underlying theory, of two massive scalar fields minimally coupled to gravity, and an effective theory of two massive scalars interacting through a four-point interaction potential, which we denote by $\widetilde{V}^D(\vec{p},\vec{p}\,')$ in momentum space.

In this approach, one starts by making an ansatz for the effective potential: to 2PM order and in momentum space one has

$$\widetilde{V}^D(\vec{p},\vec{p}\,') = G_N c_1\left(\tfrac{p^2+p'^2}{2}\right)\left(\frac{q^2}{\hbar^2}\right)^{-1} + G_N^2 c_2\left(\tfrac{p^2+p'^2}{2}\right)\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} + \cdots\,, \tag{290}$$

where c_1 and c_2 are unknown coefficients. Since the fundamental and the effective theory should give rise to the same dynamics for the massive scalar particles, a valid matching condition between the two is the equality of scattering amplitudes order by order in the coupling, or equivalently in the PM counting

$$\widetilde{\mathcal{M}}_{(n-1)-\text{loop}}(\vec{p},\vec{p}\,') = \mathcal{M}^{\text{EFT}}_{n\text{PM}}(\vec{p},\vec{p}\,')\,, \tag{291}$$

where the left hand side of Eq. (291) is computed in the full theory with the normalization of Eq.(272), while the right hand side is computed in the effective theory by a perturbative expansion in iterated bubbles as done in [12]. At 1PM order, comparing the coefficient of G_N in (291) with the tree amplitude (267), as dictated by the matching condition

$$\widetilde{\mathcal{M}}_{\text{tree}}(\vec{p}, \vec{p}\,') = \mathcal{M}^{\text{EFT}}_{1\text{PM}}(\vec{p}, \vec{p}\,')\,, \tag{292}$$

gives

$$c_1(p^2) = A_1(p^2) \tag{293}$$

with $A_1(p^2)$ as in (277).

At 2PM order, the EFT amplitude is made by two contributions, a contact term proportional to the potential and a bubble: truncating at G_N^2 order one finds

$$\begin{aligned}\mathcal{M}^{\text{EFT}}_{2\text{PM}}(\vec{p}, \vec{p}\,') = {} & G_N^2 c_2\left(p^2\right)\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} \\ & + G_N^2 \int \frac{d^d\vec{k}}{(2\pi\hbar)^d} \frac{\hbar^4 c_1^2\left(\frac{p^2+k^2}{2}\right)}{(E_p - E_k + i\epsilon)|\vec{p}-\vec{k}|^2|\vec{p}\,'-\vec{k}|^2} + \cdots\,,\end{aligned} \tag{294}$$

At this point one needs to evaluate the integral appearing in the second line of (294) and then compare this the EFT amplitude with the box and triangle contributions (269), (270) so as to derive $c_2(p^2)$. However, thanks to the condition (293), the second line of (294) equals $-\widetilde{\mathcal{M}}_{Born}(\vec{p}, \vec{p}\,')$, namely the Born subtraction (275) except for the overall sign. Therefore the matching condition

$$\widetilde{\mathcal{M}}_{1-\text{loop}}(\vec{p}, \vec{p}\,') = \mathcal{M}^{\text{EFT}}_{2\text{PM}}(\vec{p}, \vec{p}\,') \tag{295}$$

is equivalent to

$$\widetilde{\mathcal{M}}_{\triangleleft}(\vec{p},\vec{p}\,') + \widetilde{\mathcal{M}}_{\triangleright}(\vec{p},\vec{p}\,') + \widetilde{\mathcal{M}}_{\Box,s}(\vec{p},\vec{p}\,') + \widetilde{\mathcal{M}}_{\Box,u}(\vec{p},\vec{p}\,') = \widetilde{V}^{D}_{\text{2PM}}(\vec{p},\vec{p}\,') - \widetilde{\mathcal{M}}_{Born}(\vec{p},\vec{p}\,') \,. \tag{296}$$

We thus see that the EFT matching condition is in fact identical to Eq. (274), which was at the basis of the calculation of the previous subsection, and thus leads to the same answer for the 2PM potential (284).

Let us once again briefly stress the new features arising in this analysis in higher dimensions. We find that the box topologies not only provide a super-classical term that is compensated by a corresponding contribution in the effective theory, but also possess a subleading term which is non vanishing and classical in $D > 4$. This term is not removed by a similar contribution from $\mathcal{M}_{Born}(\vec{p},\vec{p}\,')$ and this leaves a term in the 2PM potential which is proportional to the difference in the total energy and masses. This term vanishes at $D = 4$, as can be seen from the last line of Eq. (289).

More on the EFT matching and the Lippmann-Schwinger equation

At 2PM and in arbitrary dimensions the classical post-Minkwoskian potential describing a binary system in isotropic coordinates is equivalent if computed using the Lippmann-Schwinger equation or the EFT matching. Restricting to the conservative sector, we can easily show the equivalence to hold to all orders in G_N

and in arbitrary dimensions. To this extent, let's go back to Eq. (271) and let's find a formal solution for a given scattering amplitude $\mathcal{M}(\vec{p},\vec{p}\,')$. Similar to Eq. (273), the potential will be given by a formal series

$$\begin{aligned}\widetilde{\mathcal{M}}(\vec{p},\vec{p}\,') &= \widetilde{V}^D(\vec{p},\vec{p}\,') \\ &+ \sum_{n=1}^{\infty}\int \frac{d^d\vec{k}_1}{(2\pi\hbar)^d}\frac{d^d\vec{k}_2}{(2\pi\hbar)^d}\cdots\frac{d^d\vec{k}_n}{(2\pi\hbar)^d}\frac{\widetilde{V}^D(\vec{p},\vec{k}_1)\cdots\widetilde{V}^D(\vec{k}_n,\vec{p}\,')}{(E_p-E_{k_1}+i\epsilon)\cdots(E_{k_{n-1}}-E_{k_n}+i\epsilon)}\,.\end{aligned} \tag{297}$$

At this point, we can recast each propagator in Eq. (297) as being an "effective two body propagator" so as to rewrite each of them as a couple of matter propagators

$$\frac{1}{E_{k_i}-E_{k_j}} = i\int\frac{dk_0}{2\pi}\frac{1}{k_0-\sqrt{k_j^2+m_1^2}}\frac{1}{E_{k_i}-k_0-\sqrt{k_j^2+m_2^2}}\,. \tag{298}$$

If we now plug back Eq. (298) into Eq. (297) we can easily recognize on the right hand side of the latter the same scattering amplitude computed in [12], where the n^{th} term of the series corresponds to the n^{th} loop in an effective field theory of only scalar fields. Using this observation, we get

$$\widetilde{\mathcal{M}}(\vec{p},\vec{p}\,') = \widetilde{\mathcal{M}}^{EFT}(\vec{p},\vec{p}\,') \tag{299}$$

thus showing the equivalence between EFT matching and the Lippmann-Schwinger equation. It would be interesting to understand if the equivalence persists once introducing radiative effects in the potential, which are expected to first appear at 4PM [10].

FROM THE CLASSICAL AMPLITUDE TO KINEMATICS

In the previous section we have used the classical limit of the scattering amplitude to derive the classical potential at 2PM order. Including the kinetic terms this brings us to the following Hamiltonian describing the interaction between the two objects with mass m_1 and m_2:

$$H(r,p) = \sum_{i=1,2} \sqrt{p^2 + m_i^2} + V^D_{1\mathrm{PM}}(r,p) + V^D_{2\mathrm{PM}}(r,p) = E\,. \tag{300}$$

Since E is a constant of motion the previous equation implicitly determines the quantity $p^2 = p^2(r, G_N)$ as a function of r and G_N. Knowledge of this function is crucial in order to compute the scattering angle θ in the center-of-mass frame. Going to polar coordinates we can write p^2 as follows:

$$p^2(r, G_N) = p_r^2 + \frac{L^2}{r^2}\,, \tag{301}$$

where $p_\varphi \equiv L$ is the conserved angular momentum of the system. Then, the deflection angle is given by the relation:

$$\theta = -2\int_{r_{\min}}^{+\infty} \frac{\partial p_r}{\partial L}\, dr - \pi = 2L \int_{r_{\min}}^{\infty} \frac{dr}{r^2 p_r} - \pi\,, \tag{302}$$

$r_{\min}$ being the positive root of p_r closest to zero. As noticed in Refs. [10, 17, 73] for $D = 4$ one has the remarkable relation

$$p^2(r, G_N) = p_\infty^2 - 2E_{p_\infty}\xi(p_\infty)\widetilde{\mathcal{M}}(r, p_\infty)\,, \tag{303}$$

where $\widetilde{\mathcal{M}}(r, p_\infty)$ is the Fourier transform of the amplitude given by

$$\widetilde{\mathcal{M}}^{\text{cl.}}(r, p) \equiv \int \frac{d^d\vec{q}}{(2\pi\hbar)^d} \widetilde{\mathcal{M}}^{\text{cl.}}(\vec{p}, \vec{p}\,') e^{i\frac{\vec{q}}{\hbar}\cdot\vec{x}} . \tag{304}$$

Working as usual in the center-of-mass frame, we find it convenient here to emphasize the difference between the momentum evaluated along the classical trajectory, $p^2(r, G_N)$, and the asymptotic momentum by the denoting the latter by p_∞, although it had been simply called p in Sect. 8. For instance, the relation (258) between the asymptotic momentum and the energy now reads

$$p_\infty^2 = \frac{(m_1^2 + m_2^2 - E_{p_\infty}^2)^2 - 4m_1^2 m_2^2}{4E_{p_\infty}^2} . \tag{305}$$

We shall now generalize Eq. (303) to the D-dimensional case. Starting from Eq. (300), we expand the function $p^2(r, G_N)$, whose existence is ensured by the implicit function theorem, order by order in the coupling G_N. This allows us to write

$$p^2(r, G_N) = p_\infty^2 + G_N \, (p^2)'_{G_N=0}(r) + \frac{G_N^2}{2} \, (p^2)''_{G_N=0}(r) + \cdots , \tag{306}$$

where for brevity

$$(p^2)'_{G_N=0}(r) = \frac{\partial}{\partial G_N} p^2(r, G_N)\big|_{G_N=0} , \qquad \frac{1}{2}(p^2)''_{G_N=0}(r) = \frac{1}{2}\frac{\partial^2}{\partial G_N^2} p^2(r, G_N)\big|_{G_N=0} \tag{307}$$

denote the first two coefficients of said expansion in powers of G_N. Note that (306) is a D-independent expression. We then extend the analysis of Ref. [17], substituting (306) in (300) and solving order by order in G_N, to get

$$G_N(p^2)'_{G_N=0}(r) = -2E_{p_\infty}\xi(p_\infty)V^D_{1\text{PM}}(r,p)\big|_{p^2=p^2_\infty} \tag{308}$$

and

$$\begin{aligned}\frac{G_N^2}{2}(p^2)''_{G_N=0}(r) = -2E_{p_\infty}\xi(p_\infty)\Big[V^D_{2\text{PM}}(r,p)\\ - 2E_p\xi(p)V^D_{1\text{PM}}(r,p)\partial_{p^2}V^D_{1\text{PM}}(r,p) + \left(\frac{3\xi(p)-1}{2E_p\xi(p)}\right)(V^D_{1\text{PM}})^2(r,p)\Big]_{p^2=p^2_\infty}.\end{aligned} \tag{309}$$

Using the fact that $\gamma(p^2)$ in Eq. (260) can be written as follows,

$$\gamma(p^2) = 2E_p^2p^2 + 2m_1^2m_2^2\frac{D-3}{D-2}, \tag{310}$$

we can easily get

$$\partial_{p^2}\left(\frac{\gamma(p^2)}{E_1(p)E_2(p)}\right) = -\frac{\gamma(p^2)(1-2\xi(p))}{2\xi^3(p)E_p^4} + \frac{2}{\xi(p)}\left(1+\frac{p^2}{\xi(p)E_p^2}\right). \tag{311}$$

Inserting then in Eqs. (308) and (309) the potential in Eq. (289), we find:

$$G_N(p^2)'_{G_N=0} = -2E_{p_\infty}\xi(p_\infty)\left[-\frac{\gamma(p_\infty^2)G_N}{E^2_{p_\infty}\xi(p_\infty)}\frac{\Gamma(\frac{D-3}{2})}{\pi^{\frac{D-3}{2}}}\frac{1}{r^{D-3}}\right] = -2E_{p_\infty}\xi(p_\infty)\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}}(r,p_\infty) \tag{312}$$

together with

$$\begin{aligned}\frac{G_N^2}{2}(p^2)''_{G_N=0} &= -2E_{p_\infty}\xi(p_\infty)\Big[-\frac{G_N^2}{\pi^{D-3}}\frac{\Gamma^2(\frac{D-3}{2})}{r^{2D-6}}\frac{(m_1+m_2)}{E_p^2\xi(p)}\Big(4(p_1\cdot p_2)^2-\frac{4m_1^2m_2^2}{(D-2)^2}\\
&-\frac{(D-3)E_p^2p^2}{(D-2)^2}\Big)+\frac{G_N^2\gamma^2(p^2)(E_p-m_1-m_2)}{E_p^4\,p^2\xi(p)\,\pi^{D-3}}\frac{\Gamma^2(\frac{D-3}{2})}{\Gamma(D-4)}\frac{\Gamma(D-3)}{r^{2D-6}}\Big]_{p=p_\infty}\\
&= -2E_{p_\infty}\xi(p_\infty)\Big(\widetilde{\mathcal{M}}^{\text{cl.}}{}_{\triangleleft,\triangleright}(r,p_\infty)+(\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}})^2(r,p_\infty)\frac{\xi(p_\infty)(E_{p_\infty}-m_1-m_2)}{p_\infty^2}\frac{\Gamma(D-3)}{\Gamma(D-4)}\Big)\\
&= -2E_{p_\infty}\xi(p_\infty)\Big(\widetilde{\mathcal{M}}^{\text{cl.}}_{1-\text{loop}}(r,p_\infty)+(\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}})^2(r,p_\infty)\frac{\xi(p_\infty)E_{p_\infty}}{p_\infty^2}\frac{\Gamma(D-3)}{\Gamma(D-4)}\Big),\end{aligned} \tag{313}$$

where the Fourier transform of the classical part of the scattering amplitude is defined by Eq. (304). Inserting Eqs. (312) and (313) in Eq. (306), we get

$$\begin{aligned}p^2(r,G_N) = p_\infty^2 - 2E_{p_\infty}\xi(p_\infty)\Big(&\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}}(r,p_\infty)+\widetilde{\mathcal{M}}^{\text{cl.}}_{1-\text{loop}}(r,p_\infty)\\
&+(\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}})^2(r,p_\infty)\frac{\xi(p_\infty)E_{p_\infty}}{p_\infty^2}\frac{\Gamma(D-3)}{\Gamma(D-4)}\Big)+\cdots,\end{aligned} \tag{314}$$

which of course reduces to Eq. (303) for $D=4$.

It was argued in Ref. [17] that the simpler relation in four dimensions nicely aligned with our expectations that the effective potential describing the scattering of particles from flat space at minus infinity to flat space at plus infinity should depend only on the classical part of the scattering amplitude. We note that this expectation, although slightly modified due to the new term proportional to the square of the tree-level amplitude at 2PM order, is still borne out by this new result for $D>4$.

An alternative derivation

An alternative derivation of the modified relation (314) for $D > 4$ that directly points towards a generalization to any order in the post-Minkowskian expansion proceeds via Damour's effective Hamiltonian defined by the solution to the energy equation (300) [23, 24].

To apply this strategy, let us start with the following ansatz $p^2(r, G_N)$ for the solution of Eq. (300)

$$p^2(r, G_N) = p_\infty^2 + \sum_{n=1}^{2} \frac{G_N^n f_n^D(p_\infty^2)}{r^{n(D-3)}}, \tag{315}$$

where the constants f_n^D are found by solving Eq. (300) iteratively. As discussed in detail in Refs. [17, 24, 73], one can consider the energy-momentum relation (315) as an effective nonrelativistic "Hamiltonian" for the scattering problem, in which the term p_∞^2 is regarded as the kinetic term, *i.e.* the unperturbed Hamiltonian, while

$$V_{\text{eff}} \equiv -\sum_{n=1}^{2} \frac{G_N^n f_n^D(p_\infty^2)}{r^{n(D-3)}} \tag{316}$$

plays the role of an effective small perturbation. Notice however that the "potential" V_{eff} has the dimension of an energy squared by (315). It is crucial that here the coefficients of the potential are constants, only depending on the total conserved energy E.

The associated Lippmann–Schwinger equation then reads

$$\widetilde{\mathcal{M}}_{\text{eff}}(\vec{p},\vec{p}\,') = \widetilde{V}_{\text{eff}}(\vec{p},\vec{p}\,') + \int \frac{d^d\vec{k}}{(2\pi\hbar)^d} \frac{\widetilde{\mathcal{M}}_{\text{eff}}(\vec{p},\vec{k}\,)\widetilde{V}_{\text{eff}}(\vec{k},\vec{p}\,')}{\vec{p}\,^2 - \vec{k}^2 + i\epsilon}\,, \tag{317}$$

where we have rescaled the amplitude by a normalization factor according to

$$\widetilde{\mathcal{M}}_{\text{eff}}(r,p_\infty) = 2E_{p_\infty}\xi(p_\infty)\widetilde{\mathcal{M}}(r,p_\infty) \tag{318}$$

as in (385) and $\widetilde{V}_{\text{eff}}$ denotes the effective potential in momentum space. In four dimensions the perturbative iteration of Eq. (317) produces only super-classical terms. For example, at 2PM order, the perturbative expansion of Eq. (317)

$$\widetilde{\mathcal{M}}_{\text{eff}}(\vec{p},\vec{p}\,') = \widetilde{V}_{\text{eff}}(\vec{p},\vec{p}\,') + \int \frac{d^3\vec{k}}{(2\pi\hbar)^3} \frac{\widetilde{V}_{\text{eff}}(\vec{p},\vec{k}\,)\widetilde{V}_{\text{eff}}(\vec{k},\vec{p}\,')}{\vec{p}\,^2 - \vec{k}^2 + i\epsilon} + \cdots \tag{319}$$

implies

$$\widetilde{\mathcal{M}}_{\text{eff}}(\vec{p},\vec{p}\,') = \widetilde{V}_{\text{eff}}(\vec{p},\vec{p}\,') + \int \frac{d^3\vec{k}}{(2\pi\hbar)^3} \frac{16\pi^2(f_1)^2G_N^2\hbar^4}{(\vec{p}\,^2 - \vec{k}^2 + i\epsilon)(\vec{k}-\vec{p}\,)^2(\vec{k}-\vec{p}\,')^2} + \cdots\,, \tag{320}$$

where f_1 stands for f_1^D for $D=4$ and we have used that the Fourier transform of $\frac{1}{r}$ is equal to $\frac{4\pi\hbar^2}{q^2}$ (see Eq. (286)). From Eq. (444) one can see that the integral in the previous equation has only super-classical and quantum contributions in $D=4$, or in other words that its classical piece vanishes in four dimensions.

However, this argument does not apply for arbitrary dimensions $D > 4$. Working again to 2PM order, the integral involved is now

$$\widetilde{\mathcal{M}}_{\text{eff}}(\vec{p},\vec{p}\,') = \widetilde{V}_{\text{eff}}(\vec{p},\vec{p}\,') + \frac{1}{\Gamma\left(\frac{D-3}{2}\right)^2}\int \frac{d^d\vec{k}}{(2\pi\hbar)^d}\frac{16\pi^{D-1}G_N^2(f_1^D)^2\hbar^4}{(\vec{p}^{\,2}-\vec{k}^2+i\epsilon)(\vec{k}-\vec{p}\,)^2(\vec{k}-\vec{p}\,')^2} + \cdots, \tag{321}$$

where we employed (286). Using Eq. (444) and restricting ourselves to just the classical part of this equation, we get in position space,

$$\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{eff}}(r,p) = V_{\text{eff}}(r,p) - \frac{1}{2p^2}\frac{\Gamma(D-3)}{\Gamma(D-4)}\frac{G_N^2(f_1^D)^2}{r^{2(D-3)}} \tag{322}$$

from which

$$V_{\text{eff}}(r,p) = \widetilde{\mathcal{M}}^{\text{cl.}}_{\text{eff}}(r,p) + \frac{1}{2p^2}\frac{\Gamma(D-3)}{\Gamma(D-4)}(\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{eff,tree}})^2(r,p)\,. \tag{323}$$

Inserting the proportionality relation $\widetilde{\mathcal{M}}_{\text{eff}}(r,p_\infty) = 2E_{p_\infty}\xi(p_\infty)\widetilde{\mathcal{M}}(r,p_\infty)$, we obtain that the effective potential at 2PM order for $p = p_\infty$ is

$$V_{\text{eff}}(r,p_\infty) \equiv 2E_{p_\infty}\xi(p_\infty) \left(\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}}(r,p_\infty) + \widetilde{\mathcal{M}}^{\text{cl.}}_{1-\text{loop}}(r,p_\infty) + (\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}})^2(r,p_\infty)\frac{\xi(p_\infty)E_{p_\infty}}{p_\infty^2}\frac{\Gamma(D-3)}{\Gamma(D-4)}\right) \tag{324}$$

as well as the relation

$$\begin{aligned} p^2(r,G_N) = p_\infty^2 - 2E_{p_\infty}\xi(p_\infty)\Big(&\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}}(r,p_\infty) \\ &+ \widetilde{\mathcal{M}}^{\text{cl.}}_{1-\text{loop}}(r,p_\infty) + (\widetilde{\mathcal{M}}^{\text{cl.}}_{\text{tree}})^2(r,p_\infty)\frac{\xi(p_\infty)E_{p_\infty}}{p_\infty^2}\frac{\Gamma(D-3)}{\Gamma(D-4)}\Big)\,, \end{aligned} \tag{325}$$

confirming the previous derivation of Eq. (314). The advantage of this alternative derivation is that it is more suitable to generalization to higher orders in the PM expansion. Further corrections of arbitrarily high order in G_N will in general appear in the relation when $D > 4$.

THE SCATTERING ANGLE IN ARBITRARY DIMENSIONS

In this section we compute the deflection angle and in particular we see how the new terms that appear in the quantity $p^2(r, G_N)$ reproduce the deflection angle already obtained from the eikonal in dimensions greater than four [47].

For the calculation of the scattering angle using $p^2(r, G_N)$, one could in principle employ Eqs. (301) and (302), which however involves computing the root $r_{\min}$ of a polynomial in G_N of increasing complexity. A more convenient strategy, as seen in [17], is to express the scattering angle only in terms of $p^2(r, G_N)$ and the impact parameter b as [4]

$$\theta^D = \sum_{k=1}^{\infty} \widetilde{\theta}_k(b)\,, \qquad \widetilde{\theta}_k(b) = \frac{2b}{k!} \int_0^{\infty} du \left(\frac{d}{db^2}\right)^k \frac{(V_{\text{eff}}(r, p_\infty))^k r^{2(k-1)}}{p_\infty^{2k}}\,, \tag{326}$$

where $r^2 = u^2 + b^2$, while the effective potential is given by

$$V_{\text{eff}}(r, p_\infty) = -\sum_{n=1}^{\infty} \frac{G_N^n f_n^D(p_\infty^2)}{r^{n(D-3)}}\,, \tag{327}$$

which avoids the need to evaluate $r_{\min}$. Since $p^2(r, G_N) = p_\infty^2 - V_{\text{eff}}$, one can always

4 For an alternative way to relate $p^2(r, G_N)$ to the scattering angle, see Ref. [73].

read the f_n^D coefficients from Eq. (324).[5]

At 2PM order the D-dimensional scattering angle is thus provided by

$$\theta_{2\mathrm{PM}}^D = \widetilde{\theta}_1(b) + \widetilde{\theta}_2(b)\,, \tag{328}$$

where

$$\widetilde{\theta}_1(b) = \frac{2b}{p_\infty^2}\int_0^{+\infty} du \frac{dV_{\mathrm{eff}}}{db^2}(r, p_\infty)\,, \tag{329}$$

$$\widetilde{\theta}_2(b) = \frac{b}{p_\infty^4}\int_0^{+\infty} du \left(\frac{d}{db^2}\right)^2 \left[r^2 V_{\mathrm{eff}}^2(r, p_\infty)\right]. \tag{330}$$

From Eq. (324) we can read off the f_n^D coefficients in terms of the amplitudes, namely

$$f_1^D(p_\infty) = \frac{2\gamma(p_\infty^2)}{E_{p_\infty}\pi^{\frac{D-3}{2}}}\Gamma\left(\frac{D-3}{2}\right) \tag{331}$$

and

$$\begin{aligned} f_2^D(p_\infty) = {} & \frac{2(m_1+m_2)\Gamma^2\left(\frac{D-3}{2}\right)}{E_{p_\infty}\pi^{D-2}}\left(4(p_1\cdot p_2)^2 - \frac{4m_1^2m_2^2 + (D-3)p^2E_p^2}{(D-2)^2}\right)_{p=p_\infty} \\ & + \frac{2\gamma^2(p_\infty)(m_1+m_2-E_{p_\infty})}{E^3 p_\infty^2 \pi^{D-3}}\Gamma^2\left(\frac{D-3}{2}\right)\frac{\Gamma(D-3)}{\Gamma(D-4)}\,. \end{aligned} \tag{332}$$

5 In certain dimensions particular combinations of f_n^D terms in the expansion of the scattering angle may vanish [17]. This phenomenon occurs already at 2PM order in four dimensions, where the expansion of the scattering angle exceptionally does not involve f_1^2. This is not so in dimensions $D > 4$.

The integrals in Eqs. (329)–(330) are elementary. The first contribution to the scattering angle gives

$$\widetilde{\theta}_1(b) = \frac{G_N f_1^D(p_\infty)}{p_\infty^2}\frac{\sqrt{\pi}}{b^{D-3}}\frac{\Gamma(\frac{D-2}{2})}{\Gamma(\frac{D-3}{2})} + \frac{G_N^2 f_2^D(p_\infty)}{p_\infty^2}\frac{\sqrt{\pi}}{b^{2D-6}}\frac{\Gamma(D-\frac{5}{2})}{\Gamma(D-3)}\,. \tag{333}$$

Inserting Eqs. (331)–(332), this becomes

$$\begin{aligned}\widetilde{\theta}_1(b) =& \frac{2\gamma(p_\infty)G_N}{p_\infty^2\,E_{p_\infty}\,b^{D-3}}\frac{\Gamma(\frac{D}{2})}{\pi^{\frac{D-4}{2}}}\\ &+\frac{2G_N^2\Gamma(D-\frac{5}{2})\Gamma^2(\frac{D-3}{2})}{p_\infty^2\,E_{p_\infty}\,b^{2D-6}\pi^{D-\frac{7}{2}}}\frac{(m_1+m_2)}{\Gamma(D-3)}\left(4(p_1\cdot p_2)^2-\frac{4m_1^2m_2^2+(D-3)p^2E_p^2}{(D-2)^2}\right)_{p=p_\infty}\\ &+\frac{2\gamma^2(p_\infty)(m_1+m_2-E_{p_\infty})}{E_{p_\infty}^3 p_\infty^4\pi^{D-\frac{7}{2}}}\Gamma^2\left(\frac{D-3}{2}\right)\frac{\Gamma(D-\frac{5}{2})}{\Gamma(D-4)}\frac{G_N^2}{b^{2D-6}}\,.\end{aligned} \tag{334}$$

The remaining contribution gives

$$\begin{aligned}\widetilde{\theta}_2(b) &= \frac{bG_N^2(f_1^D)^2}{p_\infty^4}\int_0^{+\infty}du\left(\frac{d}{db^2}\right)^2 r^2\left(\frac{1}{r^{2d-4}}\right)\\ &= \frac{2\gamma^2(p_\infty)}{E_{p_\infty}^2 p_\infty^4}\frac{\Gamma(D-\frac{5}{2})}{\Gamma(D-4)}\frac{\Gamma^2(\frac{D-3}{2})}{\pi^{D-\frac{7}{2}}}\frac{G_N^2}{b^{2D-6}}\,.\end{aligned} \tag{335}$$

Note that this additional term vanishes in four space-time dimensions $D = 4$. Adding these pieces together, we find the D-dimensional scattering angle at 2PM order to be

$$\begin{aligned}\theta_{\rm 2PM}^D =& \frac{2\gamma(p_\infty)G_N}{p_\infty^2\,E_{p_\infty}\,b^{D-3}}\frac{\Gamma(\frac{D}{2})}{\pi^{\frac{D-4}{2}}}\\ &+\frac{2G_N^2\Gamma(D-\frac{5}{2})\Gamma^2(\frac{D-3}{2})}{p_\infty^2\,E_{p_\infty}\,b^{2D-6}\pi^{D-\frac{7}{2}}}\frac{(m_1+m_2)}{\Gamma(D-3)}\left(4(p_1\cdot p_2)^2-\frac{4m_1^2m_2^2+(D-3)E_p^2p^2}{(D-2)^2+}\right)_{p=p_\infty}\\ &+\frac{2\gamma^2(p_\infty)(m_1+m_2)}{E_{p_\infty}^3 p_\infty^4\pi^{D-\frac{7}{2}}}\Gamma^2\left(\frac{D-3}{2}\right)\frac{\Gamma(D-\frac{5}{2})}{\Gamma(D-4)}\frac{G_N^2}{b^{2D-6}}\end{aligned} \tag{336}$$

in complete agreement with the eikonal calculation [47].

It is also interesting to see how this agreement comes about. On the one hand, the new classical pieces from the box and crossed-box diagrams in $D > 4$ dimensions yield a contribution proportional to $(m_1 + m_2 - E_{p_\infty})$ in the last line of Eq. (334). On the other hand, for $D > 4$ there is a new term in the formula for the scattering angle that is proportional to E_{p_∞} (and f_1^2) in Eq. (335). Adding these two contributions one gets the last line of Eq. (336) where we see that the two terms proportional to E_{p_∞} have cancelled each other leaving only the term proportional to $m_1 + m_2$.

Finally, let us consider an alternative route to the computation of the scattering angle which also can be phrased in terms of amplitude evaluations and which has been described in Ref. [21]. As shown there, one can express the change in four-momentum of a particle in two-body scattering by means of

$$\langle \Delta p_1^\mu \rangle = \langle \psi | \, S^\dagger \mathbb{P}_1^\mu S \, | \psi \rangle - \langle \psi | \, \mathbb{P}_1^\mu \, | \psi \rangle \tag{337}$$

where S denotes the S-matrix and the two particle state is given by a suitable $|\psi\rangle$. Re-expressing the S-matrix in terms of the T-operator, one gets [21]

$$\langle \Delta p_1^\mu \rangle = I_{(1)}^\mu + I_{(2)}^\mu \tag{338}$$

$$I_{(1)}^\mu \equiv \langle \psi \left| i \left[\mathbb{P}_1^\mu, T \right] \right| \psi \rangle \quad , \quad I_{(2)}^\mu \equiv \langle \psi | \, T^\dagger \left[\mathbb{P}_1^\mu, T \right] | \psi \rangle \tag{339}$$

In the center-of-mass frame, it is now straightforward to relate the scattering angle θ to Eq.(338) by means of

$$\sin\theta = \frac{\langle \Delta p_1^\mu \rangle b_\mu}{p_\infty b} \tag{340}$$

where $b^\mu = (0, \vec{b})$ denotes the impact parameter as in [21]. In the case of classical General Relativity and to second Post-Minkwoskian order the scattering angle can be read off from of Eq. (340) and (338),

$$\theta_{2PM} = \frac{I_1^\mu b_\mu}{p_\infty b} + \frac{I_2^\mu b_\mu}{p_\infty b}, \tag{341}$$

since $\sin\theta \simeq \theta$ at this level of approximation.

To this order the scattering angle arises from two contributions, one linear and one quadratic in the involved scattering amplitudes. The term quadratic in the amplitude plays a role somewhat analogous of the Born subtraction needed to define the potential as in Eq. (275). Indeed, the quadratic term removes a classically singular term coming from I_1^μ [21], thus rendering a well-defined classical observable in the same way as the Born subtraction of Eq. (275) removes super-classical pieces, and thus allowing the $\hbar \to 0$ limit. It would be interesting to understand the precise relationship between these two methods, and in particular to see how the method of Ref. [21] leads to the same result as the two other amplitude methods, also for $D > 4$.

Eikonal exponentiation and unitarity

As we have already pointed out in the introduction, the computation of the scattering angle to a certain fixed order in the expansion parameter G_N requires the calculation of an infinite series of terms of the scattering amplitude, in the eikonal approach. This is needed in order to ensure the exponentiation of terms in impact-parameter space. In contrast, the fixed-order calculation that uses the Hamiltonian language needs only the amplitude computed up to the given order in G_N. It is therefore instructive to further explore the connection between unitarity, as encoded in Eq. (370) and the eikonal exponentiation.

To analyze this issue, let us consider again the identity (283) for two-to-two scattering in the center-of-mass frame, which we may recast as

$$\mathrm{Im}\,\mathcal{M}_{1-\mathrm{loop}}(\vec{p},\vec{p}\,') = -\frac{\pi}{2E_p}\int \frac{d^d\vec{k}}{(2\pi\hbar)^d}\delta(\vec{p}^{\,2}-\vec{k}^2)\mathcal{M}_{\mathrm{tree}}(\vec{p},\vec{k}\,)\mathcal{M}_{\mathrm{tree}}(\vec{k},\vec{p}\,')\,, \tag{342}$$

(note that we are dealing here with the invariant amplitude $\mathcal{M}$ instead of $\widetilde{\mathcal{M}}$) or

$$\mathrm{Im}\,\mathcal{M}_{1-\mathrm{loop}}(\vec{p},\vec{p}\,') = \frac{1}{2E_p}\,\mathrm{Im}\int \frac{d^d\vec{k}}{(2\pi\hbar)^d}\frac{\mathcal{M}_{\mathrm{tree}}(\vec{p},\vec{k}\,)\mathcal{M}_{\mathrm{tree}}(\vec{k},\vec{p}\,')}{\vec{p}^{\,2}-\vec{k}^2+i\epsilon}\,. \tag{343}$$

The integral appearing on the right-hand side is the same as that in the first line of Eq. (278), thus immediately giving us

$$\mathrm{Im}\,\mathcal{M}_{1-\mathrm{loop}}(\vec{p},\vec{p}\,') = \frac{G_N^2c_1^2(p^2)\pi^{1-\frac{D}{2}}}{2^{D+1}pE_p}\frac{\Gamma\left(\frac{6-D}{2}\right)\Gamma^2\left(\frac{D-4}{2}\right)}{\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}}\,. \tag{344}$$

Transforming to impact parameter space b by means of a Fourier transform in $D-2$ dimensions yields

$$\mathrm{Im}\,\mathcal{M}_{1-\mathrm{loop}}(b) = \frac{1}{2}\frac{G_N^2 c_1^2(p^2)}{64E_p p}\frac{\Gamma^2(\frac{D-4}{2})}{(b^2)^{D-4}}\pi^{2-D}\,, \tag{345}$$

while the same Fourier transform for the tree level amplitude (277) gives

$$\mathcal{M}_{\mathrm{tree}}(b) = \frac{G_N c_1(p^2)}{4}\frac{\Gamma(\frac{D-4}{2})}{b^{D-4}}\pi^{\frac{2-D}{2}} \tag{346}$$

and hence, dividing by the normalization factor $4E_p p$ as in [47] (see also Eq. (387)), we find

$$\mathrm{Im}\,\frac{\mathcal{M}_{1-\mathrm{loop}}(b)}{4E_p p} = \frac{1}{2}\left(\frac{\mathcal{M}_{\mathrm{tree}}(b)}{4E_p p}\right)^2 . \tag{347}$$

This is the first identity needed to ensure exponentiation of the tree-level amplitude in the eikonal limit and we see that it follows from unitarity alone.

We interpret this as further evidence that, even at higher orders, unitarity indeed lies behind the eikonal exponentiation. A remarkable phenomenon is that in this approach super-classical terms of increasingly high inverse of powers of $\hbar$ are needed to achieve the exponentiation in impact-parameter space that eventually, at the saddle point, leads to the *classical* scattering angle.

SIMPLE EXPRESSIONS FOR THE DEFLECTION ANGLE

In this section we show that, if the potential is just given by the contribution of the tree diagram, then we can obtain a closed expression for the deflection angle in D dimensions. Let us now assume that the effective potential in D dimensions is only given by the tree-level contribution:

$$V_{\text{eff}}(r) = -\frac{G_N f_1^D}{r^{D-3}}, \qquad f_1^D(p_\infty) = \frac{2\gamma(p_\infty^2)}{E_{p_\infty}\pi^{\frac{D-3}{2}}}\Gamma\left(\frac{D-3}{2}\right), \tag{348}$$

where f_1^D is given in Eq. (331). The deflection angle is computed from Eq. (326) which, for the potential in Eq. (348), implies

$$\begin{aligned}\theta^D_{\text{tree}} &= \sum_{k=1}^{\infty}\frac{2b}{k!}\left(-\frac{G_N f_1^D}{p_\infty^2}\right)^k \int_0^{+\infty} du\, \partial_{b^2}^{(k)}\left[(u^2+b^2)^{k\frac{(5-D)}{2}-1}\right] \\ &= \sum_{k=1}^{\infty}\frac{2b}{k!}\left(\frac{-G_N f_1^D}{p_\infty^2}\right)^k \prod_{l=0}^{k-1}\left(k\frac{(5-D)}{2}-1-l\right)\int_0^{+\infty} du\,\frac{1}{(u^2+b^2)^{1+\frac{k(D-3)}{2}}}.\end{aligned} \tag{349}$$

The integral over the variable u can be easily computed and one gets

$$\theta^D_{\text{tree}} = \sum_{k=1}^{\infty}\frac{2b}{k!}\left(\frac{-G_N f_1^D}{p_\infty^2}\right)^k \prod_{l=0}^{k-1}\left(k\frac{(5-D)}{2}-1-l\right)\frac{\sqrt{\pi}}{b^{k(D-3)+1}}\frac{1}{k(D-3)}\frac{\Gamma(\frac{k(D-3)+1}{2})}{\Gamma(\frac{k(D-3)}{2})}, \tag{350}$$

which we may finally recast in the form

$$\theta^D_{\text{tree}} = \sqrt{\pi}\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\frac{\Gamma(\frac{k(D-3)+1}{2})}{\Gamma(\frac{k(D-5)}{2}+1)} \tag{351}$$

with

$$\alpha_D = \frac{G_N f_1^D}{p_\infty^2 b^{D-3}} \,. \tag{352}$$

In some particular case, such as $D = 4, 5$, the sum of the series (351) evaluates to simple functions. For $D = 4$ one gets [6]

$$\theta^4 = 2 \arctan\left(\frac{\alpha_4}{2}\right) \Longrightarrow \tan\frac{\theta^4}{2} = \frac{\alpha_4}{2} \,, \tag{353}$$

while for $D = 5$ one finds

$$\theta^5 = \frac{\pi}{\sqrt{1-\alpha_5}} - \pi \,. \tag{354}$$

The two previous deflection angles have the same form as the deflection angles in Eq. (4.5) of Ref. [99] corresponding to the scattering of a massless scalar particle on a maximally supersymmetric $D6$-brane and on a $D5$-brane, respectively. For $D = 7$ we get

$$\theta^7 = \frac{2K\left(\frac{4\sqrt{\alpha_7}}{2\sqrt{\alpha_7}+1}\right)}{\sqrt{2\sqrt{\alpha_7}+1}} - \pi \,, \tag{355}$$

6 A closed expression for the scattering angle in $D = 4$ up to 2PM included has been given in [15, 73].

where K is the complete elliptic integral of first kind. Also this expression agrees with the one in Eq. (4.6) of Ref. [99] for the $D3$-brane. Finally, for $D = 6, 8, 9$ and $D = 10$ we can write the deflection angle in terms of hypergeometric functions:

$$\theta^6 = 2\alpha_6\, {}_3F_2\left(\frac{2}{3}, 1, \frac{4}{3}; \frac{3}{2}, \frac{3}{2}; \frac{27\alpha_6^2}{4}\right) + \pi\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{27\alpha_6^2}{4}\right) - \pi\,, \tag{356}$$

$$\theta^8 = \pi\, {}_4F_3\left(\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}; \frac{1}{3}, \frac{2}{3}, 1; \frac{3125\alpha_8^2}{108}\right) \tag{357}$$

$$+ \frac{8}{3}\alpha_8\, {}_5F_4\left(\frac{3}{5}, \frac{4}{5}, 1, \frac{6}{5}, \frac{7}{5}; \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \frac{3}{2}; \frac{3125\alpha_8^2}{108}\right) - \pi\,, \tag{358}$$

$$\theta^9 = \pi\; {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{27\alpha_9}{4}\right) - \pi\,, \tag{359}$$

$$\theta^{10} = \pi\, {}_6F_5\left(\frac{1}{14}, \frac{3}{14}, \frac{5}{14}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14}; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1; \frac{823543\alpha_{10}^2}{12500}\right) \tag{360}$$

$$+ \frac{16}{5}\alpha_{10}\;\, {}_7F_6\left(\frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1, \frac{8}{7}, \frac{9}{7}, \frac{10}{7}; \frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}, \frac{3}{2}, \frac{3}{2}; \frac{823543\alpha_{10}^2}{12500}\right) - \pi\,. \tag{361}$$

The power-series expansions of these results (up to order α_D^2) again agree with Eq. (4.8) of Ref. [99] with the following identification of the variables involved in the two cases:

$$\alpha_D \Longleftrightarrow \left(\frac{R_p}{b}\right)^{7-p}, \qquad p + D = 10\,. \tag{362}$$

An alternative way to show the equivalence between our approach with only the tree diagram potential and that of Ref. [99] is using Eq. (302). In fact in this case

$p^2(r, G_N)$ in Eq. (315) contains only the term with $n = 1$ and taking into account Eq. (301) one gets the following expression for the deflection angle in Eq. (302):

$$\theta^D(b) = 2\int_{r_{\min}}^{\infty} \frac{dr}{r^2} \frac{b}{\sqrt{1 + \left(\frac{R_D}{r}\right)^{D-3} - \frac{b^2}{r^2}}} - \pi \tag{363}$$

with

$$b \equiv \frac{L}{p_\infty}, \qquad R_D^{D-3} \equiv \frac{G_N f_1^D}{p_\infty^2} = \frac{2G_N \gamma(p_\infty^2)}{E_{p_\infty} p_\infty^2} \frac{\Gamma(\frac{D-3}{2})}{\pi^{\frac{D-3}{2}}}, \tag{364}$$

where in the last step we have used Eq. (348). On the other hand Eq. (4.4) of Ref. [99] can be easily rewritten as follows,

$$\theta^p(b) = 2\int_{r_{\min}}^{\infty} \frac{dr}{r^2} \frac{b}{\sqrt{1 + \left(\frac{R_p}{r}\right)^{7-p} - \frac{b^2}{r^2}}} - \pi, \tag{365}$$

where R_p is a quantity defined in Ref. [99]. The two equations give the same deflection angle if we make the following identification:

$$R_p^{7-p} \Longleftrightarrow R_D^{D-3}, \qquad p + D = 10. \tag{366}$$

CONCLUSIONS

Starting from the elastic scattering amplitude of two scalar particles with arbitrary masses in Einstein gravity in an arbitrary number D of space-time dimensions, we isolated the terms that contribute in the classical limit by the method of regions. We then extracted from them the long-range classical effective potential between the two scalar particles for arbitrary D by means of the Lippmann–Schwinger equation or, equivalently, by the technique of EFT matching.

We then used the Hamiltonian consisting of the sum of the relativistic kinetic terms for the two particles and the potential to determine the conjugate momentum $p^2(r, G_N)$. It turns out that, unlike the case $D = 4$, for arbitrary D this relation contains an extra term proportional to the square of the tree scattering amplitude that, of course, vanishes for $D = 4$. We then used it to compute the deflection angle, finding complete agreement with the one obtained using the eikonal approach [47].

The approach of this paper is not only different from the one of Ref. [47] because here we use the Hamiltonian approach to derive the deflection angle, while Ref. [47] was based on the the eikonal approach, but also because the box and crossed box integrals are computed using two different methods. It turns out that, if we use the method of the regions directly on the fully relativistic expression for the box and crossed box diagrams, as explained in Appendix 8, we get the same result for the subleading term as in Ref. [47], while, if we first go to the potential region and then compute the subleading term, we get the same result only in the nonrelativistic limit, where the energy of the two particles becomes equal to their mass. Since we use the fully relativistic expression for the sum of the box and crossed box diagrams in the underlying fundamental theory, while the nonrelativistic expression for those diagrams emerges in the EFT, from the matching between the two theories we get the important result that, for $D > 4$, these diagrams leave a nonzero contribution to the potential that, of course, vanishes for $D = 4$.

APPENDIX

NORMALIZATION OF THE AMPLITUDE

In this Appendix we fix the conventions that we adopt for the normalization of scattering amplitudes. We decompose the S-matrix according to

$$S = 1 - \frac{i}{\hbar} T \,. \tag{367}$$

The operator T has therefore the dimension of an action, EL, where E stands for an energy scale and L for a length scale. Its matrix elements $T_{ba} = \langle b|T|a\rangle$ between asymptotic states $|b\rangle$ and $|a\rangle$ define the standard scattering amplitudes $\mathcal{M}_{ba}$ according to

$$T_{ba} = (2\pi\hbar)^D \delta(P_a - P_b)\mathcal{M}_{ab} \,, \tag{368}$$

where P_b and P_a denote the total outgoing and incoming D-momenta. The unitarity of the S-matrix $SS^\dagger = 1 = S^\dagger S$ also implies the following identity among T-matrix elements involving the sum over a complete set of intermediate asymptotic states

$$T_{ba} - (T^\dagger)_{ba} = -\frac{i}{\hbar} \sum_c T_{bc} (T^\dagger)_{ca} \,, \tag{369}$$

or, at the level of scattering amplitudes,

$$\mathcal{M}_{ab} - \overline{\mathcal{M}_{ba}} = -i2\pi \sum_c (2\pi\hbar)^{D-1} \delta(P_a - P_c) \, \overline{\mathcal{M}_{ca}} \, \mathcal{M}_{cb} \tag{370}$$

for states such that $P_a = P_b$.

We are interested in asymptotic states containing two kinds of scalar particles with masses m_1 and m_2 although we shall suppress the subscripts 1, 2 for simplicity. The associated free Hermitian scalar fields $\varphi(x)$ are described by the action

$$S_{\text{free}} = -\frac{1}{2} \int d^D x \left(\hbar^2 \partial^\mu \varphi \, \partial_\mu \varphi + m^2 \varphi^2 \right) . \tag{371}$$

The Fock expansion for $\varphi(x)$ can be taken as

$$\varphi(x) = \int \frac{d^D p}{(2\pi\hbar)^{D-1}} \, \delta(p^2 + m^2) \widetilde{\varphi}(p) e^{\frac{ip\cdot x}{\hbar}} , \tag{372}$$

where $\widetilde{\varphi}(p) = a(\vec{p}\,)$ and $\widetilde{\varphi}(-p) = a^\dagger(\vec{p}\,)$ for $p = (p^0, \vec{p}\,)$ and $p^0 > 0$, while the canonical commutation relations read

$$[a(\vec{p}), a^\dagger(\vec{p}\,')] = 2E(p)(2\pi\hbar)^{D-1} \delta(\vec{p} - \vec{p}\,') , \tag{373}$$

with $E(p) = \sqrt{\vec{p}^{\,2} + m^2}$ denoting the single-particle energy. The field $\varphi(x)$ has dimension $E^{-\frac{1}{2}} L^{\frac{1-D}{2}}$ and the creation/annihilation operators $\widetilde{\varphi}(p)$ have dimension

$E^{\frac{1}{2}}L^{\frac{D-1}{2}}$. Single-particle states are obtained acting with the creation operator $a^\dagger(\vec{p}\,)$ on the Fock vacuum $|0\rangle$,

$$a(\vec{p}\,)|0\rangle = 0\,, \qquad |\vec{p}\,\rangle = a^\dagger(\vec{p}\,)|0\rangle\,, \qquad \langle\vec{p}\,|\vec{p}\,'\rangle = 2E(p)(2\pi\hbar)^{D-1}\delta^{(D-1)}(\vec{p}-\vec{p}\,')\,, \tag{374}$$

so that their normalization is Lorentz invariant. The completeness relation for asymptotic states reads

$$\sum_{n=1}^{\infty}\int \frac{d^{D-1}\vec{p}_1}{(2\pi\hbar)^{D-1}}\frac{1}{2E(p_1)}\cdots\frac{d^{D-1}\vec{p}_n}{(2\pi\hbar)^{D-1}}\frac{1}{2E(p_n)}\,|\vec{p}_n,\ldots,\vec{p}_1\rangle\langle\vec{p}_n,\ldots,\vec{p}_1| = 1\,. \tag{375}$$

The invariant amplitude $\mathcal{M}(\vec{p}_1,\ldots,\vec{p}_M,\vec{p}_1{}',\ldots,\vec{p}_N{}')$ for the scattering of M incoming and N outgoing massive scalars is then given by the relation

$$\langle\vec{p}_N{}',\ldots,\vec{p}_1{}'|T|\vec{p}_M,\ldots,\vec{p}_1\rangle = (2\pi\hbar)^D\delta(P-P')\mathcal{M}(\vec{p}_1,\ldots,\vec{p}_M,\vec{p}_1{}',\ldots,\vec{p}_N{}') \tag{376}$$

with

$$P = \sum_{i=1}^{M} p_i\,, \qquad P' = \sum_{i=1}^{N} p_i{}' \tag{377}$$

and has the physical dimension $EL^{1-D}(EL^{D-1})^{\frac{M+N}{2}}$. This is a direct consequence of the fact that the creation and annihilation operators have dimension $E^{\frac{1}{2}}L^{\frac{D-1}{2}}$.

For the specific case of two-to-two scattering of particles with mass m_1 and m_2, which we describe at the beginning of Section 8, one has

$$\begin{aligned}\langle\vec{p}_4,\vec{p}_3|S|\vec{p}_2,\vec{p}_1\rangle &= 2E_1(p_1)(2\pi\hbar)^{D-1}\delta(\vec{p}_1-\vec{p}_3)2E_2(p_2)(2\pi\hbar)^{D-1}\delta(\vec{p}_2-\vec{p}_4)\\ &\quad - i2\pi(2\pi\hbar)^{D-1}\delta(p_1+p_2-p_3-p_4)\mathcal{M}(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)\,,\end{aligned} \tag{378}$$

and we adopt a simplified notation for the invariant amplitude evaluated in the center-of-mass frame

$$\mathcal{M}(\vec{p},\vec{p}\,') = \mathcal{M}(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)\,, \tag{379}$$

which has dimension E^3L^{D-1}. We also consider a reduced S-matrix, s, which relates to the standard S-matrix by

$$\langle\vec{p}_4,\vec{p}_3|S|\vec{p}_2,\vec{p}_1\rangle = 4E_1(p_1)E_2(p_2)(2\pi\hbar)^{D-1}\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\langle\vec{p}\,'|s|\vec{p}\,\rangle\,, \tag{380}$$

with

$$\vec{p} = \frac{m_2\vec{p}_1 - m_1\vec{p}_2}{m_1+m_2}\,, \qquad \vec{p}\,' = \frac{m_2\vec{p}_3 - m_1\vec{p}_4}{m_1+m_2}\,, \tag{381}$$

and reads

$$\langle\vec{p}\,'|s|\vec{p}\,\rangle = (2\pi\hbar)^{D-1}\delta(\vec{p}-\vec{p}\,') - i2\pi\delta(E_p - E_{p'})\widetilde{\mathcal{M}}(\vec{p},\vec{p}\,') \tag{382}$$

in the center-of-mass frame. Therefore the reduced amplitude in the center-of-mass frame $\widetilde{\mathcal{M}}(\vec{p},\vec{p}\,')$ is related to the invariant amplitude by

$$\widetilde{\mathcal{M}}(\vec{p},\vec{p}\,') = \frac{\mathcal{M}(\vec{p},\vec{p}\,')}{4E_1(p)E_2(p)} \tag{383}$$

and has dimension EL^{D-1}. Eq. (382) for the reduced S-matrix can be also written as

$$\langle\vec{p}\,'|s|\vec{p}\,\rangle = (2\pi\hbar)^{D-1}\delta(\vec{p}-\vec{p}\,') - i2\pi\delta(p^2 - p'^2)\widetilde{\mathcal{M}}_{\text{eff}}(\vec{p},\vec{p}\,') \tag{384}$$

with

$$\widetilde{\mathcal{M}}_{\text{eff}}(\vec{p}, \vec{p}\,') = 2E_p \xi(p) \widetilde{\mathcal{M}}(\vec{p}, \vec{p}\,') , \tag{385}$$

or as

$$\langle \vec{p}\,' | s | \vec{p}\, \rangle = (2\pi\hbar)^{D-1} \delta(\vec{p} - \vec{p}\,') - i2\pi\delta(p - p') \widetilde{\mathcal{M}}_{\text{eik}}(\vec{p}, \vec{p}\,') , \tag{386}$$

with

$$\mathcal{M}_{\text{eik}}(\vec{p}, \vec{p}\,') = \frac{\mathcal{M}(\vec{p}, \vec{p}\,')}{4E_p p} . \tag{387}$$

We should also mention that the T matrix is often defined in the following *alternative* way:

$$S = 1 + i\, T . \tag{388}$$

In this case one would get a scattering amplitude that differs from the previous one by a factor $-\hbar$. This alternative normalization was employed in [47] to retrace the dependence on $\hbar$ of the eikonal factor that one extracts from the scattering amplitude.

ONE-LOOP INTEGRALS IN THE $\hbar \to 0$ LIMIT

In this Appendix we explicitly discuss the evaluation of triangle and box integrals in the classical limit $\hbar \to 0$, *i.e.* the limit of small transferred momentum q. We employ a technique that can be used to extract the asymptotic expansion of Feynman integrals in certain limits known as the method of regions [105], which consists in splitting the domain of integration into sectors defined by suitable scaling relations.

In the examples we shall consider, the asymptotic expansions of Feynman integrals will emerge in particular from the soft region, in which the integrated momentum k scales as $k \sim \mathcal{O}(q)$, and from the hard region, $k \sim \mathcal{O}(1)$. The non-analytic contributions in momentum space giving rise to long-range effects in position space, on which we focus in the main body of the paper, are those obtained from the soft region. We will then comment on the relation between the results obtained from these regions and the potential region. This region involves both the classical limit of small q and the nonrelativistic limit of small v, where v is the relative velocity in the center-of-mass frame, and can be characterized by the scaling relations $k^0 \sim \mathcal{O}(qv)$ and $\vec{k} \sim \mathcal{O}(q)$.

Triangle integrals

Let us first consider the scalar triangle integral (262)

$$I_{\triangleright} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2 - i\epsilon)\left((q-k)^2 - i\epsilon\right)\left(k^2 - 2p_1 \cdot k - i\epsilon\right)}, \tag{389}$$

which we may recast as

$$I_{\triangleright} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2 - i\epsilon)\left((q-k)^2 - i\epsilon\right)\left(k^2 - (q_\perp + q) \cdot k - i\epsilon\right)} \tag{390}$$

introducing, together with the momentum transfer $q = p_1 - p_3$, the additional variable

$$q_\perp = p_1 + p_3 \,. \tag{391}$$

Note in particular that $q \cdot q_\perp = 0$.

The classical limit consists in letting $\hbar \to 0$ in such a way that the momentum transfer q vanishes, while the transferred wave-vector $\frac{1}{\hbar} q$ and the average momentum $\frac{1}{2} q_\perp$ of the massive particle are kept fixed. We schematically identify this situation by writing

$$q \sim \mathcal{O}(\hbar)\,, \qquad q_\perp \sim \mathcal{O}(1)\,, \qquad q \ll q_\perp \,. \tag{392}$$

We note that this limit requires the mass m_1 to be nonzero, in view of the relation

$$-q_\perp^2 = 4m_1^2 + q^2 \,. \tag{393}$$

We shall now employ the expansion by regions to obtain an asymptotic approximation of the integral (390) in the classical limit. This method consists in splitting the integration over the loop momentum k into a soft region, characterized by the scaling $k \sim \mathcal{O}(\hbar)$ and hence $k \sim q \ll q_\perp$, and a hard region, in which $k \sim \mathcal{O}(1)$ and hence $k \sim q_\perp \gg q$, namely

$$I_\triangleright = I_\triangleright^{(s)} + I_\triangleright^{(h)}, \tag{394}$$

with

$$I_{\triangleright}^{(s)} = \int_{k\sim q} \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2 - i\epsilon)\,((q-k)^2 - i\epsilon)\,(k^2 - (q_\perp + q)\cdot k - i\epsilon)}\,, \tag{395}$$

$$I_{\triangleright}^{(h)} = \int_{k\sim q_\perp} \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2 - i\epsilon)\,((q-k)^2 - i\epsilon)\,(k^2 - (q_\perp + q)\cdot k - i\epsilon)}\,. \tag{396}$$

One then considers the Taylor expansion of the integrands according to the appropriate scaling relations, thus obtaining two asymptotic series for $I^{(s)}$ and $I^{(h)}$,

$$\begin{aligned} I^{(s)} &= I^{(1s)} + I^{(2s)} + \cdots\,, \\ I^{(h)} &= I^{(1h)} + I^{(2h)} + \cdots\,. \end{aligned} \tag{397}$$

The first two contributions to the soft region thus read

$$I_{\triangleright}^{(1s)} = \int_{k\sim q} \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2 - i\epsilon)((q-k)^2 - i\epsilon)(-q_\perp\cdot k - i\epsilon)}\,, \tag{398}$$

$$I_{\triangleright}^{(2s)} = \int_{k\sim q} \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5(-k^2 + q\cdot k)}{(k^2 - i\epsilon)((q-k)^2 - i\epsilon)(-q_\perp\cdot k - i\epsilon)^2}\,, \tag{399}$$

while for the hard contribution one has

$$I_{\triangleright}^{(1h)} = \int_{k\sim q_\perp} \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2 - i\epsilon)^2(k^2 - q_\perp\cdot k - i\epsilon)}\,, \tag{400}$$

$$I_{\triangleright}^{(2h)} = \int_{k\sim q_\perp} \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5 q\cdot k(3k^2 - 2q_\perp\cdot k)}{(k^2 - i\epsilon)^3(k^2 - q_\perp\cdot k - i\epsilon)^2}\,. \tag{401}$$

The integration can be then extended to the whole D-dimensional space in both regions in view of the fact that the error $R_\triangleright$ thus introduced always takes the form of a scaleless integral and is therefore identically vanishing in dimensional regularization: to leading order, for instance,

$$R_{\triangleright} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(k^2 - i\epsilon)^2(-q_\perp \cdot k - i\epsilon)} = 0\,. \tag{402}$$

By means of the above expansion we have reduced the problem to the evaluation of simpler Feynman integrals, which can be directly calculated introducing Feynman parameters and exploiting the orthogonality between q and $q_\perp$, as detailed in Section 8 below. The leading contribution (398) to the soft region can be read from the general integral (458) and takes the form

$$I_{\triangleright}^{(1s)} = \frac{i\sqrt{\pi}}{m_1 (4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(\frac{D-3}{2}\right)^2 \Gamma\left(\frac{5-D}{2}\right)}{2\Gamma(D-3)} \left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}}, \tag{403}$$

since $-q_\perp^2 = 4m_1^2 + \mathcal{O}(\hbar^2)$ thanks to (393), while the leading hard contribution (400) reads, by (450),

$$I_{\triangleright}^{(1h)} = \frac{i\Gamma\left(\frac{6-D}{2}\right)}{(4-D)(5-D)(4\pi)^{\frac{D}{2}}\hbar} \left(\frac{m_1^2}{\hbar^2}\right)^{\frac{D-6}{2}}. \tag{404}$$

We note that the leading soft term behaves as $\mathcal{O}(1)$ as $\hbar \to 0$ and is therefore classical, while the hard term scales like $\hbar^{\frac{5-D}{2}}$. Furthermore, the latter is analytic (in fact, constant) in the transferred momentum and therefore corresponds to a local term in position space, while the former gives rise to a power-law dependence on r via (286). Actually, the whole hard asymptotic expansion is just a power series expansion in q^2 and this leads us to focus on the terms arising from the soft region in the discussion of the long-range potential.

Considering now the subleading soft integral (399), we note that the first term in the numerator gives rise to a scaleless integral, after sending $k \to q - k$, and thus can be discarded. The remaining integral is then given by (459), namely

$$I_{\triangleright}^{(2s)} = -\frac{i\hbar}{m_1^2(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(\frac{D-2}{2}\right)^2 \Gamma\left(\frac{4-D}{2}\right)}{2\Gamma(D-3)} \left(\frac{q^2}{\hbar^2}\right)^{\frac{D-4}{2}}, \tag{405}$$

which is $\mathcal{O}(\hbar)$ and hence quantum. Interestingly, we note that this term of the expansion is divergent as $D \to 4$, despite the fact that the original integral (390) is clearly finite in four dimensions. The appearance of such spurious divergences is a standard feature of the expansion by regions and indicates the presence of cancellations between the soft and the hard series. In this case, the pole at $\epsilon = 0$ for $D = 4 - 2\epsilon$ cancels in the sum of the leading hard term (404) and subleading soft term (405), leaving behind the finite contribution

$$\left(I_{\triangleright}^{(1h)} + I_{\triangleright}^{(2s)}\right)\Big|_{D=4} = \frac{i\hbar}{2m_1^2(4\pi)^2} \left(\log \frac{q^2}{m_1^2} - 2\right). \tag{406}$$

This can be regarded as a quantum contribution since it contains terms scaling as $\mathcal{O}(\hbar \log \hbar)$ and $\mathcal{O}(\hbar)$ in the classical limit.

A similar strategy also applies to tensor integrals associated to the triangle diagram, such as

$$I_{\triangleright}^{\mu} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^4 k^\mu}{(k^2 - i\epsilon)\left[(q-k)^2 - i\epsilon\right](k^2 - (q_\perp + q)\cdot k - i\epsilon)} \tag{407}$$

and the one appearing in (263),

$$I_{\triangleright}^{\mu\nu} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^3 k^\mu k^\nu}{(k^2 - i\epsilon)\left[(q-k)^2 - i\epsilon\right]\left(k^2 - (q_\perp + q)\cdot k - i\epsilon\right)} . \tag{408}$$

After performing a tensor decomposition in terms of q^μ, $q_\perp^\mu$ and $\eta^{\mu\nu}$, these integrals can be evaluated directly in the soft region by means of Feynman parameters, (see (451), (458), (459), (460)). To leading order as $\hbar \to 0$, one finds

$$\begin{aligned} I_{\triangleright}^{(s)\mu} &= \frac{i\sqrt{\pi}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(\frac{5-D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(\frac{D-3}{2}\right)}{2\Gamma(D-2)} \frac{q^\mu}{\hbar m_1}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} \\ &\quad + \frac{i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(\frac{4-D}{2}\right)\Gamma\left(\frac{D-2}{2}\right)^2}{2\Gamma(D-2)} \frac{p_1^\mu}{m_1^2}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-4}{2}} \end{aligned} \tag{409}$$

and

$$\begin{aligned} I_{\triangleright}^{(s)\mu\nu} &= \frac{i}{4m_1(4\pi)^{\frac{D}{2}}\Gamma(D-1)} \\ &\times \left[\left(\eta^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{m_1^2} - (D-1)\frac{q^\mu q^\nu}{q^2}\right)\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-3}{2}} \sqrt{\pi}\,\Gamma\left(\frac{3-D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)^2\right. \\ &\left. + \frac{2(q^\mu p_1^\nu + q^\nu p_1^\mu)}{\hbar m_1}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-4}{2}} \Gamma\left(\frac{4-D}{2}\right)\Gamma\left(\frac{D-2}{2}\right)\Gamma\left(\frac{D}{2}\right)\right] . \end{aligned} \tag{410}$$

The analogous results for $I_{\triangleleft}$, $I_{\triangleleft}^\mu$, $I_{\triangleleft}^{\mu\nu}$ can be obtained by replacing $m_1 \leftrightarrow m_2$ in the above expressions (403), (405), (409) and (410).

Box integrals

Let us now turn to the scalar box integral (264), leaving the $-i\epsilon$ prescription implicit for the time being,

$$I_{\Box,s} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{k^2(k-q)^2(k^2-2p_1\cdot k)(k^2+2p_2\cdot k)}\,. \tag{411}$$

Introducing the variables

$$q_\perp = p_1 + p_3\,, \qquad Q = p_1 + p_2 \tag{412}$$

allows us to recast the desired integral as follows

$$I^{(1s)}_{\Box,s} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{k^2(k-q)^2(k^2+(2Q-q_\perp-q)\cdot k)(k^2-(q_\perp+q)\cdot k)}\,. \tag{413}$$

These new variables satisfy in particular

$$q\cdot q_\perp = 0 = q\cdot Q\,, \qquad q_\perp\cdot Q = Q^2-(m_1^2-m_2^2)\,. \tag{414}$$

We are interested in the classical limit described by the scaling

$$q \sim \mathcal{O}(\hbar)\,, \qquad q_\perp, Q \sim \mathcal{O}(1)\,, \qquad q \ll q_\perp, Q\,, \tag{415}$$

as $\hbar \to 0$, which implicitly requires a nonzero mass because

$$-q_\perp^2 = 4m_1^2 + q^2\,. \tag{416}$$

The leading soft term then reads

$$I^{(1s)}_{\Box,s} = \int \frac{d^Dk}{(2\pi\hbar)^D} \frac{\hbar^5}{k^2(k-q)^2((2Q-q_\perp)\cdot k)(-(q_\perp \cdot k))}\,, \tag{417}$$

where, following the same strategy detailed for the triangle diagram, we have performed a Taylor expansion of the integrand of (413) to leading order for $k \sim \mathcal{O}(\hbar)$, namely $k \sim q \ll q_\perp, Q$. Introducing a Feynman parameter x for the two linear factors in the denominator, we then have

$$I^{(1s)}_{\Box,s} = \int_0^1 dx \int \frac{d^Dk}{(2\pi\hbar)^D} \frac{\hbar^5}{k^2(k-q)^2((2xQ-q_\perp)\cdot k)^2}\,. \tag{418}$$

Since $2xQ - q_\perp$ is orthogonal to q, we can apply (458), which thus yields

$$I^{(1s)}_{\Box,s} = \frac{i\Gamma\left(\frac{D-4}{2}\right)^2 \Gamma\left(\frac{6-D}{2}\right)}{2(4\pi)^{\frac{D}{2}}\Gamma(D-4)} \frac{1}{\hbar} \left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}} \int_0^1 \frac{dx}{-\left(xQ - \frac{1}{2}q_\perp\right)^2 - i\epsilon}\,, \tag{419}$$

where we have reinstated the $-i\epsilon$ prescription. The roots of the polynomial

$$-\left(xQ - \frac{q_\perp}{2}\right)^2 - i\epsilon \tag{420}$$

appearing in the denominator are given up to $\mathcal{O}(\hbar^2)$ by

$$x_\pm = \frac{m_1^2 - p_1\cdot p_2 \pm \sqrt{(p_1\cdot p_2)^2 - (m_1m_2)^2}}{m_1^2 + m_2^2 - 2p_1\cdot p_2} \pm i\epsilon \tag{421}$$

and their real parts both lie in the integration interval, namely between 0 and 1. We thus obtain[7]

$$I^{(1s)}_{\Box,s} = \frac{i\Gamma\left(\frac{D-4}{2}\right)^2 \Gamma\left(\frac{6-D}{2}\right)}{2\hbar(4\pi)^{\frac{D}{2}}\Gamma(D-4)} \frac{i\pi - \cosh^{-1}\left(-\frac{p_1\cdot p_2}{m_1 m_2}\right)}{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}} . \tag{422}$$

The crossed box diagram is related to the one we just discussed by $p_1 \mapsto -p_3$, which corresponds to exchanging $p_1 \cdot p_2 \leftrightarrow -p_1 \cdot p_2$ up to $\mathcal{O}(\hbar^2)$. The real parts of the roots analogous to (421) then no longer fall between 0 and 1 and the resulting integral gives

$$I^{(1s)}_{\Box,u} = \frac{i\Gamma\left(\frac{D-4}{2}\right)^2 \Gamma\left(\frac{6-D}{2}\right)}{2\hbar(4\pi)^{\frac{D}{2}}\Gamma(D-4)} \frac{\cosh^{-1}\left(-\frac{p_1\cdot p_2}{m_1 m_2}\right)}{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}} . \tag{423}$$

The sum of the leading box and crossed box diagrams finally reads

$$I^{(1s)}_{\Box,s} + I^{(1s)}_{\Box,u} = \frac{\Gamma\left(\frac{D-4}{2}\right)^2 \Gamma\left(\frac{6-D}{2}\right)}{2\hbar(4\pi)^{\frac{D}{2}}\Gamma(D-4)} \frac{-\pi}{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}} . \tag{424}$$

The subleading term in the soft expansion for the box integral is instead

$$I^{(2s)}_{\Box,s} = 2\hbar^5 \int_0^1 dx \int \frac{d^D k}{(2\pi\hbar)^D} \frac{q\cdot k - k^2}{k^2 (q-k)^2 \left[(2xQ - q_\perp)\cdot k\right]^3} , \tag{425}$$

where we have considered the second term in the Taylor expansion of the integrand of (413) for $k \sim \mathcal{O}(\hbar)$, namely $k \sim q \ll q_\perp, Q$. Recognizing that the second term in

7 $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$.

the numerator gives rise to a scaleless integral, this expression can be evaluated by the help of formula (459) to

$$I^{(2s)}_{\square,s} = -\frac{i\sqrt{\pi}\,\Gamma\left(\frac{5-D}{2}\right)\Gamma\left(\frac{D-3}{2}\right)^2}{4(4\pi)^{\frac{D}{2}}\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}}\int_0^1 \frac{dx}{\left[-\left(xQ-\frac{q_\perp}{2}\right)^2-i\epsilon\right]^{\frac{3}{2}}}\,. \tag{426}$$

Performing the integral over x then yields, to leading order in $\hbar$,

$$I^{(2s)}_{\square,s} = \frac{i\sqrt{\pi}\,\Gamma\left(\frac{5-D}{2}\right)\Gamma\left(\frac{D-3}{2}\right)^2}{8(4\pi)^{\frac{D}{2}}\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}}\frac{\left[s\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+(m_1^2-m_2^2)\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\right]}{(p_1\cdot p_2)^2-m_1^2m_2^2}\,, \tag{427}$$

where $s=-(p_1+p_2)^2$. Adding this expression, corresponding to the s-channel, to the one obtained from the u-channel yields in particular

$$I^{(2s)}_{\square,s}+I^{(2s)}_{\square,u} = \frac{i\sqrt{\pi}\,\Gamma\left(\frac{5-D}{2}\right)\Gamma\left(\frac{D-3}{2}\right)^2}{2(4\pi)^{\frac{D}{2}}\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}}\frac{m_1+m_2}{(p_1\cdot p_2)^2-m_1^2m_2^2}\,. \tag{428}$$

As mentioned for the case of triangle integrals, we have focused on the soft-region expansion of box diagrams because it is the one containing terms with a non-analytic dependence on q^2 for generic D. The hard region, obtained expanding the original integral (413) for $k\sim\mathcal{O}(1)$, namely $k\sim q_\perp$, $Q\gg q$, gives rise instead to terms with positive integer powers of q^2. For instance, the leading hard term for the box integral is given by

$$I^{(1h)}_{\square} = \int\frac{d^Dk}{(2\pi\hbar)^D}\frac{\hbar^5}{(k^2)^2(k^2+(2Q-q_\perp)\cdot k)(k^2-q_\perp\cdot k)} \tag{429}$$

so that, employing again Feynman parameters to rewrite the linear factors in the denominator in terms of a single one and using (450),

$$I_{\square}^{(1h)} = \frac{i\Gamma\left(\frac{8-D}{2}\right)\Gamma\left(D-6\right)}{(4\pi)^{\frac{D}{2}}\Gamma(D-4)} \int_0^1 \frac{\hbar^{5-D}dx}{\left[-\left(xQ-\frac{q_\perp}{2}\right)^2 - i\epsilon\right]^{\frac{8-D}{2}}} . \tag{430}$$

This contribution is thus analytic in q^2 and finite in four dimensions. However, it is infrared divergent in, say, $D = 5$. The box integral (413) is however finite in five dimensions and this means that such a divergence must cancel out when adding the soft and the hard contributions: indeed, comparing (430) with the subleading soft term (426) we see that the two divergent contributions cancel as $D \to 5$ leading to a finite limit for $I_{\square}^{(1h)} + I_{\square}^{(2s)}$.

The potential region

Another region which can be useful for the expansion of Feynman integrals in the classical limit is the so-called potential region, as also argued in [10, 12]. To describe it, let us again consider the scalar triangle (389), which we write in the center-of-mass frame as

$$\begin{aligned} I_{\triangleright} = \int \frac{d^D k}{(2\pi\hbar)^D} &\frac{\hbar^5}{(-(k^0)^2 + |\vec{k}\,|^2 - i\epsilon)(-(k^0)^2 + |\vec{k} + \vec{q}\,|^2 - i\epsilon)} \\ &\frac{1}{(-(k^0)^2 + |\vec{k}|^2 - 2E_1(p)k^0 + 2\vec{p}\cdot\vec{k} - i\epsilon)} , \end{aligned} \tag{431}$$

where we have sent $k \to -k$ and adopted the same notation as in Section 8.

As before, we are interested in the limit in which the transferred momentum $\vec{q}$ is of order $\hbar$ and is hence small with respect to the mass. We also consider the nonrelativistic limit, *i.e.* the regime $|\vec{p}\,| \ll m_1$ in which the relative velocity v is much smaller than the speed of light. The potential region is then defined by the following scaling relations

$$k^0 \sim qv\,, \qquad \vec{k} \sim q\,, \tag{432}$$

which break Lorentz invariance as they prescribe the time-component k^0 of the loop momentum to be negligible with respect to its spatial components $\vec{k}$. The leading potential term is then obtained by simply neglecting the $(k^0)^2$ terms in the propagators,

$$I_{\triangleright}^{(1p)} = \int \frac{d^{D-1}\vec{k}}{(2\pi\hbar)^{D-1}} \frac{\hbar^4}{|\vec{k}|^2|\vec{k}+\vec{q}\,|^2} \int \frac{dk^0}{2\pi} \frac{1}{(-2E_1(p)k^0 + |\vec{k}|^2 + 2\vec{p}\cdot\vec{k} - i\epsilon)}\,. \tag{433}$$

The resulting integral over dk^0 is in principle ill defined, but can be evaluated by prescribing the application of the standard formula for the passage near a simple pole $\frac{1}{x-i\epsilon} = \mathrm{PV}\,\frac{1}{x} + i\pi\delta(x)$. We thus obtain

$$I_{\triangleright}^{(1p)} = \frac{i}{4E_1(p)} \int \frac{d^{D-1}\vec{k}}{(2\pi\hbar)^{D-1}} \frac{\hbar^4}{|\vec{k}|^2|\vec{k}+\vec{q}\,|^2}\,. \tag{434}$$

The remaining integral is elementary and can be evaluated by means of Feynman parameters, yielding

$$I_{\triangleright}^{(1p)} = \frac{i\sqrt{\pi}}{E_1(p)(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(\frac{D-3}{2}\right)^2 \Gamma\left(\frac{5-D}{2}\right)}{2\Gamma(D-3)} \left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}}\,. \tag{435}$$

Taking into account the fact that $E_1(p) \approx m_1$ up to terms of order v^2 in the nonrelativistic limit, this is the same as the leading soft result (403).

It would be interesting to reproduce the subleading soft term (405) from the subleading potential expansion, which is obtained from the higher-order terms in Taylor series of the integrand in (431) for small $(k^0)^2$. However, the resulting integral in dk^0 presents further difficulties, in particular due to appearance of a double pole.

Let us now turn to the potential-region expansion of the massive box (411). We go to the center-of-mass frame, adopting the same conventions as in Section 8, so that

$$I_\Box = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{(-(k^0)^2 + \vec{k}^{\,2} - i\epsilon)(-(k^0)^2 + |\vec{k} - \vec{q}\,|^2 - i\epsilon)} \\ \frac{1}{(-(k^0)^2 + \vec{k}^2 + 2E_1 k^0 - 2\vec{p}\cdot\vec{k} - i\epsilon)(-(k^0)^2 + \vec{k}^2 - 2E_2 k^0 - 2\vec{p}\cdot\vec{k} - i\epsilon)} \,. \tag{436}$$

In addition to the classical limit, which consists here in sending $\hbar \to 0$ in such a way that

$$\vec{q} \sim \mathcal{O}(\hbar)\,, \qquad \vec{q}_\perp \sim \mathcal{O}(1)\,, \tag{437}$$

where $\vec{q}_\perp = \vec{p} + \vec{p}\,'$, we also consider the nonrelativistic limit of small v, as we did for the triangle. We then adopt the scaling relations

$$k^0 \sim qv \qquad \vec{k} \sim q\,, \tag{438}$$

which characterize the potential region for the loop momentum. We are thus justified in neglecting the $(k^0)^2$ appearing in the denominator, to leading order,

$$I_{\square}^{(1p)} = \int \frac{d^D k}{(2\pi\hbar)^D} \frac{\hbar^5}{\vec{k}^{\,2}|\vec{k}-\vec{q}\,|^2(2E_1 k^0 + \vec{k}^2 - 2\vec{p}\cdot\vec{k} - i\epsilon)(-2E_2 k^0 + \vec{k}^2 - 2\vec{p}\cdot\vec{k} - i\epsilon)} \,. \tag{439}$$

The integral in dk^0 can be performed with the help of the residue theorem, leading to

$$I_{\square}^{(1p)} = \frac{i}{2E_p} \int \frac{d^{D-1}\vec{k}}{(2\pi\hbar)^{D-1}} \frac{\hbar^4}{\vec{k}^2|\vec{k}-\vec{q}\,|^2(\vec{k}^2 - 2\vec{p}\cdot\vec{k} - i\epsilon)} \,. \tag{440}$$

Letting $\vec{k} \to \vec{p} - \vec{k}$, we have

$$I_{\square}^{(1p)} = \frac{i}{2E_p} \int \frac{d^{D-1}\vec{k}}{(2\pi\hbar)^{D-1}} \frac{\hbar^4}{|\vec{k}-\vec{p}\,|^2|\vec{k}-\vec{p}\,'|^2(\vec{k}^2 - |\vec{p}\,|^2 - i\epsilon)} \,, \tag{441}$$

so that we have reduced the problem to the evaluation of a Euclidan version of the triangle integral with an effective "squared mass" $m^2 = -|\vec{p}\,|^2 - i\epsilon$. Indeed, with an appropriate choice of routing for the loop momentum, the triangle integral (389) can be written as follows

$$I_{\triangleright} = i \int \frac{d^D k_E}{(2\pi\hbar)^D} \frac{\hbar^5}{(k_E - p_{1E})^2 (k_E - p_{3E})^2 \left(k_E^2 + \frac{m^2}{\hbar^2}\right)} \,, \tag{442}$$

after Wick rotation, and therefore the above integral can be obtained from this one by the identifications

$$D \to D - 1 \,, \qquad m \to -i\hbar|\vec{p}\,| \,. \tag{443}$$

Consequently, thanks to (403) and (405), we find

$$\int \frac{d^{D-1}\vec{k}}{(2\pi\hbar)^d} \frac{\hbar^4}{|\vec{k}-\vec{p}\,|^2|\vec{k}-\vec{p}\,'|^2(\vec{k}^2-\vec{p}\,^2-i\epsilon)}$$
$$= \frac{i\pi}{\hbar\,(4\pi)^{\frac{D}{2}}|\vec{p}|} \frac{\Gamma\left(\frac{6-D}{2}\right)\Gamma^2\left(\frac{D-4}{2}\right)}{\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}} + \frac{1}{2|\vec{p}\,|^2(4\pi)^{\frac{D-1}{2}}} \frac{\Gamma\left(\frac{5-D}{2}\right)\Gamma^2\left(\frac{D-3}{2}\right)}{\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} + \cdots . \tag{444}$$

We thus have, retaining the first two nontrivial orders for the soft-region expansion of (441),

$$\begin{aligned} I_{\square}^{(1p)} &= -\frac{\pi}{\hbar|\vec{p}\,|E_p} \frac{\Gamma\left(\frac{D-4}{2}\right)^2\Gamma\left(\frac{6-D}{2}\right)}{2(4\pi)^{\frac{D}{2}}\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-6}{2}} \\ &+ \frac{i\sqrt{\pi}}{|\vec{p}\,|^2E_p} \frac{\Gamma\left(\frac{D-3}{2}\right)^2\Gamma\left(\frac{5-D}{2}\right)}{2(4\pi)^{\frac{D}{2}}\Gamma(D-4)}\left(\frac{q^2}{\hbar^2}\right)^{\frac{D-5}{2}} + \cdots . \end{aligned} \tag{445}$$

Note that the first line coincides with the leading order (424) for the soft expansion of the sum of box and crossed box diagrams written in the center-of-mass frame, where $|\vec{p}\,|E_p = \sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}$. Indeed, in the potential region, the crossed box diagram gives zero to leading order since the poles in k^0 both lie in the upper half plane.

However, the subleading order does not coincide with (428). It is in fact proportional to it, but instead of the total mass m_1+m_2 it displays a factor E_p, the center-of-mass energy, so that the two results do agree in the nonrelativistic limit $v \ll 1$. This is in general to be expected, since the leading potential contribution $I_{\square}^{(1p)}$ is only reliable to first order in the nonrelativistic limit.

A more complete comparison between the results coming from the potential region and the ones obtained from the soft region for *generic* velocities, *i.e.* beyond the nonrelativistic regime, should be performed after resumming the potential series to all orders in v. However, the evaluation of subleading potential integrals is quite complicated due to the fact that they are in principle ill defined, as we have already seen for the triangle integral. A viable alternative to the evaluation of such integrals could be provided by an extension of the nonrelativistic integration techniques discussed in [10] to the case of generic dimensions.

In conclusion the potential region provides an expression for the non-analytic terms terms in the small-q expansion of the relevant Feynman integrals that agrees with the one furnished by the soft region at least to leading order in the nonrelativistic limit. In contrast, the soft region directly provides the non-analytic terms in the small-q expansion in a fully relativistic manner. Let us also mention once more that the soft region gives rise to the needed cancellation of the spurious divergences appearing in the hard region, again without involving the nonrelativistic limit, as for instance between (426) and (430) as $D \to 5$.

Auxiliary integrals

In this subsection we collect a number of useful standard techniques and results that allow one to explicitly evaluate the Feynman integrals presented above. To simplify the presentation, all quantities appearing in this section are understood to be dimensionless. We first recall that, in D-dimensional Euclidean space, we have the general formula

$$\int \frac{d^D \ell_E}{(2\pi)^D} \frac{(\ell_E^2)^\beta}{(\ell_E^2 + \Delta_E^2)^\alpha} = \frac{\Gamma\left(\beta + \frac{D}{2}\right)\Gamma\left(\alpha - \beta - \frac{D}{2}\right)}{(4\pi)^{\frac{D}{2}}\Gamma(\alpha)\Gamma\left(\frac{D}{2}\right)} (\Delta_E^2)^{\frac{D}{2}-\alpha+\beta} . \quad (446)$$

Let us consider

$$I(p^2) = \int \frac{d^D \ell}{(\ell^2 - i\epsilon)^{\lambda_1}(\ell^2 - 2p \cdot \ell - i\epsilon)^{\lambda_2}} , \quad (447)$$

where p^μ is a time–like vector, $(-p^2) > 0$. Introducing Feynman parameters we have

$$I(p^2) = \frac{\Gamma(\lambda_1 + \lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^1 dx\,(1-x)^{\lambda_1 - 1} x^{\lambda_2 - 1} \int \frac{d^D \ell}{(\ell^2 - 2xp \cdot \ell - i\epsilon)^{\lambda_1 + \lambda_2}} . \quad (448)$$

Shifting ℓ by xp so as to complete the square in the denominator, performing the Wick rotation $(\ell^0, \vec{\ell}\,) = (i\ell_E^0, \vec{\ell}_E)$ and employing equation (446), one then obtains

$$I(p^2) = i\pi^{\frac{D}{2}} \frac{\Gamma(\lambda_1 + \lambda_2 - \frac{D}{2})}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^1 (1-x)^{\lambda_1 - 1} x^{D - 2\lambda_1 - \lambda_2 - 1} dx\,(-p^2)^{\frac{D}{2} - \lambda_1 - \lambda_2} . \quad (449)$$

Finally, recognizing the Beta function appearing in the last equation, we get the formula (cf. [105, eq. (A.13)])

$$\int \frac{d^D \ell}{(\ell^2 - i\epsilon)^{\lambda_1}(\ell^2 - 2p \cdot \ell - i\epsilon)^{\lambda_2}} = i\pi^{\frac{D}{2}} \frac{\Gamma(\lambda_1 + \lambda_2 - \frac{D}{2})\Gamma(D - 2\lambda_1 - \lambda_2)}{\Gamma(\lambda_2)\Gamma(D - \lambda_1 - \lambda_2)(-p^2)^{\lambda_1 + \lambda_2 - \frac{D}{2}}} . \quad (450)$$

In a very similar way, one can also derive (cf. [105, eq. (A.7)])

$$\int \frac{d^D\ell}{(\ell^2-i\epsilon)^{\lambda_1}\left((\ell-q)^2-i\epsilon\right)^{\lambda_2}} = i\pi^{\frac{D}{2}}\,\frac{\Gamma\left(\lambda_1+\lambda_2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\lambda_1\right)\Gamma\left(\frac{D}{2}-\lambda_2\right)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(D-\lambda_1-\lambda_2)(q^2)^{\lambda_1+\lambda_2-\frac{D}{2}}}\,. \tag{451}$$

Let us now consider the following integral

$$I_\perp(q^2,r^2) = \int \frac{d^D\ell}{(\ell^2-i\epsilon)^{\lambda_1}((q-\ell)^2-i\epsilon)^{\lambda_2}(2r\cdot\ell-i\epsilon)^{\lambda_3}}\,, \tag{452}$$

where r^μ is time–like, $(-r^2)>0$, and $q\cdot r=0$, so that q^μ is space–like, $q^2>0$. Proceeding as in the previous case, we obtain

$$\begin{aligned} I_\perp(q^2,r^2) = i\pi^{\frac{D}{2}}\,&\frac{\Gamma(\lambda_1+\lambda_2+\lambda_3-\frac{D}{2})}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}\int_0^\infty dx\,x^{\lambda_1-1}\int_0^\infty dy\,y^{\lambda_2-1}\int_0^\infty dz\,z^{\lambda_3-1} \\ &\times\delta(1-x-y-z)\frac{(z^2(-r^2)+xy\,q^2)^{\frac{D}{2}-\lambda_1-\lambda_2-\lambda_3}}{(x+y)^{D-\lambda_1-\lambda_2-\lambda_3}}\,, \end{aligned} \tag{453}$$

where x, y and z are Feynman parameters. We change variables according to

$$x=\lambda x_1\sqrt{\frac{(-r^2)}{q^2}}\,,\qquad y=\lambda x_2\sqrt{\frac{(-r^2)}{q^2}}\,,\qquad z=\lambda\,, \tag{454}$$

which simplifies the integral to

$$I_\perp(r^2,q^2) = i\pi^{\frac{D}{2}}\,\frac{\Gamma(\lambda_1+\lambda_2+\lambda_3-\frac{D}{2})}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}\frac{I'}{(q^2)^{\lambda_1+\lambda_2+\frac{\lambda_3-D}{2}}(-r^2)^{\frac{\lambda_3}{2}}}\,, \tag{455}$$

where I' is an integral which does not depend on q^2 nor on r^2,

$$I' = \int_0^\infty dx_1\,x_1^{\lambda_1-1}\int_0^\infty dx_2\,x_2^{\lambda_2-1}\,\frac{(1+x_1x_2)^{\frac{D}{2}-\lambda_1-\lambda_2-\lambda_3}}{(x_1+x_2)^{D-\lambda_1-\lambda_2-\lambda_3}}\,. \tag{456}$$

This can be evaluated performing the substitution $x_1 = uv$ and $x_2 = \frac{u}{v}$, which factorizes it into two integrals of the type

$$\int_0^\infty u^\alpha (1+u^2)^\beta du = \frac{\Gamma(-\frac{\alpha+2\beta+1}{2})\Gamma(\frac{\alpha+1}{2})}{2\Gamma(-\beta)}, \tag{457}$$

conveniently evaluated letting $x = \frac{1}{1+u^2}$.

In conclusion, for the two orthogonal vectors $q \cdot r = 0$, we obtain (cf. [105, eq. (A.27)])

$$\begin{aligned} I_\perp(q^2, r^2) &= \int \frac{d^D \ell}{(\ell^2 - i\epsilon)^{\lambda_1}((q-\ell)^2 - i\epsilon)^{\lambda_2}(2r\cdot\ell - i\epsilon)^{\lambda_3}} \\ &= i\pi^{\frac{D}{2}} \frac{\Gamma(\lambda_1+\lambda_2+\frac{\lambda_3-D}{2})\Gamma(\frac{\lambda_3}{2})}{2\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(D-\lambda_1-\lambda_2-\lambda_3)} \frac{\Gamma(\frac{D-\lambda_3}{2}-\lambda_1)\Gamma(\frac{D-\lambda_3}{2}-\lambda_2)}{(q^2)^{\lambda_1+\lambda_2+\frac{\lambda_3-D}{2}}(-r^2)^{\frac{\lambda_3}{2}}}. \end{aligned} \tag{458}$$

Variants of the above integral that can be evaluated in a similar fashion, still under the assumption $q \cdot r = 0$, are

$$\begin{aligned} I^{(1)}_\perp(q^2, r^2) &= \int \frac{(q\cdot\ell)\, d^D \ell}{(\ell^2 - i\epsilon)^{\lambda_1}((q-\ell)^2 - i\epsilon)^{\lambda_2}(2r\cdot\ell - i\epsilon)^{\lambda_3}} \\ &= i\pi^{\frac{D}{2}} \frac{\Gamma(\lambda_1+\lambda_2+\frac{\lambda_3-D}{2})\Gamma(\frac{\lambda_3}{2})}{2\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(D-\lambda_1-\lambda_2-\lambda_3+1)} \frac{\Gamma(\frac{D-\lambda_3}{2}-\lambda_2)\Gamma(\frac{D-\lambda_3}{2}-\lambda_1+1)}{(q^2)^{\lambda_1+\lambda_2+\frac{\lambda_3-D}{2}-1}(-r^2)^{\frac{\lambda_3}{2}}} \end{aligned} \tag{459}$$

and

$$\begin{aligned} I^{(2)}_\perp(q^2, r^2) &= \int \frac{(q\cdot\ell)^2\, d^D \ell}{(\ell^2 - i\epsilon)^{\lambda_1}((q-\ell)^2 - i\epsilon)^{\lambda_2}(2r\cdot\ell - i\epsilon)^{\lambda_3}} \\ &= i\pi^{\frac{D}{2}} \frac{\Gamma(\lambda_1+\lambda_2+\frac{\lambda_3-D}{2})\Gamma(\frac{\lambda_3}{2})}{2\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(D-\lambda_1-\lambda_2-\lambda_3+2)} \frac{\Gamma(\frac{D-\lambda_3}{2}-\lambda_1+1)\Gamma(\frac{D-\lambda_3}{2}-\lambda_2+1)}{(q^2)^{\lambda_1+\lambda_2+\frac{\lambda_3-D}{2}-2}(-r^2)^{\frac{\lambda_3}{2}}} \\ &\times \left(\frac{D-2\lambda_1-\lambda_3+2}{D-2\lambda_2-\lambda_3} - \frac{1}{D+2-2\lambda_1-2\lambda_2-\lambda_3} \right). \end{aligned} \tag{460}$$

Part III

SCATTERING AMPLITUDE METHODS

9

SCALAR-GRAVITON AMPLITUDES

Using the CHY-formalism and its extension to a double cover we provide covariant expressions for tree-level amplitudes with two massive scalar legs and an arbitrary number of gravitons in D dimensions. Using unitarity methods, such amplitudes are needed inputs for the computation of post-Newtonian and post-Minkowskian expansions in classical general relativity.

INTRODUCTION

Recently it has been realized that modern methods for amplitude computations at loop level may provide a powerful new way to compute post-Newtonian and post-Minkowskian expansions in classical general relativity [10, 12, 14, 15, 23, 24, 32, 37, 40, 47, 72, 73]. This builds on the observation that the quantum mechanical scattering matrix for matter interacting gravitationally contains classical pieces at arbitrarily high order in the loop expansion [21,31,46] and the fact that the sought-for long-distance contributions are non-analytic in the exchanged momentum [44,45], thus making them straightforwardly accessible through unitarity cuts. All needed contributions being classical, one would not expect it to be necessary to regularize the loops dimensionally. However, since infrared 'super-classical' (see, e.g., ref. [21]) terms appear at intermediate steps it is nevertheless convenient to use dimensional regularization.

For the scattering of two massive objects at large distances the needed tree-level amplitudes are those of two massive scalars and, at n-loop order, $(n+1)$ on-shell gravitons. Using the Kawai-Lewellen-Tye (KLT) relations [106–109] these can conveniently be constructed from the corresponding amplitudes with the $(n+1)$ gravitons replaced by gluons, amplitudes that are given in the literature on the basis of recursion relations [110, 111] in four space-time dimensions, using the spinor-helicity formalism. More recently, Naculich [112] has suggested an alternative and more direct method for the computation of such amplitudes based on the Cachazo-He-Yuan (CHY) formalism [113, 114]. One advantage of using the CHY-formalism is that it immediately provides the amplitudes 'covariantly', in terms of general polarization tensors for the gravitons, and hence not restricted to four space-time dimensions.

From a practical point of view, it suffices to evaluate amplitudes with two scalar legs and $(n+1)$ gluons and subsequently turning them into scalar-graviton amplitudes by KLT-squaring. This is our approach here. One key point of the present calculation is the computation of a factorized expression for the amplitudes of two massive scalars coupled to Yang-Mills theory, expressing them as sums over lower-point amplitudes which are combinations of scalar-gluon amplitudes and pure gluon amplitudes. Each of these has one gluon leg off-shell and an associated polarization vector of both transverse and longitudinal components. In this way, we can iteratively construct amplitudes of an arbitrarily high order. Crucial for this factorized form is the insight gained from the double-cover version [115–119] of the CHY-formalism. This double-cover description naturally splits amplitudes into two lower-point amplitudes, each with one leg off-shell. These vector currents, contracted with polarization vectors, are glued together by the polarization sum. A subtlety here is the contribution from longitudinal modes that need to be dealt with carefully. Useful relations that short-cut the evaluations of some of the color-ordered amplitudes needed for the recursive evaluation of higher n-point amplitudes are provided by simple identities [120–122] among these partly massive amplitudes.

The outline of this paper is as follows. In sections 2 and 3 we show how to compute amplitudes with two scalars and n gluons using different methods. In section 4 we briefly discuss the straightforward application of Kawai-Lewellen-Tye relations to replace the gluons with gravitons. Some technical details and a proof of an important theorem regarding vanishing longitudinal contributions are provided in appendices.

PRELUDE: TWO MASSIVE SCALARS AND n GLUONS

We first present a simple way to obtain explicit expressions for the scattering amplitudes of two massive scalars and n gluons. Since our method is based on the CHY approach, we give a very brief review of this formalism. We then apply the factorization method developed in [119] to obtain, up to six-point, analytical expressions for the scattering of gluons where two of them, suitably defined, are massive. Next, we turn the two massive gluons into massive scalars, thus providing the scattering amplitudes for two massive scalars and in principle any number of massless gluons.

Massive Yang-Mills Amplitudes

We start by presenting a simple recursive formula that computes pure Yang-Mills amplitudes with up to three massive gluons. The method we will use was developed by one of us in a different context [115, 119]. We shall show explicit expressions up to six points but it is straightforward to extend the method to any higher number of external legs. In the following, we will denote massive particles with the capital letter "P_α" and the massless ones with the lower-case letter "k_a". Unless otherwise mentioned we will work under the assumption of implicit momentum conservation,

$$K_1 + K_2 + \cdots + K_n = 0 . \tag{461}$$

Let us first recall how to extend the CHY approach to the massive case following the method of Naculich [112]. We have $\{P_1, ..., P_i\}$ as momenta of the massive particles ($P_\alpha^2 \neq 0$) and $\{k_{i+1}, ..., k_n\}$ as momenta the massless gluons ($k_a^2 = 0$). A generic momentum vector is thus $K_A \in \{P_1, ..., P_i, k_{i+1}, ..., k_n\}$. We define as well

$$\begin{aligned} P_{AB...D} &\equiv K_A + K_B + \cdots + K_D, \\ P_{A:A+j} &\equiv K_A + K_{A+1} + \cdots + K_{A+j} . \end{aligned} \tag{462}$$

The modified CHY scattering equations are then given by

$$S_A = \sum_{\substack{B=1 \\ B\neq A}}^{n} \frac{2\,K_A \cdot K_B + 2\,\Delta_{AB}}{\sigma_{AB}} = 0 , \quad A = 1, 2, ..., n , \tag{463}$$

where the matrix Δ_{AB} is still to be determined. In order to guarantee $SL(2, \mathbb{C})$ invariance, i.e., $\sum_{A=1}^{n} \sigma_A^m S_A = 0$ for $m = 0, 1, 2$, the matrix Δ_{AB} must be symmetric, $\Delta_{AB} = \Delta_{BA}$, and it must satisfy the conditions

$$\begin{aligned} &\sum_{\substack{\beta=1 \\ \beta\neq a}}^{i} \Delta_{\alpha\beta} + \sum_{b=i+1}^{n} \Delta_{\alpha b} = P_\alpha^2 , \quad \alpha = 1, \ldots, i, \\ &\sum_{\beta=1}^{i} \Delta_{a\beta} + \sum_{\substack{b=i+1 \\ b\neq a}}^{n} \Delta_{ab} = 0 , \qquad a = i+1, \ldots, n. \end{aligned} \tag{464}$$

Since we are interested in at most up to three massive gluons of momenta $\{P_1, P_2, P_3\}$, it is sufficient to consider only $\Delta_{12}, \Delta_{13}, \Delta_{23}$. Therefore, we therefore have the simple conditions

$$\begin{aligned} \Delta_{12} + \Delta_{13} &= P_1^2\,, \\ \Delta_{12} + \Delta_{23} &= P_2^2\,, \\ \Delta_{13} + \Delta_{23} &= P_3^2\,, \end{aligned} \tag{465}$$

that have a unique solution given by

$$\Delta_{12} = \frac{P_1^2 + P_2^2 - P_3^4}{2}\,, \quad \Delta_{13} = \frac{P_1^2 - P_2^2 + P_3^4}{2}\,, \quad \Delta_{23} = \frac{-P_1^2 + P_2^2 + P_3^4}{2}\,. \tag{466}$$

When two masses are degenerate, e.g. , $P_1^2 = P_2^2 \neq 0$ and $P_3^2 = 0$, it is straightforward to see from (466) that $\Delta_{12} = P_1^2$ and $\Delta_{13} = \Delta_{23} = 0$, which, not surprisingly, is in agreement with the one-loop scattering equations formulated in refs. [116, 123–125]. On the other hand, when only one of the legs is massive, e.g. $P_1^2 \neq 0$ and $P_2^2 = P_3^2 = 0$, then $\Delta_{12} = P_1^2/2$, $\Delta_{13} = P_1^2/2$ and $\Delta_{23} = -P_1^2/2$, i.e., in order to describe one massive particle it is necessary to use at least three Δ_{AB} parameters.

After having described the massive scattering equations let us now remind that the CHY prescription for color ordered amplitudes of the scattering of gluons at tree-level is given by [112, 113, 126]

$$A_n(P_1, ..., P_i, i+1, ..., n) = \int d\mu_n\, PT(1, 2, ..., n) \times Pf'\Psi_n\,, \tag{467}$$

where $d\mu_n$ is the usual CHY measure

$$d\mu_n = (\sigma_{jk}\sigma_{kl}\sigma_{lj}) \prod_{\substack{A=1 \\ A\neq j,k,l}}^{n} d\sigma_A \times (\sigma_{mr}\sigma_{rs}\sigma_{sm}) \prod_{\substack{B=1 \\ B\neq m,r,s}}^{n} \delta(S_B)\,, \tag{468}$$

and $PT(1,...,n)$ and $Pf'\Psi_n$ are the usual Parke-Taylor and reduced Pfaffian factors

$$PT(1,...,n) \equiv \frac{1}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}}\,, \quad Pf'\Psi_n \equiv \frac{(-1)^{A+B}}{\sigma_{AB}}\, Pf[(\Psi_n)^{AB}_{AB}]\,. \tag{469}$$

The $2n \times 2n$ matrix, Ψ_n, is defined as

$$\Psi_n \equiv \begin{pmatrix} \mathsf{A} & -\mathsf{C}^T \\ \mathsf{C} & \mathsf{B} \end{pmatrix}, \tag{470}$$

with,

$$\mathsf{A}_{AB} \equiv \begin{cases} \dfrac{2\,K_A\cdot K_B + 2\,\Delta_{AB}}{\sigma_{AB}}\,, \\ 0\,, \end{cases} \quad \mathsf{B}_{AB} \equiv \begin{cases} \dfrac{\epsilon_A\cdot\epsilon_A}{\sigma_{AB}} & A\neq B\,, \\ 0 & A = B\,, \end{cases} \tag{471}$$

and

$$\mathsf{C}_{AB} \equiv \begin{cases} \dfrac{\sqrt{2}\,\epsilon_A\cdot K_B}{\sigma_{AB}}\,, & A\neq B\,, \\ -\displaystyle\sum_{\substack{C=1 \\ C\neq A}}^{n} \dfrac{\sqrt{2}\,\epsilon_A\cdot K_C}{\sigma_{AC}}\,, & A = B\,. \end{cases} \tag{472}$$

The matrix, $(\Psi_n)^{AB}_{AB}$, denotes the reduced matrix obtained by removing the rows and columns A, B from Ψ_n, where $1 \leq A < B \leq n$.

Since we are interested in the case of at most three massive particles of momenta $\{P_1, P_2, P_3\}$ we can avoid dealing with the Δ_{AB}-matrix in the scattering equations altogether by choosing the labels $\{j,k,l\}$ and $\{m,r,s\}$ in (468) to match with the massive ones, i.e., $\{j,k,l\} = \{m,r,s\} = \{1,2,3\}$.

It is useful to recall that the reduced Pfaffian ($Pf'\Psi_n = \frac{(-1)^{A+B}}{\sigma_{AB}} Pf[(\Psi_n)^{AB}_{AB}]$) is independent of the choice of A and B, and that the $SL(2,\mathbb{C})$ symmetry is guaranteed by the transversality of the external polarization vectors, $(\epsilon_C \cdot K_C) = 0$. However, we note that the terms C_{AA} and C_{BB} do not appears in the reduced matrix, $(\Psi_n)^{AB}_{AB}$. It follows that the transversality conditions on ϵ_A and ϵ_B are not needed to obtain an integrand invariant under the action of $SL(2,\mathbb{C})$ [127]. We can therefore consistently define the integral with these two legs being off mass-shell and with arbitrary polarization vectors for $(\epsilon_A \cdot K_A) \neq 0$ and $(\epsilon_B \cdot K_B) \neq 0$. We now use the double-cover method ref. [119] to obtain compact recursive expressions for these massive and/or off-shell scattering amplitudes as defined above. The results clearly reduce to the usual expressions when all external legs are massless and on-shell.

First, let us consider the basic building block of three legs. We take all three particles to be massive and choose the polarization vectors ϵ_1 and ϵ_2 as not necessarily transverse so that we do not impose $(\epsilon_1 \cdot P_1) = 0 = (\epsilon_2 \cdot P_2)$. We are going to denote with a ***bold*** source in the amplitude (as in. [112, 113, 119]), e.g.

$$A_n(\ldots, \boldsymbol{P}_\alpha, \ldots, \boldsymbol{P}_\beta, \ldots)\,, \tag{473}$$

the rows/columns that are removed from its reduced Pfaffian. In the above amplitude the reduced Pfaffian is given by, $Pf'\Psi_n = \frac{(-1)^{\alpha+\beta}}{\sigma_{\alpha\beta}} Pf[(\Psi_n)^{\alpha\beta}_{\alpha\beta}]$. Particles P_α and

P_β can thus be off-shell, so that $(\epsilon_\alpha \cdot P_\alpha) \neq 0$ and $(\epsilon_\beta \cdot P_\beta) \neq 0$.

Therefore, using the CHY prescription given in (467) one has

$$\begin{aligned} A_3(\boldsymbol{P}_1, \boldsymbol{P}_2, P_3) &= (\sigma_{12}\,\sigma_{23}\,\sigma_{23})^2\, PT(1,2,3)\, \frac{(-1)}{\sigma_{12}}\, Pf \begin{bmatrix} 0 & -\frac{\epsilon_1\cdot\sqrt{2}\,P_3}{\sigma_{13}} & -\frac{\epsilon_2\cdot\sqrt{2}\,P_3}{\sigma_{23}} & -\mathsf{C}_{33} \\ \frac{\epsilon_1\cdot\sqrt{2}\,P_3}{\sigma_{13}} & 0 & \frac{\epsilon_1\cdot\epsilon_2}{\sigma_{12}} & \frac{\epsilon_1\cdot\epsilon_3}{\sigma_{13}} \\ \frac{\epsilon_2\cdot\sqrt{2}\,P_3}{\sigma_{23}} & \frac{\epsilon_2\cdot\epsilon_1}{\sigma_{21}} & 0 & \frac{\epsilon_2\cdot\epsilon_3}{\sigma_{23}} \\ \mathsf{C}_{33} & \frac{\epsilon_3\cdot\epsilon_1}{\sigma_{31}} & \frac{\epsilon_3\cdot\epsilon_2}{\sigma_{32}} & 0 \end{bmatrix} \\ &= \sqrt{2}\,\{(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot P_1) - (\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot P_3) + (\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot P_3)\}\,, \end{aligned} \tag{474}$$

where we have used

$$\mathsf{C}_{33} = -\sqrt{2}\left(\frac{\epsilon_3\cdot P_1}{\sigma_{31}} + \frac{\epsilon_3\cdot P_2}{\sigma_{32}}\right) = \sqrt{2}\,(\epsilon_3\cdot P_1)\times\frac{\sigma_{12}}{\sigma_{31}\,\sigma_{23}}\,, \tag{475}$$

due to the momentum conservation constraint $P_1 + P_2 + P_3 = 0$ and the transversality condition $(\epsilon_3 \cdot P_3) = 0$. Although the amplitude itself is independent of the choice of rows/columns that are removed in the Pfaffian, the intermediate expressions do depend on the choice and we have therefore introduced a notation where we indicate which rows and columns are removed.

We consider next a computation with three massive gluons of momenta $\{P_1, P_2, P_3\}$ and one massless gluon of momentum $\{k_4\}$. Using the factorization method described in [117, 119], this four-point calculation can be expressed in terms of the $A_3(\boldsymbol{P}_a, \boldsymbol{P}_b, P_c)$ building-blocks,

$$
\begin{aligned}
&A_4(\boldsymbol{P}_1, P_2, \boldsymbol{P}_3, 4) \\
&= \sum_M \left[\frac{A_3(\boldsymbol{P}_{34}^{\epsilon^M}, \boldsymbol{P}_1, P_2)\, A_3(\boldsymbol{P}_{12}^{\epsilon^M}, \boldsymbol{P}_3, 4)}{s_{P_3 4}} + \frac{A_3(\boldsymbol{P}_1, \boldsymbol{P}_{23}^{\epsilon^M}, 4)\, A_3(\boldsymbol{P}_3, \boldsymbol{P}_{41}^{\epsilon^M}, P_2)}{s_{4 P_1}} \right] \\
&\qquad\qquad -2 \sum_L \left[\frac{A_3(\boldsymbol{P}_{13}^{\epsilon^L}, \boldsymbol{P}_2, 4)}{s_{P_2 4}} \times A_3(\boldsymbol{P}_{24}^{\epsilon^L}, \boldsymbol{P}_1, P_3) \right], \qquad (476)
\end{aligned}
$$

where the notation $P_i^{\epsilon^M}$ ($P_i^{\epsilon^L}$) means the particle with momentum P_i has as polarization vector ϵ_i^M (ϵ_i^L). The sums over the polarizations are given by the relations

$$\sum_M \epsilon_i^{M\,\mu} \epsilon_j^{M\,\nu} = \eta^{\mu\nu}\,, \qquad (477)$$

$$\sum_L \epsilon_i^{L\,\mu} \epsilon_j^{L\,\nu} = \frac{P_i^\mu\, P_j^\nu}{P_i \cdot P_j + P_1^2 - P_3^2}\,. \qquad (478)$$

The unusual normalization factor of the longitudinal modes is precisely what is needed to recover the correct four-point amplitude [118, 128]. The polarization vectors of all massive on-shell legs of course still satisfy $\epsilon_i \cdot P_i = 0$. Using that condition it is easy to see that the last term in (476) evaluates to

$$-2 \sum_L \left[\frac{A_3(\boldsymbol{P}_{13}^{\epsilon^L}, \boldsymbol{P}_2, 4)}{s_{P_2 4}} \times A_3(\boldsymbol{P}_{24}^{\epsilon^L}, \boldsymbol{P}_1, P_3) \right] = (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4)\,. \qquad (479)$$

The full four-point amplitude is thus remarkably simple.

Finally, in order to calculate higher-point amplitudes we will also need $A_4(\boldsymbol{P}_1, \boldsymbol{P}_2, P_3, 4)$. Using the BCJ-like identity [120–122],

$$s_{4P_3}\, PT(3,4,1,2) + s_{4P_{13}}\, PT(3,1,4,2) = 0\,,$$

it is straightforward to deduce

$$A_4(\boldsymbol{P}_1, \boldsymbol{P}_2, P_3, 4) = -\left(1 + \frac{s_{4P_1}}{s_{4P_3}}\right) \times A_4(\boldsymbol{P}_1, P_3, \boldsymbol{P}_2, 4)\,. \tag{480}$$

The calculation of higher-point amplitudes with massive gluons now proceeds recursively. We illustrate a few cases in the appendix.

Turning massive gluons into scalars

Now, using the prescriptions of Naculich [112] and Cachazo, He, and Yuan [114] we can compute the amplitudes of interest which also involve massive scalar legs. The basic idea is to consider the massive gluon theory in one extra dimension (i.e., in $D+1$ dimensions) with "polarizations" and momenta of massive scalars chosen to be

$$\left.\begin{array}{ll} P_1^\mu = (\vec{p}_1, 0), & \epsilon_1^\mu = (\vec{0}, 1) \\ P_n^\mu = (\vec{p}_n, 0), & \epsilon_n^\mu = (\vec{0}, 1) \end{array}\right\} \quad \text{Massive scalars} \;\; (P_1^2 = P_n^2 = m^2)\,, \tag{481}$$

$$\left.\begin{array}{ll} k_a^\mu = (\vec{k}_a, 0), & \epsilon_a^\mu = (\vec{\epsilon}_a, 0) \end{array}\right\} \quad \text{Massless gluons} \;\; (a = 2, \ldots, n-2, \; \textit{and} \; k_a^2 = 0)\,.$$

In this set-up all external particles satisfy $P_i \cdot \epsilon_i = 0 = k_a \cdot \epsilon_a$ and, additionally, it is easy to see that $\Delta_{1n} = \Delta_{n1} = m^2$ in Naculich's notation as a consequence of equation (466). The CHY prescription for the scattering of two massive scalars with $n-2$ gluons can thus be written as

$$A_n(1_\varphi, 2_g, ..., (n-1)_g, n_\varphi) = \int d\mu_n \; PT(1,2,...,n) \times Pf'\Psi_n \Big|_{\epsilon_1, \epsilon_n = (\vec{0},1)}, \tag{482}$$

where the massive scattering equations, the reduced Pfaffian and the measure as defined above. It is useful to note that these ordered amplitudes are invariant under cyclic permutations, i.e.,

$$A_n(1,2,...,\boldsymbol{P}_\alpha,...,\boldsymbol{P}_\beta,...,n) = A_n(n,1,2,...,\boldsymbol{P}_\alpha,...,\boldsymbol{P}_\beta,...,n-1)\,, \tag{483}$$

and also satisfy

$$A_n(1,2,...,\boldsymbol{P}_\alpha,...,\boldsymbol{P}_\beta,...,n-1,n) = (-1)^n A_n(n,n-1,...,\boldsymbol{P}_\beta,...,\boldsymbol{P}_\alpha,...,2,1)\,.$$

As a first step we note that when P_1 and P_2 are associated with scalar legs the three-point amplitude reads

$$A_3(\boldsymbol{P}_1, \boldsymbol{P}_2, P_3)\Big|_{\epsilon_1, \epsilon_2 = (\vec{0},1)} = \sqrt{2}\,(\epsilon_3 \cdot P_1)\,. \tag{484}$$

We illustrate our method by evaluating the four-point function of two massive scalars and two gluons. Using the above conditions and the cyclicity property (483) we immediately infer this amplitude from eqs. (476):

$$A_4(1_\varphi, 2_g, 3_g, 4_\varphi) = A_4(\boldsymbol{P}_4, P_1, \boldsymbol{2}, 3)\Big|_{\epsilon_1, \epsilon_4 = (\vec{0}, 1)} \tag{485}$$
$$= \sum_M \left[\frac{A_3(\boldsymbol{P}_{23}^{\epsilon^M}, \boldsymbol{P}_4^{\varphi}, P_1^{\varphi})\, A_3(\boldsymbol{P}_{41}^{\epsilon^M}, \boldsymbol{2}, 3)}{s_{23}} + \frac{A_3(\boldsymbol{P}_4^{\varphi}, \boldsymbol{P}_{12}^{\epsilon^M}, 3)\, A_3(\boldsymbol{2}, \boldsymbol{P}_{34}^{\epsilon^M}, P_1^{\varphi})}{s_{3P_4}} \right],$$

where the superscript (or subscript) "φ" refers to one of the massive scalars. We note that the term (479) does not contribute at all, (we shall return to this point later).

Remark: Since $\epsilon_1^\mu = \epsilon_4^\mu = (\vec{0}, 1)$ the contraction relation for the second term in (485),

$$\sum_M (\epsilon_{P_{34}}^M \cdot \epsilon_1)(\epsilon_{P_{12}}^M \cdot V) = (\epsilon_1 \cdot V)\,, \tag{486}$$

is non-vanishing only when V^μ has a non-zero projection on ϵ_4. Therefore, it is equivalent to choosing $\epsilon_{P_{34}}^{M\,\mu} = \epsilon_{P_{12}}^{M\,\mu} = (\vec{0}, 1)$, i.e., the internal lines corresponding to momenta P_{12} and P_{34} turn out to be propagating scalars as expected due to current conservation. In other words,

$$\sum_M \frac{A_3(\boldsymbol{P}_4^{\varphi}, \boldsymbol{P}_{12}^{\epsilon^M}, 3)\, A_3(\boldsymbol{2}, \boldsymbol{P}_{34}^{\epsilon^M}, P_1^{\varphi})}{s_{3P_4}} = \frac{A_3(\boldsymbol{P}_4^{\varphi}, \boldsymbol{P}_{12}^{\varphi}, 3)\, A_3(\boldsymbol{2}, \boldsymbol{P}_{34}^{\varphi}, P_1^{\varphi})}{s_{3P_4}}\,. \tag{487}$$

The same phenomenon occurs for higher n-point amplitudes. Let us now introduce some convenient notation:

$$F_A^{\mu\nu} \equiv K_A^\mu \, \epsilon_A^\nu - K_A^\nu \, \epsilon_A^\mu \,,$$
$$(V\, W)_{ab} \equiv V_a^\mu \, \eta_{\mu\nu} \, W_b^\nu \,, \tag{488}$$

as well as

$$(V\,F \ldots F\, W)_{aA_1 \ldots A_j b} \equiv V_a^\mu \, \eta_{\mu\gamma} \, F_{A_1}^{\gamma\nu} \, \eta_{\nu\sigma} \, F_{A_2}^{\sigma\alpha} \cdots F_{A_j}^{\rho\delta} \, \eta_{\delta\beta} W_b^\beta \,, \tag{489}$$
$$s_{A_1 A_2 \ldots A_j} \equiv (K_{A_1} + K_{A_2} + \cdots + K_{A_j})^2 - (K_{A_1}^2 + K_{A_2}^2 + \cdots + K_{A_j}^2) \,,$$

where V_a^μ and W_b^ν are two generic vectors. From (477) and (474) it is straightforward to compute

$$\sum_M \frac{A_3(\boldsymbol{P}_{23}^{\epsilon^M}, \boldsymbol{P}_4^\varphi, P_1^\varphi) \, A_3(\boldsymbol{P}_{41}^{\epsilon^M}, \boldsymbol{2}, 3)}{s_{23}} = \frac{2\,(\epsilon P)_{21}\,(\epsilon k)_{32} - 2\,(\epsilon F P)_{231}}{s_{23}} \,, \tag{490}$$

as well as

$$\frac{A_3(\boldsymbol{P}_4^\varphi, \boldsymbol{P}_{12}^\varphi, 3) \, A_3(\boldsymbol{2}, \boldsymbol{P}_{34}^\varphi, P_1^\varphi)}{s_{3P_4}} = -\frac{2(\epsilon P)_{21}\,(\epsilon P)_{34}}{s_{P_1 2}} \,. \tag{491}$$

The four-point covariant amplitude of two massive scalars and two gluons is thus given by the simple expression

$$A_4(1_\varphi, 2_g, 3_g, 4_\varphi) = \frac{2\,(\epsilon P)_{21}\,(\epsilon k)_{32} - 2\,(\epsilon F P)_{231}}{s_{23}} - \frac{2(\epsilon P)_{21}\,(\epsilon P)_{34}}{s_{P_1 2}} \,. \tag{492}$$

Specializing to four dimensions, this is in agreement with the result found in the literature on the basis of the spinor-helicity formalism [111].

In an analogous way, the five-point amplitude becomes

$$
\begin{aligned}
&A_5(1_\varphi, 2_g, 3_g, 4_g, 5_\varphi) = A_5(\boldsymbol{P}_5, P_1, \mathbf{2}, 3, 4)\Big|_{\epsilon_1, \epsilon_5 = (\vec{0}, 1)} \\
&= (-1) \times \sum_M \left[\frac{A_3(\boldsymbol{P}_{4:1}^{\epsilon^M}, \mathbf{2}, 3)\, A_4(\boldsymbol{P}_5^{\varphi}, P_1^{\varphi}, \boldsymbol{P}_{23}^{\epsilon^M}, 4)}{s_{23}} + \frac{A_3(\boldsymbol{P}_{2:4}^{\epsilon^M}, \boldsymbol{P}_5^{\varphi}, P_1^{\varphi})\, A_4(\boldsymbol{P}_{51}^{\epsilon^M}, \mathbf{2}, 3, 4)}{s_{234}} \right] \\
&\qquad + (-1) \times \frac{A_3(\mathbf{2}, \boldsymbol{P}_{3:5}^{\varphi}, P_1^{\varphi}) \times A_4(\boldsymbol{P}_5^{\varphi}, \boldsymbol{P}_{12}^{\varphi}, 3, 4)}{s_{34P_5}}\,,
\end{aligned}
\tag{493}
$$

where eq. (513) has been used. As in the four-point case, the purely longitudinal contributions vanish on account of the orthogonality conditions for the polarization vectors associated with external scalar legs, $(\epsilon_1 \cdot \epsilon_3) = (\epsilon_5 \cdot \epsilon_2) = 0$. In appendix 9, we prove the vanishing of these longitudinal contributions for any number of external gluons.

Applying the identity (477) and using (474), (476) and (480) we finally find an explicit covariant expression for $A_5(1_\varphi, 2_g, 3_g, 4_g, 5_\varphi)$:

$$
\begin{aligned}
\frac{A_5(1_\varphi,2_g,3_g,4_g,5_\varphi)}{\sqrt{2}} &= (\epsilon P)_{21}\frac{(\epsilon\epsilon)_{34}\, s_{34} - 2\,(\epsilon k)_{34}\,(\epsilon P)_{45} + 2\,(\epsilon k)_{43}\,(\epsilon P)_{35}}{s_{P_1 2}\, s_{34}} \\
&+ \frac{s_{23}[(\epsilon\epsilon)_{23}\,(\epsilon k)_{43} - (\epsilon\epsilon)_{34}\,(\epsilon P)_{21} - (\epsilon F \epsilon)_{342}] - (\epsilon\epsilon)_{23}\,(\epsilon P)_{41}\, s_{34}}{s_{23}\, s_{34}} + \frac{(\epsilon\epsilon)_{34}\,(\epsilon P)_{21}\, s_{4P_5}}{s_{P_1 2} s_{34}} \\
&+ 2(\epsilon P)_{21}(\epsilon P)_{45}\frac{(\epsilon P)_{31} + (\epsilon k)_{32}}{s_{P_1 2}\, s_{4P_5}} + (\epsilon P)_{45}\frac{2(\epsilon k)_{21}(\epsilon P)_{32} - 2(\epsilon k)_{23}(\epsilon P)_{31} - (\epsilon\epsilon)_{23} s_{23}}{s_{23}\, s_{4P_5}} \\
&+ \frac{s_{P_1 2}(\epsilon F \epsilon)_{243} + (\epsilon\epsilon)_{23}\,(\epsilon k)_{43}\, s_{P_1 2} - (\epsilon\epsilon)_{34}\,[(\epsilon P)_{21}\, s_{P_1 4} + (\epsilon P)_{25}\, s_{P_1 4} - (\epsilon P)_{21}\, s_{23}]}{s_{34}\, s_{234}} \\
&+ \frac{(\epsilon\epsilon)_{23}(\epsilon P)_{41} s_{34}}{s_{23} s_{234}} - \frac{(\epsilon\epsilon)_{23}\,(\epsilon P)_{45}\, s_{P_1 2}}{s_{23} s_{4P_5}} + \frac{(\epsilon\epsilon)_{34}(\epsilon P)_{21} - (\epsilon\epsilon)_{24}(\epsilon P)_{31} + (\epsilon\epsilon)_{23}(\epsilon P)_{41}}{s_{234}} \\
&+ \frac{(\epsilon\epsilon)_{34}(\epsilon k)_{23} s_{P_1 4} - (\epsilon\epsilon)_{24}(\epsilon k)_{32} s_{P_1 4} + (\epsilon\epsilon)_{23}[(\epsilon k)_{42} s_{P_1 4} - (\epsilon P)_{41} s_{P_1 2} - (\epsilon P)_{45} s_{P_1 2}]}{s_{23} s_{234}} \\
&+ 2\,\frac{(\epsilon P)_{25}(\epsilon P)_{31}(\epsilon k)_{42} + (\epsilon P)_{45}[(\epsilon P)_{21}(\epsilon k)_{32} + (\epsilon P)_{25}(\epsilon P)_{31} + (\epsilon P)_{21}(\epsilon P)_{31}] - (1 \leftrightarrow 5)}{s_{34}\, s_{234}} \\
&+ 2\,\frac{(\epsilon k)_{23}(\epsilon P)_{35}(\epsilon P)_{41} + (\epsilon k)_{32}(\epsilon P)_{45}(\epsilon P)_{21} - (1 \leftrightarrow 5)}{s_{23} s_{234}}\,.
\end{aligned}
\tag{494}
$$

Specializing to four dimensions, this matches the spinor-helicity result provided in [111]. We note that this five-point amplitude $A_5(1_\varphi, 2_g, 3_g, 4_g, 5_\varphi)$ can also be computed using eq. (516) so that, alternatively,

$$
A_5(1_\varphi, 2_g, 3_g, 4_g, 5_\varphi) = A_5(\mathbf{3}, 4, \boldsymbol{P}_5, P_1, 2)\Big|_{\epsilon_1, \epsilon_5 = (\vec{0}, 1)}\,. \tag{495}
$$

It is now straightforward to move to any higher number of points, recursively. Using the result of the appendix we find the six-point amplitude

$$
\begin{aligned}
&A_6(1_\varphi, 2_g, 3_g, 4_g, 5_g, 6_\varphi) = A_6(\boldsymbol{P}_6, P_1, \mathbf{2}, 3, 4, 5)\Big|_{\epsilon_1,\epsilon_6=(\vec{0},1)} = \\
&\frac{A_3(\mathbf{2}, \boldsymbol{P}^{\varphi}_{3:6}, P^{\varphi}_1)\, A_5(\boldsymbol{P}^{\varphi}_6, \boldsymbol{P}^{\varphi}_{12}, 3, 4, 5)}{s_{345P_6}} + \sum_M \left[\frac{A_3(\boldsymbol{P}^{\epsilon_M}_{2:5}, \boldsymbol{P}^{\varphi}_6, P^{\varphi}_1)\, A_5(\boldsymbol{P}^{\epsilon_M}_{61}, \mathbf{2}, 3, 4, 5)}{s_{2345}} \right. \\
&\left. + \frac{A_3(\boldsymbol{P}^{\epsilon_M}_{4:1}, \mathbf{2}, 3)\, A_5(\boldsymbol{P}^{\varphi}_6, P^{\varphi}_1, \boldsymbol{P}^{\epsilon_M}_{23}, 4, 5)}{s_{23}} - \frac{A_4(\boldsymbol{P}^{\varphi}_6, P^{\varphi}_1, \boldsymbol{P}^{\epsilon_M}_{2:4}, 5)\, A_4(\boldsymbol{P}^{\epsilon_M}_{5:1}, \mathbf{2}, 3, 4)}{s_{234}} \right] .
\end{aligned}
\tag{496}
$$

The longitudinal pieces have again cancelled, leaving a simple sum over intermediate polarizations and a very intuitive recursive structure, as shown. Although the explicit evaluation of this expression is straightforward, the resulting expression is lengthy and we do not reproduce it here.

KLEISS-KUIJF DECOMPOSITION

While the method described in the previous section is straightforward and immediately generalizable to any number of gluons n, we wish to point out that an alternative track based on an expansion with analytically computed BCJ-numerators is of comparable simplicity. The trick is to compute the scattering of two massive scalar fields with massless gluons (eventually gravitons) by decomposing the reduced Pffafian in terms of a Kleiss-Kuijf (KK) basis [129] by using the Bern-Carrasco-Johansson (BCJ) numerators [130] for Yang-Mills theory. This useful technique was developed in [126, 131, 132].

Let us recall that our first main goal is to calculate the amplitude

$$A_n(1_\varphi, 2_g, 3_g, \ldots, (n-1)_g, n_\varphi)\,, \tag{497}$$

and in order to avoid dealing with the terms

$$\frac{2P_1\cdot P_n + 2\Delta_{1n}}{\sigma_{1n}}\quad,\quad \frac{2P_n\cdot P_1 + 2\Delta_{n1}}{\sigma_{n1}}\,,$$

we remove from the reduced Pfaffian the rows/columns $\{1, n\}$. Thus, we are looking for the following KK expansion

$$\frac{(-1)^{1+n}}{\sigma_{1n}} \times Pf\left[(\Psi)^{1n}_{1n}\right] = \sum_{\rho\in S_{n-2}} N_{(1,\rho(2,\cdots,n-1),n)}\, PT(1,\rho(2,\cdots,n-1),n)\,, \tag{498}$$

where $N_{(1,\rho(2,\cdots,n-1),n)}$ are the BCJ Yang-Mills numerators and S_{n-2} is the group of the $(n-2)!$ permutations of the set $\{2,3,\cdots,n-1\}$. As argued in [133], since the Pfaffian $Pf\left[(\Psi)^{1n}_{1n}\right]$, is independent of the products $P_1\cdot P_n$ and $\Delta_{1n} = \Delta_{n1} = P_1^2$ the algorithm proposed in ref. [132] can be applied. Therefore, the scattering between two massive scalar fields with (n-2) gluons can be written as

$$A_n(1_\varphi, 2_g, ..., n_\varphi) = \sum_{\rho\in S_{n-2}} m_n[12\cdots n|1\rho(2\cdots n-1)n] \times N_{(1,\rho(2,\cdots,n-1),n)}\Big|_{\epsilon_1,\epsilon_n=(\vec{0},1)}\,, \tag{499}$$

where the BCJ numerators $N_{(1,\rho(2,\cdots,n-1),n)}$ can be obtained by the algorithm developed in [132] and where $m_n[\alpha|\beta]$ is defined by

$$m_n[\alpha_1\cdots\alpha_n|\beta_1\cdots\beta_n] \equiv \int d\mu_n\, PT(\alpha_1,\ldots,\alpha_n)\times PT(\beta_1,\ldots,\beta_n)\,, \tag{500}$$

with the massive measure $d\mu_n$ as given in (468).

To illustrate, let us consider the four-point amplitude $A_4(1_\varphi, 2_g, 3_g, 4_\varphi)$. From (499) we arrive at

$$\begin{aligned} A_4(1_\varphi, 2_g, 3_g, 4_\varphi) &= N_{(1,2,3,4)}\Big|_{\epsilon_1,\epsilon_4=(\vec{0},1)} \int d\mu_4\, PT(1,2,3,4) \times PT(1,2,3,4) \\ &+ N_{(1,3,2,4)}\Big|_{\epsilon_1,\epsilon_4=(\vec{0},1)} \int d\mu_4\, PT(1,2,3,4) \times PT(1,3,2,4) \end{aligned} \tag{501}$$

Now applying the method of ref. [132], the BCJ numerators are readily found to be given by

$$N_{(1,2,3,4)}\Big|_{\epsilon_1,\epsilon_4=(\vec{0},1)} = -2\,(\epsilon P)_{21}\,(\epsilon P)_{34}, \quad N_{(1,3,2,4)}\Big|_{\epsilon_1,\epsilon_4=(\vec{0},1)} = 2\,(\epsilon P)_{21}(\epsilon P)_{31} + 2\,(\epsilon F P)_{231}\,, \tag{502}$$

where we have fixed the reference ordering to be $(1,2,3,4)$. The massive integrals obtained in (501) are straightforward to do using the Λ-algorithm [115]. We find

$$\begin{aligned} m_4[1234|1234] &= \int d\mu_4\, PT(1,2,3,4) \times PT(1,2,3,4) = \frac{1}{s_{P_1 2}} + \frac{1}{s_{23}}\,, \\ m_4[1234|1324] &= \int d\mu_4\, PT(1,2,3,4) \times PT(1,3,2,4) = -\frac{1}{s_{23}}\,. \end{aligned} \tag{503}$$

For the four-point amplitude we therefore get

$$A_4(1_\varphi, 2_g, 3_g, 4_\varphi) = \frac{-2\,(\epsilon P)_{21}\,(\epsilon P)_{34}}{s_{P_1 2}} + \frac{-2\,(\epsilon P)_{21}\,(\epsilon P)_{34} - (2\,(\epsilon P)_{21}(\epsilon P)_{31} + 2\,(\epsilon F P)_{231})}{s_{23}}\,,$$

which agrees with the result we found in equation (492).

Explicit BCJ numerators at five points

This method easily generalizes. For two massive scalar legs and three gluons we need to evaluate

$$A_5(1_\varphi, 2_g, 3_g, 4_g, 5_\varphi) = \sum_{\rho\in S_3} m_5[12345|1\rho(234)5] \times N_{(1,\rho(2,3,4),5)}\Big|_{\epsilon_1,\epsilon_5=(\vec{0},1)} . \quad (504)$$

The six BCJ-numerators $N_{(1,\rho(2,3,4),5)}\Big|_{\epsilon_1,\epsilon_5=(\vec{0},1)}$ are given by

$$\begin{aligned}
\left(\frac{1}{\sqrt{2}}\right) N_{(1,2,3,4,5)}\Big|_{\epsilon_1,\epsilon_5=(\vec{0},1)} &= -2\,(\epsilon P)_{21}\,(\epsilon P)_{45}\,\left[(\epsilon P)_{31} + (\epsilon k)_{32}\right] , \qquad (505)\\
\left(\frac{1}{\sqrt{2}}\right) N_{(1,4,2,3,5)}\Big|_{\epsilon_1,\epsilon_5=(\vec{0},1)} &= -2\,(\epsilon P)_{25}\,(\epsilon P)_{41}\,[(\epsilon P)_{21} + (\epsilon k)_{24}] - (\epsilon\epsilon)_{34}(\epsilon P)_{21}s_{P_14}\\
&\quad + (\epsilon\epsilon)_{24}(\epsilon P)_{35}s_{P_14} ,\\
\left(\frac{1}{\sqrt{2}}\right) N_{(1,3,4,2,5)}\Big|_{\epsilon_1,\epsilon_5=(\vec{0},1)} &= 2(\epsilon P)_{25}(\epsilon P)_{31}\,[(\epsilon k)_{42} + (\epsilon P)_{45}] + (\epsilon\epsilon)_{23}\,[(\epsilon k)_{42} + (\epsilon P)_{45}]\,s_{P_13}\\
&\quad - (\epsilon\epsilon)_{43}(\epsilon k)_{24}s_{P_13} - (\epsilon\epsilon)_{24}(\epsilon P)_{31}s_{P_134} - (\epsilon\epsilon)_{24}\,[(\epsilon k)_{32} + (\epsilon P)_{35}]\,s_{P_13} ,\\
\left(\frac{1}{\sqrt{2}}\right) N_{(1,2,4,3,5)}\Big|_{\epsilon_1,\epsilon_5=(\vec{0},1)} &= -2(\epsilon P)_{21}(\epsilon P)_{35}\,[(\epsilon P)_{41} + (\epsilon k)_{42}] - (\epsilon\epsilon)_{34}(\epsilon P)_{21}(s_{P_14} + s_{24}) ,\\
\left(\frac{1}{\sqrt{2}}\right) N_{(1,3,2,4,5)}\Big|_{\epsilon_1,\epsilon_5=(\vec{0},1)} &= -2(\epsilon P)_{31}(\epsilon P)_{45}\,[(\epsilon P)_{21} + (\epsilon k)_{23}] + (\epsilon\epsilon)_{23}(\epsilon P)_{45}s_{P_13} ,\\
\left(\frac{1}{\sqrt{2}}\right) N_{(1,4,3,2,5)}\Big|_{\epsilon_1,\epsilon_5=(\vec{0},1)} &= 2(\epsilon P)_{41}(\epsilon P)_{25}\,[(\epsilon k)_{32} + (\epsilon P)_{35}] - (\epsilon\epsilon)_{34}\,[(\epsilon P)_{21} + (\epsilon k)_{23}]\,s_{P_14}\\
&\quad + (\epsilon\epsilon)_{24}\,[(\epsilon k)_{32} + (\epsilon P)_{35}]\,s_{P_14} - (\epsilon\epsilon)_{23}\,[(\epsilon k)_{42} + (\epsilon P)_{45}]\,s_{P_14} - (\epsilon\epsilon)_{23}(\epsilon P)_{41}s_{P_134} ,
\end{aligned}$$

where we have fixed the reference ordering to be $(1, 2, 3, 4, 5)$.

Using again the Λ-algorithm [115], it is straightforward to compute, with two massive legs,

$$
\begin{aligned}
m_5[12345|12345] &= \frac{1}{s_{234}\, s_{34}} + \frac{1}{s_{P_1 2}\, s_{34}} + \frac{1}{s_{23}\, s_{4P_5}} + \frac{1}{s_{4P_5}\, s_{P_1 2}} + \frac{1}{s_{23}\, s_{234}}, \qquad (506)\\
m_5[12345|14235] &= -\frac{1}{s_{23}\, s_{234}},\\
m_5[12345|13425] &= -\frac{1}{s_{234}\, s_{34}},\\
m_5[12345|12435] &= -\frac{1}{s_{234}\, s_{34}} - \frac{1}{s_{P_1 2}\, s_{34}},\\
m_5[12345|13245] &= -\frac{1}{s_{23}\, s_{4P_5}} - \frac{1}{s_{23}\, s_{234}},\\
m_5[12345|14325] &= \frac{1}{s_{234}\, s_{34}} + \frac{1}{s_{23}\, s_{234}}.
\end{aligned}
$$

After substituting eqs. (505) and (506) into (504) one can check that the result matches the one given in eq. (494).

This method does have the drawback for n large that the number of BCJ numerators grow in a factorial way. For instance, to compute the six and seven-point amplitudes one needs to calculate $4! = 24$ and $5! = 120$ numerators, respectively.

TWO MASSIVE SCALARS AND GRAVITONS

In the previous sections, we have shown different methods for efficient evaluation of scattering amplitudes of two massive scalar fields and $(n-2)$ gluons. Staying within the CHY-framework as in section 9, one could similarly express the amplitude of the scattering among two massive scalars (φ) and gravitons (h_a) through [126, 134, 135],

$$\mathcal{M}_n(2\varphi,(n-2)h) = \int d\mu_n \; Pf'\Psi_n\Big|_{\epsilon_1,\epsilon_n=(\vec{0},1)} \times Pf'\Psi_n\Big|_{\epsilon_1,\epsilon_n=(\vec{0},1)}, \tag{507}$$

where the gravitons are identified as, $h_a^{\mu\nu} \equiv \epsilon_a^\mu \epsilon_a^\nu$ and using the same massive measure defined in (468). Similarly, one can use a KK-decomposition analogous to what we explained above for the case of gluons in (499), and write

$$\mathcal{M}_n(2\varphi,(n-2)h) = \sum_{\substack{\rho\in S_{n-2}\\ \delta\in S_{n-2}}} N_{(1,\rho,n)}\Big|_{\epsilon_1,\epsilon_n=(\vec{0},1)} \times m_n[1\,\rho\,n|1\,\delta\,n] \times N_{(1,\delta,n)}\Big|_{\epsilon_1,\epsilon_n=(\vec{0},1)}. \tag{508}$$

However, by using the Kawai-Lewellen-Tye (KLT) [106] relations at the amplitude level, it seems much more straightforward to find the scattering between two massive scalar and $(n-2)$ gravitons by use of the momentum kernel [108, 109], *i.e*,

$$\mathcal{M}_n(2\varphi,(n-2)h) = (-1)^{n-3} \sum_{\substack{\alpha\in S_{n-3}\\ \beta\in S_{n-3}}} A_n(1_\varphi,\alpha_g,(n-1)_g,n_\varphi) \times \mathcal{S}[\alpha|\beta]_{k_1} \times \tag{509}$$

$$A_n(n_\varphi,(n-1)_g,\beta_g,1_\varphi)\,. \tag{510}$$

Here A_n is an amplitude of two massive scalars and $(n-2)$ gluons as defined in (482), and the momentum kernel $\mathcal{S}[\alpha|\beta]$ is

$$\mathcal{S}[i_1,\ldots,i_k|j_1,\ldots,j_k]_{k_1} \equiv \prod_{t=1}^{k}\left(s_{i_t 1} + \sum_{q>t}^{k}\Theta(i_t,i_q)s_{i_t,i_q}\right), \tag{511}$$

where Θ is the step function. For instance, for the four-point amplitude we immediately get

$$\mathcal{M}_4(2\varphi, 2h) = A_4(1_\varphi, 3_g, 2_g, 4_\varphi) \times \mathcal{S}[3|2] \times A_4(1_\varphi, 2_g, 3_g, 4_\varphi)\,, \tag{512}$$

where $\mathcal{S}[3|2] = -s_{23}$, thus using the result found in (492), one has

$$\begin{aligned}
\mathcal{M}_4(2\varphi, 2h) &= \left[\frac{2\,(\epsilon P)_{31}(\epsilon k)_{23}\, s_{P_1 3} - 2\,(\epsilon F P)_{321}\, s_{P_1 3} - 2\,(\epsilon P)_{31}(\epsilon P)_{24}\, s_{23}}{s_{P_1 3}\, s_{23}}\right] \times (-s_{23}) \\
&\quad \times \left[\frac{2\,(\epsilon P)_{21}(\epsilon k)_{32}\, s_{P_1 2} - 2\,(\epsilon F P)_{231}\, s_{P_1 2} - 2\,(\epsilon P)_{21}(\epsilon P)_{34}\, s_{23}}{s_{P_1 2}\, s_{23}}\right] \\
&= -\frac{\left[2\,(\epsilon P)_{24}(\epsilon P)_{31} s_{P_1 2} + 2\,(\epsilon P)_{21}(\epsilon P)_{34} s_{2P_4} + (\epsilon\epsilon)_{23} s_{P_1 2} s_{2P_4}\right]^2}{s_{P_1 2}\, s_{23}\, s_{2P_4}}\,,
\end{aligned}$$

which is the correct 4-point amplitude. Higher order amplitudes follow by KLT-squaring analogously.

CONCLUSION

We have presented different methods to compute the tree-level scattering amplitudes of two massive scalars and an in principle arbitrary number of gravitons in D-dimensions. These are the tree-level amplitudes needed to obtain the classical two-body scattering of two massive objects without spin in general relativity through the use of unitarity. The most economical method appears to be the one based on a new set of recursive relations that can be derived from the so-called Λ-algorithm (or double cover) in the CHY-formalism. In this method one first defines an extension of scattering amplitudes where one external leg is taken off-shell (defining, effectively, a current in the case of Yang-Mills theory) and then glues off-shell legs together by an appropriate polarization sum. We have proven a particular simplification in comparison to the pure Yang-Mills case when the amplitude contains two massive scalar legs: a sum over longitudinal polarizations cancels exactly. The resulting amplitude relations for two massive scalars and any number of on-shell gluons thus becomes surprisingly simple.

Although a similar technique can be used to compute amplitudes of two massive scalars with an arbitrary number of gravitons we have found it economical to simply use KLT-squaring in order to obtain these. Again, they are then provided in D-dimensions and with arbitrary polarization tensors.

We have checked our general recursive formula up to six points with existing expressions in the literature for the case $D = 4$, always finding complete agreement. An interesting observation is the possibility of establishing a new on-shell set of recursion relations for these amplitudes based on BCFW-recursion combined with the double-cover analysis of the Λ-algorithm. This will be discussed elsewhere.

APPENDIX

HIGHER-POINT YANG-MILLS AMPLITUDES WITH MASSIVE LEGS

Here we present details of the main ingredients that go into the computation of the massive five-point gluon amplitudes: $A_5(\boldsymbol{P}_1, P_2, \boldsymbol{P}_3, 4, 5)$, $A_5(\boldsymbol{P}_1, 2, \boldsymbol{P}_3, P_4, 5)$ and $A_5(\boldsymbol{P}_1, \boldsymbol{P}_2, P_3, 4, 5)$.

Using the method developed in [119], the factorization decomposition of $A_5(\boldsymbol{P}_1, P_2, \boldsymbol{P}_3, 4, 5)$ becomes

$$\begin{aligned} A_5(\boldsymbol{P}_1, P_2, \boldsymbol{P}_3, 4, 5) = (-1) \times \sum_M \Bigg\{ & \frac{A_3(\boldsymbol{P}_{5:2}^{\epsilon^M}, \boldsymbol{P}_3, 4) \times A_4(\boldsymbol{P}_1, P_2, \boldsymbol{P}_{34}^{\epsilon^M}, 5)}{s_{P_34}} + \\ & \frac{A_3(\boldsymbol{P}_{3:5}^{\epsilon^M}, \boldsymbol{P}_1, P_2) \times A_4(\boldsymbol{P}_{12}^{\epsilon^M}, \boldsymbol{P}_3, 4, 5)}{s_{P_345}} + \frac{A_3(\boldsymbol{P}_3, \boldsymbol{P}_{4:1}^{\epsilon^M}, P_2) \times A_4(\boldsymbol{P}_1, \boldsymbol{P}_{23}^{\epsilon^M}, 4, 5)}{s_{45P_1}} \Bigg\} \\ +2 \sum_L \Bigg\{ & \frac{A_3(\boldsymbol{P}_{513}^{\epsilon^L}, \boldsymbol{P}_2, 4)}{s_{P_24}} \times A_4(\boldsymbol{P}_1, P_3, \boldsymbol{P}_{24}^{\epsilon^L}, 5) + \frac{A_4(\boldsymbol{P}_{13}^{\epsilon^L}, \boldsymbol{P}_2, 4, 5)}{s_{P_245}} \times \\ & A_3(\boldsymbol{P}_{245}^{\epsilon^L}, \boldsymbol{P}_1, P_3) \Bigg\}, \end{aligned} \tag{513}$$

where we have written $A_5(\boldsymbol{P}_1, P_2, \boldsymbol{P}_3, 4, 5)$ in terms of the smaller amplitudes, $A_3(\boldsymbol{P}_a, \boldsymbol{P}_b, P_c)$, $A_4(\boldsymbol{P}_a, P_b, \boldsymbol{P}_c, d)$ and $A_4(\boldsymbol{P}_a, \boldsymbol{P}_b, P_c, d)$. As in the four-point case, we must use the identities in (477) and (478).

It is straightforward to find the longitudinal contributions,

$$-2\sum_L\left[\frac{A_3(\boldsymbol{P}^{\epsilon^L}_{513},\boldsymbol{P}_2,4)}{s_{P_24}}\times A_4(\boldsymbol{P}_1,P_3,\boldsymbol{P}^{\epsilon^L}_{24},5)\right]=$$
$$-\sqrt{2}\,(\epsilon_2\cdot\epsilon_4)\times\frac{s_{5P_{13}}(\epsilon_1\cdot F_5\cdot\epsilon_3)+2(\epsilon_1\cdot\epsilon_3)(P_1\cdot F_5\cdot P_3)}{s_{5P_{13}}\,s_{P_15}},\tag{514}$$

and

$$-2\sum_L\left[\frac{A_4(\boldsymbol{P}^{\epsilon^L}_{13},\boldsymbol{P}_2,4,5)}{s_{P_245}}\times A_3(\boldsymbol{P}^{\epsilon^L}_{245},\boldsymbol{P}_1,P_3)\right]=$$
$$-\sqrt{2}\,(\epsilon_1\cdot\epsilon_3)\times\frac{s_{5P_{24}}\,(\epsilon_2\cdot F_5\cdot\epsilon_4)+2(\epsilon_2\cdot\epsilon_4)(P_2\cdot F_5\cdot k_4)}{s_{5P_{24}}\,s_{45}}.\tag{515}$$

Similarly, the amplitude $A_5(\boldsymbol{P}_1,2,\boldsymbol{P}_3,P_4,5)$ is factorized according to

$$A_5(\boldsymbol{P}_1,2,\boldsymbol{P}_3,P_4,5)=(-1)\times\sum_M\left\{\frac{A_3(\boldsymbol{P}^{\epsilon^M}_{5:2},\boldsymbol{P}_3,P_4)\times A_4(\boldsymbol{P}_1,2,\boldsymbol{P}^{\epsilon^M}_{34},5)}{s_{P_3P_4}+2\,\Delta_{34}}+\right.$$
$$\left.\frac{A_3(\boldsymbol{P}^{\epsilon^M}_{3:5},\boldsymbol{P}_1,2)\times A_4(\boldsymbol{P}^{\epsilon^M}_{12},\boldsymbol{P}_3,P_4,5)}{s_{P_3P_45}+2\,\Delta_{34}}+\frac{A_3(\boldsymbol{P}_3,\boldsymbol{P}^{\epsilon^M}_{4:1},2)\times A_4(\boldsymbol{P}_1,\boldsymbol{P}^{\epsilon^M}_{23},P_4,5)}{s_{P_45P_1}+2\,\Delta_{14}}\right\}$$
$$+2\sum_L\left\{\frac{A_3(\boldsymbol{P}^{\epsilon^L}_{513},\boldsymbol{2},P_4)}{s_{2P_4}+2\,\Delta_{14}+2\,\Delta_{34}}\,A_4(\boldsymbol{P}_1,P_3,\boldsymbol{P}^{\epsilon^L}_{24},5)+\frac{A_4(\boldsymbol{P}^{\epsilon^L}_{13},\boldsymbol{2},P_4,5)}{s_{2P_45}+2\,\Delta_{14}+2\,\Delta_{34}}\times\right.$$
$$\left.A_3(\boldsymbol{P}^{\epsilon^L}_{245},\boldsymbol{P}_1,P_3)\right\},\tag{516}$$

where we have used (466), namely

$$\Delta_{13}=\frac{P_1^2+P_3^2-P_4^2}{2},\quad \Delta_{14}=\frac{P_1^2-P_3^2+P_4^2}{2},\quad \Delta_{34}=\frac{-P_1^2+P_3^2+P_4^2}{2}.\tag{517}$$

We also recall that $\Delta_{14} + \Delta_{34} = P_4^2$. It is now straightforward to verify that the longitudinal contributions in (516) are identical to the one evaluated above, i.e.,

$$-2\sum_L \left[\frac{A_3(\boldsymbol{P}_{513}^{\epsilon^L}, \mathbf{2}, P_4)}{s_{P_4 2} + 2\,\Delta_{14} + 2\,\Delta_{34}} \times A_4(\boldsymbol{P}_1, P_3, \boldsymbol{P}_{24}^{\epsilon^L}, 5) \right] = (514)$$

and

$$-2\sum_L \left[\frac{A_4(\boldsymbol{P}_{13}^{\epsilon^L}, \mathbf{2}, P_4, 5)}{s_{P_4 25} + 2\,\Delta_{14} + 2\,\Delta_{34}} \times A_3(\boldsymbol{P}_{245}^{\epsilon^L}, \boldsymbol{P}_1, P_3) \right] = (515).$$

Finally, we are able to expand the amplitude, $A_5(\boldsymbol{P}_1, \boldsymbol{P}_2, P_3, P_4, 5)$, in terms of the two previous ones, $A_5(\boldsymbol{P}_a, P_b, \boldsymbol{P}_c, d, e)$ and $A_5(\boldsymbol{P}_a, b, \boldsymbol{P}_c, P_d, , e)$. Using the BCJ-like identity [120, 121],

$$s_{P_3 45}\, PT(3,4,5,1,2) + (s_{P_3 45} + s_{P_1 5})\, PT(3,4,1,5,2) + (s_{P_3 45} + s_{P_1 P_{45}})\, PT(3,1,4,5,2) = 0,$$

the amplitude, $A_5(\boldsymbol{P}_1, \boldsymbol{P}_2, P_3, 4, 5)$, turns into

$$\begin{aligned} &A_5(\boldsymbol{P}_1, \boldsymbol{P}_2, P_3, 4, 5) = \\ &-\left(1 + \frac{s_{P_1 P_{45}}}{s_{P_3 45}}\right) A_5(\boldsymbol{P}_2, P_3, \boldsymbol{P}_1, 4, 5) - \left(1 + \frac{s_{P_1 2 6}}{s_{P_3 45}}\right) A_5(\boldsymbol{P}_1, 5, \boldsymbol{P}_2, P_3, 4). \end{aligned} \tag{518}$$

Finally, let us show how to compute the six-point amplitude, $A_6(\boldsymbol{P}_1, P_2, \boldsymbol{P}_3, 4, 5, 6)$. The factorization decomposition of $A_6(\boldsymbol{P}_1, P_2, \boldsymbol{P}_3, 4, 5, 6)$ is given by

$$
\begin{aligned}
A_6(\boldsymbol{P}_1, P_2, \boldsymbol{P}_3, 4, 5, 6) = \sum_M \Bigg\{ & \frac{A_3(\boldsymbol{P}_3, \boldsymbol{P}^{\epsilon^M}_{4:1}, P_2) \times A_5(\boldsymbol{P}_1, \boldsymbol{P}^{\epsilon^M}_{23}, 4, 5, 6)}{s_{456P_1}} + \\
& \frac{A_3(\boldsymbol{P}^{\epsilon^M}_{3:6}, \boldsymbol{P}_1, P_2) \times A_5(\boldsymbol{P}^{\epsilon^M}_{12}, \boldsymbol{P}_3, 4, 5, 6)}{s_{P_3456}} + \frac{A_3(\boldsymbol{P}^{\epsilon^M}_{5:2}, \boldsymbol{P}_3, 4) \times A_5(\boldsymbol{P}_1, P_2, \boldsymbol{P}^{\epsilon^M}_{34}, 5, 6)}{s_{P_34}} \\
& - \frac{A_4(\boldsymbol{P}_1, P_2, \boldsymbol{P}^{\epsilon^M}_{3:5}, 6) \times A_4(\boldsymbol{P}^{\epsilon^M}_{6:2}, \boldsymbol{P}_3, 4, 5)}{s_{P_345}} \Bigg\} \\
-2 \sum_L \Bigg\{ & \frac{A_3(\boldsymbol{P}^{\epsilon^L}_{3:6}, \boldsymbol{P}_1, P_2) \times A_5(\boldsymbol{P}^{\epsilon^L}_{12}, \boldsymbol{P}_3, 4, 5, 6)}{s_{P_3456}} + \frac{A_3(\boldsymbol{P}^{\epsilon^L}_{5:2}, \boldsymbol{P}_3, 4) \times A_5(\boldsymbol{P}_1, P_2, \boldsymbol{P}^{\epsilon^L}_{34}, 5, 6)}{s_{P_34}} \\
& - \frac{A_4(\boldsymbol{P}_1, P_2, \boldsymbol{P}^{\epsilon^L}_{3:5}, 6) \times A_4(\boldsymbol{P}^{\epsilon^L}_{6:2}, \boldsymbol{P}_3, 4, 5)}{s_{P_345}} \Bigg\} \Bigg|_{\substack{2 \leftrightarrow 3 \\ (\epsilon_\alpha \cdot P_A) = -(\epsilon_\alpha \cdot P_{\bar{A}}) \\ (\epsilon_\alpha \cdot P_\alpha) = 0}} ,
\end{aligned}
\tag{519}
$$

where $2 \leftrightarrow 3$ means the changing of the two labels, $\alpha = 1, 3$ and $P_{\bar{A}}$ is the complement of P_A (by the momentum conservation condition, $P_A + P_{\bar{A}} = 0$). For example, P_A is given by P_{2456}, P_{24} and P_{245} in the last three term in (519), respectively, therefore, $P_{\bar{A}}$ is P_{13}, P_{1356} and P_{136}. Additionally, the identities in (477) and (478) must be used in the above factorization expansion.

LONGITUDINAL CONTRIBUTIONS

As we have observed in all special cases worked out in this paper, the longitudinal contributions to the factorized amplitudes with massive scalars always vanish identically. In this section we prove this important fact in all generality.

Let us consider a Yang-Mills n-point amplitude with up to three massive legs $A_n(\boldsymbol{P}_n, P_1, \boldsymbol{P}_2, 3, ..., n-1)$. Applying the factorization method, a generic longitudinal contribution is given by

$$\sum_L \frac{A_{(n-i)+2}(\boldsymbol{P}_n, P_2, \boldsymbol{P}^{\epsilon^L}_{134...i}, i+1, ..., n-1) \times A_i(\boldsymbol{P}^{\epsilon^L}_{i+1...n2}, \boldsymbol{P}_1, 3, 4, ..., i)}{s_{P_1 34...i}}, \tag{520}$$

where the the two amplitudes are sewn together by the rule

$$\sum_L \epsilon_i^{L\,\mu} \epsilon_j^{L\,\nu} = \frac{P_i^\mu \, P_j^\nu}{P_i \cdot P_j + P_n^2 - P_2^2} \,. \tag{521}$$

We can now show the following:

Under the condition $\epsilon_1 = \epsilon_n = (\vec{0}, 1)$, the amplitudes, $A_{(n-i)+2}(\boldsymbol{P}_n, P_2, \boldsymbol{P}^{\epsilon^L}_{134...i}, i+1, ..., n-1)$ and $A_i(\boldsymbol{P}^{\epsilon^L}_{i+1...n2}, \boldsymbol{P}_1, 3, 4, ..., i)$ vanish identically.

The proof of this proposition is straightforward.

Let us consider the amplitude, $A_i(\boldsymbol{P}^{\epsilon^L}_{i+1...n2}, \boldsymbol{P}_1, 3, 4, ..., i)$. From the notation introduced in the main text, it is clear that the reduced matrix $\left[(\Psi_i)^{P_{i+1...n2}P_1}_{P_{i+1...n2}P_1}\right]$, has a row (column) given by the vector

$$\left(\frac{\epsilon_1 \cdot k_2}{\sigma_{12}}, \; \cdots, \; \frac{\epsilon_1 \cdot k_i}{\sigma_{1i}}, \; \frac{\epsilon_1 \cdot \epsilon^L_{i+1...n2}}{\sigma_{1P_{i+1...n2}}}, \; 0, \; \frac{\epsilon_1 \cdot \epsilon_2}{\sigma_{12}}, \cdots, \; \frac{\epsilon_1 \cdot \epsilon_i}{\sigma_{1i}} \right) \Bigg|_{\epsilon_1, \epsilon_n = (\vec{0}, 1)} \tag{522}$$

$$= \left(0, \; \cdots, \; 0, \; \frac{\epsilon_1 \cdot \epsilon^L_{i+1...n2}}{\sigma_{1P_{i+1...n2}}}, \; 0, \cdots, \; 0 \right).$$

Since $\epsilon^L_{i+1...n2}$ is proportional to $P_{i+1...n2} = k_{i+1} + \cdots + k_n + P_2$ it follows that

$$\frac{\epsilon_1 \cdot \epsilon^L_{i+1...n2}}{\sigma_{1P_{i+1...n2}}} \propto \frac{\epsilon_1 \cdot P_{i+1...n2}}{\sigma_{1P_{i+1...n2}}} = 0, \tag{523}$$

using that $(\epsilon_1 \cdot k_i) = 0$. Therefore, $A_i(\boldsymbol{P}^{\epsilon^L}_{i+1...n2}, \boldsymbol{P}_1, 3, 4, ..., i)$ vanishes trivially for, $\epsilon_1 = \epsilon_n = (\vec{0}, 1)$. The essential property that makes these contributions vanish is the fact that the polarization vectors associated with what become massive scalars live in a higher dimensional space with no overlap with the momenta of the D-dimensional space.

The same argument works for $A_{(n-i)+2}(\boldsymbol{P}_n, P_2, \boldsymbol{P}^{\epsilon^L}_{134...i}, i+1, ..., n-1)$.

Part IV

CLASSICAL WAVE PHYSICS

10

GRAVITATIONAL SHOCK WAVES AND SCATTERING AMPLITUDES

We study gravitational shock waves using scattering amplitude techniques. After first reviewing the derivation in General Relativity as an ultrarelativistic boost of a Schwarzschild solution, we provide an alternative derivation by exploiting a novel relation between scattering amplitudes and solutions to Einstein field equations. We prove that gravitational shock waves arise from the classical part of a three point function with two massless scalars and a graviton. The region where radiation is localized has a distributional profile and it is now recovered in a natural way, thus bypassing the introduction of singular coordinate transformations as used in General Relativity. The computation is easily generalized to arbitrary dimensions and we show how the exactness of the classical solution follows from the absence of classical contributions at higher loops.

A classical double copy between gravitational and electromagnetic shock waves is also provided and for a spinning source, using the exponential form of three point amplitudes, we infer a remarkable relation between gravitational shock waves and spinning ones, also known as gyratons. Using this property, we infer a family of exact solutions describing gravitational shock waves with spin. We then compute the phase shift of a particle in a background of shock waves finding agreement with an earlier computation by Amati, Ciafaloni and Veneziano for particles in the high energy limit. Applied to a gyraton, it provides a result for the scattering angle to all orders in spin.

INTRODUCTION

The study of General Relativity using scattering amplitudes techniques is in a golden era thanks to state of the art computations for interacting black holes and the possibility to relate classical gravitational observables to scattering amplitudes [10, 12, 21, 24, 32]. Nowadays, the literature is vast and includes different approaches to deal with post-Newtonian and post-Minkwoskian black holes [15, 18, 19, 47, 85, 136, 137], including also classical spin effects for Kerr black holes [65–70, 72, 87, 88, 138–144] and tidal effects [145, 146]. The existence of this literature might seem surprising, given that we are trading General Relativity for an even more complicated quantum gravitational system and its classical limit. However, the introduction of concepts such as unitarity and double-copy [130, 147–159] has made possible not only the computation of observables relevant for LIGO/Virgo [14] but also to unravel new structures in classical field theory, proving that quantum mechanics can help us in elucidating the essence of classical physics. In fact, the EOB approach [7], which led to accurate models of gravitational wave signals for a binary system, was inspired by these ideas [8]. Along this line, this paper describes perturbative solutions in General Relativity using the scattering amplitude approach recently developed by Kosower, Maybee and O'Connell [21]. We focus on the Aichelburg-Sexl metric describing a gravitational shock wave sourced by a massless particle [160]. Derived almost simultaneously by Aichelburg, Sexl, Penrose [161] and Bonnor [162], it has been central to our understanding of graviton dominance in high energy scattering [74], and in the past years it has been studied in different settings [75, 163–166]. As we will see, an alternative derivation is also possible using a novel relation between perturbative solutions to Einstein's field equations and scattering amplitudes. Several authors have conjectured a similar connection and in the case of a static massive source, it has led to the computation at second order in G_N of the Schwarzschild[1] and Kerr-Newman solution [11, 76, 168, 169]. However, the lack of a covariant framework has made it impossible to treat more general cases such as those described by an energy momentum tensor sourced by

1 See also [167] for a derivation up to G_N^3 using EFT methods.

massless particles with spin. This work is a step toward this direction. We start by first reviewing the original derivation by Aichelburg and Sexl of a gravitational shock wave employing an ultrarelativistic boost of a Schwarzschild solution. We then present a relation between classical solutions in General Relativity and the classical part of three point functions from quantum field theory. Using a massless particle coupled to a graviton, we derive the complete Aichelburg-Sexl metric as an exact solution to Einstein field equations. The region where the radiation is localized has a well known distributional profile and it is now recovered from the amplitude itself, bypassing the introduction of singular coordinate transformations as used in General Relativity. We also generalize the computation to arbitrary D dimensions finding agreement in the literature [170, 171] with the ultrarelativistic boost of the so called Tangherlini metric [172, 173]. We then extend the classical double copy for static black holes to gravitational shock waves, showing that their single copy is described by electromagnetic ones. For a spinning source, using the exponential form of three point amplitudes, we infer a remarkable relation between gravitational shock waves and spinning ones, also known as gyratons. From this, we obtain solutions describing spinning gravitational shock waves directly from the spinless case, avoiding the use of ultrarelativistic boosts on Kerr black holes [174]. To our knowledge, the existence of such a relation between exact solutions in General Relativity was previously unknown. Interestingly, this relation resembles the Newman-Janis algorithm [175] which provides a Kerr solution from a complex deformation of Schwarzschild, recently studied by Arkani-Hamed, Huang and O'Connell in [69]. We then compute the phase shift of a particle in a background of shock waves, finding agreement with earlier computations for particles in the high energy limit [176, 177]. Applied to a gyraton, it provides a result for the scattering angle valid to all orders in the spin.

We will work throughout in natural units and in mostly negative signature.

THE AICHELBURG-SEXL METRIC

Aichelburg and Sexl derived for the first time an exact solution to Einstein field equations describing the gravitational field generated by a massless particle [160]. Their procedure employed the use of an ultrarelativistic boost of a Schwarzschild solution, previously used by D'Eath to address the scattering of two ultrarelativistic black holes [178]. Let us review their original derivation. We start by introducing the Schwarzschild metric in isotropic coordinates [179]

$$ds^2 = \frac{(1-A)^2}{(1+A)^2}\mathrm{d}t^2 - (1+A)^4\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right) \quad , \quad A = \frac{mG_N}{2\sqrt{x^2+y^2+z^2}}\,, \tag{524}$$

and we decompose it as

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + \left[\frac{(1-A)^2}{(1+A)^2} - 1\right]\mathrm{d}t^2 - \left[(1+A)^4 - 1\right]\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)\,. \tag{525}$$

If we apply a Lorentz transformation to eq.(524) on the x -direction,

$$t=\frac{\bar{t}-v\bar{x}}{\sqrt{1-v^2}} \quad , \quad x=\frac{\bar{x}-v\bar{t}}{\sqrt{1-v^2}} \quad , \quad y=\bar{y} \quad , \quad z=\bar{z}\,, \tag{526}$$

the previous line element changes to

$$\begin{aligned} ds^2 = d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 + \left[\frac{(1-A')^2}{(1+A')^2}-1\right]\frac{(d\bar{t}-vd\bar{x})^2}{1-v^2} \\ -\left[(1+A')^4-1\right]\left(\frac{(d\bar{x}-vd\bar{t})^2}{1-v^2}+\mathrm{d}\bar{y}^2+\mathrm{d}\bar{z}^2\right)\,, \end{aligned} \tag{527}$$

$$A' = \frac{mG_N\sqrt{1-v^2}}{2\left\{(\bar{x}-v\bar{t})^2+(1-v^2)\left(\bar{y}^2+\bar{z}^2\right)\right\}^{1/2}}\,. \tag{528}$$

We write $m=p\sqrt{1-v^2}$ and expand (527-528) around $v=1$ for a fixed value of p to find

$$ds^2 = d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 - \frac{4pG_N}{|\bar{t}-\bar{x}|}(d\bar{t}-d\bar{x})^2 \quad , \quad \bar{x}\neq\bar{t}\,. \tag{529}$$

In order to include also the missing region given by $\bar{x}=\bar{t}$, Aichelburg and Sexl proposed a coordinate transformation which becomes singular in the limit for $v=1$

$$\begin{aligned} x'-vt' &= \bar{x}-v\bar{t} \\ x'+vt' &= \bar{x}+v\bar{t}-4pG_N\log\left[\sqrt{(\bar{x}-v\bar{t})^2+(1-v^2)}-(\bar{x}-\bar{t})\right]. \end{aligned} \tag{530}$$

Using the following relation

$$\lim_{v\to 1}\left[\frac{1}{\sqrt{(x'-vt')^2+(1-v^2)\rho}}-\frac{1}{\sqrt{(x'-vt')^2+(1-v^2)}}\right]=-2\delta(t'-x')\log(\rho)\,, \tag{531}$$

the line element assumes the usual form of an impulsive pp-wave

$$ds^2 = dt'^2 - dx'^2 - dy'^2 - dz'^2 + 4pG_N\delta(t'-x')\log(y'^2+z'^2)(dt'-dx')^2\,. \tag{532}$$

The latter defines a global solution given by two copies of Minkwoski space connected by a singularity along a light cone coordinate. Among the relevant properties of this solution we can notice that in going from eq.(524) to eq.(532) we have changed the algebraic type of the Weyl tensor from Petrov type D to the radiative type N [180], a property first discovered by Pirani [181]. Moreover, from the computation of the associated Einstein tensor we can infer that the energy momentum tensor is simply that of a massless particle, thus confirming the physical interpretation of the metric.

GRAVITATIONAL SHOCK WAVES FROM SCATTERING AMPLITUDES

The idea to perturbatively solve Einstein field equations using quantum field theory techniques dates back to a paper by Duff [168] where the Schwarzschild solution was derived up to G_N^2 order. After, several authors used known relations among off-shell scattering amplitudes and form factors so as to include quantum effects in the latter, confirming the same results for the classical part [169]. Both approaches require the knowledge of the Einstein-Hilbert action expanded around a fixed background which becomes intractable already after few iterations in the coupling $k = \sqrt{32\pi G_N}$. In order to have a better control on the complexity of the calculation, it would be desirable to relate scattering amplitudes directly to the metric tensor in the same way as these have been related to classical observables in [21]. To this end we start by considering a Riemannian manifold and a Minkwoskian background. We then introduce an off-shell continuation of the second quantized solution to the linearized Einstein field equations

$$\hat{h}_{\mu\nu}(x) = \frac{k}{2}\sum_{\lambda}\int d\Phi_{off}(q)\left[\epsilon^{\lambda}_{\mu\nu}(q)\hat{a}^{\lambda}_{q}e^{-iq\cdot x} + (\epsilon^{\lambda}_{\mu\nu})^{\dagger}(q)(\hat{a}^{\lambda}_{q})^{\dagger}e^{iq\cdot x}\right]. \tag{533}$$

To ensure the gauge dependence of the metric, the sum runs over longitudinal polarizations and the measure of integration used in [21] has been replaced with

$$d\Phi_{on}(q) = \frac{d^D q}{(2\pi)^D}2\pi\delta(q^2)\theta(q_0) \quad \to \quad d\Phi_{off}(q) = \frac{d^D q}{(2\pi)^D}\frac{1}{q^2}. \tag{534}$$

We have added the subscripts *on-off* to denote that every integral carrying such measure of integration will lead to an integrand with a momentum which is respectively on-shell or off-shell. In our case, eq.(534) ensures the off-shellness of the graviton and the fact we are not looking for radiative modes of the metric tensor. The $i\epsilon$ prescription is implicitly assumed. We then propose the following wave-function describing our system in absence of interactions

$$|\Psi_{\text{in}}\rangle = \int d\Phi_{on}(p)\varphi(p)\,|p\rangle \otimes |0\rangle \quad , \quad d\Phi_{on}(p) = \frac{d^D p}{(2\pi)^D}\hat{\delta}^{(+)}(p^2 - m^2)\,, \tag{535}$$

where $\hat{\delta}^{(+)}(p^2 - m^2) = 2\pi\delta(p^2 - m^2)\theta(p_0)$. The state $|0\rangle$ denotes the vacuum state of the gravitational field while $\varphi(p)$ is a proper wave-packet describing the source. We now define the metric tensor satisfying the non linear Einstein field equations as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad , \quad h_{\mu\nu}(x) = \langle\Psi_I(t)|\,\hat{h}^I_{\mu\nu}(x)\,|\Psi_I(t)\rangle\ . \tag{536}$$

The operator $\hat{h}^I_{\mu\nu}(x)$ is defined as the action of $U_{int}(+\infty, t)$ on (533), while the state $|\Psi_I(t)\rangle$ is defined as the evolution at time t of the initial state under $U_{int}(t, -\infty)$. We can now express (536) as follows

$$h_{\mu\nu}(x) = \langle\Psi_I(t)|\,U^\dagger_{\text{int}}(+\infty, t)\hat{h}_{\mu\nu}(x)U_{\text{int}}(+\infty, t)\,|\Psi_I(t)\rangle \tag{537}$$

$$= \langle_{\text{in}}\Psi|\,U^\dagger_{\text{int}}(t, -\infty)U^\dagger_{\text{int}}(+\infty, t)\hat{h}_{\mu\nu}(x)U_{\text{int}}(+\infty, t)U_{\text{int}}(t, -\infty)\,|\Psi_{\text{in}}\rangle \tag{538}$$

$$= \langle_{\text{in}}\Psi|\,S^\dagger\hat{h}_{\mu\nu}(x)S\,|\Psi_{\text{in}}\rangle\ \ , \tag{539}$$

where we have the introduced the S matrix of the system. In doing so, we have been able to relate the solution to the complete Einstein field equations with a plane wave operator, by encoding all non linearities in the S matrix alone. Using then

$S = 1 + iT$ we can expand eq.(539) neglecting for the moment terms proportional to $TT^\dagger$,

$$h_{\mu\nu}(x) = i \langle\Psi_{\rm in}| \left(\hat{h}_{\mu\nu}(x)T - T^\dagger\hat{h}_{\mu\nu}(x)\right) |\Psi_{\rm in}\rangle \ . \tag{540}$$

From which

$$h_{\mu\nu}(x) = \frac{ik}{2}\sum_\lambda \int d\Phi_{off}(q)d\Phi_{on}(p)d\Phi_{on}(p')\varphi(p)\varphi^\dagger(p')\times$$
$$[\langle p'q^\lambda| T |p\rangle \epsilon^\lambda_{\mu\nu}(q)e^{-iq\cdot x} - \langle p'| T^\dagger |pq^\lambda\rangle (\epsilon^\lambda_{\mu\nu})^\dagger(q)e^{iq\cdot x}] \tag{541}$$

$$= -k\sum_\lambda \int d\Phi_{off}(q)d\Phi_{on}(p)d\Phi_{on}(p') \,\mathrm{Im}\Big[\varphi(p)\varphi^\dagger(p') \langle p'q^\lambda| T |p\rangle \epsilon^\lambda_{\mu\nu}(q)e^{-iq\cdot x}\Big] . \tag{542}$$

Matrix elements in eq.(542) usually describe on-shell scattering amplitudes thanks to the covariant measures which have a Dirac delta in each integrated momentum. Having assumed instead an off-shell integration measure for gravitons, the term in eq.(542) won't be an on-shell scattering amplitude but an off-shell three point function given by

$$\langle p'q^\lambda| T |p\rangle = (2\pi)^D\delta^D(p-q-p') \,(\epsilon^\lambda_{\alpha\beta})^\dagger(q)\mathcal{M}^{\alpha\beta}(p,p',q) \, . \tag{543}$$

We now choose harmonic coordinates which amounts to requiring the following identity to hold

$$\sum_{\lambda} \epsilon^{\lambda}_{\mu\nu}(q)(\epsilon^{\lambda}_{\alpha\beta})^{\dagger}(q) = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta}) \equiv P_{\mu\nu\alpha\beta}\,. \tag{544}$$

Using this, eq.(542) becomes

$$h_{\mu\nu}(x) = -k \int d\Phi_{off}(q) d\Phi_{on}(p) d\Phi_{on}(p') \,\mathrm{Im}\Big[\varphi(p)\varphi^{\dagger}(p') \times (2\pi)^D \delta^D(p' + q - p)\, P_{\mu\nu\alpha\beta}\mathcal{M}^{\alpha\beta}(p,p',q)e^{-iq\cdot x}\Big]\,. \tag{545}$$

We now proceed by making explicit the integration measure for the source particle. Integrating over p' we obtain

$$h_{\mu\nu}(x) = -k \int d\Phi_{off}(q) \frac{d^D p}{(2\pi)^D}\frac{d^D p'}{(2\pi)^D}\hat{\delta}^{(+)}(p^2 - m^2)\hat{\delta}^{(+)}(p'^2 - m^2) \times \mathrm{Im}\Big[\varphi(p)\varphi^{\dagger}(p')(2\pi)^D\delta^D(p - q - p')\, P_{\mu\nu\alpha\beta}\mathcal{M}^{\alpha\beta}(p,p',q)e^{-iq\cdot x}\Big]\,, \tag{546}$$

$$h_{\mu\nu}(x) = -k \int d\Phi_{off}(q) \frac{d^D p}{(2\pi)^D}\hat{\delta}^{(+)}(p^2 - m^2)\hat{\delta}^{(+)}((p-q)^2 - m^2) \times \mathrm{Im}\Big[\varphi(p)\varphi^{\dagger}(p - q)\, P_{\mu\nu\alpha\beta}\mathcal{M}^{\alpha\beta}(p,p',q)e^{-iq\cdot x}\Big]\,, \tag{547}$$

where it is implicitly assumed that $\mathcal{M}^{\alpha\beta}(p, p', q)$ is constrained with $p' = p - q$. Making also explicit the off-shell integration measure we obtain,

$$h_{\mu\nu}(x) = -k \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2} \int \frac{d^D p}{(2\pi)^D} \hat{\delta}^{(+)}(p^2 - m^2) \hat{\delta}^{(+)}(q^2 - 2q \cdot p) \times \\ \text{Im}\left[\varphi(p) \varphi^\dagger(p-q)\, P_{\mu\nu\alpha\beta} \mathcal{M}^{\alpha\beta}(p, p', q) e^{-iq\cdot x} \right]. \quad (548)$$

For a wave-packet sharply peaked around a given momentum p_0[2] we obtain

$$h_{\mu\nu}(x) = -k \int \frac{d^D q}{(2\pi)^D} \frac{\hat{\delta}^{(+)}(q^2 - 2q \cdot p_0)}{q^2} \text{Im}[P_{\mu\nu\alpha\beta} \mathcal{M}^{\alpha\beta}(p_0, p' = p_0 - q, q) e^{-iq\cdot x}] . \quad (549)$$

We are thus left with a remarkable relation between the classical metric tensor satisfying Einstein field equation and three point functions with an external graviton, valid both for massive and massless sources

$$h_{\mu\nu}(x) = -k \int \frac{d^D q}{(2\pi)^D} \frac{\hat{\delta}^{(+)}(q^2 - 2q \cdot p_0)}{q^2} \text{Im}[P_{\mu\nu\alpha\beta} \mathcal{M}^{\alpha\beta}(p_0, p' = p_0 - q, q) e^{-iq\cdot x}] . \quad (550)$$

Let us consider the massless case. Taking advantage of this covariant relation, we can proceed to explore which space-time corresponds to a three point function with an off-shell graviton and a massless source. Based on what has been discussed in the previous section, this should correspond to a gravitational shock wave. We start

2 For further details, see [21], Section 4.

at tree level from the interaction of a graviton with a massless source

$$\mathcal{M}^{\mu\nu}(p_1, p_2) = \frac{ik}{2}\left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \eta^{\mu\nu} p_1 \cdot p_2\right), \tag{551}$$

$$P_{\mu\nu\alpha\beta}\mathcal{M}^{\alpha\beta}(p, q) = \frac{ik}{2}\left[2p_\mu p_\nu - p_\mu q_\nu - p_\nu q_\mu + \eta_{\mu\nu} p \cdot q\right], \tag{552}$$

where in the last equation we have expressed the whole contributions in terms of p^μ and q^μ, being the former the incoming momenta. The whole metric tensor depends on only two functions

$$h_{\mu\nu}(x) = -\frac{k^2}{2}\left[2p_\mu p_\nu \Theta(x) - p_\mu \Theta_\nu(x) - p_\nu \Theta_\mu(x) + \eta_{\mu\nu} p^\alpha \Theta_\alpha(x)\right], \tag{553}$$

$$\Theta(x) = \int \frac{d^D q}{(2\pi)^D} \hat{\delta}^{(+)}(q^2 - 2q \cdot p) \frac{\cos(q \cdot x)}{q^2}, \tag{554}$$

$$\Theta_\mu(x) = \int \frac{d^D q}{(2\pi)^D} \hat{\delta}^{(+)}(q^2 - 2q \cdot p) \frac{q_\mu}{q^2} \cos(q \cdot x). \tag{555}$$

In the classical limit we implement the limit for small q by considering the integration domain where $p \gg q$. This amounts to disregarding the Heaviside theta in eq.(550) as well as the q^2 term in its Dirac delta

$$\hat{\delta}^{(+)}(q^2 - 2q \cdot p) \quad \to \quad 2\pi\delta(2q \cdot p). \tag{556}$$

In this limit $\Theta_\mu(x)$ is vanishing being the integrand an odd and real valued function and we are thus left with the computation of $\Theta(x)$. Using then the integral representation for a Dirac delta together with the Schwinger parametrization we obtain

$$\Theta(x) = -i \int_{\mathbb{R}} ds \int_{\mathbb{R}_+} dt \int \frac{d^D q}{(2\pi)^D} e^{-iq\cdot(x-2ps)+iq^2 t} . \tag{557}$$

The latter is a complex Gaussian integral and its computation gives

$$\Theta(x) = -\left(\frac{i}{4\pi}\right)^{\frac{D}{2}} \int_{\mathbb{R}} ds \int_{\mathbb{R}_+} dt \, \frac{e^{-i\frac{(x-2ps)^2}{4t}}}{t^{\frac{D}{2}}} . \tag{558}$$

Expanding the square in the exponential,

$$\begin{aligned} \Theta(x) &= -\left(\frac{i}{4\pi}\right)^{\frac{D}{2}} \int_{\mathbb{R}} ds \int_{\mathbb{R}_+} dt \, \frac{e^{-i\frac{(x-2ps)^2}{4t}}}{t^{\frac{D}{2}}} && (559) \\ &= -2\pi\left(\frac{i}{4\pi}\right)^{\frac{D}{2}} \int_{\mathbb{R}_+} \frac{dt}{t^{\frac{D}{2}}} e^{-\frac{ix^2}{4t}} \delta\left(\frac{p\cdot x}{t}\right) && (560) \\ &= -2\pi\left(\frac{i}{4\pi}\right)^{\frac{D}{2}} \delta(p\cdot x) \int_{\mathbb{R}_+} \frac{dt}{t^{\frac{D}{2}}} e^{\frac{-ix^2}{4y}} |t| . && (561) \end{aligned}$$

Changing variables to $t = \frac{1}{u}$, we get

$$\Theta(x) = -2\pi \left(\frac{i}{4\pi} \right)^{\frac{D}{2}} \delta(p \cdot x) \int_{\mathbb{R}_+} du \, \frac{e^{\frac{-iux^2}{4}}}{u^{\frac{6-D}{2}}} \, . \tag{562}$$

At this point, we should carefully distinguish the computation in $D = 4$ from other dimensions. One can realize it by computing separately the two cases and by a comparison afterwords. We start from the case with $D = 4$,

$$\Theta(x) = \frac{1}{8\pi} \delta(p \cdot x) \int_{\mathbb{R}_+} \frac{du}{u} e^{-\frac{iux^2}{4}} \, . \tag{563}$$

In order to compute this integral, we consider its partie finie (Pf) to find

$$\begin{aligned} \Theta(x) &= \frac{1}{8\pi} \delta(p \cdot x) \, \mathrm{Pf} \lim_{z \to 0} \int_z^{+\infty} \frac{du}{u} e^{-\frac{iux^2}{4}} \\ &= \frac{1}{8\pi} \delta(p \cdot x) \, \mathrm{Pf} \lim_{z \to 0} \int_1^{+\infty} \frac{du}{u} e^{-\frac{izux^2}{4}} \\ &= \frac{1}{8\pi} \delta(p \cdot x) \, \mathrm{Pf} \lim_{z \to 0} E_1 \left(\frac{zix^2}{4} \right) \, , \end{aligned} \tag{564}$$

where we have introduced the exponential integral $E_1(x)$. Using the Puiseux series

$$E_1(z) = -\gamma - \log z - \sum_{k=1}^{\infty} \frac{(-z)^k}{kk!} \quad , \quad |arg(z) < \pi| \, . \tag{565}$$

The result is

$$\Theta(x) = -\frac{1}{8\pi} \delta(p \cdot x) \log(|x^2|) \quad , \quad D = 4 \, . \tag{566}$$

As for the case $D \neq 4$, we evaluate it by first rescaling eq.(562),

$$\Theta(x) = -2\pi \left(\frac{i}{4\pi}\right)^{\frac{D}{2}} \delta(p \cdot x) \left(\frac{x^2}{4}\right)^{\frac{4-D}{2}} \int_{\mathbb{R}_+} du\, e^{-iu} u^{\frac{D-6}{2}} . \tag{567}$$

After a Wick rotation, the remaining integral defines a Gamma function, from which

$$\Theta(x) = \frac{\pi^{\frac{2-D}{2}}}{4} \frac{\Gamma(\frac{D-2}{2})}{D-4} \frac{\delta(p \cdot x)}{(x^2)^{\frac{D-4}{2}}} \quad , \quad D > 4 . \tag{568}$$

We can thus summarize our results,[3]

$$\Theta(x) = \begin{cases} -\frac{1}{8\pi}\delta(p \cdot x) \log(|x^2|) & , \quad D = 4 \\ \\ \frac{\pi^{\frac{2-D}{2}}}{4} \frac{\Gamma(\frac{D-2}{2})}{D-4} \frac{\delta(p \cdot x)}{(x^2)^{\frac{D-4}{2}}} & , \quad \text{otherwise} \end{cases} \tag{569}$$

Using eq.(553) we can read the metric tensor related to a three point function with a massless source and an off-shell graviton. The final result is

$$h_{\mu\nu}(x) = \begin{cases} 4G_N p_\mu p_\nu \delta(p \cdot x) \log(|x^2|) & , \quad D = 4 \\ -8\pi^{\frac{4-D}{2}} G_N p_\mu p_\nu \frac{\Gamma\left(\frac{D-2}{2}\right) \delta(p \cdot x)}{(D-4)(x^2)^{\frac{D-4}{2}}} & , \quad \text{otherwise} \end{cases} \tag{570}$$

3 One could also infer the $D = 4$ case from the following regularization $\frac{\Gamma(\frac{D-4}{2})}{x^{D-4}} \to -2\log(x)$. This amounts to remove a divergent quantity from the metric tensor with a gauge transformation.

We will shortly argue that contributions from higher loops produce only divergences which are removed from the cut terms proportional to $TT^\dagger$ in (539). This procedure provides an exact solution to Einstein field equations already at linear order in G_N. In $D = 4$ the line element reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 + 4G_N\delta(p\cdot x)\log(|x^2|)p_\mu p_\nu dx^\mu dx^\nu \,. \tag{571}$$

For a massless particle moving along the x direction we recover the Aichelburg-Sexl metric (532) for a gravitational shock wave

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + 4pG_N\delta(t-x)\log(y^2+z^2)(dt-dx)^2 \,. \tag{572}$$

In D dimensions, the metric is in agreement with earlier computations describing the ultrarelativistic boost of the Schwarzschild-Tangherlini metric in D dimensions [171]. As for the coordinates associated with this metric, we notice that eq.(570) satisfies the harmonic gauge condition, equivalent at linear order in G_N with the linear harmonic,

$$\eta^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0 \quad \rightarrow \quad \partial_\alpha h^{\mu\alpha} = \frac{1}{2}\partial^\mu h \,, \tag{573}$$

which can be easily seen to be satisfied thanks to eq.(570) being traceless. In $D = 4$

$$\partial_\alpha h^{\mu\alpha} = 4G_N p^\mu p^\alpha \partial_\alpha \delta(p\cdot x)\log(|x^2|) + 8G_N p^\mu p^\alpha \delta(p\cdot x)\frac{x_\mu}{x^2} = 0 \,, \tag{574}$$

with the same result in higher dimensions. This is consistent with the harmonic gauge choice made in eq.(544). The advantages of this computation with respect to the derivation from classical General Relativity are several. The exactness of the solution already at linear in G_N can now be explained in light of the absence of classical contributions to higher loops in three point functions with two massless scalars and a graviton. This is in contrast with the computation for a massive three point function where the classical part from higher loop orders is non vanishing and needed in order to reproduce the expansion of Schwarzschild in G_N [169]. Remarkably, the distributional profile emerges in a natural way from the amplitude itself, with no need to introduce singular coordinate transformations as those in eq.(530). As we will see, this property is more general: it is valid also for gravitational shock waves carrying a spin dependence.

A CLASSICAL DOUBLE COPY FOR GRAVITATIONAL SHOCK WAVES

In the previous section we have shown a relation between perturbative solutions to Einstein field equations and scattering amplitudes. The latter can be introduced also for a gauge theory as classical electromagnetism. We start by introducing the following operator for a gauge potential

$$\hat{A}^{\mu}(x) = \sum_{\lambda} \int d\Phi_{off}(q) \left[\epsilon^{\lambda}_{\mu}(q) \hat{a}^{\lambda}_{q} e^{-iq\cdot x} + (\epsilon^{\lambda}_{\mu})^{\dagger}(q) (\hat{a}^{\lambda}_{q})^{\dagger} e^{iq\cdot x} \right] . \tag{575}$$

Following the same steps seen before and working in Feynman gauge we can easily derive a relation between a gauge potential $A^\mu(x)$ and three point functions in scalar QED with an external photon,

$$A^\mu(x) = \int \frac{d^D q}{(2\pi)^D} \frac{\hat{\delta}^{(+)}(q^2 - 2q \cdot p_0)}{q^2} \text{Im}[\mathcal{M}^\mu(p_0, p' = p_0 - q, q) e^{-iq\cdot x}] . \tag{576}$$

We now consider electromagnetic shock waves [182]. We find natural to relate these to a three point amplitude of a massless scalar particle coupled to a photon,

$$\mathcal{M}^\mu = -ie(2p^\mu - q^\mu) . \tag{577}$$

Using this, we can express the gauge potential $A^\mu(x)$ in terms of (554, 555),

$$A^\mu(x) = -e\left[2p^\mu \Theta(x) - \Theta^\mu(x)\right] . \tag{578}$$

The final result for an electromagnetic shock wave is

$$A^\mu(x) = \begin{cases} \frac{e}{4\pi} p^\mu \delta(p\cdot x) \log(|x^2|) & , \quad D = 4 \\ \\ -\frac{e}{2\pi} p^\mu \, \pi^{\frac{4-D}{2}} \frac{\Gamma\left(\frac{D-2}{2}\right)\delta(p\cdot x)}{(D-4)(x^2)^{\frac{D-4}{2}}} & , \quad \text{otherwise} \end{cases} \tag{579}$$

We may now consider the classical double copy procedure shown in [149] in order to construct a solution in General Relativity. This amounts to the following replacement

$$e \to 16\pi G_N \quad , \quad p^\mu \to p^\mu p^\nu \, . \tag{580}$$

Interestingly, this gives the correct gravitational shock wave in arbitrary D dimensions of eq.(570) showing that gravitational and electromagnetic shock waves are related by a classical double copy, in agreement with [150].

SPINNING GRAVITATIONAL SHOCK WAVES

Having studied in depth the relation between massless particles and gravitational shock waves, we find natural to investigate the same relation for the case of a spinning source. As we will see, this leads to a family of classical solutions also known in the literature as gyratons [183, 184]. For ease of discussion we restrict ourselves to the case $D = 4$. In order to perform the computation, we take advantage of the exponential representation of three point functions for a spinning massive particle emitting a graviton [67, 88]

$$\mathcal{M}^S_{\mu\nu} = \mathcal{M}_{\mu\nu} \, e^{a\cdot q} \quad , \quad a^\mu = \frac{1}{2m^2} \epsilon^\mu{}_{\nu\alpha\beta} S^{\nu\alpha} p^\beta \, . \tag{581}$$

being $\mathcal{M}_{\mu\nu}$ the associated spinless three point amplitude and a^μ the rescaled spin vector of the source. For ease of discussion we restrict ourselves to the case $D = 4$. In the massless limit we assume a^μ to be fixed. We start from the associated metric tensor

$$h^S_{\mu\nu}(x) = -\frac{k^2}{2}\left(2p_\mu p_\nu \Theta^S(x) - p_\mu \Theta^S_\nu(x) - p_\nu \Theta^S_\mu(x) + \eta_{\mu\nu}\Theta^S_\alpha(x)p^\alpha\right), \qquad (582)$$

$$\Theta^S(x) = \int \frac{d^4q}{(2\pi)^4}\hat{\delta}^{(+)}(q^2 - 2q\cdot p)\frac{\cos(q\cdot x)e^{a\cdot q}}{q^2}, \qquad (583)$$

$$\Theta^S_\mu(x) = \int \frac{d^4q}{(2\pi)^4}\hat{\delta}^{(+)}(q^2 - 2q\cdot p)\frac{q_\mu}{q^2}\cos(q\cdot x)e^{a\cdot q}. \qquad (584)$$

We can now prove a relation between spinless gravitational shock waves and gyratons. We restrict to the classical limit by considering the integration region where $p \gg q$, thus

$$\Theta^S(x) = \frac{\Theta(x - ia) + \Theta(x + ia)}{2} \quad , \quad \Theta^S_\mu(x) = \frac{\Theta_\mu(x - ia) + \Theta_\mu(x + ia)}{2}. \qquad (585)$$

We now consider the behavior of $\Theta(x)$ under a complex shift. Introducing $a = \sqrt{-a^\mu a_\mu}$ we obtain the following expression

$$\Theta(x - ia) = -\frac{1}{8\pi}\delta(p\cdot x)\log(|x^2 + a^2|). \qquad (586)$$

Eq.(586) is real valued due to the absence of linear terms in $a \cdot x$. It follows from the Dirac delta in $p \cdot x$ and the orthogonality of p^μ with respect to the spin tensor. Thus,

$$\begin{cases} \Theta^S(x) = \Theta(x - ia) \\ \Theta^S_\mu(x) = \Theta_\mu(x - ia) \end{cases} \qquad \rightarrow \qquad h^S_{\mu\nu}(x) = h_{\mu\nu}(x - ia)\,. \tag{587}$$

Remarkably, thanks to the exponential form of the three point amplitude we can now read the metric tensor sourced by a massless spinning source directly from the spinless case using the shift $x^\mu \rightarrow x^\mu - ia^\mu$. To our knowledge, this property between spinless shock waves and gyratons was previously unknown and it relates to the exponential form of the energy momentum tensor for linearized Kerr black holes [65]. The line element is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + 4G_N\delta(p \cdot x)\log(|(x^2 + a^2|)p_\mu p_\nu dx^\mu dx^\nu\,. \tag{588}$$

In particular, for a spinning particle moving along the x direction

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + 4G_N p\delta(x - t)\log(|y^2 + z^2 - a^2|)(dt - dx)^2\,. \tag{589}$$

in agreement with an earlier computation by Ferrari and Pendenza [174] describing the ultrarelativistic boost of a Kerr black hole. The derivation by a simple shift in a^μ is remarkable, since the same in classical General Relativity is much more complicated.

Interestingly, this procedure resembles the Newman-Janis algorithm [175] which provides a Kerr solution from a complex deformation of Schwarzschild, this last recently studied by Arkani-Hamed, Huang and O'Connell in [69]. As for the singularity at $y^2 + z^2 = a^2$, we interpret this as the remnant of the singularity in the equatorial plane.

THE SCATTERING ANGLE IN THE HIGH ENERGY LIMIT

The computation of geodesics in a gravitational shock wave background has been explored by several authors [186, 187]. Since the whole space-time is Minkwoskian up to a region defined by a null light cone coordinate, geodesics are fully determined from the net change in momentum of a particle

$$\Delta p_0^\mu = \frac{1}{2} \int_{\mathbb{R}} d\sigma \partial^\mu h_{\alpha\beta}(x(\sigma)) p_0^\alpha p_0^\beta \,, \tag{590}$$

where the subscript 0 denotes the particle, while σ the affine parameter of its world-line. To leading order in G_N we assume free motion

$$x_0^\mu(\sigma) = p_0^\mu \sigma + b^\mu \quad , \quad b \cdot p_0 = 0 \quad , \quad b \cdot p = 0 \,, \tag{591}$$

being b^μ a covariant impact parameter and p^μ the momentum associated with the shock wave. The resulting change of impulse in $D = 4$ reads

$$\Delta p_0^\mu = \frac{1}{2} p_0^\alpha p_0^\beta \int_{\mathbb{R}} d\sigma 8 G_N p_\alpha p_\beta \left[\delta\left(p \cdot x_0(\sigma)\right) \frac{x_0^\mu(\sigma)}{x_0^2(\sigma)} \right] = \frac{4 G_N p \cdot p_0}{b \cdot b} \, b^\mu \,. \tag{592}$$

Having computed the change of momentum experienced by the particle, we can compute the associated phase shift using

$$\sin(\theta) = \frac{\Delta p_0^\mu b_\mu}{p_0 \, b} \,, \tag{593}$$

where we have introduced $b = \sqrt{-b_\mu b^\mu}$. The result is

$$\sin(\theta) = \frac{4 G_N p \cdot p_0}{p_0 \, b} \,. \tag{594}$$

Let's now consider the massless limit,

$$p \cdot p_0 = p \, p_0 - \vec{p} \cdot \vec{p}_0 = 2 p_0^2 \quad , \quad s = 4 p_0^2 \,. \tag{595}$$

If we now apply the small angle approximation we obtain

$$\theta = \frac{4 G_N \sqrt{s}}{b} \,, \tag{596}$$

in agreement with an earlier computation by Dray and t'Hooft [163]. Interestingly, as shown by Amati, Ciafaloni and Veneziano, the same result agrees with the leading order scattering angle between particles in the high energy limit [176]. We can generalize this result including effects to all orders in spin using as a source the metric tensor for a gyraton derived in eq.(587). This provides the following result

$$\Delta p^{\mu} = \frac{4G_N p \cdot p_0 \, b^{\mu}}{a^2 - b^2} \,. \tag{597}$$

The scattering angle in the massless limit and including effects to all order in spin reads

$$\theta = \frac{2G_N\sqrt{s}}{b-a} + \frac{2G_N\sqrt{s}}{b+a} \,. \tag{598}$$

Interestingly, this scattering angle in the high energy limit resembles a striking similarity with the all order in spin result by Vines [65] including the pole at $b = a$.

CONCLUSION

We have derived a relation between perturbative solutions to Einstein field equations and off-shell scattering amplitudes thanks to a covariant framework developed by Kosower, Maybee and O'Connell [21]. We have studied to which gravitational field corresponds a scattering amplitude with an off-shell graviton and two massless particles finding that the latter describes a gravitational shock wave also known as Aichelburg-Sexl metric [160]. The result has been easily generalized to arbitrary

D dimensions finding agreement with previous computation of D dimensional shock waves in General Relativity [171]. The advantage of this computation are several. We have been able to avoid singular coordinate transformations which were used in General Relativity to deal with the singular behavior of the gravitational field along a light cone coordinate. Remarkably, the distributional profile emerges in a natural way from the amplitude itself, while the exactness of the classical solution at linear in G_N can now be explained in light of the absence of classical contributions at higher loops for three point functions with massless particles. We have also shown that a classical double copy is satisfied between gravitational and electromagnetic shock waves and for a spinning source, using the exponential form of three point amplitudes, we have inferred a remarkable relation between gravitational shock waves and spinning ones, also known as gyratons. Using this property, we have been able to infer solutions describing spinning gravitational shock waves directly from the spinless case, thus bypassing the derivation in General Relativity involving an ultrarelativistic boost of a Kerr black hole. We have computed the phase shift of a particle in a background of shock waves finding agreement with earlier computations for the scattering angle of particles in the high energy limit [176, 177]. Applied to a gyraton, it has provided a result to all orders in the spin.

11

WAVEFORMS FROM AMPLITUDES

We show how to compute classical wave observables using quantum scattering amplitudes. We discuss observables both with incoming and with outgoing waves. The required classical limits are naturally described by coherent states of massless bosons. We recompute the classic gravitational deflection of light, and also show how to rederive Thomson scattering. We introduce a new class of local observables, which includes the asymptotic electromagnetic and gravitational Newman–Penrose scalars. As an example, we compute a simple radiated waveform: the expectation of the electromagnetic field in charged-particle scattering. At leading order, the waveform is trivially related to the five-point scattering amplitude.

INTRODUCTION

Theoretical waveforms play an important role in the LIGO/Virgo Collaboration's observational program of gravitational-wave events from binary mergers [6, 81–84]. These waveforms provide templates that enable the detection of events against otherwise overwhelming noise backgrounds. They also allow observers to extract the masses and spins of the binaries' constituents [188]. To date, theorists have computed waveforms (or equivalently, spectral functions for decaying binaries) using long-established effective-one-body (EOB) methods [189] and numerical-relativity approaches [190], in addition to methods based on the 'traditional' Arnowitt-Deser-Misner Hamiltonian formalism [191], direct post-Newtonian solutions in harmonic gauge [192], and computations in the effective-field theory approach pioneered by Goldberger and Rothstein [167, 193, 194].

The start of the gravitational-wave observational era has spurred theorists to explore new approaches to computing classical observables for the two-body problem in gravity, in particular using quantum scattering amplitudes. The connection between the quantum S-matrix and observables in classical General Relativity (GR) was first explored nearly fifty years ago by Iwasaki [31]. More recently, renewed interest has been driven by modern on-shell techniques for computing amplitudes and the double-copy relation between Yang–Mills and gravitational amplitudes [130, 147–154, 195–211], as well as the bounty of observations. Earlier investigations included applications to the two-body potential [11] and study of quantum corrections to gravity [212].

An important step was taken by Cheung, Solon, and Rothstein [12], who showed how to match effective field theories (EFTs) to scattering amplitudes above threshold in order to extract a classical potential. The classical potential can then be used in the EOB or other frameworks to make predictions for bound-state quantities. Bern, Cheung, Roiban, Shen, Solon, and Zeng used [10] this approach to compute the third-order corrections (G^3) to the conservative potential. This milestone computation went beyond what had been known from direct classical GR calculations, and provided the first concrete fulfillment of the promise of the scattering-amplitudes class of methods. It used a two-loop scattering amplitude for massive particles, and was followed by many new calculations using amplitude methods [15, 17, 26, 68, 69, 72, 86, 139, 144, 213–221]. New EFT-based results have also emerged [70, 73, 222–244]. In this context, Kälin and Porto have pointed out an interesting analytic continuation from scattering to bound-state observables [73, 141]. Several groups have pursued an eikonal approach [47, 245–250], and connections to it [251]. Another approach which has seen recent attention is the world-line formalism [193, 252, 253]. In the context of EFT, this world-line approach is particularly important since it makes immediate sense classically. Treated as an effective quantum field theory, this means that it organizes quantum corrections particularly simply. Finally, two of the present authors have examined light-ray operators [254] and shock waves [20]. Researchers working within a traditional GR framework have also continued to produce new results [14, 25, 85, 87, 146, 255–264].

In a previous paper [21], two of the present authors and Maybee outlined an observables-based approach to computing classical quantities. It starts with an observable in the quantum theory, expressing it in terms of scattering amplitudes; and then uses an efficient and controlled method for taking the classical limit. In this approach, rather than trying to compute intermediate quantities such as the conservative potential, we write down a formal expression for an observable of interest — for example, the total change in the momentum of one of two scattered particles, *aka* its impulse — in the quantum theory. With an appropriate wavefunction for the initial state, we can express the chosen observable in terms of quantum scattering amplitudes. We further restore powers of $\hbar$ via dimensional analysis. At this stage, the $\hbar$ scaling is naively bad, as the observable may be seemingly divergent in the classical, $\hbar \to 0$ limit, and loop corrections appear to be increasingly divergent with increasing order.

The original paper [21] focused on scattering two massive particles. Appropriate wavefunctions were necessary to localize each incoming particle. This localization will sharpen in the classical limit, when we are focusing on point particles. The localization will in turn lead us to retain momenta for the scattering particles in the expression for the observable, but to use *wavenumbers* for exchanged, emitted, or virtual massless particles (photons or gravitons). The change of variables from momenta to wavenumbers for the latter reveals additional powers of $\hbar$ that then yield a finite classical limit at each perturbative order. Herrmann, Parra-Martinez, Ruf, and Zeng [26, 215], and separately Bautista and Guevara [220] have applied this approach in their calculations.

Ref. [21] did not discuss massless bosonic particles, in particular in the initial state. We remedy that in this article. Furthermore, ref. [21] focused only on global observables, which require surrounding an event with a detector of 4π coverage. We remedy this as well with a discussion of local observables, such as electromagnetic and gravitational waveforms. Newman–Penrose [265] scalars provide a natural language for these quantities. We will introduce these two principal topics of our article in the remainder of this introduction.

Let us begin with the question of initial-state massless bosons. In the classical limit, one describes *massive* particles as superpositions of single-particle states. They ultimately appear as point-like particles or extended bodies. In contrast, *massless* bosons appear as waves or wave packets. It is no longer possible to describe them as superpositions of single-particle states. Instead, we shall see that they emerge most naturally from coherent states of the corresponding quantum fields. Such states are inherently superpositions of multiparticle states.

The significance of coherent states was emphasized by Glauber from 1963 on. He proved that every quantum state of radiation — that is, every density matrix — can be described as a suitable superposition of coherent states [266, 267]. In particular, in the classical limit one can describe these density matrices using the so-called Glauber–Sudarshan P-representation [267, 268]. In this representation, there is a classical probability distribution in the space of coherent states. The application of coherent states to the classical limit of quantum scattering amplitudes started soon afterwards in the work of Frantz, Kibble, and Brown [269–271], but a systematic analysis of the question was still lacking [272]. Most calculations were limited to the solvable model of the linear interactions of a current (or a stress tensor) with the associated field [273]: in this case the S-matrix is solvable to all orders in perturbation theory, and its structure is exactly equivalent to a coherent state. Yaffe later showed [274] that coherent states are very convenient for understanding the emergence of the classical approximation from quantum physics quite generally. Concrete applications are nonetheless rare in the literature, especially outside the case of a single particle interacting with a fixed coherent background (see ref. [275] and references therein)[1]. Coherent states have a close connection to soft limits and infrared divergences, which provide a natural arena for their emergence in the late-time dynamics of QED and linearized gravity [277–282].

Let us turn next to the question of local observables. In ref. [21], the authors studied time-integrated observables, in the context of scalar electrodynamics, and validated the amplitude-based approach through comparisons with direct calculations in classical electromagnetism. What is of more direct interest to observers,

1 Ref. [276] offers a notable exception in the context of the superradiance problem.

however, are time-*dependent* observables such as radiation waveforms. These are examples of a class of observables which are *local* in the sense that they describe a measurement at one spacetime point (or in a small region of spacetime). The time-integrated observables of [21] in principle require an apparatus which completely surrounds a scattering event, so that (for example) the impulse of any incoming particle can be measured. We describe this class of observables as *global* as a result.

In this article, we establish a direct connection between local observables, such as waveforms, and scattering amplitudes. We validate our approach with a calculation of a simple waveform, arising from the scattering of two charged particles in scalar QED. We will see that waveforms are effectively amplitudes for detecting massless particles, or waves in the classical limit. We show how to write appropriate quantum observables, and how to take their limits. Finally, we provide a direct connection between the celebrated Newman–Penrose formalism [265] and scattering amplitudes.

As our work has progressed, we have become aware of a parallel line of investigation by Bautista, Guevara, Kavanagh and Vines [283]. Their work is broadly complementary to ours, but touches on some of the same themes: the connection between the Compton amplitude and classical wave scattering, for example, and the close connection between the Newman-Penrose scalars and helicity amplitudes.

Our article is organized as follows. We begin in the next section with a review of the formalism of ref. [21]. In Sect. 11, we review coherent states for the electromagnetic field, show how they correspond to classical fields, and give a simple example of a light beam built from them. In Sect. 11, we discuss global observables with massless waves in the initial state, concentrating on the impulse in this context. As examples, we discuss Thomson scattering and its relation to the Compton amplitude, and we examine the calculation of the gravitational deflection of light within our formalism. We turn to the second major topic of our article in Sect. 11 with a discussion of the general form of local observables far from some event. Sect. 11 follows with an introduction to spectral aspects of local observables, leading to the Newman–Penrose projection formalism. In Sect. 11, we pause the general development to give example of a local observable: the scattered radiation field in Thomson scattering. In Sect. 11, we present the general form of the emission waveform when two massive particles scatter, and in Sect. 11 we give explicit results for electromagnetic emission in charged-particle scattering to leading order. We discuss the connection between the waveform and the total radiated momentum in Sect. 11, and end with concluding remarks in Sect. 11.

REVIEW OF FORMALISM

We use relativistic units, retaining $c = 1$, even as we restore $\hbar$ explicitly. This means that we must distinguish units of energy and length, which we denote by $[M]$ and $[L]$ respectively. In this article, we will use a different normalization than the conventions of ref. [21] (which are also the conventions of Peskin and Schroeder [284]). Here, we normalize the annihilation and creation operators such that,

$$[a_p, a^{\dagger}_{p'}] = (2\pi)^3 2E_p \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')\,. \tag{599}$$

(Bold symbols denote spatial three-vectors.) Accordingly, n-point scattering amplitudes continue to have dimension $[M]^{4-n}$.

We keep $[M]^{-1}$ as the dimension of single-particle states $|p\rangle$,

$$|p\rangle \equiv a^{\dagger}_p |0\rangle\,, \tag{600}$$

with the vacuum state being dimensionless. We define n-particle plane-wave states as simply the tensor product of normalized single-particle states. (The normalization of the single-particle states is the same as in ref. [21].) The state $|p\rangle$ represents a particle of momentum p and positive energy, while $\langle p| = \langle 0|a_p$ is the conjugate state.

We find it convenient to define an n-fold Dirac δ distribution with normalization absorbing 2πs,

$$\hat{\delta}^{(n)}(p) \equiv (2\pi)^n \delta^{(n)}(p)\,. \tag{601}$$

The scattering matrix S and the transition matrix T are both dimensionless. Scattering amplitudes are matrix elements of the latter between plane-wave states,

$$\langle p_1' \cdots p_m' | T | p_1 \cdots p_n \rangle = \mathcal{A}(p_1 \cdots p_n \to p_1' \cdots p_m') \hat{\delta}^{(4)}(p_1 + \cdots p_n - p_1' - \cdots - p_m') \,. \tag{602}$$

As our formalism encompasses both QED and gravity, as well as other theories with massless force carriers, we denote the coupling by g. In electrodynamics, it corresponds to e, while in gravity to $\kappa = \sqrt{32\pi G}$. It is not dimensionless once we have restored the factors of $\hbar$; rather, it is $g/\sqrt{\hbar}$ that is the dimensionless coupling.

We start by taking the momenta of all particles as the primary variables; but as explained in the introduction, for most massless momenta, wavenumbers are the variables of interest. We introduce a notation for the wavenumber $\bar{p}$ associated to the momentum p,

$$\bar{p} \equiv p/\hbar \,. \tag{603}$$

We use the notation of ref. [21] for the on-shell phase-space measure,

$$d\Phi(p_i) \equiv \hat{d}^4 p_i \, \hat{\delta}^{(+)}(p_i^2 - m_i^2) \,. \tag{604}$$

We will leave the mass implicit, along with the designation of the integration variable as the first summand when the argument is a sum. The notation for the measure again absorbs factors of 2π,

$$\hat{d}^4 p \equiv \frac{d^4 p}{(2\pi)^4} \,, \tag{605}$$

and as usual,

$$\delta^{(+)}(p^2 - m^2) = \Theta(p^t)\,\delta(p^2 - m^2)\,, \tag{606}$$

so that,

$$\hat{\delta}^{(+)}(p^2 - m^2) = 2\pi\Theta(p^t)\,\delta(p^2 - m^2)\,. \tag{607}$$

(p^t is the energy component of the four-vector.)

Given our convention for normalizing single-particle states, their inner product is,

$$\langle p'|p\rangle = 2E_p\hat{\delta}^{(3)}(\boldsymbol{p} - \boldsymbol{p}'). \tag{608}$$

The expression on the right-hand side is the appropriately normalized delta function for the on-shell measure, which is convenient to express in more compact notation,

$$\hat{\delta}_\Phi(p_1 - p_1') \equiv 2E_{p_1}\hat{\delta}^{(3)}(\boldsymbol{p}_1 - \boldsymbol{p}_1')\,. \tag{609}$$

We should understand the argument on the left-hand side as a function of four-vectors. In this notation, eq. (608) is simply,

$$\langle p'|p\rangle = \hat{\delta}_\Phi(p - p')\,. \tag{610}$$

With this notation, we can also rewrite the normalization of creation and annihilation operations (599) in a natural form,

$$[a_p, a^\dagger_{p'}] = \hat{\delta}_\Phi(p - p')\,. \tag{611}$$

We will also employ the notation $a(k) \equiv a_k$ and $a^\dagger(k) \equiv a^\dagger_k$ to allow for additional indices.

Ref. [21] exclusively considered the scattering of two massive point-like particles. In this article we go beyond that discussion to consider initial states which may involve massless radiation. However, when appropriate we will continue to use the notation of ref. [21] for initial states involving only massive particles: we take the initial momenta to be p_1 and p_2, initially separated by a transverse impact parameter b. The latter is transverse in that $p_1 \cdot b = 0 = p_2 \cdot b$.

In the quantum theory, the system of massive particles is described by wave functions, which we build out of plane waves. In the classical limit, these wave functions must localize the two point-like particles, and must separate them clearly. We describe the incoming particles in the far past by wave functions $\varphi_i(p_i)$, which we take to have reasonably well-defined positions and momenta. We will review the requirements on the wave packets, discussed in detail in sect. 4 of ref. [21], below.

We express the initial state in terms of plane waves $|p_1\, p_2\rangle_{\text{in}}$,

$$\begin{aligned} |\psi\rangle_{\text{in}} &= \int \hat{d}^4 p_1 \hat{d}^4 p_2\, \hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2)\, \varphi_1(p_1)\varphi_2(p_2)\, e^{ib\cdot p_1/\hbar}|p_1\, p_2\rangle_{\text{in}} \\ &= \int d\Phi(p_1) d\Phi(p_2)\; \varphi_1(p_1)\varphi_2(p_2)\, e^{ib\cdot p_1/\hbar}|p_1\, p_2\rangle_{\text{in}}\,. \end{aligned} \tag{612}$$

We require each wave function φ_i to be normalized to unity,

$$\int d\Phi(p_1)\; |\varphi_1(p_1)|^2 = 1\,, \tag{613}$$

so that our incoming state is also normalized to unity,

$$
\begin{aligned}
{}_{\text{in}}\langle\psi|\psi\rangle_{\text{in}} &= \int d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')e^{ib\cdot(p_1-p_1')/\hbar} \\
&\qquad \times \varphi_1(p_1)\varphi_1^*(p_1')\,\varphi_2(p_2)\varphi_2^*(p_2')\,\hat{\delta}_\Phi(p_1-p_1')\,\hat{\delta}_\Phi(p_2-p_2') \\
&= \int d\Phi(p_1)d\Phi(p_2)\,|\varphi_1(p_1)|^2|\varphi_2(p_2)|^2 \\
&= 1\,.
\end{aligned}
\tag{614}
$$

Finally, we turn to a review of the classical limit. As discussed in ref. [21], there are three scales we must consider in the context of massive particle scattering: the Compton wavelengths of the particles, $\ell_c^{(i)} \equiv \hbar/m_i$; the intrinsic spread of the two particles' wavepackets, given by ℓ_w; and the scattering length, ℓ_s. Taking the classical limit requires that we impose the 'Goldilocks' conditions,

$$
\ell_c^{(i)} \ll \ell_w \ll \ell_s\,. \tag{615}
$$

The calculation of the scattering reveals that $\ell_s \sim \sqrt{-b^2}$.

In order to expand in the $\hbar \to 0$ limit and extract the leading, classical, term for any observable, as mentioned above we must make the powers of $\hbar$ explicit. These arise from two sources: powers ordinarily hidden inside electromagnetic or gravitational couplings; and powers arising from keeping the wavenumbers of massless particles fixed rather than their momenta. This is true both for emitted and virtual particles, when considering quantities such as the total emitted radiation.

CLASSICAL LIMIT FOR MASSLESS PARTICLES

We are now ready to address the first major topic of this article: how to include initial-state massless classical waves in the formalism of ref. [21]. A naive extension of the considerations of ref. [21] to massless particles is clearly impossible. A particle's Compton wavelength diverges when its mass goes to zero, making it impossible to satisfy the required conditions (615). It doesn't make sense to treat messengers (photons or gravitons) as point-like particles. Indeed, Newton and Wigner [285] and Wightman [286] proved rigorously long ago that a strict localization of known massless particles in position space is impossible[2]. A proper treatment instead relies on coherent states. We begin such a treatment in the following subsection by discussing general aspects of coherent states, focusing on the electromagnetic case. We then describe the kind of coherent states of interest to us.

Coherent States of the Electromagnetic Field

We can write the electromagnetic field operator as,

$$\mathbb{A}_\mu(x) = \frac{1}{\sqrt{\hbar}} \sum_\eta \int d\Phi(k) \left[a_{(\eta)}(k) \varepsilon_\mu^{(\eta)*}(k)\, e^{-ik\cdot x/\hbar} + a^\dagger_{(\eta)}(k) \varepsilon_\mu^{(\eta)}(k)\, e^{+ik\cdot x/\hbar} \right], \quad (616)$$

2 The proof holds for vector bosons and gravitons.

where $\eta = \pm$ labels the helicity, and the polarization vectors satisfy,

$$\left[\varepsilon_{\mu}^{(\eta)}(k)\right]^{*} = \varepsilon_{\mu}^{(-\eta)}(k)\,. \tag{617}$$

We follow the usual amplitudes convention of representing an outgoing positive-helicity photon of momentum k by $\varepsilon_{\mu}^{(+)}(k)$, which also corresponds to an incoming negative-helicity photon of the opposite momentum. To understand the helicity flip for an incoming state, note that we can analytically continue an incoming momentum k to an outgoing momentum $k' = -k$. The energy component k'^{t} of the outgoing momentum is now negative. Thus, in an all-outgoing convention, positive-helicity photons of momentum k with $k^{t} > 0$ are represented by the polarization vector $\varepsilon_{\mu}^{(+)}(k)$, while positive-helicity photons of momentum k with $k^{t} < 0$ are represented by the polarization vector $\varepsilon_{\mu}^{(-)}(k)$.

More generally, $a^{\dagger}_{(\eta)}(k)$ creates a single-messenger state of momentum k and helicity η, while $a_{(\eta)}(k)$ destroys such a state. Equivalently, the latter operator creates a conjugate state of momentum k and helicity η.

The commutation relations are

$$\left[a_{(\eta)}(k), a^{\dagger}_{(\eta')}(k')\right] = \delta_{\eta,\eta'}\hat{\delta}_{\Phi}(k-k')\,. \tag{618}$$

For example, a single-particle positive-helicity state is

$$|k^{+}\rangle \equiv a^{\dagger}_{(+)}(k)|0\rangle = \left[a_{(+)}(k)\right]^{\dagger}|0\rangle\,. \tag{619}$$

The conjugate state is $\langle k^{+}|$.

Using the form of the electromagnetic field in eq. (616), the electromagnetic field strength operator is,

$$\mathbb{F}_{\mu\nu}(x) = -\frac{2i}{\hbar^{3/2}}\sum_{\eta}\int d\Phi(k)\,\left[a_{(\eta)}(k)\,k_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(k)\,e^{-ik\cdot x/\hbar} - a^{\dagger}_{(\eta)}(k)\,k_{[\mu}\varepsilon^{(\eta)}_{\nu]}(k)\,e^{+ik\cdot x/\hbar}\right], \tag{620}$$

where as usual the subscripted brackets denote antisymmetrization,

$$A_{[\mu}B_{\nu]} = \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)\,. \tag{621}$$

Introduce the coherent-state operator,

$$\mathbb{C}_{\alpha,(\eta)} \equiv \mathcal{N}_\alpha \exp\left[\int d\Phi(k)\,\alpha(k)a^{\dagger}_{(\eta)}(k)\right], \tag{622}$$

where the normalization $\mathcal{N}_\alpha$ will be given below. We can build coherent states of the electromagnetic field using this operator, such as a positive-helicity one,

$$|\alpha^+\rangle = \mathbb{C}_{\alpha,(+)}|0\rangle\,. \tag{623}$$

More generally, we could consider coherent states containing both helicities. Since coherent-state operators for different helicities commute and every polarization vector can be decomposed in the helicity basis, there is no loss of generality in making a specific helicity choice for the coherent states we consider. The coherent state operators are unitary,

$$(\mathbb{C}_{\alpha,(\eta)})^{\dagger} = (\mathbb{C}_{\alpha,(\eta)})^{-1}\,. \tag{624}$$

The normalization factor $\mathcal{N}_\alpha$ is determined by the condition $\langle\alpha^+|\alpha^+\rangle = 1$, that is,

$$\mathcal{N}_\alpha = \exp\left[-\frac{1}{2}\int d\Phi(k)\,|\alpha(k)|^2\right], \tag{625}$$

as can be seen by using the Baker–Campbell–Hausdorff formula.

At this stage, the function $\alpha(k)$ is quite general, however in specific examples, we may take it to be real. We will see below that it is subject to certain restrictions in the classical limit. We will also see that its functional form will determine the physical shape of the corresponding state, so we will call it the 'waveshape' function.

The coherent-state creation operator acting on the vacuum can be rewritten using the Baker-Campbell-Hausdorff identity as a displacement operator [269, 270] yielding

$$\mathbb{C}_{\alpha,(\eta)}|0\rangle = \exp\left[\int d\Phi(k)\alpha(k)(a^\dagger_{(\eta)}(k) - a_{(\eta)}(k))\right]|0\rangle\,. \tag{626}$$

Its action on creation and annihilation operators is given by,

$$\begin{aligned}\mathbb{C}^\dagger_{\alpha,(\eta)}a_{(\rho)}(k)\mathbb{C}_{\alpha,(\eta)} &= a_{(\rho)}(k) + \delta_{\eta\rho}\,\alpha(k)\,,\\ \mathbb{C}^\dagger_{\alpha,(\eta)}a^\dagger_{(\rho)}(k)\mathbb{C}_{\alpha,(\eta)} &= a^\dagger_{(\rho)}(k) + \delta_{\eta\rho}\,\alpha^*(k)\,.\end{aligned} \tag{627}$$

To interpret the state, let us compute $\langle\alpha^+|\mathbb{A}^\mu(x)|\alpha^+\rangle$. It is useful to note,

$$
\begin{aligned}
a_{(+)}(k)|\alpha^+\rangle &= \alpha(k)|\alpha^+\rangle\,,\\
a_{(-)}(k)|\alpha^+\rangle &= 0\,,\\
\langle\alpha^+|a^\dagger_{(+)}(k) &= \langle\alpha^+|\alpha^*(k)\,,\\
\langle\alpha^+|a^\dagger_{(-)}(k) &= 0\,,
\end{aligned}
\tag{628}
$$

which incidentally imply that the dimension of $\alpha(k)$ is the same as the dimension of the annihilation operator: inverse mass. It is then easy to see that,

$$
\begin{aligned}
\langle\alpha^+|\mathbb{A}_\mu(x)|\alpha^+\rangle &= \frac{1}{\sqrt{\hbar}}\int d\Phi(k)\,\left[\alpha(k)\varepsilon^{(+)*}_\mu(k)e^{-ik\cdot x/\hbar} + \alpha^*(k)\varepsilon^{(+)}_\mu(k)e^{+ik\cdot x/\hbar}\right]\\
&= \int d\Phi(\bar{k})\,\left[\bar{\alpha}(\bar{k})\varepsilon^{(+)*}_\mu(\bar{k})e^{-i\bar{k}\cdot x} + \bar{\alpha}^*(\bar{k})\varepsilon^{(+)}_\mu(\bar{k})e^{+i\bar{k}\cdot x}\right]\\
&\equiv A_{\mathrm{cl}\,\mu}(x)\,,
\end{aligned}
\tag{629}
$$

where we define

$$
\bar{\alpha}(\bar{k}) \equiv \hbar^{3/2}\alpha(k)\,. \tag{630}
$$

Additional constraints on $\bar{\alpha}$ will emerge below from the consideration of correlators in the classical limit. Note that the polarization vector is invariant under the rescaling from a momentum to a wavevector: $\varepsilon^{(\eta)}(\bar{k}) = \varepsilon^{(\eta)}(k)$ is independent of $\hbar$.

Now, the most general solution of the classical Maxwell equation in empty space is,

$$
\sum_\eta A_{\mathrm{cl}}^{(\eta)\,\mu}(x) = \sum_\eta \int d\Phi(\bar{k})\left[\widetilde{A}_\eta(\bar{k})\varepsilon^{(\eta)*\mu}(\bar{k})e^{-i\bar{k}\cdot x} + \widetilde{A}^*_\eta(\bar{k})\varepsilon^{(\eta)\mu}(\bar{k})e^{+i\bar{k}\cdot x}\right], \tag{631}
$$

in terms of Fourier coefficients $\widetilde{A}_\eta(\bar{k})$, which we can identify as $\bar{\alpha}(\bar{k})$. Evidently our state $|\alpha^+\rangle$ contributes only the terms of positive helicity ($\eta = +$); a more general coherent state involving creation operators of both helicities would generate this most general solution of the free Maxwell equations. In examples we will consider, the simpler state $|\alpha^+\rangle$ will suffice.

To further illuminate the meaning of coherent states, we may consider scattering amplitudes in the presence of a non-trivial background field $A_{\text{cl}}(x)$. The scattering matrix in the presence of this background field depends on it. We denote this dependence by $S(A_{\text{cl}})$. Using the properties of the coherent state operator it can be shown that,

$$\mathbb{C}^\dagger_{\alpha,(\eta)} S(A) \mathbb{C}_{\alpha,(\eta)} = S(A + A^{(\eta)}_{\text{cl}})\,. \tag{632}$$

Coherent states thus allow us to capture the physics of a specific background field based on vacuum scattering amplitudes:

$$\mathbb{C}^\dagger_{\alpha,(\eta)} S(0) \mathbb{C}_{\alpha,(\eta)} = S(A^{(\eta)}_{\text{cl}})\,. \tag{633}$$

The formulation of the perturbation theory in a fixed background is particularly convenient when the Feynman rules — or the scattering amplitudes — in the background are known exactly [287].

Classical Coherent States

The coherence of a state does not suffice for it to behave classically. We must also require factorization of expectation values,

$$\langle \alpha^+ | \mathbb{A}^\mu(x)\mathbb{A}^\nu(y) | \alpha^+ \rangle \simeq \langle \alpha^+ | \mathbb{A}^\mu(x) | \alpha^+ \rangle \langle \alpha^+ | \mathbb{A}^\nu(y) | \alpha^+ \rangle \,. \tag{634}$$

A straightforward calculation in a light-cone gauge defined by a light-like vector q shows that,

$$\begin{aligned}
&\langle \alpha^+ | \mathbb{A}^\mu(x)\mathbb{A}^\nu(y) | \alpha^+ \rangle = \\
&\quad \langle \alpha^+ | \mathbb{A}^\mu(x) | \alpha^+ \rangle \langle \alpha^+ | \mathbb{A}^\nu(y) | \alpha^+ \rangle \;+ \frac{1}{\hbar}\int d\Phi(k) \left[\eta^{\mu\nu} - \frac{k^\mu q^\nu + k^\nu q^\mu}{k\cdot q + i\delta} \right] e^{-ik\cdot(x-y)/\hbar} \\
&\quad = \langle \alpha^+ | \mathbb{A}^\mu(x) | \alpha^+ \rangle \langle \alpha^+ | \mathbb{A}^\nu(y) | \alpha^+ \rangle \;+ \hbar \int d\Phi(\bar{k}) \left[\eta^{\mu\nu} - \frac{\bar{k}^\mu q^\nu + \bar{k}^\nu q^\mu}{\bar{k}\cdot q + i\delta} \right] e^{-i\bar{k}\cdot(x-y)} \,.
\end{aligned} \tag{635}$$

Similarly for the field strengths, in a gauge independent way using eq. (620), we obtain

$$\begin{aligned}
\langle \alpha^+ | \mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\rho\sigma}(y) | \alpha^+ \rangle &= \langle \alpha^+ | \mathbb{F}^{\mu\nu}(x) | \alpha^+ \rangle \langle \alpha^+ | \mathbb{F}^{\rho\sigma}(y) | \alpha^+ \rangle \\
&\quad + 4\hbar \partial^{[\mu}\eta^{\nu][\sigma}\partial^{\rho]} \int d\Phi(\bar{k})\, e^{-i\bar{k}\cdot(x-y)} \,.
\end{aligned} \tag{636}$$

For classical behavior, the second term on the right-hand side of eq. (636) must be negligible compared to the first term. Writing $F_{\text{cl}}^{\mu\nu}(x) \equiv \langle \alpha^+ | \mathbb{F}^{\mu\nu}(x) | \alpha^+ \rangle$, the right-hand side becomes,

$$F_{\text{cl}}^{\mu\nu}(x) F_{\text{cl}}^{\rho\sigma}(y) + \frac{\hbar}{\pi^2} \partial^{[\mu}\eta^{\nu][\sigma}\partial^{\rho]} \frac{1}{(\mathbf{x}-\mathbf{y})^2 - (x^0 - y^0 - i\delta)^2} \,. \tag{637}$$

The first term has a nontrivial limit as $\hbar \to 0$, whereas the second term goes to zero in the limit, consistent with our expectations. For $\hbar \neq 0$, it is not possible to satisfy the inequality in the full spacetime region due to the divergence on the light-cone $(x^0 - y^0)^2 = |\boldsymbol{x} - \boldsymbol{y}|^2$ of the massless photon propagator: causally connected measurements cannot be disentangled. We expect these contributions to fade away in the classical limit of a physical observable [270]. The factorization condition, which is trivial in the classical limit, has been dubbed the "complete coherence condition" in the literature[3], a term coined by Glauber [267].

As usual, we define the operator measuring the number of photons to be,

$$\mathbb{N}_\gamma = \sum_\eta \int d\Phi(k)\, a^\dagger_{(\eta)}(k) a_{(\eta)}(k)\,. \tag{638}$$

A short computation shows that the expectation number N_γ of photons in our coherent state is,

$$\begin{aligned} N_\gamma &= \langle \alpha^+ | \mathbb{N}_\gamma | \alpha^+ \rangle \\ &= \int d\Phi(k) |\alpha(k)|^2 \\ &= \frac{1}{\hbar} \int d\Phi(\bar{k}) |\bar{\alpha}(\bar{k})|^2\,. \end{aligned} \tag{639}$$

The classical limit $\hbar \to 0$ thus corresponds to the limit of a large number of photons, that is a limit of large occupation number [274]. The desired factorization property eq. (634) will thus hold when,

$$N_\gamma \gg 1\,. \tag{640}$$

3 In the quantum optics literature the normal-ordered correlator of the electric field at different spatial locations can have various degrees of coherence [288].

We must choose the waveshape α such that the integral in the last line of eq. (639) is not parametrically small as $\hbar \to 0$. A simple way to do so is to chose $\bar{\alpha}$ independent of $\hbar$.

Similarly, the momentum carried by the coherent state is,

$$\begin{aligned} K^{\mu}_{\odot} &= \langle \alpha^{+} | \mathbb{K}^{\mu} | \alpha^{+} \rangle \\ &= \int d\Phi(k) |\alpha(k)|^2 \, k^{\mu} \\ &= \int d\Phi(\bar{k}) |\bar{\alpha}(\bar{k})|^2 \, \bar{k}^{\mu} \, . \end{aligned} \tag{641}$$

This quantity ("K beam") is finite in the classical limit, as expected.

We emphasize that this coherent-state construction and its connection to classical states generalizes to any massless particle, including gravitons. Finally, it is worth remarking on the important and familiar case of geometric optics. This is a purely classical approximation to wave phenomena, valid in situations where the wavelength is negligible in comparison to other physical scales. An important example, which we discuss below, is of the gravitational bending of light.

Localized Beams of Light

In this paper, one of our foci will be on phenomena associated with scattering light from a point-like object. For problems of this type to be well-defined, the incoming wave must be spatially separated from the incoming particle in the far past. Consequently, we need to understand how to describe a localized incoming beam of light. We can choose the beam to be moving in the z direction, localized around the origin of the x–y plane. To see how to do this, let's consider some examples.

The simplest option for the waveshape is,

$$\alpha(k) = \alpha_\odot \hat{\delta}_\Phi(k - \hbar\bar{k}_\odot)\,, \tag{642}$$

where $\bar{k}_\odot$ ("k-bar beam") is the overall wavevector of the wave, and $\alpha_\odot$ ("α beam") is a constant which scales like $\sqrt{\hbar}$. Defining $\bar{\alpha}_\odot = \hbar^{-1/2}\alpha_\odot$, this choice implies that,

$$\bar{\alpha}(\bar{k}) = \bar{\alpha}_\odot \hat{\delta}_\Phi(\bar{k} - \bar{k}_\odot)\,, \tag{643}$$

and that the classical field takes the form,

$$A^\mu_{\text{cl}}(x) = 2\,\text{Re}\,\bar{\alpha}_\odot\, \varepsilon^{*\mu}_\odot(\bar{k}_\odot)e^{-i\bar{k}_\odot\cdot x}\,. \tag{644}$$

It is worth pointing out here that the expectation value of the gauge potential between coherent states is always a real quantity: a physical field which can be measured. We can choose

$$\begin{aligned} \bar{k}^\mu_\odot &= (\omega, 0, 0, \omega) \\ \varepsilon^\mu_\odot &= \frac{1}{\sqrt{2}}(0, 1, i, 0)\,, \end{aligned} \tag{645}$$

to provide an explicit example. If we pick the normalization of $\bar{\alpha}$ to be given by $\bar{\alpha}_\odot = A_\odot/\sqrt{2}$ with $A_\odot$ real, the classical field for this example is,

$$A^\mu_{\rm cl}(x) = A_\odot\,(0, \cos\omega(t-z), -\sin\omega(t-z), 0)\,, \tag{646}$$

which is a plane wave of circular polarization[4] moving in the z-direction with angular frequency ω. This wave is completely delocalized, which is a disadvantage for our purposes: we wish to have a clean separation between the incoming wave and point-like particle states.

To localize the wave, we may "broaden" the delta function in eq. (642). Define,

$$\delta_\sigma(\bar{k}) \equiv \frac{1}{\sigma\sqrt{\pi}}\exp\left[-\frac{\bar{k}^2}{\sigma^2}\right], \tag{647}$$

which is normalized so that

$$\int_{-\infty}^{\infty} d\bar{k}\,\delta_\sigma(\bar{k}) = 1\,. \tag{648}$$

The peak width is controlled by the parameter σ. As $\bar{k}$ is a wavenumber, σ has dimensions of inverse length. We may choose our incoming wave, moving along the z-axis, to be symmetric under a rotation about that axis. Consider the choice,

$$\alpha(k) = \frac{1}{\hbar^3}|\boldsymbol{k}|(2\pi)^3 A_\odot\sqrt{2\hbar}\,\delta_{\sigma_\parallel}(\omega - k^z/\hbar)\delta_{\sigma_\perp}(k^x/\hbar)\delta_{\sigma_\perp}(k^y/\hbar)\,; \tag{649}$$

or equivalently,

$$\bar{\alpha}(\bar{k}) = \sqrt{2}|\bar{\boldsymbol{k}}|(2\pi)^3 A_\odot\,\delta_{\sigma_\parallel}(\omega - \bar{k}^z)\delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y)\,, \tag{650}$$

4 The wave $\langle\alpha^-|\mathbb{A}^\mu|\alpha^-\rangle$ is circularly polarized in the opposite sense.

with $A_\odot$ real. (We use the superscripts t, x, y, and z to denote the corresponding components of $\bar{k}$.) We have introduced two measures of beam spread, $\sigma_\parallel$ and $\sigma_\perp$, along and transverse to the wave direction respectively. The corresponding classical field is,

$$A^\mu_{\rm cl}(x) = \sqrt{2}A_\odot \,\mathrm{Re} \int d^3\bar{k}\, \varepsilon^{*\mu}_\odot(\bar{k})\, \delta_{\sigma_\parallel}(\omega - \bar{k}^z) \\ \times \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y) e^{-i\bar{k}\cdot x}\Big|_{\bar{k}^t=\sqrt{(\bar{k}^x)^2+(\bar{k}^y)^2+(\bar{k}^z)^2}}\,. \tag{651}$$

We emphasize that other choices of wave shape are available in the classical theory: the only constraint is that N_γ must be large.

Let us further refine our example by taking $\sigma_\parallel$ to be very small compared to the other two scales, $\sigma_\perp$ and $\omega = \bar{k}^t_\odot$. We are thus considering a monochromatic beam, for which we can replace $\delta_{\sigma_\parallel}$ by a Dirac delta distribution. Doing so, we obtain,

$$A^\mu_{\rm cl}(x) = \sqrt{2}A_\odot \,\mathrm{Re} \int d^2\bar{k}_\perp\, \varepsilon^{*\mu}_\odot(\bar{k})\, \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y)\, e^{-it\sqrt{\omega^2+(\bar{k}^x)^2+(\bar{k}^y)^2}} e^{i\omega z} e^{i\bar{k}^x x} e^{i\bar{k}^y y}\,. \tag{652}$$

We can simplify this expression with the following considerations. For the beam to be moving in the z-direction, the photons in the beam should dominantly have their momenta, or equivalently their wavenumbers, aligned in the z-direction. However, the broadened distribution $\delta_{\sigma_\perp}$ does allow small components of momentum in the x and y directions. These components should be subdominant. The corresponding

x and y wavenumbers are of order $\sigma_\perp$ while the wavenumber in the z direction is of order ω. Let us define the (reduced) wavelength $\lambda \equiv \omega^{-1}$. We must thus require,

$$\lambda^{-1} \gg \sigma_\perp\,. \tag{653}$$

We can also define a transverse size of the beam,

$$\ell_\perp = \sigma_\perp^{-1}\,, \tag{654}$$

along with a 'pulse length',

$$\ell_\parallel = \sigma_\parallel^{-1}\,. \tag{655}$$

We see that we must require,

$$\lambda \ll \ell_\perp\,. \tag{656}$$

In other words, a collimated monochromatic beam must have a transverse size which is large in units of the beam's wavelength. The requirement (656) is in some respects analogous to the first part of the 'Goldilocks' condition (615). However, we emphasize that eq. (656) arises from our desire to localize the wave in the far past. In particular, waves violating the requirement (656) may still be classical.

Turning back to eq. (652), we may now simplify the time-dependent exponential factor. The broadened delta distribution $\delta_{\sigma_\perp}$ forces,

$$(\bar{k}^x)^2 + (\bar{k}^y)^2 \lesssim \sigma_\perp^2 = \ell_\perp^{-2}\,, \tag{657}$$

so that,

$$\sqrt{\omega^2 + (\bar{k}^x)^2 + (\bar{k}^y)^2} \lesssim \sqrt{\omega^2 + \ell_\perp^{-2}} \simeq \omega + \mathcal{O}(\ell_\perp^{-2}\omega^{-2}) \simeq \omega \,. \tag{658}$$

For the wave's field, we obtain, in this approximation,

$$\begin{aligned} A^\mu_{\rm cl}(x) &= \sqrt{2}\mathcal{A}_\odot \,\mathrm{Re}\Big\{ e^{-i\omega(t-z)} \int d^2\bar{k}_\perp \,\varepsilon^{*\mu}_\odot(\bar{k})\, \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y) e^{i\bar{k}^x x} e^{i\bar{k}^y y} \Big\} \\ &= \sqrt{2}\mathcal{A}_\odot \,\mathrm{Re}\Big\{ e^{-i\omega(t-z)}\, \varepsilon^{*\mu}_\odot(\bar{k}_\odot) \int d^2\bar{k}_\perp \,\delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y) e^{i\bar{k}^x x} e^{i\bar{k}^y y} \Big\}\,, \end{aligned} \tag{659}$$

where we can replace $\varepsilon^\mu_\odot(\bar{k})$ by $\varepsilon^\mu_\odot(\bar{k}_\odot)$ because of the smallness of the transverse components of $\bar{k}$. (Recall that $\bar{k}^\mu_\odot = (\omega, 0, 0, \omega)$.) To continue, we may note that the integral,

$$\int_{-\infty}^{\infty} d\bar{q}\, e^{i\bar{q}x} \delta_\sigma(\bar{q}) = e^{-x^2\sigma^2/4}\,, \tag{660}$$

so that we finally obtain,

$$A^\mu_{\rm cl}(x) = \sqrt{2}\mathcal{A}_\odot \,\mathrm{Re}\Big[e^{-i\omega(t-z)} \varepsilon^{*\mu}_\odot(\bar{k}_\odot)\, e^{-(x^2+y^2)/(4\ell_\perp^2)} \Big]\,. \tag{661}$$

This is indeed a wave of circular polarization along the z-axis, with finite size in the x–y plane.

Our approximation that $\sigma_\parallel$ is negligible gives us a beam of *infinite* spatial extent along the direction of propagation (here, the z axis). Were we to stop short of the $\sigma_\parallel \to 0$ limit, we would find a finite size in this direction too. The occupation number, which is divergent for infinite extent in the z-direction, would also become finite for nonvanishing $\sigma_\parallel$.

The classical field in eq. (661) describes a beam of light that does not spread in the transverse direction, in apparent contradiction to the non-zero transverse momenta the integral contains. This seeming contradiction is lifted when we compute the field of eq. (652) to the next order in $1/(\omega\ell_\perp)$ and $t/\ell_\perp$, as described in App. 11. The result for short enough times is,

$$
\begin{aligned}
A^\mu_{\text{cl}}(x) = \sqrt{2}A_\odot \operatorname{Re}\Bigg\{ & \frac{\exp[-i\omega(t-z)]}{1+i\frac{t}{2\omega\ell_\perp^2}}\varepsilon_\odot^{*\mu}(\bar{k}_\odot)\exp\left[-\frac{(x^2+y^2)}{4\ell_\perp^2[1+it/(2\omega\ell_\perp^2)]}\right]\Bigg\} \\
& + \frac{A_\odot}{\sqrt{2}}\operatorname{Re}\Bigg\{\exp[-i\omega(t-z)]\left[i\frac{x}{\ell_\perp^2}\partial_{\bar{k}^x}\varepsilon_\odot^{*\mu}(\bar{k})\Big|_{\bar{k}=\bar{k}_\odot}+i\frac{y}{\ell_\perp^2}\partial_{\bar{k}^y}\varepsilon_\odot^{*\mu}(\bar{k})\Big|_{\bar{k}=\bar{k}_\odot}\right] \\
& \qquad \times\exp\left[-\frac{(x^2+y^2)}{4\ell_\perp^2}\right]\Bigg\}+\cdots.
\end{aligned}
\tag{662}
$$

GLOBAL OBSERVABLES WITH INCOMING RADIATION

In the previous section, we examined the use of coherent states to describe waves built up of massless messengers (photons or gravitons), and understood that the classical limit emerges in the limit of large occupation number. In this section, we turn to dynamics: we will consider the scattering of a messenger wave and a scalar point particle. Real-life examples are the classical scattering of a light beam off a charged point particle; a light beam scattering gravitationally off a point particle; or a gravitational wave scattering off a point particle.

Our focus in this section will be on *global* observables, obtained by surrounding the scattering event with a distant sphere of detectors. These detectors can register the total change in momentum (or impulse) of the particle, or of the wave, during scattering. These are the same kinds of observables considered in ref. [21]. The main novelty in this section will be the computation of global observables for scattering with incoming classical radiation, which we will describe using the coherent states discussed in the previous section. In the following sections we will discuss local observables.

Two examples will allow us to explore different aspects of the dynamics: the electromagnetic impulse on a charge in a spatially localized beam of light (Thomson scattering); and the General-Relativistic deflection of light in the geometric-optics limit. We begin by discussing the details of the requirements imposed by the dynamics in the classical limit, and the nature of the initial state.

Setup

In the classical limit, the Compton wavelength ℓ_c of a point-like particle must be unobservably small. However, there is (in general) no need for the wavelength of massless waves to be small. On the contrary, finite-wavelength classical waves are quotidian phenomena, and propagate along the pages of many classical-physics textbooks.

In the scattering of two point-like particles, this requirement on ℓ_c would be violated if the particles approach at distances smaller than (or of order of) their

Compton wavelength, because then the underlying wave nature of the particles becomes important. Thus we arrive at the conclusion that classical scattering of two particles obtains only when the impact parameter $b \neq 0$.

In contrast, for a wave of wavelength λ interacting with a particle, we simply require that λ be much larger than the Compton wavelength ℓ_c of the particle. When this is the case, the messengers comprising the wave cannot resolve the quantum structure of the particle. For the classical point-particle approximation to be valid, we further require that λ should be large compared to the finite size ℓ_w of the particle's wave packet. We thus have the requirement,

$$\ell_c \ll \ell_w \ll \lambda\,, \tag{663}$$

for classical interactions of a wave with a particle of Compton wavelength ℓ_c. There is no *a priori* constraint on the impact parameter b.

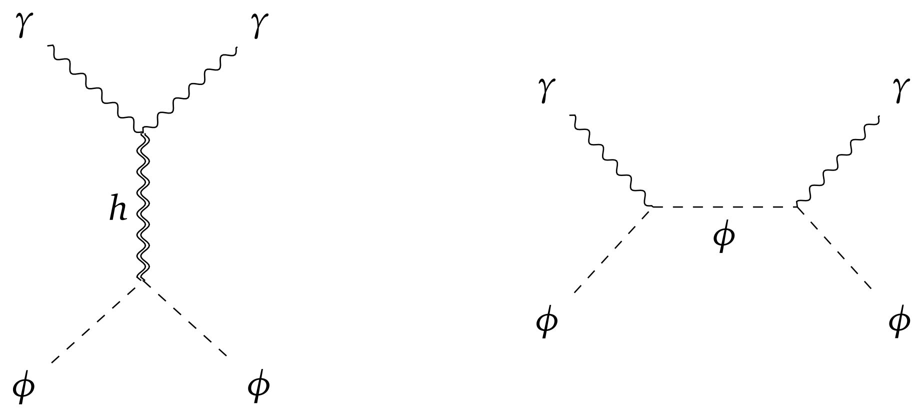


Figure 1: While the t-channel graviton exchange contribution exists for a photon interacting gravitationally with a scalar, this is not true in electromagnetic case

As exemplified in Fig. 1, in the electromagnetic scattering of a photon off a charged particle, there is no t-channel contribution. Correspondingly we are primarily interested in the $b \simeq 0$ case. (More precisely, we are interested in b smaller than the transverse size of the beam.) We will explore this in more detail below. In contrast, in the gravitational scattering of a photon off a neutral particle, there are both s- and t-channel contributions. In this case, we are interested in general b.

The interaction between our particle and our wave introduces another length scale to consider, namely the scattering length ℓ_s. Let $q = \hbar \bar{q}$ be a characteristic momentum exchange associated with the interaction; then the scattering length is defined to be,

$$\ell_s = \frac{1}{\sqrt{|\bar{q}^2|}} . \tag{664}$$

The value of the scattering length depends on the details of the scattering process. In the case where two point-like particles scatter, for instance, one finds that $\ell_s \sim b$. In the case at hand where a particle interacts with a wave this need not be the case. Indeed for an s channel processes it is more natural to expect ℓ_s to be determined by the off-shellness of intermediate propagators such as $s - m^2$. For definiteness let us take the momentum of the incoming particle to be $p_1 = m_1 u_1$ while the incoming wave has characteristic wavenumber $\bar{k}_\odot$. Then $s - m_1^2 = 2\hbar \bar{k}_\odot \cdot p_1$, so that the scattering length should be,

$$\ell_s \sim \frac{1}{\bar{k}_\odot \cdot u_1} . \tag{665}$$

This is simply of the order of the wavelength of the incoming wave.

We turn next to the construction of the incoming state. As in ref. [21] and in eq. (612), we write the point particle as a superposition of plane-wave states weighted by a wavefunction $\varphi(p)$. Following the discussion in the previous section, we write the messenger wave as a coherent state of helicity η characterized by the waveshape $\alpha(k)$. We start with a basis of states constructed out of coherent states (623) of definite helicity $|\alpha^\eta\rangle$ for the messenger and plane-wave states for the massive particle,

$$|p_1\,\alpha_2^\eta\rangle_{\text{in}} = |p_1\rangle|\alpha_2^\eta\rangle\,. \tag{666}$$

Our initial state then takes the form,

$$|\psi_w\rangle_{\text{in}} = \int d\Phi(p_1)\;\varphi_1(p_1)\,e^{ib\cdot p_1/\hbar}|p_1\,\alpha_2^\eta\rangle_{\text{in}}\,. \tag{667}$$

The impact parameter b now separates the particle from the center of the beam in the far past. As in the earlier discussion, the state is normalized to unity, ${}_{\text{in}}\langle\psi_w|\psi_w\rangle_{\text{in}} = 1$. (We will leave the 'in' subscript implicit going forward.)

Information about the classical four-velocity of the point particle is hidden inside $\varphi(p)$. The explicit example studied in ref. [21] made use of a linear exponential (which slightly counter-intuitively reduces to a Gaussian in the nonrelativistic limit). In the same way, the information about the overall momentum $K_\odot$ of the messenger wave is hidden inside $\alpha(k)$.

In the following, we will make use of the coherent wave shape $\alpha(k)$ chosen in eq. (649) corresponding to the choice of $\bar{\alpha}(k)$ of eq. (650), independent of $\hbar$ as desired. We will elucidate inequalities between the various parameters defining the beam below, where relevant.

General Expression for the Impulse

Before we discuss the details of specific examples, let us investigate the general structure of the impulse, $\langle \Delta p_1 \rangle$, on a massive particle during a scattering event with a classical wave. We can carry over the expression from ref. [21],

$$\begin{aligned} \langle \Delta p_1^\mu \rangle &= \langle \psi_w | \, i[\mathbb{P}_1^\mu, T] \, |\psi_w\rangle + \langle \psi_w | \, T^\dagger [\mathbb{P}_1^\mu, T] \, |\psi_w\rangle \\ &= I^\mu_{w(1)} + I^\mu_{w(2)} \, . \end{aligned} \tag{668}$$

Compared to ref. [21], only the initial state is different.

Before studying the expansion of this expression, we remark that there is an equivalent formulation in terms of the background field,

$$\begin{aligned} \langle \Delta p_1^\mu \rangle = &\int d\Phi(p_1) d\Phi(p_1') \, \varphi_1(p_1) \varphi_1^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \langle p_1' | \, \mathbb{C}^\dagger_{\alpha,(\eta)} i[\mathbb{P}_1^\mu, T] \mathbb{C}_{\alpha,(\eta)} \, |p_1\rangle \\ &+ \int d\Phi(p_1) d\Phi(p_1') \, \varphi_1(p_1) \varphi_1^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \langle p_1' | \, \mathbb{C}^\dagger_{\alpha,(\eta)} T^\dagger [\mathbb{P}_1^\mu, T] \mathbb{C}_{\alpha,(\eta)} \, |p_1\rangle \\ = &\int d\Phi(p_1) d\Phi(p_1') \, \varphi_1(p_1) \varphi_1^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \langle p_1' | \, i[\mathbb{P}_1^\mu, T(A_{\rm cl}^{(\eta)})] \, |p_1\rangle \\ &+ \int d\Phi(p_1) d\Phi(p_1') \, \varphi_1(p_1) \varphi_1^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \langle p_1' | \, T^\dagger(A_{\rm cl}^{(\eta)}) [\mathbb{P}_1^\mu, T(A_{\rm cl}^{(\eta)})] \, |p_1\rangle \, , \end{aligned} \tag{669}$$

where the scattering matrix computed from the background $A_{\rm cl}^{(\eta)}$ is denoted by

$T(A_{\rm cl}^{(\eta)})$, and we have used the relation $\mathbb{C}^{\dagger}_{\alpha,(\eta)}\mathbb{C}_{\alpha,(\eta)} = \mathbb{1}$. While we will focus on the formulation (668), it is intriguing to notice the linear term of the impulse $I^{\mu}_{w(1)}$ is closely related to the two-point function of the massive scalar field in the coherent state background. As a consequence, we should expect a resummation of all higher-order results.

Returning to eq. (668), we note that — just as in the scattering of two massive particles — only the first term contributes at leading order (LO) in the generic coupling g. This LO contribution arises at $\mathcal{O}(g^2)$; the second term only contributes starting at $\mathcal{O}(g^4)$. Let us focus on the $I^{\mu}_{w(1)}$ term, and write out the details of the wavefunction (667),

$$I^{\mu}_{w(1)} = \int d\Phi(p_1)d\Phi(p_1')\, e^{-ib\cdot(p_1'-p_1)/\hbar}\varphi_1(p_1)\varphi_1^*(p_1')\, i(p_1'-p_1)^{\mu}\langle p_1'\,\alpha_2^{\eta}|T|p_1\,\alpha_2^{\eta}\rangle\,. \tag{670}$$

The matrix elements of coherent states are not of definite order in perturbation theory. In order to isolate the contributions at each order, one would ordinarily introduce a complete set of states of definite particle number on each side of the T matrix,

$$\begin{aligned} I^{\mu}_{w(1)} = \sum_{X,X'}\sum_{\zeta,\zeta'=\pm}\int & d\Phi(p_1)d\Phi(p_1')d\Phi(r_1)d\Phi(r_1')d\Phi(k_2)d\Phi(k_2') \\ & \times e^{-ib\cdot(p_1'-p_1)/\hbar}\varphi_1(p_1)\varphi_1^*(p_1')\, i(p_1'-p_1)^{\mu} \\ & \times \langle p_1'\,\alpha_2^{\eta}|r_1'\,k_2'^{\zeta'}\,X'\rangle\langle r_1'\,k_2'^{\zeta'}\,X'|T|r_1\,k_2^{\zeta}\,X\rangle\langle r_1\,k_2^{\zeta}\,X|p_1\,\alpha_2^{\eta}\rangle\,. \end{aligned} \tag{671}$$

The sums over X and X' are over different numbers of messengers, including none, and include the phase-space integrals over their momenta. Charge conservation implies that each intermediate state must contain one net massive-particle number; we drop additional particle–antiparticle pairs as their effects will disappear in the classical limit, and we denote the massive-particle momenta by r_1 and r_1'. Moreover, in order to satisfy on-shell conditions of the T matrix element, each intermediate state must contain at least one messenger, whose momenta are denoted by k_2 and k_2'.

The LO contribution to $I^{\mu}_{w(1)}$ is the simplest. One may be tempted to believe that it arises from terms with $X = X' = \varnothing$, but this is not quite right: that would omit disconnected parts of the S-matrix. In the situation at hand, a great many photons are present in the initial state; the dominant contribution to the interaction occurs when most photons pass directly from the initial to the final state. Thus rather than taking $X = X' = \varnothing$, we instead need to sum over additional messengers in the coherent states. These sums over non-interacting messengers, contributing disconnected S-matrix terms, are necessary to recover the correct normalization.

One can carry out these sums explicitly, but it is convenient instead to introduce an alternate representation for the T matrix in terms of creation and annihilation operators. As the incoming state $|\psi_w\rangle$ given in eq. (667) contains one massive particle and an arbitrary number of photons (or messengers more generally), we must consider terms with a pair of massive-particle annihilation and creation operators, and an arbitrary nonzero number of messenger annihilation and creation operators (not necessarily paired). That representation has the form,

$$T = \sum_{\widetilde{\eta},\widetilde{\eta}'} \int d\Phi(\widetilde{r}_1, \widetilde{r}_1', \widetilde{k}_2, \widetilde{k}_2') \, \langle \widetilde{r}_1' \widetilde{k}_2'^{\widetilde{\eta}'} | T | \widetilde{r}_1 \widetilde{k}_2^{\widetilde{\eta}} \rangle \, a^\dagger_{(\widetilde{\eta}')}(\widetilde{k}_2') a^\dagger(\widetilde{r}_1') \, a(\widetilde{r}_1) a_{(\widetilde{\eta})}(\widetilde{k}_2) + \cdots , \quad (672)$$

where the ellipsis indicates higher order terms in the coupling g as well as amplitudes which do not contribute in the classical limit. We will summarily drop all these terms in the following, retaining only the explicit $\mathcal{O}(g^2)$ term. The measure here is a shorthand,

$$d\Phi(\widetilde{r}_1, \widetilde{r}_1', \widetilde{k}_2, \widetilde{k}_2') = d\Phi(\widetilde{r}_1) d\Phi(\widetilde{r}_1') d\Phi(\widetilde{k}_2) d\Phi(\widetilde{k}_2') \, . \quad (673)$$

The advantage of the representation (672) is that the creation and annihilation operators act simply on coherent states, yielding factors of $\alpha(k_2)$ and $\alpha^*(k_2')$, and taking care of the normalization for us. Each term within this representation contains an ordinary (connected) amplitude with a definite number of external messengers.

The required matrix element for the integrand term in eq. (672) can be computed easily,

$$\begin{aligned} \langle p_1' \, \alpha_2^\eta | T | p_1 \, \alpha_2^\eta \rangle &= \langle \widetilde{r}_1' \widetilde{k}_2'^{\widetilde{\eta}'} | T | \widetilde{r}_1 \widetilde{k}_2^{\widetilde{\eta}} \rangle \, \langle p_1' \, \alpha_2^\eta | a^\dagger_{(\widetilde{\eta}')}(\widetilde{k}_2') a^\dagger(\widetilde{p}') \, a_{(\widetilde{\eta})}(\widetilde{k}_2) a(\widetilde{p}) | p_1 \, \alpha_2^\eta \rangle \\ &= \hat{\delta}_\Phi(\widetilde{r}_1 - p_1) \, \hat{\delta}_\Phi(\widetilde{r}_1' - p_1') \, \delta_{\widetilde{\eta},\eta} \delta_{\widetilde{\eta}',\eta} \alpha_2(\widetilde{k}_2) \alpha_2^*(\widetilde{k}_2') \, \langle \widetilde{r}_1' \widetilde{k}_2'^{\widetilde{\eta}'} | T | \widetilde{r}_1 \widetilde{k}_2^{\widetilde{\eta}} \rangle \, , \end{aligned} \quad (674)$$

where we neglected all the terms in the ellipsis of eq. (672). Notice that we encountered the matrix element $\langle \alpha_2^{\eta} | \alpha_2^{\eta} \rangle = 1$: this conveniently takes care of all the disconnected diagrams. The remaining matrix element introduces the desired scattering amplitude,

$$\langle \widetilde{r}_1' \, \widetilde{k}_2'^{\eta'} | T | \widetilde{r}_1 \, \widetilde{k}_2^{\widetilde{\eta}} \rangle = \mathcal{A}(\widetilde{r}_1 \, \widetilde{k}_2^{\widetilde{\eta}} \to \widetilde{r}_1' \, k_2'^{\widetilde{\eta}'}) \, \hat{\delta}^4(\widetilde{r}_1 + \widetilde{k}_2 - \widetilde{r}_1' - \widetilde{k}_2') \,. \tag{675}$$

As usual, the superscripts on the messenger momenta denote the corresponding physical helicity. To write it in the usual amplitudes convention, $A(0 \to p_1, p_2, \ldots)$, we must cross the momenta to the other side. This flips the helicity of incoming messengers.

Using the results of eqs. (674) and (675) in eq. (670) and carrying out the sums over $\widetilde{\eta}, \widetilde{\eta}'$, we obtain,

$$\begin{aligned} I^{\mu}_{w(1)} = \int d\Phi(p_1) d\Phi(p_1') d\Phi(k_2) d\Phi(k_2') \; \varphi_1(p_1) \varphi_1^*(p_1') \alpha_2(k_2) \alpha_2^*(k_2') \\ \times e^{-ib\cdot(p_1' - p_1)/\hbar} \, i(p_1' - p_1)^{\mu} \\ \times \mathcal{A}(p_1 \, k_2^{\eta} \to p_1' \, k_2'^{\eta}) \, \hat{\delta}^4(p_1 + k_2 - p_1' - k_2') \,, \end{aligned} \tag{676}$$

where we have dropped the tildes on k_2 and k_2'.

If we make the usual change of variables to the momentum mismatches $q_{1,2}$,

$$\begin{aligned} q_1 &= p_1' - p_1 \,, \\ q_2 &= k_2' - k_2 \,; \end{aligned} \tag{677}$$

use the delta function to integrate over q_2; and drop the subscript on q_1, we find,

$$\begin{aligned} I^{\mu}_{w(1)} = \int d\Phi(p_1) d\Phi(k_2) \hat{d}^4 q \; \hat{\delta}(2q \cdot p_1 + q^2) \hat{\delta}(2q \cdot k_2 - q^2) \Theta(p_1^t + q^t) \Theta(k_2^t - q^t) \\ \times \varphi_1(p_1) \varphi_1^*(p_1 + q) \, \alpha_2^*(k_2 - q) \alpha_2(k_2) \\ \times e^{-ib \cdot q/\hbar} \, i q^{\mu} \, \mathcal{A}(p_1 \, k_2^{\eta} \to p_1 + q, \, (k_2 - q)^{\eta}) \, . \end{aligned} \tag{678}$$

The analysis of the classical limit as far as the $\varphi_1(p_1)\varphi_1^*(p_1+q)$ factor is concerned is the same as in ref. [21]. It requires us to take the wavenumber mismatch as our integration variable in lieu of the momentum mismatch. At leading order, we do not have to worry about terms singular in $\hbar$, so the evaluation as far as the massive particle is concerned will take,

$$\begin{aligned} \hat{\delta}(2q \cdot p_1 + q^2) \to \hbar^{-1} \hat{\delta}(2\bar{q} \cdot p_1) \, , \\ \varphi(p_1 + q) \to \varphi(p_1) \, . \end{aligned} \tag{679}$$

Removing the coupling from inside the scattering amplitude (as in ref. [21], the reduced amplitude is denoted by $\bar{\mathcal{A}}$), we find for the classical limit,

$$\begin{aligned} I^{\mu,\text{cl}}_{w(1)} = g^2 \left\langle\!\!\left\langle \int d\Phi(\bar{k}_2) \hat{d}^4 \bar{q} \; \hat{\delta}(2\bar{q} \cdot p_1) \hat{\delta}(2\bar{q} \cdot \bar{k}_2 - \bar{q}^2) \, \Theta(\bar{k}_2^t - \bar{q}^t) \; \bar{\alpha}_2^*(\bar{k}_2 - \bar{q}) \bar{\alpha}_2(\bar{k}_2) \right.\right. \\ \left.\left. \times \; e^{-ib \cdot \bar{q}} \; i \bar{q}^{\mu} \; \bar{\mathcal{A}}(p_1 \, \hbar \bar{k}_2^{\eta} \to p_1 + \hbar \bar{q}, \, \hbar(\bar{k}_2 - \bar{q})^{\eta}) \right\rangle\!\!\right\rangle . \end{aligned} \tag{680}$$

As in ref. [21], the double-angle brackets indicate an average over the wave function of the point-like particle. Classically, this is a function of the momentum p_1 with a very sharp peak at $p_1 = m_1 u_1$ where u_1 is the classical (proper) velocity and m_1 is the particle's mass.

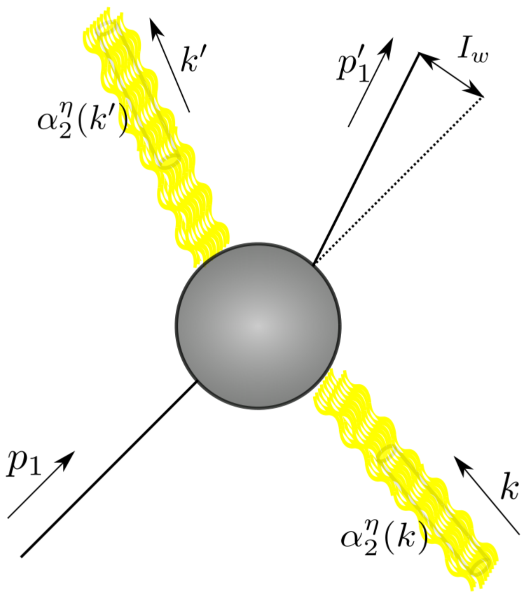


Figure 2: Impulse in scattering of a massive object off a coherent state background.

We can now apply this general result in a variety of specific cases. We shall describe two examples in detail: Thomson scattering of a charge by a wave, with $b \simeq 0$, and gravitational scattering of light by a mass in the geometric-optics limit.

Impulse in Thomson Scattering

Our first application is to Thomson scattering, of a particle of charge Qe and mass m, by a collimated beam of light. We take the light beam to have positive helicity, corresponding to the coherent state $|\alpha^+\rangle$. We need the four-point tree Compton amplitude in scalar QED,

$$\bar{\mathcal{A}}(p_1, k_2^\eta \to p_1', k_2'^{\eta'}) = 2Q^2\, \varepsilon^{(\eta)*}(k_2) \cdot \varepsilon^{(\eta')}(k_2') = 2Q^2\, \varepsilon^{(-\eta)}(k_2) \cdot \varepsilon^{(\eta')}(k_2')\,, \quad (681)$$

where we have chosen the gauge,

$$\varepsilon^{(\eta)} \cdot p_1 = 0\,, \tag{682}$$

for both photons. Alternatively, in spinor variables, we have a gauge-invariant expression for the helicity amplitude, namely

$$\bar{\mathcal{A}}(p_1, k_2^+ \to p_1', k_2'^+) = -\frac{Q^2}{2} \frac{\langle k_2|p_1|k_2']^2}{k_2 \cdot p_1\, k_2' \cdot p_1}\,. \tag{683}$$

This form of the amplitude is manifestly gauge independent, but it depends explicitly on spinors $|k_2'\rangle$ and $|k_2]$ associated with photon momenta. As usual, in the classical limit we prefer to work with photon wavenumbers. We therefore introduce rescaled spinors,

$$|\bar{k}_2'\rangle \equiv \hbar^{-1/2}\,|k_2'\rangle\,, \quad |\bar{k}_2] \equiv \hbar^{-1/2}\,|k_2]\,, \tag{684}$$

which are directly associated with the photon wavenumbers. The amplitude then has the expression,

$$\bar{\mathcal{A}}(p_1, k_2^+ \to p_1', k_2'^+) = -\frac{Q^2}{2} \frac{\langle \bar{k}_2|p_1|\bar{k}_2']^2}{\bar{k}_2 \cdot p_1\, \bar{k}_2' \cdot p_1}\,. \tag{685}$$

Choosing $b = 0$, and for a more symmetric presentation, writing $k = k_2$ and $k' = k_2 - q$, the impulse eq. (680) takes the form,

$$\langle \Delta p^\mu \rangle = \frac{Q^2 e^2}{2} \int d\Phi(\bar{k}) d\Phi(\bar{k}')\, \hat{\delta}(2p \cdot (\bar{k} - \bar{k}'))\, \bar{\alpha}^*(\bar{k}')\bar{\alpha}(\bar{k})\, i(\bar{k}' - \bar{k})^\mu \frac{\langle \bar{k}|p|\bar{k}']^2}{(\bar{k} \cdot p)^2}\,. \tag{686}$$

This expression may be compared with the classical electromagnetic result, obtained by iterating the classical Lorentz force twice. Thus we see in an explicit example that a vanishing impact parameter is perfectly acceptable in the classical scattering of waves off matter, in contrast to the situation for two massive particles scattering.

It is interesting that the Compton amplitude appears at tree level in the classical physics of wave scattering off massive particles. This amplitude is also relevant [67] for purely massive particle scattering, though at one loop order. While the amplitude is very simple for spinless particles, it is considerably more complicated [71] for particles with large spins. Currently we do not have a clear understanding of the appropriate Compton amplitude for the Kerr black hole, or of what principle we could use to determine it. This is an important area for further research. Our work suggests one angle of attack: information about the classical part of the Compton amplitude could be extracted by a purely classical analysis of the impulse on a massive spinning object in scattering off a messenger wave. This is one topic under independent study in ref. [283].

Light Deflection in Gravitational Scattering

A second interesting application of the formulas derived in the previous section is to the gravitational deflection of light by a massive object. We may access this observable by computing the change in momentum of a narrow (small $\ell_\perp$) beam of light passing with non-zero impact parameter b past a massive point-like particle. At leading order, there is no radiation of momentum, so the change in momentum of the wave is simply the negative of the change in momentum of the massive point source: our starting point is once again eq. (680).

Before we discuss the details of the calculation, it is worth dwelling for a moment on our setup. Eddington's famous observations demonstrated that starlight is deflected by the sun in accordance with General Relativity. Near the sun, light emitted by a distant star is essentially a spherical wave, and so the incoming wave is extremely delocalized. In contrast, we have chosen to study a collimated, narrow beam of light. Nevertheless, the difference between our setup and Eddington's case is immaterial. We work in the situation where the wavelength λ of the light is very small compared to the impact parameter: this is the domain of geometric optics, and also applies to Eddington's case. It is in the context of geometric optics that the bending is well-defined; the geometric bending does not depend on the details of the wave.

For our purposes the setup of a narrow beam in the far past is just a simpler place to start. The reason is that we can then determine the bending of light by computing the impulse on the beam: this impulse is directly the change in direction of the wave. By contrast the impulse on starlight due to the sun involves integrating over the whole incoming spherical wavefront: this is not related in a simple manner to the bending of light.

In the geometric-optics regime, we need the wavelength of the light λ to be small. At the same time we must suppress all quantum effects, so we choose λ to be large compared to the Compton wavelength ℓ_c of our point source. To keep our beam collimated, eq. (656) requires that $\ell_\perp \gg \lambda$. The requirement that our beam is narrow is $\ell_\perp \ll b$. Thus there is a series of inequalities:

$$\ell_c \ll \lambda \ll \ell_\perp \ll \ell_s \sim b\,. \tag{687}$$

Note that the scattering length ℓ_s is expected to be of order of the impact parameter in this case, as we are considering a t channel process. For simplicity, we consider a monochromatic beam with $\sigma_{||} \to 0$. The final length scale to consider is the size ℓ_w of the point-particle's wave packet. As usual we require $\ell_c \ll \ell_w \ll \ell_s$. Once these conditions are met, there will be little overlap between the beam and the wave packet, so we do not anticipate that the values of the ratios λ/ℓ_w or $\ell_\perp/\ell_w$ will be important.

The impulse given in eq. (680) simplifies due to the constraints of eq. (687). Note that the quantity $|\bar{q}\cdot\bar{k}_2| \gg |\bar{q}^2|$ in the second delta function, as $\bar{k}_2 \sim 1/\lambda$ while $\bar{q} \sim 1/\ell_s$. The wavenumber $\bar{q}$ is then dominantly in the plane of scattering. In this plane, the coherent waveshape $\bar{\alpha}_2$ is of width $1/\ell_\perp$ so that we may approximate $\bar{\alpha}_2^*(\bar{k}_2 - \bar{q}) \simeq \bar{\alpha}_2^*(\bar{k}_2)$. For the same reason, the explicit theta function in the impulse simplifies: $\Theta(\bar{k}_2^t - \bar{q}^t) = 1$. Taking into account the sign demanded by momentum balance, the impulse on the wave is,

$$\langle \Delta p_2^\mu \rangle = -g^2 \left\langle\!\!\left\langle \int d\Phi(\bar{k}_2) \hat{d}^4\bar{q}\; \hat{\delta}(2\bar{q}\cdot p_1)\hat{\delta}(2\bar{q}\cdot\bar{k}_2)\;\; |\bar{\alpha}_2(\bar{k}_2)|^2 \right.\right.$$
$$\left.\left. \times\; e^{-ib\cdot\bar{q}}\, i\bar{q}^\mu\, \bar{\mathcal{A}}(p_1\, \hbar\bar{k}_2^\eta \to p_1 + \hbar\bar{q}, \hbar(\bar{k}_2 - \bar{q})^\eta) \right\rangle\!\!\right\rangle . \tag{688}$$

The integral over $\bar{k}_2$ is now in a great many respects analogous to the integral over the massive particle wave function which is hidden in our double-angle brackets. In the geometric optics limit, $\bar{\alpha}_2(\bar{k}_2)$ is a steeply-peaked function of the wave number peaked at $\bar{k}_2 = \bar{k}_\odot$; in view of eq. (639), its normalization is related to the number of photons in the beam. The amplitude, meanwhile, is a smooth function in this region. The $\bar{k}_2$ integral then has the structure,

$$\int d\Phi(\bar{k}_2)\, \hat{\delta}(2\bar{q}\cdot\bar{k}_2)\, |\bar{\alpha}_2(\bar{k}_2)|^2 f(\bar{k}_2) \;\simeq f(\bar{k}_\odot) \int d\Phi(\bar{k}_2)\, \hat{\delta}(2\bar{q}\cdot\bar{k}_2)\, |\bar{\alpha}_2(\bar{k}_2)|^2 , \tag{689}$$

where f is a slowly-varying function. We thus encounter the convolution of a delta function and the sharply-peaked $|\alpha_2(k)|^2$. The result of the convolution is a broadened delta function centered at $\bar{k}_2 = \bar{k}_\odot$. Neglecting the width (of order $\sigma_\perp$) of this function we have,

$$\int d\Phi(\bar{k}_2)\, \hat{\delta}(2\bar{q}\cdot\bar{k}_2)\, |\bar{\alpha}_2(\bar{k}_2)|^2 f(\bar{k}_2) \;\simeq f(\bar{k}_\odot)\, N_\gamma \hbar\, \hat{\delta}(2\bar{q}\cdot\bar{k}_\odot) . \tag{690}$$

Notice the appearance of the number of photons N_γ in the beam: this normalization constant emerges from the integral over $|\alpha_2(k)|^2$. The classical geometric optics approximation does not have access to this number of photons, and correspondingly it will cancel in our expression for the deflection angle below. Certain other physical quantities do involve this number of photons: for example, the total momentum of the beam is,

$$K^\mu_\odot = \int d\Phi(\bar{k}) |\bar{\alpha}(\bar{k})|^2 \, \bar{k}^\mu \\ \simeq N_\gamma \hbar \, \bar{k}^\mu_\odot \,. \tag{691}$$

Returning to the impulse on the beam, use of eq. (690) leads to the expression,

$$\langle \Delta p^\mu_{\text{geom}} \rangle = -N_\gamma \hbar \, g^2 \left\langle\!\!\left\langle \int \hat{d}^4\bar{q} \; \hat{\delta}(2\bar{q}\cdot p_1)\hat{\delta}(2\bar{q}\cdot\bar{k}_\odot) \right.\right. \\ \left.\left. \times \; e^{-ib\cdot\bar{q}} \, i\bar{q}^\mu \, \bar{\mathcal{A}}(p_1 \, \hbar\bar{k}^\eta_\odot \to p_1 + \hbar\bar{q}, \, \hbar(\bar{k}_\odot - \bar{q})^\eta) \right\rangle\!\!\right\rangle . \tag{692}$$

The subscript reminds us that the approximation is valid in the geometric-optics limit.

At leading order, we only need the four-point tree-level amplitude. As there are no contributions singular in $\hbar$ at this order, we can simply retain only the terms that survive in the classical limit:

$$\bar{\mathcal{A}}(p_1 \, k_2^\eta \to p_1', \, k_2'^\eta) = \frac{p_1\cdot k_2 \, p_1\cdot k_2'}{q^2} \, \varepsilon^{(\eta)*}(k_2)\cdot\varepsilon^{(\eta)}(k_2') + \cdots \,, \\ = \frac{p_1\cdot \bar{k}_2 \, p_1\cdot \bar{k}_2'}{\bar{q}^2} \, \varepsilon^{(\eta)*}(\bar{k}_2)\cdot\varepsilon^{(\eta)}(\bar{k}_2') + \cdots \,, \tag{693}$$

where we have chosen the gauge $p_1\cdot\varepsilon^{(\eta)}(k) = 0$ for each polarization vector, and the ellipsis indicates terms which are suppressed by powers of $\hbar$.

This amplitude simplifies further in the geometric-optics limit. The inequalities eq. (687) require in particular that the wave number $\bar{q} \sim 1/b \ll \bar{k}_2$. We may therefore replace the scalar product $p\cdot\bar{k}_2'$ with $p\cdot\bar{k}_2$ in eq. (693), up to terms which are neglected in the geometric-optics limit. At the same time, we may replace the polarization vector $\varepsilon^{(\eta)}(\bar{k}_2')$ with $\varepsilon^{(\eta)}(\bar{k}_2)$ to the same order of approximation. The amplitude is then simply,

$$\bar{\mathcal{A}}(p_1 k_2^{\eta} \to p_1', k_2'^{\eta}) = -\frac{(p_1 \cdot \bar{k}_2)^2}{\bar{q}^2} + \cdots \tag{694}$$

We note that the geometric-optics limit of the amplitude for the scattering of a photon off a massive scalar is helicity-independent. Up to constant factors, it reduces to the amplitude between one massless and one massive scalar[5]. This is as expected from the equivalence principle: if the classical limit weren't universal, the impulse and hence the scattering angle would have helicity-dependent contributions.

In order to the evaluate the impulse, we insert the geometric-optics amplitude (694) into the expression (692) for the impulse in the geometric-optics limit. We obtain,

$$\begin{aligned} \langle \Delta p^{\mu}_{\text{geom}} \rangle &= i\kappa^2 \, N_\gamma \hbar \, (p_1 \cdot \bar{k}_\odot)^2 \int \hat{d}^4 \bar{q} \; \hat{\delta}(2\bar{q} \cdot p_1) \hat{\delta}(2\bar{q} \cdot \bar{k}_\odot) \; e^{-ib\cdot\bar{q}} \frac{\bar{q}^{\mu}}{\bar{q}^2} \\ &= i\kappa^2 \, (p_1 \cdot K_\odot)^2 \int \hat{d}^4 \bar{q} \; \hat{\delta}(2\bar{q} \cdot p_1) \hat{\delta}(2\bar{q} \cdot K_\odot) \; e^{-ib\cdot\bar{q}} \frac{\bar{q}^{\mu}}{\bar{q}^2} \, . \end{aligned} \tag{695}$$

Here, we have replaced the general coupling g by the appropriate gravitational coupling κ, and the wavenumber $\bar{k}_\odot$ by the total beam momentum $K_\odot$. The second line of this equation is strikingly similar to the impulse in a scattering process between two *massive* classical objects. Indeed, the integral remaining in eq. (695) is essentially the same as the integral appearing in the LO impulse in ref. [21]. It can

5 See the beautiful and pedagogical discussion in ref. [289] for more details.

easily be performed by taking the light beam in the z direction, $K_\odot^\mu = (E, 0, 0, E)$. The result is,

$$\langle \Delta p^\mu_{\rm geom} \rangle = -\kappa^2 \, \frac{p_1 \cdot K_\odot}{8\pi \, b^2} \, b^\mu \, . \tag{696}$$

The impact parameter b^μ is directed from the massive particle towards the wave, so the sign above indicates that the interaction is attractive.

The scattering angle θ is then determined geometrically in terms of the impulse,

$$\sin\theta = \frac{|b \cdot \Delta p|}{|\boldsymbol{b}| \, E} \, , \tag{697}$$

once we have fixed a frame. We have taken the absolute value to drop the sign of the angle, understanding that the bending is towards the scatterer. Working in the rest frame of the massive scalar, and using $\kappa^2 = 32\pi G_N$, we reproduce the well-known value for the gravitational bending of light,

$$\theta = \frac{4 G_N m}{|\boldsymbol{b}|} + \cdots \, . \tag{698}$$

As a final comment, it is satisfying that the impulse we have obtained in eq. (695) is essentially the same as the impulse on massive point particles as discussed in ref. [21]. This occurred as the inequalities eq. (687) greatly simplified the impulse. These inequalities themselves are very similar to the Goldilocks conditions eq. (615) for classical point-like particles. The fact that the dynamics of massive particles is so similar to the behavior of waves in the geometric-optics regime was a celebrated aspect of nineteenth and early twentieth century physics, known as the Hamiltonian analogy. This analogy was highlighted by Schrödinger [290] and others as an important consideration in the early days of quantum mechanics.

Higher Orders

Although in sections 11 and 11 we focused on leading-order applications, our formalism is completely general and eq. (668) holds to all perturbative orders. As we have seen, the leading-order contribution arises at $\mathcal{O}(g^2)$. The second term, $I^{\mu}_{w(2)}$, in the impulse of eq. (668) involves one-loop amplitudes, and therefore contributes only starting at $\mathcal{O}(g^4)$. Consequently, we can identify a further contribution, at $\mathcal{O}(g^3)$, which receives no contribution from $I^{\mu}_{w(2)}$ but only from $I^{\mu}_{w(1)}$. It arises from the leading corrections to eq. (672),

$$\begin{aligned}\delta T_3 \equiv \sum_{\widetilde{\eta},\widetilde{\eta}',\widetilde{\eta}''} \int d\Phi(\widetilde{r}_1,\widetilde{r}'_1,\widetilde{k}_2,\widetilde{k}'_2,\widetilde{k}_3) \\ \times \big[\langle \widetilde{r}'_1 \widetilde{k}'^{\widetilde{\eta}'}_2 | T | \widetilde{r}_1 \widetilde{k}^{\widetilde{\eta}}_2 \widetilde{k}^{\widetilde{\eta}''}_3 \rangle \, a^{\dagger}_{(\widetilde{\eta}')}(\widetilde{k}'_2) a^{\dagger}(\widetilde{r}'_1)\, a(\widetilde{r}_1) a_{(\widetilde{\eta})}(\widetilde{k}_2) a_{(\widetilde{\eta}'')}(\widetilde{k}_3) \\ + \langle \widetilde{r}'_1 \widetilde{k}'^{\widetilde{\eta}'}_2 \widetilde{k}^{\widetilde{\eta}''}_3 | T | \widetilde{r}_1 \widetilde{k}^{\widetilde{\eta}}_2 \rangle \, a^{\dagger}_{(\widetilde{\eta}'')}(\widetilde{k}_3) a^{\dagger}_{(\widetilde{\eta}')}(\widetilde{k}'_2) a^{\dagger}(\widetilde{r}'_1)\, a(\widetilde{r}_1) a_{(\widetilde{\eta})}(\widetilde{k}_2) \big] ,\end{aligned} \tag{699}$$

where the additional argument in the measure corresponds to a factor of $d\Phi(\widetilde{k}_3)$.

Inserting the integrand of δT_3 into the matrix element in eq. (670), we obtain (analogously to eq. (674)),

$$
\begin{aligned}
&\langle p_1'\, \alpha_2^{\eta} | \delta T_3 | p_1\, \alpha_2^{\eta} \rangle = \\
&\qquad \big[\langle \widetilde{r}_1' \widetilde{k}_2'^{\widetilde{\eta}'} | T | \widetilde{r}_1 \widetilde{k}_2^{\widetilde{\eta}} \widetilde{k}_3^{\widetilde{\eta}''} \rangle \; \langle p_1'\, \alpha_2^{\eta} | a^{\dagger}_{(\widetilde{\eta}')}(\widetilde{k}_2') a^{\dagger}(\widetilde{r}_1')\, a(\widetilde{r}_1) a_{(\widetilde{\eta})}(\widetilde{k}_2) a_{(\widetilde{\eta}'')}(\widetilde{k}_3) | p_1\, \alpha_2^{\eta} \rangle \\
&\qquad + \langle \widetilde{r}_1' \widetilde{k}_2'^{\widetilde{\eta}'} \widetilde{k}_3^{\widetilde{\eta}''} | T | \widetilde{r}_1 \widetilde{k}_2^{\widetilde{\eta}} \rangle \; \langle p_1'\, \alpha_2^{\eta} | a^{\dagger}_{(\widetilde{\eta}'')}(\widetilde{k}_3) a^{\dagger}_{(\widetilde{\eta}')}(\widetilde{k}_2') a^{\dagger}(\widetilde{r}_1')\, a(\widetilde{r}_1) a_{(\widetilde{\eta})}(\widetilde{k}_2) | p_1\, \alpha_2^{\eta} \rangle \big] \\
&\quad = \hat{\delta}_{\Phi}(\widetilde{r}_1 - p_1)\, \hat{\delta}_{\Phi}(\widetilde{r}_1' - p_1')\, \delta_{\widetilde{\eta},\eta} \delta_{\widetilde{\eta}',\eta} \alpha_2(\widetilde{k}_2) \alpha_2^*(\widetilde{k}_2') \\
&\qquad \times \big[\delta_{\widetilde{\eta}'',\eta}\, \alpha_2(k_3) \langle \widetilde{r}_1' \widetilde{k}_2'^{\widetilde{\eta}'} | T | \widetilde{r}_1 \widetilde{k}_2^{\widetilde{\eta}} \widetilde{k}_3^{\widetilde{\eta}''} \rangle + \delta_{\widetilde{\eta}'',\eta}\, \alpha_2^*(k_3) \langle \widetilde{r}_1' \widetilde{k}_2'^{\widetilde{\eta}'} \widetilde{k}_3^{\widetilde{\eta}''} | T | \widetilde{r}_1 \widetilde{k}_2^{\widetilde{\eta}} \rangle \big] \,.
\end{aligned}
\tag{700}
$$

The scattering matrix elements in this expression introduce five-point amplitudes,

$$
\begin{aligned}
\langle \widetilde{r}_1'\, \widetilde{k}_2'^{\widetilde{\eta}'} | T | \widetilde{r}_1\, \widetilde{k}_2^{\widetilde{\eta}}\, \widetilde{k}_3^{\widetilde{\eta}''} \rangle &= \mathcal{A}(\widetilde{r}_1\, \widetilde{k}_2^{\widetilde{\eta}}\, \widetilde{k}_3^{\widetilde{\eta}''} \to \widetilde{r}_1'\, \widetilde{k}_2'^{\widetilde{\eta}'})\, \hat{\delta}^4(\widetilde{r}_1 + \widetilde{k}_2 + \widetilde{k}_3 - \widetilde{r}_1' - \widetilde{k}_2') \,, \\
\langle \widetilde{r}_1'\, \widetilde{k}_2'^{\widetilde{\eta}'}\, \widetilde{k}_3^{\widetilde{\eta}''} | T | \widetilde{r}_1\, \widetilde{k}_2^{\widetilde{\eta}} \rangle &= \mathcal{A}(\widetilde{r}_1\, \widetilde{k}_2^{\widetilde{\eta}} \to \widetilde{r}_1'\, \widetilde{k}_2'^{\widetilde{\eta}'}\, \widetilde{k}_3^{\widetilde{\eta}''})\, \hat{\delta}^4(\widetilde{r}_1 + \widetilde{k}_2 - \widetilde{r}_1' - \widetilde{k}_2' - \widetilde{k}_3) \,.
\end{aligned}
\tag{701}
$$

By crossing, we could choose to identify,

$$
\mathcal{A}(\widetilde{r}_1\, \widetilde{k}_2^{\widetilde{\eta}}\, \widetilde{k}_3^{\widetilde{\eta}''} \to \widetilde{r}_1'\, \widetilde{k}_2'^{\widetilde{\eta}'}) = \mathcal{A}(\widetilde{r}_1\, \widetilde{k}_2^{\widetilde{\eta}} \to \widetilde{r}_1', \widetilde{k}_2'^{\widetilde{\eta}'}, (-\widetilde{k}_3)^{-\widetilde{\eta}''}) \,. \tag{702}
$$

Substituting these expressions into eq. (670) and dropping tildes, we obtain,

$$
\begin{aligned}
I^{\mu}_{w(1)}\big|_{g^3} = \int & d\Phi(p_1) d\Phi(p_1') d\Phi(k_2) d\Phi(k_2') d\Phi(k_3)\; \alpha_2^*(k_2') \alpha_2(k_2) \\
& \times e^{-ib\cdot(p_1' - p_1)/\hbar} \varphi_1(p_1) \varphi_1^*(p_1')\, i(p_1' - p_1)^{\mu} \\
& \times \Big[\alpha_2(k_3) \mathcal{A}(p_1\, k_2^{\eta}\, k_3^{\eta} \to p_1'\, k_2'^{\eta})\, \hat{\delta}^4(p_1 + k_2 + k_3 - p_1' - k_2') \\
& \quad + \alpha_2^*(k_3) \mathcal{A}(p_1\, k_2^{\eta} \to p_1'\, k_2'^{\eta}\, k_3^{\eta})\, \hat{\delta}^4(p_1 + k_2 - p_1' - k_2' - k_3) \Big] \,.
\end{aligned}
\tag{703}
$$

This $\mathcal{O}(g^3)$ term is interesting as it differs in structure from contributions to the impulse for massive-particle scattering studied in ref. [21]. In that case, the first corrections arise at $\mathcal{O}(g^4)$, from one-loop amplitudes in $I^{\mu}_{(1)}$ and cut one-loop amplitudes in $I^{\mu}_{(2)}$. We leave an investigation of the new contributions (703) to future work.

Another difference between purely massive scattering and particle-on-wave scattering relates to the radiation reaction. In the massive case [21], radiation reaction first occurs at next-to-next-to-leading order, that is at $\mathcal{O}(g^6)$. In contrast, radiation reaction arises at $\mathcal{O}(g^4)$ in wave–particle scattering. This radiation reaction must contain contributions from the second term in the impulse, $I^{\mu}_{w(2)}$, which contributes at that order.

POINT-LIKE OBSERVABLES

In the previous section, we built on ref. [21] to analyze what we may call *global* observables, requiring an array of detectors covering the celestial sphere at infinity in order to measure the quantity. This is most manifest for the total radiated momentum, defined by eq. (3.33) of ref. [21],

$$R^{\mu} \equiv \langle k^{\mu} \rangle = {}_{\text{in}}\langle \psi | \, S^{\dagger} \mathbb{K}^{\mu} S \, | \psi \rangle_{\text{in}} = {}_{\text{in}}\langle \psi | \, T^{\dagger} \mathbb{K}^{\mu} T \, | \psi \rangle_{\text{in}} \, . \tag{704}$$

Even in electromagnetic scattering, achieving 4π coverage would make this a challenging measurement. In the gravitational context, where we would be looking to detect emission from scattering of distant black holes, such a measurement would be hopelessly impractical. Instead, for the remainder of this article, we turn to what we may call *local* observables, which can be measured with a localized detector, albeit still sitting somewhere on the celestial sphere, say at x. The paradigm for such a measurement is that of the waveform $W(t, \hat{n}; x)$ of radiation emitted during a scattering event in direction $\hat{n}$ from an event at the coordinate origin. (That is, we adopt the convention that $-\hat{n}$ points back from the observer towards the scattering event.) We will focus on electromagnetic radiation here, but much of the formalism will carry over to the gravitational case. Let us keep in mind that we will be interested in several detectors, all nearby x, though with separations that are completely negligible compared to the distance from the origin.

Local observables have a general structure which, as we will see, is determined by some source (the scattering event) and the propagation of messengers over very large distances. In fact it is convenient to break up our discussion of these observables along these lines. In the present section we will discuss this overall structure in more detail, with a focus on the crucial aspect of propagation. In the following sections, we will extract general expressions for local observables from quantum field theory, and connect to the Newman-Penrose formalism. Then we will examine global observables in cases where a classical wave scatters off a massive particle before turning to the physically important case where two massive particles scatter and radiate.

It will be easier to discuss and manipulate the Fourier transform of the waveform with respect to time. We will refer to this as the spectral waveform $f(\omega, \hat{n}; x)$:

$$f(\omega, \hat{n}; x) = \int_{-\infty}^{+\infty} dt\, W(t, \hat{n}; x)\, e^{i\omega t}\,. \tag{705}$$

Given a result for the spectral waveform, we can of course recover the time-dependent waveform via an inverse Fourier transform. Because we are interested in radiation produced by long-range forces, the idealized waveforms for the scattering processes we will consider stretch infinitely far back and forward in time. The idealization is implicit in the infinite limits for the integral in eq. (705). In an actual measurement, however, the waveform would be below the noise floor of the detector for all times before a 'signal start time' preceding the moment of closest approach, and likewise for all times after a 'signal end time' following that moment. We can then take the theoretical waveforms to be approximations to actual ones cut off at the start and end times. Label the interval between the two by Δt_s.

Let us imagine that the point of closest approach during the scattering event is at the coordinate origin, $(t, x) = (0, 0)$. When a massless wave scatters off a point particle, the wave may overlap the particle; we take a suitable event of maximum overlap as the origin. We can treat the scattering as occurring in a box of temporal length Δt_s, and of spatial size Δx_s. Radiation is emitted inside the box during the scattering event, and then spreads out. We will take an (idealized) measurement of the radiation in some direction $\hat{n}$, at a much later time and at a point very far away in that direction. The details of the scattering — the particles' interaction and spins — will determine the radiation emitted inside the box. Modifying those details could radically change the emission. Those details, however, will have no effect on the propagation of the radiation out to the distant measuring apparatus. Only the spin of the radiated field can have any effect. We thus expect the form of the result to be a Green's function convoluted with a source. More precisely, given that we have only outgoing radiation, we expect a retarded Green's function G_{ret}. We can then expand the Green's function in the large-distance limit to obtain the connection between the observable and the emitted radiation inside the box.

The details of the scattering inside the box around $(0,0)$ define a current for our radiation. In a real-world context, we are interested in electromagnetic or gravitational radiation, but we can equally well treat the case of (massless) scalar radiation as well. The details of the scattering inside the box give rise to a wavenumber-space field-strength current, $\widetilde{J}_{\vec{\mu}}(\bar{k})$, where the notation $\vec{\mu}$ denotes a number of indices appropriate to the radiated messenger: none for a scalar, two for a photon, and four for a graviton,

$$\begin{aligned} \widetilde{J}(\bar{k}) : &\qquad \text{scalar}\,, \\ \widetilde{J}_{\mu\nu}(\bar{k}) : &\qquad \text{electromagnetism}\,, \\ \widetilde{J}_{\mu\nu\rho\sigma}(\bar{k}) : &\qquad \text{gravity}\,. \end{aligned} \tag{706}$$

In a slight abuse of language, we will refer to these quantities simply as currents.

They will satisfy appropriate conservation conditions. We will later obtain an expression for such a current in terms of scattering amplitudes.

Given this current, the usual position-space current can of course be obtained by taking a Fourier transform,

$$J_{\vec{\mu}}(x) = \int \hat{d}^4\bar{k}\, \widetilde{J}_{\vec{\mu}}(\bar{k})\, e^{-i\bar{k}\cdot x}\,. \tag{707}$$

Clearly we can also write $\widetilde{J}_{\vec{\mu}}(\bar{k})$ in terms of $J_{\vec{\mu}}(x)$ via an inverse transform,

$$\widetilde{J}_{\vec{\mu}}(\bar{k}) = \int d^4x\, J_{\vec{\mu}}(x)\, e^{i\bar{k}\cdot x}\,. \tag{708}$$

Both of these forms of the current will be helpful for us below.

As we will show in detail in the next section, we obtain an x-dependent radiation observable in the general form,

$$R_{\vec{\mu}}(x) = i \int d\Phi(\bar{k})\, \left[\widetilde{J}_{\vec{\mu}}(\bar{k})\, e^{-i\bar{k}\cdot x} - \widetilde{J}^*_{\vec{\mu}}(\bar{k})\, e^{+i\bar{k}\cdot x}\right], \tag{709}$$

that is, as an integral of the source $\widetilde{J}_{\vec{\mu}}(\bar{k})$ over the on-shell massless phase space for the radiated messenger. Examples will include expectations of hermitian operators, such as the field-strength operator in electromagnetism, or the Riemann tensor in gravity.

The hermiticity properties of our radiation observables is manifest in eq. (709). But notice that the observables are defined as integrals over positive frequencies $\bar{k}^t \geq 0$. Yet in writing the innocuous-seeming Fourier transform in eq. (707), we have assumed knowledge of the current for both positive *and* negative frequency.

So we must fill a gap: what do we mean by the current for negative frequency? In fact, the reality condition provides the necessary information. Our currents are real in position space, and we may note that,

$$J_{\vec{\mu}}(x) = \int \hat{d}^4\bar{k}\, \theta(\bar{k}^t) \left[\widetilde{J}_{\vec{\mu}}(\bar{k})\, e^{-i\bar{k}\cdot x} + \widetilde{J}_{\vec{\mu}}(-\bar{k}) e^{i\bar{k}\cdot x}\right] . \tag{710}$$

The reality condition then leads to the relation,

$$\widetilde{J}_{\vec{\mu}}(-\bar{k}) = \widetilde{J}^*_{\vec{\mu}}(\bar{k}) . \tag{711}$$

We use this relation to define the current for negative frequency.

A key simplification arises because the source event, occurring in our box, is sourced in a comparatively localized region compared to the very large propagation distance of the outgoing radiation. To access this simplification, we follow a well-trodden path [291] by rewriting our radiation observables as integrals over the spatial extent of the source. Thus, we express the observable of eq. (709) in terms of the spatial current $J_{\vec{\mu}}(x)$, yielding

$$R_{\vec{\mu}}(x) = i \int d\Phi(\bar{k})\, d^4y\; J_{\vec{\mu}}(y) \left[e^{-i\bar{k}\cdot(x-y)} - e^{+i\bar{k}\cdot(x-y)}\right] . \tag{712}$$

Next, we interchange orders of integration. Judicious forethought reveals the combination of phase space integrals to be a difference of retarded and advanced Green's functions,

$$R_{\vec{\mu}}(x) = \int d^4y\; J_{\vec{\mu}}(y) \left[G_{ret}(x-y) - G_{adv}(x-y)\right] . \tag{713}$$

In the far future, where the observer measures the wavetrain emitted from the scattering event, G_{adv} will vanish. Put in an explicit form for G_{ret}, and switch back to the wavenumber-space current in order to make the complete dependence of the integrand on x and y manifest. The result is,

$$\begin{aligned} R_{\vec{\mu}}(x) &= \int \hat{d}\omega \hat{d}^3\bar{\boldsymbol{k}}\, d^4y\, \widetilde{J}_{\vec{\mu}}(\bar{k})\, e^{-i\bar{k}\cdot y}\, \frac{\delta(x^0 - y^0 - |\boldsymbol{x} - \boldsymbol{y}|)}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \\ &= \int \hat{d}\omega \hat{d}^3\bar{\boldsymbol{k}}\, d^3\boldsymbol{y}\, \widetilde{J}_{\vec{\mu}}(\bar{k})\, \frac{e^{-i\omega x^0}\, e^{+i\omega|\boldsymbol{x}-\boldsymbol{y}|}\, e^{+i\bar{\boldsymbol{k}}\cdot\boldsymbol{y}}}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\,. \end{aligned} \tag{714}$$

Notice that the integral is now over *all* wavenumbers. We have split the four-dimensional momentum integration into integrals over spatial and frequency components for later convenience.

From the earlier discussion, we know that $J_{\vec{\mu}}(y)$ is concentrated around $y \simeq 0$, whereas x is far away ($x \gg y$). Accordingly we can expand the integrand there, using,

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{y}| &\sim \left[\boldsymbol{x}^2 - 2\boldsymbol{x}\cdot\boldsymbol{y}\right]^{1/2} \\ &\sim |\boldsymbol{x}|\left(1 - \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{y}}{|\boldsymbol{x}|}\right). \end{aligned} \tag{715}$$

We must be careful in performing this expansion: while it is sufficient to retain the leading term in the denominator, we must retain formally subleading terms that contribute to nontrivial phases. Even in those exponents, we can of course still drop terms beyond the subleading, as they give rise to no nontrivial phases.

Substituting the expansion (715) into eq. (714), we obtain,

$$R_{\vec{\mu}}(x) = \int \hat{d}\omega \hat{d}^3\bar{\boldsymbol{k}}\, d^3\boldsymbol{y}\, \widetilde{J}_{\vec{\mu}}(\bar{k})\, \frac{e^{-i\omega x^0}\, e^{+i\omega|\boldsymbol{x}|} e^{-i\omega\hat{n}\cdot\boldsymbol{y}}\, e^{+i\bar{\boldsymbol{k}}\cdot\boldsymbol{y}}}{4\pi|\boldsymbol{x}|}\,; \tag{716}$$

performing in turn the $\boldsymbol{y}$ and $\boldsymbol{k}$ integrals, we finally obtain,

$$\begin{aligned} R_{\vec{\mu}}(x) &= \frac{(2\pi)^3}{4\pi|\boldsymbol{x}|}\int \hat{d}\omega \hat{d}^3\bar{\boldsymbol{k}}\, \widetilde{J}_{\vec{\mu}}(\bar{k})\, e^{-i\omega x^0}\, e^{+i\omega|\boldsymbol{x}|}\, \delta^3(\bar{\boldsymbol{k}}-\omega\hat{n}) \\ &= \frac{1}{4\pi|\boldsymbol{x}|}\int \hat{d}\omega\, \widetilde{J}_{\vec{\mu}}(\omega,\omega\hat{n})\, e^{-i\omega(x^0-|\boldsymbol{x}|)}\,. \end{aligned} \tag{717}$$

We can thus identify the waveform with the coefficient of the leading-power term $|\boldsymbol{x}|^{-1}$,

$$W_{\vec{\mu}}(t,\hat{n};x) = \frac{1}{4\pi}\int \hat{d}\omega\, \widetilde{J}_{\vec{\mu}}(\omega,\omega\hat{n})\, e^{-i\omega(x^0-|\boldsymbol{x}|)} \quad . \tag{718}$$

In this equation, t represents the observer's clock time. We could take it to be x^0, or $x^0 - |\boldsymbol{x}|$, or some other convenient time. We must nonetheless retain the separate dependence on x^0 and $|\boldsymbol{x}|$, because these quantities will differ between the cluster of nearby observers in which we are interested. That is, the absolute phase of the waveform at any given observer's location is not measurable and is therefore irrelevant, but the relative phases between nearby observers are measurable.

Choosing $t = x^0 - |\boldsymbol{x}|$, the corresponding spectral waveform is then simply,

$$f_{\vec{\mu}}(\omega,\hat{n}) = \frac{1}{4\pi}\widetilde{J}_{\vec{\mu}}(\omega,\omega\hat{n})\,. \tag{719}$$

More precisely, eq. (719) is the waveform for positive frequencies. For negative frequencies, the waveform follows from eq. (711),

$$f_{\vec{\mu}}(\omega,\hat{n}) = \frac{1}{4\pi}\widetilde{J}^*_{\vec{\mu}}(-\omega,-\omega\hat{n})\,; \tag{720}$$

notice that $-\omega$ is now positive. In both cases, once we know the current $\widetilde{J}_{\vec{\mu}}(\bar{k})$, we can immediately write down the spectral waveform.

SPECTRAL WAVEFORMS

As we have seen, the waveform is directly related to the current $\widetilde{J}_{\vec{\mu}}(\bar{k})$ generated by the scattering event. We must choose a specific local radiation observable to determine this current using its definition, eq. (709). In this section we will study examples in both electrodynamics and gravity. Let us begin with a simple case: the field-strength tensor (620) in electrodynamics.

We choose an observer at x, in the far future of the event, equipped to measure the expectation value of the electric and magnetic field at the point x. The observable is therefore

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle \equiv {}_{\text{out}}\langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle_{\text{out}}\,. \tag{721}$$

We can rewrite the outgoing state in terms of the incoming state using the time-evolution operator or S-matrix,

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle = {}_{\text{in}}\langle\psi|S^{\dagger}\mathbb{F}_{\mu\nu}(x)S|\psi\rangle_{\text{in}}\,, \tag{722}$$

where (as usual) $|\psi\rangle_{\text{in}}$ is the incoming state in the far past. This state could contain, for example, two isolated massive point-like particles, or a single isolated massive particle and a coherent state describing incoming radiation. A state of the former type would be appropriate to study radiation emitted as two particles scatter, while a state of the latter type can be used to study the scattered radiation field in a Thomson scattering process. We will study both of these examples in detail later in this article.

Inserting the expression for the field-strength tensor (620) into this expectation value, and converting to integrals over wavenumbers, we learn that,

$$\begin{aligned}\langle F^{\text{out}}_{\mu\nu}(x)\rangle = -2i\hbar^{3/2}\sum_\eta \int d\Phi(\bar{k})\, \big[\langle\psi|S^\dagger a_{(\eta)}(k)S|\psi\rangle\, \bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k})\, e^{-i\bar{k}\cdot x} \\ - \langle\psi|S^\dagger a^\dagger_{(\eta)}(k)S|\psi\rangle\, \bar{k}_{[\mu}\varepsilon^{(\eta)}_{\nu]}(\bar{k})\, e^{+i\bar{k}\cdot x}\big]\,,\end{aligned} \tag{723}$$

where we have again dropped the 'in' subscript, leaving it implicit in the rest of our discussion. (Recall that k is just a label for the creation and annihilation operators, and we can use $\bar{k}$ interchangeably for this purpose.)

We now see the virtue of our definition of the general class of radiation observables in eq. (709). Evidently the expectation value $\langle F^{\text{out}}_{\mu\nu}(x)\rangle$ is of precisely this form, and we can read off the current $\widetilde{J}_{\vec{\mu}}(\bar{k})$ as

$$\widetilde{J}_{\mu\nu}(\bar{k}) = -2\hbar^{3/2}\sum_\eta \langle\psi|S^\dagger a_{(\eta)}(k)S|\psi\rangle\, \bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k})\,. \tag{724}$$

The discussion of the previous section therefore applies, and we see from eq. (719) that the corresponding spectral waveform is,

$$f_{\mu\nu}(\omega,\hat{n}) = -\frac{1}{2\pi}\hbar^{3/2}\sum_\eta \langle\psi|S^\dagger a_{(\eta)}(k)S|\psi\rangle\, \bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k})\Big|_{\bar{k}=(\omega,\omega\hat{n})}\,, \tag{725}$$

for positive frequency ($\omega > 0$). For negative frequency ($\omega < 0$) the waveform is,

$$f_{\mu\nu}(\omega,\hat{n}) = -\frac{1}{2\pi}\hbar^{3/2}\sum_\eta \langle\psi|S^\dagger a^\dagger_{(\eta)}(k)S|\psi\rangle\, \bar{k}_{[\mu}\varepsilon^{(\eta)}_{\nu]}(\bar{k})\Big|_{\bar{k}=-(\omega,\omega\hat{n})} \tag{726}$$

This result holds to all orders in perturbation theory.

It is straightforward to extend this result to gravity. We work in Einstein gravity, and assume that the spacetime is asymptotically Minkowskian. In this case our observer at x is very far from the source of gravitational waves, and is equipped to measure the expectation value of the local spacetime curvature $\langle R^{\text{out}}_{\mu\nu\rho\sigma}(x)\rangle$. The corresponding spectral waveform is nothing but the double copy of eq. (725),

$$f_{\mu\nu\rho\sigma}(\omega,\hat{n}) = \frac{i\kappa}{2\pi}\hbar^{3/2}\sum_{\eta}\langle\psi|S^{\dagger}a_{(\eta)}(k)S|\psi\rangle\,\bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k})\,\bar{k}_{[\rho}\varepsilon^{(\eta)*}_{\sigma]}(\bar{k})\Big|_{\bar{k}=(\omega,\omega\hat{n})}, \tag{727}$$

for $\omega > 0$. In this equation, the operator $a_{(\eta)}(k)$ annihilates perturbative gravitational states. We have included a factor $\kappa/2$ so that the Riemann tensor has the conventional normalization. Noting that the metric perturbation falls off as inverse distance, it follows that non-linear terms in the Riemann tensor produce corrections which fall off faster than inverse distance. Consequently, we have neglected them. Notice that all possible traces of eq. (727) vanish, consistent with the fact that the Riemann tensor in vacuum equals the Weyl tensor. The waveform for negative frequency is,

$$f_{\mu\nu\rho\sigma}(\omega,\hat{n}) = -\frac{i\kappa}{2\pi}\hbar^{3/2}\sum_{\eta}\langle\psi|S^{\dagger}a^{\dagger}_{(\eta)}(k)S|\psi\rangle\,\bar{k}_{[\mu}\varepsilon^{(\eta)}_{\nu]}(\bar{k})\,\bar{k}_{[\rho}\varepsilon^{(\eta)}_{\sigma]}(\bar{k})\Big|_{\bar{k}=-(\omega,\omega\hat{n})}. \tag{728}$$

The Lorentz indices on these observables reflects the tensor structure of electrodynamics and gravity. In both cases, however, there are only two possible polarizations of the outgoing radiation. It is helpful to project the waveform onto one of these polarizations. Classically, a convenient way to do so is to use the Newman–Penrose

(NP) [265] formalism, which is intimately connected to the spinor-helicity method of scattering amplitudes [210, 211, 283]. We can adopt the same idea in the present context. For us, a simple route to the NP formalism is to pick a complex basis of vectors which is aligned with our setup. We choose the vectors[6]

$$L^\mu = \bar{k}^\mu/\omega = (1, \hat{n})^\mu, \quad N^\mu = \zeta^\mu, \quad M^\mu = \varepsilon^{(+)\mu}, \quad M^{*\mu} = \varepsilon^{(-)\mu}. \tag{729}$$

The null vector ζ is simply a gauge choice, satisfying $\zeta \cdot \varepsilon^{(\pm)} = 0$ and $L \cdot N = L \cdot \zeta = 1$. Furthermore note that $M \cdot M^* = -1$. The scaling of the NP vector L ensures that it does not depend on frequency ω, and is dimensionless. Indeed the polarization vectors $\varepsilon^{(\pm)}$ do not depend on the scaling of $\bar{k}$ so they are also independent of frequency. These vectors therefore make sense as a spacetime basis, not merely as a basis in Fourier space.

It is easy to check that the only non-zero components of $f_{\mu\nu}$ in the NP basis are $f_{\mu\nu} M^{*\mu} N^\nu$ and $f_{\mu\nu} M^\mu N^\nu$. These are the leading radiative NP scalar, traditionally [292] denoted Φ_2^0, and its conjugate. We can write these NP scalars as Fourier transforms:

$$\Phi_2^0(t, \hat{n}) = \int \hat{d}\omega \, e^{-i\omega t} \, \widetilde{\Phi}_2^0(\omega, \hat{n}) \,. \tag{730}$$

Notice that we commuted the NP basis vectors through the frequency integration

6 We use capital letters to denote the elements of our NP basis rather than the more traditional lower case symbols in order to distinguish the vectors from loop momenta, masses, et cetera.

sign. This is permissible as the basis vectors are independent of frequency. For positive frequency ω, we find,

$$\widetilde{\Phi}_2^0(\omega,\hat{n}) = -\frac{\omega}{4\pi}\hbar^{3/2}\langle\psi|S^\dagger a_{(-)}(k)S|\psi\rangle\Big|_{\bar{k}=(\omega,\omega\hat{n})}, \tag{731}$$

while for negative frequency, the corresponding expression reads,

$$\widetilde{\Phi}_2^0(\omega,\hat{n}) = +\frac{\omega}{4\pi}\hbar^{3/2}\langle\psi|S^\dagger a^\dagger_{(+)}(k)S|\psi\rangle\Big|_{\bar{k}=-(\omega,\omega\hat{n})}. \tag{732}$$

Combining these results, we find that the time-domain NP scalar is,

$$\begin{aligned}\Phi_2^0(t,\hat{n}) = -\frac{\hbar^{3/2}}{4\pi}\int \hat{d}\omega\,\Theta(\omega)\,\omega\big[&e^{-i\omega t}\langle\psi|S^\dagger a_{(-)}(k)S|\psi\rangle \\ &+ e^{+i\omega t}\langle\psi|S^\dagger a^\dagger_{(+)}(-k)S|\psi\rangle\big]\Big|_{\bar{k}=(\omega,\omega\hat{n})}.\end{aligned} \tag{733}$$

In gravity, the corresponding radiative NP scalar is defined by

$$\Psi_4(x) = -N_\mu M^*_\nu N_\rho M^*_\sigma \langle W^{\mu\nu\rho\sigma}(x)\rangle\,, \tag{734}$$

where $W^{\mu\nu\rho\sigma}(x)$ is the Weyl tensor, equal to the Riemann tensor in our case. Expanded at large distances, the leading term in the NP scalar is Ψ_4^0:

$$\Psi_4(x) = \frac{1}{|\boldsymbol{x}|}\Psi_4^0 + \cdots\,. \tag{735}$$

This object is directly relevant to gravitational waveforms [190, 293]. We find that the spectral version of the NP scalar is,

$$\widetilde{\Psi}_4^0(\omega,\hat{n}) = -i\frac{\kappa\,\omega^2}{8\pi}\hbar^{3/2}\langle\psi|S^\dagger a_{(--)}(k)S|\psi\rangle\Big|_{\bar{k}=(\omega,\omega\hat{n})}, \tag{736}$$

for positive ω. Let us emphasize once again that these results hold to all orders of perturbation theory.

NP scalars are particularly well-suited for comparison with helicity amplitudes in quantum field theory. However, they may be slightly less familiar than the more elementary field strengths; field strengths also have the virtue of being hermitian quantities. Therefore, in the remainder of this article, we will also study the expectation of the radiative field-strength tensor in perturbation theory. This entails rewriting the scattering matrix in terms of the transition matrix T, $S = 1 + iT$,

$$\begin{aligned} \langle F^{\text{out}}_{\mu\nu}(x)\rangle &= \langle\psi|(1 - iT^\dagger)\mathbb{F}_{\mu\nu}(x)\,(1 + iT)|\psi\rangle \\ &= \langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle + 2\,\text{Re}\,i\langle\psi|\mathbb{F}_{\mu\nu}(x)T|\psi\rangle + \langle\psi|T^\dagger\mathbb{F}_{\mu\nu}(x)T|\psi\rangle\,. \end{aligned} \tag{737}$$

The first term in eq. (737) is the expectation value of the field strength due to any incoming radiation which may be present in $|\psi\rangle_{\text{in}}$; the following term is linear in amplitudes, and thus of $\mathcal{O}(g^3)$ (or higher); the last term is quadratic in amplitudes (or equivalently, linear in a cut amplitude), and contains terms of $\mathcal{O}(g^5)$ and higher.

Using unitarity, we can rewrite eq. (737),

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle(x) = \langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle + i\langle\psi|[\mathbb{F}_{\mu\nu}(x), T]|\psi\rangle + \langle\psi|T^\dagger[\mathbb{F}_{\mu\nu}(x), T]|\psi\rangle\,. \tag{738}$$

The commutator in the second term of this expression is reminiscent of the form of the impulse Δp (although in case of the field strength, the first term above need not vanish). This second form of the field strength can be both instructive and useful, but it has a slight disadvantage that reality properties are somewhat obscured compared to eq. (737). When taking the classical limit, we are interested in the leading term in the large-distance expansion as well; for such radiation observables, we will understand the $\langle\!\langle \cdots \rangle\!\rangle$ notation to impose that expansion as well.

We will use this observable to analyze emitted radiation in the scattering of two charged particles in Sect. 11. We first continue our analysis of Thomson scattering in the next section.

from compton scattering to thomson scattering

In Sect. 11, we considered the Thomson scattering process: electromagnetic scattering of a classical beam off of a massive point charge. In our earlier discussion we studied the impulse suffered by the massive particle during the process. We are now equipped to deepen our analysis by determining the scattered light generated during Thomson scattering. We will do so by using the results of the previous section to compute the NP scalar Φ_2^0 which describes that scattered light at very large distances.

In this situation, our initial state eq. (667) describes an isolated massive particle, and a localized beam of incoming classical radiation described as in Sect. 11 by a coherent state with an appropriate waveshape function. Correspondingly, the

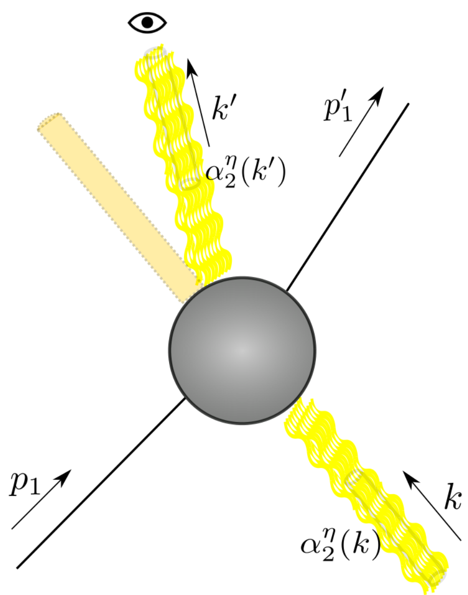


Figure 3: The observer measures the field strength of the outgoing wave.

incoming state generates a non-vanishing expectation value for the electromagnetic field strength tensor. This is the incoming classical radiation $\langle F^{\text{in}}_{\mu\nu}(x)\rangle$:

$$\langle F^{\text{in}}_{\mu\nu}(x)\rangle = \langle\psi_w|\mathbb{F}_{\mu\nu}|\psi_w\rangle\,. \tag{739}$$

In particular, there is a non-vanishing NP scalar Φ_2^0 in the far past.

To focus attention on the scattered light, it is convenient to study the overall change in the NP scalar during the process,

$$\Delta\Phi_2^0(\omega,\hat{n}) = -\frac{\omega}{4\pi}\hbar^{3/2}\left[\langle\psi_w|S^\dagger a_{(-)}(k)S|\psi_w\rangle - \langle\psi_w|a_{(-)}(k)|\psi_w\rangle\right]\Big|_{\bar{k}=(\omega,\omega\hat{n})}. \tag{740}$$

This simply subtracts the contribution of the incoming beam to the radiation field in the future. We will compute this quantity at leading order, focusing on the positive-frequency part throughout.

Using unitarity of the S matrix, we may write $\Delta\Phi_2^0$ in terms of a commutator,

$$\Delta\Phi_2^0(\omega,\hat{n}) = -\frac{i}{4\pi}\omega\hbar^{3/2}\langle\psi_w|[a_{(-)}(k_2'),T]|\psi_w\rangle\Big|_{\bar{k}_2'=(\omega,\omega\hat{n})} . \tag{741}$$

We relabeled the quantity $\bar{k}$ appearing in eq. (740) as $\bar{k}_2'$ because, as we will see below, it has the interpretation of the wavevector associated with the outgoing wave which was denoted $\bar{k}_2'$ in Sect. 11.

To compute the commutator $[a_{(-)}(k_2'),T]$, we make use of eq. (672) to expand the T matrix in terms of creation and annihilation operators. Dropping the terms in the ellipsis of eq. (672), the commutator is easily computed to be,

$$[a_{(-)}(k_2'),T] = \sum_{\widetilde{\eta}}\int d\Phi(\widetilde{r}_1,\widetilde{r}_1',\widetilde{k}_2)\; \langle\widetilde{r}_1'k_2'^{-}|T|\widetilde{r}_1\widetilde{k}_2^{\widetilde{\eta}}\rangle\; a^\dagger(\widetilde{r}_1')a(\widetilde{r}_1)\,a_{(\widetilde{\eta})}(\widetilde{k}_2)\,. \tag{742}$$

Inserting this result in eq. (741), and expanding the state $|\psi_w\rangle$ using its definition eq. (667) specialized to the case $b=0$ we easily find that,

$$\begin{aligned}\Delta\Phi_2^0(\omega,\hat{n}) &= \\ &-\frac{i}{4\pi}\omega\hbar^{3/2}\sum_{\eta}\int d\Phi(p_1,p_1',k_2)\;\varphi^*(p_1')\varphi(p_1)\;\langle p_1'k_2'^{-}|T|p_1k_2^{\eta}\rangle\;\langle\alpha^+|a_{(\eta)}(k_2)|\alpha^+\rangle \\ &= -\frac{i}{4\pi}\omega\hbar^{3/2}\int d\Phi(p_1,p_1',k_2)\;\varphi^*(p_1')\varphi(p_1)\;\langle p_1'k_2'^{-}|T|p_1k_2^{+}\rangle\;\alpha(k_2)\,.\end{aligned} \tag{743}$$

The matrix element of the transition operator yields the Compton amplitude, as well as the usual delta function enforcing overall momentum conservation. We may perform the p_1' integral using this delta function to find that,

$$\Delta\Phi_2^0(\omega,\hat{n}) = \\ -\frac{i}{4\pi}\omega\hbar^{3/2}\int d\Phi(p_1,k_2)\,\hat{\delta}(2p_1\cdot(k_2'-k_2))\,|\varphi(p_1)|^2\,\alpha(k_2)\,\mathcal{A}(p_1k_2^+\to p_1'k_2'^-)\,. \tag{744}$$

We replaced the (conjugated) wavefunction $\varphi^*(p_1'+k_2-k_2')$ by $\varphi^*(p_1')$ because the difference $(k_2-k_2')/\hbar = \bar{k}_2-\bar{k}_2'$ is small (of order $1/\lambda$) compared to the width of the wavefunction (which is of order $1/\ell_w$). The integral over the wavefunction is now precisely of the form required for the double-angle-bracket notation of ref. [21] so that we arrive at,

$$\Delta\Phi_2^0(\omega,\hat{n}) = -\frac{i}{4\pi}\left\langle\!\!\left\langle \omega\hbar^{3/2}\int d\Phi(k_2)\,\hat{\delta}(2p_1\cdot(k_2'-k_2))\,\alpha(k_2)\,\mathcal{A}(p_1k_2^+\to p_1'k_2'^-)\right\rangle\!\!\right\rangle. \tag{745}$$

Finally, we insert the explicit Compton amplitude of eq. (681), and replace the remaining integral over k_2 with an integral over the associated wavenumber $\bar{k}_2$ to learn that the LO NP scalar due to the scattering process is,

$$\Delta\Phi_2^0(\omega,\hat{n}) = i\frac{Q^2e^2}{16\pi}\left\langle\!\!\left\langle \omega\int d\Phi(\bar{k}_2)\,\hat{\delta}(2p_1\cdot(\bar{k}_2'-\bar{k}_2))\,\bar{\alpha}(k_2)\,m^2\frac{\langle\bar{k}_2\bar{k}_2'\rangle}{[\bar{k}_2\bar{k}_2']\,\bar{k}_2\cdot p_1}\right\rangle\!\!\right\rangle. \tag{746}$$

The same result would also be obtained from a classical analysis of the leading order radiation field of a point charge moving under the influence of an incoming classical wave.

Alternatively, it is possible to compute the expectation value of the field strength in the very far future. Focusing again on the change in the field strength,

$$\langle \Delta F_{\mu\nu}(x) \rangle \equiv \langle F^{\text{out}}_{\mu\nu}(x) \rangle - \langle F^{\text{in}}_{\mu\nu}(x) \rangle \,, \tag{747}$$

it is straightforward to use eq. (737) and find that,

$$\langle \Delta F_{\mu\nu}(x) \rangle = i \langle \psi_w | [\mathbb{F}_{\mu\nu}(x), T] | \psi_w \rangle + \cdots \,. \tag{748}$$

We have indicated higher order terms are present in the ellipsis. It may be worth emphasizing once again that this result is the same as one would find be direct computation using background field methods:

$$\begin{aligned}\langle \Delta F^{\mu\nu}(x) \rangle &= \langle \psi_w | S^\dagger \mathbb{F}^{\mu\nu}(x) S | \psi_w \rangle - \langle \psi_w | \mathbb{F}^{\mu\nu}(x) | \psi_w \rangle \\ &= \int d\Phi(p_1) d\Phi(p_1) \varphi(p_1) \varphi^*(p_1') e^{-ib\cdot(p_1'-p_1)/\hbar} \\ &\qquad \times \Big\{ i \langle p_1' | [\mathbb{F}^{\mu\nu}(x), T(A^{(\eta)}_{\text{cl}})] | p_1 \rangle + \langle p_1' | T^\dagger(A^{(\eta)}_{\text{cl}}) \mathbb{F}^{\mu\nu}(x) T(A^{(\eta)}_{\text{cl}}) | p_1 \rangle \Big\} \,,\end{aligned} \tag{749}$$

where $A^{(\eta)}_{\text{cl}}(x)$ denotes the classical background field corresponding to our coherent state, and we once again used the relation $\mathbb{C}^\dagger_{\alpha,(\eta)} \mathbb{C}_{\alpha,(\eta)} = \mathbb{1}$.

Returning to the LO computation of the scattered field strength, by inserting the definition of the field strength operator, we now encounter two commutators:

$$\begin{aligned}\langle \Delta F_{\mu\nu}(x) \rangle = \frac{2}{\hbar^{3/2}} \sum_{\eta'} \int d\Phi(k') \Big[&\langle \psi_w | [a_{(\eta')}(\bar{k}'), T] | \psi_w \rangle \, \bar{k}'_{[\mu} \varepsilon^{(\eta')*}_{\nu]}(k') \, e^{-ik'\cdot x/\hbar} \\ &- \langle \psi_w | [a^\dagger_{(\eta')}(\bar{k}'), T] | \psi_w \rangle \, \bar{k}'_{[\mu} \varepsilon^{(\eta')}_{\nu]}(k') \, e^{+ik'\cdot x/\hbar} \Big] \,.\end{aligned} \tag{750}$$

The first of these was computed explicitly above; the second is very similar. After a short computation, the field strength can be expressed as,

$$\langle \Delta F_{\mu\nu}(x) \rangle = \text{Re} \left\langle\!\!\left\langle 2g^2 \sum_{\eta'} \int d\Phi(\bar{k}_2, \bar{k}_2')\, \hat{\delta}(p_1 \cdot (\bar{k}_2' - \bar{k}_2))\, \bar{\alpha}(\bar{k}_2) \right.\right. \\ \left.\left. \times \bar{\mathcal{A}}(p_1 k_2^+ \to p_1' k'^{\eta'})\, \bar{k}'_{2[\mu} \varepsilon^{(\eta')*}_{\nu]}(\bar{k}_2')\, e^{-i\bar{k}_2' \cdot x} \right\rangle\!\!\right\rangle . \tag{751}$$

Comparison with the NP scalar is facilitated by performing the $\bar{k}_2'$ integral using the methods of Sect. 11. Indeed, the field strength change of eq. (751) is of the general form of the radiation observable eq. (709). The corresponding current is,

$$\widetilde{J}_{\mu\nu}(\bar{k}_2) = \\ -4i \left\langle\!\!\left\langle \sum_{\eta'} \int d\Phi(\bar{k}_2')\, \hat{\delta}(2p_1 \cdot (\bar{k}_2' - \bar{k}_2))\, \bar{\alpha}(\bar{k}_2) \bar{\mathcal{A}}(p_1 k_2^+ \to p_1' k_2'^{\eta'})\, \bar{k}'_{2[\mu} \varepsilon^{(\eta')*}_{\nu]}(\bar{k}_2') \right\rangle\!\!\right\rangle . \tag{752}$$

The NP scalar can be obtained directly from this current as,

$$\Delta\Phi_2^0(\omega, \hat{n}) = \frac{1}{4\pi} \widetilde{J}_{\mu\nu}(\bar{k}) M^{*\mu} N^{\nu} . \tag{753}$$

Performing the dot products, we recover our earlier result, eq. (746).

Earlier, we identified incoming classical radiation with coherent states. The reader may wonder then about the nature of outgoing radiation. A necessary condition for the outgoing radiation to be represented by a coherent state is that expectation values of observables, such as the field strength, should factorize. We have proved this explicitly earlier, see eq. (636). Perhaps surprisingly, it turns out that this is also a sufficient condition. Indeed, one can work out the constraints on the probability density of the outgoing (pure) radiation: in the coherent state space (also called the Glauber–Sudarshan representation), the classical factorization of observables implies that the distribution has zero variance. In turn, this makes the distribution degenerate, *i.e.* supported on isolated points. But as shown by Hillery [294], the normalization condition together with the purity constraint suffices to reduce the sum of delta functions in the coherent state space to just a single delta function. That is, we have only a single outgoing coherent state in the classical limit. In appendix 11, we prove that the factorization condition holds at the lowest order in the coupling constant, which makes the outgoing radiation state of the Thomson scattering coherent up to order g^2. A more detailed discussion on this point will appear in forthcoming work [295].

EMISSION WAVEFORM

We turn now to photon emission in the scattering of two charged point particles. At leading order in perturbation theory, only the second term in eq. (737) (or similarly, in eq. (738)) contributes. It will be of order $\mathcal{O}(g^3)$, whereas the second term will be of $\mathcal{O}(g^5)$.

If we now substitute the expression (620), along with that (612) for the initial-state wavefunction for the scattering particles into the first term of eq. (737), we obtain,

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_1 = \frac{4}{\hbar^{3/2}}\,\mathrm{Re}\sum_\eta \int & d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\\ &\times e^{-ib\cdot(p_1'-p_1)/\hbar}\varphi(p_1)\varphi^*(p_1')\varphi(p_2)\varphi^*(p_2')\\ &\times k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'|a_{(\eta)}(k)\,T|p_1\,p_2\rangle\\ = \frac{4}{\hbar^{3/2}}\,\mathrm{Re}\sum_\eta \int & d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\\ &\times e^{-ib\cdot(p_1'-p_1)/\hbar}\varphi(p_1)\varphi^*(p_1')\varphi(p_2)\varphi^*(p_2')\\ &\times k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'\,k^\eta|T|p_1\,p_2\rangle\,.\end{aligned} \tag{754}$$

We can identify the matrix element as a five-point amplitude,

$$\langle p_1'\,p_2'\,k^\eta|T|p_1\,p_2\rangle = \mathcal{A}(p_1,p_2\to p_1',p_2',k^\eta)\hat{\delta}^4(p_1+p_2-p_1'-p_2'-k)\,. \tag{755}$$

At leading order, we replace the amplitude by its LO contribution, given by a tree-level expression. To compute the required waveform, we must identify the expectation of $F^{\mu\nu}(x)$ as the spatial current $J_{\vec{\mu}}(x)$ in eqs. (707) and (708), and via eq. (708), in eq. (718).

Beyond leading order, the expectation of $F^{\mu\nu}(x)$ will receive higher-order contributions to the amplitudes in eq. (755), alongside contributions from the last term in eq. (738),

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_2 = -\frac{2i}{\hbar^{3/2}}\sum_\eta \int & d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k) \\ & \times e^{-ib\cdot(p_1'-p_1)/\hbar}\varphi(p_1)\varphi^*(p_1')\varphi(p_2)\varphi^*(p_2') \\ & \times \Big[k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'|T^\dagger a_{(\eta)}(k)\,T|p_1\,p_2\rangle \\ & \quad - k^{[\mu}\varepsilon^{(\eta)\nu]}e^{+ik\cdot x/\hbar}\langle p_1'\,p_2'|T^\dagger a^\dagger_{(\eta)}(k)T|p_1\,p_2\rangle\Big]\end{aligned} \tag{756}$$

Insert a complete set of states to the right of each $T^\dagger$,

$$\langle\psi|T^\dagger \mathbb{F} T|\psi\rangle = \sum_X \int d\Phi(r_1)d\Phi(r_2)\;\langle\psi|T^\dagger|r_1\,r_2\,X\rangle\langle r_1\,r_2\,X|\mathbb{F}T|\psi\rangle\,, \tag{757}$$

where the sum over X is over all states, including no additional particles, and includes an implicit integral over momenta of any particles in X and a sum over any other quantum numbers. As in ref. [21], we assume that each of the incoming massive particles carries a separately conserved global charge, so that each intermediate state has one net particle of each type. We can ignore additional particle-antiparticle pairs of the massive particles, as these contributions will disappear in the classical limit. As there are no messengers in the initial state, and hence no coherent states, there is no need to sum over arbitrary numbers of messengers. Accordingly, we do not need to switch to a coherent-friendly representation (672) of the T matrix. We obtain,

$$
\begin{aligned}
\langle F^{\mu\nu}(x)\rangle_2 = -\frac{2i}{\hbar^{3/2}}\sum_X\sum_\eta\int & d\Phi(r_1)d\Phi(r_2)d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\\
&\times e^{-ib\cdot(p_1'-p_1)/\hbar}\varphi(p_1)\varphi^*(p_1')\varphi(p_2)\varphi^*(p_2')\\
&\times \big[k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1', p_2'|T^\dagger|r_1\, r_2\, X\rangle\langle r_1\, r_2\, X|a_{(\eta)}(k)\, T|p_1\, p_2\rangle\\
&\quad - k^{[\mu}\varepsilon^{(\eta)\nu]}e^{+ik\cdot x/\hbar}\langle p_1', p_2'|T^\dagger|r_1\, r_2\, X\rangle\langle r_1\, r_2\, X|a^\dagger_{(\eta)}(k)T|p_1\, p_2\rangle\big]\\
= -\frac{2i}{\hbar^{3/2}}\sum_X\sum_\eta\int & d\Phi(r_1)d\Phi(r_2)d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\\
&\times e^{-ib\cdot(p_1'-p_1)/\hbar}\varphi(p_1)\varphi^*(p_1')\varphi(p_2)\varphi^*(p_2')\\
&\times \big[k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1', p_2'|T^\dagger|r_1\, r_2\, X\rangle\langle r_1\, r_2\, k^\eta\, X|T|p_1\, p_2\rangle\\
&\quad - k^{[\mu}\varepsilon^{(\eta)\nu]}e^{+ik\cdot x/\hbar}\langle p_1'\, p_2'|T^\dagger|r_1\, r_2\, k^\eta\, X\rangle\langle r_1\, r_2\, X|T|p_1\, p_2\rangle\big]\,.
\end{aligned}
\tag{758}
$$

In the second term within brackets, the creation operator requires a photon in the intermediate state, and eliminates it from the bra. We then relabeled X to exclude it. Note as well that at next-to-next-leading order and beyond, we necessarily require amplitudes with three incoming particles. These can just as easily be obtained by crossing. The term (758) has the interpretation of a cut of an amplitude, just as for the second term in the impulse in ref. [21], as seen in eqs. (3.26–3.31) therein.

This contribution first appears at next-to-leading order. At this order, we are interested in contributions with $X = \varnothing$, and we can identify the required matrix elements as a combination of four- and five-point amplitudes,

$$\begin{aligned}
\langle r_1\, r_2|T|p_1\, p_2\rangle &= \mathcal{A}(p_1\, p_2 \to r_1\, r_2)\hat{\delta}^4(p_1 + p_2 - r_1 - r_2)\,,\\
\langle p_1', p_2'|T^\dagger|r_1\, r_2\rangle &= \mathcal{A}^*(p_1', p_2' \to r_1, r_2)\hat{\delta}^4(p_1' + p_2' - r_1 - r_2)\,,\\
\langle r_1\, r_2\, k^\eta|T|p_1\, p_2\rangle &= \mathcal{A}(p_1, p_2 \to r_1, r_2, k^\eta)\hat{\delta}^4(p_1 + p_2 - r_1 - r_2 - k)\,,\\
\langle p_1'\, p_2'|T^\dagger|r_1\, r_2\, k^\eta\rangle &= \mathcal{A}^*(p_1', p_2' \to r_1, r_2, k^\eta)\hat{\delta}^4(p_1' + p_2' - r_1 - r_2 - k)\,.
\end{aligned} \tag{759}$$

For the next-to-leading order contribution to $\langle F^{\mu\nu}(x)\rangle$, we use tree-level amplitudes in eq. (759).

THE DETECTED WAVE AT LEADING ORDER

The leading-order contribution to the waveform will arise at $\mathcal{O}(g^3)$, as described in the previous section. We apply the approach of ref. [21] to eq. (754). Similarly to that reference, and to Sect. 11, we define the momentum mismatches,

$$\begin{aligned}
q_1 &= p_1' - p_1\,,\\
q_2 &= p_2' - p_2\,;
\end{aligned} \tag{760}$$

and trade the integrals over the p_i' for integrals over the q_i,

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_1 = \\ \frac{4}{\hbar^{3/2}}\,\mathrm{Re}\sum_\eta \int d\Phi(p_1)d\Phi(p_2)\hat{d}^4q_1\hat{d}^4q_2 d\Phi(k)\;\hat{\delta}(2p_1\cdot q_1+q_1^2)\hat{\delta}(2p_2\cdot q_2+q_2^2) \\ \times e^{-ib\cdot q_1/\hbar}\Theta(p_1^t+q_1^t)\Theta(p_2^t+q_2^t)\varphi(p_1)\varphi^*(p_1+q_1)\varphi(p_2)\varphi^*(p_2+q_2) \\ \times k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\mathcal{A}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\eta)\hat{\delta}^4(q_1+q_2+k)\,.\end{aligned} \tag{761}$$

We can take the classical limit, and change to the required wavenumber variables for the q_i and k,

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_{1,\mathrm{cl}} = \\ g^3\bigg\langle\!\!\bigg\langle \hbar^2\mathrm{Re}\sum_\eta\int d\Phi(\bar{k})\bar{k}^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-i\bar{k}\cdot x} \\ \times\prod_{i=1,2}\int \hat{d}^4\bar{q}_i\;\hat{\delta}(p_i\cdot\bar{q}_i)\;e^{-ib\cdot\bar{q}_1}\hat{\delta}^4(\bar{q}_1+\bar{q}_2+\bar{k}) \\ \times\bar{\mathcal{A}}(p_1,p_2\to p_1+\hbar\bar{q}_1,p_2+\hbar\bar{q}_2,\hbar\bar{k}^\eta)\bigg\rangle\!\!\bigg\rangle\,.\end{aligned} \tag{762}$$

We have also extracted powers of $\hbar$ from the coupling, and dropped the $\hbar$-suppressed terms inside the on-shell delta functions as well as the positive-energy theta functions. We recognize the inner integral in the second term as the radiation kernel defined in eq. (4.42) of ref. [21] (after changing variables there $p_i\to p_i-\hbar\bar{w}_i$ and $\bar{w}_i\to-\bar{q}_i$),

$$\begin{aligned}\mathcal{R}^{(0)}(\bar{k}^\eta;b)\equiv\hbar^2\prod_{i=1,2}\int\hat{d}^4\bar{q}_i\;\hat{\delta}(p_i\cdot\bar{q}_i)\;e^{-ib\cdot\bar{q}_1}\hat{\delta}^4(\bar{q}_1+\bar{q}_2+\bar{k}) \\ \times\bar{\mathcal{A}}(p_1,p_2\to p_1+\hbar\bar{q}_1,p_2+\hbar\bar{q}_2,\hbar\bar{k}^\eta)\,.\end{aligned} \tag{763}$$

We have made the impact parameter an explicit argument here. At LO, we can then write,

$$\langle F^{\mu\nu}(x)\rangle_{1,\text{cl}} = g^3 \left\langle\!\!\left\langle \text{Re}\sum_\eta \int d\Phi(\bar{k})\bar{k}^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-i\bar{k}\cdot x}\mathcal{R}^{(0)}(\bar{k}^\eta;b)\right\rangle\!\!\right\rangle . \tag{764}$$

The integrand has the form of the radiation observables introduced in Sect. 11. The spectral waveform is then,

$$\begin{aligned} f_{\mu\nu}(\omega,\hat{n}) = -\frac{ig^3}{8\pi}\sum_\eta \big[&\Theta(\omega)\bar{k}^{[\mu}\varepsilon^{(\eta)\nu]*}\mathcal{R}^{(0)}(\bar{k}^\eta;b)\big|_{\bar{k}=\omega(1,\hat{n})} \\ &- \Theta(-\omega)\bar{k}^{[\mu}\varepsilon^{(\eta)\nu]}\mathcal{R}^{(0)*}(\bar{k}^\eta;b)\big|_{\bar{k}=-\omega(1,\hat{n})}\big] \end{aligned} \tag{765}$$

The corresponding result for the Fourier-space NP scalar is,

$$\widetilde{\Phi}_2^0(\omega,\hat{n}) = -\frac{ig^3\omega}{16\pi}\left\langle\!\!\left\langle \Theta(\omega)\mathcal{R}^{(0)}(\omega(1,\hat{n})^-;b) + \Theta(-\omega)\mathcal{R}^{(0)*}(-\omega(1,\hat{n})^+;b)\right\rangle\!\!\right\rangle . \tag{766}$$

Equivalently, we may write,

$$\begin{aligned} \Phi_2^0(t,\hat{n}) = -\frac{ig^3}{16\pi}\left\langle\!\!\left\langle \int \hat{d}\omega\,\Theta(\omega)\,\omega\big[\right.\right.&e^{-i\omega\cdot t}\mathcal{R}^{(0)}(\omega(1,\hat{n})^-;b) \\ &\left.\left.- e^{+i\omega\cdot t}\mathcal{R}^{(0)*}(\omega(1,\hat{n})^+;b)\big]\right\rangle\!\!\right\rangle . \end{aligned} \tag{767}$$

As the LO radiation kernel $\mathcal{R}^{(0)}$ is given by a five-point amplitude, the waveform as a function of frequency ω, is simply the five-point amplitude up to the additional factor of ω.

The explicit form of eq. (763) for electromagnetic scattering is given in eq. (5.46) of ref. [21], and reproduced as eq. (816). We evaluate it in appendix 11, to obtain,

$$\begin{aligned}
\mathcal{R}^{(0)}(\bar{k};b) &= \frac{Q_1^2 Q_2}{m_1\, u_1\cdot\bar{k}}\big[u_2\cdot\bar{k}\, u_1\cdot\varepsilon - u_1\cdot\bar{k}\, u_2\cdot\varepsilon\big]\, I_3 \\
&\quad - \frac{Q_1^2 Q_2 \gamma}{m_1\, u_1\cdot\bar{k}(\gamma^2-1)}\big[u_1\cdot\bar{k}\,(u_1-\gamma u_2)\cdot\varepsilon - (u_1-\gamma u_2)\cdot\bar{k}\, u_1\cdot\varepsilon\big]\, I_3 \\
&\quad + \frac{Q_1^2 Q_2\, \gamma\, e^{ib\cdot\bar{k}}}{m_1\, u_1\cdot\bar{k}}\big[u_1\cdot\bar{k}\,\widetilde{b}\cdot\varepsilon - \widetilde{b}\cdot\bar{k}\, u_1\cdot\varepsilon\big] \\
&\quad \times \frac{i}{2\pi\,(\gamma^2-1)} K_1\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\
&\quad + (1\leftrightarrow 2 \text{ modulo phases}) \\
&= \frac{Q_1^2 Q_2\, e^{ib\cdot\bar{k}}}{m_1\, u_1\cdot\bar{k}}\big[u_2\cdot\bar{k}\, u_1\cdot\varepsilon - u_1\cdot\bar{k}\, u_2\cdot\varepsilon\big] \\
&\quad \times \frac{1}{2\pi\,\sqrt{\gamma^2-1}} K_0\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\
&\quad + \frac{Q_1^2 Q_2 \gamma\, e^{ib\cdot\bar{k}}}{m_1\, u_1\cdot\bar{k}}\big[u_1\cdot\bar{k}\,\widetilde{b}\cdot\varepsilon - \widetilde{b}\cdot\bar{k}\, u_1\cdot\varepsilon\big] \\
&\quad \times \frac{i}{2\pi\,(\gamma^2-1)} K_1\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\
&\quad + (1\leftrightarrow 2 \text{ modulo phases})\,.
\end{aligned} \tag{768}$$

A side calculation shows that (with ζ a null reference momentum),

$$\begin{aligned}
u_2\cdot\bar{k}\, u_1\cdot\varepsilon - u_1\cdot\bar{k}\, u_2\cdot\varepsilon &= \\
&\frac{1}{\sqrt{2}\,\langle\zeta\,\bar{k}\rangle}\big[\langle\bar{k}|\, u_2\,|\bar{k}]\,\langle\zeta|\, u_1\,|\bar{k}] - \langle\bar{k}|\, u_1\,|\bar{k}]\,\langle\zeta|\, u_2\,|\bar{k}]\big] \\
&= \frac{1}{\sqrt{2}}\,[\bar{k}|\, u_2\, u_1\,|\bar{k}]
\end{aligned} \tag{769}$$

for positive-helicity emission, and

$$\frac{1}{\sqrt{2}} \langle \bar{k}| \, u_2 \, u_1 \, |\bar{k}\rangle \tag{770}$$

for negative-helicity emission.

Then,

$$\begin{aligned}\mathcal{R}^{(0)}(\bar{k}^+; b) = {} & \frac{Q_1^2 Q_2 \, e^{ib\cdot\bar{k}}}{2\sqrt{2}\pi m_1 \, u_1\cdot\bar{k}\,\sqrt{\gamma^2-1}} \\ & \quad \times \Bigg\{ [\bar{k}| \, u_2 \, u_1 \, |\bar{k}] \; K_0\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\ & \qquad + \frac{i\,[\bar{k}| \, b \, u_1 \, |\bar{k}]}{\sqrt{\gamma^2-1}\,\sqrt{-b^2}} K_1\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \Bigg\} \\ & + \frac{Q_1 Q_2^2}{2\sqrt{2}\pi m_2 \, u_2\cdot\bar{k}\,\sqrt{\gamma^2-1}} \\ & \quad \times \Bigg\{ [\bar{k}| \, u_1 \, u_2 \, |\bar{k}] \; K_0\big(\sqrt{-b^2}\, u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\ & \qquad + \frac{i\,[\bar{k}| \, b \, u_2 \, |\bar{k}]}{\sqrt{\gamma^2-1}\,\sqrt{-b^2}} K_1\big(\sqrt{-b^2}\, u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \Bigg\} .\end{aligned} \tag{771}$$

There is a similar result for the other photon helicity.

Using the integrals,

$$\begin{aligned}
&\int_0^\infty d\omega\ \omega e^{-i\omega(t+a_0)} K_0(\omega a_1) = \\
&\quad \frac{1}{a_1^2+(a_0+t)^2} - \frac{(t+a_0)}{[a_1^2+(a_0+t)^2]^{3/2}} \operatorname{arcsinh}\Big(\frac{1}{a_1}(t+a_0)\Big) \\
&\quad - \frac{i\pi}{2}\frac{(t+a_0)}{[a_1^2+(a_0+t)^2]^{3/2}}, \\
&\int_0^\infty d\omega\ \omega e^{-i\omega(t+a_0)} K_1(\omega a_1) = \\
&\quad \frac{\pi a_1}{2[a_1^2+(a_0+t)^2]^{3/2}} - i\frac{(a_0+t)}{a_1[a_1^2+(a_0+t)^2]} \\
&\quad - i\frac{a_1}{[a_1^2+(a_0+t)^2]^{3/2}} \operatorname{arcsinh}\Big(\frac{1}{a_1}(t+a_0)\Big);
\end{aligned} \tag{772}$$

and defining,

$$\begin{aligned}
u_{i,\hat{n}} &\equiv u_i \cdot \bar{k}/\omega = u_i \cdot (1, \hat{n}), \\
\rho_1(t) &\equiv -b^2 u_{1,\hat{n}}^2 + (\gamma^2-1)(t+b\cdot\hat{n})^2, \\
\rho_2(t) &\equiv -b^2 u_{2,\hat{n}}^2 + (\gamma^2-1)t^2,
\end{aligned} \tag{773}$$

along with,

$$\begin{aligned}
\Xi_{ia}^{\zeta}(t,\hat{n};v) &= \frac{\sqrt{\gamma^2-1}}{\rho_1(t)} - \zeta\frac{(\gamma^2-1)(t+v\cdot\hat{n})}{\rho_1^{3/2}(t)} \operatorname{arcsinh}\Bigg(\frac{\sqrt{\gamma^2-1}}{\sqrt{-b^2}u_{1,\hat{n}}}(t+v\cdot\hat{n})\Bigg) \\
&\quad - \frac{i\pi}{2}\frac{(\gamma^2-1)(t+v\cdot\hat{n})}{\rho_1^{3/2}(t)}, \\
\Xi_{ib}(t,\hat{n};v) &= \frac{\pi u_{1,\hat{n}}}{\rho_1^{3/2}(t)} + i\frac{\sqrt{\gamma^2-1}(t+v\cdot\hat{n})}{b^2 u_{1,\hat{n}}\rho_1(t)} \\
&\quad - i\frac{u_{1,\hat{n}}}{\rho_1^{3/2}(t)} \operatorname{arcsinh}\Bigg(\frac{\sqrt{\gamma^2-1}}{\sqrt{-b^2}u_{1,\hat{n}}}(t+v\cdot\hat{n})\Bigg),
\end{aligned} \tag{774}$$

we can write,

$$\begin{aligned}
&\Phi_2^0(t,\hat{n}) = \\
&\quad -\frac{ig^3 Q_1^2 Q_2}{(4\pi)^3\sqrt{2}m_1\, u_{1,\hat{n}}}\Big[\langle\hat{n}|\, u_2\, u_1\, |\hat{n}\rangle\ \Xi^+_{1a}(t,\hat{n};b) - [\hat{n}|\, u_2\, u_1\, |\hat{n}]\ \Xi^-_{1a}(t,\hat{n};b) \\
&\qquad\qquad + i\big(\langle\hat{n}|\, b\, u_1\, |\hat{n}\rangle - [\hat{n}|\, b\, u_1\, |\hat{n}]\big)\,\Xi_{1b}(t,\hat{n};b)\Big] \\
&\quad -\frac{ig^3 Q_1 Q_2^2}{(4\pi)^3\sqrt{2}m_2\, u_{2,\hat{n}}}\Big[\langle\hat{n}|\, u_1\, u_2\, |\hat{n}\rangle\ \Xi^+_{2a}(t,\hat{n};0) - [\hat{n}|\, u_1\, u_2\, |\hat{n}]\ \Xi^-_{2a}(t,\hat{n};0) \\
&\qquad\qquad + i\big(\langle\hat{n}|\, b\, u_2\, |\hat{n}\rangle - [\hat{n}|\, b\, u_2\, |\hat{n}]\big)\,\Xi_{2b}(t,\hat{n};0)\Big]\,.
\end{aligned} \tag{775}$$

Here, $|\hat{n}\rangle$ and $|\hat{n}]$ are spinors built out of the null vector $(1,\hat{n})$.

CONNECTION TO RADIATED MOMENTUM

In Sect. 11, we presented the general form for the waveform observable. We worked out the leading-order form in two-particle scattering in Sect. 11, and computed the explicit form for electromagnetic scattering in the previous section. The appearance of the radiation kernel suggests a connection to the radiated momentum previously computed in ref. [21]. Let us elucidate that connection in this section.

In eq. (3.33) of ref. [21], we find an expression for time-averaged radiated momentum,

$$R^\mu \equiv \langle k^\mu\rangle = {}_{\text{in}}\langle\psi|\, S^\dagger \mathbb{K}^\mu S\, |\psi\rangle_{\text{in}} = {}_{\text{in}}\langle\psi|\, T^\dagger \mathbb{K}^\mu T\, |\psi\rangle_{\text{in}}\,. \tag{776}$$

This quantity is also integrated over the entire celestial sphere; we need a more differential observable. Furthermore, this expression is related to the energy emitted, rather than the amplitude of the emitted wave.

We can use Mellin transforms to extract a more restricted observable, passing through the spectral waveform to relate the emitted power to the amplitude. Write the expectation of the observable $\langle (k^t)^{z-1} \rangle$,

$$R(z) \equiv \langle (k^t)^{z-1} \rangle = {}_{\text{in}}\langle \psi | \, T^\dagger (\mathbb{K}^t)^{z-1} T \, | \psi \rangle_{\text{in}} \,. \tag{777}$$

The inverse Mellin transform is related to the unpolarized energy density function,

$$f_\epsilon(E) = -iE \int_{c-i\infty}^{c+i\infty} dz \; E^{-z} R(z) \,, \tag{778}$$

where the integral is taken along a line parallel to the imaginary axis, with $c \in (0,1)$ (or a deformation of that contour that doesn't cross any poles or branch points)[7]. The total energy is given by the integral,

$$E_{\text{tot}} = \int_0^\infty dE \, f_\epsilon(E) \,. \tag{779}$$

Using the form in eq. (3.38) of ref. [21], we can write,

$$R(z) = \sum_X \int d\Phi(k) d\Phi(r_1) d\Phi(r_2) \; (k_X^t)^{z-1} \sum_\eta |\hat{\mathcal{R}}(k^\eta, r_X)|^2 \,, \tag{780}$$

for the expression in the quantum theory. In this equation, $\hat{\mathcal{R}}$ represents the quantum radiation kernel, given by the integral over wavefunctions inside the absolute square in eq. (3.38). The quantum radiation kernel is expressed directly in terms of a scattering amplitude.

7 With our conventions, the expected power of $(2\pi)^{-1}$ is in the forward rather than the inverse Mellin transform.

In the classical limit, the density function is more naturally a function of frequency rather than of energy,

$$f_{\epsilon,\text{cl}}(\omega) = -i\,\omega \int_{c-i\infty}^{c+i\infty} dz\; \omega^{-z} R_{\text{cl}}(z)\,, \tag{781}$$

so that $R_{\text{cl}}(z) = \hbar^{-z-1} R(z)$. We can use eqs. (4.40–4.41) of ref. [21] to write,

$$R_{\text{cl}}(z) = \sum_X \hbar^{-z-1} \left\langle\!\!\left\langle \int d\Phi(k)\; (k_X^t)^{z-1} \sum_\eta |\mathcal{R}(k^\eta, r_X)|^2 \right\rangle\!\!\right\rangle. \tag{782}$$

The radiation kernel here is expressed in terms of the appropriate limit of a quantum scattering amplitude.

We next need to restrict the measured radiation from the entire celestial sphere to a narrow cone in a given direction. We take the limit of the cone, and measure only the radiation in a given direction from the scattering event. We implicitly assume that the measurement distance is much larger than the impact parameter, so that there is a unique and well-defined direction. It's not clear exactly what a formal expression for the operator would be, but what we want is,

$$\mathbb{K}^\mu\, \delta^{(2)}(\hat{\mathbb{K}} - \hat{n})\,, \tag{783}$$

for radiation in the $\hat{n}$ direction. This operator is to be understood as inserting,

$$\sum_{i \in \text{messengers}} k_i^\mu\, \delta^{(2)}(\hat{k}_i - \hat{n})\,, \tag{784}$$

into a sum over states or equivalently the phase-space integral. Focusing on the energy component, this can be understood as a light ray operator [254, 296–299] given by,

$$\mathbb{E}(\hat{\mathbf{n}}) = \int_{-\infty}^{+\infty} du \lim_{r\to\infty} r^2 \mathbb{T}_{uu}\,(u, r, \hat{\mathbf{n}}) \tag{785}$$

where u denotes the light-cone time $u = t - r$ and $\mathbb{T}_{uu}\,(u, r, \hat{\mathbf{n}})$ is the (light-cone) time-time component of the stress-energy tensor (in gravity, this will be replaced by the Bondi news squared operator [254]). By applying the saddle point approximation for the fields in the energy momentum tensor, the plane wave expansion will localize to the point on the sphere in the direction of propagation. Schematically we will have (see refs. [300, 301] for further details)

$$e^{ix\cdot k/\hbar} = e^{i\omega u + i\omega r\left(1-\hat{n}\cdot\hat{k}\right)} \overset{r\to\infty}{\sim} \frac{1}{i\omega r} e^{i\omega u} \delta^{(2)}\left(\hat{n} - \hat{k}\right) \tag{786}$$

where $\omega = \bar{k}^t$. Then one finds,

$$\mathbb{E}(\hat{\mathbf{n}}) = \sum_{\eta} \int d\Phi(k)\, k^t\, \delta^{(2)}\left(\hat{n} - \hat{k}\right) \left[a^{\dagger}_{(\eta)}(k) a_{(\eta)}(k)\right] \tag{787}$$

where the action on on-shell particle states is equivalent to the time component of eq. (784). The analogous Mellin kernel for $(\mathbb{K}^t)^{z-1}$ is presumably,

$$(\mathbb{K}^t)^{z-1}\, \delta^{(2)}(\hat{\mathbb{K}} - \hat{n})\,, \tag{788}$$

which is to be understood as inserting,

$$\sum_{i\in \substack{\text{distinct}\\ \text{messengers}}} \Big(\sum_{\substack{j\| i\\ j\in\text{messengers}}} k_j^t \Big)^{z-1} \delta^{(2)}(\hat{k}_i - \hat{n})\,, \tag{789}$$

into a sum over states or the phase-space integral. The sum over distinct messengers is a sum over messengers which are not collinear; the sum over the collinear messengers is taken in the inner sum. The inner sum includes i itself.

This form is motivated by a subtlety about overlapping directions: if $\hat{k}_j = \hat{k}_l$ with the remaining directions distinct we want,

$$\sum_{\substack{i \in \text{messengers} \\ i \neq j,l}} (k_i^t)^{z-1}\, \delta^{(2)}(\hat{k}_i - \hat{n}) + (k_j^t + k_l^t)^{z-1}\, \delta^{(2)}(\hat{k}_j - \hat{n})\,, \tag{790}$$

which is what eq. (789) is designed to give. At leading order this subtlety is irrelevant.

The analog to eq. (780) is,

$$\begin{aligned} R(z,\hat{n}) = \sum_{i \in \substack{\text{distinct} \\ \text{messengers}}} \sum_X \int d\Phi(k_i) d\Phi(r_1) d\Phi(r_2) \; \Big(\sum_{\substack{j \| i \\ j \in \text{messengers}}} k_j^t \Big)^{z-1} \\ \times\, \delta^{(2)}(\hat{k}_i - \hat{n}) \sum_\eta |\hat{\mathcal{R}}(k_i^\eta, r_X)|^2\,, \end{aligned} \tag{791}$$

and to eq. (782),

$$R_{\text{cl}}(z,\hat{n}) = \sum_{i \in \substack{\text{distinct} \\ \text{messengers}}} \hbar^{-z-1} \Big\langle\!\!\Big\langle \int d\Phi(k_i) \Big(\sum_{\substack{j \| i \\ j \in \text{messengers}}} k_j^t \Big)^{z-1} \delta^{(2)}(\hat{k}_i - \hat{n}) \sum_\eta \Big| \mathcal{R}(k_i^\eta, r_X) \Big|^2 \Big\rangle\!\!\Big\rangle\,. \tag{792}$$

At LO, eq. (792) simplifies to just,

$$R_{\text{cl}}^{(0)}(z,\hat{n}) = g^6 \Big\langle\!\!\Big\langle \int d\Phi(\bar{k})\, (\bar{k}^t)^{z-1}\, \delta^{(2)}(\hat{k} - \hat{n}) \sum_\eta \Big| \mathcal{R}^{(0)}(\bar{k}^\eta; b) \Big|^2 \Big\rangle\!\!\Big\rangle\,. \tag{793}$$

The corresponding result for the spectral density in the $\hat{n}$ direction is,

$$f_{\epsilon,\mathrm{cl}}(\omega,\hat{n}) = g^6\omega\left\langle\!\!\left\langle \int d\Phi(\bar{k})\,\frac{\hat{\delta}(\ln\bar{k}^t - \ln\omega)}{\bar{k}^t}\,\delta^{(2)}(\hat{k}-\hat{n})\,\sum_\eta \left|\mathcal{R}^{(0)}(\bar{k}^\eta;b)\right|^2\right\rangle\!\!\right\rangle. \tag{794}$$

Writing out,

$$\begin{aligned} d\Phi(\bar{k}) &= \frac{d^3\bar{k}}{2(2\pi)^3\,|\bar{k}|} \\ &= \frac{|\bar{k}|d|\bar{k}|\,d\Omega_{\bar{k}}}{2(2\pi)^3}\,, \end{aligned} \tag{795}$$

we can perform the integrals in eq. (794) to obtain,

$$f_{\epsilon,\mathrm{cl}}(\omega,\hat{n}) = \frac{g^6\omega^2}{8\pi^2}\sum_\eta\left\langle\!\!\left\langle \left|\mathcal{R}^{(0)}(\omega(1,\hat{n})^\eta;b)\right|^2\right\rangle\!\!\right\rangle. \tag{796}$$

We can now compare this with the amplitude of each component of the waveform, expanded at the leading order order in the coupling: for $|f_{\mu\nu}M^{*\mu}N^\nu|$ and $|f_{\mu\nu}M^\mu N^\nu|$ we have, respectively

$$\begin{aligned} |f_{\mu\nu}(\omega(1,\hat{n}))M^{*\mu}N^\nu| &= \frac{\omega}{16\pi}g^3\left|\left\langle\!\!\left\langle\,\mathcal{R}^{(0)}(\omega(1,\hat{n})^-;b)\right\rangle\!\!\right\rangle\right| \\ |f_{\mu\nu}(\omega(1,\hat{n}))M^{\mu}N^\nu| &= \frac{\omega}{16\pi}g^3\left|\left\langle\!\!\left\langle\,\mathcal{R}^{(0)}(\omega(1,\hat{n})^+;b)\right\rangle\!\!\right\rangle\right| \end{aligned} \tag{797}$$

At LO, we can also write

$$\left\langle\!\!\left\langle \left|\mathcal{R}^{(0)}(\omega(1,\hat{n})^\eta;b)\right|^2\right\rangle\!\!\right\rangle = \left|\left\langle\!\!\left\langle\,\mathcal{R}^{(0)}(\omega(1,\hat{n})^\eta;b)\right\rangle\!\!\right\rangle\right|^2 \tag{798}$$

and therefore we can express the spectral density of emission from eq. (796) in terms of the amplitudes of the two helicity components of the waveform,

$$f_{\epsilon,\mathrm{cl}}(\omega,\hat{n}) = 32\big[|f_{\mu\nu}(\omega(1,\hat{n}))M^{*\mu}N^\nu|^2 + |f_{\mu\nu}(\omega(1,\hat{n}))M^{\mu}N^\nu|^2\big]\,. \tag{799}$$

This relation is the avatar of the relation between the energy of the wave and the squared amplitude of the wave, the only difference being that here we are measuring the momentum emitted in a given direction at a large distance r from the source. The emitted radiation observable provides information about the magnitude of the observed messenger wave, but not about its phase. The direct derivation in previous sections adds that information.

A recently proposed generalization of a standard event shape is sensitive to amplitude phases [302]. It would be interesting to explore a possible connection to the waveform.

CONCLUSIONS

In this paper, we have developed an observables-based formalism for computing classical waves from quantum scattering amplitudes. We have shown how to incorporate both outgoing and incoming narrowly sampled waves, via the "local" observables needed for the former, and scattering of waves needed for the latter.

Waveforms measured at gravitational wave observatories are "local" measurements, in the sense that the passing gravitational wave train is sampled only at the (small) spatial location of the observatory relative to the (very large) spatial extent of the gravitational wave. In this paper, our first major focus was on developing a quantum-field theoretic formalism to describe this kind of classical, local measurement. This is in contrast to previous work [21,72] on classical observables in quantum field theory, which discussed "global" observables, such as the total amount of energy-momentum radiated in a scattering event. Our formalism is very general, though in our explicit discussions we focused on the case of electromagnetic radiation, which has the pedagogical benefit of being slightly easier to work with. We look forward to applications of our formalism in gravity.

Scattering amplitudes are remarkably simple objects which can be computed efficiently. For this reason, it seems very promising that waveforms can be computed so directly in terms of amplitudes. In particular, it is clear from our work that there is no obstacle to using the double copy to compute gravitational waveforms (sourced by a scattering event) to any order of perturbation theory. It may be worth emphasizing that we do not need the BCJ formulation [130] of the double copy at loop level, which remains conjectural, to perform such a computation. The gravitational waveform at higher orders could be computed using the unitarity method, with only tree-level gravitational amplitudes required as inputs. For those amplitudes, the BCJ relations are proven. The insensitivity of the classical waveform to delta-function contributions localized on the worldlines of the particles offers another, potentially significant simplification: only a subset of all possible quantum factorization channels needs to be computed. The possibility of computing leading-order gravitational radiation using amplitudes and the double copy was previously discussed by Luna *et al.* [154], building on a leading-order worldline treatment by Goldberger and Ridgway [195]. The formalism presented here makes this computation possible to any order. Shen [196] has already computed the next-to-leading order waveform; it will be interesting to compare the efficiency of our methods, using the conventional double copy of amplitudes, with Shen's ingenious world-line implementation of the double copy.

One important theme in the calculation and exploration of scattering amplitudes is the search for the simplest forms in which to cast them. An early realization came through the focus on helicity amplitudes rather than covariant forms (in terms of polarization vectors and momenta). The former contain all physical information, and are simpler. This is especially true when they are expressed in terms of spinorial variables. The translation comes through a spinor-helicity formalism; historically, that of Xu, Zhang, and Chang [303] played an important role.

Remarkably, the same phenomenon occurs in classical field theory. Newman–Penrose (NP) scalars [265] are classical analogs of helicity amplitudes; indeed, as we have seen, the NP scalar Φ_2 is an integral over a helicity amplitude. The NP scalars can be defined by contracting tensorial quantities, such as the electromagnetic field strength $F^{\mu\nu}$, with a basis of four null vectors. This basis is a direct analog of the momentum of a particle, along with its two possible polarization vectors, and a gauge choice. Alternatively, the NP scalars can be constructed directly by passing from the tensorial field strength to its spinorial equivalent. In this formulation, a natural basis of spinors occurs classically, in an exact analogue of the spinor-helicity method in scattering amplitudes. It seems likely that further study will reveal more close connections between sophisticated approaches to classical physics and the methods of scattering amplitudes.

As a concrete application of our formalism, we computed a simple waveform: the electromagnetic radiation emitted as two charges scatter. We extracted the asymptotic spectral functions, as well as the relevant asymptotic Newman-Penrose scalar. At leading order, these quantities are closely related to five-point amplitudes. In the Fourier domain, they are built out of modified Bessel functions. At higher orders, the connection to five-point amplitudes will persist. We expect that an interesting class of functions, generalizing Bessel functions, will appear. In the time domain, the functions were simpler; we suspect that this may be an accident of low orders.

Our second major focus in this paper has been developing a quantum field-theoretic description of massless classical waves readily amenable to calculations using scattering amplitudes. Coherent states are key tools in extracting classical behavior from quantum field theory [274], so it is no surprise that we found them to be very helpful. Indeed, they mesh very naturally with amplitudes, and especially with the transition operator T whose matrix elements are the amplitudes. The reason is that the T matrix can be written out in terms of amplitudes and of creation and annihilation operators. These operators, in turn, act very simply on coherent states.

As an application of massless waves, we studied the scattering of a massless electromagnetic wave off a classical charge. We showed that the resulting outgoing wave is determined, at leading order, by the classical limit of the Compton four-point amplitude. We expect this final state to also be coherent. In appendix 11 we provide evidence in favor of the coherence of this radiation.

Throughout our paper, the focus has been on scattering events. These are very naturally described using amplitudes. Scattering events in general relativity are interesting in themselves given the possibility that the tightly-bound compact binaries observed by the LIGO and Virgo collaborations are created after a scattering event with a third object [188]. Of course, a major goal for the future will be to understand how gravitational waveforms from classically bound objects can also be computed using amplitudes. This will need a new understanding, perhaps building on the work [73, 141] of Kälin and Porto in the context of conservative classical dynamics. Yet even without such a direct connection, it seems clear that our work can be used in the context of bound state physics by developing an effective action to enable the transfer of know-how from unbound to bound cases. The reader may also be interested in forthcoming work by Bautista, Guevara, Kavanagh and Vines [283] on related subjects.

The future for gravitational wave physics is data-rich and high precision. We will need every good idea we can find to calculate waveform templates at the necessary precision. By now it is clear that amplitudes and the double copy will be a useful tool. The double copy, at least in its BCJ form, was a theoretical discovery which was a by-product of the drive for precision theory for LHC physics. New theoretical discoveries may well await us as we develop our understanding of gravitational amplitudes in the drive for precision gravitational-wave physics.

APPENDIX

BEAM SPREADING

Let us obtain a more refined picture of the time dependence of the classical wave in eq. (652). Expand the square root in the exponent in that expression, keeping the next-to-leading term in the expansion,

$$\sqrt{\omega^2 + (\bar{k}^x)^2 + (\bar{k}^y)^2} = \omega + \frac{(\bar{k}^x)^2 + (\bar{k}^y)^2}{2\omega} + \cdots . \tag{800}$$

Substituting this into eq. (652), we obtain,

$$A^\mu_{\rm cl}(x) = \sqrt{2} A_\odot \, \mathrm{Re}\, \varepsilon^\mu_\odot(\bar{k}_\odot) e^{-i\omega(t-z)} \mathcal{I}(\omega, \ell_\perp) \,, \tag{801}$$

where we have introduced the following scalar integral (recall that $\sigma_\perp = \ell_\perp^{-1}$),

$$\mathcal{I}(\omega, \ell_\perp) = \int d^2\bar{k}_\perp \, \delta_{\sigma_\perp}(\bar{k}^x) \delta_{\sigma_\perp}(\bar{k}^y) e^{i\bar{k}^x x} e^{i\bar{k}^y y} e^{-it\bar{k}_x^2/(2\omega)} e^{-it\bar{k}_y^2/(2\omega)} \tag{802}$$

Integrating over the angular variable, we find,

$$\mathcal{I}(\omega, \ell_\perp) = 2\ell_\perp^2 \int_0^\infty dk \, k \, J_0(\sqrt{x^2+y^2}\, k) e^{-k^2[\ell_\perp^2 + it/(2\omega)]} \,. \tag{803}$$

Performing the integral, we obtain,

$$\mathcal{I}(\omega, l_\perp) = \frac{e^{-\frac{(x^2+y^2)}{4l_\perp^2}\left[1+i\frac{t}{2\omega l_\perp^2}\right]^{-1}}}{1+\frac{it}{2\omega l_\perp^2}}\,. \tag{804}$$

Yet higher-order contributions may be computed by noticing that the electromagnetic field can be expressed — without expanding the square root in eq. (800) — as,

$$A^\mu_{\text{cl}}(x) = \sqrt{2}A_\odot \,\text{Re}\, \varepsilon^\mu_\odot(\bar{k}_\odot)\ \ e^{i\omega z - it\hat{H}(\omega)}\left[e^{-\frac{(x^2+y^2)}{4l_\perp^2}}\right], \tag{805}$$

where we have introduced the operator $\hat{H}(\omega) = \sqrt{\omega^2 - \nabla^2_{(x,y)}}$. In this reformulation, the problem is now equivalent to computing the time evolution — for a relativistic Hamiltonian with effective mass ω — of a Gaussian wavepacket. Restricting the time evolution to the nonrelativistic limit, we obtain the well-known result for the spread of a Gaussian wavepacket in two dimensions, in agreement with eq. (804). In a similar way, we can easily generalize the computation by adding contributions from the expansion of the polarization vectors in the integrand as in eq. (662).

FACTORIZATION AND UNITARITY IN THE CLASSICAL LIMIT

Our framework allows the computation of classical phenomena such as the electromagnetic field generated by the scattering of an incoming beam of light with a massive particle. In this appendix, we address the question of whether the final state is coherent, in the context of a perturbative calculation. For coherence to hold, we must show that the mean value of the electromagnetic field operator on the final state factorizes. The final state is given by the evolution of the initial state,

$$|\psi\rangle_{\text{out}} = \int d\Phi(p)\, \varphi(p)\, e^{ib\cdot p/\hbar}\, S\, |p\, \alpha^{+}\rangle_{\text{in}} \ . \tag{806}$$

We say that the final state is coherent if the following correlation function vanishes in the classical limit,

$$\Delta = {}_{\text{out}}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\text{out}} - {}_{\text{out}}\langle\psi|\mathbb{F}^{\mu\nu}(x)|\psi\rangle_{\text{out}}\, {}_{\text{out}}\langle\psi|\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\text{out}} \tag{807}$$

where the electromagnetic field operator is given by eq. (620). Let us prove that the previous correlation function vanishes at the first nontrivial order in the coupling g. The second term in eq. (807) is already known to this order as it matches the value of the electromagnetic field in Thomson scattering times its free counterpart. What is left is to compute is the first term. We can safely disregard contributions quadratic in the transfer matrix, leaving us to compute the classical limit of,

$${}_{\text{out}}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\text{out}} = F^{\mu\nu,(0)}(x)F^{\alpha\beta,(0)}(y) + i\, {}_{\text{in}}\langle\psi|\, [\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y), T]\, |\psi\rangle_{\text{in}} \ , \tag{808}$$

where $F^{(0)}_{\mu\nu}(x)$ denotes the free field. Expanding the electromagnetic field operator in terms of annihilation and creation operators,

$$
\begin{aligned}
&\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y) = \\
&-\frac{4}{\hbar^3}\sum_{\eta_1,\eta_2}\int d\Phi(k_1)d\Phi(k_2)\Big[a_{(\eta_1)}(k_1)a_{(\eta_2)}(k_2)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}\,k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]*}e^{-i(k_1\cdot x+k_2\cdot y)/\hbar} \\
&\qquad + a^\dagger_{(\eta_1)}(k_1)a^\dagger_{(\eta_2)}(k_2)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]}\,k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{i(k_1\cdot x+k_2\cdot y)/\hbar} \\
&\qquad - a^\dagger_{(\eta_2)}(k_2)a_{(\eta_1)}(k_1)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}\,k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{-i(k_1\cdot x-k_2\cdot y)/\hbar} \\
&\qquad - a^\dagger_{(\eta_1)}(k_1)a_{(\eta_2)}(k_2)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]}\,k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]*}e^{i(k_1\cdot x-k_2\cdot y)/\hbar} \\
&\qquad - \delta_\Phi(k_1-k_2)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}\,k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{i(k_1\cdot x-k_2\cdot y)/\hbar}\Big]\,.
\end{aligned}
\tag{809}
$$

At leading order in the coupling, the T matrix reads

$$
T=\sum_{\eta,\eta'}\int d\Phi(\widetilde{k}',\widetilde{k},\widetilde{p}',\widetilde{p})\;\langle\widetilde{k}'^{\eta'}\widetilde{p}'|T|\widetilde{k}^{\eta}\widetilde{p}\rangle\;a^\dagger_{(\eta')}(\widetilde{k}')a^\dagger(\widetilde{p}')\,a_{(\eta)}(\widetilde{k})a(\widetilde{p})+\cdots\,, \tag{810}
$$

We can now evaluate the correlation function. The first two terms inside the bracket in eq. (809) can contribute only at higher order in the coupling, and can be safely neglected in the evaluation of the correlation function. As for the last term in eq. (809), we can see it is similar to (636), providing a quantum contribution at leading order in the coupling which will disappear in the classical limit. We are left with the following,

$$
\begin{aligned}
[a^\dagger_{(\eta_1)}(k_1)a_{(\eta_2)}(k_2), T] &= \sum_{\eta} \int d\Phi(\widetilde{p}, \widetilde{p}', \widetilde{k}) \, \langle k_2^{\eta_2} \widetilde{p}' | \, T \, | \widetilde{k}^{\eta} \widetilde{p} \rangle \, a^\dagger_{(\eta_1)}(k_1) a^\dagger(\widetilde{p}') \, a_{(\eta)}(\widetilde{k}) a(\widetilde{p}) \\
&\quad - \sum_{\eta'} \int d\Phi(\widetilde{p}, \widetilde{p}', \widetilde{k}') \, \langle \widetilde{k}'^{\eta'} \widetilde{p}' | \, T \, | k_1^{\eta_1} \widetilde{p} \rangle \, a^\dagger_{(\eta')}(\widetilde{k}') a^\dagger(\widetilde{p}') \, a_{(\eta_2)}(k_2) a(\widetilde{p}) \\
[a^\dagger_{(\eta_2)}(k_2)a_{(\eta_1)}(k_1), T] &= \sum_{\eta} \int d\Phi(\widetilde{p}, \widetilde{p}', \widetilde{k}) \, \langle k_1^{\eta_1} \widetilde{p}' | \, T \, | \widetilde{k}^{\eta} \widetilde{p} \rangle \, a^\dagger_{(\eta_2)}(k_2) a^\dagger(\widetilde{p}') \, a_{(\eta)}(\widetilde{k}) a(\widetilde{p}) \\
&\quad - \sum_{\eta'} \int d\Phi(\widetilde{p}, \widetilde{p}', \widetilde{k}') \, \langle \widetilde{k}'^{\eta'} \widetilde{p}' | \, T \, | k_2^{\eta_2} \widetilde{p} \rangle \, a^\dagger_{(\eta')}(\widetilde{k}') a^\dagger(\widetilde{p}') \, a_{(\eta_1)}(k_1) a(\widetilde{p}) \, .
\end{aligned}
\tag{811}
$$

These results imply that,

$$
\begin{aligned}
&{}_{\text{out}}\langle \psi | \mathbb{F}^{\mu\nu}(x) \mathbb{F}^{\alpha\beta}(y) | \psi \rangle_{\text{out}} = \\
&\quad F^{\mu\nu,(0)}(x) F^{\alpha\beta,(0)}(y) \\
&\quad + \frac{8}{\hbar^3} \operatorname{Re} \sum_{\eta,\eta_1,\eta_2} \int d\Phi(k_1, k_2, \widetilde{k}, p, p') \varphi(p) \varphi^*(p') \\
&\quad \times \left[i \, \langle k_1^{\eta_1} p' | \, T \, | \widetilde{k}^{\eta} p \rangle \, \langle \alpha^+ | \, a^\dagger_{(\eta_2)}(k_2) a_{(\eta)}(\widetilde{k}) \, | \alpha^+ \rangle \, k_1^{[\mu} \varepsilon^{(\eta_1)\nu]*} \, k_2^{[\alpha} \varepsilon^{(\eta_2)\beta]} e^{-i(k_1 \cdot x - k_2 \cdot y)/\hbar} \right] \\
&\quad + \frac{8}{\hbar^3} \operatorname{Re} \sum_{\eta,\eta_1,\eta_2} \int d\Phi(k_1, k_2, \widetilde{k}, p, p') \varphi(p) \varphi^*(p') \\
&\quad \times \left[i \, \langle k_2^{\eta_2} p' | \, T \, | \widetilde{k}^{\eta} p \rangle \, \langle \alpha^+ | \, a^\dagger_{(\eta_1)}(k_1) a_{(\eta)}(\widetilde{k}) \, | \alpha^+ \rangle \, k_1^{[\mu} \varepsilon^{(\eta_1)\nu]} \, k_2^{[\alpha} \varepsilon^{(\eta_2)\beta]*} e^{i(k_1 \cdot x - k_2 \cdot y)/\hbar} \right] \, .
\end{aligned}
\tag{812}
$$

After some simple algebra, we find

$$
\begin{aligned}
&{}_{\text{out}}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\text{out}} = \\
&\quad F^{\mu\nu,(0)}(x)F^{\alpha\beta,(0)}(y) \\
&\quad + \frac{8}{\hbar^3}\,\text{Re}\sum_{\eta_1,\eta_2}\int d\Phi(k_1,\widetilde{k},p,p')\varphi(p)\varphi^*(p') \\
&\qquad \times \left[i\,\langle k_1^{\eta_1}p'|\,T\,|\widetilde{k}^{\eta}p\rangle\,\alpha(\widetilde{k})k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}e^{-ik_1\cdot x/\hbar}\int d\Phi(k_2)\,\alpha^*(k_2)k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{ik_2\cdot y/\hbar}\right] \\
&\quad + \frac{8}{\hbar^3}\,\text{Re}\sum_{\eta_1,\eta_2}\int d\Phi(k_1,k_2,\widetilde{k},p,p')\varphi(p)\varphi^*(p') \\
&\qquad \times \left[i\,\langle k_2^{\eta_2}p'|\,T\,|\widetilde{k}^{\eta}p\rangle\,\alpha(\widetilde{k})\,k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]*}e^{-ik_2\cdot y/\hbar}\int d\Phi(k_1)\,\alpha^*(k_1)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]}\,e^{ik_1\cdot x/\hbar}\right];
\end{aligned}
\tag{813}
$$

reorganizing the terms we then obtain, as expected,

$$
\begin{aligned}
&{}_{\text{out}}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\text{out}} = \\
&\qquad F^{\mu\nu,(0)}(x)F^{\alpha\beta,(0)}(y) \\
&\qquad + F^{\alpha\beta,(0)}(y)\frac{4}{\hbar^{3/2}}\,\text{Re}\sum_{\eta_1}\int d\Phi(k_1,\widetilde{k},p,p')\varphi(p)\varphi^*(p') \\
&\qquad\qquad \times \left[i\,\langle k_1^{\eta_1}p'|\,T\,|\widetilde{k}^{\eta}p\rangle\,\alpha(\widetilde{k})k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}e^{-ik_1\cdot x/\hbar}\right] \\
&\qquad + F^{\mu\nu,(0)}(x)\frac{4}{\hbar^{3/2}}\,\text{Re}\sum_{\eta_2}\int d\Phi(k_2,\widetilde{k},p,p')\varphi(p)\varphi^*(p') \\
&\qquad\qquad \times \left[i\,\langle k_2^{\eta_2}p'|\,T\,|\widetilde{k}^{\eta}p\rangle\,\alpha(\widetilde{k})k_2^{[\mu}\varepsilon^{(\eta_2)\nu]*}e^{-ik_2\cdot y/\hbar}\right].
\end{aligned}
\tag{814}
$$

From this result we conclude that,

$$
\Delta\big|_{g^2} = 0\,. \tag{815}
$$

This demonstrates that the semiclassical state generated in Thomson scattering is a coherent state to this nontrivial order in the coupling.

INTEGRALS

We require explicit expressions for the integrals appearing in the leading-order radiation kernel, eq. (5.46) of ref. [21]. The integral is,

$$\mathcal{R}^{(0)}(\bar{k};b) = 4\int \hat{d}^4\overline{w}_1\hat{d}^4\overline{w}_2\ \hat{\delta}(2p_1\cdot\overline{w}_1)\hat{\delta}(2p_2\cdot\overline{w}_2)\hat{\delta}^{(4)}(\bar{k}-\overline{w}_1-\overline{w}_2)\,e^{i\overline{w}_1\cdot b} \\ \times\left\{\frac{Q_1^2Q_2}{\overline{w}_2^2}\left[-p_2\cdot\varepsilon+\frac{(p_1\cdot p_2)(\overline{w}_2\cdot\varepsilon)}{p_1\cdot\bar{k}}+\frac{(p_2\cdot\bar{k})(p_1\cdot\varepsilon)}{p_1\cdot\bar{k}} \right.\right. \\ \left.\left. -\frac{(\bar{k}\cdot\overline{w}_2)(p_1\cdot p_2)(p_1\cdot\varepsilon)}{(p_1\cdot\bar{k})^2}\right]+(1\leftrightarrow 2)\right\}. \tag{816}$$

We replace p_i^μ by $m_iu_i^\mu$, and introduce a fourth basis vector,

$$v_\mu = 4\epsilon_{\mu\nu\lambda\rho}u_1^\nu u_2^\lambda b^\rho\,. \tag{817}$$

Its square is given by,

$$v^2 = -2G(u_1,u_2,b)\,, \tag{818}$$

where G is the Gram determinant

$$G(\{p_i\}) = \det(2p_i\cdot p_j)\,. \tag{819}$$

The only nontrivial Lorentz invariants that can be built out of the u_i^μ, b^μ, and v^μ are,

$$\gamma = u_1 \cdot u_2 \,, \tag{820}$$

and b^2, as $u_i^2 = 1$.

We note that,

$$v^2 = 16b^2(\gamma^2 - 1)\,. \tag{821}$$

It is convenient to introduce two rescaled four-vectors,

$$\begin{aligned} \widetilde{b}^\mu &= b^\mu / \sqrt{-b^2}\,, \\ \widetilde{v}^\mu &= v^\mu / \sqrt{-v^2} = v^\mu / (4\sqrt{-b^2(\gamma^2-1)})\,. \end{aligned} \tag{822}$$

Let us also introduce as well new coordinates $z_{1,2,b,v}^{[i]}$,

$$\overline{w}_i^\mu = z_1^{[i]} u_1^\mu + z_2^{[i]} u_2^\mu + z_b^{[i]} \widetilde{b}^\mu + z_v^{[i]} \widetilde{v}^\mu\,. \tag{823}$$

The Jacobian from the change of variables in each $\overline{w}_i$ is,

$$|\epsilon_{\mu\nu\lambda\rho} \widetilde{v}^\mu u_1^\nu u_2^\lambda \widetilde{b}^\rho| = -\frac{v^2}{4\sqrt{-v^2}\sqrt{-b^2}} = \sqrt{\gamma^2 - 1}\,. \tag{824}$$

We also have the following expression for each square,

$$\overline{w}_i^2 = (z_1^{[i]})^2 + 2\gamma z_1^{[i]} z_2^{[i]} + (z_2^{[i]})^2 - (z_b^{[i]})^2 - (z_v^{[i]})^2\,. \tag{825}$$

There are four elementary integrals we need to evaluate,

$$
\begin{aligned}
I_1 &= \int \hat{d}^4\overline{w}_1 \hat{d}^4\overline{w}_2 \; \hat{\delta}(u_1\cdot\overline{w}_1)\hat{\delta}(u_2\cdot\overline{w}_2)\hat{\delta}^{(4)}(\bar{k}-\overline{w}_1-\overline{w}_2)\,\frac{e^{i\overline{w}_1\cdot b}}{\overline{w}_1^2}\,, \\
I_2^\mu &= \int \hat{d}^4\overline{w}_1 \hat{d}^4\overline{w}_2 \; \hat{\delta}(u_1\cdot\overline{w}_1)\hat{\delta}(u_2\cdot\overline{w}_2)\hat{\delta}^{(4)}(\bar{k}-\overline{w}_1-\overline{w}_2)\,\frac{e^{i\overline{w}_1\cdot b}\overline{w}_1^\mu}{\overline{w}_1^2}\,, \\
I_3 &= \int \hat{d}^4\overline{w}_1 \hat{d}^4\overline{w}_2 \; \hat{\delta}(u_1\cdot\overline{w}_1)\hat{\delta}(u_2\cdot\overline{w}_2)\hat{\delta}^{(4)}(\bar{k}-\overline{w}_1-\overline{w}_2)\,\frac{e^{i\overline{w}_1\cdot b}}{\overline{w}_2^2}\,, \\
I_4^\mu &= \int \hat{d}^4\overline{w}_1 \hat{d}^4\overline{w}_2 \; \hat{\delta}(u_1\cdot\overline{w}_1)\hat{\delta}(u_2\cdot\overline{w}_2)\hat{\delta}^{(4)}(\bar{k}-\overline{w}_1-\overline{w}_2)\,\frac{e^{i\overline{w}_1\cdot b}\overline{w}_2^\mu}{\overline{w}_2^2}\,.
\end{aligned}
\tag{826}
$$

Start evaluating I_1 by using the four-fold delta function to evaluate the $\overline{w}_2$ integral,

$$
I_1 = \int \hat{d}^4\overline{w}_1 \; \hat{\delta}(u_1\cdot\overline{w}_1)\hat{\delta}(u_2\cdot\overline{w}_1 - u_2\cdot\bar{k})\,\frac{e^{i\overline{w}_1\cdot b}}{\overline{w}_1^2}\,,
\tag{827}
$$

and then make the change of variables (823),

$$
\begin{aligned}
&\frac{\sqrt{\gamma^2-1}}{(2\pi)^2}\int dz_1^{[1]}dz_2^{[1]}dz_b^{[1]}dz_v^{[1]}\;\delta(z_1^{[1]}+\gamma z_2^{[1]})\delta(\gamma z_1^{[1]}+z_2^{[1]}-u_2\cdot\bar{k}) \\
&\qquad\times\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}}{(z_1^{[1]})^2+2\gamma z_1^{[1]}z_2^{[1]}+(z_2^{[1]})^2-(z_b^{[1]})^2-(z_v^{[1]})^2}\,.
\end{aligned}
\tag{828}
$$

Use the delta functions to perform the $z_{1,2}^{[1]}$ integrals,

$$
\frac{1}{(2\pi)^2\sqrt{\gamma^2-1}}\int dz_b^{[1]}dz_v^{[1]}\;\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}}{-(u_2\cdot\bar{k})^2/(\gamma^2-1)-(z_b^{[1]})^2-(z_v^{[1]})^2}\,.
\tag{829}
$$

Perform the $z_v^{[1]}$ integral to obtain,

$$
-\frac{1}{4\pi\sqrt{\gamma^2-1}}\int dz_b^{[1]}\;\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}}{\sqrt{(z_b^{[1]})^2+(u_2\cdot\bar{k})^2/(\gamma^2-1)}}\,.
\tag{830}
$$

This can be evaluated as a Fourier transform,

$$I_1 = -\frac{1}{2\pi\sqrt{\gamma^2-1}} K_0\big(\sqrt{-b^2}\, u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\,, \tag{831}$$

where K_0 is a modified Bessel function of the second kind.

The first two steps are the same for I_2^μ,

$$\begin{aligned} I_2^\mu &= \int \hat{d}^4\overline{w}_1\; \hat{\delta}(u_1\cdot\overline{w}_1)\hat{\delta}(u_2\cdot\overline{w}_1 - u_2\cdot\bar{k})\,\frac{e^{i\overline{w}_1\cdot b}\overline{w}_1^\mu}{\overline{w}_1^2} \\ &= \frac{\sqrt{\gamma^2-1}}{(2\pi)^2}\int dz_1^{[1]}dz_2^{[1]}dz_b^{[1]}dz_v^{[1]}\; \delta(z_1^{[1]}+\gamma z_2^{[1]})\delta(\gamma z_1^{[1]}+z_2^{[1]}-u_2\cdot\bar{k}) \\ &\qquad\times \frac{e^{-iz_b^{[1]}\sqrt{-b^2}}(z_1^{[1]}u_1^\mu + z_2^{[1]}u_2^\mu + z_b^{[1]}\widetilde{b}^\mu + z_v^{[1]}\widetilde{v}^\mu)}{(z_1^{[1]})^2 + 2\gamma z_1^{[1]}z_2^{[1]} + (z_2^{[1]})^2 - (z_b^{[1]})^2 - (z_v^{[1]})^2}\,. \end{aligned} \tag{832}$$

The $\widetilde{v}^\mu$ term will vanish because of the antisymmetry in $z_v^{[1]}$; the $u_{1,2}^\mu$ terms will yield a result proportional to I_1,

$$\begin{aligned} I_{2a}^\mu &= \frac{u_2\cdot\bar{k}}{\gamma^2-1}(\gamma u_1^\mu - u_2^\mu)I_1 \\ &= -\frac{u_2\cdot\bar{k}}{2\pi\,(\gamma^2-1)^{3/2}}(\gamma u_1^\mu - u_2^\mu)K_0\big(\sqrt{-b^2}\,u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\,. \end{aligned} \tag{833}$$

The remaining ($\widetilde{b}^\mu$) term is,

$$\begin{aligned} I_{2b}^\mu &= -\frac{\widetilde{b}^\mu}{4\pi\sqrt{\gamma^2-1}}\int dz_b^{[1]}\;\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}z_b^{[1]}}{\sqrt{(z_b^{[1]})^2 + (u_2\cdot\bar{k})^2/(\gamma^2-1)}}\,. \\ &= \frac{iu_2\cdot\bar{k}\,\widetilde{b}^\mu}{2\pi\,(\gamma^2-1)}K_1\big(\sqrt{-b^2}\,u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\,, \end{aligned} \tag{834}$$

where we have dropped a delta-function contribution. The total is,

$$I_2^\mu = I_{2a}^\mu + I_{2b}^\mu\,. \tag{835}$$

In I_3, start by using the four-fold delta function to integrate out $\overline{w}_1$,

$$I_3 = e^{ib\cdot\bar{k}} \int \hat{d}^4\overline{w}_2\, \hat{\delta}(u_1\cdot\overline{w}_2 - u_1\cdot\bar{k})\hat{\delta}(u_2\cdot\overline{w}_2)\, \frac{e^{i\overline{w}_2\cdot b}}{\overline{w}_2^2}\,. \tag{836}$$

This is proportional to I_1, with the exchange $u_1 \leftrightarrow u_2$,

$$I_3 = -\frac{e^{ib\cdot\bar{k}}}{2\pi\sqrt{\gamma^2-1}} K_0\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\,. \tag{837}$$

Similarly for I_4^μ,

$$I_4^\mu = I_{4a}^\mu + I_{4b}^\mu\,, \tag{838}$$

with,

$$\begin{aligned} I_{4a}^\mu &= -\frac{u_1\cdot\bar{k}}{\gamma^2-1}\big(u_1^\mu - \gamma u_2^\mu\big) I_3 \\ &= \frac{u_1\cdot\bar{k}\, e^{ib\cdot\bar{k}}}{2\pi\,(\gamma^2-1)^{3/2}}\big(u_1^\mu - \gamma u_2^\mu\big) K_0\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\,, \\ I_{4b}^\mu &= \frac{i u_1\cdot\bar{k}\, e^{ib\cdot\bar{k}}\, \widetilde{b}^\mu}{2\pi\,(\gamma^2-1)} K_1\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\,. \end{aligned} \tag{839}$$

Part V

CONCLUSION AND FUTURE WORK

12

CONCLUSION

Black holes are extraordinarily simple. According to the no-hair theorem, they are entirely described only by three externally observable parameters: their mass, angular momentum and electric charge. This thesis shows that their dynamics shares an equal remarkable simplicity since we can uniquely describe it in terms of scattering amplitudes, the most perfect microscopic structures available in quantum field theory. To outline this incredible fact, in the first part of the thesis we have focused on the two-body problem in General Relativity, with an attention toward the post-Minkowskian approximation for spinless binary black holes. We have started in Chapter 4 by presenting the complexity of the classical post-Minkowskian approximation in General Relativity. Tracing a close analogy with a quantum scattering, we have introduced the Lippmann-Schwinger equation as a tool to reformulate the classical scattering in terms of amplitudes. Although usually presented in the non-relativistic approximation, the Lippmann-Schwinger equation can be easily generalized to capture the two-body dynamics in the fully relativistic sector. We have proven this in Chapter 5 by deriving the Hamiltonian of a spinless binary system up to second order in G_N and all orders in the velocities, finding agreement with an earlier calculation for the scattering angle by Westpfahl. We have then applied this general framework to the case of a modified theory of gravity in Chapter 6, deriving the same observable for binary systems. We have then focused in Chapter 7 on a remarkable linear relation between the classical momentum of a binary system - precisely the square of the derivative of the radial action - and on-shell amplitudes. Using this relation, first introduced by Damour, we have provided a direct formula relating the scattering angle in the post-Minkowskian approximation to scattering amplitudes, valid to all orders in the coupling and in the conservative regime. Interestingly, this simple relation holds at a linear level only in $D = 4$ dimensions. We have shown this in Chapter 8, where non-trivial information coming from box diagrams in dimensions higher than 4 is crucial to agree with earlier calculations from eikonal approaches. Chapter 9 has introduced the CHY formalism providing the needed input for on-shell calculations relevant to derive post-Minkowskian observables from amplitudes at higher orders. We have presented different methods to compute tree-level amplitudes of two massive scalars and an arbitrary number of gravitons in D-dimensions, the most economical one being the so-called Λ-algorithm. Chapters 10 and 11 have then considered the problem left apart in the previous chapters of describing wave phenomena and radiation generated by a binary. The primary tool used has been the KMOC formalism, which provides the most rigorous map between classical observables and quantum scattering amplitudes. In Chapter 10, we have presented an off-shell generalization of the underlying ideas in KMOC, which has been used to derive gravitational and electromagnetic shock waves solutions in a simple manner, including higher-dimensional ones and spin effects. In Chapter 11, we have extended the realm of application of the original work in KMOC by showing how classical wave phenomena, particularly those related to the two-body problem, are accessible once we consider coherent states in scattering processes. Equipped with this generalized framework, we have derived the bending of light and the Thomson scattering from on-shell amplitudes in the most rigorous way. Finally, we have shown a deep relation between waveforms and on-shell amplitudes. The latter has been considered in two cases: waveforms generated in the Thomson scattering and in a binary encounter. In both cases, we have adopted the Newman-Penrose formalism, providing an elegant expression for the final emitted waveform as an integral over helicity amplitudes.

13

FUTURE WORK

There are many avenues for research based on the ideas presented in this thesis. Self-force results for binary systems can provide an essential improvement to waveform templates, especially considering the proposed Laser Interferometer Space Antenna and its sensitivity to low-frequency signals. From this perspective, it would be interesting to explore the application of scattering amplitudes to self-force effects, especially in light of the recent success in tackling the inspiral phase of the general relativistic two-body problem. Another interesting possibility is the study of alternative wave and fluid solutions in General Relativity from the scattering amplitude approach outlined in Chapter 10. Based on the work in Chapter 11, which treats unbound states, it would be of particular interest also to understand how gravitational waveforms from classically bound objects can be computed using amplitudes. Hopefully, these research directions will be pursued in the upcoming years.

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