UNIVERSITY OF COPENHAGEN

A thesis presented to the Faculty of Science in partial fulfillment of the requirements for the degree

Doctor of Philosophy in Physics

Effective Field Theory

&

Scattering Amplitudes

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ABSTRACT

This thesis is devoted to theoretical developments in effective field theories and scattering amplitudes. We discuss several effective field theories, ranging from the effective-field-theory extension of the Standard Model to an effective field theory describing gravitational interactions of black holes. We also develop modern methods for calculating scattering amplitudes, and apply them to effective field theories.

RESUMÉ

Denne afhandling er viet til teoretiske udviklinger inden for effektive feltteorier og spredningsamplituder. Vi diskuterer adskillige effektive feltteorier, lige fra den effektive feltteori-udvidelse af standardmodellen til en effektiv feltteori, der beskriver gravitationsinteraktioner mellem sorte huller. Vi udvikler også moderne metoder til beregning af spredningsamplituder og anvender dem på effektive feltteorier.

PREFACE

This thesis contains results originating from a three-year-long research project in theoretical physics. The main topics are effective field theory and scattering amplitudes. The thesis is divided in two parts:

Part II: Effective Field Theory

Part III: Scattering Amplitudes

The main text consists of reprints of preprints and published journal articles.

REPRINTS

The thesis is based on, and consists of reprints of, the following publications and preprints:

- A. Helset and M. Trott, *On interference and non-interference in the smeft*, *JHEP* **04** (2018) 038 [1711.07954]
- A. Helset, M. Paraskevas and M. Trott, *Gauge fixing the standard model effective field theory*, *Phys.Rev.Lett.* **120** (2018) 251801 [1803.08001]
- N. Bjerrum-Bohr, H. Gomez and A. Helset, *New factorization relations for nonlinear sigma model amplitudes*, *Phys.Rev.D* **99** (2019) 045009 [1811.06024]
- A. Helset and M. Trott, *Equations of motion, symmetry currents and eft below the electroweak scale*, *Phys.Lett.B* **795** (2019) 606 [1812.02991]
- H. Gomez and A. Helset, *Scattering equations and a new factorization for amplitudes. part ii. effective field theories*, *JHEP* **05** (2019) 129 [1902.02633]
- P. H. Damgaard, K. Haddad and A. Helset, *Heavy black hole effective theory*, *JHEP* **11** (2019) 070 [1908.10308]
- A. Helset and A. Kobach, *Baryon number, lepton number, and operator dimension in the smeft with flavor symmetries*, *Phys.Lett.B* **800** (2020) 135132 [1909.05853]
- T. Corbett, A. Helset and M. Trott, *Ward identities for the standard model effective field theory*, *Phys.Rev.D* **101** (2020) 013005 [1909.08470]
- R. Aoude and A. Helset, *Soft Theorems and the KLT-Relation*, *JHEP* **04** (2020) 044 [1911.09962]
- A. Helset, A. Martin and M. Trott, *The Geometric Standard Model Effective Field Theory*, *JHEP* **03** (2020) 163 [2001.01453]
- R. Aoude, K. Haddad and A. Helset, *On-shell heavy particle effective theories*, *JHEP* **05** (2020) 051 [2001.09164]
- K. Haddad and A. Helset, *The double copy for heavy particles*, 2005.13897

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Also, as my supervisor, I thank N. Emil J. Bjerrum-Bohr. I am grateful for our many relaxed discussions about physics and other matters. Also, I am thankful for being allowed a high degree of independence, as well as always being provided help when I needed it.

A special thanks is in order for my friend and collaborator, Kays Haddad. We have worked so closely together that it at times felt like we were our own little research group.

Another close friend and collaborator is Rafael Aoude. Even though we have to work non-locally, we have had a fruitful collaboration, which I hope will continue in the future. Our long discussions have been for me the highlights of several conferences which we both attended.

I would like to thank my other collaborators, Tyler Corbett, Poul Henrik Damgaard, Humberto Gomez, Andrew Kobach, Adam Martin, and Michael Paraskevas, for the stimulating discussions we had, either in person or over email, which lead to the papers we wrote together.

Most of my time I spent at the Niels Bohr International Academy. The environment was both friendly and intellectually stimulating. In particular, numerous discussions in the coffee lounge about important and less important topics were some of the highlights of the work day. These encounters were both spontaneous, and structured via the daily tea time.

As part of the PhD program, I was able to take advantage of a large budget for travel. In particular, I was highly encouraged to have a stay abroad. I spent three months at the University of California, San Diego. I thank Aneesh Manohar and Ben Grinstein for their hospitality. I enjoyed the group lunches in the California sun, and our discussions. The time spent in San Diego was educational.

My fellow PhD students at the Niels Bohr Institute, Kays Haddad, Andrea Cristofoli, Laurie Walk, Joeri de Bruijckere, Anagha Vasudevan, Meera Machado, Anna Suliga, and many others, have made the time in Copenhagen a joy, both at an academic as well as social level.

I am privileged to have a family who fully supports me. I thank them for always being there for me the way I try to be there for them.

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CONTENTS

Part I

INTRODUCTION

1

MOTIVATION

Quantum field theory successfully describes many different phenomena, ranging in scale from the shortest scales we can experimentally probe at the Large Hadron Collider, to the scales of black hole mergers. Many of the quantum field theories which describe the observed phenomena are effective field theories. Improving our calculational abilities of the effective field theories is of great value, both practically and theoretically. Modern methods have been developed which enable many calculations which were previously intractable with traditional methods. One of the main objectives of this thesis is to apply the ever-increasing toolbox of modern methods for calculating scattering amplitudes to effective field theories. This would simultaneously improve our ability to perform hard calculations as well as illuminate the underlying structure of the effective field theory.

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THESIS OUTLINE

Chapter 3 contains an introduction to effective field theory and scattering amplitudes.

The main results of the thesis are divided in two parts;

Part II: Effective Field Theory

Part III: Scattering Amplitudes

Part II: Effective Field Theory contains the following chapters:

Chapter 4 is a reprint of A. Helset and M. Trott, *On interference and non-interference in the smeft*, *JHEP* **04** (2018) 038 [1711.07954]

Chapter 5 is a reprint of A. Helset, M. Paraskevas and M. Trott, *Gauge fixing the standard model effective field theory*, *Phys.Rev.Lett.* **120** (2018) 251801 [1803.08001]

Chapter 6 is a reprint of T. Corbett, A. Helset and M. Trott, *Ward identities for the standard model effective field theory*, *Phys.Rev.D* **101** (2020) 013005 [1909.08470]

Chapter 7 is a reprint of A. Helset, A. Martin and M. Trott, *The Geometric Standard Model Effective Field Theory*, *JHEP* **03** (2020) 163 [2001.01453]

Chapter 8 is a reprint of A. Helset and A. Kobach, *Baryon number, lepton number, and operator dimension in the smeft with flavor symmetries*, *Phys.Lett.B* **800** (2020) 135132 [1909.05853]

Chapter 9 is a reprint of A. Helset and M. Trott, *Equations of motion, symmetry currents and eft below the electroweak scale*, *Phys.Lett.B* **795** (2019) 606 [1812.02991]

Chapter 10 is a reprint of P. H. Damgaard, K. Haddad and A. Helset, *Heavy black hole effective theory*, *JHEP* **11** (2019) 070 [1908.10308]

Part III: Scattering Amplitudes contains the following chapters:

Chapter 11 is a reprint of K. Haddad and A. Helset, *The double copy for heavy particles*, 2005.13897

Chapter 12 is a reprint of R. Aoude, K. Haddad and A. Helset, *On-shell heavy particle effective theories*, *JHEP* **05** (2020) 051 [2001.09164]

Chapter 13 is a reprint of R. Aoude and A. Helset, *Soft Theorems and the KLT-Relation*, *JHEP* **04** (2020) 044 [1911.09962]

Chapter 14 is a reprint of N. Bjerrum-Bohr, H. Gomez and A. Helset, *New factorization relations for nonlinear sigma model amplitudes*, *Phys.Rev.D* **99** (2019) 045009 [1811.06024]

Chapter 15 is a reprint of H. Gomez and A. Helset, *Scattering equations and a new factorization for amplitudes. part ii. effective field theories*, *JHEP* **05** (2019) 129 [1902.02633]

Chapter 16 contains conclusions, while future work is discussed chapter 17.

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INTRODUCTION

The first key concept of this thesis is *effective field theory* (EFT).[1] EFTs are consistent quantum field theories. The power of EFTs comes from the feature that one does not need to know the full theory in order to perform calculations; we do not need to know about the mass of the Higgs boson or quantum gravity in order to calculate the trajectory of a baseball.

In order to mathematically implement the ideas of effective field theory, we first need to know how to describe a physical system. A general description of a physical system might consist of the following elements:

- Degrees of freedom
- Symmetries
- Expansion parameter

The relevant degrees of freedom for a chemist are atoms and electrons, not muons and B-mesons. The key principle here is that in order to describe physics at one length scale, it is sufficient to know the field content on that length scale. What happens at a much smaller length scale is *irrelevant*.

Once we have specified the degrees of freedom for our system, we can study how they transform under certain symmetries. Much of the success of physics is based on the use of symmetries. Thus, symmetries are important for effective field theories. Sometimes the effective theory will have different symmetries than the underlying theory.

Lastly, one of the hallmarks of an effective field theory is the expansion parameter. Often can we not calculate the full answer to the complicated interactions of sub-atomic particles. But we can get pretty close to the full answer by organizing the calculation with an expansion parameter. We can then obtain the leading-order description and subsequently calculate corrections to the result.

[1] See e.g. refs. [13–16] for reviews on effective field theories.

The second key concept of this thesis is *scattering amplitudes*. The scattering amplitude $\mathcal{A}$ is a key ingredient in the differential cross-section, $\frac{d\sigma}{\delta\Omega} \sim |\mathcal{A}|^2$. The differential cross-section, and the cross-section (which is found by integrating the differential cross-section over angles), are experimental observables. Thus, the scattering amplitude is one connection between theoretical and experimental particle physics. However, calculating the scattering amplitude can be a lot of work. In the traditional formulation of quantum field theory, we start with a Lagrangian and derive Feynman rules. These Feynman rules are used to calculate various Feynman diagrams. However, when the number of particles involved increases, so does the number of Feynman diagrams we need to calculate. For example, a scattering amplitude with 10 gluons[2] at tree level involves more than 1 million diagrams [17]. This becomes cumbersome very quickly, and alternative methods for performing the calculation should be applied.

In recent years, much progress has been made to find alternatives to the Feynman-diagram expansion. In particular, on-shell scattering amplitude methods dramatically reduce the complexity of calculating multi-gluon scattering amplitudes. Some keywords from this toolbox are: on-shell recursion relations, spinor-helicity formalism, generalized unitarity, the double-copy relation between gauge and gravity amplitudes, etc. For reviews of the scattering-amplitude program, see refs. [18–21].

The synergy of effective field theory and modern methods for scattering amplitudes has currently not been fully exploited. Effective-field-theory calculations are often performed with the use of more traditional methods like Feynman diagrams. Many practitioners of modern methods for scattering amplitudes, on the other hand, have put a large focus on simpler theories as e.g. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. That leaves a vast landscape of effective field theories which could benefit from being more interconnected to these modern methods. This thesis attempts to start bridging the gap.

[2] For scattering amplitudes involving gravitons the situation is even worse.

Part II

EFFECTIVE FIELD THEORY

4

ON INTERFERENCE AND NON-INTERFERENCE IN THE SMEFT

We discuss interference in the limit $\hat{m}_W^2/s \to 0$ in the Standard Model Effective Field Theory (SMEFT). Dimension six operators that contribute to $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ scattering events can experience a suppression of interference effects with the Standard Model in this limit. This occurs for subsets of phase space in some helicity configurations. We show that approximating these scattering events by $2 \to 2$ on-shell scattering results for intermediate unstable gauge bosons, and using the narrow width approximation, can miss interference terms present in the full phase space. Such interference terms can be uncovered using off-shell calculations as we explicitly show and calculate. We also study the commutation relation between the SMEFT expansion and the narrow width approximation, and discuss some phenomenological implications of these results.

4.1 INTRODUCTION

When physics beyond the Standard Model (SM) is present at scales larger than the Electroweak scale, the SM can be extended into an Effective Field Theory (EFT). This EFT can characterize the low energy limit (also known as the infrared (IR) limit) of such physics relevant to the modification of current experimental measurements. Assuming that there are no light hidden states in the spectrum with appreciable couplings in the SM, and that a $\mathrm{SU_L}(2)$ scalar doublet with hypercharge $\mathrm{y}_h = 1/2$ is present in the IR limit of a new physics sector, the theory that results from expanding in the Higgs vacuum expectation value $\sqrt{2\langle H^\dagger H\rangle} \equiv \bar{v}_T$ over the scale of new physics $\sim \Lambda$ is the Standard Model Effective Field Theory (SMEFT).

When the SMEFT is formulated using standard EFT techniques, this theoretical framework is a well defined and rigorous field theory that can consistently describe and characterize the breakdown of the SM emerging from experimental measurements, in the presence of a mass gap ($\bar{v}_T/\Lambda < 1$). For a review of such a formulation of the SMEFT see Ref. [22]. The SMEFT is as useful as it is powerful as it can be systematically improved, irrespective of its UV completion, to ensure that its theoretical precision can match or exceed the experimental accuracy of such measurements.

Calculating in the SMEFT to achieve this systematic improvement can be subtle. Well known subtleties in the SM predictions of cross sections can be present, and further subtleties can be introduced due to the presence of the EFT expansion parameter $\bar{v}_T/\Lambda < 1$. Complications due to the combination of these issues can also be present. As the SMEFT corrections to the SM cross sections are expected to be small $\lesssim$ % level perturbations, it is important to overcome these issues with precise calculations, avoiding approximations or assumptions that introduce theoretical errors larger than the effects being searched for, to avoid incorrect conclusions. For this reason, although somewhat counterintuitive, rigour and precise analyses on a firm field theory footing are as essential in the SMEFT as in the SM.

In this paper we demonstrate how subtleties of this form are present when calculating the leading interference effect of some $\mathcal{L}^{(6)}$ operators as $\hat{m}^2_{W/Z}/s \to 0$. We demonstrate how this limit can be modified from a naive expectation formed through on-shell calculations due to off-shell contributions to the cross section. Furthermore, we show[1] how to implement the narrow width approximation in a manner consistent with the SMEFT expansion.

These subtleties are relevant to recent studies of the interference of the leading SMEFT corrections in the $\hat{m}^2_{W/Z}/s \to 0$ limit, as they lead to a different estimate of interference effects than has appeared in the literature when considering experimental observables.

4.2 CC03 APPROXIMATION OF $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$

The Standard Model Effective Field Theory is constructed out of $\mathrm{SU_C}(3) \times \mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$ invariant higher dimensional operators built out of SM fields. The Lagrangian is given as

$$\mathcal{L}_{SMEFT} = \mathcal{L}_{SM} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \mathcal{L}^{(7)} + ..., \qquad \mathcal{L}^{(d)} = \sum_{i=1}^{n_d} \frac{C_i^{(d)}}{\Lambda^{d-4}} Q_i^{(d)} \quad \text{for } d > 4. \tag{1}$$

We use the Warsaw basis [25] for the operators ($Q_i^{(6)}$) in $\mathcal{L}^{(6)}$, that are the leading SMEFT corrections studied in this work. We absorb factors of $1/\Lambda^2$ into the Wilson coefficients below. We use the conventions of Refs. [22, 26] for the SMEFT; defining Lagrangian parameters in the canonically normalized theory with a bar superscript, and Lagrangian parameters inferred from experimental measurements at tree level with hat superscripts. These quantities differ (compared to the SM) due to

[1] For past discussions see Refs. [22–24].

the presence of higher dimensional operators. We use the generic notation $\delta X = \bar{X} - \hat{X}$ for these differences for a Lagrangian parameter X. See Refs. [22, 26] and the Appendix for more details on notation.

Consider $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ scattering in the SMEFT with leptonic $\bar{\psi}\psi$ and quark $\bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ fields. The differential cross section for this process in the SM can be approximated by the CC03 set of Feynman diagrams,[2] where the $W^\pm$ bosons are considered to be on-shell. This defines the related differential cross section $d\sigma(\bar{\psi}\psi \to W^+W^-)/d\Omega$, which is useful to define as an approximation to the observable, but it is formally unphysical as the $W^\pm$ bosons decay. The lowest order results of this form were determined in Refs. [27–34] and the CC03 diagrams are shown in Fig. 1. The amplitude for $\bar{\psi}\psi \to W^+W^- \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ in this approximation is defined as

$$\sum_{\substack{X=\{\nu,A,Z\}\\ \lambda^\pm=\{+,-\}}} M_X^{\lambda^\pm} = \bar{D}_W(s_{12})\bar{D}_W(s_{34})\mathcal{M}_X^{\lambda_i}\mathcal{M}_{W^+}^{\lambda_{12}}\mathcal{M}_{W^-}^{\lambda_{34}}, \quad \bar{D}_W(s_{ij}) = \frac{1}{s_{ij} - \bar{m}_W^2 + i\bar{\Gamma}_W\bar{m}_W + i\epsilon}, \tag{2}$$

where a constant s-independent width for the $W^\pm$ propagators $\bar{D}_W(s_{ij})$ is introduced[3] and

$$\mathcal{M}_\nu^{\lambda_i} = \mathcal{M}_{ee\to WW,\nu}^{\lambda_{12}\lambda_{34}\lambda_+\lambda_-}\delta^+_{\lambda_+}\delta^-_{\lambda_-}, \qquad \mathcal{M}_V^{\lambda_i} = \mathcal{M}_{ee\to WW,V}^{\lambda_{12}\lambda_{34}\lambda_+\lambda_-},$$

$$\mathcal{M}_{W^+}^{\lambda_{12}} = \mathcal{M}_{W^+\to f_1\bar{f}_2}^{\lambda_{12}}, \qquad \mathcal{M}_{W^-}^{\lambda_{34}} = \mathcal{M}_{W^-\to f_3\bar{f}_4}^{\lambda_{34}},$$

where $V = \{A, Z\}$. Here λ_{12} and λ_{34} label helicities of the intermediate $W^\pm$ bosons with four momenta s_{12}, s_{34}, and $\lambda_\pm$ label helicities of the $\bar{\psi}\psi$ initial state fermions. Transversely polarized massive vector bosons are labeled as $\lambda_{12/34} = \pm$ and the remaining polarization (in the massless fermion limit) is labeled as $\lambda_{12/34} = 0$. The individual sub-amplitudes are taken from Ref. [23] where the complete SMEFT result was reported (see also Refs. [35–45]). The total spin averaged differential cross section is defined as

$$\frac{d\sigma}{d\Omega\, ds_{12} ds_{34}} = \frac{\sum |M_X^{\lambda^\pm}|^2}{(2\pi)^2\, 8s}, \qquad \sum |M_X^{\lambda^\pm}|^2 = |\bar{D}_W(s_{12})\bar{D}_W(s_{34})|^2 \sum_{\substack{X=\{\nu,A,Z\}\\ \lambda^\pm=\{+,-\}}} M_X^{\lambda^\pm}(M_X^{\lambda^\pm})^*, \tag{3}$$

where $d\Omega = d\cos\theta_{ab}\, d\phi_{ab}\, d\cos\theta_{cd}\, d\phi_{cd}\, d\cos\theta\, d\phi$, with θ, ϕ the angles between the W^+ and ℓ^- in the center of mass frame. The remaining angles describing the two body decays of the $W^\pm$

[2] So named as CC indicates charged current.

[3] We have checked and confirmed that the novel interference effects we discuss below persist if an s dependent width is used.

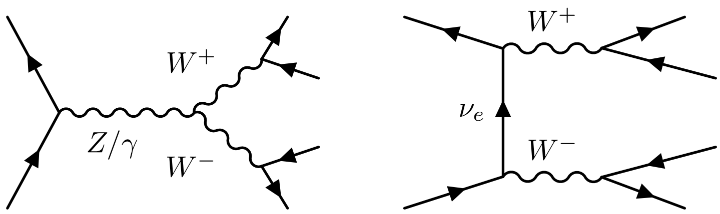


Figure 1: The CC03 Feynman diagrams contributing to $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ with leptonic initial states.

are in the rest frames of the respective bosons. The integration ranges for $\{s_{12}, s_{34}\}$ are $s_{34} \in \left[0, (\sqrt{s} - \sqrt{s_{12}})^2\right]$, $s_{12} \in [0, s]$. It is instructive to consider the decomposition of the general amplitude in terms of helicity labels of the initial state fermions, and the intermediate $W^{\pm}$ bosons in the limit $\hat{m}^2_{W/Z}/s \to 0$ [31, 33, 40, 46–48]. Note that the results we report below are easily mapped to other initial and finals states, so long as these states are distinct.

4.2.1 *Near on-shell phase space*

First, consider the near on-shell region of phase space for the $W^{\pm}$ bosons defined by

$$\textbf{Case 1}: \qquad s_{12} = s_1\, \bar{m}^2_W, \qquad s_{34} = s_3\, \bar{m}^2_W. \tag{4}$$

This expansion is limited to the near on-shell region of phase space for the intermediate $W^{\pm}$ bosons ($s_1 \sim s_3 \sim 1$) by construction. Introducing $x = \hat{m}_W/\sqrt{s}$ and $y = s/\Lambda^2$ an expansion in $x, y < 1$ can be performed by expressing the dimensionful parameters in terms of these dimensionless variables, times the appropriate coupling constant when required. The δX parameters were rescaled to extract these dimensionful scales as $x^2 y\, \delta X = \bar{X} - \hat{X}$ where required. This gives the results shown in Table 1.

Table 1 shows an interesting pattern of suppressions to $\mathcal{L}^{(6)}$ operator corrections dependent upon the helicity configuration of the intermediate $W^{\pm}$ polarizations. This result is consistent with recent discussions in Refs. [40, 46–48]. In the near on-shell region of phase space a relative suppression of interference terms by x^2 for amplitudes with a $\pm$ polarized $W^{\pm}$ compared to the corresponding case with a 0 polarization is present. These results for the $\lambda_{12}\lambda_{34}\lambda_+\lambda_- = \pm\pm+-$ and $\pm\pm-+$ helicity

$\lambda_{12}\lambda_{34}\lambda_+\lambda_-$	$\sum_X M_X^{\lambda^\pm}/4\pi\hat{\alpha}$
$00-+$	$\frac{\sin\theta}{2\sqrt{s_1 s_3}}\left[\frac{1}{c_{\hat\theta}^2}+\left(\delta\kappa^{Z\alpha}-\delta F_2^{Z\alpha}\right)y\right]$
$\pm\pm-+$	$-\sin\theta\left[\frac{x^2}{c_{\hat\theta}^2}+\frac{y\,\delta\lambda^{Z\alpha}}{2}+\left(\delta g_1^{Z\alpha}-\delta F_2^{Z\alpha}-(s_1+s_3)\frac{\lambda_{Z\alpha}}{2}+\frac{\delta\lambda_Z}{2c_{\hat\theta}^2}\right)y\,x^2\right]$
$\pm 0-+$	$-\frac{(1\pm\cos\theta)\,x}{\sqrt{2s_3}}\left[\frac{1}{c_{\hat\theta}^2}+\frac{y}{2}\left(\delta g_1^{Z\alpha}-2\delta F_2^{Z\alpha}+\delta\kappa^{Z\alpha}+s_3\,\delta\lambda^{Z\alpha}\right)\right]$
$0\pm-+$	$\frac{(1\mp\cos\theta)\,x}{\sqrt{2s_1}}\left[\frac{1}{c_{\hat\theta}^2}+\frac{y}{2}\left(\delta g_1^{Z\alpha}-2\delta F_2^{Z\alpha}+\delta\kappa^{Z\alpha}+s_1\delta\lambda^{Z\alpha}\right)\right]$
$00+-$	$\frac{\sin\theta}{2\sqrt{s_1 s_3}}\left[\left(\frac{s_{\hat\theta}^2-c_{\hat\theta}^2}{2c_{\hat\theta}^2 s_{\hat\theta}^2}\right)+\frac{s_1+s_3}{2s_{\hat\theta}^2}+\left(\delta\kappa^{Z\alpha}-\frac{\delta\kappa_Z}{2s_{\hat\theta}^2}+\frac{2\,\delta\bar{g}_W^\ell}{s_{\hat\theta}^2}-\delta F_2^{Z\alpha}\right)y\right]$
$\pm\pm+-$	$-\frac{\sin\theta}{2}\left[\left(1-\frac{1}{2s_{\hat\theta}^2}\right)\delta\lambda_Z-\delta\lambda_\alpha\right]y$
$\pm 0+-$	$\frac{(1\mp\cos\theta)\,x}{2\sqrt{2s_3}}\left[\left(\frac{s_{\hat\theta}^2-c_{\hat\theta}^2}{c_{\hat\theta}^2 s_{\hat\theta}^2}\right)+\frac{s_1}{s_{\hat\theta}^2}+\frac{s_3}{s_{\hat\theta}^2}\frac{1\pm 2+3\cos\theta}{1+\cos\theta}-y\frac{\left(\delta g_1^z+\delta\kappa_z+s_3\,\delta\lambda_z\right)}{2s_{\hat\theta}^2}\right.$ $\left.-y\left(\delta F_1^{Z\alpha}-\frac{4\,\delta\bar{g}_W^\ell}{s_{\hat\theta}^2}-\left(\delta g_1^{Z\alpha}+\delta\kappa^{Z\alpha}+s_3\,\delta\lambda^{Z\alpha}\right)\right)\right]$
$0\pm+-$	$-\frac{(1\pm\cos\theta)\,x}{2\sqrt{2s_1}}\left[\left(\frac{s_{\hat\theta}^2-c_{\hat\theta}^2}{c_{\hat\theta}^2 s_{\hat\theta}^2}\right)+\frac{s_3}{s_{\hat\theta}^2}+\frac{s_1}{s_{\hat\theta}^2}\frac{1\mp 2+3\cos\theta}{1+\cos\theta}-y\frac{\left(\delta g_1^z+\delta\kappa_z+s_1\,\delta\lambda_z\right)}{2s_{\hat\theta}^2}\right.$ $\left.-y\left(\delta F_1^{Z\alpha}-\frac{4\,\delta\bar{g}_W^\ell}{s_{\hat\theta}^2}-\left(\delta g_1^{Z\alpha}+\delta\kappa^{Z\alpha}+s_1\,\delta\lambda^{Z\alpha}\right)\right)\right]$
$\pm\mp+-$	$\frac{(\mp 1+\cos\theta)\sin\theta}{2\,s_{\hat\theta}^2(1+\cos\theta)}$

Table 1: Expansion in $x, y < 1$ for the near on-shell region of phase space of the CC03 diagrams approximating $\bar\psi\psi \to \bar\psi'_1\psi'_2\bar\psi'_3\psi'_4$. For exactly on-shell intermediate $W^\pm$ bosons $s_1 = s_3 = 1$. We have used the notation $\delta F^i_{Z\alpha} = (\delta F^Z_i + \delta F^\alpha_i)/4\pi\hat\alpha$, $\delta\lambda^{Z\alpha} = \delta\lambda_Z - \delta\lambda_\alpha$, $\delta\kappa^{Z\alpha} = \delta\kappa_Z - \delta\kappa_\alpha$ and $\delta g_1^{Z\alpha} = \delta g_1^Z - \delta g_1^\alpha$.

terms (which correspond to initial state left and right handed leptons respectively) involve an intricate cancellation of a leading SM contribution between the CC03 diagrams as

$$
\begin{aligned}
\frac{\mathcal{A}_{\pm\pm-+}}{4\,\pi\,\hat\alpha} &\simeq -\sin\theta\,\left[\left(1+\delta\lambda_\alpha\,\frac{y}{2}\right)_\alpha-\left(1+\delta\lambda_Z\,\frac{y}{2}\right)_z\right]+\cdots,\\
&\simeq -\frac{\sin\theta}{2}\,(\delta\lambda_\alpha-\delta\lambda_Z)\,y, \qquad (5)\\
\frac{\mathcal{A}_{\pm\pm+-}}{4\,\pi\,\hat\alpha} &\simeq -\,\sin\theta\left[\left(1+\delta\lambda_\alpha\,\frac{y}{2}\right)_{\alpha\,pole}-\left(\left(1-\frac{1}{2\,s_{\hat\theta}^2}\right)\left(1+\delta\lambda_Z\,\frac{y}{2}\right)\right)_{z\,pole}-\left(\frac{1}{2\,s_{\hat\theta}^2}\right)_\nu\right]+\ldots\\
&\simeq \frac{\sin\theta}{2}\left[\left(1-\frac{1}{2s_{\hat\theta}^2}\right)\delta\lambda_Z-\delta\lambda_\alpha\right]y. \qquad (6)
\end{aligned}
$$

Here we have labeled the contributions by the internal states contributing to $M_X^{\lambda^\pm}$. The $\{\nu, \alpha, Z\}$ contributions to the scattering events populate phase space in a different manner in general. These differences are trivialized away in the near on-shell limit, leading to the cancellation shown of the leading SM contributions in the expansion in x, but can be uncovered by considering different limits of s_{12}, s_{34} and considering off-shell phase space.

4.2.2 *Both $W^{\pm}$ bosons off-shell phase space*

For example, consider the off-shell region of phase space defined through

$$\textbf{Case 2}: \qquad s_{12} = s_1\, s, \qquad s_{34} = s_3\, s, \tag{7}$$

with $s_1 \lesssim 1, s_3 \lesssim 1$. In this limit, one finds the expansions of the CC03 results

$$\mathcal{A}^{s_1,s_3}_{\pm\pm-+} \simeq -4\,\pi\,\hat{\alpha}\,\sin\theta\,\sqrt{\tilde{\lambda}(s_1,s_3)}\left[\left(1+\delta\lambda_\alpha\,\frac{y}{2}\right)_\alpha - \left(1+\delta\lambda_Z\,\frac{y}{2}\right)_z\right] + \cdots, \tag{8}$$

$$\begin{aligned}\mathcal{A}_{\pm\pm+-} \simeq{}& -4\,\pi\,\hat{\alpha}\,\sin\theta\sqrt{\tilde{\lambda}(s_1,s_3)}\left[\left(1+\delta\lambda_\alpha\,\frac{y}{2}\right)_{\alpha\,pole} - \left(\left(1-\frac{1}{2\,s^2_{\hat{\theta}}}\right)\left(1+\delta\lambda_Z\,\frac{y}{2}\right)\right)_{z\,pole}\right] \\ &+\left[\left(\frac{4\,\pi\,\hat{\alpha}\,\sin\theta}{2\,s^2_{\hat{\theta}}\,\sqrt{\tilde{\lambda}(s_1,s_3)}}\right)\left(1+\frac{-(s_1+s_3)+(s_1-s_3)(s_1-s_3\mp\sqrt{\tilde{\lambda}(s_1,s_3)})}{1-s_1-s_3+\sqrt{\tilde{\lambda}(s_1,s_3)}\cos\theta}\right)\right]_{\nu\,pole}.\end{aligned} \tag{9}$$

Here we have defined $\sqrt{\tilde{\lambda}(s_1,s_3)} = \sqrt{1-2\,s_1-2s_3-2s_1\,s_3+s_1^2+s_3^2}$. In the case of left handed electrons, the differences in the way the various t and s channel poles populate phase space are no longer trivialized away, and a SM contribution exists at leading order in the x expansion. This SM term can then interfere with the contribution due to a $\mathcal{L}^{(6)}$ operator correction in the SMEFT. The complete results in this limit for the helicity eigenstates are reported in Table 2.

4.2.3 *One $W^{\pm}$ boson off-shell phase space*

One can define the region of phase space where one $W^{\pm}$ boson is off-shell as

$$\begin{aligned}\textbf{Case 3a}: &\qquad s_{12} = s_1\, s, &\qquad s_{34} = s_3\,\bar{m}^2_W, \\ \textbf{Case 3b}: &\qquad s_{12} = s_1\,\bar{m}^2_W, &\qquad s_{34} = s_3\, s,\end{aligned}$$

λ_i	$\sum_X M_X^{\lambda^\pm}/4\pi\hat{\alpha}$
$00-+$	$\frac{\sqrt{\tilde{\lambda}}\,\sin\theta}{2\sqrt{s_1 s_3}}\left[\frac{1}{c_{\hat{\theta}}^2}(1+s_1+s_3)+\left(\delta\kappa^{Z\alpha}-\delta F_2^{Z\alpha}(1+s_1+s_3)+\delta g_1^{Z\alpha}(s_1+s_3)\right)y\right]x^2$
$\pm\pm-+$	$-\sin\theta\sqrt{\tilde{\lambda}}\left[\frac{x^2}{c_{\hat{\theta}}^2}+\frac{y\,\delta\lambda^{Z\alpha}}{2}+\left(\delta g_1^{Z\alpha}-\delta F_2^{Z\alpha}+\frac{\delta\lambda_Z}{2\,c_{\hat{\theta}}^2}\right)y\,x^2\right]$
$\pm 0-+$	$-\frac{(1\pm\cos\theta)}{\sqrt{2 s_3}}\left[\frac{x^2}{c_{\hat{\theta}}^2}+\frac{y\,s_3\,\delta\lambda^{Z\alpha}}{2}+\frac{y\,x^2}{2}\left(\delta g_1^{Z\alpha}-2\delta F_2^{Z\alpha}+\delta\kappa^{Z\alpha}+s_3\,\delta\lambda^{Z\alpha}\right)\right]$
$0\pm-+$	$\frac{(1\mp\cos\theta)}{\sqrt{2 s_1}}\left[\frac{x^2}{c_{\hat{\theta}}^2}+\frac{y\,s_1\,\delta\lambda^{Z\alpha}}{2}+\frac{y\,x^2}{2}\left(\delta g_1^{Z\alpha}-2\delta F_2^{Z\alpha}+\delta\kappa^{Z\alpha}+s_1\delta\lambda^{Z\alpha}\right)\right]$
$00+-$	$-\frac{\sin\theta\,\sqrt{\tilde{\lambda}}}{4\sqrt{s_1 s_3}\,s_{\hat{\theta}}^2}\left[1+s_1+s_3-\frac{1}{\tilde{\lambda}}\left(1-(s_1-s_3)^2-\frac{8\,s_1\,s_3}{1-s_1-s_3+\sqrt{\tilde{\lambda}}\cos\theta}\right)\right]$
$\pm\pm+-$	$\frac{\sin\theta\,\sqrt{\tilde{\lambda}}}{2\,s_{\hat{\theta}}^2}\left[1-\frac{1}{\tilde{\lambda}}\left(1+\frac{-(s_1+s_3)+(s_1-s_3)(s_1-s_3\mp\sqrt{\tilde{\lambda}})}{1-s_1-s_3+\sqrt{\tilde{\lambda}}\cos\theta}\right)-s_{\hat{\theta}}^2 F_3(\lambda_\alpha,\lambda_Z)y\right]$
$\pm 0+-$	$-\frac{(1\mp\cos\theta)\sqrt{\tilde{\lambda}}}{2\sqrt{2 s_3}\,s_{\hat{\theta}}^2}\left[1-\frac{1}{\tilde{\lambda}}\left(1-s_1+s_3-\frac{2\,s_3(1+s_1-s_3\mp\sqrt{\tilde{\lambda}})}{1-s_1-s_3+\sqrt{\tilde{\lambda}}\cos\theta}\right)-s_{\hat{\theta}}^2 s_3 F_3(\lambda_\alpha,\lambda_Z)y\right]$
$0\pm+-$	$\frac{(1\pm\cos\theta)\sqrt{\tilde{\lambda}}}{2\sqrt{2 s_1}\,s_{\hat{\theta}}^2}\left[1-\frac{1}{\tilde{\lambda}}\left(1+s_1-s_3-\frac{2\,s_1(1-s_1+s_3\pm\sqrt{\tilde{\lambda}})}{1-s_1-s_3+\sqrt{\tilde{\lambda}}\cos\theta}\right)-s_{\hat{\theta}}^2 s_1 F_3(\lambda_\alpha,\lambda_Z)y\right]$
$\pm\mp+-$	$\frac{(\mp 1+\cos\theta)\sin\theta}{2\,s_{\hat{\theta}}^2\left(1-s_1-s_3+\sqrt{\tilde{\lambda}}\,\cos\theta\right)}$

Table 2: Expansion in $x, y < 1$ for the off-shell region of phase space of the CC03 diagrams in when $s_{12} = s_1\,s, s_{34} = s_3\,s$. Here we have used a short hand notation $\tilde{\lambda} = \tilde{\lambda}(s_1, s_3)$ and $F_3(\lambda_\alpha, \lambda_Z) = \left(\left(\frac{2s_{\hat{\theta}}^2-1}{2s_{\hat{\theta}}^2}\right)\delta\lambda_Z - \delta\lambda_\alpha\right)$ to condense results.

with $s_1 \lesssim 1, s_3 \sim 1$ for Case 3a, and $s_1 \sim 1, s_3 \lesssim 1$ for Case 3b. In these limits, the expansions of the CC03 results are as follows. In Case 3a one has $\mathcal{A}^{s_1,0}_{\pm\pm-+}$ and

$$\mathcal{A}_{\pm\pm+-} \simeq -4\,\pi\,\hat{\alpha}\,\sin\theta\sqrt{\tilde{\lambda}(s_1,0)}\left[\left(1+\delta\lambda_\alpha\,\frac{y}{2}\right)_{\alpha\,pole}-\left(\left(1-\frac{1}{2\,s_{\hat{\theta}}^2}\right)\left(1+\delta\lambda_Z\,\frac{y}{2}\right)\right)_{z\,pole}\right] + \left[\left(\frac{4\,\pi\,\hat{\alpha}\,\sin\theta}{2\,s_{\hat{\theta}}^2\,\sqrt{\tilde{\lambda}(s_1,0)}}\right)\left(1+s_1-\frac{2\,s_1(1-s_1\pm\sqrt{\tilde{\lambda}(s_1,0)})}{1-s_1+\sqrt{\tilde{\lambda}(s_1,0)}\cos\theta}\right)\right]_{\nu\,pole}. \tag{10}$$

While in Case 3b one finds $\mathcal{A}^{0,s_3}_{\pm\pm-+}$ and

$$\mathcal{A}_{\pm\pm+-} \simeq -4\,\pi\,\hat{\alpha}\,\sin\theta\sqrt{\tilde{\lambda}(0,s_3)}\left[\left(1+\delta\lambda_\alpha\,\frac{y}{2}\right)_{\alpha\,pole}-\left(\left(1-\frac{1}{2\,s_{\hat{\theta}}^2}\right)\left(1+\delta\lambda_Z\,\frac{y}{2}\right)\right)_{z\,pole}\right] + \left[\left(\frac{4\,\pi\,\hat{\alpha}\,\sin\theta}{2\,s_{\hat{\theta}}^2\,\sqrt{\tilde{\lambda}(0,s_3)}}\right)\left(1+s_3-\frac{2\,s_3(1-s_3\mp\sqrt{\tilde{\lambda}(0,s_3)})}{1-s_3+\sqrt{\tilde{\lambda}(0,s_3)}\cos\theta}\right)\right]_{\nu\,pole}. \tag{11}$$

Again, the SM term for left handed initial states does not vanish and can interfere with the contribution due to a $\mathcal{L}^{(6)}$ operator correction in the SMEFT in these regions of phase space. The complete results in this limit for the helicity eigenstates are reported in Table 3,4.

λ_i	$\sum_X M_X^{\lambda^\pm}/4\pi\hat{\alpha}$
$00-+$	$\frac{\sqrt{\tilde{\lambda}}\,\sin\theta}{2\sqrt{s_1 s_3}}\left[\frac{1}{c^2_{\hat{\theta}}}(1+s_1)+\left(\delta\kappa^{Z\alpha}-\delta F_2^{Z\alpha}(1+s_1)+\delta g_1^{Z\alpha}s_1\right)y\right]x$
$\pm\pm-+$	$-\sin\theta\sqrt{\tilde{\lambda}}\left[\frac{x^2}{c^2_{\hat{\theta}}}+\frac{y\,\delta\lambda^{Z\alpha}}{2}+\left(\delta g_1^{Z\alpha}-\delta F_2^{Z\alpha}+\frac{\delta\lambda_Z}{2\,c^2_{\hat{\theta}}}-\frac{\delta\lambda^{Z\alpha}(1+s_1)s_3}{2\,\tilde{\lambda}}\right)y\,x^2\right]$
$\pm 0-+$	$-\frac{(1\pm\cos\theta)\,\sqrt{\tilde{\lambda}}\,x}{\sqrt{2s_3}}\left[\frac{1}{c^2_{\hat{\theta}}}+\frac{y}{2}\left(\delta g_1^{Z\alpha}-2\delta F_2^{Z\alpha}+\delta\kappa^{Z\alpha}+s_3\,\delta\lambda^{Z\alpha}\right)\right]$
$0\pm-+$	$\frac{(1\mp\cos\theta)\sqrt{\tilde{\lambda}}}{\sqrt{2s_1}}\left[\frac{x^2}{c^2_{\hat{\theta}}}+\frac{y\,s_1\,\delta\lambda^{Z\alpha}}{2}+\frac{y\,x^2}{2}\left(\delta g_1^{Z\alpha}-2\delta F_2^{Z\alpha}+\delta\kappa^{Z\alpha}+\frac{s_1\delta\lambda_Z}{c^2_{\hat{\theta}}}-\frac{s_1\,s_3\,(1+s_1)}{(1-s_1)^2}\,\delta\lambda^{Z\alpha}\right)\right]$
$00+-$	$-\frac{\sin\theta}{\sqrt{s_1\,s_3}\,\sqrt{\tilde{\lambda}}\,s^2_{\hat{\theta}}}\,\frac{s_1(s_1^2-1)}{4\,x}$
$\pm\pm+-$	$\frac{\sin\theta\,\sqrt{\tilde{\lambda}}}{2\,s^2_{\hat{\theta}}}\left[1-\frac{1}{\tilde{\lambda}}\left(1-\frac{s_1(1-s_1\pm s_1\sqrt{\tilde{\lambda}})}{1-s_1+\sqrt{\tilde{\lambda}}\cos\theta}\right)-s^2_{\hat{\theta}}F_3(\lambda_\alpha,\lambda_Z)y\right]$
$\pm 0+-$	$-\frac{s_1(s_1-1)(1\mp\cos\theta)}{2\,\sqrt{2\,s_3}\,s^2_{\hat{\theta}}\,\sqrt{\tilde{\lambda}}}\,\frac{1}{x}$
$0\pm+-$	$\frac{(1\pm\cos\theta)\sqrt{\tilde{\lambda}}}{2\sqrt{2s_1}\,s^2_{\hat{\theta}}}\left[1-\frac{1}{\tilde{\lambda}}\left(1+s_1-\frac{2\,s_1(1-s_1\pm\sqrt{\tilde{\lambda}})}{1-s_1+\sqrt{\tilde{\lambda}}\cos\theta}\right)-s^2_{\hat{\theta}}s_1F_3(\lambda_\alpha,\lambda_Z)y\right]$
$\pm\mp+-$	$\frac{(\mp 1+\cos\theta)\,\sin\theta}{2\,s^2_{\hat{\theta}}\left(1-s_1+\sqrt{\tilde{\lambda}}\,\cos\theta\right)}$

Table 3: Expansion in $x,y<1$ for the off-shell region of phase space of the CC03 diagrams. Here we have used a short hand notation $\tilde{\lambda}=\tilde{\lambda}(s_1,0)$.

These results make clear that non-interference arguments based on on-shell simplifications of the kinematics of decaying $W^\pm$ bosons get off-shell corrections for an LHC observable that includes off-shell intermediate $W^\pm$ kinematics. (Admittedly a somewhat obvious result.) Such kinematics are parametrically suppressed by the small width of the unstable gauge boson, but are generically included in LHC observables due to realistic experimental cuts.[4]

[4]In some cases, off-shell effects are not relevant for physical conclusions. For example, Ref. [49] used helicity arguments similar to those employed here to study the approximate holomorphy of the anomalous dimension matrix of the SMEFT [50]. Ref. [49] was focused on the cut-constructable part of the amplitude related to logarithmic terms and the corresponding divergences. As noted in Ref. [49] such reasoning does not apply to finite contributions, which can come about due to off-shell effects.

λ_i	$\sum_X M_X^{\lambda^\pm}/4\pi\hat{\alpha}$
$00-+$	$\frac{-\sqrt{\tilde{\lambda}}\sin\theta}{2\sqrt{s_1 s_3}}\left[\frac{1}{c_{\hat{\theta}}^2}(1+s_3)-\left(\delta\kappa^{Z\alpha}+\delta F_2^{Z\alpha}(1+s_3)+\delta g_1^{Z\alpha}s_3\right)y\right]x$
$\pm\pm-+$	$-\sin\theta\sqrt{\tilde{\lambda}}\left[\frac{x^2}{c_{\hat{\theta}}^2}+\frac{y\,\delta\lambda^{Z\alpha}}{2}+\left(\delta g_1^{Z\alpha}-\delta F_2^{Z\alpha}+\frac{\delta\lambda_Z}{2\,c_{\hat{\theta}}^2}-\frac{\delta\lambda^{Z\alpha}(1+s_3)s_1}{2\,\tilde{\lambda}}\right)y\,x^2\right]$
$\pm 0-+$	$\frac{-(1\pm\cos\theta)\sqrt{\tilde{\lambda}}}{\sqrt{2s_3}}\left[\frac{x^2}{c_{\hat{\theta}}^2}+\frac{y\,s_3\,\delta\lambda^{Z\alpha}}{2}+\frac{y\,x^2}{2}\left(\delta g_1^{Z\alpha}-2\delta F_2^{Z\alpha}+\delta\kappa^{Z\alpha}+\frac{s_3\delta\lambda_Z}{c_{\hat{\theta}}^2}-\frac{s_1\,s_3\,(1+s_3)}{(1-s_3)^2}\,\delta\lambda^{Z\alpha}\right)\right]$
$0\pm-+$	$\frac{(1\mp\cos\theta)\sqrt{\tilde{\lambda}}x}{\sqrt{2s_1}}\left[\frac{1}{c_{\hat{\theta}}^2}+\frac{y}{2}\left(\delta g_1^{Z\alpha}-2\delta F_2^{Z\alpha}+\delta\kappa^{Z\alpha}+s_1\,\delta\lambda^{Z\alpha}\right)\right]$
$00+-$	$-\frac{\sin\theta}{\sqrt{s_1\,s_3}\,\sqrt{\tilde{\lambda}}\,s_{\hat{\theta}}^2}\,\frac{s_3(s_3^2-1)}{4\,x}$
$\pm\pm+-$	$\frac{\sin\theta\,\sqrt{\tilde{\lambda}}}{2\,s_{\hat{\theta}}^2}\left[1-\frac{1}{\tilde{\lambda}}\left(1-\frac{s_3(1-s_3\pm s_3\sqrt{\tilde{\lambda}})}{1-s_3+\sqrt{\tilde{\lambda}}\cos\theta}\right)-s_{\hat{\theta}}^2F_3(\lambda_\alpha,\lambda_Z)y\right]$
$\pm 0+-$	$\frac{-(1\mp\cos\theta)\sqrt{\tilde{\lambda}}}{2\sqrt{2s_3}\,s_{\hat{\theta}}^2}\left[1-\frac{1}{\tilde{\lambda}}\left(1+s_3-\frac{2\,s_3(1-s_3\pm\sqrt{\tilde{\lambda}})}{1-s_3+\sqrt{\tilde{\lambda}}\cos\theta}\right)-s_{\hat{\theta}}^2 s_3F_3(\lambda_\alpha,\lambda_Z)y\right]$
$0\pm+-$	$\frac{s_3(s_3-1)(1\pm\cos\theta)}{2\,\sqrt{2\,s_1}\,s_{\hat{\theta}}^2\,\sqrt{\tilde{\lambda}}\,x}\,\frac{1}{x}$
$\pm\mp+-$	$\frac{(\mp 1+\cos\theta)\sin\theta}{2\,s_{\hat{\theta}}^2\left(1-s_3+\sqrt{\tilde{\lambda}}\,\cos\theta\right)}$

Table 4: Expansion in $x, y < 1$ for the off-shell region of phase space of the CC03 diagrams and $\tilde{\lambda} = \tilde{\lambda}(0, s_3)$.

4.3 MAPPING TO PAST RESULTS

The results in Table 1,2,3,4 are input parameter scheme independent, and can be applied to more than one basis for $\mathcal{L}^{(6)}$. Specializing to the Warsaw basis of operators, and the electroweak input parameter scheme $\{\hat{\alpha}_{ew}, \hat{m}_Z, \hat{G}_F\}$ the (re-scaled) $x^2 y\,\delta X$ parameters are given by

$$
\begin{aligned}
&\frac{\hat{m}_W^2}{\Lambda^2}\,\delta g_1^\alpha = 0, && \frac{\hat{m}_W^2}{\Lambda^2}\,\delta\kappa_\alpha = \frac{1}{\sqrt{2}\hat{G}_F}\frac{c_{\hat{\theta}}}{s_{\hat{\theta}}}C_{HWB},\\
&\frac{\hat{m}_W^2}{\Lambda^2}\,\delta\lambda_\alpha = 6s_{\hat{\theta}}\frac{\hat{m}_W^2}{\sqrt{4\,\pi\,\hat{\alpha}}}C_W, && \frac{\hat{m}_W^2}{\Lambda^2}\,\delta\lambda_Z = 6s_{\hat{\theta}}\frac{\hat{m}_W^2}{\sqrt{4\,\pi\,\hat{\alpha}}}C_W,\\
&\frac{\hat{m}_W^2}{\Lambda^2}\,\delta F_{1,2}^\alpha = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{\hat{m}_W^2}{\Lambda^2}\frac{\delta F_1^Z}{4\pi\hat{\alpha}} &= \delta\bar{g}_Z\,(g_L^\ell)_{ss}^{SM} - \frac{1}{2\sqrt{2}\hat{G}_F}\left(C^{(1)}_{\substack{H\ell\\ ss}} + C^{(3)}_{\substack{H\ell\\ ss}}\right) - \delta s_\theta^2,\\
-\frac{\hat{m}_W^2}{\Lambda^2}\frac{\delta F_2^Z}{4\pi\hat{\alpha}} &= \delta\bar{g}_Z\,(g_R^\ell)_{ss}^{SM} - \frac{1}{2\sqrt{2}\,\hat{G}_F}C_{\substack{He\\ ss}} - \delta s_\theta^2,\\
\frac{\hat{m}_W^2}{\Lambda^2}\,\delta g_1^Z &= \frac{1}{2\sqrt{2}\hat{G}_F}\left(\frac{s_{\hat{\theta}}}{c_{\hat{\theta}}} + \frac{c_{\hat{\theta}}}{s_{\hat{\theta}}}\right)C_{HWB} + \frac{1}{2}\delta s_\theta^2\left(\frac{1}{s_{\hat{\theta}}^2} + \frac{1}{c_{\hat{\theta}}^2}\right),\\
\frac{\hat{m}_W^2}{\Lambda^2}\,\delta\kappa_Z &= \frac{1}{2\sqrt{2}\hat{G}_F}\left(-\frac{s_{\hat{\theta}}}{c_{\hat{\theta}}} + \frac{c_{\hat{\theta}}}{s_{\hat{\theta}}}\right)C_{HWB} + \frac{1}{2}\delta s_\theta^2\left(\frac{1}{s_{\hat{\theta}}^2} + \frac{1}{c_{\hat{\theta}}^2}\right),
\end{aligned}
$$

with $\delta\bar{g}_Z, \delta s_\theta^2$ defined in the Appendix. The left and right handed couplings are $(g_L^\ell)_{ss}^{SM} = -1/2 + s_{\hat{\theta}}^2$, and $(g_R^\ell)_{ss}^{SM} = s_{\hat{\theta}}^2$. Here $s = \{1,2,3\}$ is a flavour index labeling the initial state leptons. The results in Table 1 can be more directly compared to Refs. [40, 46–48, 51] using this procedure, finding agreement in the subset of terms that were reported in these works. This comparison also utilizes the naive narrow width limit to simplify the amplitudes as follows. In the sense of a distribution over phase space, the following replacement is made

$$
|\bar{D}_W(s_{12})\bar{D}_W(s_{34})|^2\,ds_{12}\,ds_{34} \to \frac{\pi^2}{\bar{m}_W^2\,\bar{\Gamma}_W^2}\,\delta(s_{12} - \bar{m}_W^2)\,\delta(s_{34} - \bar{m}_W^2)\,ds_{12}\,ds_{34}. \tag{14}
$$

The result of this replacement is a factorizing of the diboson production mechanism $d\sigma(\bar{\psi}\psi \to W^+W^-)/d\Omega$ and the branching ratios of the $W^\pm$ decays into specified final states as $s_1 = s_3 = 1$ is fixed in Table 1. This approximation holds up to $\mathcal{O}(\Gamma_W/M_W)$ corrections to Eqn. 14. The corrections in Tables 2,3,4 are present and should not be overlooked by the construction of a simplified high energy expansion, that is formally unphysical. It is not advisable to extrapolate the limited phase space results of Table 1 to the full phase space.

Another key difference between more recent studies of interference in the SMEFT in the high energy limit, compared to the past studies of interference of higher dimensional operators in the high energy limit for gluonic operators [52, 53], is the presence of an unstable massive gauge boson. Such massive gauge bosons have been studied using the narrow width approximation. However, a too naive version of the narrow width approximation does not commute with the SMEFT expansion.

This non-commutation can be seen as follows. Expanding the propagator of the intermediate W boson in the SMEFT

$$
\frac{1}{(p^2 - \bar{m}_W^2)^2 + \bar{\Gamma}_W^2\bar{m}_W^2} = \frac{1}{(p^2 - \hat{m}_W^2)^2 + \hat{\Gamma}_W^2\hat{m}_W^2}(1 + \delta D_W(p^2) + \delta D_W(p^2)^*) \tag{15}
$$

one has

$$\delta D_W(p^2) = \frac{1}{p^2 - \hat{m}_W^2 + i\hat{\Gamma}_W \hat{m}_W} \times \left[\left(1 - \frac{i\hat{\Gamma}_W}{2\hat{m}_W}\right) \delta m_W^2 - i\hat{m}_W \delta\Gamma_W \right]. \tag{16}$$

By first doing the narrow width approximation, and then doing the SMEFT expansion, one obtains

$$\begin{aligned} \frac{dp^2}{(p^2 - \bar{m}_W^2)^2 + \bar{\Gamma}_W^2 \bar{m}_W^2} &\to \frac{\pi dp^2}{\bar{\Gamma}_W \bar{m}_W} \delta(p^2 - \bar{m}_W^2) \\ &= \frac{\pi dp^2}{\hat{\Gamma}_W \hat{m}_W} \left(1 - \frac{\delta m_W^2}{2\hat{m}_W^2} - \frac{\delta\Gamma_W}{\hat{\Gamma}_W}\right) \delta(p^2 - \bar{m}_W^2). \end{aligned} \tag{17}$$

Reversing the order of operations, we square the expanded propagators and then do the narrow width approximation. For a general function $f(p^2)$, we find that after integrating

$$\begin{aligned} \frac{f(p^2) dp^2}{(p^2 - \hat{m}_W^2)^2 + \hat{\Gamma}_W^2 \hat{m}_W^2} & \left(1 + \delta D_W(p^2) + \delta D_W(p^2)^*\right) \\ &\to \frac{f(\hat{m}_W^2)\pi}{\hat{\Gamma}_W \hat{m}_W} \left(1 - \frac{\delta m_W^2}{2\hat{m}_W^2} - \frac{\delta\Gamma_W}{\hat{\Gamma}_W}\right) + \frac{f'(\hat{m}_W^2)\pi}{\hat{\Gamma}_W \hat{m}_W} \delta m_W^2. \end{aligned} \tag{18}$$

In a naive version of the narrow width approximation, we simply replace $\bar{m}_W$ by $\hat{m}_W$ in Eqn. (17). The operations of expanding in the SMEFT and doing the naive narrow width approximation don't commute in general. The reason is that the naive narrow width approximation assumes that the part of the integrand that is odd in its dependence on the invariant mass cancels out in the near on-shell region. With the SMEFT corrections, this is no longer the case, as the real part of δD_W gives a finite contribution to this part of the integrand. This difference is proportional to the shift of the mass of the $W^{\pm}$ boson. The correct way to implement the narrow width approximantion in the SMEFT is to use Eqn. (17) and expand the general fuction $f(p^2)$ in the SMEFT expansion after integration. We then obtain Eqn. (18), and see that the commutation property is restored. Furthermore, we note that the x expansion parameter itself can be chosen to be $\hat{m}_W/\sqrt{s}$ or $\bar{m}_W/\sqrt{s}$ when studying the high energy limit (we choose the former expansion parameter). This is another ambiguity that can be introduced into studies of this form, when using a $\{\hat{\alpha}, \hat{m}_Z, \hat{G}_F\}$ scheme.

4.4 SINGLE CHARGE CURRENT RESONANT CONTRIBUTIONS (CC11)

It is well known in the SM literature, that the CC03 diagrams, with $W^{\pm}$ bosons fixed to be on-shell, are an insufficient approximation to a $\bar{\psi}\psi \to \bar{\psi}'_1 \psi'_2 \bar{\psi}'_3 \psi'_4$ cross section to describe the full phase space of

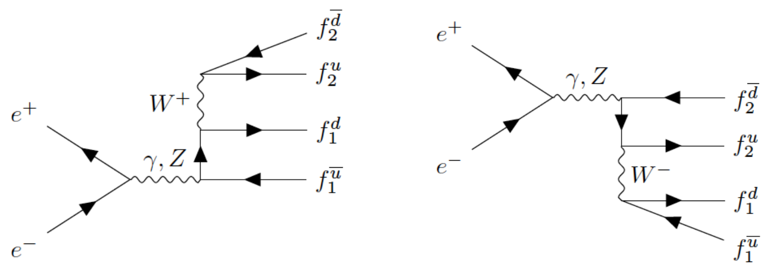


Figure 2: A subset of the CC11 Feynman diagrams contributing to $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ with leptonic initial states.

scattering events [54–60]. Such scattering events need not proceed through the CC03 set of diagrams, so limiting an analysis to this set of diagrams is formally unphysical. This issue can be overcome using the standard techniques of expanding around the poles of the process [61, 62] and including more contributions to the physical scattering process due to single resonant or non-resonant diagrams. Including the effect of single resonant diagrams allows one to develop gauge invariant results for such scattering events [54–59] when considering the full phase space (so long as the initial and final states are distinct). Including the single resonant diagrams is frequently referred to as calculating the set of CC11 diagrams in the literature. Some of the additional diagrams required are shown in Fig. 2.[5]

Considering the results in the previous sections, it is of interest to check if single resonant diagrams contribute to the physical $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ observables in a manner that potentially cancels the contributions for the off-shell phase space results in Tables 2,3,4. We find this is not the case, as can be argued on general grounds, and demonstrated in explicit calculations which we report below.

In general, an expansion of a SM Lagrangian parameter with a SMEFT correction is generically considered to be a correction of the form

$$\bar{X} = \hat{X} + x^2\, y\, \delta X \tag{19}$$

in the high energy limit considered, and one expects the SMEFT shifts to enter at two higher orders in the x expansion compared to a SM result. In addition the SMEFT can introduce new operator forms that directly lead to high energy growth and scale as a y correction to the amplitude, such as the effect of the operator $\mathcal{Q}_W$ in $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ scattering.

The CC03 diagram results are quite unusual due to the intricate cancellation present between the leading terms in the x expansion in the SM, at least in some regions of phase space. This leads to the

[5]Note that the CC03 diagrams are a (gauge dependent) subset of the CC11 diagrams [34] which can be seen considering the differences found in CC03 results comparing axial and R_ξ gauges.

SM and SMEFT terms occurring in some cases at the same order in x, contrary to the expectation formed by Eqn. 19. Conversely, the CC11 diagram contributions[6] follow the expectation in Eqn. 19.

4.4.1 *Single charge current resonant contributions - the SM*

We use the results of Refs. [54–59], in particular Ref. [55], for the SM results of the CC11/CC03 diagrams. We neglect contributions suppressed by light fermion masses. The generic SM amplitude is defined to have the form

$$\begin{aligned} i\,\mathcal{M}^{\sigma_a\,\sigma_b\,\sigma_c\,\sigma_d\,\sigma_e\,\sigma_f}_{V_1\,V_2}(p_a,p_b,p_c,p_d,p_e,p_f) &= -4i\,\bar{e}^4\,\delta_{\sigma_a,-\sigma_b}\delta_{\sigma_c,-\sigma_d}\delta_{\sigma_e,-\sigma_f}\bar{g}^{\sigma_b}_{V_1\bar{f}_a f_g}\bar{g}^{\sigma_b}_{V_2\bar{f}_g f_b}\bar{g}^{\sigma_d}_{V_1\bar{f}_c f_d}\bar{g}^{\sigma_f}_{V_2\bar{f}_e f_f} \\ &\quad\times\frac{\bar{D}_{V_1}(p_c+p_d)\bar{D}_{V_2}(p_e+p_f)}{(p_b+p_e+p_f)^2}A_2^{\sigma_a,\sigma_c,\sigma_e}(p_a,p_b,p_c,p_d,p_e,p_f). \end{aligned} \tag{20}$$

We have adopted the conventions of Ref. [55], and the initial and final states are labelled as $a\,b \to c\,d\,e\,f$. See the Appendix for more notational details. The functions $A_2^{\sigma_a,\sigma_c,\sigma_e}$ are given in terms of spinor products as [55, 63],

$$A_2^{+++}(p_a,p_b,p_c,p_d,p_e,p_f) = \langle p_a p_c\rangle\langle p_b p_f\rangle^*\left(\langle p_b p_d\rangle^*\langle p_b p_e\rangle + \langle p_d p_f\rangle^*\langle p_e p_f\rangle\right), \tag{21}$$

and satisfy [55, 63]

$$A_2^{++-}(p_a,p_b,p_c,p_d,p_e,p_f) = A_2^{+++}(p_a,p_b,p_c,p_d,p_f,p_e), \tag{22}$$

$$A_2^{+-+}(p_a,p_b,p_c,p_d,p_e,p_f) = A_2^{+++}(p_a,p_b,p_d,p_c,p_e,p_f), \tag{23}$$

$$A_2^{+--}(p_a,p_b,p_c,p_d,p_e,p_f) = A_2^{+++}(p_a,p_b,p_d,p_c,p_f,p_e), \tag{24}$$

$$A_2^{-,\sigma_c,\sigma_d}(p_a,p_b,p_c,p_d,p_e,p_f) = \left(A_2^{+,-\sigma_c,-\sigma_d}(p_a,p_b,p_c,p_d,p_e,p_f)\right)^*. \tag{25}$$

[6] Modulo the CC03 diagrams which we indicate with CC11/CC03.

The CC11/CC03 results are

$$\begin{aligned}\mathcal{M}^{\sigma_+,\sigma_-,\sigma_1,\sigma_2,\sigma_3,\sigma_4} = \sum_{V=A,Z} [\, & \mathcal{M}_{VW}^{-\sigma_1,-\sigma_2,\sigma_+,\sigma_-,-\sigma_3,-\sigma_4}(-k_1,-k_2,p_+,p_-,-k_3,-k_4) \\ & + \mathcal{M}_{VW}^{-\sigma_3,-\sigma_4,\sigma_+,\sigma_-,-\sigma_1,-\sigma_2}(-k_3,-k_4,p_+,p_-,-k_1,-k_2) \\ & + \mathcal{M}_{WV}^{-\sigma_1,-\sigma_2,-\sigma_3,-\sigma_4,\sigma_+,\sigma_-}(-k_1,-k_2,-k_3,-k_4,p_+,p_-) \\ & + \mathcal{M}_{WV}^{-\sigma_3,-\sigma_4,-\sigma_1,-\sigma_2,\sigma_+,\sigma_-}(-k_3,-k_4,-k_1,-k_2,p_+,p_-)\,]\,. \end{aligned} \tag{26}$$

As the final state fermions couple to one $W^\pm$ boson, and fermion masses are neglected, $\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\} = \{-+-+\}$. We denote the amplitude by the helicities of the incoming fermions, $\mathcal{M}^{\sigma_+,\sigma_-,\sigma_1,\sigma_2,\sigma_3,\sigma_4} = \mathcal{M}^{\sigma_+,\sigma_-}$ and find using [55] in the $x < 1$ limit for Case 1 and right handed electrons

$$\mathcal{M}^{-+} = \frac{\hat{e}^4 Q_l \sin\theta \sin\tilde{\theta}_{12} \sin\tilde{\theta}_{34}}{4 s_{\hat{\theta}}^2\, c_{\hat{\theta}}^2\, s\, x^2} \left[\frac{Q_{f_1} - I_{f_1}^3 - Q_{f_2} + I_{f_2}^3}{s_3 - 1 + i\hat{\gamma}_W} + \frac{Q_{f_4} - I_{f_4}^3 - Q_{f_3} + I_{f_3}^3}{s_1 - 1 + i\hat{\gamma}_W} \right], \tag{27}$$

and for left handed electrons

$$\begin{aligned}\mathcal{M}^{+-} = & \frac{\hat{e}^4 \sin\theta \sin\tilde{\theta}_{12} \sin\tilde{\theta}_{34}}{4 s_{\hat{\theta}}^4\, c_{\hat{\theta}}^2\, s\, x^2} \times \\ & \left[\frac{[Q_{f_1} s_{\hat{\theta}}^2 (Q_l - I_l^3) + I_{f_1}^3 (I_l^3 - Q_l\, s_{\hat{\theta}}^2)] - [Q_{f_2}\, s_{\hat{\theta}}^2 (Q_l - I_l^3) + I_{f_2}^3 (I_l^3 - Q_l\, s_{\hat{\theta}}^2)]}{s_3 - 1 + i\hat{\gamma}_W} \right. \\ & \left. + \frac{[Q_{f_4} s_{\hat{\theta}}^2 (Q_l - I_l^3) + I_{f_4}^3 (I_l^3 - Q_l\, s_{\hat{\theta}}^2)] - [Q_{f_3}\, s_{\hat{\theta}}^2 (Q_l - I_l^3) + I_{f_3}^3 (I_l^3 - Q_l\, s_{\hat{\theta}}^2)]}{s_1 - 1 + i\hat{\gamma}_W} \right]. \end{aligned} \tag{28}$$

Here $\hat{\gamma}_W = \hat{\Gamma}_W / \hat{m}_W$, Q_{f_i} is the electric charge and $I_{f_i}^3 = \pm 1/2$ is the isospin of the fermion f_i. Similarly for Case 2 we find using [55] the results for right handed electrons

$$\begin{aligned}\mathcal{M}^{-+} = \frac{4\hat{e}^4 Q_l}{s_{\hat{\theta}}^2 c_{\hat{\theta}}^2 s} & \left[\frac{I_{f_1}^3 - Q_{f_1}}{s_3(1 - s_1 + s_3 - \tilde{\lambda}\cos\tilde{\theta}_{12})} R_1 - \frac{I_{f_2}^3 - Q_{f_2}}{s_3(1 - s_1 + s_3 + \tilde{\lambda}\cos\tilde{\theta}_{12})} R_2 \right. \\ & \left. + \frac{I_{f_3}^3 - Q_{f_3}}{s_1(1 + s_1 - s_3 - \tilde{\lambda}\cos\tilde{\theta}_{34})} R_3 - \frac{I_{f_4}^3 - Q_{f_4}}{s_1(1 + s_1 - s_3 + \tilde{\lambda}\cos\tilde{\theta}_{34})} R_4 \right], \end{aligned} \tag{29}$$

and for left handed electrons

$$\mathcal{M}^{+-} = \frac{-4\hat{e}^4}{s_{\hat{\theta}}^4 c_{\hat{\theta}}^2 s}\left[\frac{Q_{f_1}s_{\hat{\theta}}^2(Q_l - I_l^3) + I_{f_1}^3(I_l^3 - Q_l s_{\hat{\theta}}^2)}{s_3(1 - s_1 + s_3 - \tilde{\lambda}\cos\tilde{\theta}_{12})}L_1 - \frac{Q_{f_2}s_{\hat{\theta}}^2(Q_l - I_l^3) + I_{f_2}^3(I_l^3 - Q_l s_{\hat{\theta}}^2)}{s_3(1 - s_1 + s_3 + \tilde{\lambda}\cos\tilde{\theta}_{12})}L_2 \right.$$
$$\left. + \frac{Q_{f_3}s_{\hat{\theta}}^2(Q_l - I_l^3) + I_{f_3}^3(I_l^3 - Q_l s_{\hat{\theta}}^2)}{s_1(1 + s_1 - s_3 - \tilde{\lambda}\cos\tilde{\theta}_{34})}L_3 - \frac{Q_{f_4}s_{\hat{\theta}}^2(Q_l - I_l^3) + I_{f_4}^3(I_l^3 - Q_l s_{\hat{\theta}}^2)}{s_1(1 + s_1 - s_3 + \tilde{\lambda}\cos\tilde{\theta}_{34})}L_4\right]. \tag{30}$$

The functions R_i,L_i, $i = 1, .., 4$ are given in the Appendix, along with additional definitions. For Case 3a one finds for right handed electrons

$$\mathcal{M}^{-+} = \frac{\hat{e}^4 Q_l \sin\tilde{\theta}_{34}}{4s_{\hat{\theta}}^2 c_{\hat{\theta}}^2 s x^2(s_3 - 1 + i\gamma_W)}\left[(Q_{f_1} - I_{f_1}^3) - (Q_{f_2} - I_{f_2}^3)\right] \times \tag{31}$$
$$\left(\sin\theta\sin\tilde{\theta}_{12}(1 + s_1) + \sqrt{s_1}e^{-i\tilde{\phi}_{12}}(1 - \cos\theta)(1 + \cos\tilde{\theta}_{12}) + \sqrt{s_1}e^{i\tilde{\phi}_{12}}(1 + \cos\theta)(1 - \cos\tilde{\theta}_{12})\right),$$

and for left-handed electrons

$$\mathcal{M}^{+-} = \frac{\hat{e}^4 \sin\tilde{\theta}_{34}}{4s_{\hat{\theta}}^4 c_{\hat{\theta}}^2 s x^2(s_3 - 1 + i\gamma_W)} \times \tag{32}$$
$$\left[(Q_{f_1}s_{\hat{\theta}}^2(Q_l - I_l^3) + I_{f_1}^3(I_l^3 - Q_l s_{\hat{\theta}}^2)) - (Q_{f_2}s_{\hat{\theta}}^2(Q_l - I_l^3) + I_{f_2}^3(I_l^3 - Q_l s_{\hat{\theta}}^2))\right] \times$$
$$\left[\sin\theta\sin\tilde{\theta}_{12}(1 + s_1) - \sqrt{s_1}e^{-i\tilde{\phi}_{12}}(1 + \cos\theta)(1 + \cos\tilde{\theta}_{12}) - \sqrt{s_1}e^{i\tilde{\phi}_{12}}(1 - \cos\theta)(1 - \cos\tilde{\theta}_{12})\right],$$

and finally for Case 3b one finds for right-handed electrons

$$\mathcal{M}^{-+} = \frac{\hat{e}^4 Q_l \sin\tilde{\theta}_{12}}{4s_{\hat{\theta}}^2 c_{\hat{\theta}}^2 s x^2(s_1 - 1 + i\gamma_W)}\left[(Q_{f_4} - I_{f_4}^3) - (Q_{f_3} - I_{f_3}^3)\right] \times \tag{33}$$
$$\left(\sin\theta\sin\tilde{\theta}_{34}(1 + s_3) - \sqrt{s_3}e^{-i\tilde{\phi}_{34}}(1 - \cos\theta)(1 - \cos\tilde{\theta}_{34}) - \sqrt{s_3}e^{i\tilde{\phi}_{34}}(1 + \cos\theta)(1 + \cos\tilde{\theta}_{34})\right),$$

and for left-handed electrons

$$\mathcal{M}^{+-} = \frac{\hat{e}^4 \sin\tilde{\theta}_{12}}{4s_{\hat{\theta}}^4 c_{\hat{\theta}}^2 s x^2(s_1 - 1 + i\gamma_W)} \times \tag{34}$$
$$\left[(Q_{f_4}s_{\hat{\theta}}^2(I_l^3 - Q_l s_{\hat{\theta}}^2) + I_{f_4}^3(I_l^3 - Q_l s_{\hat{\theta}}^2)) - (Q_{f_3}s_{\hat{\theta}}^2(I_l^3 - Q_l s_{\hat{\theta}}^2) + I_{f_3}^3(I_l^3 - Q_l s_{\hat{\theta}}^2))\right] \times$$
$$\left[\sin\theta\sin\tilde{\theta}_{34}(1 + s_3) + \sqrt{s_3}e^{-i\tilde{\phi}_{34}}(1 + \cos\theta)(1 - \cos\tilde{\theta}_{34}) + \sqrt{s_3}e^{i\tilde{\phi}_{34}}(1 - \cos\theta)(1 + \cos\tilde{\theta}_{34})\right].$$

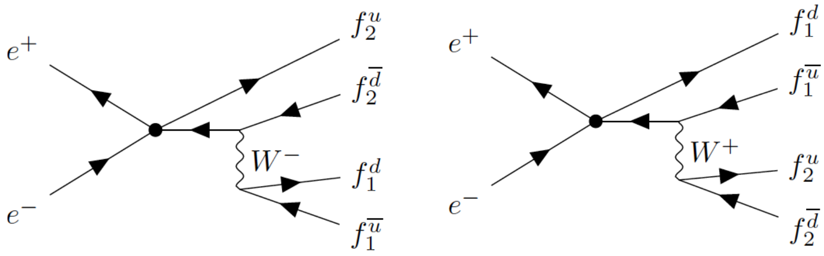


Figure 3: A subset of ψ^4 operator insertions contributing to $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ scattering.

4.4.2 *Single resonant contributions - the SMEFT*

The SMEFT corrections to the single resonant charged current contributions to $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$, follow directly from the results in the previous section. These corrections follow the scaling in x expectation formed by Eqn. 19, and the spinor products are unaffected by these shifts. As the charges of the initial and final states through neutral currents are fairly explicit in the previous section, it is easy to determine the coupling shifts and the SMEFT corrections to the propagators ($\delta D_{W,Z}$) by direct substitution.

We find that the single resonant contributions are distinct in their kinematic dependence compared to the novel interference results we have reported in section 4.2. The direct comparison of the results is somewhat challenged by the lack of a meaningful decomposition of the single resonant diagrams into helicity eigenstates of two intermediate charged currents, when only one charged current is present. Furthermore, we also note that the angular dependence shown in the single resonant results in Eqns. 29-34 reflects the fact that the center of mass frame relation to the final state phase space in the case of the CC03 diagrams is distinct from the CC11/CC03 results. This is the case despite both contributions being required for gauge independence in general [34].

To develop a complete SMEFT result including single resonant contributions, it is also required to supplement the results in the previous section with four fermion diagrams where a near on-shell charged current is present. For diagrams of this form see Fig. 3. These contributions introduce dependence on $\mathcal{L}^{(6)}$ operators that are not present in the CC03 diagrams, and once again the angular dependence in the phase space is distinct from the CC03 results.

4.5 CONCLUSIONS

In this paper, we have shown that off-shell effects in CC03 diagrams contributing to $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ observables lead to interference between the SM and $\mathcal{L}^{(6)}$ operators in the high energy limit. These effects can be overlooked when studying a simplified limit of these scattering events, as defined by the CC03 diagrams and the narrow width approximation. We have determined the results of the CC03 diagrams in several novel regions of phase space, compared to recent SMEFT literature, and have shown that single resonant diagrams do not change these conclusions when included into the results. We have also illustrated how to make the narrow width approximation consistent with the SMEFT expansion.

The off-shell phase space of the CC03 diagrams considered, and the phase space of the single resonant diagrams, is parametrically suppressed in an inclusive $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ observable. The full phase space is dominated by the near on-shell contributions of the CC03 diagrams which can be parametrically larger by $\sim (\hat{\Gamma}_W\,\hat{m}_W/\bar{v}_T^2)^{-1}$ or $\sim (\hat{\Gamma}_W\,\hat{m}_W/p_i^2)^{-1}$ where p_i^2 is a Lorentz invariant of mass dimension two. The exact degree of suppression that the off-shell region of phase space experiences strongly depends on the experimental cuts defining the inclusive $\bar{\psi}\psi \to \bar{\psi}'_1\psi'_2\bar{\psi}'_3\psi'_4$ observables, which should be studied in a gauge independent manner including all diagrams that contribute to the experimental observable, i.e. including all CC11 diagrams.

In some sense, our results coincide with the overall thrust of the discussion of Ref. [46], which emphasizes that searching for the effects of $\mathcal{L}^{(6)}$ operators interfering with the SM in tails of distributions (i.e. in the $\hat{m}_W^2/s \to 0$ limit) can be challenged in some helicity configurations, by the smallness of such interference effects. Arguably, this encourages prioritizing SMEFT studies on "pole observables" and makes such LHC studies a higher priority compared to pursuing such suppressed "tail observables".[7] At the same time, we stress that the results of this work indicate that the strong statements on non-interference of the SM and $\mathcal{L}^{(6)}$ operators, in subsets of phase space, and for some helicity configurations, are tempered by finite width effects, in addition to perturbative corrections [46, 53] and finite mass suppressions [46]. Finally, our results also demonstrate that a careful examination of historical and rigorous SM results, in the well developed SM literature, are an essential foundation to precise and accurate SMEFT studies.

[7] For a recent discussion on a systematic SMEFT pole program see Ref. [26]. One of the comparative strengths of the pole observable program is that observables can be optimized so that interference suppression effects *enhance* theoretical control of a process for SMEFT studies.

APPENDIX

4.A CONVENTIONS AND NOTATION

We use the generic notation $\delta X = \bar{X} - \hat{X}$ for the differences for a Lagrangian parameter X [22, 26] due to $\mathcal{L}^{(6)}$ corrections in the SMEFT and define

$$\delta G_F = \frac{1}{\sqrt{2}\,\hat{G}_F}\left(\sqrt{2}\,C_{Hl}^{(3)} - \frac{C_{ll}}{\sqrt{2}}\right), \tag{35}$$

$$\delta m_Z^2 = \frac{1}{2\sqrt{2}}\frac{\hat{m}_Z^2}{\hat{G}_F}C_{HD} + \frac{2^{1/4}\sqrt{\pi\hat{\alpha}}\,\hat{m}_Z}{\hat{G}_F^{3/2}}C_{HWB}, \tag{36}$$

$$\delta\bar{g}_Z = -\frac{\delta G_F}{\sqrt{2}} - \frac{\delta m_Z^2}{2\hat{m}_Z^2} + \frac{s_{\hat{\theta}}\,c_{\hat{\theta}}}{\sqrt{2}\hat{G}_F}\,C_{HWB}, \tag{37}$$

$$\delta g_1 = \frac{\hat{g}_1}{2c_{2\hat{\theta}}}\left[s_{\hat{\theta}}^2\left(\sqrt{2}\delta G_f + \frac{\delta m_Z^2}{\hat{m}_Z^2}\right) + c_{\hat{\theta}}^2\,s_{2\hat{\theta}}\bar{v}_T^2 C_{HWB}\right],$$

$$\delta g_2 = -\frac{\hat{g}_2}{2c_{2\hat{\theta}}}\left[c_{\hat{\theta}}^2\left(\sqrt{2}\delta G_f + \frac{\delta m_Z^2}{\hat{m}_Z^2}\right) + s_{\hat{\theta}}^2\,s_{2\hat{\theta}}\bar{v}_T^2 C_{HWB}\right], \tag{38}$$

$$\delta s_\theta^2 = 2c_{\hat{\theta}}^2 s_{\hat{\theta}}^2\left(\frac{\delta g_1}{\hat{g}_1} - \frac{\delta g_2}{\hat{g}_2}\right) + \bar{v}_T^2\frac{s_{2\hat{\theta}}c_{2\hat{\theta}}}{2}C_{HWB}, \tag{39}$$

$$\frac{\hat{m}_W^2}{\Lambda^2}\delta\bar{g}_W^\ell = \frac{1}{2\sqrt{2}\hat{G}_F}\left(C_{H\ell}^{(3)} + \frac{1}{2}\frac{c_{\hat{\theta}}}{s_{\hat{\theta}}}C_{HWB}\right) - \frac{1}{4}\frac{\delta s_\theta^2}{s_{\hat{\theta}}^2}. \tag{40}$$

$$\begin{aligned} R_1 = &\left[-\gamma_{12}^{-}e^{i\tilde{\phi}_{12}}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2} - \sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2}\right]\left[\gamma_{12}^{-}e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^{+}e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right] \\ &\left\{\sqrt{s_1}\left[e^{-i\tilde{\phi}_{12}}\sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} + \gamma_{12}^{+}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\left[-\gamma_{12}^{-}e^{i\tilde{\phi}_{12}}\cos\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^{+}e^{i\tilde{\phi}_{34}}\sin\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\right. \\ &\left. - \sqrt{s_3}\left[-e^{i\tilde{\phi}_{34}}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^{+}\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\right\} e^{-i(\tilde{\phi}_{12}+\tilde{\phi}_{34})}\sqrt{s_1 s_3}\gamma_{12}^{+}\gamma_{34}^{-} \end{aligned} \tag{41}$$

$$R_2 = \left[\gamma_{12}^- \sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} + e^{i\tilde{\phi}_{12}}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\left[-\gamma_{12}^- e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2} - \gamma_{34}^+ e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]$$
$$\left\{\sqrt{s_1}\left[\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} - \gamma_{12}^+ e^{-i\tilde{\phi}_{12}}\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\left[\gamma_{12}^- e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^+ e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\right.$$
$$\left. + \left[\gamma_{34}^+ \sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2} - e^{i\tilde{\phi}_{34}}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\right\} e^{-i(\tilde{\phi}_{12}+\tilde{\phi}_{34})}\sqrt{s_1 s_3}\gamma_{12}^+\gamma_{34}^- \tag{42}$$

$$R_3 = \left[\gamma_{34}^+ e^{i\tilde{\phi}_{34}}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2} - \sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\left[\gamma_{12}^- e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^+ e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]$$
$$\left\{\sqrt{s_3}\left[e^{-i\tilde{\phi}_{34}}\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2} - \gamma_{34}^-\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\left[-\gamma_{12}^- e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^+ e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\right.$$
$$\left. + \sqrt{s_1}\left[\gamma_{12}^- \sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} + e^{i\tilde{\phi}_{12}}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\right\} e^{-i(\tilde{\phi}_{12}+\tilde{\phi}_{34})}\sqrt{s_1 s_3}\gamma_{12}^+\gamma_{34}^- \tag{43}$$

$$R_4 = \left[\gamma_{34}^+ \sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2} - e^{i\tilde{\phi}_{34}}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\left[\gamma_{12}^- e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^+ e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]$$
$$\left\{\sqrt{s_3}\left[\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^- e^{-i\tilde{\phi}_{34}}\sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\left[-\gamma_{12}^- e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} - \gamma_{34}^+ e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\right.$$
$$\left. + \left[\gamma_{12}^- \sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} + e^{i\tilde{\phi}_{12}}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\right\} e^{-i(\tilde{\phi}_{12}+\tilde{\phi}_{34})}\sqrt{s_1 s_3}\gamma_{12}^+\gamma_{34}^- \tag{44}$$

$$L_1 = \left[-\gamma_{12}^- e^{i\tilde{\phi}_{12}}\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2} + \cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2}\right]\left[\gamma_{12}^- e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^+ e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]$$
$$\left\{\sqrt{s_1}\left[e^{-i\tilde{\phi}_{12}}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} - \gamma_{12}^+\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\left[-\gamma_{12}^- e^{i\tilde{\phi}_{12}}\cos\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^+ e^{i\tilde{\phi}_{34}}\sin\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\right.$$
$$\left. - \sqrt{s_3}\left[e^{i\tilde{\phi}_{34}}\sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^+\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\right\} e^{-i(\tilde{\phi}_{12}+\tilde{\phi}_{34})}\sqrt{s_1 s_3}\gamma_{12}^+\gamma_{34}^- \tag{45}$$

$$
\begin{aligned}
L_2 = &\left[-\gamma_{12}^{-}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} + e^{i\tilde{\phi}_{12}}\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\left[\gamma_{12}^{-}e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^{+}e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right] \\
&\left\{\sqrt{s_1}\left[\sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} + \gamma_{12}^{+}e^{-i\tilde{\phi}_{12}}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\left[\gamma_{12}^{-}e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^{+}e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\right. \\
&\left. - \left[\gamma_{34}^{+}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2} + e^{i\tilde{\phi}_{34}}\sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\right\} e^{-i(\tilde{\phi}_{12}+\tilde{\phi}_{34})}\sqrt{s_1 s_3}\gamma_{12}^{+}\gamma_{34}^{-}
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
L_3 = &\left[\gamma_{34}^{+}e^{i\tilde{\phi}_{34}}\sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\left[-\gamma_{12}^{-}e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} - \gamma_{34}^{+}e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right] \\
&\left\{-\sqrt{s_3}\left[e^{-i\tilde{\phi}_{34}}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^{-}\sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\left[-\gamma_{12}^{-}e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} + \gamma_{34}^{+}e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\right. \\
&\left. + \sqrt{s_1}\left[-\gamma_{12}^{-}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} + e^{i\tilde{\phi}_{12}}\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\right\} e^{-i(\tilde{\phi}_{12}+\tilde{\phi}_{34})}\sqrt{s_1 s_3}\gamma_{12}^{+}\gamma_{34}^{-}
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
L_4 = &\left[\gamma_{34}^{+}\cos\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2} + e^{i\tilde{\phi}_{34}}\sin\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\left[-\gamma_{12}^{-}e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2} - \gamma_{34}^{+}e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right] \\
&\left\{-\sqrt{s_3}\left[\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{34}}{2} - \gamma_{34}^{-}e^{-i\tilde{\phi}_{34}}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{34}}{2}\right]\left[-\gamma_{12}^{-}e^{i\tilde{\phi}_{34}}\cos\frac{\tilde{\theta}_{12}}{2}\cos\frac{\tilde{\theta}_{34}}{2} - \gamma_{34}^{+}e^{i\tilde{\phi}_{12}}\sin\frac{\tilde{\theta}_{12}}{2}\sin\frac{\tilde{\theta}_{34}}{2}\right]\right. \\
&\left. - \left[-\gamma_{12}^{-}\cos\frac{\theta}{2}\cos\frac{\tilde{\theta}_{12}}{2} + e^{i\tilde{\phi}_{12}}\sin\frac{\theta}{2}\sin\frac{\tilde{\theta}_{12}}{2}\right]\right\} e^{-i(\tilde{\phi}_{12}+\tilde{\phi}_{34})}\sqrt{s_1 s_3}\gamma_{12}^{+}\gamma_{34}^{-}
\end{aligned}
\tag{48}
$$

4.A.1 *Phase space*

The four momenta are defined as $p_+^\mu = \frac{\sqrt{s}}{2}(1, \sin\theta, 0, -\cos\theta)$, $p_-^\mu = \frac{\sqrt{s}}{2}(1, -\sin\theta, 0, \cos\theta)$ with $s = (p_+ + p_-)^2$ and $s_{ij} = (k_i + k_j)^2$ while the final state momenta (boosted to a common center of mass frame) are

$$\frac{2k_1^\mu}{\sqrt{s_{12}}} = \left(\gamma_{12,0} - \gamma_{12}\cos\tilde{\theta}_{12}, -\sin\tilde{\theta}_{12}\cos\tilde{\phi}_{12}, -\sin\tilde{\theta}_{12}\sin\tilde{\phi}_{12}, \gamma_{12,0}\cos\tilde{\theta}_{12} + \gamma_{12}\right), \tag{49}$$

$$\frac{2k_2^\mu}{\sqrt{s_{12}}} = \left(\gamma_{12,0} + \gamma_{12}\cos\tilde{\theta}_{12}, \sin\tilde{\theta}_{12}\cos\tilde{\phi}_{12}, \sin\tilde{\theta}_{12}\sin\tilde{\phi}_{12}, \gamma_{12,0}\cos\tilde{\theta}_{12} + \gamma_{12}\right), \tag{50}$$

$$\frac{2k_3^\mu}{\sqrt{s_{34}}} = \left(\gamma_{34,0} - \gamma_{34}\cos\tilde{\theta}_{34}, \sin\tilde{\theta}_{34}\cos\tilde{\phi}_{34}, \sin\tilde{\theta}_{34}\sin\tilde{\phi}_{34}, \gamma_{34,0}\cos\tilde{\theta}_{34} - \gamma_{34}\right), \tag{51}$$

$$\frac{2k_4^\mu}{\sqrt{s_{34}}} = \left(\gamma_{34,0} + \gamma_{34}\cos\tilde{\theta}_{34}, -\sin\tilde{\theta}_{34}\cos\tilde{\phi}_{34}, -\sin\tilde{\theta}_{34}\sin\tilde{\phi}_{34}, -\gamma_{34,0}\cos\tilde{\theta}_{34} - \gamma_{34}\right). \tag{52}$$

We use the definitions $\lambda = s^2 + s_{12}^2 + s_{34}^2 - 2ss_{12} - 2ss_{34} - 2s_{12}s_{34}$

$$\begin{aligned}
\gamma_{12} &= \frac{\sqrt{\lambda}}{2\sqrt{ss_{12}}}, & \gamma_{12,0} &= \frac{s + s_{12} - s_{34}}{2\sqrt{ss_{12}}}, \\
\gamma_{34} &= \frac{\sqrt{\lambda}}{2\sqrt{ss_{34}}}, & \gamma_{34,0} &= \frac{s + s_{34} - s_{12}}{2\sqrt{ss_{34}}}, \\
\gamma_{12}^\pm &= \gamma_{12,0} \pm \gamma_{12}, & \gamma_{34}^\pm &= \gamma_{34,0} \pm \gamma_{34}.
\end{aligned}$$

Useful identities are $\gamma_{12,0}^2 - \gamma_{12}^2 = \gamma_{12}^+\gamma_{12}^- = 1$ and $\gamma_{34,0}^2 - \gamma_{34}^2 = \gamma_{34}^+\gamma_{34}^- = 1$. A phase convention choice on $\phi_{12,34}$ in the spinors is required to be the same in the CC03 and CC11 results.

5

GAUGE FIXING THE STANDARD MODEL EFFECTIVE FIELD THEORY

We gauge fix the Standard Model Effective Field Theory in a manner invariant under background field gauge transformations using a geometric description of the field connections.

5.1 INTRODUCTION

When physics beyond the Standard Model (SM) is present at scales (Λ) larger than the Electroweak scale ($\sqrt{2\,\langle H^\dagger H\rangle} \equiv \bar{v}_T$), the SM can be extended into an effective field theory (EFT). The Standard Model Effective Field Theory (SMEFT), defined by a power counting expansion in the ratio of scales $\bar{v}_T/\Lambda < 1$, extends the SM with higher dimensional operators $\mathcal{Q}_i^{(d)}$ of mass dimension d. The Lagrangian is

$$\begin{aligned}
\mathcal{L}_{\rm SMEFT} &= \mathcal{L}_{\rm SM} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \mathcal{L}^{(7)} + \dots, \\
\mathcal{L}^{(d)} &= \sum_i \frac{C_i^{(d)}}{\Lambda^{d-4}} \mathcal{Q}_i^{(d)} \text{ for } d > 4.
\end{aligned} \tag{53}$$

The SMEFT is a model independent and consistent low energy parameterization of heavy physics beyond the SM, so long as its defining assumptions are satisfied: that there are no light hidden states in the spectrum with couplings to the SM; and a $\mathrm{SU}(2)_{\rm L}$ scalar doublet with hypercharge $\mathrm{y}_h = 1/2$ is present in the EFT.

The SMEFT has the same $\mathrm{SU}(3)_{\rm C} \times \mathrm{SU}(2)_{\rm L} \times \mathrm{U}(1)_{\rm Y}$ global symmetry as the SM. The SMEFT also has a Higgsed phase of $\mathrm{SU}(2)_{\rm L} \times \mathrm{U}(1)_{\rm Y} \to \mathrm{U}(1)_{\rm em}$. A difference between these theories is that additional couplings and interactions between the fields come about due to the $\mathcal{Q}_i^{(d)}$. Some of these interactions are bilinear in the SM fields in the Higgsed phase. These terms are important for gauge fixing and the presence of these interactions introduce technical challenges to the usual gauge fixing approach.

The bilinear field interactions in the SMEFT are usefully thought of in terms of connections in the field space manifolds of the theory [64, 65]. The purpose of this paper is to show that gauge fixing the SMEFT, taking into account the field space metrics, directly resolves many of the technical challenges that have been identified to date. The approach we develop generalizes directly to higher orders in the SMEFT power counting expansion.

The difficulties in gauge fixing the SMEFT are also present when the Background Field Method (BFM) [66–72] is used [73]. The BFM splits the fields in the theory into quantum and classical fields ($F \to F + \hat{F}$), with the latter denoted with a hat superscript. One performs a gauge fixing procedure that preserves background field gauge invariance while breaking explicitly the quantum field gauge invariance. This allows a gauge choice for the quantum fields to be made to one's advantage, while still benefiting from the simplifications that result from naive Ward identities [74] due to the preserved background field gauge invariance.[1]

In this paper, we show how to perform gauge fixing with the BFM taking into account the field space metrics that are present due to the SMEFT power counting expansion. The usual R_ξ gauge fixing approach in the BFM for the Standard Model [69–72] is a special case of this approach.[2] Conceptually one can understand that this procedure is advantageous as it preserves the background $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ invariance on the curved field spaces present due to the power counting expansion. The latter is trivialized away in the Standard Model.

5.2 SCALAR SPACE

The operators that lead to scalar kinetic terms in the Higgsed phase of the theory up to $\mathcal{L}^{(6)}$ are [25]

$$
\begin{aligned}
\mathcal{L}_{\text{scalar,kin}} = & \left(D_\mu H\right)^\dagger \left(D^\mu H\right) + \frac{C_{H\Box}}{\Lambda^2}\left(H^\dagger H\right)\Box\left(H^\dagger H\right) \\
& + \frac{C_{HD}}{\Lambda^2}\left(H^\dagger D_\mu H\right)^*\left(H^\dagger D^\mu H\right), \\
\equiv & \frac{1}{2} h_{IJ}(\phi)\left(D_\mu \phi\right)^I \left(D^\mu \phi\right)^J . \qquad (54)
\end{aligned}
$$

[1] The Ward identities result from considering BRST invariance [75] when the BFM is not used, which can be more cumbersome when extending results to higher orders in the SMEFT power counting expansion.

[2] For a R_ξ gauge SMEFT formulation with three distinct ξ parameters see Ref. [76].

Our covariant derivative sign convention is given by $D^\mu H = (\partial^\mu + i\, g_2 W^{a,\mu}\, \sigma_a/2 + i\, g_1\, \mathrm{y}_h B^\mu Y)H$ and $(D^\mu \phi)^I = (\partial^\mu \delta^I_J - \frac{1}{2} W^{A,\mu} \tilde{\gamma}^I_{A,J})\phi^J$, with definitions given below. Defining

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_2 + i\phi_1 \\ \phi_4 - i\phi_3 \end{bmatrix}, \tag{55}$$

the scalar field connections can be described by a $\mathbb{R}^4$ field manifold with the metric $h_{IJ}(\phi)$. Our notation is that the latin capital letters $I, J, K, L \cdots$ run over $\{1,2,3,4\}$, while lower case latin letters $i, j, k, l \cdots$ run over $\{1,2\}$. The metric takes the form

$$h_{IJ}(\phi) = \delta_{IJ} - 2\frac{C_{H\Box}}{\Lambda^2}\phi_I \phi_J + \frac{1}{2}\frac{C_{HD}}{\Lambda^2} f_{IJ}(\phi), \tag{56}$$

where

$$f_{IJ}(\phi) = \begin{bmatrix} a & 0 & d & c \\ 0 & a & c & -d \\ d & c & b & 0 \\ c & -d & 0 & b \end{bmatrix}, \qquad \begin{aligned} a &= \phi_1^2 + \phi_2^2, \\ b &= \phi_3^2 + \phi_4^2, \\ c &= \phi_1\phi_4 + \phi_2\phi_3, \\ d &= \phi_1\phi_3 - \phi_2\phi_4. \end{aligned} \tag{57}$$

The Riemann curvature tensor calculated from the scalar field metric is non-vanishing [64, 65, 77]. The scalar manifold is curved due to the power counting expansion. An interesting consequence is that there does not exist a gauge independent field redefinition which sets $h_{IJ} = \delta_{IJ}$ when considering $\mathcal{L}^{(6)}$ corrections [77]. As a result, demanding that the Higgs doublet field to be canonically normalized in the SMEFT to $\mathcal{L}^{(6)}$ cannot be used as a defining condition for operator bases [22, 77–79].

5.3 GAUGE BOSON SPACE

The operators that lead to CP even bilinear interactions for the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ spin one fields up to $\mathcal{L}^{(6)}$ are

$$\begin{aligned}\mathcal{L}_{\mathrm{WB}} &= -\frac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{C_{HB}}{\Lambda^2} H^\dagger H B_{\mu\nu} B^{\mu\nu} \\ &\quad + \frac{C_{HW}}{\Lambda^2} H^\dagger H W^a_{\mu\nu} W^{a,\mu\nu} + \frac{C_{HWB}}{\Lambda^2} H^\dagger \sigma^a H W^a_{\mu\nu} B^{\mu\nu} \\ &\equiv -\frac{1}{4} g_{AB}(H) \mathcal{W}^A_{\mu\nu} \mathcal{W}^{B,\mu\nu}, \end{aligned} \tag{58}$$

where $a, b \cdots$ run over $\{1,2,3\}$, $A, B, C \cdots$ run over $\{1,2,3,4\}$. Here $\mathcal{W}^4_{\mu\nu} = B_{\mu\nu}$. Analogous to the scalar sector, we have introduced a metric $g_{AB}[H(\phi_i)]$, taking the form

$$\begin{aligned} g_{ab} &= \left(1 - 4\frac{C_{HW}}{\Lambda^2} H^\dagger H\right) \delta_{ab}, \qquad\qquad g_{44} = 1 - 4\frac{C_{HB}}{\Lambda^2} H^\dagger H, \\ g_{a4} &= g_{4a} = -2\frac{C_{HWB}}{\Lambda^2} H^\dagger \sigma_a H. \end{aligned} \tag{59}$$

The Riemann curvature tensor for the gauge fields can be calculated from g_{AB} and is nonvanishing; the (CP even) $\mathbb{R}^4$ spin one field manifold is also curved.[3] A physical consequence is that, as in the case of the scalar manifold, there does not exist a gauge independent field redefinition that sets $g_{AB} = \delta_{AB}$ including $\mathcal{L}^{(6)}$ corrections.[4,5] The power counting expansion of the SMEFT is relevant for gauge fixing and cannot be removed with gauge independent field redefinitions, which is a novel feature compared to more familiar EFTs without a Higgsed phase. The particular form of the field space metrics depends on the operator basis used, but the utility of the geometric approach developed here does not. This argues for a modified gauge fixing procedure using the BFM in the SMEFT.

---

[3] $\mathrm{SU}(2)_\mathrm{L}$ is self adjoint. As a result, one can define a $\mathbb{G}^{AB}$ tensor of the same form as g_{AB} through $\mathbb{G}^{AB}(H)\, \mathcal{W}^{\mu\nu}_A\, \mathcal{W}_{B,\mu\nu}$. This $\mathbb{G}^{AB}$ is not the tensor g^{AB} defined through the relation $g^{AB} g_{BC} = \delta^A_C$ and used in the gauge fixing term.

[4] A rotation to the mass eigenstate basis for the field bilinear interactions can be made, and this is consistent with the curvature of the gauge manifold.

[5] Field redefiniton invariant quantities are more directly connected to S-matrix elements. For a similar discussion of how field redefintion invariant beta functions can be defined in the SMEFT, see [80].

5.4 GAUGE FIXING

Eliminating bilinear kinetic mixing between the gauge bosons and the Goldstone bosons in an efficient gauge fixing procedure is advantageous. A simpler LSZ procedure [81] to construct S-matrix elements results from this condition being imposed. R_ξ gauge [82] in the SM when $\xi_W = \xi_B$ has some further advantages in eliminating contact operators that complicate calculations in intermediate steps. Using the BFM combined with R_ξ gauge fixing, the gauge fixing term for the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ fields in the SM takes the form [69–72]

$$
\begin{aligned}
\mathcal{L}_{\mathrm{GF}} = &-\frac{1}{2\xi_W}\sum_a \left[\partial_\mu W^{a,\mu} - g_2 \epsilon^{abc}\hat{W}_{b,\mu}W_c^\mu + i g_2 \frac{\xi_W}{2}\left(\hat{H}_i^\dagger (\sigma^a)^i_j H^j - H_i^\dagger (\sigma^a)^i_j \hat{H}^j\right)\right]^2 \\
&-\frac{1}{2\xi_B}\left[\partial_\mu B^\mu + i g_1 \frac{\xi_B}{2}\left(\hat{H}_i^\dagger H^i - H_i^\dagger \hat{H}^i\right)\right]^2,
\end{aligned}
\tag{60}
$$

where the background fields are denoted by a hat.

The $\mathrm{SU}(2)_\mathrm{L}$ Pauli matrix representation in Eq. 60 is inconvenient for characterizing the gauge fixing term as g_{AB} is defined on $\mathbb{R}^4$. The Pauli matrix algebra is isomorphic to the Clifford algebra $C(0,3)$, and the latter can be embedded in the $\mathbb{R}^4$ field space using the real representations $\gamma_{1,2,3}$ such that

$$
\gamma^I_{1,J} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad
\gamma^I_{2,J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
$$
$$
\gamma^I_{3,J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad
\gamma^I_{4,J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{61}
$$

The γ_4 generator is used for the $\mathrm{U}(1)_\mathrm{Y}$ embedding. As $\mathrm{SU}(2)_\mathrm{L}$ is self adjoint we can also define this algebra for the adjoint fields, using the same real representations. $\gamma_{1,2,3,4}\langle\phi\rangle \neq 0$ and the unbroken combination of generators $(\gamma_3 + \gamma_4)\langle\phi\rangle = 0$ corresponds to $\mathrm{U}(1)_{\mathrm{em}}$. We absorb the couplings into the structure constants and gamma matrices,

$$\tilde{\epsilon}^{A}_{\ BC} = g_2\, \epsilon^{A}_{\ BC}, \quad \text{with } \tilde{\epsilon}^{1}_{\ 23} = +g_2,$$
$$\tilde{\gamma}^{I}_{A,J} = \begin{cases} g_2\, \gamma^{I}_{A,J}, & \text{for } A = 1,2,3 \\ g_1 \gamma^{I}_{A,J}, & \text{for } A = 4. \end{cases} \tag{62}$$

The different couplings g_1, g_2 enter as the group defined on the $\mathbb{R}^4$ field space is not simple. The $\gamma^{I}_{a,J}$ matricies satisfy the algebra $[\tilde{\gamma}_a, \tilde{\gamma}_b] = 2\,\tilde{\epsilon}^{c}_{\ ab}\,\tilde{\gamma}_c$ and $[\tilde{\gamma}_a, \tilde{\gamma}_4] = 0$. The structure constants vanish when any of $A, B, C = 4$. Note also that $\hat{H}^\dagger \sigma_A H - H^\dagger \sigma_A \hat{H} = -i\phi\gamma_A\hat{\phi}$, with $\sigma_4 = Y = \mathbb{I}_{2\times 2}$. The gauge fixing term in the background field gauge takes the form

$$\mathcal{L}_{\mathrm{GF}} = -\frac{\hat{g}_{AB}}{2\,\xi} \mathcal{G}^A\, \mathcal{G}^B, \tag{63}$$
$$\mathcal{G}^X \equiv \partial_\mu \mathcal{W}^{X,\mu} - \tilde{\epsilon}^{X}_{\ CD} \hat{\mathcal{W}}^{C}_{\mu} \mathcal{W}^{D,\mu} + \frac{\xi}{2} \hat{g}^{XC} \phi^I\, \hat{h}_{IK}\, \tilde{\gamma}^{K}_{C,J} \hat{\phi}^J.$$

The R_ξ gauge fixing term follows when replacing the background fields with their vacuum expectation values. The gauge fixing term is bilinear in the quantum fields. The field space metrics in Eq. 63 are denoted with a hat superscript indicating they are defined to depend only on the background fields. Contracting with the field space metrics is a basis independent feature of the gauge fixing term.

It is useful to note the following background field gauge transformations ($\delta\hat{F}$), with infinitesimal local gauge parameters $\delta\hat{\alpha}_A(x)$ when verifying explicitly the background field gauge invariance of this expression

$$
\begin{aligned}
\delta\,\hat{\phi}^I &= -\delta\hat{\alpha}^A\,\frac{\tilde{\gamma}^I_{A,J}}{2}\hat{\phi}^J,\\
\delta\,(D^\mu\hat{\phi})^I &= -\delta\hat{\alpha}^A\,\frac{\tilde{\gamma}^I_{A,J}}{2}(D^\mu\hat{\phi})^J,\\
\delta\hat{\mathcal{W}}^{A,\mu} &= -\partial^\mu(\delta\hat{\alpha}^A) - \tilde{\epsilon}^A_{\;BC}\,\delta\hat{\alpha}^B\,\hat{\mathcal{W}}^{C,\mu},\\
\delta\hat{h}_{IJ} &= \hat{h}_{KJ}\,\frac{\delta\hat{\alpha}^A\,\tilde{\gamma}^K_{A,I}}{2} + \hat{h}_{IK}\,\frac{\delta\hat{\alpha}^A\,\tilde{\gamma}^K_{A,J}}{2},\\
\delta\hat{\mathcal{W}}^A_{\mu\nu} &= -\tilde{\epsilon}^A_{\;BC}\,\delta\hat{\alpha}^B\,\hat{\mathcal{W}}^C_{\mu\nu},\\
\delta\hat{g}_{AB} &= \hat{g}_{CB}\,\tilde{\epsilon}^C_{\;DA}\,\delta\hat{\alpha}^D + \hat{g}_{AC}\,\tilde{\epsilon}^C_{\;DB}\,\delta\hat{\alpha}^D.
\end{aligned} \tag{64}
$$

The background field gauge invariance is established by using these transformations in conjuction with a linear change of variables on the quantum fields

$$
\begin{aligned}
\mathcal{W}^{A,\mu} &\to \mathcal{W}^{A,\mu} - \tilde{\epsilon}^A_{\;BC}\,\delta\hat{\alpha}^B\,\mathcal{W}^{C,\mu},\\
\phi^I &\to \phi^I - \frac{\delta\hat{\alpha}^B\,\tilde{\gamma}^I_{B,K}}{2}\phi^K.
\end{aligned} \tag{65}
$$

The transformation of the gauge fixing term is

$$
\delta\mathcal{G}^X = -\tilde{\epsilon}^X_{\;AB}\,\delta\hat{\alpha}^A\mathcal{G}^B. \tag{66}
$$

With these transformations, the background field gauge invariance of the gauge fixing term is directly established.

The background field generating functional (Z) depends on the background fields $\hat{F} \equiv \{\hat{\mathcal{W}}^A, \hat{\phi}^I\}$ and the sources $J_F \equiv \{J^A, J^I_\phi\}$. The source terms transform as

$$
\delta J^A_\mu = -\tilde{\epsilon}^A_{\;BC}\,\delta\hat{\alpha}^B\,J^C_\mu, \qquad \delta J^I_\phi = -\frac{\delta\hat{\alpha}^B\,\tilde{\gamma}^I_{B,K}}{2}J^K_\phi. \tag{67}
$$

The background field generating functional dependence on the source terms is invariant under the background field gauge transformations, as they are contracted with the field space metrics in $Z[\hat{F}, J_F]$ defined by

$$\int \mathcal{D}F \det\left[\frac{\Delta \mathcal{G}^A}{\Delta \alpha^B}\right] e^{i\left(S[F+\hat{F}]+\mathcal{L}_{\rm GF}+\hat{g}_{CD}J^C_\mu \mathcal{W}^{D,\mu}+\hat{h}_{IJ}J^I_\phi \phi^J\right)}.$$

The integration over dx^4 is implicit in this expression. Here a quantum field gauge transformation is indicated with a Δ. The action is manifestly invariant under the gauge transformation of $F+\hat{F}$. This establishes the background field invariance of the generating functional.

The quantum fields gauge transformations are

$$\begin{aligned}
\Delta \mathcal{W}^A_\mu &= -\partial_\mu \Delta\alpha^A - \tilde{\epsilon}^A_{\,\,BC}\, \Delta\alpha^B\, (\mathcal{W}^C_\mu + \hat{\mathcal{W}}^C_\mu), \\
\Delta \phi^I &= -\Delta\alpha^A\, \frac{\tilde{\gamma}^I_{A,J}}{2}\, (\phi^J + \hat{\phi}^J).
\end{aligned} \tag{68}$$

As the field metrics in Eq. 63 depend only on the background fields and do not transform under quantum field gauge transformations, the Faddeev-Popov [83] ghost term still follows directly; we find

$$\begin{aligned}
\mathcal{L}_{\rm FP} = &-\hat{g}_{AB}\bar{u}^B \Big[-\partial^2 \delta^A_C - \overleftarrow{\partial}_\mu \tilde{\epsilon}^A_{\,\,DC}(\mathcal{W}^{D,\mu} + \hat{\mathcal{W}}^{D,\mu}) \\
&+ \tilde{\epsilon}^A_{\,\,DC} \hat{\mathcal{W}}^D_\mu \overrightarrow{\partial}^\mu - \tilde{\epsilon}^A_{\,\,DE} \tilde{\epsilon}^E_{\,\,FC} \hat{\mathcal{W}}^D_\mu (\mathcal{W}^{F,\mu} + \hat{\mathcal{W}}^{F,\mu}) \\
&- \frac{\xi}{4} \hat{g}^{AD} (\phi^J + \hat{\phi}^J) \tilde{\gamma}^I_{C,J}\, \hat{h}_{IK}\, \tilde{\gamma}^K_{D,L}\, \hat{\phi}^L) \Big]\, u^C.
\end{aligned} \tag{69}$$

The form of this expression follows from the convention choice in Eq. 74, and the descendent convention in Eq. 63. The mass eigenstate $\mathcal{Z}_\mu$, $\mathcal{A}_\mu$ fields are defined by

$$\begin{bmatrix} W^3_\mu \\ B_\mu \end{bmatrix} = \begin{bmatrix} 1 + \frac{C_{HW}\bar{v}_T^2}{\Lambda^2} & -\frac{C_{HWB}\bar{v}_T^2}{2\Lambda^2} \\ -\frac{C_{HWB}\bar{v}_T^2}{2\Lambda^2} & 1 + \frac{C_{HB}\bar{v}_T^2}{\Lambda^2} \end{bmatrix} \begin{bmatrix} c_{\bar{\theta}} & s_{\bar{\theta}} \\ -s_{\bar{\theta}} & c_{\bar{\theta}} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_\mu \\ \mathcal{A}_\mu \end{bmatrix},$$

where the introduced rotation angles $s_{\bar{\theta}}, c_{\bar{\theta}}$ are [84, 85]

$$t_{\bar{\theta}} \equiv \frac{s_{\bar{\theta}}}{c_{\bar{\theta}}} = \frac{\bar{g}_1}{\bar{g}_2} + \frac{\bar{v}_T^2}{2} \frac{C_{HWB}}{\Lambda^2} \left(1 - \frac{\bar{g}_1^2}{\bar{g}_2^2}\right), \tag{70}$$

and $\bar{g}_2 = g_2(1 + C_{HW}\bar{v}_T^2/\Lambda^2)$, $\bar{g}_1 = g_1(1 + C_{HB}\bar{v}_T^2/\Lambda^2)$. This removes mixing terms as well as making the kinetic term of the spin one electroweak fields canonically normalized. This results in a simplified LSZ procedure to construct S-matrix elements. Ghost fields associated with the mass eigenstates follow from the linear rotation to the mass eigenstate fields. Feynman rules can be extracted directly from these expressions. Corrections from the Wilson coefficients $(C_{H\Box}, C_{HD}, C_{HWB}, C_{HB}, C_{HW})$ enter in ghost interactions and couple to the sources through the gauge and scalar metrics.

5.5 CONCLUSIONS

In this paper we have defined an approach to gauge fixing the SMEFT that preserves background field gauge invariance. This approach directly generalizes to higher orders in the SMEFT power counting. The key point is to gauge fix the fields on the curved field space due to the power counting expansion.

6

WARD IDENTITIES FOR THE STANDARD MODEL EFFECTIVE FIELD THEORY

We derive Ward identities for the Standard Model Effective Field Theory using the background field method. The resulting symmetry constraints on the Standard Model Effective Field Theory are basis independent, and constrain the perturbative and power-counting expansions. A geometric description of the field connections, and real representations for the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ generators, underlies the derivation.

6.1 INTRODUCTION

The Standard Model (SM) is an incomplete description of observed phenomena in nature. However, explicit evidence of new long-distance propagating states is lacking. Consequently, the SM is usefully thought of as an Effective Field Theory (EFT) for measurements and data analysis, with characteristic energies proximate to the Electroweak scale ($\sqrt{2\,\langle H^\dagger H \rangle} \equiv \bar{v}_T$) – such as those made at the LHC or lower energies.

The Standard Model Effective Field Theory (SMEFT) is based on assuming that physics beyond the SM is present at scales $\Lambda > \bar{v}_T$. The SMEFT also assumes that there are no light hidden states in the spectrum with couplings to the SM; and a $\mathrm{SU}(2)_{\mathrm{L}}$ scalar doublet (H) with hypercharge $\mathsf{y}_h = 1/2$ is present in the EFT.

A power-counting expansion in the ratio of scales $\bar{v}_T/\Lambda < 1$ defines the SMEFT Lagrangian as

$$\mathcal{L}_{\rm SMEFT} = \mathcal{L}_{\rm SM} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \mathcal{L}^{(7)} + \dots, \tag{71}$$
$$\mathcal{L}^{(d)} = \sum_i \frac{C_i^{(d)}}{\Lambda^{d-4}} \mathcal{Q}_i^{(d)} \quad \text{for } d > 4.$$

The higher-dimensional operators $\mathcal{Q}_i^{(d)}$ are labelled with a mass dimension d superscript, and multiply unknown, dimensionless Wilson coefficients $C_i^{(d)}$. The sum over i, after non-redundant operators are removed with field redefinitions of the SM fields, runs over the operators in a particular operator basis. In this paper we use the Warsaw basis [25]. However, the main results are formulated in a basis independent manner and constrain relationships between Lagragian parameters due to the linear realization of $\mathrm{SU}(2)_{\rm L} \times \mathrm{U}(1)_{\rm Y}$ in the SMEFT.

The SMEFT is a powerful practical tool, but it is also a well-defined field theory. Many formal field-theory issues also have a new representation in the SMEFT. This can lead to interesting subtleties, particularly when developing SMEFT analyses beyond leading order. When calculating beyond leading order in the loop ($\hbar$) expansion, renormalization is required. The counterterms for the SMEFT at dimension five [86, 87], and six [85, 88–90] are known and preserve the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry of the SM. Such unbroken (but non-manifest in some cases) symmetries are also represented in the naive Ward-Takahashi identities [74, 91] when the Background Field Method (BFM) [66–70, 72] is used to gauge fix the theory. In Ref. [2] it was shown how to gauge fix the SMEFT in the BFM in R_ξ gauges, and we use this gauge-fixing procedure in this work.

The BFM splits the fields in the theory into quantum and classical background fields ($F \to F + \hat{F}$), with the latter denoted with a hat superscript. By performing a gauge-fixing procedure that preserves the background-field gauge invariance, while breaking explicitly the quantum-field gauge invariance, the Ward identities [74] are present in a “naive manner" – i.e. the identities are related to those that would be directly inferred from the classical Lagrangian. This approach is advantageous, as otherwise the gauge-fixing term, and ghost term, of the theory can make symmetry constraints non-manifest in intermediate steps of calculations.

The BFM gauge-fixing procedure in the SMEFT relies on a geometric description of the field connections, and real representations for the $\mathrm{SU}(2)_{\rm L} \times \mathrm{U}(1)_{\rm Y}$ generators. Using this formulation of the SMEFT allows a simple Ward-Takahashi identity to be derived, that constrains the n-point vertex functions. The purpose of this paper is to report this result and derivation.[1]

[1]Modified Ward identities in the SMEFT have been discussed in an on-shell scheme in Ref. [92].

6.2 PATH INTEGRAL FORMULATION

The BFM generating functional of the SMEFT is given by

$$Z[\hat{F}, J] = \int \mathcal{D}F \det\left[\frac{\Delta\mathcal{G}^A}{\Delta\alpha^B}\right] e^{i\left(S[F+\hat{F}]+\mathcal{L}_{\rm GF}+\text{source terms}\right)}.$$

The integration over d^4x is implicit in $\mathcal{L}_{\rm GF}$. The generating functional is integrated over the quantum field configurations via $\mathcal{D}F$, with F field coordinates describing all long-distance propagating states. J stands for the dependence on the sources which only couple to the quantum fields [93]. The background fields also effectively act as sources of the quantum fields. S is the action, initially classical, and augmented with a renormalization prescription to define loop corrections.

The scalar Higgs doublet is decomposed into field coordinates $\phi_{1,2,3,4}$, defined with the normalization

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_2 + i\phi_1 \\ \phi_4 - i\phi_3\end{bmatrix}. \tag{72}$$

The scalar kinetic term is defined with a field space metric introduced as

$$\mathcal{L}_{\rm scalar,kin} = \frac{1}{2}h_{IJ}(\phi)\left(D_\mu\phi\right)^I \left(D^\mu\phi\right)^J, \tag{73}$$

where $(D^\mu\phi)^I = (\partial^\mu\delta^I_J - \frac{1}{2}\mathcal{W}^{A,\mu}\tilde{\gamma}^I_{A,J})\phi^J$, with real generators ($\tilde{\gamma}$) and structure constants ($\tilde{\epsilon}^A_{\ BC}$) defined in the Appendix. The corresponding kinetic term for the $\mathrm{SU}(2)_{\rm L}\times\mathrm{U}(1)_{\rm Y}$ spin-one fields is

$$\mathcal{L}_{\rm gauge,kin} = -\frac{1}{4}g_{AB}(\phi)\mathcal{W}^A_{\mu\nu}\mathcal{W}^{B,\mu\nu}, \tag{74}$$

where $A, B, C, \ldots$ run over $\{1,2,3,4\}$, (as do I, J) and $\mathcal{W}^4_{\mu\nu} = B_{\mu\nu}$. Extending this definition to include the gluons is straight-forward.

A quantum-field gauge transformation involving these fields is indicated with a Δ, with an infinitesimal quantum gauge parameter $\Delta\alpha^A$. Explicitly, the transformations are

$$
\begin{aligned}
\Delta\mathcal{W}^A_\mu &= -\tilde{\epsilon}^A_{\ BC}\,\Delta\alpha^B\left(\hat{\mathcal{W}}^{C,\mu}+\mathcal{W}^{C,\mu}\right)-\partial^\mu(\Delta\alpha^A),\\
\Delta\phi^I &= -\Delta\alpha^A\,\frac{\tilde{\gamma}^I_{A,J}}{2}\,(\phi^J+\hat{\phi}^J).
\end{aligned}
\tag{75}
$$

The BFM gauge-fixing term *of the quantum fields* $\mathcal{W}^A$ is [2]

$$
\begin{aligned}
\mathcal{L}_{\rm GF} &= -\frac{\hat{g}_{AB}}{2\,\xi}\mathcal{G}^A\,\mathcal{G}^B, \qquad (76)\\
\mathcal{G}^A &\equiv \partial_\mu\mathcal{W}^{A,\mu}-\tilde{\epsilon}^A_{\ CD}\hat{\mathcal{W}}^C_\mu\mathcal{W}^{D,\mu}+\frac{\xi}{2}\hat{g}^{AC}\phi^I\,\hat{h}_{IK}\,\tilde{\gamma}^K_{C,J}\hat{\phi}^J.
\end{aligned}
$$

The introduction of field space metrics in the kinetic terms reflects the geometry of the field space due to the power-counting expansion. These metrics are the core conceptual difference of the relation between Lagrangian parameters, compared to the SM, in the Ward identities we derive. The field spaces defined by these metrics are curved, see Refs. [64, 65, 77]. The background-field gauge fixing relies on the basis independent transformation properties of g_{AB} and h_{IJ},[2] and the fields, under background-field gauge transformations ($\delta\hat{F}$) with infinitesimal local gauge parameters $\delta\hat{\alpha}_A(x)$ given by

$$
\begin{aligned}
\delta\,\hat{\phi}^I &= -\delta\hat{\alpha}^A\,\frac{\tilde{\gamma}^I_{A,J}}{2}\hat{\phi}^J,\\
\delta\hat{\mathcal{W}}^{A,\mu} &= -(\partial^\mu\delta^A_B+\tilde{\epsilon}^A_{\ BC}\,\hat{\mathcal{W}}^{C,\mu})\delta\hat{\alpha}^B,\\
\delta\hat{h}_{IJ} &= \hat{h}_{KJ}\,\frac{\delta\hat{\alpha}^A\,\tilde{\gamma}^K_{A,I}}{2}+\hat{h}_{IK}\,\frac{\delta\hat{\alpha}^A\,\tilde{\gamma}^K_{A,J}}{2},\\
\delta\hat{g}_{AB} &= \hat{g}_{CB}\,\tilde{\epsilon}^C_{\ DA}\,\delta\hat{\alpha}^D+\hat{g}_{AC}\,\tilde{\epsilon}^C_{\ DB}\,\delta\hat{\alpha}^D,\\
\delta\mathcal{G}^X &= -\tilde{\epsilon}^X_{\ AB}\,\delta\hat{\alpha}^A\mathcal{G}^B,\\
\delta f_i &= \Lambda^j_{A,i}\,\hat{\alpha}^A\,f_j,\\
\delta\bar{f}_i &= \hat{\alpha}^A\,\bar{f}_j\bar{\Lambda}^j_{A,i},
\end{aligned}
\tag{77}
$$

[2] The explicit forms of g_{AB} and h_{IJ} are basis dependent. The forms of the corrections for the Warsaw basis at $\mathcal{L}^{(6)}$ are given in Ref. [2].

where we have left the form of the transformation of the fermion fields implicit. Here i, j are flavour indicies. The background-field gauge invariance of the generating functional, i.e.

$$\frac{\delta Z[\hat{F}, J]}{\delta \hat{\alpha}^A} = 0, \tag{78}$$

is established by using these gauge transformations in conjunction with the linear change of variables on the quantum fields.

The generating functional of connected Green's functions is given by

$$W[\hat{F}, J] = -i \log Z[\hat{F}, J], \tag{79}$$

where $J = \{J_\mu^A, J_\phi^I, J_f, J_{\bar{f}}\}$. As usual the effective action is the Legendre transform

$$\Gamma[\hat{F}, \tilde{F}] = W[\hat{F}, J] - \int dx^4 J \cdot \tilde{F}|_{\tilde{F}=\frac{\delta W}{\delta J}}. \tag{80}$$

Here our notation is chosen to match Ref. [94]. S-matrix elements are constructed via [94–96]

$$\Gamma^{\text{full}}[\hat{F}, 0] = \Gamma[\hat{F}, 0] + i \int d^4x \mathcal{L}_{\text{GF}}^{\text{BF}}. \tag{81}$$

The last term in Eq. (81) is a gauge-fixing term for the background fields, formally independent from Eq. (76), and introduced to define the propagators of the background fields.

Finally, we define a generating functional of connected Green's functions $W_c[\hat{J}]$ as a further Legendre transform [96]

$$W_c[\hat{J}] = \Gamma^{\text{full}}[\hat{F}] + i \int d^4x \left[\sum_{\hat{F}} \hat{J}_{\hat{F}^\dagger} \hat{F} + \sum_f (\bar{f} \hat{J}_{\bar{f}} + \hat{J}_f f) \right]. \tag{82}$$

with $\hat{F} = \{\mathcal{W}^A, \phi^I\}$ and

$$\begin{aligned}
i\hat{J}_{\hat{F}^\dagger} &= -\frac{\delta \Gamma^{\text{full}}}{\delta \hat{F}}, & i\hat{J}_f &= -\frac{\delta \Gamma^{\text{full}}}{\delta \bar{f}}, & i\hat{J}_{\bar{f}} &= \frac{\delta \Gamma^{\text{full}}}{\delta f}, \\
\hat{F} &= \frac{\delta W_c}{i\delta \hat{J}_{\hat{F}^\dagger}}, & f &= \frac{\delta W_c}{i\delta \hat{J}_{\bar{f}}}, & \bar{f} &= -\frac{\delta W_c}{i\delta \hat{J}_f}.
\end{aligned} \tag{83}$$

6.3 WEAK EIGENSTATE WARD IDENTITIES

The BFM Ward identities follow from the invariance of $\Gamma[\hat{F},0]$ under background-field gauge transformations,

$$\frac{\delta\Gamma[\hat{F},0]}{\delta\hat{\alpha}^B} = 0. \tag{84}$$

In position space, the identities are

$$\begin{aligned} 0 = &\left(\partial^\mu \delta^A_B - \tilde{\epsilon}^A_{\ BC}\ \hat{\mathcal{W}}^{C,\mu}\right)\frac{\delta\Gamma}{\delta\hat{\mathcal{W}}^\mu_A} - \frac{\tilde{\gamma}^I_{B,J}}{2}\hat{\phi}^J\frac{\delta\Gamma}{\delta\hat{\phi}^I} \\ &+ \sum_j\left(\bar{f}_j\bar{\Lambda}^j_{B,i}\frac{\delta\Gamma}{\delta\bar{f}_i} - \frac{\delta\Gamma}{\delta f_i}\Lambda^i_{B,j}f_j\right). \end{aligned} \tag{85}$$

For some n-point function Ward identities, the background fields are set to their vacuum expectation values. When this is defined through the minimum of the classical action S, where the scalar potential is a function of $H^\dagger H$, which we denote as $\langle\,\rangle$. For example, the scalar vev defined in this manner is through $\sqrt{2\langle H^\dagger H\rangle} \equiv \bar{v}_T$ and explicitly $\langle\phi^J\rangle$ with an entry set to the numerical value of the vev does not transform via $\tilde{\gamma}^I_{A,J}$.

A direct relation follows between the tadpoles (i.e. the one point functions $\delta\Gamma/\delta\hat{\phi}^I$) and, $\langle\hat{\phi}^J\rangle$, given by

$$0 = \partial^\mu\frac{\delta\Gamma}{\delta\hat{\mathcal{W}}^{B,\mu}} - \frac{\tilde{\gamma}^I_{B,J}}{2}\langle\hat{\phi}^J\rangle\frac{\delta\Gamma}{\delta\hat{\phi}^I}. \tag{86}$$

Requiring a Lorentz-invariant vacuum sets the tadpoles for the gauge fields to zero. Thus, for the scalars

$$0 = \frac{\tilde{\gamma}^I_{B,J}}{2}\langle\hat{\phi}^J\rangle\frac{\delta\Gamma}{\delta\hat{\phi}^I}. \tag{87}$$

$\gamma_B\langle\phi^J\rangle \neq 0$ and the unbroken combination $(\gamma_3+\gamma_4)\langle\phi^J\rangle = 0$ corresponds to $\mathrm{U}(1)_{\mathrm{em}}$. Eq. (87) with $B = 3,4$ does not given linearly independent constraints. This leads to the requirement of a further renormalization condition to define the tadpole $\delta\Gamma/\delta\hat{\phi}^4$ to vanish.

The Ward identities for the two-point functions are

$$0 =\partial^\mu \frac{\delta^2\Gamma}{\delta\hat{\mathcal{W}}^{A,\nu}\delta\hat{\mathcal{W}}^{B,\mu}} - \frac{\tilde{\gamma}^I_{B,J}}{2}\langle\hat{\phi}^J\rangle\frac{\delta^2\Gamma}{\delta\hat{\mathcal{W}}^{A,\nu}\delta\hat{\phi}^I}, \tag{88}$$

$$0 =\partial^\mu \frac{\delta^2\Gamma}{\delta\hat{\phi}^K\delta\hat{\mathcal{W}}^{B,\mu}} - \frac{\tilde{\gamma}^I_{B,J}}{2}\left(\langle\hat{\phi}^J\rangle\frac{\delta^2\Gamma}{\delta\hat{\phi}^K\delta\hat{\phi}^I} + \delta^J_K\frac{\delta\Gamma}{\delta\hat{\phi}^I}\right). \tag{89}$$

The three-point Ward identities are

$$\begin{aligned}0 =&\partial^\mu \frac{\delta^3\Gamma}{\delta\bar{f}_k\delta f_l\delta\hat{\mathcal{W}}^{B,\mu}} - \frac{\tilde{\gamma}^I_{B,J}}{2}\langle\hat{\phi}^J\rangle\frac{\delta^3\Gamma}{\delta\bar{f}_k\delta f_l\delta\hat{\phi}^I}\\ &+\overline{\Lambda}^k_{B,i}\frac{\delta^2\Gamma}{\delta\bar{f}_i\delta f_l} - \frac{\delta^2\Gamma}{\delta\bar{f}_k\delta f_i}\Lambda^i_{B,l},\end{aligned} \tag{90}$$

$$\begin{aligned}0 =&\partial^\mu \frac{\delta^3\Gamma}{\delta\hat{W}^{A,\nu}\delta\hat{W}^{B,\mu}\delta\hat{W}^{C,\rho}} - \tilde{\epsilon}^D_{\ BC}\frac{\delta^2\Gamma}{\delta\hat{\mathcal{W}}^{D,\rho}\delta\hat{\mathcal{W}}^{A,\nu}}\\ &-\frac{\tilde{\gamma}^I_{B,J}}{2}\langle\hat{\phi}^J\rangle\frac{\delta^3\Gamma}{\delta\hat{\phi}^I\delta\hat{\mathcal{W}}^{A,\nu}\delta\hat{\mathcal{W}}^{C,\rho}},\end{aligned} \tag{91}$$

$$\begin{aligned}0 =&\partial^\mu \frac{\delta^3\Gamma}{\delta\hat{\mathcal{W}}^{A,\nu}\delta\hat{\mathcal{W}}^{B,\mu}\delta\hat{\phi}^K} - \tilde{\epsilon}^D_{\ BA}\frac{\delta^2\Gamma}{\delta\hat{\mathcal{W}}^{D,\nu}\delta\hat{\phi}^K}\\ &-\frac{\tilde{\gamma}^I_{B,J}}{2}\left(\langle\hat{\phi}^J\rangle\frac{\delta^3\Gamma}{\delta\hat{\mathcal{W}}^{A,\nu}\delta\hat{\phi}^I\delta\hat{\phi}^K} + \delta^J_K\frac{\delta^2\Gamma}{\delta\hat{\mathcal{W}}^{A,\nu}\delta\hat{\phi}^I}\right),\end{aligned} \tag{92}$$

$$\begin{aligned}0 =&\partial^\mu \frac{\delta^3\Gamma}{\delta\hat{\mathcal{W}}^{B,\mu}\delta\hat{\phi}^K\delta\hat{\phi}^L} - \frac{\tilde{\gamma}^I_{B,J}}{2}\langle\hat{\phi}^J\rangle\frac{\delta^3\Gamma}{\delta\hat{\phi}^I\delta\hat{\phi}^K\delta\hat{\phi}^L}\\ &-\frac{\tilde{\gamma}^I_{B,J}}{2}\left(\delta^J_K\frac{\delta^2\Gamma}{\delta\hat{\phi}^I\delta\hat{\phi}^L} + \delta^J_L\frac{\delta^2\Gamma}{\delta\hat{\phi}^I\delta\hat{\phi}^K}\right).\end{aligned} \tag{93}$$

6.4 MASS EIGENSTATE WARD IDENTITIES

The mass eigenstate SM Ward identities in the BFM are summarized in Ref. [72]. The tranformation of the gauge fields, gauge parameters and scalar fields into mass eigenstates in the SMEFT is

$$\hat{\mathcal{W}}^{A,\nu} = \sqrt{g}^{AB}U_{BC}\hat{\mathcal{A}}^{C,\nu}, \tag{94}$$

$$\hat{\alpha}^A = \sqrt{g}^{AB}U_{BC}\hat{\beta}^C, \tag{95}$$

$$\hat{\phi}^J = \sqrt{h}^{JK}V_{KL}\hat{\Phi}^L, \tag{96}$$

with $\hat{\mathcal{A}}^C = (\hat{\mathcal{W}}^+, \hat{\mathcal{W}}^-, \hat{\mathcal{Z}}, \hat{\mathcal{A}})$, $\hat{\Phi}^L = \{\hat{\Phi}^+, \hat{\Phi}^-, \hat{\chi}, \hat{H}^0\}$. This follows directly from the formalism in Ref. [2] (see also Ref. [97]). The matrices U, V are unitary, with $\sqrt{g}^{AB}\sqrt{g}_{BC} \equiv \delta^A_C$ and $\sqrt{h}^{AB}\sqrt{h}_{BC} \equiv \delta^A_C$. The square root metrics are understood to be matrix square roots and the entries are $\langle\rangle$ of the field space metrics entries. The combinations $\sqrt{g}U$ and $\sqrt{h}V$ perform the mass eigenstate rotation for the vector and scalar fields, and bring the corresponding kinetic term to canonical form, including higher-dimensional-operator corrections. We define the mass-eigenstate transformation matrices

$$\mathcal{U}^A_C = \sqrt{g}^{AB} U_{BC}, \qquad (\mathcal{U}^{-1})^D_F = U^{DE}\sqrt{g}_{EF},$$
$$\mathcal{V}^A_C = \sqrt{h}^{AB} V_{BC}, \qquad (\mathcal{V}^{-1})^D_F = V^{DE}\sqrt{h}_{EF},$$

to avoid a proliferation of index contractions. The structure constants and generators, transformed to those corresponding to the mass eigenstates, are defined as

$$\boldsymbol{\epsilon}^C_{\ GY} = (\mathcal{U}^{-1})^C_A \tilde{\epsilon}^A_{\ DE}\, \mathcal{U}^D_G\, \mathcal{U}^E_Y, \qquad \boldsymbol{\gamma}^I_{G,L} = \frac{1}{2}\tilde{\gamma}^I_{A,L}\, \mathcal{U}^A_G,$$
$$\boldsymbol{\Lambda}^i_{X,j} = \Lambda^i_{A,j}\, \mathcal{U}^A_X.$$

The background-field gauge transformations in the mass eigenstate are

$$\begin{aligned} \delta\hat{\mathcal{A}}^{C,\mu} &= -\left[\partial^\mu \delta^C_G + \boldsymbol{\epsilon}^C_{\ GY}\hat{\mathcal{A}}^{Y,\mu}\right]\delta\hat{\beta}^G, \\ \delta\hat{\Phi}^K &= -(\mathcal{V}^{-1})^K_I\, \boldsymbol{\gamma}^I_{G,L}\, \mathcal{V}^L_N \hat{\Phi}^N \delta\hat{\beta}^G. \end{aligned} \tag{97}$$

The Ward identities are then expressed compactly as

$$\begin{aligned} 0 &= \frac{\delta\Gamma}{\delta\hat{\beta}^G} \qquad (98) \\ &= \partial^\mu \frac{\delta\Gamma}{\delta\hat{\mathcal{A}}^{X,\mu}} + \sum_j \left(\bar{f}_j \overline{\boldsymbol{\Lambda}}^j_{X,i}\frac{\delta\Gamma}{\delta\bar{f}_i} - \frac{\delta\Gamma}{\delta f_i}\boldsymbol{\Lambda}^i_{X,j} f_j\right) \\ &- \frac{\delta\Gamma}{\delta\hat{\mathcal{A}}^{C\mu}}\boldsymbol{\epsilon}^C_{\ XY}\hat{\mathcal{A}}^{Y\mu} - \frac{\delta\Gamma}{\delta\hat{\Phi}^K}(\mathcal{V}^{-1})^K_I\boldsymbol{\gamma}^I_{X,L}\mathcal{V}^L_N\hat{\Phi}^N. \end{aligned}$$

In this manner, the “naive" form of the Ward identities is maintained. The BFM Ward identities in the SMEFT take the same form as those in the SM up to terms involving the tadpoles. This is the case once a consistent redefinition of couplings, masses and fields is made.

6.5 TWO-POINT FUNCTION WARD IDENTITIES

The Ward identities for the two-point functions take the form

$$\begin{aligned} 0 &= \partial^\mu \frac{\delta^2\Gamma}{\delta\hat{\mathcal{A}}^{X\mu}\delta\hat{\mathcal{A}}^{Y\nu}} - \frac{\delta^2\Gamma}{\delta\hat{\mathcal{A}}^{Y\nu}\delta\hat{\Phi}^K}(\mathcal{V}^{-1})^K_I \gamma^I_{X,L}\mathcal{V}^L_N\langle\hat{\Phi}^N\rangle, \\ 0 &= \partial^\mu \frac{\delta^2\Gamma}{\delta\hat{\mathcal{A}}^{X\mu}\delta\hat{\Phi}^O} - \frac{\delta^2\Gamma}{\delta\hat{\Phi}^K\delta\hat{\Phi}^O}(\mathcal{V}^{-1})^K_I \gamma^I_{X,L}\mathcal{V}^L_N\langle\hat{\Phi}^N\rangle \\ &- \frac{\delta\Gamma}{\delta\hat{\Phi}^K}(\mathcal{V}^{-1})^K_I \gamma^I_{X,L}\mathcal{V}^L_O. \end{aligned} \tag{99}$$

6.6 PHOTON IDENTITIES

The Ward identities for the two-point functions involving the photon are given by

$$0 = \partial^\mu \frac{\delta^2\Gamma}{\delta\hat{\mathcal{A}}^{4\mu}\delta\hat{\mathcal{A}}^{Y\nu}}, \qquad\qquad 0 = \partial^\mu \frac{\delta^2\Gamma}{\delta\hat{\mathcal{A}}^{4\mu}\delta\hat{\Phi}^I}. \tag{100}$$

Using the convention of Ref. [72] for the decomposition of the vertex function

$$\begin{aligned} -i\Gamma^{\hat{\mathcal{V}},\hat{\mathcal{V}}'}_{\mu\nu}(k,-k) &= \left(-g_{\mu\nu}k^2 + k_\mu k_\nu + g_{\mu\nu}M^2_{\hat{\mathcal{V}}}\right)\delta^{\hat{\mathcal{V}}\hat{\mathcal{V}}'}, \\ &+ \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\Sigma^{\hat{\mathcal{V}},\hat{\mathcal{V}}'}_T - \frac{k_\mu k_\nu}{k^2}\Sigma^{\hat{\mathcal{V}},\hat{\mathcal{V}}'}_L, \end{aligned}$$

an overall normalization factors out of the photon two-point Ward identities compared to the SM, and

$$\Sigma^{\hat{\mathcal{A}},\hat{\mathcal{A}}}_{L,\mathrm{SMEFT}}(k^2) = 0, \qquad\qquad \Sigma^{\hat{\mathcal{A}},\hat{\mathcal{A}}}_{T,\mathrm{SMEFT}}(0) = 0. \tag{101}$$

The latter result follows from analyticity at $k^2 = 0$.

6.7 $\mathcal{W}^{\pm}, \mathcal{Z}$ IDENTITIES

Directly, one finds the identities

$$
\begin{aligned}
0 &= \partial^{\mu} \frac{\delta^2 \Gamma}{\delta \hat{\mathcal{A}}^{3\mu} \delta \hat{\mathcal{A}}^{Y\nu}} - \bar{M}_Z \frac{\delta^2 \Gamma}{\delta \hat{\Phi}^3 \delta \hat{\mathcal{A}}^{Y\nu}}, && (102) \\
0 &= \partial^{\mu} \frac{\delta^2 \Gamma}{\delta \hat{\mathcal{A}}^{3\mu} \delta \hat{\Phi}^I} - \bar{M}_Z \frac{\delta^2 \Gamma}{\delta \hat{\Phi}^3 \delta \hat{\Phi}^I} && (103) \\
&+ \frac{\bar{g}_Z}{2} \frac{\delta \Gamma}{\delta \hat{\Phi}^4} \left(\sqrt{h}_{[4,4]} \sqrt{h}^{[3,3]} - \sqrt{h}_{[4,3]} \sqrt{h}^{[4,3]} \right) \delta^3_I \\
&- \frac{\bar{g}_Z}{2} \frac{\delta \Gamma}{\delta \hat{\Phi}^4} \left(\sqrt{h}_{[4,4]} \sqrt{h}^{[3,4]} - \sqrt{h}_{[4,3]} \sqrt{h}^{[4,4]} \right) \delta^4_I,
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \partial^{\mu} \frac{\delta^2 \Gamma}{\delta \hat{\mathcal{W}}^{\pm\mu} \delta \hat{\mathcal{A}}^{Y\nu}} \pm i \bar{M}_W \frac{\delta^2 \Gamma}{\delta \hat{\Phi}^{\pm} \delta \hat{\mathcal{A}}^{Y\nu}}, && (104) \\
0 &= \partial^{\mu} \frac{\delta^2 \Gamma}{\delta \hat{\mathcal{W}}^{\pm\mu} \delta \hat{\Phi}^I} \pm i \bar{M}_W \frac{\delta^2 \Gamma}{\delta \hat{\Phi}^{\pm} \delta \hat{\Phi}^I} && (105) \\
&\mp \frac{i \bar{g}_2}{4} \frac{\delta \Gamma}{\delta \hat{\Phi}^4} \left(\sqrt{h}_{[4,4]} \mp i \sqrt{h}_{[4,3]} \right) \times \\
&\quad \left[(\sqrt{h}^{[1,1]} + \sqrt{h}^{[2,2]} \mp i \sqrt{h}^{[1,2]} \pm i \sqrt{h}^{[2,1]}) \delta^{\mp}_I \right. \\
&- \left. (\sqrt{h}^{[1,1]} - \sqrt{h}^{[2,2]} \pm i \sqrt{h}^{[1,2]} \pm i \sqrt{h}^{[2,1]}) \delta^{\pm}_I \right].
\end{aligned}
$$

These identities have the same structure as in the SM. The main differences are the factors multiplying the tadpole terms. By definition, the vev is defined as $\sqrt{2\,\langle H^\dagger H \rangle} \equiv \bar{v}_T$. The substitution of the vev leading to the $\hat{\mathcal{Z}}$ boson mass in the SMEFT ($\bar{M}_Z$) absorbs a factor in the scalar mass-eigenstate transformation matrix as $\sqrt{2\,\langle H^\dagger H \rangle} = \sqrt{2\,\langle H^\dagger \mathcal{V}^{-1} \mathcal{V} H \rangle}$. If a scheme is chosen so that $\delta\Gamma/\delta\hat{\phi}^4$ vanishes, then rotation to the mass eigenstate basis of the one-point vector $\delta\Gamma/\delta\hat{\phi}^i$ are still vanishing in each equation above. One way to tackle tadpole corrections is to use the FJ tadpole scheme, for discussion see Ref. [98, 99].

6.8 $\mathcal{A}, \mathcal{Z}$ IDENTITIES

The mapping of the SM Ward identites for Γ_{AZ} in the background field method given in Ref. [72] to the SMEFT is

$$0 = \partial^{\mu} \frac{\delta^2 \Gamma}{\delta \hat{\mathcal{A}}^{\nu} \delta \hat{\mathcal{Z}}^{\mu}}. \tag{106}$$

As an alternative derivation, the mapping between the mass eigenstate (Z, A) fields in the SM and the SMEFT $(\mathcal{Z}, \mathcal{A})$ reported in Ref. [26] directly follows from Eq. (97). Input parameter scheme dependence drops out when considering the two-point function Γ_{AZ} in the SM mapped to the SMEFT and a different overall normalization factors out. One still finds $\Sigma_{L,\mathrm{SMEFT}}^{\hat{\mathcal{A}},\hat{\mathcal{Z}}}(k^2) = 0$ and, as a consequence of analyticity at $k^2 = 0$, $\Sigma_{T,\mathrm{SMEFT}}^{\hat{\mathcal{A}},\hat{\mathcal{Z}}}(0) = 0$. This result has been used in the BFM calculation reported in Ref. [73, 100].

6.9 CONCLUSIONS

We have derived Ward identities for the SMEFT, constraining both the perturbative and power-counting expansions. The results presented already provide a clarifying explanation to some aspects of the structure of the SMEFT that has been determined at tree level. The utility of these results is expected to become clear as studies of the SMEFT advance to include sub-leading corrections.

APPENDIX

6.A NOTATION

The metric forms and rotations to $\mathcal{L}^{(6)}$ in the Warsaw basis are explicitly [84, 85]

$$\sqrt{g}^{AB} = \begin{bmatrix} 1+\tilde{C}_{HW} & 0 & 0 & 0 \\ 0 & 1+\tilde{C}_{HW} & 0 & 0 \\ 0 & 0 & 1+\tilde{C}_{HW} & -\frac{\tilde{C}_{HWB}}{2} \\ 0 & 0 & -\frac{\tilde{C}_{HWB}}{2} & 1+\tilde{C}_{HB} \end{bmatrix},$$

$$\sqrt{h}^{IJ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\frac{1}{4}\tilde{C}_{HD} & 0 \\ 0 & 0 & 0 & 1+\tilde{C}_{H\Box}-\frac{1}{4}\tilde{C}_{HD} \end{bmatrix},$$

$$U_{BC} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & c_{\overline{\theta}} & s_{\overline{\theta}} \\ 0 & 0 & -s_{\overline{\theta}} & c_{\overline{\theta}} \end{bmatrix}, \qquad V_{JK} = \begin{bmatrix} \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{107}$$

The notation for dimensionless Wilson coefficients is $\tilde{C}_i = \bar{v}_T^2 C_i/\Lambda^2$. The convention for $s_{\bar{\theta}}$ here has a sign consistent with Ref. [85], which has an opposite sign compared to Ref. [72]. For details and explicit results on couplings for the SMEFT including $\mathcal{L}^{(6)}$ corrections in the Warsaw basis, we note that we are consistent in notational conventions with Ref. [85].

The generators are given as

$$\gamma^I_{1,J} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \gamma^I_{2,J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$\gamma^I_{3,J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \gamma^I_{4,J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{108}$$

The γ_4 generator is used for the $\mathrm{U}(1)_\mathrm{Y}$ embedding. The couplings are absorbed into the structure constants and generators leading to tilde superscripts,

$$\begin{aligned} \tilde{\epsilon}^A{}_{BC} &= g_2\, \epsilon^A{}_{BC}, \quad \text{with } \tilde{\epsilon}^1{}_{23} = +g_2, \\ \tilde{\gamma}^I_{A,J} &= \begin{cases} g_2\, \gamma^I_{A,J}, & \text{for } A = 1,2,3 \\ g_1 \gamma^I_{A,J}, & \text{for } A = 4. \end{cases} \end{aligned} \tag{109}$$

In mass eigenstate basis, the transformed generators are

$$\gamma^I_{1,J} = \frac{\overline{g}_2}{2\sqrt{2}} \begin{bmatrix} 0 & 0 & i & -1 \\ 0 & 0 & -1 & -i \\ -i & 1 & 0 & 0 \\ 1 & i & 0 & 0 \end{bmatrix}, \qquad \gamma^I_{2,J} = \frac{\overline{g}_2}{2\sqrt{2}} \begin{bmatrix} 0 & 0 & -i & -1 \\ 0 & 0 & -1 & i \\ i & 1 & 0 & 0 \\ 1 & -i & 0 & 0 \end{bmatrix},$$

$$\gamma^I_{3,J} = \frac{\bar{g}_Z}{2}\begin{bmatrix} 0 & -(c^2_{\bar\theta} - s^2_{\bar\theta}) & 0 & 0 \\ (c^2_{\bar\theta} - s^2_{\bar\theta}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \gamma^I_{4,J} = \bar{e}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{110}$$

6.B CONNECTED GREEN'S FUNCTIONS FORMULATION

An alternative approach is to derive the Ward identities in terms of the generating functional for connected Green's functions – W_c. The non-invariance of $\mathcal{L}^{\rm BF}_{\rm GF}$ under background-field gauge transformations leads to

$$\frac{\delta W_c}{\delta \alpha^B} = i\int d^4x\, \frac{\delta}{\delta\hat{\alpha}^B}\mathcal{L}^{\rm BF}_{\rm GF}. \tag{111}$$

We choose the gauge-fixing term *for the background fields*

$$\begin{aligned}\mathcal{L}^{\rm BF}_{\rm GF} &= -\frac{1}{2\xi}\langle g_{AB}\rangle G^A G^B, \\ G^X &= \partial_\mu \hat{\mathcal{W}}^{X,\mu} + \frac{\xi}{2}\langle g^{XC}\rangle(\hat\phi^I - \langle\hat\phi^I\rangle)\langle h_{IK}\rangle\tilde\gamma^K_{C,J}\langle\hat\phi^J\rangle.\end{aligned} \tag{112}$$

The variation of the gauge-fixing term with respect to the background-gauge parameter is

$$\begin{aligned}\frac{\delta}{\delta\hat\alpha^B}\mathcal{L}^{\rm BF}_{\rm GF} = \frac{1}{\xi}\langle g_{AD}\rangle\Bigg(&\Box\delta^A_B + i\partial^\mu\tilde\epsilon^A_{\ BC}\frac{\delta W_c}{\delta J_{\hat{\mathcal{W}}^{C,\mu}}} \\ &+\frac{\xi}{2}\langle g^{AE}\rangle\frac{\tilde\gamma^I_{B,J}}{2}\left(-i\frac{\delta W_c}{\delta J_{\hat\phi^J}}\right)\langle h_{IK}\rangle\tilde\gamma^K_{E,L}\langle\phi^L\rangle\Bigg)G^D_{\mathcal{J}},\end{aligned} \tag{113}$$

where

$$G^D_{\mathcal{J}} = -i\partial^\nu\frac{\delta W_c}{\delta J_{\hat{\mathcal{W}}^{D,\nu}}} - i\frac{\xi}{2}\langle g^{DX}\rangle\frac{\delta W_c}{\delta J_{\hat\phi^I}}\langle h_{IK}\rangle\tilde\gamma^K_{X,J}\langle\phi^J\rangle.$$

Consider the difference between the vev defined by $\langle\,\rangle$ and an alternate vev denoted by $\langle\phi^J\rangle'$ where the minimum of the action still dictates the numerical value, but in addition $\langle\phi^J\rangle'$ transforms as

$\delta\langle\phi^I\rangle' = \tilde{\gamma}^I_{A,J}\langle\phi^J\rangle'\,\hat{\alpha}^A$. Replacing all instances of $\langle\rangle$ in the above equations with this expectation value, and related transformation properties on the modified metrics, one finds

$$\frac{\delta}{\delta\hat{\alpha}^B}\mathcal{L}^{\rm BF}_{\rm GF} = \frac{1}{\xi}\langle g_{BD}\rangle'\Box G^D_{\mathcal{J}}. \tag{114}$$

The two results coincide for on-shell observables, for further discussion this point, and tadpole schemes, see Ref. [96]. We postpone a detailed discussion of these two approaches to a future publication.

7

THE GEOMETRIC STANDARD MODEL EFFECTIVE FIELD THEORY

We develop the geometric formulation of the Standard Model Effective Field Theory (SMEFT). Using this approach we derive all-orders results in the $\sqrt{2\langle H^\dagger H\rangle}/\Lambda$ expansion relevant for studies of Electroweak Precision and Higgs data.

7.1 INTRODUCTION

The Higgs field in the Standard Model (SM) defines a set of field connections of the SM states. The mass scales of the SM states are dictated by the vaccum expectation value (vev) of the theory, which is defined to be $\sqrt{2\langle H^\dagger H\rangle} = \bar{v}_T$. When the SM is generalized to the Standard Model Effective Field Theory (SMEFT) [25, 101], the Lagrangian contains two characteristic power counting expansions. The SMEFT is of interest when physics beyond the SM is present at scales $\Lambda > \bar{v}_T$. One of the power counting expansions in the SMEFT is in the ratio of scales $\bar{v}_T/\Lambda < 1$. This ratio defines the nature of the SMEFT operator expansion for measurements with phase space populations dictated by SM resonances (i.e. near SM poles). The SMEFT is well-defined and useful when such effects are perturbations to SM predictions.

A second power counting expansion is present in the SMEFT. This expansion is in $(p^2/\Lambda^2)^{d-4} \lesssim 1$, with p^2 a kinematic Lorentz invariant. It is linked to the novel Lorentz-invariant connections between SM fields, due to higher-dimensional (and frequently derivative) operators. This expansion is most relevant when studying measurements with phase space populations away from the poles of the SM states (when $p^2 \neq m_{SM}^2$), i.e. in tails of kinematic distributions.

For the SMEFT to be a predictive and meaningful theory, it is necessary that both of these expansions are under control.[1] In this paper, we develop the geometric approach to the SMEFT. This approach is

[1]For this reason, the fact that the number of parameters present in the SMEFT operator basis expansion also grows exponentially on general grounds [102, 103] is a challenge for the SMEFT. We return to this point below.

useful as it makes the effects of these two distinct power counting expansions more manifest. Here, we advance this approach by systematically defining connections that depend on the scalar field coordinates, defining a scalar field space geometry, that is factorized from composite operator forms. These connections depend on the vev and functionally this is useful as it (largely) factorizes out the $\bar{v}_T/\Lambda$ power counting expansion from the remaining part of the composite operator, and the derivative expansion. The propagating degrees of freedom, including the Higgs field, then interact on field manifolds, which encode the effects of higher-dimensional operators. The scalar field space is curved, and the degree of curvature is linked to the size of the ratio of scales $\bar{v}_T/\Lambda$ [2, 8, 22, 64, 65, 77, 104]. This curved field space modifies correlation functions, and the definition of SM Lagrangian parameters such as gauge couplings, mixing angles, and masses. The flat field space limit of the Lagrangian parameters simplifies to the definitions in the SM as $\bar{v}_T/\Lambda \to 0$.

In this paper, we also introduce a consistent all-orders general definition of SM Lagrangian parameters (in this expansion) embedded into the SMEFT. This is possible by taking into account the geometry of the field space defined by the Higgs field. This definition is limited in form by consistency of the parameter definitions in the SMEFT $\bar{v}_T/\Lambda$ expansion. These constraints due to self-consistency allow several all-orders results to be derived for critical experimental observables in Electroweak precision and Higgs data, which we also report.

7.2 THE GEOMETRIC SMEFT

The SMEFT Lagrangian is defined as

$$\mathcal{L}_{\rm SMEFT} = \mathcal{L}_{\rm SM} + \mathcal{L}^{(d)}, \qquad\qquad \mathcal{L}^{(d)} = \sum_i \frac{C_i^{(d)}}{\Lambda^{d-4}} \mathcal{Q}_i^{(d)} \quad \text{for } d > 4. \tag{115}$$

The particle spectrum has masses $m \sim g_{SM}\sqrt{\langle H^\dagger H\rangle}$, and includes a $\rm SU(2)_L$ scalar doublet (H) with Hypercharge $\mathsf{y}_h = 1/2$, distinguishing this theory from the Higgs Effective Field Theory (HEFT) [105–108], where only a singlet scalar is in the spectrum.[2]

The higher-dimensional operators $\mathcal{Q}_i^{(d)}$ in the SMEFT are labelled with a mass dimension d superscript and multiply unknown Wilson coefficients $C_i^{(d)}$. The sum over i, after non-redundant operators are removed with field redefinitions of the SM fields, runs over the operators in a particular operator basis. We use the Warsaw basis [25] in this paper for $\mathcal{L}^{(6)}$. The operators defined in Ref. [111] are frequently used for $\mathcal{L}^{(8)}$ results, when basis dependent results are quoted. We frequently absorb

[2] The direct meaning of this assumption of including a $\rm SU(2)_L$ scalar doublet in the theory, is that the local operators are *analytic* functions of the field H. The analyticity of the operator expansion was reviewed in Ref. [22]. See also Refs. [64, 109, 110] for some discussion on the HEFT/SMEFT distinction.

powers of $1/\Lambda^2$ into the definition of the Wilson coefficients for brevity of presentation and use $\tilde{C}_i^{(6)} \equiv C_i^{(6)} \bar{v}_T^2/\Lambda^2$ as a short-hand notation at times for $\mathcal{L}^{(6)}$ operators. We generalize this notation to $\mathcal{L}^{(2n)}$ operators, so that $\tilde{C}_i^{(2n)} \equiv C_i^{(2n)} \bar{v}_T^{2n-4}/\Lambda^{2n-4}$. Our remaining notation is largely consistent with Ref. [22].

Field space metrics have been studied and developed outside the SMEFT in many works.[3] These techniques are particularly useful in the SMEFT, due to the presence of the Higgs field which takes on a vev. When this occurs, a tower of high-order field interactions multiplies a particular composite operator form. For low n-point interactions, the field space metrics defined in Refs. [2, 64, 65, 77, 104] are sufficient to describe this physics. It has been shown that this approach can be used to understand what operator forms cannot be removed in operator bases [77], how scalar curvature invariants and the scalar geometry is related to experiment and the distinction between SMEFT, HEFT and the SM [64, 65, 104], and how to gauge fix the SMEFT in a manner invariant under background field transformations [2]. (See also Ref. [97].) This approach also gives all-orders SMEFT (background field) Ward identities [8].

The generalization of this approach to arbitrary n-point functions is via the decomposition

$$\mathcal{L}_{\rm SMEFT} = \sum_i f_i(\alpha \cdots) G_i(I, A \cdots), \tag{116}$$

where $f_i(\alpha \cdots)$ indicates all explicit Lorentz-index-carrying building blocks of the Lagrangian, while the G_i depend on group indicies A, I for the (non-spacetime) symmetry groups that act on the scalar fields, and the scalar field coordinates themselves. By factorizing systematically the dependence on the scalar field coordinates from the remaining parts of a composite operator, the expectation value of $G_i(I, A \cdots)$ reduces to a number, and emissions of h. This collapses a tower of higher-order interactions into a numerical coefficient for a composite operator – when considering matrix elements without propagating h fields. The f_i are built out of the combinations of fields and derivatives that are outputs of the Hilbert series characterizing and defining a set of higher-dimensional operators, see Refs. [103, 111, 115–117]. This introduces a basis dependence into the results. The Hilbert series generates operator bases with minimal sets of explicit derivatives, consistent with reductions of operators in an operator basis by the Equation of Motion (EOM). For example, the Warsaw basis for $\mathcal{L}^{(6)}$ is consistent with the output of a Hilbert series expansion.[4] The f_i retain a minimal scalar field coordinate dependence, and vev dependence, through powers of $(D^\mu H)$ and at higher orders through

[3] See for example Refs. [112, 113]. It is remarkable that the similar theoretical techniques to those we develop here also enable studies of GR as an EFT, see Ref. [114].

[4] Such a basis also offers a number of other benefits when calculating in the SMEFT, that are most apparent beyond leading order in the operator expansion; see the review [22] for more details.

symmetric derivatives acting on H. As these operator forms depend on powers of $\partial_\mu h$ they do not collapse to just a number when a scalar expectation value is taken.

7.2.1 *Mass eigenstates*

The field coordinates of the Higgs doublet are put into a convenient form with a common set of generators for $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$, by using the real scalar field coordinates $\phi_I = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ introduced with normalization

$$H(\phi_I) = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_2 + i\phi_1 \\ \phi_4 - i\phi_3 \end{bmatrix}, \quad \tilde{H}(\phi_I) = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_4 + i\phi_3 \\ -\phi_2 + i\phi_1 \end{bmatrix}. \tag{117}$$

ϕ_4 is expanded around the vacuum expectation value with the replacement $\phi_4 \to \phi_4 + \bar{v}_T$. The gauge boson field coordinates are similarly unified into $\mathcal{W}^A = \{W^1, W^2, W^3, B\}$ with $A = \{1, 2, 3, 4\}$. The corresponding general coupling is defined as $\alpha_A = \{g_2, g_2, g_2, g_1\}$.

We define short-hand notation as in Ref. [8] for the transformation matrices that lead to the canonically normalized mass eigenstate fields

$$\mathcal{U}^A_C = \sqrt{g}^{AB} U_{BC}, \qquad \mathcal{V}^I_K = \sqrt{h}^{IJ} V_{JK}.$$

Here $\sqrt{g}^{AB}$ and $\sqrt{h}^{IJ}$ are square-root metrics, which are understood to be matrix square roots of the expectation value – $\langle\rangle$ – of the field space connections for the bilinear terms in the SMEFT. These connections are defined below in Section 7.2.3. The matrices U, V are unitary, and given by

$$U_{BC} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & c_{\bar{\theta}} & s_{\bar{\theta}} \\ 0 & 0 & -s_{\bar{\theta}} & c_{\bar{\theta}} \end{bmatrix}, \qquad V_{JK} = \begin{bmatrix} \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Also, $\sqrt{h}^{IJ} \sqrt{h}_{JK} \equiv \delta^I_K$ and $\sqrt{g}^{AB} \sqrt{g}_{BC} \equiv \delta^A_C$. The rotation angles $c_{\bar{\theta}}, s_{\bar{\theta}}$ are functions of α_A and $\langle g^{AB} \rangle$ and are defined geometrically in Section 7.4.3.

The SMEFT weak/mass eigenstate dynamical fields[5] and related couplings are then given by [2] (see also Refs. [76, 85, 92, 118])

$$\alpha^A = \mathcal{U}^A_C\, \beta^C, \quad \mathcal{W}^{A,\mu} = \mathcal{U}^A_C\, \mathcal{A}^{C,\mu}, \quad \phi^J = \mathcal{V}^J_K\, \Phi^K \tag{118}$$

where in the SM limit

$$\alpha^A = \{g_2\, g_2, g_2, g_1\}, \qquad \mathcal{W}^A = \{W_1, W_2, W_3, B\},$$
$$\beta^C = \left\{ \frac{g_2\,(1-i)}{\sqrt{2}}, \frac{g_2\,(1+i)}{\sqrt{2}}, \sqrt{g_1^2+g_2^2}(c^2_{\bar\theta} - s^2_{\bar\theta}), \frac{2\, g_1\, g_2}{\sqrt{g_1^2+g_2^2}} \right\}, \qquad \mathcal{A}^C = (\mathcal{W}^+, \mathcal{W}^-, \mathcal{Z}, \mathcal{A})\,.$$

and $\phi^J = \{\phi_1, \phi_2, \phi_3, \phi_4\}, \Phi^K = \{\Phi^-, \Phi^+, \chi, h\}$ for the scalar fields. All-orders results in the $\bar{v}_T/\Lambda$ expansion can be derived as the relationship between the mass and weak eigenstate fields is always given by Eqn. (118). Remarkably, the remaining field space connections for two- and three-point functions can also be defined at all-orders in the $\bar{v}_T/\Lambda$ expansion.

7.2.2 *Classifying field space connections for two- and three-point functions*

We first classify the operators contributing to two- and three-point functions. The arguments used here build on those in Refs. [25, 111]. Consider a generic three-point function, including the effects of a tower of higher-dimensional operators. We denote a SM field, defined in the weak eigenstate basis, as $F = \{H, \psi, \mathcal{W}^{\mu\nu}\}$ for the discussion to follow. Recall the SM EOM for the Higgs field,

$$D^2 H_k - \lambda v^2 H_k + 2\lambda (H^\dagger H) H_k + \bar{q}^j\, Y_u^\dagger\, u (i\sigma_2)_{jk} + \bar{d}\, Y_d\, q_k + \bar{e}\, Y_e\, l_k = 0\,, \tag{119}$$

indicating that dependence on $D^2 H_k$ can be removed in a set of operator forms contributing to three-point functions, in favour of just H_k, and higher-point functions. Further, using the Bianchi identity

$$D_\mu \mathcal{W}_{\alpha\beta} + D_\alpha \mathcal{W}_{\beta\mu} + D_\beta \mathcal{W}_{\mu\alpha} = 0, \tag{120}$$

[5] The vev $\bar{v}_T$ is subtracted from ϕ_4 in the equation below involving ϕ^J.

one can also reduce $D^2\,\mathcal{W}_{\alpha\beta}$ to EOM-reducible terms and higher-point interactions via

$$\begin{aligned} D^2\,\mathcal{W}^A_{\alpha\beta} &= D_\mu D_\nu g^{\mu\nu}\mathcal{W}^A_{\alpha\beta}, \\ &= -D_\mu g^{\mu\nu}\left(D_\alpha \mathcal{W}^A_{\beta\nu} + D_\beta \mathcal{W}^A_{\nu\alpha}\right), \\ &= -\frac{1}{2}D_{\{\nu,\alpha\}}\mathcal{W}^A_{\beta\nu} - \frac{1}{2}D_{\{\nu,\beta\}}\mathcal{W}^A_{\nu\alpha} - \frac{1}{2}\mathcal{W}^A_{\nu\alpha}\mathcal{W}^A_{\beta\nu} - \frac{1}{2}\mathcal{W}^A_{\nu\beta}\mathcal{W}^A_{\nu\alpha}, \\ &\Rightarrow \boxed{\text{EOM}}\ \text{and higher-points} \end{aligned} \tag{121}$$

Here $D_{\{\nu,\alpha\}}$ is the symmetric combination of covariant derivatives. An explicit appearance of $D_{[\mu,\nu]}F$ is reduced to $\mathcal{W}^A_{\mu\nu}F$, where A is dictated by the SM charge of F.

Similarly, $D^2\psi$ can be reduced as

$$D^2\psi = D_\mu D_\nu g^{\mu\nu}\psi = D_\mu D_\nu(\gamma^\mu\gamma^\nu + i\sigma^{\mu\nu})\psi \quad \Rightarrow \quad \boxed{\text{EOM}}\ \text{and higher-points}, \tag{122}$$

where $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. In what follows, when D^2F appears, it is replaced in terms of EOM terms and higher-point functions for these reasons. Explicitly reducing operator forms by the EOM, when possible, in favour of other composite operators, has a key role in these arguments.

Now consider higher-derivative contributions to three-point functions. Explicit appearances of D^2F are removed due to the proceeding argument. Further, a general combination of derivatives, acting on three general SM fields $F_{1,2,3}$,

$$f(H)(D_\mu F_1)(D_\nu F_2)D_{\{\mu\nu\}}F_3, \tag{123}$$

is removable in terms of EOM terms and higher-point functions, using integration by parts:

$$\begin{aligned} &f(H)(D_\mu F_1)(D_\nu F_2)D_{\{\mu\nu\}}F_3 \qquad (124)\\ &= -f(H)\left[(D^2F_1)(D_\nu F_2) + (D_\mu F_1)(D_\mu D_\nu F_2) + (D_\mu D_\nu F_1)(D_\mu F_2) + (D_\nu F_1)(D^2F_2)\right](D_\nu F_3) \\ &\quad - \left(D_\mu f(H)\right)\left[(D_\mu F_1)(D_\nu F_2) + (D_\nu F_1)(D_\mu F_2)\right](D_\nu F_3) \\ &\Rightarrow -f(H)\left[(D_\mu F_1)(D_\mu D_\nu F_2) + (D_\mu D_\nu F_1)(D_\mu F_2)\right](D_\nu F_3) + \boxed{\text{EOM}}\ \text{and higher-points} \\ &\Rightarrow -f(H)(D_{[\mu,\nu]}F_1)(D_\mu F_2)(D_\nu F_3) + f(H)(D_\mu F_1)(D_\mu F_2)(D^2F_3) + \boxed{\text{EOM}}\ \text{and higher-points} \\ &\Rightarrow \boxed{\text{EOM}}\ \text{and higher-points.} \end{aligned}$$

As a result, in general, an operator with four or more derivatives acting on three (possibly different) fields F_i can be reduced out of three-point amplitudes.

When considering field space connections that can reduce to three-point functions when a vacuum expectation value is taken, we also use

$$f(\phi)\,F_1\,(D_\mu F_2)\,(D_\mu F_3) \Rightarrow (D_\mu f(\phi))\,(D_\mu F_1)\,F_2\,F_3 + \frac{1}{2}(D^2 f(\phi))\,F_1\,F_2\,F_3 + \boxed{\mathrm{EOM}}\,, \quad (125)$$

to conventionally move derivative terms onto scalar fields. After reducing the possible field space connections using these arguments systematically, and integrating by parts, a minimal generalization of field space connections for CP even electroweak bosonic two- and three-point amplitudes is composed of

$$h_{IJ}(\phi)(D_\mu\phi)^I(D_\mu\phi)^J, \quad g_{AB}(\phi)\mathcal{W}^A_{\mu\nu}\mathcal{W}^{B,\mu\nu}, \quad k^A_{IJ}(\phi)(D_\mu\phi)^I(D_\nu\phi)^J\,\mathcal{W}^{\mu\nu}_A,$$
$$f_{ABC}(\phi)\mathcal{W}^A_{\mu\nu}\mathcal{W}^{B,\nu\rho}\mathcal{W}^{C,\mu}_\rho,$$

and the scalar potential $V(\phi)$.

The minimal set of field space connections involving fermionic field in two- and three-point functions is

$$Y(\phi)\bar{\psi}_1\psi_2, \quad L_{I,A}(\phi)\bar{\psi}_1\gamma^\mu\sigma_A\psi_2(D_\mu\phi)^I, \quad d_A(\phi)\bar{\psi}_1\sigma^{\mu\nu}\psi_2\mathcal{W}^A_{\mu\nu},$$

where flavour indicies are suppressed. Here we have defined $\sigma_A = \{\sigma_i, \mathbb{I}\}$, and use this notation below. The corresponding connections in the case of the gluon field are

$$k_{\mathcal{AB}}(\phi)G^{\mathcal{A}}_{\mu\nu}G^{\mathcal{B},\mu\nu}, \quad k_{\mathcal{ABC}}(\phi)G^{\mathcal{A}}_{\nu\mu}G^{\mathcal{B},\rho\nu}G^{\mathcal{C},\mu\rho}, \quad c(\phi)\bar{\psi}_1\sigma^{\mu\nu}T_{\mathcal{A}}\psi_2G^{\mathcal{A}}_{\mu\nu}. \quad (126)$$

When considering two- or three-point functions the expectation values of the scalar field connections are taken with $\langle\rangle$. Although we are focusing our presentation on CP even field space connections, the case of CP odd connections is analogous and an additional connection can be defined for g_{AB}, f_{ABC}, $k_{\mathcal{AB}}$, and $k_{\mathcal{ABC}}$. The connections h_{IJ}, g_{AB} are symmetric and real, while f_{ABC} and $k_{\mathcal{ABC}}$ are anti-symmetric. The $Y(\phi)$, $d_A(\phi)$, and $c(\phi)$ connections are complex. $L_{I,A}$ is real for the SM fields, and complex in general in the case of the right-handed charged current connection. k^A_{IJ} is antisymmetric in the subscript indicies.

7.2.3 *Definition of field space connections*

The scalar functions include the potential $G_V = V(H^\dagger H)$, with corresponding $f_V \equiv 1$.

$$V(\phi) = -\mathcal{L}_{\rm SMEFT}|_{\mathcal{L}(\alpha,\beta\cdots\to 0)} \tag{127}$$

Going beyond the potential, we define field space connections from the Lagrangian for a series of composite operator forms. The field space metric for the scalar field bilinear, dependent on the SM field coordinates, is defined via

$$h_{IJ}(\phi) = \frac{g^{\mu\nu}}{d} \frac{\delta^2 \mathcal{L}_{\rm SMEFT}}{\delta(D_\mu\phi)^I\,\delta(D_\nu\phi)^J}\bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0}. \tag{128}$$

The notation $\mathcal{L}(\alpha,\beta\cdots)$ corresponds to non-trivial Lorentz-index-carrying Lagrangian terms and spin connections, i.e. $\{\mathcal{W}^A_{\mu\nu}, (D^\mu\Phi)^K, \bar\psi\sigma^\mu\psi, \bar\psi\psi\cdots\}$. This definition reduces the connection h_{IJ} to a function of $\mathrm{SU(2)_L}\times\mathrm{U(1)_Y}$ generators, scalar fields coordinates ϕ_i and $\bar v_T$.

The CP even gauge field scalar manifolds, for the $\mathrm{SU(2)_L}\times\mathrm{U(1)_Y}$ fields interacting with the scalar fields, are defined as

$$g_{AB}(\phi) = \frac{-2\,g^{\mu\nu}\,g^{\sigma\rho}}{d^2}\,\frac{\delta^2\mathcal{L}_{\rm SMEFT}}{\delta\mathcal{W}^A_{\mu\sigma}\,\delta\mathcal{W}^B_{\nu\rho}}\Bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0,\text{CP-even}}, \tag{129}$$

and (here $\mathcal{A},\mathcal{B}$ run over $1\cdots 8$)

$$k_{\mathcal{AB}}(\phi) = \frac{-2\,g^{\mu\nu}\,g^{\sigma\rho}}{d^2}\,\frac{\delta^2\mathcal{L}_{\rm SMEFT}}{\delta G^{\mathcal{A}}_{\mu\sigma}\,\delta G^{\mathcal{B}}_{\nu\rho}}\Bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0,\text{CP-even}}. \tag{130}$$

We also have

$$k^A_{IJ}(\phi) = \frac{g^{\mu\rho}g^{\nu\sigma}}{2d^2}\,\frac{\delta^3\mathcal{L}_{\rm SMEFT}}{\delta(D_\mu\phi)^I\delta(D_\nu\phi)^J\delta\mathcal{W}^A_{\rho\sigma}}\Bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0} \tag{131}$$

and

$$f_{ABC}(\phi) = \frac{g^{\nu\rho}g^{\sigma\alpha}g^{\beta\mu}}{3!d^3}\frac{\delta^3\mathcal{L}_{\rm SMEFT}}{\delta\mathcal{W}^A_{\mu\nu}\delta\mathcal{W}^B_{\rho\sigma}\delta\mathcal{W}^C_{\alpha\beta}}\bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0,\text{CP-even}},$$
$$k_{ABC}(\phi) = \frac{g^{\nu\rho}g^{\sigma\alpha}g^{\beta\mu}}{3!d^3}\frac{\delta^3\mathcal{L}_{\rm SMEFT}}{\delta G^A_{\mu\nu}\delta G^B_{\rho\sigma}\delta G^C_{\alpha\beta}}\bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0,\text{CP-even}}. \tag{132}$$

We also define the fermionic connections

$$Y^{\psi_1}_{pr}(\phi_I) = \frac{\delta\mathcal{L}_{\rm SMEFT}}{\delta(\bar{\psi}^I_{2,p}\psi_{1,r})}\bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0}, \quad L^{\psi,pr}_{J,A} = \frac{\delta^2\mathcal{L}_{\rm SMEFT}}{\delta(D^\mu\phi)^J\delta(\bar{\psi}_p\gamma_\mu\sigma_A\psi_r)}\bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0}, \tag{133}$$

and

$$d^{\psi_1,pr}_A(\phi_I) = \frac{\delta^2\mathcal{L}_{\rm SMEFT}}{\delta(\bar{\psi}^I_{2,p}\sigma_{\mu\nu}\psi_{1,r})\delta\mathcal{W}^A_{\mu\nu}}\bigg|_{\mathcal{L}(\alpha,\beta\cdots)\to 0}. \tag{134}$$

7.2.4 *Hilbert series counting*

The Hilbert series is a compact mathematical tool that uses character orthonormality to count group invariants. As shown in Refs. [103, 115–117], it can be adapted to count SMEFT operators up to arbitrary mass dimension while accounting for EOM and integration by parts (IBP) redundancies. The ingredients required are simply the SMEFT field content and each field's representation under the SM gauge groups and 4-d conformal symmetry. The output of the Hilbert series is the number of SMEFT operators with a given mass dimension and field/derivative content. To convert this output into something useful for phenomenology, one must make a choice of how to contract indices and where to apply any derivatives. This choice introduces basis dependence.

The results from the Section 7.2.2 (combined with similar results from Ref. [97] for two-point vertices) show that it is possible to construct a basis where the two- and three-point vertices are particularly simple – meaning that they are impacted by a minimal set of higher-dimensional operator effects. Following Eqns. (124) and (125), three-point (electroweak) bosonic vertices are captured entirely by operators of the form $D^2(H^\dagger H)^n$, $(H^\dagger H)^n X^2$, $D^2(H^\dagger H)^n X$, $(H^\dagger H)^n X^3$ and $(H^\dagger H)^n$ (n an integer), with $X_{L/R} = \{W^a \pm \tilde{W}^a, B \pm \tilde{B}, G \pm \tilde{G}\}$. The Lorentz group representation is $SO(4) \simeq \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$, so that $X_{L/R}$ are in the $(1,0)$ and $(0,1)$ representations.

Studying the Hilbert series output for this restricted set, we find that the number of invariants in each category approaches a fixed value, and then remains fixed independent of mass dimension: there are 2 operators of the form $D^2(H^\dagger H)^n$ for all n, 2 operators $(H^\dagger H)^n\, W^2$, 1 operator $(H^\dagger H)^n\, WB$, etc. The fact that the number of operators relevant to the field connections for the two- and three-point vertices saturates can be proven in each case using techniques from Ref. [111]. As one example, take $(H^\dagger H)^n\, W_L^2$ and suppress all indices other than Lorentz, in the form $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{SU}(2)_\mathrm{R}$, and $\mathrm{SU}(2)_\mathrm{w}$: being bosonic, the H^n and $H^{\dagger,n}$ terms must be completely symmetric and therefore in representations $(0, 0, \frac{n}{2})$ of $(\mathrm{SU}(2)_\mathrm{L}, \mathrm{SU}(2)_\mathrm{R}, \mathrm{SU}(2)_\mathrm{w})$. Their product lies in $(0, 0, 0 \oplus 1 \oplus 2 \oplus \cdots n)$. W_L^2 must also be symmetric, but it is more complicated as W_L contains both Lorentz and $\mathrm{SU}(2)_\mathrm{w}$ indices (here we use the notation $\mathrm{SU}(2)_\mathrm{w}$ to avoid a double use of $\mathrm{SU}(2)_\mathrm{L}$). Keeping all symmetric combinations, we find $(0 \oplus 2, 0, 0 \oplus 2) + (1, 0, 1)$. Combining the two pieces, the product $(H^\dagger H)^n W_L^2$ clearly contains two invariants, one where the $(H^\dagger H)^n$ form a net $\mathrm{SU}(2)_\mathrm{w}$ singlet, and one where $(H^\dagger H)^n$ lie in a quintuplet (spin-2).[6] Since B_L transforms under Lorentz symmetry alone, there is only one operator of the form $(H^\dagger H)^n B_L^2$, and the $\mathrm{SU}(2)_\mathrm{w}$ triplet component of $(H^\dagger H)^n$ combines with the Lorentz singlet piece of $W_L\, B_L$ to form one operator of the form $(H^\dagger H)^n W_L\, B_L$. Together, these make up the 4 terms in the $g_{AB}(\phi)\mathcal{W}^A_{\mu\nu}\mathcal{W}^{B,\mu\nu}$ entry of Table 7.2.1 for mass dimension ≥ 8.[7] Similar arguments can be made for the other operator categories in Table 7.2.1, which are also consistent with the results reported in Ref. [103].

The argument can also be made using on-shell amplitude methods for counting higher-dimensional operators, and there is clearly a profitable connection between SMEFT geometry and the recent developments using on-shell methods to study the SMEFT to exploit. See Refs. [119–126] for recent developments of this form.

Because the number of terms of each operator form for the field connections saturates to a fixed value, the expressions for the connections for the two- and three-point vertices at all orders in the $\bar{v}_T/\Lambda$ expansion of the SMEFT can be written compactly and exactly. This implies that the general exponential nature of the operator basis expansion [102, 103] is more strongly expressed in the growth of higher-point functions and the SMEFT derivative expansion.[8]

---

[6] This second possibility requires at least four Higgs fields ($n \geq 2$), and therefore total operator mass dimension ≥ 8.

[7] In addition to the X_L^2 operators, there are an identical number of hermitian conjugate terms involving X_R. Only one combination of the X_L^2, X_R^2 terms are CP conserving.

[8] The very simple form of the resulting field space connections can clearly be examined using Borel re-summation, once assumptions on perturbativity of the Wilson coefficients are made. This offers the potential to construct error estimates due to the series truncation on the field space connection. We leave an exploration of this observation to a future publication.

	Mass Dimension				
Field space connection	6	8	10	12	14
$h_{IJ}(\phi)(D_\mu\phi)^I(D^\mu\phi)^J$	2	2	2	2	2
$g_{AB}(\phi)\mathcal{W}^A_{\mu\nu}\mathcal{W}^{B,\mu\nu}$	3	4	4	4	4
$k_{IJA}(\phi)(D^\mu\phi)^I(D^\nu\phi)^J\mathcal{W}^A_{\mu\nu}$	0	3	4	4	4
$f_{ABC}(\phi)\mathcal{W}^A_{\mu\nu}\mathcal{W}^{B,\nu\rho}\mathcal{W}^{C,\mu}_\rho$	1	2	2	2	2
$Y^u_{pr}(\phi)\bar{Q}u+$ h.c.	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$
$Y^d_{pr}(\phi)\bar{Q}d+$ h.c.	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$
$Y^e_{pr}(\phi)\bar{L}e+$ h.c.	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$	$2\,N_f^2$
$d^{e,pr}_A(\phi)\bar{L}\sigma_{\mu\nu}e\mathcal{W}^{\mu\nu}_A+$ h.c.	$4\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$
$d^{u,pr}_A(\phi)\bar{Q}\sigma_{\mu\nu}u\mathcal{W}^{\mu\nu}_A+$ h.c.	$4\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$
$d^{d,pr}_A(\phi)\bar{Q}\sigma_{\mu\nu}d\mathcal{W}^{\mu\nu}_A+$ h.c.	$4\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$	$6\,N_f^2$
$L^{\psi_R}_{pr,A}(\phi)(D^\mu\phi)^J(\bar{\psi}_{p,R}\gamma_\mu\sigma_A\psi_{r,R})$	N_f^2	N_f^2	N_f^2	N_f^2	N_f^2
$L^{\psi_L}_{pr,A}(\phi)(D^\mu\phi)^J(\bar{\psi}_{p,L}\gamma_\mu\sigma_A\psi_{r,L})$	$2\,N_f^2$	$4\,N_f^2$	$4\,N_f^2$	$4\,N_f^2$	$4\,N_f^2$

Table 7.2.1: Counting of operators contributing to two- and three-point functions from Hilbert series. These results are consistent with Ref. [103].

7.3 FIELD SPACE CONNECTIONS

The explicit forms of the field space connections are basis dependent. In this section we give results in a specific operator basis set, the Warsaw basis at $\mathcal{L}^{(6)}$, and some operators at $\mathcal{L}^{(8)}$ defined in Ref. [111].

The potential is defined in a power counting expansion as

$$V(H^\dagger H) = \lambda\left(H^\dagger H - \frac{v^2}{2}\right)^2 - C_H^{(6)}(H^\dagger H)^3 - C_H^{(8)}(H^\dagger H)^4\cdots \tag{135}$$

The minimum is redefined order by order in the power counting expansion

$$\langle H^\dagger H\rangle = \frac{v^2}{2}\left(1 + \frac{3\,C_H^{(6)}\,v^2}{4\lambda} + v^4\frac{9\left(C_H^{(6)}\right)^2 + 4C_H^{(8)}\lambda}{8\lambda^2} + \cdots\right) \equiv \frac{\bar{v}_T^2}{2}. \tag{136}$$

This generalization of the expectation value simplifies at leading order in $1/\Lambda^2$ to the vev of the SM. Including the leading $1/\Lambda^2$ correction, the result is that of Ref. [85], the $1/\Lambda^4$ correction is as given in Ref. [111], etc. At higher orders in the polynomial expansion of $H^\dagger H$ that results from taking the derivative of the potential, numerical methods must be used to find a minimum due to the Abel–Ruffini theorem. Note that this also means that expanding out the vev dependence in a formal all-orders result to a fixed order necessarily requires numerical methods.

The expectation values of the field space connections is also denoted by $\langle\rangle$ and a critical role is played by $\sqrt{h}^{IJ} = \langle h^{IJ}\rangle^{1/2}$, and $\sqrt{g}^{AB} = \langle g^{AB}\rangle^{1/2}$. The $\sqrt{h}$ and $\sqrt{g}$ depend on $\bar{v}_T$.

7.3.1 *Scalar bilinear metric: $h_{IJ}(\phi)$*

The relevant terms in $\mathcal{L}^{(6,8)}$ for the scalar metric are [111]

$$\begin{aligned}\mathcal{L}^{(6,8)} \supseteq\; & C^{(6)}_{H\Box}(H^\dagger H)\Box(H^\dagger H) + C^{(6)}_{HD}(H^\dagger D_\mu H)^\star(H^\dagger D^\mu H) \\ + \; & C^{(8)}_{HD}(H^\dagger H)^2(D_\mu H)^\dagger(D^\mu H) + C^{(8)}_{H,D2}(H^\dagger H)(H^\dagger\sigma_a H)\left[(D_\mu H)^\dagger\,\sigma^a\,(D^\mu H)\right].\end{aligned} \tag{137}$$

For the Warsaw basis [97], extended with the $\mathcal{L}^{(8)}$ defined in Ref. [111], h_{IJ} is

$$h_{IJ} = \left[1 + \frac{\phi^4}{4}(C^{(8)}_{HD} - C^{(8)}_{H,D2})\right]\delta_{IJ} - 2C^{(6)}_{H\Box}\phi_I\phi_J + \frac{\Gamma^I_{A,J}\phi_K\Gamma^K_{A,L}\phi^L}{4}\left(C^{(6)}_{HD} + \phi^2 C^{(8)}_{H,D2}\right). \tag{138}$$

We note $\delta_{IJ} = \Gamma^I_{A,K}\Gamma^K_{A,J}$ for all A and $\phi^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 + (\phi_4 + \bar{v}_T)^2$. As we define h_{IJ} as in Eqn. (128), the choice in the Warsaw basis to integrate by parts and retain an explicit $\Box(H^\dagger H)$ derivative form is notationally awkward. The integration by parts operator identity

$$Q_{H\Box} = (H^\dagger\, i\overleftrightarrow{D}^\mu H)(H^\dagger\, i\overleftrightarrow{D}_\mu H) - 4(H^\dagger D_\mu H)^\star(H^\dagger D^\mu H) \tag{139}$$

can be used with the results in the Appendix to write

$$\begin{aligned}h_{IJ} =\; & \left[1 + \frac{\phi^4}{4}(C^{(8)}_{HD} - C^{(8)}_{H,D2})\right]\delta_{IJ} + \frac{\Gamma^I_{A,J}\phi_K\Gamma^K_{A,L}\phi^L}{4}\left(C^{(6)}_{HD} - 4C^{(6)}_{H\Box} + \phi^2 C^{(8)}_{H,D2}\right) \\ - \; & 2(\phi\gamma_4)_J(\gamma_4\phi)^I C^{(6)}_{H\Box}.\end{aligned} \tag{140}$$

Alternatively, one can use field redefinitions, expressed through the EOM operator identity for $\mathcal{L}^{(6)}$ for the Higgs,[9] to exchange $Q_{H\Box}$ for $H^\dagger H(D^\mu H)^\dagger(D_\mu H)$. This leads to a redefinition of the Wilson coefficient dependence of the vev and

$$h_{IJ} = \left[1 + \phi^2 C^{(6)}_{H\Box} + \frac{\phi^4}{4}(C^{(8)}_{HD} - C^{(8)}_{H,D2})\right]\delta_{IJ} + \frac{\Gamma^I_{A,J}\phi_K\Gamma^K_{A,L}\phi^L}{4}\left(C^{(6)}_{HD} + \phi^2 C^{(8)}_{H,D2}\right). \tag{141}$$

[9] See the Appendix and Eqn. (5.3) of Ref. [25].

Although the dependence on $C^{(6)}_{H\Box}$ coincides in $\langle h_{IJ}\rangle$ in Eqns. (138), (140) a different dependence on $C^{(6)}_{H\Box}$ is present in $\langle h_{IJ}\rangle$ in Eqn. (141). There is also a redefined vev in this case, and a further correction to the Wilson coefficient dependence in modified Class five operators in the Warsaw basis, etc. It is important to avoid overinterpreting the specific, operator basis, and gauge dependent, form of an individual field space connection. Such a quantity, like a particular Wilson coefficient, in a particular operator basis, is unphysical on its own. (See Appendix 7.B for more discussion.) Despite this, a geometric formulation[10] of the SMEFT exists in any basis, and still dictates a consistent relationship between the mass eigenstate field and the weak eigenstate fields. This relationship also allows all-orders results in the $\bar{v}_T/\Lambda$ expansion to be derived.

The general form of the scalar metric with $d = 8 + 2n$ dimensional two derivative operators, can be defined as having the form

$$Q^{(8+2n)}_{HD} = (H^\dagger H)^{n+2}(D_\mu H)^\dagger(D^\mu H), \tag{142}$$
$$Q^{(8+2n)}_{H,D2} = (H^\dagger H)^{n+1}(H^\dagger\sigma_a H)(D_\mu H)^\dagger\,\sigma^a\,(D^\mu H), \tag{143}$$

which leads to the result

$$\begin{aligned} h_{IJ} &= \left[1+\phi^2 C^{(6)}_{H\Box} + \sum_{n=0}^{\infty}\left(\frac{\phi^2}{2}\right)^{n+2}\left(C^{(8+2n)}_{HD} - C^{(8+2n)}_{H,D2}\right)\right]\delta_{IJ} \\ &+ \frac{\Gamma^I_{A,J}\phi_K\Gamma^K_{A,L}\phi^L}{2}\left(\frac{C^{(6)}_{HD}}{2} + \sum_{n=0}^{\infty}\left(\frac{\phi^2}{2}\right)^{n+1} C^{(8+2n)}_{H,D2}\right). \end{aligned} \tag{144}$$

The scalar field space metric defines a curved field space.

7.3.2 *Gauge bilinear metric: $g_{AB}(\phi)$*

The relevant $\mathcal{L}^{(6+2n)}$ terms for the Gauge Higgs interactions are

$$Q^{(6+2n)}_{HB} = (H^\dagger H)^{n+1} B^{\mu\nu}\,B_{\mu\nu}, \tag{145}$$
$$Q^{(6+2n)}_{HW} = (H^\dagger H)^{n+1} W^{\mu\nu}_a\,W^a_{\mu\nu}, \tag{146}$$
$$Q^{(6+2n)}_{HWB} = (H^\dagger H)^{n}(H^\dagger\sigma^a H) W^{\mu\nu}_a\,B_{\mu\nu}, \tag{147}$$
$$Q^{(8+2n)}_{HW,2} = (H^\dagger H)^{n}(H^\dagger\sigma^a H)(H^\dagger\sigma^b H) W^{\mu\nu}_a\,W_{b,\mu\nu}, \tag{148}$$
$$Q^{(6+2n)}_{HG} = (H^\dagger H)^{n+1} G^{\mu\nu}_{\mathcal{A}}\,G^{\mathcal{A}}_{\mu\nu}. \tag{149}$$

[10] Christoffel symbols can be derived from the field space metrics.

The Gauge-Higgs field space metric is given by

$$
\begin{aligned}
g_{AB}(\phi_I) &= \left[1 - 4\sum_{n=0}^{\infty}\left(C_{HW}^{(6+2n)}(1-\delta_{A4}) + C_{HB}^{(6+2n)}\delta_{A4}\right)\left(\frac{\phi^2}{2}\right)^{n+1}\right]\delta_{AB} \\
&- \sum_{n=0}^{\infty} C_{HW,2}^{(8+2n)}\left(\frac{\phi^2}{2}\right)^{n}\left(\phi_I \Gamma^I_{A,J}\phi^J\right)\left(\phi_L \Gamma^L_{B,K}\phi^K\right)(1-\delta_{A4})(1-\delta_{B4}) \\
&+ \left[\sum_{n=0}^{\infty} C_{HWB}^{(6+2n)}\left(\frac{\phi^2}{2}\right)^{n}\right]\left[(\phi_I \Gamma^I_{A,J}\phi^J)\,(1-\delta_{A4})\delta_{B4} + (A \leftrightarrow B)\right], \qquad (150)
\end{aligned}
$$

and for the gluon fields $G^{\mathcal{A},\mu} = \sqrt{k}^{\mathcal{AB}}\,\mathcal{G}^{\mu}_{\mathcal{B}}$, where

$$
k_{\mathcal{AB}}(\phi) = \left(1 - 4\sum_{n=0}^{\infty} C_{HG}^{(6+2n)}\left(\frac{\phi^2}{2}\right)^{n}\right)\delta_{\mathcal{AB}}. \qquad (151)
$$

7.3.3 *Yukawa couplings: $Y(\phi)$*

The Yukawa interactions of the Higgs field are extended in interpretation in a straightforward manner. Here the relevant $\mathcal{L}^{(6+2n)}$ operators are

$$
\begin{aligned}
Q^{(6+2n)}_{\substack{eH\\pr}} &= (H^\dagger H)^{n+1}(\bar{\ell}_p\, e_r\, H), && (152)\\
Q^{(6+2n)}_{\substack{uH\\pr}} &= (H^\dagger H)^{n+1}(\bar{q}_p\, u_r\, \tilde{H}), && (153)\\
Q^{(6+2n)}_{\substack{dH\\pr}} &= (H^\dagger H)^{n+1}(\bar{q}_p\, d_r\, H). && (154)
\end{aligned}
$$

We define the Yukawa connection in Eqn. (133), where

$$
\begin{aligned}
Y^e_{pr}(\phi_I) &= -H(\phi_I)[Y_e]^\dagger_{pr} + H(\phi_I)\sum_{n=0}^{\infty} C^{(6+2n)}_{\substack{eH\\pr}}\left(\frac{\phi^2}{2}\right)^{n+1}, && (155)\\
Y^d_{pr}(\phi_I) &= -H(\phi_I)[Y_d]^\dagger_{pr} + H(\phi_I)\sum_{n=0}^{\infty} C^{(6+2n)}_{\substack{dH\\pr}}\left(\frac{\phi^2}{2}\right)^{n+1}, && (156)\\
Y^u_{pr}(\phi_I) &= -\tilde{H}(\phi_I)[Y_u]^\dagger_{pr} + \tilde{H}(\phi_I)\sum_{n=0}^{\infty} C^{(6+2n)}_{\substack{uH\\pr}}\left(\frac{\phi^2}{2}\right)^{n+1}. && (157)
\end{aligned}
$$

7.3.4 $(D^\mu\phi)^I\,\bar\psi\,\Gamma_\mu\psi$

The class seven operators in the Warsaw basis, and extended to higher mass dimensions, are of the form

$$\begin{aligned}
\mathcal{Q}^{1,(6+2n)}_{\substack{H\psi\\pr}} &= (H^\dagger H)^n H^\dagger\, i\overleftrightarrow{D}^\mu H\bar\psi_p\gamma_\mu\psi_r,\\
\mathcal{Q}^{3,(6+2n)}_{\substack{H\psi\\pr}} &= (H^\dagger H)^n\, H^\dagger\, i\overleftrightarrow{D}^\mu_a H\bar\psi_p\gamma_\mu\sigma_a\psi_r,\\
\mathcal{Q}^{2,(8+2n)}_{\substack{H\psi\\pr}} &= (H^\dagger H)^n(H^\dagger\sigma_a H)\, H^\dagger\, i\overleftrightarrow{D}^\mu H\bar\psi_p\gamma_\mu\sigma_a\psi_r,\\
\mathcal{Q}^{\epsilon,(8+2n)}_{\substack{H\psi\\pr}} &= \epsilon^a_{bc}\,(H^\dagger H)^n\,(H^\dagger\sigma_c H)\,H^\dagger\, i\overleftrightarrow{D}^\mu_b H\bar\psi_p\gamma_\mu\sigma_a\psi_r.
\end{aligned} \tag{158}$$

where $\overleftrightarrow{D}^\mu_a = (\sigma_a D^\mu - \overleftarrow{D}^\mu\,\sigma_a)$. Connections corresponding to these operators are defined as

$$\begin{aligned}
L^{\psi,pr}_{J,A} &= -(\phi\,\gamma_4)_J\delta_{A4}\sum_{n=0}^\infty C^{1,(6+2n)}_{\substack{H\psi\\pr}}\left(\frac{\phi^2}{2}\right)^n - (\phi\,\gamma_A)_J(1-\delta_{A4})\sum_{n=0}^\infty C^{3,(6+2n)}_{\substack{H\psi_L\\pr}}\left(\frac{\phi^2}{2}\right)^n\\
&+ \frac{1}{2}(\phi\,\gamma_4)_J(1-\delta_{A4})\left(\phi_K\Gamma^K_{A,L}\phi^L\right)\sum_{n=0}^\infty C^{2,(8+2n)}_{\substack{H\psi_L\\pr}}\left(\frac{\phi^2}{2}\right)^n\\
&+ \frac{\epsilon^A_{BC}}{2}(\phi\,\gamma_B)_J\left(\phi_K\Gamma^K_{C,L}\phi^L\right)\sum_{n=0}^\infty C^{\epsilon,(8+2n)}_{\substack{H\psi_L\\pr}}\left(\frac{\phi^2}{2}\right)^n.
\end{aligned} \tag{159}$$

Similarly one can define the right-handed charged current connection

$$L^{ud,pr}_J = \frac{\delta^2\mathcal{L}}{\delta(D^\mu\phi)^J\delta(\bar u_p\gamma_\mu d_r)} = \frac{\tilde\phi_I}{2}(-\Gamma^I_{4,J} + i\gamma^I_{4,J})\sum_{n=0}^\infty C^{(6+2n)}_{\substack{Hud\\pr}}\left(\frac{\phi^2}{2}\right)^n, \tag{160}$$

where $\mathcal{Q}^{(6+2n)}_{\substack{Hud\\pr}} = (H^\dagger H)^n(\tilde H iD^\mu H)\bar u_p\gamma_\mu d_r$.

7.3.5 $\mathcal{W}_A^{\mu\nu}\,\bar{\psi}\sigma_{\mu\nu}\sigma^A\psi$

The class six operators in the Warsaw basis, and extended to higher mass dimensions, are of the form

$$
\begin{aligned}
\mathcal{Q}^{(6+2n)}_{\substack{eW\\pr}} &= (H^\dagger H)^n \bar{\ell}_p \sigma_{\mu\nu}\sigma^A e_r \mathcal{W}_A^{\mu\nu} H(1-\delta_{A4}), &\quad \mathcal{Q}^{(6+2n)}_{\substack{eB\\pr}} &= (H^\dagger H)^n \bar{\ell}_p \sigma_{\mu\nu}\sigma^A e_r \mathcal{W}_A^{\mu\nu} H\,\delta_{A4},\\
\mathcal{Q}^{(6+2n)}_{\substack{dW\\pr}} &= (H^\dagger H)^n \bar{q}_p \sigma_{\mu\nu}\sigma^A d_r \mathcal{W}_A^{\mu\nu} H(1-\delta_{A4}), &\quad \mathcal{Q}^{(6+2n)}_{\substack{dB\\pr}} &= (H^\dagger H)^n \bar{q}_p \sigma_{\mu\nu}\sigma^A d_r \mathcal{W}_A^{\mu\nu} H\,\delta_{A4},\\
\mathcal{Q}^{(6+2n)}_{\substack{uW\\pr}} &= (H^\dagger H)^n \bar{q}_p \sigma_{\mu\nu}\sigma^A u_r \mathcal{W}_A^{\mu\nu} \tilde{H}\,(1-\delta_{A4}), &\quad \mathcal{Q}^{(6+2n)}_{\substack{uB\\pr}} &= (H^\dagger H)^n \bar{q}_p \sigma_{\mu\nu}\sigma^A u_r W_A^{\mu\nu} \tilde{H}\,\delta_{A4},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{Q}^{(8+2n)}_{\substack{eW\\pr}} &= (H^\dagger H)^n\,(H^\dagger\sigma^A H)\,\bar{\ell}_p\sigma_{\mu\nu} e_r \mathcal{W}_A^{\mu\nu} H(1-\delta_{A4}),\\
\mathcal{Q}^{(8+2n)}_{\substack{dW\\pr}} &= (H^\dagger H)^n\,(H^\dagger\sigma^A H)\bar{q}_p\sigma_{\mu\nu} d_r \mathcal{W}_A^{\mu\nu} H(1-\delta_{A4}).
\end{aligned}
\tag{161}
$$

The dipole operator connections are given by

$$
\begin{aligned}
d_A^{e,pr} &= \sum_{n=0}^{\infty}\left(\frac{\phi^2}{2}\right)^n \left[\delta_{A4}\,C^{(6+2n)}_{\substack{eB\\pr}} + \sigma_A(1-\delta_{A4})\,C^{(6+2n)}_{\substack{eW\\pr}} - \left[\phi_K\Gamma^K_{A,L}\phi^L\right](1-\delta_{A4})C^{(8+2n)}_{\substack{eW2\\pr}}\right] H,\\
d_A^{d,pr} &= \sum_{n=0}^{\infty}\left(\frac{\phi^2}{2}\right)^n \left[\delta_{A4}\,C^{(6+2n)}_{\substack{dB\\pr}} + \sigma_A(1-\delta_{A4})\,C^{(6+2n)}_{\substack{dW\\pr}} - \left[\phi_K\Gamma^K_{A,L}\phi^L\right](1-\delta_{A4})C^{(8+2n)}_{\substack{dW2\\pr}}\right] H,\\
d_A^{u,pr} &= \sum_{n=0}^{\infty}\left(\frac{\phi^2}{2}\right)^n \left[\delta_{A4}\,C^{(6+2n)}_{\substack{uB\\pr}} + \sigma_A(1-\delta_{A4})\,C^{(6+2n)}_{\substack{uW\\pr}} - \left[\phi_K\Gamma^K_{A,L}\phi^L\right](1-\delta_{A4})C^{(8+2n)}_{\substack{uW2\\pr}}\right] \tilde{H}.
\end{aligned}
$$

As the Higgs does not carry colour charge, the corresponding connections to gluons are simply

$$
c^{u,pr}(\phi) = \tilde{H}\sum_{n=0}^{\infty} C^{(6+2n)}_{\substack{uG\\pr}}\left(\frac{\phi^2}{2}\right)^n, \quad c^{d,pr}(\phi) = H\sum_{n=0}^{\infty} C^{(6+2n)}_{\substack{dG\\pr}}\left(\frac{\phi^2}{2}\right)^n. \tag{162}
$$

7.3.6 $\mathcal{W}^{\nu}_{A\mu}\,\mathcal{W}^{\rho}_{B\nu}\,\mathcal{W}^{\mu}_{C\rho}$, $\mathcal{G}^{\nu}_{A\mu}\,\mathcal{G}^{\rho}_{B\nu}\,\mathcal{G}^{\mu}_{C\rho}$

The relevant operators are

$$\begin{aligned}
Q_W^{(6+2n)} &= \epsilon_{abc}(H^\dagger H)^n W^a_{\mu\nu} W^{\nu\rho,b} W_\rho{}^{\mu,c}, &(163)\\
Q_{W2}^{(8+2n)} &= \epsilon_{abc}(H^\dagger H)^n (H^\dagger \sigma^a H) W^b_{\mu\nu} W^{\nu\rho,c} B_\rho{}^{\mu}, &(164)\\
Q_G^{(6+2n)} &= f_{\mathcal{ABC}}(H^\dagger H)^n G^{\mathcal{A}}_{\mu\nu} G^{\nu\rho,\mathcal{B}} G_\rho{}^{\mu,\mathcal{C}}. &(165)
\end{aligned}$$

The connection for the electroweak fields is given by

$$f_{ABC}(\phi) = \epsilon_{ABC}\sum_{n=0}^{\infty} C_W^{(6+2n)}\left(\frac{\phi^2}{2}\right)^n - \frac{1}{2}\delta_{A4}\epsilon_{BCD}(\phi_K \Gamma^K_{D,L}\phi^L)\sum_{n=0}^{\infty} C_{W2}^{(8+2n)}\left(\frac{\phi^2}{2}\right)^n. \quad (166)$$

While in the case of gluon fields it is

$$k_{\mathcal{ABC}}(\phi) = f_{\mathcal{ABC}}\sum_{n=0}^{\infty} C_G^{(6+2n)}\left(\frac{\phi^2}{2}\right)^n. \quad (167)$$

For both of these connections, there is also a corresponding CP odd connection of a similar form.

7.3.7 $(D_\mu\phi)^I \sigma_A (D_\nu\phi)^J \mathcal{W}^A_{\mu\nu}$

In the Warsaw basis operators of the form $(D_\mu H)^\dagger \sigma_A (D_\nu H)\mathcal{W}^A_{\mu\nu}$ are removed using field redefinitions. This connection is however populated by operator forms that cannot be removed using field redefinitions, and a derivative reduction algorithm leading to an operator basis, at higher dimensions.

The form of the connection is given by

$$\begin{aligned}
k^A_{IJ}(\phi) &= -\frac{1}{2}\gamma^I_{4,J}\delta_{A4}\sum_{n=0}^{\infty} C_{HDHB}^{(8+2n)}\left(\frac{\phi^2}{2}\right)^{n+1} - \frac{1}{2}\gamma^I_{A,J}(1-\delta_{A4})\sum_{n=0}^{\infty} C_{HDHW}^{(8+2n)}\left(\frac{\phi^2}{2}\right)^{n+1}\\
&- \frac{1}{8}(1-\delta_{A4})\left[\phi_K\Gamma^K_{A,L}\phi^L\right]\left[\phi_M\Gamma^M_{B,L}\phi^N\right]\gamma^I_{B,J}\sum_{n=0}^{\infty} C_{HDHW,3}^{(10+2n)}\left(\frac{\phi^2}{2}\right)^n \qquad (168)\\
&+ \frac{1}{4}\epsilon_{ABC}\left[\phi_K\Gamma^K_{B,L}\phi^L\right]\gamma^I_{C,J}\sum_{n=0}^{\infty} C_{HDHW,2}^{(8+2n)}\left(\frac{\phi^2}{2}\right)^n.
\end{aligned}$$

Here, the operator forms are defined as

$$\begin{aligned} Q_{HDHB}^{(8+2n)} &= i\,(H^\dagger H)^{n+1}(D_\mu H)^\dagger (D_\nu H) B^{\mu\nu}, \\ Q_{HDHW}^{(8+2n)} &= i\delta_{ab}(H^\dagger H)^{n+1}(D_\mu H)^\dagger \sigma^a (D_\nu H) W_b^{\mu\nu}, \\ Q_{HDHW,2}^{(8+2n)} &= i\epsilon_{abc}(H^\dagger H)^n (H^\dagger \sigma^a H)(D_\mu H)^\dagger \sigma^b (D_\nu H) W_c^{\mu\nu}, \\ Q_{HDHW,3}^{(10+2n)} &= i\delta_{ab}\delta_{cd}(H^\dagger H)^n (H^\dagger \sigma^a H)(H^\dagger \sigma^c H)(D_\mu H)^\dagger \sigma^b (D_\nu H) W_d^{\mu\nu}. \end{aligned} \tag{169}$$

7.4 PHENOMENOLOGY

7.4.1 *Higgs mass, and scalar self couplings*

The Higgs mass follows from the potential and is defined as

$$\left.\frac{\delta^2 V(\Phi\cdot\Phi)}{(\delta h)^2}\right|_{\Phi\to 0} = 2(\sqrt{h}^{44})^2 \bar{v}_T^2 \left[\frac{\lambda}{2}\left(3 - \frac{v^2}{\bar{v}_T^2}\right) - \sum_{n=3}^{\infty}\frac{1}{2^n}\begin{pmatrix}2n\\2\end{pmatrix}\tilde{C}_H^{(2n)}\right]. \tag{170}$$

This result follows from $\sqrt{h}^{34}$ vanishing, due to the pseudo-goldstone nature of ϕ^3. Similarly the three-, four-, and m-point ($m \geq 5$) functions are given by

$$\begin{aligned} -\left.\frac{\delta^3 V(\Phi\cdot\Phi)}{(\delta h)^3}\right|_{\Phi\to 0} &= (\sqrt{h}^{44})^3\,\bar{v}_T\left(-6\,\lambda + \sum_{n=3}^{\infty}\frac{1}{2^n}\begin{pmatrix}2n\\3\end{pmatrix}\tilde{C}_H^{(2n)}\right), \\ -\left.\frac{\delta^4 V(\Phi\cdot\Phi)}{(\delta h)^4}\right|_{\Phi\to 0} &= (\sqrt{h}^{44})^4\left(-6\,\lambda + \sum_{n=3}^{\infty}\frac{1}{2^n}\begin{pmatrix}2n\\4\end{pmatrix}\tilde{C}_H^{(2n)}\right), \\ -\left.\frac{\delta^m V(\Phi\cdot\Phi)}{(\delta h)^m}\right|_{\Phi\to 0} &= (\sqrt{h}^{44})^m \sum_{n=3}^{\infty}\frac{1}{2^n}\begin{pmatrix}2n\\m\end{pmatrix}\tilde{C}_H^{(2n)}. \end{aligned} \tag{171}$$

7.4.2 *Fermion masses, and Yukawa couplings*

The fermion masses characterise the intersection of the scalar coordinates with the colour singlet, Hypercharge $1/2$ fermion bilinears that can be constructed out of the SM fermions. The corresponding mass matrices are the expectation value of these field connections

$$[M_\psi]_{rp} = \langle (Y^\psi_{pr})^\dagger \rangle, \tag{172}$$

while the Yukawa interactions are

$$[\mathcal{Y}^\psi]_{rp} = \left.\frac{\delta (Y^\psi_{pr})^\dagger}{\delta h}\right|_{\phi_i \to 0} = \frac{\sqrt{h}^{44}}{\sqrt{2}} \left([Y^\psi]_{rp} - \sum_{n=3}^{\infty} \frac{2n-3}{2^{n-2}} \tilde{C}^{(2n),\star}_{\substack{\psi H \\ pr}} \right). \tag{173}$$

7.4.3 *Geometric definition of gauge couplings*

The covariant derivative acting on the scalar fields is

$$(D^\mu \phi)^I = \left(\partial^\mu \delta^I_J - \frac{1}{2} \mathcal{W}^{A,\mu} \tilde{\gamma}^I_{A,J} \right) \phi^J, \tag{174}$$

with the real generators $\gamma^I_{A,J}$ given in Ref. [2], and also in the Appendix. The tilde superscript on γ indicates that a coupling dependence has been absorbed into the definition of the generator. The bilinear terms in the covariant derivative in coupling and field dependence $g_2 \mathcal{W}^{1,2}_\mu = \bar{g}_2 \mathcal{W}^\pm_\mu$ etc. remain unchanged due to $\mathcal{L}^{(6)}$ transforming to the mass eigenstate canonically normalized terms [85]. This corresponds to the invariant $\alpha_A \mathcal{W}^A = \alpha \cdot \mathcal{W}$ being unchanged by these transformations. This also holds for corresponding transformations of the QCD coupling and field $g_3 G^\mu = \bar{g}_3 \mathcal{G}^\mu$. At higher orders in the SMEFT expansion an invariant of this form is also present by construction. The bar notation is introduced on the couplings to indicate couplings in $\mathcal{L}_{\rm SMEFT}$ that are canonically normalized as in Ref. [85]. Here this notation also indicates the theory is canonically normalized due to terms from $\mathcal{L}^{(d>6)}$ that appear in g^{AB}.

The geometric definition of the canonically normalized mass eigenstate gauge couplings are

$$\begin{aligned}
\bar{g}_2 &= g_2\sqrt{g}^{11} = g_2\sqrt{g}^{22}, &(175)\\
\bar{g}_Z &= \frac{g_2}{c^2_{\bar{\theta}_Z}}\left(c_{\bar{\theta}}\sqrt{g}^{33} - s_{\bar{\theta}}\sqrt{g}^{34}\right) = \frac{g_1}{s^2_{\bar{\theta}_Z}}\left(s_{\bar{\theta}}\sqrt{g}^{44} - c_{\bar{\theta}}\sqrt{g}^{34}\right), &(176)\\
\bar{e} &= g_2\left(s_{\bar{\theta}}\sqrt{g}^{33} + c_{\bar{\theta}}\sqrt{g}^{34}\right) = g_1\left(c_{\bar{\theta}}\sqrt{g}^{44} + s_{\bar{\theta}}\sqrt{g}^{34}\right), &(177)
\end{aligned}$$

with corresponding mass eigenstate generators listed in the Appendix. Here we have used the fact that as $\sqrt{g}^{11} = \sqrt{g}^{22}$ due to $\mathrm{SU}(2)_{\mathrm{L}}$ gauge invariance, it also follows that $\sqrt{g}^{12} = 0$. These definitions are geometric and follow directly from the consistency of the SMEFT description with mass eigenstate fields. These redefinitions hold at all orders in the SMEFT power counting expansion. Similarly, consistency also dictates the field space geometric definitions of the mixing angles

$$s^2_{\theta_Z} = \frac{g_1(\sqrt{g}^{44}s_{\bar{\theta}} - \sqrt{g}^{34}c_{\bar{\theta}})}{g_2(\sqrt{g}^{33}c_{\bar{\theta}} - \sqrt{g}^{34}s_{\bar{\theta}}) + g_1(\sqrt{g}^{44}s_{\bar{\theta}} - \sqrt{g}^{34}c_{\bar{\theta}})}, \qquad (178)$$

$$s^2_{\bar{\theta}} = \frac{(g_1\sqrt{g}^{44} - g_2\sqrt{g}^{34})^2}{g_1^2[(\sqrt{g}^{34})^2 + (\sqrt{g}^{44})^2] + g_2^2[(\sqrt{g}^{33})^2 + (\sqrt{g}^{34})^2] - 2g_1g_2\sqrt{g}^{34}(\sqrt{g}^{33} + \sqrt{g}^{44})}. \qquad (179)$$

The gauge boson masses are also defined in a geometric manner as

$$\bar{m}^2_W = \frac{\bar{g}_2^2}{4}\sqrt{h_{11}}^2\bar{v}^2_T, \qquad \bar{m}^2_Z = \frac{\bar{g}_Z^2}{4}\sqrt{h_{33}}^2\bar{v}^2_T \qquad \bar{m}^2_A = 0. \qquad (180)$$

To utilize these definitions, and map to a particular operator basis, one must expand out to a fixed order in $\bar{v}^2_T/\Lambda^2$. Nevertheless, such all-order definitions are of value. The relations hold in any operator basis to define the Lagrangian parameters incorporating SMEFT corrections in $\bar{v}^2_T/\Lambda^2$ and clarify the role of these Lagrangian terms in the SMEFT expansion.

When the covariant derivative acts on fermion fields, the Pauli matrices $\sigma_{1,2,3}$ for the $\mathrm{SU}(2)_{\mathrm{L}}$ generators[11], and the 2×2 identity matrix $\mathbb{I}$ for the $\mathrm{U}(1)_{\mathrm{Y}}$ generator are used. This is a more convenient generator set for chiral spinors. The covariant derivative acting on the fermion fields ψ, expressed in terms of these quantities, is

$$D_\mu\psi = \left[\partial_\mu + i\bar{g}_3\,\mathcal{G}^\mu_{\mathcal{A}}\,T^{\mathcal{A}} + i\frac{\bar{g}_2}{\sqrt{2}}\left(\mathcal{W}^+T^+ + \mathcal{W}^-T^-\right) + i\bar{g}_Z\left(T_3 - s^2_{\theta_Z}Q_\psi\right)\mathcal{Z}^\mu + i\,Q_\psi\,\bar{e}\,\mathcal{A}^\mu\right]\psi. \qquad (181)$$

[11]Defined in the Appendix.

Here $Q_\psi = \sigma_3/2 + Y_\psi$ and the positive sign convention on the covariant derivative is present and the convention $\sqrt{2}\,\mathcal{W}^\pm = \mathcal{W}^1 \mp i\mathcal{W}^2$ and $\sqrt{2}\,\Phi^\pm = \phi^2 \mp i\phi^1$ is used. Here $T_3 = \sigma_3/2$ and $2T^\pm = \sigma_1 \pm i\,\sigma_2$ and $Y_\psi = \{1/6, 2/3, -1/3, -1/2, -1\}$ for $\psi = \{q_L, u_R, d_R, \ell_L, e_R\}$. Note that the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ generators of the fermion fields do not need to be the same as those for the scalar and vector fields for the parameter redefinitions to consistently modify the covariant derivative parameters in the SMEFT.

The covariant derivative acting on the vector fields is defined as

$$D_\mu \mathcal{W}^A_\nu = \partial_\mu \mathcal{W}^A_\nu - \tilde{\epsilon}^A_{\,BC}\,\mathcal{W}^B_\mu\,\mathcal{W}^C_\nu, \tag{182}$$

where the covariant derivative sign convention is consistent with the definition, and also $\mathcal{W}^A_{\mu\nu} = \partial_\mu \mathcal{W}^A_\nu - \partial_\nu \mathcal{W}^A_\mu - \tilde{\epsilon}^A_{\,BC}\,\mathcal{W}^B_\mu\,\mathcal{W}^C_\nu$.

7.4.4 $\mathcal{W}, \mathcal{Z}$ couplings to $\bar{\psi}\psi$

The mass eigenstate coupling of the $\mathcal{Z}$ and $\mathcal{W}$ to $\bar{\psi}\psi$ are obtained by summing over more than one field space connection. For couplings to fermion fields of the same chirality, the sum is over $L^{\psi,pr}_{J,A}$ and the modified $\bar{\psi}i\not{D}\psi$, that includes the tower of SMEFT corrections in $\mathcal{U}^A_C$. A compact expression for the mass eigenstate connection is

$$-\mathcal{A}^{A,\mu}(\bar{\psi}_p\gamma_\mu\bar{\tau}_A\psi_r)\delta_{pr} + \mathcal{A}^{C,\mu}(\bar{\psi}_p\gamma_\mu\sigma_A\psi_r)\langle L^{\psi,pr}_{I,A}\rangle(-\gamma^I_{C,4})\bar{v}_T, \tag{183}$$

where the fermions are in the weak eigenstate basis. Rotating the fermions to the mass eigenstate basis is straightforward, where the V_{CKM} and U_{PMNS} matrices are introduced as usual. The generators are

$$\bar{\tau}_{1,2} = \frac{\bar{g}_2}{\sqrt{2}}\frac{\sigma_1 \pm i\sigma_2}{2}, \quad \bar{\tau}_3 = \bar{g}_Z(T_3 - s^2_{\theta_Z}Q_\psi), \quad \bar{\tau}_4 = \bar{e}Q_\psi. \tag{184}$$

Expanding out to make the couplings explicit, the Lagrangian effective couplings for $\{\mathcal{Z}, \mathcal{A}, \mathcal{W}^\pm\}$ are

$$\begin{aligned}
\langle \mathcal{Z}|\bar{\psi}_p\psi_r\rangle &= \frac{\bar{g}_Z}{2}\,\bar{\psi}_p\,\not{\epsilon}_{\mathcal{Z}}\left[(2s^2_{\theta_Z}Q_\psi - \sigma_3)\delta_{pr} + \sigma_3\bar{v}_T\langle L^{\psi,pr}_{3,3}\rangle + \bar{v}_T\langle L^{\psi,pr}_{3,4}\rangle\right]\psi_r, && (185)\\
\langle \mathcal{A}|\bar{\psi}_p\psi_r\rangle &= -\bar{e}\,\bar{\psi}_p\,\not{\epsilon}_{\mathcal{A}}\,Q_\psi\,\delta_{pr}\,\psi_r, && (186)\\
\langle \mathcal{W}_\pm|\bar{\psi}_p\psi_r\rangle &= -\frac{\bar{g}_2}{\sqrt{2}}\bar{\psi}_p(\not{\epsilon}_{\mathcal{W}^\pm})\,T^\pm\left[\delta_{pr} - \bar{v}_T\langle L^{\psi,pr}_{1,1}\rangle \pm i\bar{v}_T\langle L^{\psi,pr}_{1,2}\rangle\right]\psi_r. && (187)
\end{aligned}$$

The last expressions simplify due to $\mathrm{SU}(2)_\mathrm{L}$ gauge invariance. Similarly the SMEFT has the right-handed $W^{\pm}$ couplings to (weak eigenstate) quark fields.

$$\langle \mathcal{W}_+^\mu | \bar{u}_p d_r \rangle = \bar{v}_T \langle L_1^{ud,pr} \rangle \frac{\bar{g}_2}{\sqrt{2}} \bar{u}_p \, \not{\epsilon}_{\mathcal{W}^+} d_r, \qquad \langle \mathcal{W}_-^\mu | \bar{d}_r u_p \rangle = \bar{v}_T \langle L_1^{ud,pr} \rangle \frac{\bar{g}_2}{\sqrt{2}} \bar{d}_r \, \not{\epsilon}_{\mathcal{W}^-} u_p.$$

7.4.5 *Dipole connection of $\mathcal{W}, \mathcal{Z}$ to $\bar{\psi}\psi$*

The dipole operators generate a coupling of the $\mathcal{Z}$ and $\mathcal{W}$ to $\bar{\psi}\psi$ that is distinct from the couplings above, due to the fermion fields being of opposite chirality. Interference between the dipole connection and the connections in the previous section requires a mass insertion. The dipole couplings are defined as

$$\begin{aligned}
\langle \mathcal{Z} | \bar{u}_L^p u_R^r \rangle &= -2\bar{g}_Z \bar{u}_L^p \not{p}_Z \not{\epsilon}_Z u_R^p \left(\langle d_3^{u,pr} \rangle \frac{c_{\theta_Z}^2}{g_2} - \langle d_4^{u,pr} \rangle \frac{s_{\theta_Z}^2}{g_1} \right), \\
\langle \mathcal{Z} | \bar{d}_L^p d_R^p \rangle &= -2\bar{g}_Z \bar{d}_L^p \not{p}_Z \not{\epsilon}_Z d_R^p \left(\langle d_3^{d,pr} \rangle \frac{c_{\theta_Z}^2}{g_2} - \langle d_4^{d,pr} \rangle \frac{s_{\theta_Z}^2}{g_1} \right), \\
\langle \mathcal{Z} | \bar{e}_L^p e_R^p \rangle &= -2\bar{g}_Z \bar{e}_L^p \not{p}_Z \not{\epsilon}_Z e_R^p \left(\langle d_3^{e,pr} \rangle \frac{c_{\theta_Z}^2}{g_2} - \langle d_4^{e,pr} \rangle \frac{s_{\theta_Z}^2}{g_1} \right),
\end{aligned} \tag{188}$$

and

$$\begin{aligned}
\langle \mathcal{W}_+ | \bar{q}_p \, d_r \rangle &= -\sqrt{2} \frac{\bar{g}_2}{g_2} \left(\langle d_1^{d,pr} \rangle + i \langle d_2^{d,pr} \rangle \right) \bar{u}_L^p \not{p}_W \not{\epsilon}_W \, d_R^r, \\
\langle \mathcal{W}_- | \bar{q}_p \, u_r \rangle &= -\sqrt{2} \frac{\bar{g}_2}{g_2} \left(\langle d_1^{u,pr} \rangle - i \langle d_2^{u,pr} \rangle \right) \bar{d}_L^p \not{p}_W \not{\epsilon}_W \, u_R^r, \\
\langle \mathcal{W}_+ | \bar{\ell}_p \, e_r \rangle &= -\sqrt{2} \frac{\bar{g}_2}{g_2} \left(\langle d_1^{e,pr} \rangle + i \langle d_2^{e,pr} \rangle \right) \bar{\nu}_L^p \not{p}_W \not{\epsilon}_W \, e_R^r.
\end{aligned} \tag{189}$$

Here the fermions in the dipole connections are in the weak eigenstate basis and a Hermitian conjugate connection also exists in each case. The expectation values of d_A are understood to be the upper (lower) component of an $SU(2)$ doublet for $d^e_{1,2}$, $d^d_{1,2}$, and $d^u_{3,4}$ ($d^u_{1,2}$, $d^e_{3,4}$, and $d^d_{3,4}$).

7.4.6 $h\mathcal{A}\mathcal{A}$, $h\mathcal{A}\mathcal{Z}$ couplings

The effective coupling of h-γ-γ, including the tower of $\bar{v}_T^2/\Lambda^2$ corrections, is given by

$$\langle h|\mathcal{A}(p_1)\mathcal{A}(p_2)\rangle = -\langle hA^{\mu\nu}A_{\mu\nu}\rangle \frac{\sqrt{h}^{44}}{4}\left[\langle\frac{\delta g_{33}(\phi)}{\delta\phi_4}\rangle\frac{\bar{e}^2}{g_2^2} + 2\langle\frac{\delta g_{34}(\phi)}{\delta\phi_4}\rangle\frac{\bar{e}^2}{g_1 g_2} + \langle\frac{\delta g_{44}(\phi)}{\delta\phi_4}\rangle\frac{\bar{e}^2}{g_1^2}\right], \tag{190}$$

where $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\langle hA^{\mu\nu}A_{\mu\nu}\rangle = -4(p_1\cdot p_2\epsilon_1\cdot\epsilon_2 - p_1\cdot\epsilon_2 p_2\cdot\epsilon_1)$ when $\epsilon_1(p_1), \epsilon_2(p_2)$ are the polarization vectors of the external γ's. Similarily the coupling to h-γ-Z is given by

$$\begin{aligned}&\langle h|\mathcal{A}(p_1)\mathcal{Z}(p_2)\rangle \qquad (191)\\ &= -\langle hA^{\mu\nu}Z_{\mu\nu}\rangle\frac{\sqrt{h}^{44}}{2}\bar{e}\,\bar{g}_Z\left[\langle\frac{\delta g_{33}(\phi)}{\delta\phi_4}\rangle\frac{c_{\theta_Z}^2}{g_2^2} + \langle\frac{\delta g_{34}(\phi)}{\delta\phi_4}\rangle\frac{c_{\theta_Z}^2 - s_{\theta_Z}^2}{g_1 g_2} - \langle\frac{\delta g_{44}(\phi)}{\delta\phi_4}\rangle\frac{s_{\theta_Z}^2}{g_1^2}\right],\end{aligned}$$

where $\langle hA^{\mu\nu}Z_{\mu\nu}\rangle = -2(p_1\cdot p_2\epsilon_1\cdot\epsilon_2 - p_1\cdot\epsilon_2 p_2\cdot\epsilon_1)$.

7.4.7 $h\mathcal{Z}\mathcal{Z}$, $h\mathcal{W}\mathcal{W}$ couplings

The off-shell coupling of the Higgs to $\mathcal{Z}\mathcal{Z}$ and $\mathcal{W}\mathcal{W}$ are given by summing over multiple field space connections. One finds

$$\begin{aligned}\langle h|\mathcal{Z}(p_1)\mathcal{Z}(p_2)\rangle = &-\frac{\sqrt{h}^{44}}{4}\bar{g}_Z^2\left[\langle\frac{\delta g_{33}(\phi)}{\delta\phi_4}\rangle\frac{c_{\theta_Z}^4}{g_2^2} - 2\langle\frac{\delta g_{34}(\phi)}{\delta\phi_4}\rangle\frac{c_{\theta_Z}^2\,s_{\theta_Z}^2}{g_1 g_2} + \langle\frac{\delta g_{44}(\phi)}{\delta\phi_4}\rangle\frac{s_{\theta_Z}^4}{g_1^2}\right]\langle h\mathcal{Z}_{\mu\nu}\mathcal{Z}^{\mu\nu}\rangle\\ &+\sqrt{h}^{44}\frac{\bar{g}_Z^2}{2}\left[\langle\frac{\delta h_{33}(\phi)}{\delta\phi_4}\rangle\left(\frac{\bar{v}_T}{2}\right)^2 + \langle h_{33}(\phi)\rangle\frac{\bar{v}_T}{2}\right]\langle h\mathcal{Z}_\mu\mathcal{Z}^\mu\rangle\\ &+\sqrt{h}^{44}\bar{g}_Z^2\bar{v}_T\left[\langle k_{34}^3\rangle\frac{c_{\theta_Z}^2}{g_2} - \langle k_{34}^4\rangle\frac{s_{\theta_Z}^2}{g_1}\right]\langle\partial^\nu h\mathcal{Z}_\mu\mathcal{Z}^{\mu\nu}\rangle,\end{aligned} \tag{192}$$

and

$$\begin{aligned}
\langle h|\mathcal{W}(p_1)\mathcal{W}(p_2)\rangle = &-\frac{\sqrt{h}^{44}}{2}\bar{g}_2^2\left[\langle\frac{\delta g_{11}(\phi)}{\delta\phi_4}\rangle\frac{1}{g_2^2}\right]\langle h\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}\rangle \\
&+\sqrt{h}^{44}\bar{g}_2^2\left[\langle\frac{\delta h_{11}(\phi)}{\delta\phi_4}\rangle\left(\frac{\bar{v}_T}{2}\right)^2+\langle h_{11}(\phi)\rangle\frac{\bar{v}_T}{2}\right]\langle h\mathcal{W}_\mu\mathcal{W}^\mu\rangle \\
&+2\sqrt{h}^{44}\frac{\bar{g}_2^2}{g_2}\frac{\bar{v}_T}{4}\left[i\,\langle k_{42}^1\rangle-\langle k_{42}^2\rangle\right]\langle(\partial^\mu h)(\mathcal{W}_{\mu\nu}^+W_-^\nu+\mathcal{W}_{\mu\nu}^-W_+^\nu)\rangle. \qquad (193)
\end{aligned}$$

As these couplings are off-shell, they are not directly observable.

7.4.8 $\mathcal{Z}\to\bar{\psi}\psi$, $\mathcal{W}\to\bar{\psi}\psi$ *partial widths*

A key contribution to the full width of the $\mathcal{Z},\mathcal{W}$ bosons in the SMEFT are the two-body partial widths that follow from the SMEFT couplings of the $\mathcal{Z},\mathcal{W}$ to fermions of the same chirality. These results can be defined at all orders in the $\bar{v}_T/\Lambda$ expansion as

$$\bar{\Gamma}_{Z\to\bar{\psi}\psi}=\sum_\psi\frac{N_c^\psi}{24\pi}\sqrt{\bar{m}_Z^2}|g_{\rm eff}^{Z,\psi}|^2\left(1-\frac{4\bar{M}_\psi^2}{\bar{m}_Z^2}\right)^{3/2} \qquad (194)$$

where

$$g_{\rm eff}^{Z,\psi}=\frac{\bar{g}_Z}{2}\left[(2s_{\theta_Z}^2\,Q_\psi-\sigma_3)\delta_{pr}+\bar{v}_T\langle L_{3,4}^{\psi,pr}\rangle+\sigma_3\bar{v}_T\langle L_{3,3}^{\psi,pr}\rangle\right] \qquad (195)$$

and $\psi=\{q_L,u_R,d_R,\ell_L,e_R\}$, while $\sigma_3=1$ for u_L,ν_L and $\sigma_3=-1$ for d_L,e_L. Similarly one can define

$$\bar{\Gamma}_{W\to\bar{\psi}\psi}=\sum_\psi\frac{N_c^\psi}{24\pi}\sqrt{\bar{m}_W^2}|g_{\rm eff}^{W,\psi}|^2\left(1-\frac{4\bar{M}_\psi^2}{\bar{m}_W^2}\right)^{3/2} \qquad (196)$$

with

$$\begin{aligned}
g_{\rm eff}^{W,q_L}&=-\frac{\bar{g}_2}{\sqrt{2}}\left[V_{\rm CKM}^{pr}-\bar{v}_T\langle L_{1,1}^{q_L,pr}\rangle\pm i\bar{v}_T\langle L_{1,2}^{q_L,pr}\rangle\right], \\
g_{\rm eff}^{W,\ell_L}&=-\frac{\bar{g}_2}{\sqrt{2}}\left[U_{\rm PMNS}^{pr,\dagger}-\bar{v}_T\langle L_{1,1}^{\ell_L,pr}\rangle\pm i\bar{v}_T\langle L_{1,2}^{\ell_L,pr}\rangle\right],
\end{aligned}$$

where the $V_{\rm CKM}$ and $U_{\rm PMNS}$ matrices are implicitly absorbed into $\langle L_{J,A} \rangle$.

7.4.9 *Higher-point functions*

Field space connections for higher-point functions can also be defined in a straight-forward manner. However, due to the power-counting expansion in p^2/Λ^2 and the less trivial kinematic configurations compared to two- and three-point functions, the number of independent field space connections for e.g. four-point functions is infinite. This can be seen by noting that the field space connections can be defined as variations of the Lagrangian with respect to four fields in the set $\{D_\mu \phi^I, D_{\{\mu,\nu\}}\phi^I, D_{\{\mu,\nu,\rho\}}\phi^I, \cdots\}$, or analogous sets for $W^A_{\mu\nu}$ or the fermion fields. The higher-derivative terms are the symmetric combinations of covariant derivatives.

For two- and three-point functions, we used the integration-by-parts relations in Eqns. (124) and (125). This was crucial to make the number field space connections finite and small for two- and three-point functions. These arguments fail to reduce out higher-derivative field space connections for four-point functions and higher.

The infinite set of field space connections is related to the exponential growth of operators, and poses a challenge for the practitioners of the SMEFT on general grounds.

7.5 CONCLUSIONS

In this paper we have developed the geometric formulation of the SMEFT. This approach allows all orders results in the $\bar{v}_T/\Lambda$ expansion to be determined. We have developed and reported several of these results for Electroweak Precision and Higgs data. All-orders expressions are valuable because one can expand directly from the complete result, and one need not — potentially laboriously — rederive the result at each order in the $\bar{v}_T/\Lambda$. These results make manifest the power, utility and potential of this approach to the SMEFT.

APPENDIX

7.A GENERATOR ALGEBRA

The Pauli matrices σ_a, with $a = \{1, 2, 3\}$, are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{197}$$

The generators in the real representation are defined as

$$\begin{aligned} \gamma^I_{1,J} &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ \gamma^I_{2,J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ \gamma^I_{3,J} &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ \gamma^I_{4,J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \end{aligned} \tag{198}$$

We use tilde superscripts when couplings are absorbed in the definition of generators and structure constants,

$$\begin{aligned} \tilde{\epsilon}^A{}_{BC} &= g_2\, \epsilon^A{}_{BC}, \quad \text{with } \tilde{\epsilon}^1{}_{23} = +g_2, \quad \text{and } \tilde{\epsilon}^4{}_{BC} = 0, \\ \tilde{\gamma}^I_{A,J} &= \begin{cases} g_2\, \gamma^I_{A,J}, & \text{for } A = 1,2,3 \\ g_1 \gamma^I_{A,J}, & \text{for } A = 4. \end{cases} \end{aligned} \tag{199}$$

It is also useful to define a set of matrices

$$\Gamma^I_{A,K} = \gamma^I_{A,J}\,\gamma^J_{4,K} \tag{200}$$

where

$$\Gamma^I_{1,J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \Gamma^I_{2,J} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma^I_{3,J} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma^I_{4,J} = -\mathbb{I}_{4\times 4}. \tag{201}$$

These matrices have the commutation relations $[\gamma_A, \gamma_B] = 2\epsilon^C_{\ AB}\,\gamma_C$, $[\gamma_A, \Gamma_B] = 2\epsilon^C_{\ AB}\,\Gamma_C$, $[\Gamma_A, \Gamma_B] = 2\epsilon^C_{\ AB}\,\gamma_C$. Explicitly the mapping between the generators acting on the field coordinates is $H \to \sigma_a H$ and $\phi^I \to -(\Gamma_a)^I_J\,\phi^J$ for $a = \{1,2,3\}$, while $H \to \mathbb{I}\,H$ maps to the real field basis transformation $\phi^I \to -(\Gamma_4)^I_J\,\phi$. The matrix γ_4 is used for the Hypercharge embedding, and also plays the role of i in the real representation of the scalar field. $\gamma_4^2 = -\mathbb{I}$ while $i^2 = -1$. Note that consistent with this the mapping: $H \to i\,\sigma_a H$ is related to $\phi^I \to -(\gamma_a)^I_J\phi^J$, and $H \to i\,\mathbb{I}H$ maps to $\phi^I \to -(\gamma_4)^I_J\,\phi^J$.

An equivalent to complex conjugation is given in the real field basis by

$$\gamma^I_{\star,J} \quad = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{202}$$

This generator commutes with the remaining generators and $\Gamma_\star^2 = \mathbb{I}$. Note $\tilde{\phi} = \{\phi_3, \phi_4, -\phi_1, -\phi_2\}$, and

$$\begin{aligned} H^\dagger \sigma_a H &= -\frac{1}{2}\phi_I \Gamma^I_{a,J}\phi^J, && (203)\\ H^\dagger \, i\overleftrightarrow{D}^\mu H &= -\phi_I\, \gamma^I_{4,J}(D^\mu\phi)^J = (D^\mu\phi)_I\, \gamma^I_{4,J}\phi^J, && (204)\\ H^\dagger \, i\overleftrightarrow{D}^\mu_a H &= -\phi_I\, \gamma^I_{a,J}(D^\mu\phi)^J = (D^\mu\phi)_I\, \gamma^I_{a,J}\phi^J, && (205)\\ 2\tilde{H}^\dagger D^\mu H &= \tilde{\phi}_I(-\Gamma^I_{4,J} + i\,\gamma^I_{4,J})\,(D^\mu\phi)^J. && (206) \end{aligned}$$

Expressing $\tilde{H}^\dagger D^\mu H$ in terms of ϕ and $(D^\mu\phi)$ requires the introduction of a singluar matrix, so the introduction of $\tilde{\phi}$ is preferred. When considering possible operator forms at higher orders in the SMEFT expansion, it is useful to note that $\phi_I\Gamma^I_{A,J}\phi^J \neq 0$, while $\phi_I\gamma^I_{a,J}\phi^J = \phi_I\gamma^I_{4,J}\phi^J = 0$.

The transformation of the generators to the mass eigenstate basis is given by

$$\gamma^I_{C,J} = \frac{1}{2}\tilde{\gamma}^I_{A,J}\sqrt{g}^{AB}U_{BC}. \qquad (207)$$

Expanding the results gives the mass eigenstate generators explicitly

$$\gamma^I_{1,J} = \frac{\bar{g}_2}{2\sqrt{2}}\left(\gamma^I_{1,J} + i\gamma^I_{2,J}\right), \qquad \gamma^I_{2,J} = \frac{\bar{g}_2}{2\sqrt{2}}\left(\gamma^I_{1,J} - i\gamma^I_{2,J}\right), \qquad (208)$$

$$\gamma^I_{3,J} = \frac{\bar{g}_Z}{2}\left(c^2_{\theta_Z}\gamma^I_{3,J} - s^2_{\theta_Z}\gamma^I_{4,J}\right), \qquad \gamma^I_{4,J} = \frac{\bar{e}}{2}\left(\gamma^I_{3,J} + \gamma^I_{4,J}\right). \qquad (209)$$

7.B PHYSICAL EFFECTS OF $\langle h_{IJ}\rangle$

When h_{IJ} is chosen to have the form

$$h_{IJ} = \left[1 + \frac{\phi^4}{4}(C^{(8)}_{HD} - C^{(8)}_{H,D2})\right]\delta_{IJ} - 2C^{(6)}_{H\Box}\phi_I\phi_J + \frac{\Gamma^I_{A,J}\phi_K\Gamma^K_{A,L}\phi^L}{4}\left(C^{(6)}_{HD} + \phi^2 C^{(8)}_{H,D2}\right). \quad (210)$$

then

$$\begin{aligned} \langle h_{IJ}\rangle = &\left[1 + \frac{\bar{v}_T^4}{4}(C^{(8)}_{HD} - C^{(8)}_{H,D2})\right]\delta_{IJ} - 2C^{(6)}_{H\Box}\bar{v}_T^2\delta_{I,4}\delta_{J,4}\\ &+ \frac{\bar{v}_T^2}{2}\left(\delta_{I,3}\delta_{J,3} + \delta_{I,4}\delta_{J,4}\right)\left(C^{(6)}_{HD} + \bar{v}_T^2 C^{(8)}_{H,D2}\right). \end{aligned} \qquad (211)$$

While if h_{IJ} is chosen to have the form

$$h'_{IJ} = \left[1+\phi^2 C^{(6)}_{H\Box} + \frac{\phi^4}{4}(C^{(8)}_{HD} - C^{(8)}_{H,D2})\right]\delta_{IJ} + \frac{\Gamma^I_{A,J}\phi_K\Gamma^K_{A,L}\phi^L}{4}\left(C^{(6)}_{HD} + \phi^2 C^{(8)}_{H,D2}\right). \quad (212)$$

then

$$\langle h'_{IJ}\rangle = \left[1+\bar{v}_T^2 C^{(6)}_{H\Box} + \frac{\bar{v}_T^4}{4}(C^{(8)}_{HD} - C^{(8)}_{H,D2})\right]\delta_{IJ} + \frac{\bar{v}_T^2}{2}\left(\delta_{I,3}\delta_{J,3} + \delta_{I,4}\delta_{J,4}\right)\left(C^{(6)}_{HD} + \bar{v}_T^2 C^{(8)}_{H,D2}\right). \quad (213)$$

These two cases are related by a field redefinition, expressed through an EOM operator identity at $\mathcal{L}^{(6)}$

$$\begin{aligned} H^\dagger H \Box H^\dagger H &= 2(D^\mu H)^\dagger(D_\mu H) H^\dagger H - 2\lambda v^2 (H^\dagger H) + 4\lambda (H^\dagger H)^3 \\ &+ H^\dagger H\left[\bar{q}^j\, Y_u^\dagger\, (i\sigma_2)_{jk} u + \bar{d}\, Y_d\, q_k + \bar{e}\, Y_e\, l_k + h.c.\right]. \end{aligned} \quad (214)$$

It is instructive to examine how the difference in the $\Delta\langle h_{IJ}\rangle = \langle h'_{IJ}\rangle - \langle h_{IJ}\rangle$ cancels out of quantities closely related to S matrix elements. Explicitly

$$\Delta\langle h_{IJ}\rangle = \tilde{C}^{(6)}_{H\Box}\left[\delta_{IJ} + 2\delta_{I,4}\delta_{J,4}\right]. \quad (215)$$

The modification of $\langle h_{IJ}\rangle$ can be seen to cancel in quantites closely related to S-matrix elements, as expected. For example, one finds

$$\Delta[\mathcal{Y}^\psi]_{rp} = \left[\frac{\Delta\sqrt{h}^{44}}{\sqrt{2}}[Y^\psi]_{rp} - \frac{3}{2\sqrt{2}}[Y^\psi]_{rp}\,\tilde{C}^{(6)}_{H\Box}\right] = 0. \quad (216)$$

for the Yukawa couplings, due to the correlated shift in the $\mathcal{L}^{(6)}$ Yukawa couplings. For the W and Z masses

$$\Delta\bar{m}_W^2 = \frac{\bar{g}_2^2}{4}\left[\Delta\sqrt{h_{11}}^2\,\bar{v}_T^2 + \sqrt{h_{11}}^2\,\Delta\bar{v}_T^2\right] = 0, \quad (217)$$

and

$$\Delta\bar{m}_Z^2 = \frac{\bar{g}_Z^2}{4}\left[\Delta\sqrt{h_{33}}^2\bar{v}_T^2 + \sqrt{h_{33}}^2\Delta\bar{v}_T^2\right] = 0. \tag{218}$$

with $\Delta\bar{v}_T^2 = -\tilde{C}_{H\Box}^{(6)}\bar{v}_T^2$. Conversely, quantites (such as off-shell couplings) not closely related to S-matrix elements, are not expected to demonstrate an equivalence under field redefinitions, or the transformation of $\langle h_{IJ}\rangle$, and this can be observed in several off-shell couplings.

8

BARYON NUMBER, LEPTON NUMBER, AND OPERATOR DIMENSION IN THE SMEFT WITH FLAVOR SYMMETRIES

Using group theory techniques, we investigate the mathematical relationship between baryon number, lepton number, and operator dimension in the Standard Model effective field theory (SMEFT), when flavor symmetries are present. For a large set of flavor symmetries, the lowest-dimensional baryon- or lepton-number violating operators in the SMEFT with flavor symmetry are of mass dimension 9. As a consequence, baryon- and lepton-number violating processes are further suppressed with the introduction of flavor symmetries, e.g., the allowed scale associated with proton decay is typically lowered to 10^5 GeV, which is significantly lower than the GUT scale. To illustrate these features, we discuss Minimal Flavor Violation for the Standard Model augmented by sterile neutrinos.

8.1 INTRODUCTION

The Standard Model (SM) has been spectacularly successful at describing interactions among the known elementary particles. However, it suffers from some shortcomings. An incomplete list of phenomena not fully explained by the SM could include the experimental evidence for the existence of dark matter [127–131], the observational fact of a global baryon asymmetry in the universe [132–137], and evidence of non-zero mass terms for (at least two generations of) the neutrinos from observed neutrino oscillations [138–142].

Given that the SM is incomplete, insofar as it is unable to explain the above mentioned experimental facts, and with no conclusive hint from collider experiments of new particles beyond those in the SM at the TeV scale, the scale of new physics could be much higher. If so, the SM can be extended in an agnostic, model-independent approach to an effective field theory, namely the Standard Model

Effective Field Theory (SMEFT). The SMEFT is constructed by adding a complete set of higher-dimensional operators, which give rise to independent S-matrix elements, built out of the SM field content, respecting the underlying local $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge symmetry [25, 101].

Baryon number and lepton number are accidental symmetries of the SM. Thus, within the confines of the SM, the baryon asymmetry in the universe could only come about through non-perturbative processes, e.g., high-temperature sphaleron processes [143]. Baryon-number violating processes are exponentially suppressed at low temperatures [144]. Whether lepton number is conserved or not is intimately related to neutrino masses, and, in particular, if neutrinos are Majorana fermions, this may point to the existence of an additional scale above the weak scale.

One new feature that occurs in the SMEFT is that baryon number and lepton number can be violated by higher-dimensional operators. Lepton number and baryon number are always integers. For lepton number, this follows directly from the definition, while it is a consequence of hypercharge invariance in the case of baryon number. There is a close connection between lepton number, baryon number, and the mass dimension of operators [145];

$$\frac{\Delta B - \Delta L}{2} \equiv d \mod 2, \tag{219}$$

where ΔB is the baryon number, ΔL is the lepton number and d is the mass dimension of the operator. Some consequences of Eq. (219), among many others, are that $(\Delta B - \Delta L)/2$ must be an integer, and no operator with odd mass dimension can preserve both baryon number and lepton number. See Ref. [145] for more details.

Consider the lowest-dimensional operators that violate baryon number and/or lepton number. The only effective operator at dimension 5 is the famous Weinberg operator [146]. This operator violates lepton number by two units, $|\Delta L| = 2$, and is associated with Majorana mass terms for the neutrinos below the electroweak scale, generated by, for example, the seesaw mechanism [147]. At dimension 6, all operators satisfy $\Delta B - \Delta L = 0$. Many of the effective operators preserve both baryon number and lepton number, $\Delta B = \Delta L = 0$. There are also some operators consisting of four fermion fields of the form $qqql$, where q is a generic quark field and l is a generic lepton field. These operators violate baryon number and lepton number by $\Delta B = \Delta L = \pm 1$. As a consequence, they can mediate proton decay, through the dominant two-body decay $p \to Ml$, where p is the proton and M is a meson. The experimental null result for such decay processes has pushed the allowed scale for baryon-number violating operators with mass dimension 6 to around 10^{15} GeV [148].

Eq. (219) and the results in Ref. [145] apply to the SMEFT, with one generation of fermions. They remain true with the extension to multiple generations, and with the inclusion of flavor symmetries.

A flavor symmetry is a global symmetry among the generations of fermions. We will consider what happens to the general relation between baryon number, lepton number, and operator dimension when flavor symmetries are present.

The paper is organized as follows. We start by discussing the general relation between baryon number, lepton number and operator dimension, following from the local symmetries and field content of the SM. Then, we discuss how this relation gets modified when certain flavor symmetries are present, and discuss in detail an explicit example of Minimal Flavor Violation (MFV) for the SM augmented by sterile neutrinos. We end by discussing implications for proton decay and Majorana neutrino masses.

8.2 ΔB, ΔL AND OPERATOR DIMENSION

The SM field content consists of the fermions $\{L, e^c, Q, u^c, d^c\}$ and the Hermitian conjugate fields, the gauge bosons for the gauge groups $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$, and the Higgs boson H. The fermions are in the respective representations of the gauge groups as

$$\begin{aligned} &Q \sim (\mathbf{3},\mathbf{2})_{1/6}, \quad u^c \sim (\overline{\mathbf{3}},\mathbf{1})_{-2/3}, \quad d^c \sim (\overline{\mathbf{3}},\mathbf{1})_{1/3}, \\ &L \sim (\mathbf{1},\mathbf{2})_{-1/2}, \quad e^c \sim (\mathbf{1},\mathbf{1})_{1}, \quad \nu^c \sim (\mathbf{1},\mathbf{1})_{0}, \end{aligned} \tag{220}$$

where we have also included a sterile neutrino ν^c for generality. All the fermions are the in the $(\mathbf{2},\mathbf{1})$ representation of the Lorentz group $SU(2)_L \otimes SU(2)_R$, with the Hermitian conjugate fields being in the $(\mathbf{1},\mathbf{2})$ representation. The Higgs field is in the representation

$$H \sim (\mathbf{1},\mathbf{2})_{1/2} \tag{221}$$

of the gauge groups and is a Lorentz singlet.

We will denote the number of various fermion fields and their Hermitian conjugate fields in an operator by e.g., N_e and $N_{e^\dagger}$ for the fermion fields e^c and $e^{c\dagger}$ etc. Baryon number and lepton number are defined as

$$\Delta B \equiv \frac{1}{3}\left(N_Q + N_{u^\dagger} + N_{d^\dagger}\right) - \frac{1}{3}\left(N_{Q^\dagger} + N_u + N_d\right), \tag{222}$$

$$\Delta L \equiv \left(N_L + N_{e^\dagger} + N_{\nu^\dagger}\right) - \left(N_{L^\dagger} + N_e + N_\nu\right). \tag{223}$$

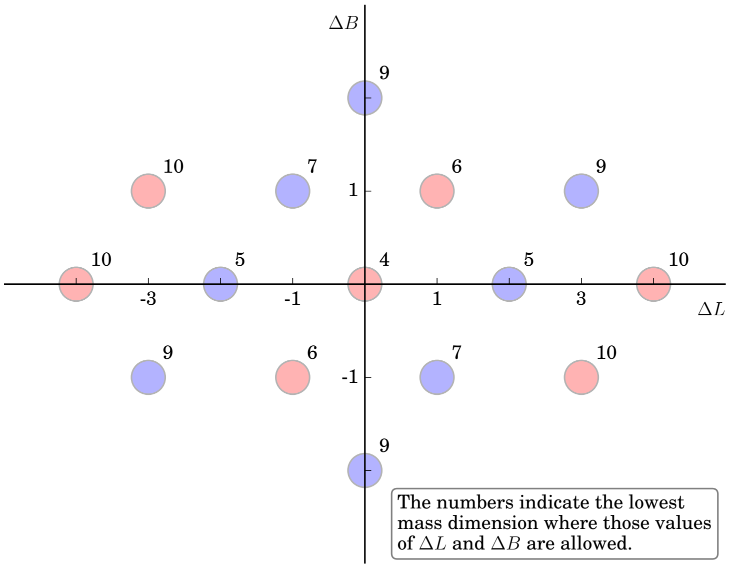


Figure 8.2.1: The $(\Delta L, \Delta B)$ values of operators with different mass dimension d without any flavor symmetry. The numbers indicate the lowest mass dimension where the $(\Delta L, \Delta B)$ value is allowed. Even (odd) dimensional operators are shown in red (blue).

Both baryon number and lepton number are integers, $\Delta B \in \mathbb{Z}$ and $\Delta L \in \mathbb{Z}$. For lepton number, this can be seen directly from Eq. (223). For baryon number, this follows from hypercharge invariance [145]. As the fermion fields have mass dimension $3/2$, it follows directly that the mass dimension of an operator which violates baryon number and/or lepton number is bounded by

$$d_{\text{min}} \geq \frac{9}{2}|\Delta B| + \frac{3}{2}|\Delta L|. \tag{224}$$

By combining Eq. (224) with Eq. (219), we show the allowed $(\Delta L, \Delta B)$ values for various mass dimensions of operators in Fig. 8.2.1, where the sterile neutrinos are excluded.

8.3 FLAVOR SYMMETRY

We consider the allowed baryon number and lepton number values of the higher-dimensional operators when the fermions in the SM transform non-trivially under a continuous flavor group G_F.[1] As the quarks and leptons are charged differently under the $SU(3)_c$ gauge group, we let the flavor group G_F

[1] This requirement excludes some prominent flavor models, see e.g. Refs. [149–151].

be factorized into a direct product of two distinct flavor groups, one for the quarks, G_q, and one for the leptons, G_l,

$$G_F = G_q \otimes G_l. \tag{225}$$

Also, we let all the generations of the fermions be charged democratically, i.e., they form irreducible representations of the flavor group.

The Yukawa terms will in general break the flavor symmetry [152]. The flavor symmetry can formally be restored by promoting the Yukawa couplings to spurion fields, transforming appropriately to form invariants under the flavor group.[2] Spurion fields are auxiliary fields with non-trivial transformation properties, but are not part of the Fock space, i.e., they do not contribute to the S-matrix.

In order to form operators which are singlets under the flavor group, more than one flavor multiplet is required (or none). The constraint that a single quark field cannot appear in an operator is already encoded in the $SU(3)_c$ invariance. The leptons, however, have no such constraint. Thus, imposing a flavor symmetry restricts the leptons to

$$N_L + N_{L^\dagger} + N_e + N_{e^\dagger} \neq 1, \tag{226}$$

where we again have excluded the sterile neutrinos. This basic fact severely restricts the allowed values of baryon and lepton number. By combining Eqs. (224) and (226), the mass dimension of a baryon- or lepton-number violating operator is bounded by

$$d_{\min} \geq \frac{9}{2}|\Delta B| + \frac{3}{2}|\Delta L| + 3\delta_{|\Delta L|,1}, \tag{227}$$

where $\delta_{|\Delta L|,1}$ is the Kronecker delta. From Eqs. (219) and (227), we find that no baryon- or lepton-number violating operator is allowed below dimension 9, except for $|\Delta L| = 2$.

This constraint excludes the dimension-6 operators $qqql$. The quarks transform under the quark flavor group and could form a singlet. However, the lepton field by itself, being in a non-trivial representation of the lepton flavor group, cannot form a singlet under the lepton flavor group by itself. Thus, the operators break the flavor symmetry. Implications for proton decay are discussed later.

The Weinberg operator also is forbidden, with one notable exception. The operator consists of two lepton fields in the same representation, and the requirement that the lepton fields transform under the

[2] In principle, spurion fields can be introduced to render any operator invariant under the flavor group. Then there is no relationship between flavor symmetry and baryon/lepton number. We consider the case where no additional spurions are introduced.

lepton flavor group as a triplet excludes this operator. The only case where two triplets could form a singlet is the case where the symmetry group is $G_l = SU(2)$ (not to be confused with the electroweak gauge group $SU(2)_L$), and the lepton fields are in the adjoint representation [153].

We now want to see which $(\Delta L, \Delta B)$ values are allowed with the inclusion of a flavor symmetry. Let us start with the quarks. By letting them transform as triplets under the flavor group, a necessary (but not sufficient) requirement of an operator being invariant under the quark flavor group and the $SU(3)_c$ gauge group is

$$\frac{1}{3}\left(N_Q + N_{u^\dagger} + N_{d^\dagger}\right) - \frac{1}{3}\left(N_{Q^\dagger} + N_u + N_d\right) \in \mathbb{Z}. \tag{228}$$

This is nothing but the result that baryon number takes integer values.

Consider the case where the leptons are in the fundamental representation of an $SU(3)$ flavor group, and not in the adjoint representation of an $SU(2)$ flavor group. They must form invariants subject to the constraint

$$\frac{1}{3}\left(N_L + N_{e^\dagger}\right) - \frac{1}{3}\left(N_{L^\dagger} + N_e\right) \in \mathbb{Z}. \tag{229}$$

From the definition of lepton number, Eq. (223), and with no sterile neutrinos, we have that

$$\frac{1}{3}\Delta L \in \mathbb{Z}. \tag{230}$$

Lepton number can only be violated in multiples of 3. From this we can immediately see that no Majorana mass term is allowed. We show the allowed baryon number and lepton number values of the operator basis with flavor symmetry in Fig. 8.3.1. The baryon- or lepton-number violating operators with the lowest mass dimension have mass dimension 9. By comparing Figs. 8.2.1 and 8.3.1, we see that the set of allowed baryon number and lepton number values has been severely restricted.

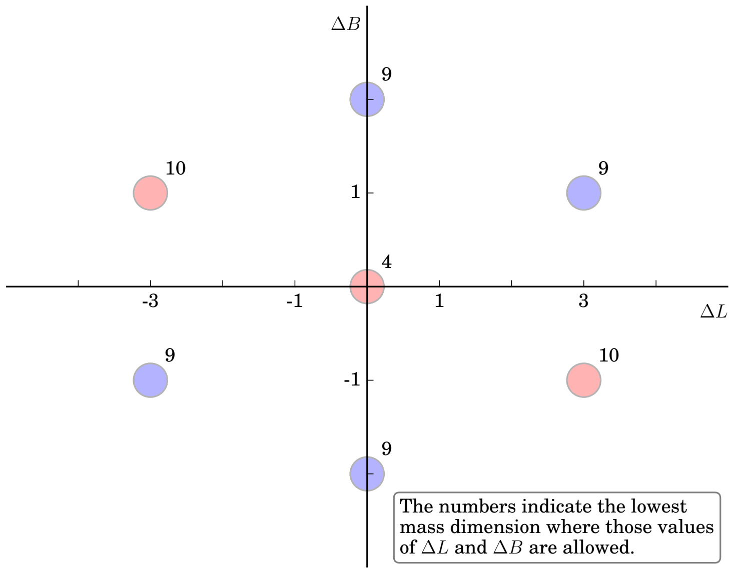


Figure 8.3.1: The $(\Delta L, \Delta B)$ values of operators with different mass dimension d with flavor symmetry, where the leptons are not in the adjoint representation of an $SU(2)$ flavor group. The numbers indicate the lowest mass dimension where the $(\Delta L, \Delta B)$ value is allowed. Even (odd) dimensional operators are shown in red (blue).

8.4 MINIMAL FLAVOR VIOLATION

We now turn to an explicit example of a flavor symmetry, namely Minimal Flavor Violation (MFV) [154, 155]. The flavor group is

$$\begin{aligned} G_F =& SU(3)_Q \otimes SU(3)_u \otimes SU(3)_d \\ & \otimes SU(3)_L \otimes SU(3)_e \otimes SU(3)_\nu, \end{aligned} \tag{231}$$

where we have allowed for the existence of three generations of sterile neutrinos. The fermions are in the fundamental or anti-fundamental representation, as

$$\begin{aligned} Q &\sim (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}), & u^c &\sim (\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}), \\ d^c &\sim (\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1}), & L &\sim (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1}), \\ e^c &\sim (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1}), & \nu^c &\sim (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\bar{\mathbf{3}}). \end{aligned} \tag{232}$$

The Yukawa terms and the Majorana mass term explicitly break the flavor symmetry. In order to preserve the symmetry, the Yukawa couplings and the Majorana mass term are promoted to spurion fields. The requirement of MFV is that all flavor breaking interactions should appear in the same pattern as for the dimension-4 SM. The Yukawa terms and the Majorana mass term take the form

$$\begin{aligned}-\mathcal{L}_{\text{spurion}} =& Y_u Q H u^c + Y_d Q H^* d^c + Y_e L H^* e^c \\ &+ Y_\nu L H \nu^c + \frac{1}{2} M_\nu \nu^c \nu^c + \text{h.c.}\end{aligned} \tag{233}$$

The spurion fields transform as

$$\begin{aligned}&Y_u \sim (\mathbf{\bar{3}}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}), \qquad Y_d \sim (\mathbf{\bar{3}}, \mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}), \\ &Y_e \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{\bar{3}}, \mathbf{3}, \mathbf{1}), \qquad Y_\nu \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{\bar{3}}, \mathbf{1}, \mathbf{3}), \\ &M_\nu \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3} \otimes \mathbf{3}).\end{aligned} \tag{234}$$

For an operator to be invariant under the MFV group, the following relations must hold,

$$\frac{1}{3}\left(N_Q + N_{Y_u^\dagger} + N_{Y_d^\dagger}\right) - \frac{1}{3}\left(N_{Q^\dagger} + N_{Y_u} + N_{Y_d}\right) \in \mathbb{Z}, \tag{235}$$

$$\frac{1}{3}\left(N_{u^\dagger} + N_{Y_u}\right) - \frac{1}{3}\left(N_u + N_{Y_u^\dagger}\right) \in \mathbb{Z}, \tag{236}$$

$$\frac{1}{3}\left(N_{d^\dagger} + N_{Y_d}\right) - \frac{1}{3}\left(N_d + N_{Y_d^\dagger}\right) \in \mathbb{Z}, \tag{237}$$

$$\frac{1}{3}\left(N_L + N_{Y_e^\dagger} + N_{Y_\nu^\dagger}\right) - \frac{1}{3}\left(N_{L^\dagger} + N_{Y_e} + N_{Y_\nu}\right) \in \mathbb{Z}, \tag{238}$$

$$\frac{1}{3}\left(N_{e^\dagger} + N_{Y_e}\right) - \frac{1}{3}\left(N_e + N_{Y_e^\dagger}\right) \in \mathbb{Z}, \tag{239}$$

$$\frac{1}{3}\left(N_{\nu^\dagger} + N_{Y_\nu}\right) - \frac{1}{3}\left(N_\nu + N_{Y_\nu^\dagger}\right) + \frac{2}{3}\left(N_{M_\nu} - N_{M_\nu^\dagger}\right) \in \mathbb{Z}. \tag{240}$$

By summing Eqs. (235)-(237), we find that baryon number must be an integer, which already followed from hypercharge invariance (or invariance under $SU(3)_c$). Adding Eqs. (238)-(240), we have that

$$\frac{1}{3}\left(N_L + N_{e^\dagger} + N_{\nu^\dagger}\right) - \frac{1}{3}\left(N_{L^\dagger} + N_e + N_\nu\right) + \frac{2}{3}\left(N_{M_\nu} - N_{M_\nu^\dagger}\right) \in \mathbb{Z}. \tag{241}$$

Using the definition of lepton number, Eq. (223), we have that

$$\frac{1}{3}\Delta L + \frac{2}{3}\left(N_{M_\nu} - N_{M_\nu^\dagger}\right) \in \mathbb{Z}. \tag{242}$$

In the case where $N_{M_\nu} = N_{M_\nu^\dagger}$, we find agreement with Eq. (230). The difference between Eqs. (230) and (242) is due to the inclusion of the sterile neutrinos, which explicitly break the flavor symmetry via the Majorana mass term and Yukawa interaction. Also, the Majorana mass term violates lepton number by two units.

8.5 PROTON DECAY

The group-theoretical considerations presented above have phenomenological consequences. Experimentally relevant are the implications for the search for proton decay. Cherenkov-radiation detectors like Super-Kamiokande are used to search for certain potential decay channels of the proton [156].

Baryon number and lepton number are accidental symmetries of the SM, and are violated in many grand unified theories [157–164] (see Ref. [165] for a discussion on MFV in grand unified theories and Ref. [166] for a discussion on MFV and baryon-number violating operators). In many of the beyond SM theories, the dominant decay channel of the proton is $p \to e^+\pi^0$ (or $p \to \mu^+\pi^0$), where the proton p decays into a charged anti-lepton and a neutral pion. The neutral pion would decay further to two photons, which could be detected by the Cherenkov-radiation detector. From an effective-field-theory perspective, the two-body decay of the proton could arise from a dimension-6 operator $qqql$ [146, 167]. The null results from the Super-Kamiokande experiment have pushed the scale of new physics associated with the dimension-6 operator $qqql$ to $\Lambda \sim 10^{15}$ GeV. This corresponds to a bound on the partial life-time of the proton of $\tau_{N\to Ml} \geq 10^{34}$ years [148, 168].

However, with the presence of certain flavor symmetries and with no sterile neutrinos, the dimension-6 operators resulting in proton decay are excluded. The baryon-number violating operators with lowest mass dimension have mass dimension 9. Thus, we need to analyze the decay channels resulting from the new leading baryon-number violating operators.

The only dimension-9 operators with $\Delta B = \Delta L/3 = 1$ are $u^{c\dagger}u^{c\dagger}u^{c\dagger}e^{c\dagger}LL$ and $u^{c\dagger}u^{c\dagger}QLLL$. However, neither contributes to three-body nucleon decay at tree-level since both contain heavier quarks, e.g., a charm or top quark [169]. At dimension-10, the operator $d^{c\dagger}d^{c\dagger}d^{c\dagger}L^\dagger L^\dagger L^\dagger H^\dagger$, with $\Delta B = -\Delta L/3 = 1$, could contribute to nucleon decay, through a four-body decay [169]. The lowest-dimensional operators with $\Delta B = \Delta L/3 = 1$ which contributes to three-body proton decay at tree-level are dimension-11 operators, such as $u^{c\dagger}d^{c\dagger}QLLLHH$ [169]. If one posits the existence of flavor symmetries at high scales, then it may be very likely that the dominant contribution to proton decay would come from such higher-dimensional operators. This could result in a three-body decay, with three leptons in the final state. The estimated decay width is

$$\Gamma \sim \frac{1}{512\pi^3}\left(\frac{\langle H\rangle^2}{\Lambda^7}\right)^2 \Lambda_{\text{QCD}}^{11}, \tag{243}$$

where Λ is the scale associated with the intermediate flavor interaction and $\langle H\rangle$ is the vacuum expectation value of the Higgs field. Current experiments would be sensitive to effects from these operators if the scale is $\Lambda \sim 10^5$ GeV.

Some searches for three-lepton decays of the proton have been performed, but not exhaustively across all possible decay channels [170, 171]. Since, on very general grounds, these three-lepton decay channels may be a positive indication of an intermediate scale associated with flavor, further experimental investigation would be valuable.

8.6 MAJORANA MASSES

Next we consider Majorana mass terms. By excluding the sterile neutrinos in the dimension-4 SM, we ask whether higher-dimensional operators resulting in Majorana mass terms for the SM neutrinos are allowed. From the discussion on MFV, by setting $N_{M_\nu} = N_{M_\nu^\dagger} = 0$, we find that lepton number can only be violated in multiples of 3, Eq. (230). This is an explicit example of a general result that, excluding fermions in the adjoint representation of $SU(2)$, no neutrino mass term is allowed. That is, if one wants to generate Majorana neutrino mass terms, and have a certain flavor symmetry, only two options are available. One could either have the leptons be in the adjoint representation of a flavor $SU(2)$ group (see e.g. Ref. [172]), or introduce some explicit violation of the flavor symmetry, e.g., as in Eq. (233).

9

EQUATIONS OF MOTION, SYMMETRY CURRENTS AND EFT BELOW THE ELECTROWEAK SCALE

The low-energy effective field theory is constructed by integrating out Standard Model states with masses proximate to the electroweak scale. We report the equations of motion for this theory, including corrections due to higher dimensional operators up to mass dimension six. We construct the corresponding symmetry currents, and discuss how the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ symmetry, and global symmetries, are manifested when Standard Model states are integrated out. Including contributions from higher dimensional operators to the equations of motion modifies the interpretation of conserved currents. We discuss the corrections to the electromagnetic current as an example, showing how modifications to the equation of motion, and corresponding surface terms, have a direct interpretation in terms of multipole charge distributions that act to source gauge fields.

9.1 INTRODUCTION

Assuming physics beyond the Standard Model (SM) at scales $\Lambda > \bar{v}_T = \sqrt{2\,\langle H^\dagger H\rangle}$, the embedding of the discovered "Higgs-like" scalar into an $\mathrm{SU}(2)_\mathrm{L}$ scalar doublet (H), and the absence of hidden states with couplings to the SM and masses $\lesssim \bar{v}_T$, the SM can be extended into the Standard Model Effective Field Theory (SMEFT). Current LHC results are consistent with interpreting data in this framework, where an infinite tower of higher dimensional operators is added to the SM. The lack of any direct discovery of new physics resonances indicating beyond the SM states with masses $\sim \bar{v}_T$ also supports the assumption that $\bar{v}_T/\Lambda < 1$. As a result, the SMEFT expansion in terms of local contact operators is a useful and predictive formalism to employ studying measurements with characteristic scales $\sim \bar{v}_T$.

The SMEFT has the same field content as the SM, and reduces to the later by taking $\Lambda \to \infty$. As the SM is falsified due to the evidence of neutrino masses from neutrino oscillations, we assume that neutrino masses are generated by the dimension five SMEFT operator.

The LHC is providing large amounts of data measured around the scale $\bar{v}_T$ to search indirectly for physics beyond the SM. These efforts are important to combine with experimental measurements at scales $\ll \bar{v}_T$, where the Low-Energy Effective Field Theory (LEFT) is the appropriate EFT description.[1] The LEFT is built out of the field content of the SM, but as the Higgs, $W^\pm$, Z, and top have masses $m_{W,Z,h,t} \sim \bar{v}_T$, these states are integrated out in sequence. The gauged and linearly realized symmetries of the LEFT are $\mathrm{U}(1)_{\mathrm{em}}$ and $\mathrm{SU}(3)_{\mathrm{c}}$. To perform EFT studies that combine data sets at scales $\sim \bar{v}_T$ and $\ll \bar{v}_T$, one matches the SMEFT onto the LEFT, and uses renormalization group evolution to run between the different scales. For recent results to this end, see Refs. [173, 174].

When considering matching onto the LEFT at sub-leading order, it is usually necessary to take into account corrections to the equations of motion (EOM) that occur due to the local contact operators present in this theory. In Ref. [175], such corrections for the SMEFT were determined. In this paper, we determine these corrections for the LEFT up to operators of mass dimension six.

The pattern of local operator corrections to the EOM encodes a (non-manifest) $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{y}}$ symmetry, when this symmetry is assumed to be present in the UV completion of the LEFT. In this paper, we also construct the corresponding symmetry currents and explain the way that the SM gauge symmetries, and global symmetries such as lepton number, are encoded in the LEFT.

Modifying the equations of motion of SM fields by higher dimensional operators challenges the standard interpretation of conserved currents which is appropriate for, and limited to, renormalizable theories. The generalized currents encode symmetry constraints that still constrain an EFT. We also discuss how higher dimensional operator corrections to the equation of motion have a direct interpretation in terms of multipole charge distributions that act to source the corresponding gauge fields. We use the electromagnetic current as an example of this phenomena, and redefine the source in Gauss's law.

[1] The notation $\bar{v}_T$ indicates that this expectation value includes the effects of possible higher dimensional operators.

9.2 EFFECTIVE FIELD THEORY TAXONOMY

This paper is concerned with the connection between three effective theories: the Standard Model, the SMEFT and the LEFT. Our SM notation is defined in Ref. [175]. The SMEFT extends the SM with higher dimensional operators $\mathcal{Q}_i^{(d)}$ of mass dimension d,

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \mathcal{L}^{(7)} + \dots \tag{244}$$
$$\mathcal{L}^{(d)} = \sum_i \frac{\mathcal{C}_i}{\Lambda^{d-4}} \mathcal{Q}_i^{(d)} \text{ for } d > 4.$$

The operators are suppressed by $d-4$ powers of the cut-off scale Λ and the $\mathcal{C}_i$ are the Wilson coefficients. The $\mathcal{Q}_i^{(d)}$ are constructed out of all of the SM fields and the mass dimension label on the operators is suppressed. We use the non-redundant Warsaw basis [25] for $\mathcal{L}^{(6)}$, which removed some redundancies in the result reported in Ref. [101]. (See also Refs. [90, 176].)

The LEFT is given by

$$L_{\text{LEFT}} = L_{\text{LEFT}}^{\text{SM}} + L^{(5)} + L^{(6)} + L^{(7)} + \dots \tag{245}$$
$$L^{(d)} = \sum_i \frac{C_i}{\bar{v}_T^{d-4}} \mathcal{P}_i^{(d)} \text{ for } d > 4,$$

where

$$\begin{aligned} L_{\text{LEFT}}^{\text{SM}} = & -\frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu} + G^A_{\mu\nu}G^{A\mu\nu}\right] + \frac{\theta_{\text{QCD}}}{32\pi^2} G^A_{\mu\nu}\tilde{G}^{A\mu\nu} \\ & + \frac{\theta_{\text{QED}}}{32\pi^2} F^A_{\mu\nu}\tilde{F}^{A\mu\nu} + \sum_\psi \overline{\psi} i \not{D} \psi + \bar{\nu}_L i \not{D} \nu_L + L^{(3)}_{\text{LEFT}}. \end{aligned} \tag{246}$$

The dual fields are defined with the convention $\widetilde{F}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ with $\epsilon_{0123} = +1$. The dimension four mass terms are

$$-L^{(3)}_{\text{LEFT}} = \sum_\psi \overline{\psi}_{\underset{r}{R}} \left[M_\psi\right]_{rs} \psi_{\underset{s}{L}} + \bar{v}_T \, C_{\underset{rs}{\nu}} \bar{\nu}^c_{\underset{r}{L}} \nu_{\underset{s}{L}} + \text{h.c.} \tag{247}$$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength of $\text{U}(1)_{\text{em}}$. Here $\psi = \{e, u, d\}$ labels the fermion fields. In the chiral basis for the γ_i we use, charge conjugation is given by $C = -i\gamma_2\,\gamma_0$. This C is not to be confused with a Wilson coefficient C_i. As chiral projection and charge conjugation do not commute, we fix notation $\psi_L^c = C\,\bar{\psi}_L^T$. C_ν has been rescaled by $\bar{v}_T$ and has mass dimension zero.

The $\mathcal{P}_i^{(\mathrm{d})}$ are constructed out of the SM fields except the Higgs, $W^\pm$, Z and the chiral top fields $t_{L,R}$. The dimensionfull cut off scale of the operators has been chosen to be $\bar{v}_T$ in the LEFT. The relative couplings required to transform this scale into the mass of a particle integrated out (or a numerical factor in the case of Λ) are absorbed by the Wilson coefficients.

9.3 EQUATIONS OF MOTION

The SM, the SMEFT and the LEFT are all consistent field theories defined by actions

$$S = \int \mathcal{L}(\chi, \partial\chi) d^{4-2\epsilon}x. \tag{248}$$

Each theory contains field variables, here generically denoted χ. The meaning of the field variables, even those with the same notational label, differs in these theories. A field is redefined order by order in an EFT power counting expansion to remove redundancies of description out of the Lagrangian. As a result, the extremum of the action under variations of field configurations,

$$0 = \delta S = \int d^{4-2\epsilon}x \left[\frac{\partial\mathcal{L}}{\partial\chi}\delta\chi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)}\right)\delta\chi\right], \tag{249}$$

is also redefined order by order. The descendent EOM for χ then depend on the local contact operators that are present in the EFT expansion. Asymptotic states can be considered to be free field solutions to the modified EOM. The Δ corrections to the EOM modify matching to sub-leading order onto an EFT [174, 175, 177], and modify the sources of gauge fields. Obviously, one must be careful to include all effects when dealing with higher orders in the power counting expansion.

For the LEFT the gauge fields have the expanded EOM

$$D_\nu F^{\nu\mu} = e\sum_\psi \overline{\psi} Q\gamma^\mu\psi + 4\frac{\theta_{\mathrm{QED}}}{32\pi^2}\partial_\nu\tilde{F}^{\nu\mu} + \sum_d \frac{\Delta_F^{\mu,(d)}}{\bar{v}_T^{d-4}}, \tag{250}$$

$$\begin{aligned}[D_\nu, G^{\nu\mu}]^A = & g_3\sum \overline{\psi}\gamma^\mu T^A\psi + 4\frac{\theta_{\mathrm{QCD}}}{32\pi^2}\left[D_\nu, \tilde{G}^{\nu\mu}\right]^A \\ & + \sum_d \frac{\Delta_G^{A\mu,(d)}}{\bar{v}_T^{d-4}}.\end{aligned} \tag{251}$$

Here we have used the adjoint derivative with definition

$$[D^\alpha, \mathcal{Q}]^A = \partial^\alpha \mathcal{Q}^A - g_3 f^{BCA} G_B^\alpha \mathcal{Q}_C. \tag{252}$$

For the fermions, the EOM take the form

$$i\not{D}\underset{p}{\psi_R} = \left[M_\psi\right]_{pr} \underset{r}{\psi_L} - \sum_{d=5}^{\infty} \frac{\Delta^{(d)}_{\psi_R,p}}{\bar{v}_T^{d-4}}, \tag{253}$$

$$i\not{D}\underset{p}{\nu_L} = - \sum_{d=3}^{\infty} \frac{\Delta^{(d)}_{\nu_L,p}}{\bar{v}_T^{d-4}}, \tag{254}$$

$$i\not{D}\underset{p}{\psi_L} = \left[M_\psi^\dagger\right]_{pr} \underset{r}{\psi_R} - \sum_{d=5}^{\infty} \frac{\Delta^{(d)}_{\psi_L,p}}{\bar{v}_T^{d-4}}. \tag{255}$$

Each $\Delta^{(d)}$ up to $L^{(6)}_{\mathrm{LEFT}}$ is given in the Appendix.

9.4 SYMMETRY CURRENTS

A continuous transformation of a field,

$$\chi(x) \to \chi'(x) = \chi(x) + \alpha \nabla \chi(x), \tag{256}$$

under a deformation $\nabla\chi(x)$, with an associated infinitesimal parameter α, is a symmetry of S if $S \to S'$ is invariant under this transformation, up to the possible generation of a surface term. The EOM defined by the variations of field configurations in the action –δS– is unchanged by this transformation. The EOM are defined with surface terms neglected, and the surface terms themselves are defined to be those of the form

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} \nabla \chi \right), \tag{257}$$

generated by δS. The Lagrangian is then invariant under $S \to S'$, up to a possible total derivative

$$\mathcal{L} \to \mathcal{L} + \alpha \partial_\mu \mathcal{K}^\mu, \tag{258}$$

for some $\mathcal{K}^\mu$. Associated with each symmetry defined in this manner is a conserved current [178]. The definition of the current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \chi\right)} \nabla \chi - \mathcal{K}^\mu. \tag{259}$$

The conservation of the current corresponds to

$$\partial_\mu J^\mu = 0. \tag{260}$$

Due to the presence of an EFT power counting expansion, it is interesting to examine how symmetry currents are defined when non-renormalizable operators are included, and how these currents encode symmetry constraints.

9.5 BASIS DEPENDENCE

The symmetry currents are basis dependent in an EFT, but still meaningful. They receive corrections due to the local contact operators in a particular basis through the modification of the EOM. The basis dependence of the symmetry currents can be made clear by considering a space-time symmetry. For an infinitesimal translation of this form

$$\begin{aligned} x^\mu &\to x^\mu - a^\mu, \\ \chi(x) &\to \chi(x+a) = \chi(x) + a^\mu \partial_\mu \chi(x), \\ \mathcal{L} &\to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu \left(\delta^\mu_\nu \mathcal{L}\right), \end{aligned} \tag{261}$$

up to $\mathcal{O}(a^2)$. Comparing to Eqn. (258) identifies $\mathcal{K}$. Four separately conserved currents result, identified as the stress-energy tensor, given by

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \chi\right)} \partial_\nu \chi - \mathcal{L} \delta^\mu_\nu. \tag{262}$$

The χ become basis dependent when redundant operators are removed from the EFT, leading to the chosen basis of operators for $\mathcal{L}$. The T^μ_ν constructed from $\{\chi, \mathcal{L}\}$ is also basis dependent as a result at the same order in the power counting. This should be unsurprising, as the currents are auxiliary operators, and sources and the related Green's functions are not invariant under field redefinitions. For more detailed discussion on this point, see Refs. [22, 179]. This basis dependence is similar to scheme dependence. It vanishes in relationships between a set of physical measured quantities (i.e. S-matrix elements constructed with an LSZ procedure) defined via the same stress-energy tensor. Symmetry constraints between S-matrix elements are basis independent, even though the symmetry current itself carries basis dependence.

9.6 NON-LINEAR GLOBAL SYMMETRIES

The effect of non-linear representations of the symmetries of the LEFT is straightforward in some cases. As a simple example, consider transforming the charged lepton fields as

$$\underset{p}{e_L} \to e^{i\alpha} \underset{p}{e_L}, \qquad \underset{p}{e_R} \to e^{i\alpha} \underset{p}{e_R}, \tag{263}$$

by some global phase α. By inspection of the LEFT operator basis, the $\Delta L = 0$ operators all respect this transformation, except $\mathcal{O}_{\nu edu}$. The charged lepton current is

$$\underset{rr}{J^\mu_e} \equiv \underset{rr}{J^\mu_{e,L}} + \underset{rr}{J^\mu_{e,R}} \equiv \underset{r}{\bar{e}_L} \gamma^\mu \underset{r}{e_L} + \underset{r}{\bar{e}_R} \gamma^\mu \underset{r}{e_R} + \ldots \tag{264}$$

The kinetic terms are taken to a flavour diagonal form

$$\underset{r}{\psi_{L/R}} \to U(\psi, L/R)_{rs} \underset{s}{\psi'_{L/R}}, \tag{265}$$

using the flavour space rotation matrix U. In the remainder of the paper, the prime superscript is suppressed. J_e descends from the kinetic terms and is also flavour diagonal after these rotations. J_e can receive contributions from higher dimensional operators in a basis, as indicated by the ellipsis in the above expression. The LEFT basis of Refs. [173, 174] removes derivative operators systematically so there are no contributions of this form due to the $L^{(6)}_{\rm LEFT}$ defined in these works. The divergence of the current including the EOM corrections $\Delta^{(6)}$ is

$$\begin{aligned} i\partial_\mu \underset{rr}{J^\mu_{e,L}} &= i \left(\partial_\mu \underset{r}{\bar{e}_L}\right) \gamma^\mu \underset{r}{e_L} + i \underset{p}{\bar{e}_L} \gamma^\mu \left(\partial_\mu \underset{p}{e_L}\right) \\ &= \left(-\underset{p}{\bar{e}_R} \underset{pr}{M_e} + \Delta^{(6)}_{\underset{r}{\bar{e}_L}}\right) \underset{r}{e_L} + \underset{p}{\bar{e}_L} \left(\underset{rp}{M_e} \underset{r}{e_R} - \Delta^{(6)}_{\underset{p}{e_L}}\right), \end{aligned} \tag{266}$$

and similarly for $i\partial_\mu J^\mu_{e,R}$. The mass terms are invariant under Eqn. (263) and cancel when the expressions are summed. We split the EOM correction and J into lepton number conserving and violating parts, $\Delta^{(6)} = \Delta^{(6,L)} + \Delta^{(6,\cancel{L})}$ and $J^\mu = J^{(L)\mu} + J^{(\cancel{L})\mu}$. First, consider the lepton number conserving part of Eqn. (266). A significant degree of cancellation occurs in the resulting expression. The only Wilson coefficient remaining corresponds to $\mathcal{P}_{\nu edu}$, an operator which is not individually invariant under the charged lepton field transformation. The explicit expression is

$$\begin{aligned}
&\Delta^{(6,L)}_{\underset{r}{\bar{e}_L}} \underset{r}{e_L} - \underset{p}{\bar{e}_L} \Delta^{(6,L)}_{\underset{p}{e_L}} + \Delta^{(6,L)}_{\underset{r}{\bar{e}_R}} \underset{r}{e_R} - \underset{p}{\bar{e}_R} \Delta^{(6,L)}_{\underset{p}{e_R}} = \left(\underset{prst}{C^{V,LL}_{\nu edu}} \underset{pr}{J^\mu_{\nu e,L}} \underset{st}{J^\nu_{du,L}} - \underset{rpts}{C^{V,LL*}_{\nu edu}} \underset{pr}{J^\mu_{e\nu,L}} \underset{st}{J^\nu_{ud,L}} \right) \eta_{\mu\nu} \\
&+ \left(\underset{prst}{C^{V,LR}_{\nu edu}} \underset{pr}{J^\mu_{\nu e,L}} \underset{st}{J^\nu_{du,R}} - \underset{rpts}{C^{V,LR*}_{\nu edu}} \underset{pr}{J^\mu_{e\nu,L}} \underset{st}{J^\nu_{ud,R}} \right) \eta_{\mu\nu} + \underset{prst}{C^{S,RR}_{\nu edu}} \underset{pr}{S_{\nu e,L}} \underset{st}{S_{du,L}} - \underset{rpts}{C^{S,RR*}_{\nu edu}} \underset{pr}{S_{e\nu,R}} \underset{st}{S_{ud,R}} \qquad (267) \\
&+ \left(\underset{prst}{C^{T,RR}_{\nu edu}} \underset{pr}{\mathcal{T}^{\mu\nu}_{\nu e,L}} \underset{st}{\mathcal{T}^{\alpha\beta}_{du,L}} - \underset{rpts}{C^{T,RR*}_{\nu edu}} \underset{pr}{\mathcal{T}^{\mu\nu}_{e\nu,R}} \underset{st}{\mathcal{T}^{\alpha\beta}_{ud,R}} \right) \eta_{\alpha\mu}\eta_{\beta\nu} + \underset{prst}{C^{S,RL}_{\nu edu}} \underset{pr}{S_{\nu e,L}} \underset{st}{S_{du,R}} - \underset{rpts}{C^{S,RL*}_{\nu edu}} \underset{pr}{S_{e\nu,R}} \underset{st}{S_{ud,L}}.
\end{aligned}$$

Similarly, we can define a neutrino current

$$\underset{rr}{J^\mu_\nu} \equiv \underset{r}{\bar{\nu}_L} \gamma^\mu \underset{r}{\nu_L} + \ldots \qquad (268)$$

The lepton number conserving contributions to the divergence of the neutrino current are such that

$$\Delta^{(6,L)}_{\underset{r}{\bar{e}_L}} \underset{r}{e_L} - \underset{p}{\bar{e}_L} \Delta^{(6,L)}_{\underset{p}{e_L}} + \Delta^{(6,L)}_{\underset{r}{\bar{e}_R}} \underset{r}{e_R} - \underset{p}{\bar{e}_R} \Delta^{(6,L)}_{\underset{p}{e_R}} + \Delta^{(6,L)}_{\underset{r}{\bar{\nu}_L}} \underset{r}{\nu_L} - \underset{p}{\bar{\nu}_L} \Delta^{(6,L)}_{\underset{p}{\nu_L}} = 0. \qquad (269)$$

This is as expected, and provides a cross check of the EOM corrections in the Appendix. The total lepton field current is conserved by the subset of $\Delta L = 0$ operators leading to

$$\partial_\mu J^{(L)\mu}_\ell = 0, \qquad (270)$$

where ℓ is the $\mathrm{SU}(2)_\mathrm{L}$ doublet field. Considering the transformation of only part of the lepton multiplet under a phase change also illustrates how a symmetry can be present in a Lagrangian, but non-linearly realized. The symmetry constraint is only made manifest when all terms corresponding to the linear symmetry multiplet are simultaneously included in the constructed symmetry current. This re-emphasizes the requirement to use a consistent LEFT with all operators retained when studying the data. Doing so ensures that the LEFT represents a consistent IR limit. Conversely, dropping operators can forbid non-linear realizations in the LEFT of UV symmetries, which can block a consistent IR limit of some UV completions being defined. For this reason (see also Ref. [180]), experimental studies of constraints on higher dimensional operators done “one at a time" can result in misleading conclusions.

9.7 LINEAR REPRESENTATIONS OF GLOBAL SYMMETRIES

Operator dimension in the SMEFT is even (odd) if $(\Delta B - \Delta L)/2$ is even (odd) [145, 181]. Here ΔB and ΔL are respectively the baryon and lepton number violation of the operator considered. In $\mathcal{L}_{\rm SM} + \mathcal{L}^{(6)}$, $B - L$ is an accidentally conserved quantity consistent with this constraint.

In the LEFT, incomplete $\rm SU(2)_L$ SM multiplets are used to construct operators, and operators are not constructed to respect hypercharge. The relationship between operator dimension and global lepton and baryon number in the LEFT is different than in the SMEFT as a result. When considering arbitrary Wilson coefficients in the LEFT, the classes of $\Delta L = 2$, $\Delta B = -\Delta L = 1$, and $\Delta L = 4$ defined in Refs. [173, 174] are present. These ψ^4 operators are not present in $\mathcal{L}^{(6)}$ in the SMEFT, and these operators violate $B - L$.

The SMEFT relationship between operator dimension and these global symmetries is projected onto the LEFT operator basis when the matching result of Ref. [173] is imposed. The corresponding $\mathcal{L}_{\rm SMEFT}$ - $L_{\rm LEFT}$ matchings that violate $B - L$ vanish exactly.

9.8 HYPERCHARGE

The fermion hypercharge current of the SM is

$$J^\mu_{\Psi \mathrm{y},\mathrm{SM}} = \sum_{\substack{\Psi = e_R, u_R, d_R,\\ \ell_L, q_L}} \mathrm{y}_\Psi \overline{\Psi} \gamma^\mu \Psi, \tag{271}$$

where $\mathrm{y}_\Psi = \{-1, 2/3, -1/3, -1/2, 1/6\}$. This current is manifestly not conserved in the LEFT

$$\partial_\mu J^\mu_{\Psi \mathrm{y},\mathrm{SM}} \neq 0. \tag{272}$$

In the LEFT, a hypercharge current can be defined as

$$J^\mu_{\Upsilon \mathrm{y}} = \sum_{\Upsilon} \mathrm{y}_\Upsilon \overline{\Upsilon} \gamma^\mu \Upsilon. \tag{273}$$

Here $\mathsf{Y} = \{\psi_R, \psi_L, \nu_L\}$ and the hypercharges are assigned as in the SM. Part of the non-conservation of the current stems from the fermion mass terms. In addition, the Δ corrections also lead to the current not being conserved when the Wilson coefficients in the LEFT take arbitrary values. When the matching conditions on the Wilson coefficients to the SMEFT are imposed [173], many of the EOM corrections generating a non-vanishing $\partial_\mu J^\mu_{\mathsf{Yy}}$ are removed. The terms that remain are

$$
\begin{aligned}
& i\partial_\mu J^\mu_{\mathsf{Yy}}\Big|_{\text{match}} \\
&= \frac{(\mathsf{y}_{u_R} - \mathsf{y}_{d_R})}{\bar{v}_T^2}\left(\underset{prst}{C^{V,LR}_{vedu}} \underset{pr}{J^\mu_{ve,L}} \underset{st}{J^\nu_{du,R}} - \underset{rpts}{C^{V,LR*}_{vedu}} \underset{pr}{J^\mu_{ev,L}} \underset{st}{J^\nu_{ud,R}} + \underset{prst}{C^{V1,LR}_{uddu}} \underset{pr}{J^\mu_{ud,L}} \underset{st}{J^\nu_{du,R}} - \underset{rpts}{C^{V1,LR*}_{uddu}} \underset{pr}{J^\mu_{du,L}} \underset{st}{J^\nu_{du,R}}\right)\eta_{\mu\nu} \\
&+ (\mathsf{y}_{\psi_R} - \mathsf{y}_{\psi_L})\left(\underset{p}{\overline{\psi}_R}\left[M_\psi\right]_{pr} \underset{r}{\psi_L} - \underset{p}{\overline{\psi}_L}\left[M_\psi^\dagger\right]_{pr} \underset{r}{\psi_R}\right) + 2\,\bar{v}_T \mathsf{y}_{\nu_L}\left[\underset{p}{\overline{\nu}_L} \underset{pr}{C^\star_\nu} \underset{r}{\nu^c_L} - \underset{p}{\overline{\nu}^c_L} \underset{pr}{C^T_\nu} \underset{r}{\nu_L}\right] \\
&+ \frac{(\mathsf{y}_{\psi_L} - \mathsf{y}_{\psi_R})}{\bar{v}_T}\sum_{\psi\neq e}\left[\underset{p}{\overline{\psi}_R}\sigma^{\alpha\beta}T^A \underset{r}{\psi_L} \underset{rp}{C^\star_{\psi G}} - \underset{p}{\overline{\psi}_L}\sigma^{\alpha\beta}T^A \underset{r}{\psi_R} \underset{rp}{C^T_{\psi G}}\right] G^{\alpha\beta}_A \\
&+ \frac{(\mathsf{y}_{\psi_L} - \mathsf{y}_{\psi_R})}{\bar{v}_T}\left[\underset{p}{\overline{\psi}_R}\sigma^{\alpha\beta} \underset{r}{\psi_L} \underset{rp}{C^\star_{\psi\gamma}} - \underset{p}{\overline{\psi}_L}\sigma^{\alpha\beta} \underset{r}{\psi_R} \underset{rp}{C^T_{\psi\gamma}}\right] F_{\alpha\beta} + \ldots
\end{aligned}
\tag{274}
$$

Here we have used the fact that in whole or in part, composite operators forms with $\sum_\Psi \mathsf{y}_\Psi = 0$ have a corresponding vanishing contribution to the current. This condition being fulfilled also provides a cross check of the $\Delta^{(3-6)}$ EOM corrections in the Appendix.

Enforcing matching constraints to the SMEFT is insufficient to make the hypercharge current manifest. The reason is that SM states are integrated out in constructing the LEFT, that carry this quantum number. Consider the definition of the full hypercharge current

$$
J^\mu_{\mathsf{y},\text{full}} = J^\mu_{\Psi\mathsf{y}} + \mathsf{y}_H H^\dagger\, i\overleftrightarrow{D}^\mu H, \tag{275}
$$

where $\mathsf{y}_H = 1/2$ for the Higgs field. Here, and later, we are using the Hermitian derivative defined by

$$
\begin{aligned}
O^\dagger\, i\overleftrightarrow{D}_\mu O &= iO^\dagger(D_\mu O) - i(D_\mu O)^\dagger O, \qquad (276)\\
O^\dagger i\overleftrightarrow{D}^I_\mu O &= iO^\dagger\tau^I\left(D_\mu O\right) - i\left(D_\mu O\right)^\dagger\tau^I O, \qquad (277)
\end{aligned}
$$

for a field O. To make hypercharge conservation manifest, we include the transformation properties of the masses associated with states integrated out that depended on $\langle H^\dagger H\rangle$. This can be done in a spurion analysis. Rescaled Wilson coefficients and mass terms are promoted to spurion fields with tilde superscripts

$$\begin{aligned}
\tilde{C}^{V,LR}_{\substack{vedu\\prst}} &= \bar{v}_T C^{V,LR}_{\substack{vedu\\prst}}, & \tilde{C}^{V1,LR}_{\substack{uddu\\prst}} &= \bar{v}_T C^{V1,LR}_{\substack{uddu\\prst}}, \\
\tilde{C}_{\substack{\psi\\pr}} &= M_{\substack{\psi\\pr}}, & \tilde{C}_{\substack{\nu\\pr}} &= 2\,\bar{v}_T\, C^{\star}_{\substack{\nu\\pr}}, \\
\tilde{C}_{\substack{\psi\gamma\\pr}} &= \bar{v}_T\, C^{\star}_{\substack{\psi\gamma\\rp}}, & \tilde{C}_{\substack{\psi G\\pr}} &= \bar{v}_T\, C^{\star}_{\substack{\psi G\\rp}}.
\end{aligned}$$

These spurion fields have the hypercharge assignments

$$\begin{aligned}
\mathsf{y}_{\tilde{C}} &= \mathsf{y}_{d_R} - \mathsf{y}_{u_R} & &\text{for } \tilde{C}^{V,LR}_{vedu},\ \tilde{C}^{V1,LR}_{uddu}, \\
\mathsf{y}_{\tilde{C}} &= -\mathsf{y}_{\nu} & &\text{for } \tilde{C}_{\nu}, \\
\mathsf{y}_{\tilde{C}} &= \mathsf{y}_{\psi_R} - \mathsf{y}_{\psi_L} & &\text{for } \tilde{C}_{\psi\gamma}, \tilde{C}_{\psi G}, \\
\mathsf{y}_{\tilde{C}} &= \mathsf{y}_{\psi_L} - \mathsf{y}_{\psi_R} & &\text{for } \tilde{C}_{\psi}.
\end{aligned}$$

As the spurions are charged under hypercharge, we need to include them in the current in the LEFT

$$J^{\mu}_{\mathsf{y},\mathrm{LEFT}} = J^{\mu}_{\mathrm{Y}\mathsf{y}} + J^{\mu}_{\mathsf{y},\mathrm{S}}, \tag{278}$$

where

$$J^{\mu}_{\mathsf{y},\mathrm{S}} = \sum_{\tilde{C}} \mathsf{y}_{\tilde{C}}\, \tilde{C}^{\dagger}\, i \overleftrightarrow{D}^{\mu} \tilde{C}. \tag{279}$$

Here the flavour indices are suppressed. When promoting the Wilson coefficients to fields, we need to include kinetic terms,

$$\mathcal{L}^{kin}_{\mathrm{S}} = \sum_{\tilde{C}} \left(D^{\mu}\tilde{C}\right)^{\dagger} \left(D_{\mu}\tilde{C}\right). \tag{280}$$

The EOM for the spurion fields are $D^2\tilde{C} = \delta L_{\mathrm{LEFT}}/\delta\tilde{C}^{\star}$. Including these contributions, the hypercharge current is conserved: $i\partial_{\mu} J^{\mu}_{\mathsf{y},\mathrm{LEFT}} = 0$.

This provides a cross check of the EOM corrections in the Appendix and the results in Ref. [173, 175].

9.9 $\mathrm{SU}(2)_\mathrm{L}$ CURRENT

The $\mathrm{SU}(2)_\mathrm{L}$ current in the SMEFT is defined as

$$J^I_\mu = \frac{1}{2}\bar{q}\tau^I\gamma_\mu q + \frac{1}{2}\bar{l}\tau^I\gamma_\mu l + \frac{1}{2}H^\dagger i\overleftrightarrow{D}^I_\mu H. \tag{281}$$

This definition of the current fixes the embedding of the LEFT states into $\mathrm{SU}(2)_\mathrm{L}$ doublets. Here τ^I are the $\mathrm{SU}(2)_\mathrm{L}$ generators (Pauli matrices) with normalization $[\tau^I, \tau^J] = 2\,i\,\epsilon_{IJK}\tau^K$ for $I = \{1, 2, 3\}$. The fields q and l are left-handed quark and lepton $\mathrm{SU}(2)_\mathrm{L}$ doublets, which are absent in the LEFT as linear multiplets. To examine the $\mathrm{SU}(2)_\mathrm{L}$ current we need to combine terms in the LEFT into reconstructed $\mathrm{SU}(2)_\mathrm{L}$ multiplets and also introduce spurions to account for the transformation properties of $\bar{v}_T$. We illustrate the constraints of the $\mathrm{SU}(2)_\mathrm{L}$ current with an operator from the class $(\overline{L}R)X + \text{h.c.}$ as an example,

$$\underset{pr}{C_{e\gamma}}\underset{p}{\bar{e}_L}\sigma^{\mu\nu}\underset{r}{e_R}F_{\mu\nu} + \text{h.c.} \to \underset{p}{\bar{l}^i_L}\sigma^{\mu\nu}\underset{r}{e_R}F_{\mu\nu}\underset{pr}{C^i_{e\gamma}} + \text{h.c.} \tag{282}$$

where

$$\underset{pr}{C^i_{e\gamma}} = \begin{pmatrix} 0 \\ C_{e\gamma} \end{pmatrix}_{pr} \qquad \text{and} \qquad \underset{p}{l^i_L} = \begin{pmatrix} \underset{p}{\nu_L} \\ \underset{p}{e_L} \end{pmatrix}. \tag{283}$$

We have promoted the Wilson coefficient to a $\mathrm{SU}(2)_{\mathrm{L}}$ doublet field, and collected the left-handed leptons into a doublet. Analogous promotions can be made for all the operators in this class. The relevant terms in the equations of motion are

$$\bar{v}_T\, i\not{D} \underset{p}{l^i_L} = -\sigma^{\mu\nu} \underset{r}{e_R} F_{\mu\nu} \underset{pr}{C^i_{e\gamma}} + \ldots \tag{284}$$

$$\bar{v}_T\, i\not{D} \underset{p}{\bar{l}^i_L} = +\underset{pr}{C^{i*}_{e\gamma}} F_{\mu\nu} \underset{r}{\bar{e}_R} \sigma^{\mu\nu} + \ldots \tag{285}$$

$$\bar{v}_T^2\, D^2 \underset{pr}{\tilde{C}^i_{e\gamma}} = F_{\mu\nu} \underset{r}{\bar{e}_R} \sigma^{\mu\nu} \underset{p}{l^i_L}, \tag{286}$$

$$\bar{v}_T^2\, D^2 \underset{pr}{\tilde{C}^{i*}_{e\gamma}} = \underset{p}{\bar{l}^i_L} \sigma^{\mu\nu} \underset{r}{e_R} F_{\mu\nu}. \tag{287}$$

The covariant derivative of J^μ_l gives

$$\begin{aligned} i\left[D_\mu, J^\mu_l\right]^I &\equiv i\partial_\mu \left(\frac{1}{2} \underset{p}{\bar{l}_L} \tau^I \gamma^\mu \underset{p}{l_L}\right) - g_2 \epsilon^{JKI} W_{\mu,J} J^\mu_{l,K} \\ &= \frac{1}{2}\left(iD_\mu \underset{p}{\bar{l}_L}\right) \tau^I \gamma^\mu \underset{p}{l_L} + \frac{1}{2} \underset{p}{\bar{l}_L} \tau^I \gamma^\mu \left(iD_\mu \underset{p}{l_L}\right) \\ &= \frac{\underset{pr}{C^*_{e\gamma}}}{2\bar{v}_T} F_{\mu\nu} \underset{r}{\bar{e}_R} \sigma^{\mu\nu} \tau^I \underset{p}{l_L} - \frac{\underset{pr}{C_{e\gamma}}}{2\bar{v}_T} \underset{p}{\bar{l}_L} \tau^I \sigma^{\mu\nu} \underset{r}{e_R} F_{\mu\nu} + \ldots \end{aligned} \tag{288}$$

To recover a conserved current, we perform a spurion analysis, similar to the one for hypercharge. We have the EOM for the spurion $C_{e\gamma}$, in Eqns. (286) and (287). The spurion current is

$$J^{\mu,I}_S = \frac{1}{2} \tilde{C}^\dagger_{e\gamma} i \overleftrightarrow{D}^{\mu,I} \tilde{C}_{e\gamma}, \tag{289}$$

with flavour indices suppressed. The covariant divergence of the spurion current is

$$\begin{aligned} i\left[D_\mu, J^\mu_S\right]^I &= -\frac{1}{2}\left[\underset{pr}{\tilde{C}_{e\gamma}}{}^* \tau^I D^2 \underset{pr}{\tilde{C}_{e\gamma}} - D^2 \underset{pr}{\tilde{C}_{e\gamma}}{}^* \tau^I \underset{pr}{\tilde{C}_{e\gamma}}\right] \\ &= -\frac{1}{2\bar{v}_T}\left[\underset{pr}{C_{e\gamma}}{}^* \tau^I F_{\mu\nu} \underset{r}{\bar{e}_R} \sigma^{\mu\nu} \underset{p}{l_L} - \underset{p}{\bar{l}_L} \sigma^{\mu\nu} \underset{r}{e_R} F_{\mu\nu} \tau^I \underset{pr}{C_{e\gamma}}\right]. \end{aligned} \tag{290}$$

Combining Eqns. (288) and (290), the new current is covariantly conserved for the chosen operator from the class $(\bar{L}R)X$,

$$i\left[D_\mu, J^\mu\right]^I \equiv i\left[D_\mu, (J^\mu_l + J^\mu_S)\right]^I = 0. \tag{291}$$

The generalization to include quarks is straightforward.

For ψ^4 operators a similar spurion analysis that also includes the promotion of all of the fermion fields into the corresponding $\mathrm{SU}(2)_\mathrm{L}$ fermion multiplet of the SM is done. The procedure is straightforward. When imposing the $\mathcal{L}_{\mathrm{SMEFT}}$ - L_{LEFT} matching and performing this spurion analysis, the $\mathrm{SU}(2)_\mathrm{L}$ current is conserved.

9.10 CONSTRAINTS DUE TO NON-MANIFEST CURRENTS

The $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{U}(1)_\mathrm{Y}$ currents are not conserved in the LEFT when the Wilson coefficients of this theory are treated as free parameters. Furthermore, the implication of these currents in the LEFT is distinct than in the SM or the SMEFT, as there is no manifest field corresponding to these currents when they are conserved. There is no direct construction of a Ward identity using a propagating gauge field as a result.

The conserved currents do constrain the LEFT by fixing relationships between otherwise free parameters of the theory. Matrix elements of the currents can be directly constructed, as they are composed of the fields of the LEFT. Constructing such a matrix element from the generalized Heisenberg current field, with a set of initial and final states denoted $\Psi_{i,f}$, and taking a total derivative gives

$$\partial^\mu \int d^4x e^{ip\cdot x} \langle \Psi_f | J_\mu(x) | \Psi_i \rangle = 0. \tag{292}$$

A series of relationships between the Wilson coefficients then follows

$$\sum_n \partial_\mu \langle \Psi_f | \mathcal{P}_n | \Psi_i \rangle^\mu (p) C_n = 0. \tag{293}$$

Formally, the measured S-matrix elements must be constructed using an LSZ reduction formula. The constraints that follow for the Wilson coefficients are trivially satisfied only if the Wilson coefficients are already fixed by a UV matching preserving the corresponding symmetry.

9.11 $\mathrm{U}(1)_{\mathrm{em}}$ AND THE LEFT MULTIPOLE EXPANSION

The classical limit of $L^{d\leq 4}_{\mathrm{LEFT}}$ reproduces the well known physics of Maxwell's equations, and in particular Gauss's law [182] (see also Ref. [183]). Gauss's law relates the time component of the electromagnetic current $J^\mu = \overline{\psi}_e \gamma^\mu \psi_e$ to

$$J^0 = \frac{\nabla \cdot \mathbf{E}}{-e_{phys}}. \tag{294}$$

Here $e_{phys} = 1.6021766208(98) \times 10^{-19}\,C$, is the electron charge in the usual SI units [184]. In the LEFT, the electromagnetic current is also expected to be conserved

$$\partial_\mu J^\mu = 0, \tag{295}$$

without any of the subtleties of the previous sections as the $\mathcal{P}_i$ are constructed to manifestly preserve $\mathrm{U}(1)_{\mathrm{em}}$.

The $\mathrm{U}(1)_{\mathrm{em}}$ current is subject to its own set of subtleties. First, the naive understanding that J^μ being conserved directly leads to its non-renormalization requires some refinement. This issue was comprehensively addressed for QED in Ref. [185], neglecting higher dimensional operators and considering a one electron state and the corresponding electron number current. Here we review the result of Refs. [185, 186] and then directly extend this result into the LEFT.

The definition of the electromagnetic current is affected by the presence of a surface term $\partial^\nu F_{\nu\mu}$ [185, 187] introducing a renormalization of this current. We define the $L^{d\leq 4}_{\mathrm{LEFT}}$ CP conserving QED Lagrangian as

$$\begin{aligned}\mathcal{L} &= \left(\overline{\psi}\,[i\gamma\cdot(\partial + eqA) - m]\,\psi\right)^{(0)} - \frac{1}{4}(F^{(0)}_{\mu\nu})^2 - \frac{1}{2\xi}(\partial\cdot A)^2 \\ &= Z_2\overline{\psi}\left[i\gamma\cdot(\partial + eq\mu^\epsilon A) - m^{(0)}\right]\psi - \frac{Z_3}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial\cdot A)^2,\end{aligned}$$

where all $()^{(0)}$ superscripted quantities are bare parameters. μ is introduced so that the renormalized coupling is dimensionless and q is the charge of ψ. We restrict our attention to $\psi = \psi_e$ for simplicity (even in loops) in the discussion below. Renormalized quantities are introduced above with a suppressed r superscript, $d = 4 - 2\epsilon$ and we use $\overline{\rm MS}$ as a subtraction scheme so that

$$A_\mu^{(0)} = \sqrt{Z_3} A_\mu^{(r)}, \quad \psi^{(0)} = \sqrt{Z_2} \psi^{(r)},$$
$$m_e^{(0)} = Z_m m_e^{(r)}, \quad e^{(0)} = Z_e \mu^\epsilon \, e^{(r)}.$$

Here $m_e^2 = [M_e]_{11}[M_e^\dagger]_{11}$. The renormalization constants in QED are given by

$$Z_3 = 1 - \frac{e^2 S_\epsilon}{12\,\pi^2\,\epsilon}, \quad Z_2 = 1 - \frac{e^2 S_\epsilon}{16\,\pi^2\,\epsilon},$$
$$Z_m = 1 - \frac{3e^2 S_\epsilon}{16\,\pi^2\,\epsilon},$$

and $Z_e = 1/\sqrt{Z_3}$ at one loop. Here $S_\epsilon = (4\pi e^{-\gamma_E})^\epsilon$, following the notation of Ref. [185]. Hereon we define our subtractions in $\overline{\rm MS}$ and suppress the corresponding constant terms, setting $S_\epsilon = 1$.

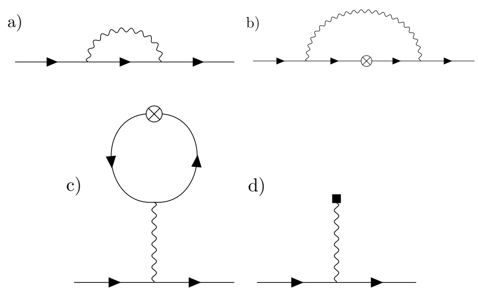


Figure 9.11.1: Figures a)-d) represent the renormalization of the electromagentic current in $L^{d\leq 4}_{\rm LEFT}$. The later two diagrams illustrate a penguin diagram c) leading to a surface counter-term in d).

Standard arguments advanced to establish the non-renormalization of J^ν are concerned with Fig. 9.11.1 a)-b). Fig. 9.11.1 a) represents wavefunction renormalization, while the insertion of the current is represented with a circled cross in Fig. 9.11.1 b)-c). The divergence and finite terms of diagrams a)-b) cancel at zero momentum transfer for an on-shell state. For a one electron state, the Noether current corresponds to the electron number current, which we label as J_N^ν consistent with Ref. [185]. The usual textbook argument then concludes

$$\mu \frac{d}{d\mu} J_N^\nu = 0, \tag{296}$$

consistent with the current being conserved. However, the penguin diagram in Fig. 9.11.1 c) is divergent. This divergence is cancelled by a counter-term of the form $\partial^\nu F_{\nu\mu}$ shown in Fig. 9.11.1 d).

This operator has a four divergence that identically vanishes (i.e. corresponds to a surface term). The EOM of the A^μ field is given by

$$0 = \frac{\delta S_{\rm LEFT}}{\delta A_\mu(x)} = e\mu^\epsilon J^\mu_N + Z_3 \partial_\nu F^{\nu\mu} + \frac{1}{\xi}\partial^\mu \partial \cdot A. \tag{297}$$

The EOM relates terms in a non-intuitive fashion when an extremum of the action is taken. J^μ_N receives a multiplicative renormalization generated from the nonzero anomalous dimension of the second term as a result. The current can be subsequently redefined to remove this effect and cancel the running, as shown in Ref. [185].

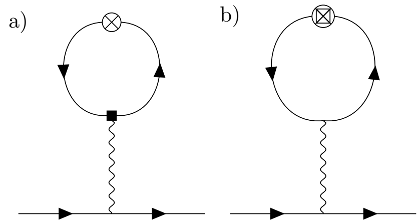


Figure 9.11.2: Figure a) shows the insertion of a dipole operator in a one loop diagram (black square) with the $d \leq 4$ LEFT electromagnetic current as a circled cross. Figure b) shows the insertion of a dipole contribution to the current as a circled cross box.

Fig. 9.11.2 shows the need to further refine this argument in the presence of higher dimensional operators. These diagrams are the direct analogy to the arguments of Ref. [185] leading to a redefinition of the current due to the mixing of the dipole operator with the counter-term multiplying $\partial_\nu F^{\nu\mu}$. Inserting the dipole operator (indicated with a black box) with the electromagnetic current, indicated with a circled cross in Fig. 9.11.2a), gives mixing proportional to M_e/v_T. Including the effect of the dipole operator in the current insertion is indicated by a "circled cross box" in Fig. 9.11.2b). Calculating the diagrams directly for an electron in the loop gives a contribution to the photon two point function of the form

$$-\Delta Z_3 = -\frac{e q_e}{2\,\pi^2\,\epsilon}\left(\underset{11}{C_{e\gamma}}[M_e]_{11} + \underset{11}{C^\star_{e\gamma}}[M_e^\dagger]_{11}\right). \tag{298}$$

This divergence is cancelled by a counter-term [174] which leads to a modification of Z_3 of the form ΔZ_3. (The generalization to other charged leptons in the loop is trivial.) This is as expected as a corresponding divergence is present in the LEFT in Fig. 9.11.3 a)-b) and the external photon does not play a role that distinguishes the divergence obtained once the current is redefined. We have calculated the diagrams in Fig. 9.11.3 and agree with the corresponding dipole operator results in Ref. [174].

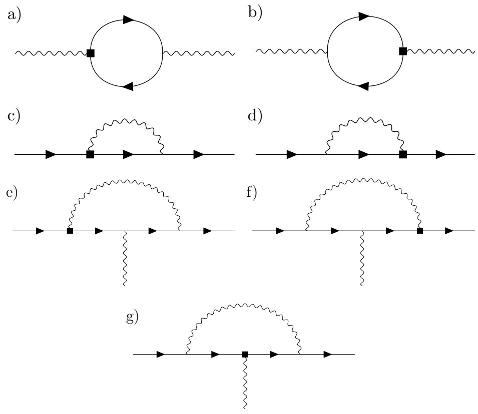


Figure 9.11.3: One loop diagrams generating the divergences of the LEFT that are removed with the renormalization reported in Ref. [174].

The interpretation of this mixing effect is subtle in the LEFT. Varying $S_{\rm LEFT}$ with respect to $A_\mu^{(r)}$ gives

$$\begin{aligned} 0 &= \frac{\delta S}{\delta A_\mu} = e\mu^\epsilon J_N^\mu + Z_3 \partial_\nu F^{\nu\mu} + \frac{1}{\xi}\partial^\mu \partial \cdot A \\ &+ \sqrt{Z_3} Z_2 \partial_\nu \left(Z_C C_{e\gamma} \left(\bar{e}_L \sigma^{\nu\mu} e_R \right) + Z_C^\star C_{e\gamma}^* \left(\bar{e}_R \sigma^{\nu\mu} e_L \right) \right) + \ldots \end{aligned} \tag{299}$$

The tree level contributions to the electron number operator of terms $\propto C_{e\gamma}, C_{e\gamma}^*$ vanish at infinity by Stokes' theorem.[2] We define a $\overline{\rm MS}$-renormalized current

$$\begin{aligned} & J_{\overline{\rm MS}}^\mu = J_N^\mu + \frac{Z_3 - 1}{e\mu^\epsilon} \partial_\nu F^{\nu\mu} \\ & + \frac{\sqrt{Z_3} Z_2}{e\mu^\epsilon} \partial_\nu \left(Z_C C_{e\gamma} \left(\bar{e}_L \sigma^{\nu\mu} e_R \right) + Z_C^* C_{e\gamma}^* \left(\bar{e}_R \sigma^{\nu\mu} e_L \right) \right) + \ldots \end{aligned} \tag{300}$$

The $\overline{\rm MS}$-renormalized current expressed in terms of bare quantities is

$$\begin{aligned} & J_{\overline{\rm MS}}^\mu = \overline{\psi}^{(0)} \gamma^\mu \psi^{(0)} + \frac{1 - Z_3^{-1}}{e_0} \partial_\nu F^{(0),\nu\mu} \\ & + \frac{1}{e^{(0)}} \partial_\nu \left(C_{e\gamma}^{(0)} \left(\bar{e}_L^{(0)} \sigma^{\nu\mu} e_R^{(0)} \right) + C_{e\gamma}^{*,(0)} \left(\bar{e}_R^{(0)} \sigma^{\nu\mu} e_L^{(0)} \right) \right) + \ldots \end{aligned} \tag{301}$$

The renormalization group flow of the current is

$$\mu \frac{d}{d\mu} J_{\overline{\rm MS}}^\mu = 2\gamma_A \frac{1}{e_0 Z_3} \partial_\nu F^{(0),\nu\mu}. \tag{302}$$

[2] We thank Mark Wise for discussions on this point.

The $\overline{\text{MS}}$-renormalized current depends on the renormalization scale μ as in the SM case. The LEFT dipole corrections to the current fall off at infinity when considering the electron number operator. They also vanish from Eqn. (302) as separate terms, which is consistent with this fact. The dipole operators mix into $\partial^\nu F_{\mu\nu}$ proportional to $M_e/\bar{v}_T$, a correction with a natural interpretation of an electron dipole charge distribution in the LEFT. In order to extract a conserved electron number which is independent of the renormalization scale, we redefine the current, including the effect of dipole operators in direct analogy to Ref. [185]. We define

$$J^\mu_{\text{LEFTphys}} = J^\mu_{\overline{\text{MS}}} - \frac{\Pi(0)}{e\mu^\epsilon}\partial_\nu F^{\nu\mu}, \tag{303}$$

where $\Pi(0)$ is the electron vacuum polarization in the LEFT, including the effects of operators of mass dimension greater than four. The electron vacuum polarization is still defined in the standard manner, and the current is modified by a redefinition at $q^2 = 0$.

It follows that

$$F^{\nu\mu}_{\text{LEFT,phys}} = [1+\Pi(0)]^{1/2}\, F^{\nu\mu}, \tag{304}$$

$$e_{\text{LEFT,phys}} = [1+\Pi(0)]^{-1/2}\, e\mu^\epsilon. \tag{305}$$

In the $\overline{\text{MS}}$ scheme

$$\begin{aligned}\Pi(0) &= -\frac{e^2}{12\pi^2}\log\frac{m_e^2}{\mu^2} \\ &+ \frac{e\,q_e}{2\,\pi^2}\left(C_{\underset{11}{e\gamma}}[M_e]_{11} + C^\star_{\underset{11}{e\gamma}}[M_e^\dagger]_{11}\right)\log\frac{m_e^2}{\mu^2} + \ldots\end{aligned} \tag{306}$$

From these results one directly defines the time component of the physical current as

$$j^0_{\text{LEFTphys}} = \frac{\nabla\cdot\mathbf{E}_{\text{LEFT,phys}}}{-e_{\text{LEFT,phys}}}, \tag{307}$$

which is the appropriate generalization of the source in Gauss's law into the LEFT. This is a numerically small effect, as the electromagentic dipole operator is constrained [188].

To summarize, higher dimensional operators in the LEFT act to change the relationship between the Lagrangian parameter e and experimental measurements in a manner that corresponds to dipole operators being present in the LEFT. This occurs through a modified source term in Gauss's law that reflects the presence of a multipole expansion in the EFT. The tree level dipole contributions to the electron number operator vanish at infinity by Stokes' theorem, but quantum effects necessitates a redefinition of the current.

9.12 CONCLUSIONS

We have reported the equations of motion for the LEFT including corrections due to dimension six operators. These results are listed in the Appendix. These corrections lead directly to questions on the meaning of conserved currents in the LEFT. We have examined how the conserved currents of the LEFT encode symmetry constraints that are manifest or non-linearly realized. We have also generalized and embedded the source in Gauss's law into the LEFT, incorporating the effects of electrically charged particles having dipole operator sources.

APPENDIX

Our operator label notation for the LEFT is largely consistent with Refs. [173, 174]. We use a different sign convention on the charge conjugation operator, here $C = -i\gamma^2\,\gamma^0$, where as in Refs. [173, 174] C is defined with opposite sign. We further introduce the current notation

$$
\begin{aligned}
&S_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}_{\underset{s}{1},L/R}\psi_{\underset{t}{2},R/L}\right), \quad S^A_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}_{\underset{s}{1},L/R}T^A\psi_{\underset{t}{2},R/L}\right), \\
&\mathcal{S}^{a,b}_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}^a_{\underset{s}{1},L/R}\psi^b_{\underset{t}{2},R/L}\right),
\end{aligned} \tag{308}
$$

$$
J^\alpha_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}_{\underset{s}{1},L/R}\gamma^\alpha\psi_{\underset{t}{2},L/R}\right), \quad J^{\alpha,A}_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}_{\underset{s}{1},L/R}\gamma^\alpha T^A\psi_{\underset{t}{2},L/R}\right), \tag{309}
$$

$$
\mathcal{T}^{\alpha\beta}_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}_{\underset{s}{1},L/R}\sigma^{\alpha\beta}\psi_{\underset{t}{2},R/L}\right), \quad \mathcal{T}^{\alpha\beta,A}_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}_{\underset{s}{1},L/R}\sigma^{\alpha\beta}T^A\psi_{\underset{t}{2},R/L}\right), \tag{310}
$$

where $J^\alpha_{\underset{st}{\psi\psi},R} \equiv J^\alpha_{\underset{st}{\psi},R}$ etc. We also define the currents where one of the fields is charge conjugated

$$
\begin{aligned}
\tilde{S}_{\underset{st}{\psi_1\psi_2},L/R} &= \left(\overline{\psi}_{\underset{s}{1},L/R}\psi_{\underset{t}{2},L/R}\right), \quad \tilde{\mathcal{S}}^{a,b}_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}^a_{\underset{s}{1},L/R}\psi^b_{\underset{t}{2},L/R}\right), \\
\tilde{J}^\alpha_{\underset{st}{\psi_1\psi_2},L/R} &= \left(\overline{\psi}_{\underset{s}{1},L/R}\gamma^\alpha\psi_{\underset{t}{2},R/L}\right), \quad \tilde{\mathcal{T}}^{\alpha\beta}_{\underset{st}{\psi_1\psi_2},L/R} = \left(\overline{\psi}_{\underset{s}{1},L/R}\sigma^{\alpha\beta}\psi_{\underset{t}{2},L/R}\right),
\end{aligned} \tag{311}
$$

and similarly for $\tilde{J}^{\alpha,A}_{\underset{st}{\psi_1\psi_2},R}$ etc.

Using these notational conventions, the EOM for the gauge fields from $L^{(5,6)}$ are

$$\frac{\Delta_F^{\mu,(5)}}{2} = \sum_{\psi\neq\nu} C_{\substack{\psi\gamma \\ pr}} \partial_\nu \mathcal{T}^{\nu\mu}_{\substack{\psi,L \\ pr}} + C_{\substack{\nu\gamma \\ pr}} \partial_\nu \tilde{\mathcal{T}}^{\nu\mu}_{\substack{\nu^c\nu,L \\ pr}} + \text{h.c.}, \tag{312}$$

$$\frac{\Delta_G^{A\mu,(5)}}{2} = \sum C_{\substack{\psi G \\ pr}} \left[D_\nu, \overline{\psi}_{\substack{L \\ p}} \sigma^{\nu\mu} T \, \psi_{\substack{R \\ r}} \right]^A + \text{h.c.}, \tag{313}$$

$$\begin{aligned}
\frac{\Delta_G^{A\mu,(6)}}{2} =& 3C_G f^{ABC} \left[\partial^\alpha \left(G_B^{\mu\beta} G_{C\beta\alpha} \right) + g f_{DEC} G^D_{\alpha\beta} G^{E\beta\mu} G^\alpha_B \right] \\
&+ C_{\tilde{G}} f^{ABC} \left[\partial^\alpha \left(G_C^{\mu\beta} \tilde{G}_{B\beta\alpha} \right) + g f_{DEB} \tilde{G}^D_{\beta\alpha} G^{E\mu\beta} G^\alpha_C \right] \\
&+ C_{\tilde{G}} f^{ABC} \left[\partial^\alpha \left(\tilde{G}_B^{\mu\beta} G_{C\alpha\beta} \right) + g f_{DEB} G^D_{\beta\alpha} \tilde{G}^{E\mu\beta} G^\alpha_C \right] \\
&+ \frac{C_{\tilde{G}}}{2} f^{ABC} \epsilon_{\alpha\beta}{}^{\gamma\mu} \left[\partial_\gamma \left(\tilde{G}_B^{\alpha\delta} G_{C\delta\beta} \right) + g f_{DEB} \tilde{G}^E_{\delta\gamma} G^{D\alpha\delta} G^\beta_C \right].
\end{aligned} \tag{314}$$

The $\Delta L, \Delta B = 0$, contributions to the EOM from $L^{(5,6)}$ are as follows

$$\Delta^{(5,B,L)}_{e_R,p} = C^*_{\substack{e\gamma \\ rp}} \sigma^{\alpha\beta} e_{\substack{L \\ r}} F_{\alpha\beta}, \tag{315}$$

$$\Delta^{(5,B,L)}_{u_R,p} = C^*_{\substack{u\gamma \\ rp}} \sigma^{\alpha\beta} u_{\substack{L \\ r}} F_{\alpha\beta} + C^*_{\substack{uG \\ rp}} \sigma^{\alpha\beta} T^A u_{\substack{L \\ r}} G^A_{\alpha\beta}, \tag{316}$$

$$\Delta^{(5,B,L)}_{d_R,p} = C^*_{\substack{d\gamma \\ rp}} \sigma^{\alpha\beta} d_{\substack{L \\ r}} F_{\alpha\beta} + C^*_{\substack{dG \\ rp}} \sigma^{\alpha\beta} T^A d_{\substack{L \\ r}} G^A_{\alpha\beta}, \tag{317}$$

$$\Delta^{(5,B,L)}_{\nu_L,p} = 0, \tag{318}$$

$$\Delta^{(5,B,L)}_{e_L,p} = C_{\substack{e\gamma \\ pr}} \sigma^{\alpha\beta} e_{\substack{R \\ r}} F_{\alpha\beta}, \tag{319}$$

$$\Delta^{(5,B,L)}_{u_L,p} = C_{\substack{u\gamma \\ pr}} \sigma^{\alpha\beta} u_{\substack{R \\ r}} F_{\alpha\beta} + C_{\substack{uG \\ pr}} \sigma^{\alpha\beta} T^A u_{\substack{R \\ r}} G^A_{\alpha\beta}, \tag{320}$$

$$\Delta^{(5,B,L)}_{d_L,p} = C_{\substack{d\gamma \\ pr}} \sigma^{\alpha\beta} d_{\substack{R \\ r}} F_{\alpha\beta} + C_{\substack{dG \\ pr}} \sigma^{\alpha\beta} T^A d_{\substack{R \\ r}} G^A_{\alpha\beta}. \tag{321}$$

$$\begin{aligned}
&\Delta^{(6,B,L)}_{e_R,p} = \\
&\gamma_\alpha e_{\substack{R \\ r}} \left(2\, C^{V,RR}_{\substack{ee \\ prst}} J^\alpha_{\substack{e,R \\ st}} + C^{V,RR}_{\substack{eu \\ prst}} J^\alpha_{\substack{u,R \\ st}} + C^{V,RR}_{\substack{ed \\ prst}} J^\alpha_{\substack{d,R \\ st}} + \sum_{\psi,\nu} C^{V,LR}_{\substack{\psi e \\ stpr}} J^\alpha_{\substack{\psi,L \\ st}} \right) \\
&+ e_{\substack{L \\ r}} \left(2\, C^{S,RR*}_{\substack{ee \\ rpts}} S_{\substack{e,R \\ st}} + C^{S,RR*}_{\substack{eu \\ rpts}} S_{\substack{u,R \\ st}} + C^{S,RR*}_{\substack{ed \\ rpts}} S_{\substack{d,R \\ st}} + C^{S,RL*}_{\substack{eu \\ rpts}} S_{\substack{u,L \\ st}} + C^{S,RL*}_{\substack{ed \\ rpts}} S_{\substack{d,L \\ st}} \right)
\end{aligned}$$

$$+\sigma_{\alpha\beta}\underset{r}{e_L}\left(\underset{rpts}{C^{T,RR*}_{eu}}\underset{st}{\mathcal{T}^{\alpha\beta}_{u,R}}+\underset{rpts}{C^{T,RR*}_{ed}}\underset{st}{\mathcal{T}^{\alpha\beta}_{d,R}}\right)+\underset{r}{\nu_L}\left(\underset{rpts}{C^{S,RL*}_{\nu edu}}\underset{st}{S_{ud,L}}+\underset{rpts}{C^{S,RR*}_{\nu edu}}\underset{st}{S_{ud,R}}\right)+\underset{rpts}{C^{T,RR*}_{\nu edu}}\sigma^{\alpha\beta}\underset{r}{\nu_L}\underset{st}{\mathcal{T}^{\alpha\beta}_{ud,R}},\tag{322}$$

$$\Delta^{(6,B,L)}_{u_R,p}=$$

$$\gamma_\alpha\underset{r}{u_R}\left(2\underset{prst}{C^{V,RR}_{uu}}\underset{st}{J^\alpha_{u,R}}+\underset{stpr}{C^{V,RR}_{eu}}\underset{st}{J^\alpha_{e,R}}+\underset{prst}{C^{V1,RR}_{ud}}\underset{st}{J^\alpha_{d,R}}+\underset{stpr}{C^{V,LR}_{\nu u}}\underset{st}{J^\alpha_{\nu,L}}+\underset{stpr}{C^{V,LR}_{eu}}\underset{st}{J^\alpha_{e,L}}+\underset{stpr}{C^{V1,LR}_{du}}\underset{st}{J^\alpha_{d,L}}+\underset{stpr}{C^{V1,LR}_{uu}}\underset{st}{J^\alpha_{u,L}}\right)$$
$$+\gamma_\alpha T^A\underset{r}{u_R}\left(\underset{prst}{C^{V8,RR}_{ud}}\underset{st}{J^{\alpha,A}_{d,R}}+\underset{stpr}{C^{V8,LR}_{uu}}\underset{st}{J^{\alpha,A}_{u,L}}+\underset{stpr}{C^{V8,LR}_{du}}\underset{st}{J^{\alpha,A}_{d,L}}\right)+T^A\underset{r}{u_L}\left(2\underset{rpts}{C^{S8,RR*}_{uu}}\underset{st}{S^A_{u,R}}+\underset{rpts}{C^{S8,RR*}_{ud}}\underset{st}{S^A_{d,R}}\right)$$
$$+\underset{r}{u_L}\left(\underset{tsrp}{C^{S,RR*}_{eu}}\underset{st}{S_{e,R}}+2\underset{rpts}{C^{S1,RR*}_{uu}}\underset{st}{S_{u,R}}+\underset{rpts}{C^{S1,RR*}_{ud}}\underset{st}{S_{d,R}}+\underset{stpr}{C^{S,RL}_{eu}}\underset{st}{S_{e,L}}\right)+\underset{tsrp}{C^{T,RR*}_{eu}}\sigma_{\alpha\beta}\underset{r}{u_L}\underset{st}{\mathcal{T}^{\alpha\beta}_{e,R}}$$
$$+\gamma_\alpha\underset{r}{d_R}\left(\underset{tsrp}{C^{V,LR*}_{\nu edu}}\underset{st}{J^\alpha_{e\nu,L}}+\underset{tsrp}{C^{V1,LR*}_{uddu}}\underset{st}{J^\alpha_{du,L}}\right)+\underset{tsrp}{C^{V8,LR*}_{uddu}}\gamma^\alpha T^A\underset{r}{d_R}\underset{st}{J^{\alpha,A}_{du,L}}$$
$$+\underset{r}{d_L}\left(\underset{tsrp}{C^{S,RR*}_{\nu edu}}\underset{st}{S_{e\nu,R}}+\underset{tsrp}{C^{S1,RR*}_{uddu}}\underset{st}{S_{du,R}}\right)+\underset{tsrp}{C^{S8,RR*}_{uddu}}T^A\underset{r}{d_L}\underset{st}{S^A_{du,R}}+\underset{tsrp}{C^{T,RR*}_{\nu edu}}\sigma_{\alpha\beta}\underset{r}{d_L}\underset{st}{\mathcal{T}^{\alpha\beta}_{e\nu,R}},\tag{323}$$

$$\Delta^{(6,B,L)}_{d_R,p}=$$

$$\gamma_\alpha\underset{r}{d_R}\left(2\underset{prst}{C^{V,RR}_{dd}}\underset{st}{J^\alpha_{d,R}}+\underset{stpr}{C^{V,RR}_{ed}}\underset{st}{J^\alpha_{e,R}}+\underset{stpr}{C^{V1,RR}_{ud}}\underset{st}{J^\alpha_{u,R}}+\underset{stpr}{C^{V,LR}_{\nu d}}\underset{st}{J^\alpha_{\nu,L}}+\underset{stpr}{C^{V,LR}_{ed}}\underset{st}{J^\alpha_{e,L}}+\underset{stpr}{C^{V1,LR}_{ud}}\underset{st}{J^\alpha_{u,L}}+\underset{stpr}{C^{V1,LR}_{dd}}\underset{st}{J^\alpha_{d,L}}\right)$$
$$+\gamma_\alpha T^A\underset{r}{d_R}\left(\underset{stpr}{C^{V8,RR}_{ud}}\underset{st}{J^{\alpha,A}_{u,R}}+\underset{stpr}{C^{V8,LR}_{ud}}\underset{st}{J^{\alpha,A}_{u,L}}+\underset{stpr}{C^{V8,LR}_{dd}}\underset{st}{J^{\alpha,A}_{d,L}}\right)+T^A\underset{r}{d_L}\left(\underset{tsrp}{C^{S8,RR*}_{ud}}\underset{st}{S^A_{u,R}}+2\underset{rpts}{C^{S8,RR*}_{dd}}\underset{st}{S^A_{d,R}}\right)$$
$$+\underset{r}{d_L}\left(\underset{tsrp}{C^{S,RR*}_{ed}}\underset{st}{S_{e,R}}+\underset{tsrp}{C^{S1,RR*}_{ud}}\underset{st}{S_{u,R}}+2\underset{rpts}{C^{S1,RR*}_{dd}}\underset{st}{S_{d,R}}+\underset{stpr}{C^{S,RL}_{ed}}\underset{st}{S_{e,L}}\right)+\underset{tsrp}{C^{T,RR*}_{ed}}\sigma_{\alpha\beta}\underset{r}{d_L}\underset{st}{\mathcal{T}^{\alpha\beta}_{e,R}}$$
$$+\gamma_\alpha\underset{r}{u_R}\left(\underset{stpr}{C^{V,LR}_{\nu edu}}\underset{st}{J^\alpha_{\nu e,L}}+\underset{stpr}{C^{V1,LR}_{uddu}}\underset{st}{J^\alpha_{ud,L}}\right)+\underset{stpr}{C^{V8,LR}_{uddu}}\gamma_\alpha T^A\underset{r}{u_R}\underset{st}{J^{\alpha,A}_{ud,L}}+\underset{r}{u_L}\left(\underset{rpts}{C^{S1,RR*}_{uddu}}\underset{st}{S_{ud,R}}+\underset{stpr}{C^{S,RL}_{\nu edu}}\underset{st}{S_{\nu e,L}}\right)$$
$$+\underset{rpts}{C^{S8,RR*}_{uddu}}T^A\underset{r}{u_L}\underset{st}{S^A_{ud,R}},\tag{324}$$

$$\Delta^{(6,B,L)}_{\nu_L,p}=$$

$$\gamma_\alpha\underset{r}{\nu_L}\left(2\underset{prst}{C^{V,LL}_{\nu\nu}}\underset{st}{J^\alpha_{\nu,L}}+\sum_{\psi\neq\nu}\underset{prst}{C^{V,LL}_{\nu\psi}}\underset{st}{J^\alpha_{\psi,L}}+\sum_{\psi\neq\nu}\underset{prst}{C^{V,LR}_{\nu\psi}}\underset{st}{J^\alpha_{\psi,R}}\right)+\gamma_\alpha\underset{r}{e_L}\left(\underset{prst}{C^{V,LL}_{\nu edu}}\underset{st}{J^\alpha_{du,L}}+\underset{prst}{C^{V,LR}_{\nu edu}}\underset{st}{J^\alpha_{du,R}}\right)$$
$$+\underset{r}{e_R}\left(\underset{prst}{C^{S,RR}_{\nu edu}}\underset{st}{S_{du,L}}+\underset{prst}{C^{S,RL}_{\nu edu}}\underset{st}{S_{du,R}}\right)+\underset{prst}{C^{T,RR}_{\nu edu}}\sigma_{\alpha\beta}\underset{r}{e_R}\underset{st}{\mathcal{T}^{\alpha\beta}_{du,L}},\tag{325}$$

$$\Delta^{(6,B,L)}_{e_L,p}=$$

$$\gamma_\alpha\underset{r}{e_L}\left(2\underset{prst}{C^{V,LL}_{ee}}\underset{st}{J^\alpha_{e,L}}+\underset{stpr}{C^{V,LL}_{\nu e}}\underset{st}{J^\alpha_{\nu,L}}+\underset{prst}{C^{V,LL}_{eu}}\underset{st}{J^\alpha_{u,L}}+\underset{prst}{C^{V,LL}_{ed}}\underset{st}{J^\alpha_{d,L}}+\sum_{\psi}\underset{prst}{C^{V,LR}_{e\psi}}\underset{st}{J^\alpha_{\psi,R}}\right)$$

$$
\begin{aligned}
&+ \underset{r}{e_R}\left(2\underset{prst}{C^{S,RR}_{ee}}\underset{st}{S_{e,L}} + \underset{prst}{C^{S,RR}_{eu}}\underset{st}{S_{u,L}} + \underset{prst}{C^{S,RR}_{ed}}\underset{st}{S_{d,L}} + \underset{prst}{C^{S,RL}_{eu}}\underset{st}{S_{u,R}} + \underset{prst}{C^{S,RL}_{ed}}\underset{st}{S_{d,R}}\right) \\
&+ \sigma_{\alpha\beta}\underset{r}{e_R}\left(\underset{prst}{C^{T,RR}_{eu}}\underset{st}{\mathcal{T}^{\alpha\beta}_{u,L}} + \underset{prst}{C^{T,RR}_{ed}}\underset{st}{\mathcal{T}^{\alpha\beta}_{d,L}}\right) + \gamma_\alpha \underset{r}{\nu_L}\left(\underset{rpts}{C^{V,LL*}_{\nu edu}}\underset{st}{J^{\alpha}_{ud,L}} + \underset{rpts}{C^{V,LR*}_{\nu edu}}\underset{st}{J^{\alpha}_{ud,R}}\right),
\end{aligned}
\tag{326}
$$

$$
\begin{aligned}
&\Delta^{(6,B,L)}_{u_L,p} = \\
&\gamma_\alpha \underset{r}{u_L}\left(2\underset{prst}{C^{V,LL}_{uu}}\underset{st}{J^{\alpha}_{u,L}} + \underset{stpr}{C^{V,LL}_{\nu u}}\underset{st}{J^{\alpha}_{\nu,L}} + \underset{stpr}{C^{V,LL}_{eu}}\underset{st}{J^{\alpha}_{e,L}} + \underset{prst}{C^{V1,LL}_{ud}}\underset{st}{J^{\alpha}_{d,L}} + \underset{prst}{C^{V,LR}_{ue}}\underset{st}{J^{\alpha}_{e,R}} + \underset{prst}{C^{V1,LR}_{uu}}\underset{st}{J^{\alpha}_{u,R}} + \underset{prst}{C^{V1,LR}_{ud}}\underset{st}{J^{\alpha}_{d,R}}\right) \\
&+ \gamma_\alpha T^A \underset{r}{u_L}\left(\underset{prst}{C^{V8,LL}_{ud}}\underset{st}{J^{\alpha,A}_{d,L}} + \underset{prst}{C^{V8,LR}_{uu}}\underset{st}{J^{\alpha,A}_{u,R}} + \underset{prst}{C^{V8,LR}_{ud}}\underset{st}{J^{\alpha,A}_{d,R}}\right) + \underset{stpr}{C^{T,RR}_{eu}}\sigma_{\alpha\beta}\underset{r}{u_R}\underset{st}{\mathcal{T}^{\alpha\beta}_{e,L}} \\
&+ \underset{r}{u_R}\left(\underset{stpr}{C^{S,RR}_{eu}}\underset{st}{S_{e,L}} + 2\underset{prst}{C^{S1,RR}_{uu}}\underset{st}{S_{u,L}} + \underset{prst}{C^{S1,RR}_{ud}}\underset{st}{S_{d,L}} + \underset{tsrp}{C^{S,RL*}_{eu}}\underset{st}{S_{e,R}}\right) + T^A\underset{r}{u_R}\left(2\underset{prst}{C^{S8,RR}_{uu}}\underset{st}{S^{A}_{u,L}} + \underset{prst}{C^{S8,RR}_{ud}}\underset{st}{S^{A}_{d,L}}\right) \\
&+ \gamma_\alpha \underset{r}{d_L}\left(\underset{tsrp}{C^{V,LL*}_{\nu edu}}\underset{st}{J^{\alpha}_{e\nu,L}} + \underset{prst}{C^{V1,LR}_{uddu}}\underset{st}{J^{\alpha}_{du,R}}\right) + \underset{prst}{C^{V8,LR}_{uddu}}\gamma_\alpha T^A \underset{r}{d_L}\underset{st}{J^{\alpha,A}_{du,R}} \\
&+ \underset{r}{d_R}\left(\underset{prst}{C^{S1,RR}_{uddu}}\underset{st}{S_{du,L}} + \underset{tsrp}{C^{S,RL*}_{\nu edu}}\underset{st}{S_{e\nu,R}}\right) + \underset{prst}{C^{S8,RR}_{uddu}}T^A\underset{r}{d_R}\underset{st}{S^{A}_{du,L}},
\end{aligned}
\tag{327}
$$

$$
\begin{aligned}
&\Delta^{(6,B,L)}_{d_L,p} = \\
&\gamma_\alpha \underset{r}{d_L}\left(2\underset{prst}{C^{V,LL}_{dd}}\underset{st}{J^{\alpha}_{d,L}} + \underset{stpr}{C^{V,LL}_{\nu d}}\underset{st}{J^{\alpha}_{\nu,L}} + \underset{stpr}{C^{V,LL}_{ed}}\underset{st}{J^{\alpha}_{e,L}} + \underset{stpr}{C^{V1,LL}_{ud}}\underset{st}{J^{\alpha}_{u,L}} + \underset{prst}{C^{V,LR}_{de}}\underset{st}{J^{\alpha}_{e,R}} + \underset{prst}{C^{V1,LR}_{du}}\underset{st}{J^{\alpha}_{u,R}} + \underset{prst}{C^{V1,LR}_{dd}}\underset{st}{J^{\alpha}_{d,R}}\right) \\
&+ \gamma_\alpha T^A \underset{r}{d_L}\left(\underset{stpr}{C^{V8,LL}_{ud}}\underset{st}{J^{\alpha,A}_{u,L}} + \underset{prst}{C^{V8,LR}_{du}}\underset{st}{J^{\alpha,A}_{u,R}} + \underset{prst}{C^{V8,LR}_{dd}}\underset{st}{J^{\alpha,A}_{d,R}}\right) + \underset{stpr}{C^{T,RR}_{ed}}\sigma_{\alpha\beta}\underset{r}{d_R}\underset{st}{\mathcal{T}^{\alpha\beta}_{e,L}} \\
&+ \underset{r}{d_R}\left(\underset{stpr}{C^{S,RR}_{ed}}\underset{st}{S_{e,L}} + \underset{stpr}{C^{S1,RR}_{ud}}\underset{st}{S_{u,L}} + 2\underset{prst}{C^{S1,RR}_{dd}}\underset{st}{S_{d,L}} + \underset{tsrp}{C^{S,RL*}_{ed}}\underset{st}{S_{e,R}}\right) + T^A\underset{r}{d_R}\left(\underset{stpr}{C^{S8,RR}_{ud}}\underset{st}{S^{A}_{u,L}} + 2\underset{prst}{C^{S8,RR}_{dd}}\underset{st}{S^{A}_{d,L}}\right) \\
&+ \gamma_\alpha \underset{r}{u_L}\left(\underset{stpr}{C^{V,LL}_{\nu edu}}\underset{st}{J^{\alpha}_{\nu e,L}} + \underset{rpts}{C^{V1,LR*}_{uddu}}\underset{st}{J^{\alpha}_{ud,R}}\right) + \underset{rpts}{C^{V8,LR*}_{uddu}}\gamma_\alpha T^A \underset{r}{u_L}\underset{st}{J^{\alpha,A}_{ud,R}} \\
&+ \underset{r}{u_R}\left(\underset{stpr}{C^{S,RR}_{\nu edu}}\underset{st}{S_{\nu e,L}} + \underset{stpr}{C^{S1,RR}_{uddu}}\underset{st}{S_{ud,L}}\right) + \underset{stpr}{C^{T,RR}_{\nu edu}}\sigma_{\alpha\beta}\underset{r}{u_R}\underset{st}{\mathcal{T}^{\alpha\beta}_{\nu e,L}} + \underset{stpr}{C^{S8,RR}_{uddu}}T^A\underset{r}{u_R}\underset{st}{S_{ud,L}}.
\end{aligned}
\tag{328}
$$

The $\Delta L \neq 0$, $\Delta B = 0$ contributions to the EOM from $L^{(6)}$ are

$$
\begin{aligned}
&\Delta^{(6,B,\not L)}_{e_R,p} = \\
&\underset{r}{e_L}\left(\underset{stpr}{C^{S,LL}_{\nu e}}\underset{st}{\tilde{S}_{\nu^c\nu,L}} + \underset{tsrp}{C^{S,LR*}_{\nu e}}\underset{st}{\tilde{S}_{\nu\nu^c,L}}\right) + \underset{stpr}{C^{T,LL}_{\nu e}}\sigma_{\alpha\beta}\underset{r}{e_L}\underset{st}{\tilde{\mathcal{T}}^{\alpha\beta}_{\nu^c\nu,L}} + \gamma_\alpha \underset{r}{\nu^c_L}\left(\underset{rpts}{C^{V,RL*}_{\nu edu}}\underset{st}{J^{\alpha}_{ud,L}} + \underset{rpts}{C^{V,RR*}_{\nu edu}}\underset{st}{J^{\alpha}_{ud,R}}\right),
\end{aligned}
\tag{329}
$$

$$
\Delta^{(6,B,\not L)}_{u_R,p} =
$$

$$
\underset{r}{u_L}\left(\underset{stpr}{C^{S,LL}_{\nu u}}\underset{st}{\tilde{S}_{\nu^c\nu,L}}+\underset{tsrp}{C^{S,LR*}_{\nu u}}\underset{st}{\tilde{S}_{\nu\nu^c,L}}\right)+\underset{stpr}{C^{T,LL}_{\nu u}}\sigma_{\alpha\beta}\underset{r}{u_L}\underset{st}{\tilde{\mathcal{T}}^{\alpha\beta}_{\nu^c\nu,L}}+\underset{tsrp}{C^{S,LR*}_{\nu edu}}\underset{r}{d_L}\underset{st}{\tilde{S}_{e\nu^c,L}}+\underset{tsrp}{C^{V,RR*}_{\nu edu}}\gamma_\alpha\underset{r}{d_R}\underset{st}{\tilde{J}^{\alpha}_{e\nu^c,R}},
\tag{330}
$$

$$
\begin{aligned}
&\Delta^{(6,B,\not{L})}_{d_R,p}=\\
&\underset{r}{d_L}\left(\underset{stpr}{C^{S,LL}_{\nu d}}\underset{st}{\tilde{S}_{\nu^c\nu,L}}+\underset{tsrp}{C^{S,LR*}_{\nu d}}\underset{st}{\tilde{S}_{\nu\nu^c,L}}\right)+\underset{stpr}{C^{T,LL}_{\nu d}}\sigma_{\alpha\beta}\underset{r}{d_L}\underset{st}{\tilde{\mathcal{T}}^{\alpha\beta}_{\nu^c\nu,L}}+\underset{stpr}{C^{S,LL}_{\nu edu}}\underset{r}{u_L}\underset{st}{\tilde{S}_{\nu^c e,L}}\\
&+\underset{stpr}{C^{T,LL}_{\nu edu}}\sigma_{\alpha\beta}\underset{r}{u_L}\underset{st}{\tilde{\mathcal{T}}^{\alpha\beta}_{\nu^c e,L}}+\underset{stpr}{C^{V,RR}_{\nu edu}}\gamma_\alpha\underset{r}{u_R}\underset{st}{\tilde{J}^{\alpha}_{\nu^c e,L}},
\end{aligned}
\tag{331}
$$

$$
\begin{aligned}
&\Delta^{(6,B,\not{L})}_{\nu_L,p}=\\
&\underset{r}{\nu^c_L}\left(2\underset{prst}{C^{S,LL*}_{\nu\nu}}\underset{st}{\tilde{S}^{*}_{\nu^c\nu,L}}+2\underset{rpts}{C^{S,LL*}_{\nu\nu}}\underset{st}{\tilde{S}_{\nu\nu^c,L}}+\sum_{\psi}\left(\underset{prst}{C^{S,LL*}_{\nu\psi}}\underset{st}{S^{*}_{\psi,R}}+\underset{rpts}{C^{S,LL*}_{\nu\psi}}\underset{st}{S_{\psi,L}}+\underset{prst}{C^{S,LR*}_{\nu\psi}}\underset{st}{S^{*}_{\psi,L}}+\underset{rpts}{C^{S,LR*}_{\nu\psi}}\underset{st}{S_{\psi,R}}\right)\right)\\
&+\sigma_{\alpha\beta}\underset{r}{\nu^c_L}\left(\underset{prst}{C^{T,LL*}_{\nu e}}\underset{st}{\mathcal{T}^{\alpha\beta*}_{e,R}}+\underset{rpts}{C^{T,LL*}_{\nu e}}\underset{st}{\mathcal{T}^{\alpha\beta}_{e,L}}+\underset{prst}{C^{T,LL*}_{\nu u}}\underset{st}{\mathcal{T}^{\alpha\beta*}_{u,R}}+\underset{rpts}{C^{T,LL*}_{\nu u}}\underset{st}{\mathcal{T}^{\alpha\beta}_{u,L}}+\underset{prst}{C^{T,LL*}_{\nu d}}\underset{st}{\mathcal{T}^{\alpha\beta*}_{d,R}}+\underset{rpts}{C^{T,LL*}_{\nu d}}\underset{st}{\mathcal{T}^{\alpha\beta}_{d,L}}\right)\\
&+\underset{r}{e^c_L}\left(\underset{prst}{C^{S,LL*}_{\nu edu}}\underset{st}{S^{*}_{du,R}}+\underset{prst}{C^{S,LR*}_{\nu edu}}\underset{st}{S^{*}_{du,L}}\right)+\underset{prst}{C^{T,LL*}_{\nu edu}}\sigma_{\alpha\beta}\underset{r}{e^c_L}\underset{st}{\mathcal{T}^{\alpha\beta*}_{du,R}}+\gamma_\alpha\underset{r}{e^c_R}\left(\underset{prst}{C^{V,RL*}_{\nu edu}}\underset{st}{J^{\alpha*}_{du,L}}+\underset{prst}{C^{V,RR*}_{\nu edu}}\underset{st}{J^{\alpha*}_{du,R}}\right),
\end{aligned}
\tag{332}
$$

$$
\begin{aligned}
&\Delta^{(6,B,\not{L})}_{e_L,p}=\\
&\underset{r}{e_R}\left(\underset{tsrp}{C^{S,LL*}_{\nu e}}\underset{st}{\tilde{S}_{\nu\nu^c,L}}+\underset{stpr}{C^{S,LR}_{\nu e}}\underset{st}{\tilde{S}_{\nu^c\nu,L}}\right)+\underset{tsrp}{C^{T,LL*}_{\nu e}}\sigma_{\alpha\beta}\underset{r}{e_R}\underset{st}{\tilde{\mathcal{T}}^{\alpha\beta}_{\nu\nu^c,L}}\\
&+\underset{r}{\nu^c_L}\left(\underset{rpts}{C^{S,LL*}_{\nu edu}}\underset{st}{S_{ud,L}}+\underset{rpts}{C^{S,LR*}_{\nu edu}}\underset{st}{S_{ud,R}}\right)+\underset{rpts}{C^{T,LL*}_{\nu edu}}\sigma_{\alpha\beta}\underset{r}{\nu^c_L}\underset{st}{\mathcal{T}^{\alpha\beta}_{ud,L}},
\end{aligned}
\tag{333}
$$

$$
\begin{aligned}
&\Delta^{(6,B,\not{L})}_{u_L,p}=\\
&\underset{r}{u_R}\left(\underset{tsrp}{C^{S,LL*}_{\nu u}}\underset{st}{\tilde{S}_{\nu\nu^c,L}}+\underset{stpr}{C^{S,LR}_{\nu u}}\underset{st}{\tilde{S}_{\nu^c\nu,L}}\right)+\underset{tsrp}{C^{T,LL*}_{\nu u}}\sigma_{\alpha\beta}\underset{r}{u_R}\underset{st}{\tilde{\mathcal{T}}^{\alpha\beta}_{\nu\nu^c,L}}+\underset{tsrp}{C^{S,LL*}_{\nu edu}}\underset{r}{d_R}\underset{st}{\tilde{S}_{e\nu^c,L}}\\
&+\underset{tsrp}{C^{T,LL*}_{\nu edu}}\sigma_{\alpha\beta}\underset{r}{d_R}\underset{st}{\tilde{\mathcal{T}}^{\alpha\beta}_{e\nu^c,L}}+\underset{tsrp}{C^{V,RL*}_{\nu edu}}\gamma_\alpha\underset{r}{d_L}\underset{st}{\tilde{J}^{\alpha}_{e\nu^c,R}},
\end{aligned}
\tag{334}
$$

$$
\begin{aligned}
&\Delta^{(6,B,\not{L})}_{d_L,p}=\\
&\underset{r}{d_R}\left(\underset{tsrp}{C^{S,LL*}_{\nu d}}\underset{st}{\tilde{S}_{\nu\nu^c,L}}+\underset{stpr}{C^{S,LR}_{\nu d}}\underset{st}{\tilde{S}_{\nu^c\nu,L}}\right)+\underset{tsrp}{C^{T,LL*}_{\nu d}}\sigma_{\alpha\beta}\underset{r}{d_R}\underset{st}{\tilde{\mathcal{T}}^{\alpha\beta}_{\nu\nu^c,L}}+\underset{stpr}{C^{S,LR}_{\nu edu}}\underset{r}{u_R}\underset{st}{\tilde{S}_{\nu^c e,L}}+\underset{stpr}{C^{V,RL}_{\nu edu}}\gamma_\alpha\underset{r}{u_L}\underset{st}{\tilde{J}^{\alpha}_{\nu^c e,L}}.
\end{aligned}
\tag{335}
$$

$\Delta L, \Delta B \neq 0$, contributions to the EOM from $L^{(6)}$ are

$$
\Delta^{(6,\not{B},\not{L})}_{e_R,p} =
$$
$$
C^{S,LR*}_{\substack{uud\\tsrp}}\epsilon_{\alpha\beta\gamma}d^{\gamma c}_{\substack{R\\r}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{u,L\\st}} + \epsilon_{\alpha\beta\gamma}u^{\gamma c}_{\substack{R\\r}}\left(C^{S,LR*}_{\substack{duu\\tsrp}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{ud,L\\st}} + C^{S,RR*}_{\substack{duu\\tsrp}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{ud,R\\st}}\right) + \epsilon_{\alpha\beta\gamma}d^{\gamma}_{\substack{L\\r}}\left(C^{S,LL}_{\substack{ddd\\stpr}}\tilde{\mathcal{S}}^{\alpha c,\beta}_{\substack{d,L\\st}} + C^{S,RL}_{\substack{ddd\\stpr}}\tilde{\mathcal{S}}^{\alpha c,\beta}_{\substack{d,R\\st}}\right),
\tag{336}
$$

$$
\Delta^{(6,\not{B},\not{L})}_{u_R,p} =
$$
$$
\begin{aligned}
&\epsilon_{\beta\gamma\alpha}e^{c}_{\substack{R\\r}}\left(C^{S,LR*}_{\substack{duu\\stpr}}\tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{du,L\\st}} + C^{S,RR*}_{\substack{duu\\stpr}}\tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{du,R\\st}}\right) + \epsilon_{\alpha\beta\gamma}u^{\beta c}_{\substack{R\\r}}\left(C^{S,RL*}_{\substack{uud\\prst}}\tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{de,L\\st}} - C^{S,RL*}_{\substack{uud\\rpts}}\tilde{\mathcal{S}}^{,\gamma c}_{\substack{ed,L\\st}}\right)\\
&+\epsilon_{\beta\alpha\gamma}d^{\beta c}_{\substack{R\\r}}\left(C^{S,RL*}_{\substack{duu\\rpts}}\tilde{\mathcal{S}}^{,\gamma c}_{\substack{eu,L\\st}} + C^{S,RR*}_{\substack{duu\\rpts}}\tilde{\mathcal{S}}^{,\gamma c}_{\substack{eu,R\\st}} + C^{S,RL*}_{\substack{dud\\rpts}}\tilde{\mathcal{S}}^{,\gamma c}_{\substack{\nu d,L\\st}} - C^{S,RR*}_{\substack{udd\\prst}}\mathcal{S}^{,\gamma *}_{\substack{\nu d,L\\st}}\right) + C^{S,LR*}_{\substack{ddu\\tsrp}}\epsilon_{\beta\gamma\alpha}\nu_{\substack{L\\r}}\tilde{\mathcal{S}}^{\gamma,\beta c}_{\substack{d,L\\st}},
\end{aligned}
\tag{337}
$$

$$
\Delta^{(6,\not{B},\not{L})}_{d_R,p} =
$$
$$
\begin{aligned}
&\epsilon_{\alpha\beta\gamma}d^{\beta c}_{\substack{R\\r}}\left(C^{S,RL*}_{\substack{ddu\\prst}}\tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{u\nu,L\\st}} - C^{S,RL*}_{\substack{ddu\\rpts}}\tilde{\mathcal{S}}^{,\gamma c}_{\substack{\nu u,L\\st}} + C^{S,RL*}_{\substack{ddd\\prst}}\mathcal{S}^{,\gamma *}_{\substack{ed,R\\st}} - C^{S,RL*}_{\substack{ddd\\rpts}}\mathcal{S}^{\gamma,}_{\substack{de,L\\st}} + C^{S,RR*}_{\substack{ddd\\prst}}\mathcal{S}^{,\gamma *}_{\substack{ed,L\\st}} - C^{S,RR*}_{\substack{ddd\\rpts}}\mathcal{S}^{\gamma,}_{\substack{de,R\\st}}\right)\\
&+\epsilon_{\alpha\beta\gamma}u^{\beta c}_{\substack{R\\r}}\left(C^{S,RL*}_{\substack{duu\\prst}}\tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{ue,L\\st}} + C^{S,RR*}_{\substack{duu\\prst}}\tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{ue,R\\st}} + C^{S,RL*}_{\substack{dud\\prst}}\tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{d\nu,L\\st}} - C^{S,RR*}_{\substack{udd\\rpts}}\mathcal{S}^{\gamma,}_{\substack{d\nu,R\\st}}\right) + C^{S,LR*}_{\substack{uud\\stpr}}\epsilon_{\beta\gamma\alpha}e^{c}_{\substack{R\\r}}\tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{u,L\\st}}\\
&+\epsilon_{\beta\gamma\alpha}e_{\substack{L\\r}}\left(C^{S,LR*}_{\substack{ddd\\tsrp}}\tilde{\mathcal{S}}^{\gamma,\beta c}_{\substack{d,L\\st}} + C^{S,RR*}_{\substack{ddd\\tsrp}}\tilde{\mathcal{S}}^{\gamma,\beta c}_{\substack{d,R\\st}}\right) + \epsilon_{\beta\gamma\alpha}\nu_{\substack{L\\r}}\left(C^{S,LR*}_{\substack{udd\\tsrp}}\tilde{\mathcal{S}}^{\gamma,\beta c}_{\substack{du,L\\st}} + C^{S,RR*}_{\substack{udd\\tsrp}}\tilde{\mathcal{S}}^{\gamma,\beta c}_{\substack{du,R\\st}}\right),
\end{aligned}
\tag{338}
$$

$$
\Delta^{(6,\not{B},\not{L})}_{\nu_L,p} =
$$
$$
\begin{aligned}
&\epsilon_{\alpha\beta\gamma}d^{\gamma c}_{\substack{L\\r}}\left(C^{S,LL*}_{\substack{udd\\tsrp}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{du,L\\st}} + C^{S,RL*}_{\substack{dud\\tsrp}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{ud,R\\st}}\right) + \epsilon_{\alpha\beta\gamma}d^{\gamma}_{\substack{R\\r}}\left(C^{S,LR}_{\substack{udd\\stpr}}\tilde{\mathcal{S}}^{\alpha c,\beta}_{\substack{ud,L\\st}} + C^{S,RR}_{\substack{udd\\stpr}}\tilde{\mathcal{S}}^{\alpha c,\beta}_{\substack{ud,R\\st}}\right)\\
&+C^{S,RL*}_{\substack{ddu\\tsrp}}\epsilon_{\alpha\beta\gamma}u^{\gamma c}_{\substack{L\\r}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{d,R\\st}} + C^{S,LR}_{\substack{ddu\\stpr}}\epsilon_{\alpha\beta\gamma}u^{\gamma}_{\substack{R\\r}}\tilde{\mathcal{S}}^{\alpha c,\beta}_{\substack{d,L\\st}},
\end{aligned}
\tag{339}
$$

$$
\Delta^{(6,\not{B},\not{L})}_{e_L,p} =
$$
$$
\epsilon_{\alpha\beta\gamma}u^{\gamma c}_{\substack{L\\r}}\left(C^{S,LL*}_{\substack{duu\\tsrp}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{ud,L\\st}} + C^{S,RL*}_{\substack{duu\\tsrp}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{ud,R\\st}}\right) + C^{S,RL*}_{\substack{uud\\tsrp}}\epsilon_{\alpha\beta\gamma}d^{\gamma c}_{\substack{L\\r}}\tilde{\mathcal{S}}^{\beta,\alpha c}_{\substack{u,R\\st}} + \epsilon_{\alpha\beta\gamma}d^{\gamma}_{\substack{R\\r}}\left(C^{S,LR}_{\substack{ddd\\stpr}}\tilde{\mathcal{S}}^{\alpha c,\beta}_{\substack{d,L\\st}} + C^{S,RR}_{\substack{ddd\\stpr}}\tilde{\mathcal{S}}^{\alpha c,\beta}_{\substack{d,R\\st}}\right),
\tag{340}
$$

$$
\Delta^{(6,\not{B},\not{L})}_{u_L,p} =
$$
$$
\epsilon_{\alpha\beta\gamma}d^{\beta c}_{\substack{L\\r}}\left(C^{S,LL*}_{\substack{udd\\prst}}\tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{d\nu,L\\st}} + C^{S,LR*}_{\substack{udd\\prst}}\mathcal{S}^{,\gamma *}_{\substack{\nu d,L\\st}} - C^{S,LL*}_{\substack{duu\\rpts}}\tilde{\mathcal{S}}^{,\gamma c}_{\substack{eu,L\\st}} - C^{S,LR*}_{\substack{duu\\rpts}}\tilde{\mathcal{S}}^{,\gamma c}_{\substack{eu,R\\st}}\right) + C^{S,RL}_{\substack{ddu\\stpr}}\epsilon_{\beta\gamma\alpha}\nu^{c}_{\substack{L\\r}}\tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{d,R\\st}}
$$

$$+\,\epsilon_{\beta\alpha\gamma} e^{c}_{\substack{L\\r}} \left(C^{S,LL*}_{\substack{duu\\stpr}} \tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{du,L\\st}} + C^{S,RL*}_{\substack{duu\\stpr}} \tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{du,R\\st}} \right) + \epsilon_{\alpha\beta\gamma} u^{\beta c}_{\substack{L\\r}} \left(C^{S,LR*}_{\substack{uud\\prst}} \tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{de,R\\st}} - C^{S,LR*}_{\substack{uud\\rpts}} \tilde{\mathcal{S}}^{\,,\gamma c}_{\substack{ed,R\\st}} \right), \tag{341}$$

$$\Delta^{(6,\not B,\not L)}_{d_L,p} =$$
$$\epsilon_{\beta\alpha\gamma} u^{\beta c}_{\substack{L\\r}} \left(C^{S,LL*}_{\substack{udd\\rpts}} \tilde{\mathcal{S}}^{\,,\gamma c}_{\substack{\nu d,L\\st}} + C^{S,LR*}_{\substack{udd\\rpts}} \mathcal{S}^{\gamma,}_{\substack{d\nu,R\\st}} - C^{S,LL*}_{\substack{duu\\prst}} \tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{ue,L\\st}} - C^{S,LR*}_{\substack{duu\\prst}} \tilde{\mathcal{S}}^{\gamma c,\,*}_{\substack{ue,R\\st}} \right) + C^{S,RL*}_{\substack{uud\\stpr}} \epsilon_{\beta\gamma\alpha} e^{c}_{\substack{L\\r}} \tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{u,R\\st}}$$
$$+\,\epsilon_{\beta\gamma\alpha} e_{\substack{R\\r}} \left(C^{S,LL*}_{\substack{ddd\\tsrp}} \tilde{\mathcal{S}}^{\gamma,\beta c}_{\substack{d,L\\st}} + C^{S,RL*}_{\substack{ddd\\tsrp}} \tilde{\mathcal{S}}^{\gamma,\beta c}_{\substack{d,R\\st}} \right) + \epsilon_{\beta\gamma\alpha} \nu^{c}_{\substack{L\\r}} \left(C^{S,LL*}_{\substack{udd\\stpr}} \tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{ud,L\\st}} + C^{S,RL*}_{\substack{dud\\stpr}} \tilde{\mathcal{S}}^{\beta c,\gamma *}_{\substack{du,R\\st}} \right)$$
$$+\,\epsilon_{\alpha\beta\gamma} d^{\beta c}_{\substack{L\\r}} \left(C^{S,LL*}_{\substack{ddd\\prst}} \mathcal{S}^{\,,\gamma *}_{\substack{ed,R\\st}} - C^{S,LL*}_{\substack{ddd\\rpts}} \mathcal{S}^{\gamma,}_{\substack{de,L\\st}} + C^{S,LR*}_{\substack{ddu\\prst}} \mathcal{S}^{\,,\gamma *}_{\substack{\nu u,L\\st}} - C^{S,LR*}_{\substack{ddu\\rpts}} \mathcal{S}^{\gamma,}_{\substack{u\nu,R\\st}} + C^{S,LR*}_{\substack{ddd\\prst}} \mathcal{S}^{\,,\gamma *}_{\substack{ed,L\\st}} - C^{S,LR*}_{\substack{ddd\\rpts}} \mathcal{S}^{\gamma,}_{\substack{de,R\\st}} \right). \tag{342}$$

Finally, the dimension 3 and 5 LEFT operators contributing to the neutrino EOM give

$$\Delta^{(3)}_{\substack{\nu_L\\p}} = -2 C^{*}_{\substack{\nu\\pr}} \nu^{c}_{\substack{L,\\r}} \tag{343}$$

$$\Delta^{(5)}_{\substack{\nu_L\\p}} = 2\, C^{*}_{\substack{\nu\gamma\\pr}} \sigma^{\alpha\beta} \nu^{c}_{\substack{L\\r}} F_{\alpha\beta}. \tag{344}$$

10

HEAVY BLACK HOLE EFFECTIVE FIELD THEORY

We formulate an effective field theory describing large mass scalars and fermions minimally coupled to gravity. The operators of this effective field theory are organized in powers of the transfer momentum divided by the mass of the matter field, an expansion which lends itself to the efficient extraction of classical contributions from loop amplitudes in both the post-Newtonian and post-Minkowskian regimes. We use this effective field theory to calculate the classical and leading quantum gravitational scattering amplitude of two heavy spin-1/2 particles at the second post-Minkowskian order.

10.1 INTRODUCTION

The direct detection of gravitational waves (GWs) from the merging of two black holes by LIGO and VIRGO in 2015 [189] has placed a spotlight on GW astronomy as a novel channel through which to test general relativity (GR). As the detection rate of GWs becomes more frequent in the years ahead, it is necessary to improve the analytical predictions on which the GW templates used in the observations are based. To do so requires knowledge of the interaction Hamiltonian of a gravitationally bound binary system to high accuracy. This necessarily entails the calculation of higher orders in the post-Newtonian (PN) and post-Minkowskian (PM) expansions.

Much of the work related to GWs has been done from the relativistic approach to GR; some notable developments are the effective-one-body approach [190–192], numerical relativity [193–195], and effective field theoretic methods [196, 197] (see Refs. [198, 199] for comprehensive reviews summarizing most of the analytical aspects of these methods). Also, there has been substantial work done using traditional and modern scattering amplitude techniques to calculate classical gravitational quantities, including the non-relativistic classical gravitational potential [200–212]. Moreover, techniques were recently presented in Refs. [213, 214] to convert fully relativistic amplitudes for scalar-scalar scattering to the classical potential, and for obtaining the scattering angle directly from the scattering amplitude [215]. The prescription of Ref. [213] was combined there with modern methods in amplitude computations to obtain the 2PM, and elsewhere the state-of-the-art 3PM Hamiltonian for classical scalar-scalar gravitational scattering [210, 212]. This large body of work, facilitated by classical effects arising at all loop orders [200, 216] (see Section 10.2), suggests that quantum field theory methods can reliably be used instead of direct computation from GR, particularly when the latter becomes intractable. Following in this vein, we apply here the machinery of effective field theory (EFT) to compute classical gravitational scattering amplitudes.

Computations of classical quantities from quantum scattering amplitudes are inherently inefficient. Entire amplitudes must first be calculated — which are comprised almost entirely of quantum contributions — and then classical terms must be isolated in a classical limit. One of the advantages of EFT methods is that they allow the contributions of certain effects to be targeted in amplitude calculations, thus excluding terms that are not of interest from the outset. From the point of view of classical gravity, it is then natural to ask whether an EFT can be formulated that isolates classical from quantum contributions already at the operator level. Indeed, we find that a reinterpretation of the operator expansion of the well-established Heavy Quark Effective Theory (HQET) [217, 218] (for a review, see, *e.g.*, Ref. [219]) leads us down the right path.

HQET has been used extensively to describe bound systems of one heavy quark — with mass M large relative to the QCD scale Λ_{QCD} — and one light quark — with mass $m \lessapprox \Lambda_{\text{QCD}}$. Interactions between the light and heavy quarks are on the order of the QCD scale, $q \sim \Lambda_{\text{QCD}}$. Thus the heavy quark can, to leading order, be treated as a point source of gluons, with corrections to the motion of the heavy quark arising from higher dimensional effective operators organized in powers of $q/M \sim \Lambda_{\text{QCD}}/M$.

A similar hierarchy of scales exists when considering the long-range (classical) gravitational scattering of two heavy bodies; for long-range scattering of macroscopic objects the momentum of an exchanged graviton q is much smaller than the mass of each object. This can be seen by noting that, once powers of $\hbar$ are restored, the transfer momentum is $q = \hbar\bar{q}$, where $\bar{q}$ is the wavenumber of the mediating boson [220]. Consequently, the expansion parameter of HQET — and its gravitational analog, which we refer to as the Heavy Black Hole Effective Theory (HBET) — can be recast as $\hbar\bar{q}/M$. The magnitude of the wavenumber is proportional to the inverse of the separation of the scattering bodies, hence for macroscopic separations and masses, $\hbar\bar{q}/M \ll 1$. The presence of this separation of scales in classical gravitational scattering further motivates the development of HBET. The explicit $\hbar$ power counting of its operators makes HBET a natural framework for the computation of classical gravitational scattering amplitudes.

This work shares conceptual similarity with the Non-Relativistic General Relativity (NRGR) EFT approach to the two-body problem introduced in Ref. [196] (extended to the case of spinning objects in Ref. [197]). As in the case of NRGR, the interacting objects of HQET and HBET are sources for the mediating bosons, and are not themselves dynamical; in HQET and HBET, this can be seen from the fact that derivatives in the Lagrangians produce residual momenta (see Sec. 10.3) in the Feynman rules, not the full momenta of the objects in the scattering. However the EFTs differ in what they describe. NRGR is organized in powers of velocity, facilitating the computation of the Post-Newtonian expansion. In contrast, the operator expansions of HQET and HBET are expansions in $\hbar$, allowing us to target terms in the amplitudes with a desired $\hbar$ scaling. Being derived directly from a relativistic quantum field theory, a Post-Minkowskian expansion is naturally produced by the amplitudes of HBET. Moreover, while NRGR computes the non-relativistic interaction potential directly, HBET is intended for the computation of the classical portions of scattering amplitudes, which must then be converted to classical observables [213–215].

In this paper, we derive HBET in two forms, describing separately the interactions of large mass scalars and fermions minimally coupled to gravity. By restoring $\hbar$ we demonstrate how to determine which operators contribute classically to $2 \to 2$ scattering at n loops. Using the developed EFT we compute the $2 \to 2$ classical scattering amplitude for both scalars and fermions up to 2PM order. We include in our calculations the leading quantum contributions to the amplitudes that originate from the non-analytic structure of the loop integrals.

The structure of this paper is as follows. In section 10.2 we explain the procedure by which we restore $\hbar$ in the amplitudes. We give a brief review of HQET in Section 10.3, and outline the derivation of the HQET Lagrangian. Our main results are presented in Sections 10.4 and 10.5. In the former we derive the HBET Lagrangians for heavy scalars and heavy fermions, whereas the latter presents the $2 \to 2$ scattering amplitudes for each theory up to 2PM. We conclude in Section 10.6. Technical details of the HQET spinors are discussed in Appendix 10.A. In Appendix 10.B we include the effective theory of a heavy scalar coupled to electromagnetism, and in Appendix 10.C we use HQET to compute the classical and leading quantum contributions to the $2 \to 2$ electromagnetic amplitude up to one-loop. Appendices 10.D and 10.E contain respectively the Feynman rules and a discussion on the one-loop integrals needed to perform the 2PM calculations. We also discuss in Appendix 10.E the circumvention of the so-called pinch singularity, which appears in some HQET loop integrals.

10.2 COUNTING $\hbar$

In quantum field theory we are accustomed to working with units where both the reduced Planck constant $\hbar$ and the speed of light c are set to unity, thus obscuring the classical limit $\hbar \to 0$. We must therefore systematically restore the powers of $\hbar$ in scattering amplitudes so that a classical limit may be taken. We follow Ref. [220] to do so.

The first place we must restore $\hbar$ is in the coupling constants such that their dimensions remain unchanged: in both gravity and QED/QCD, the coupling constants are accompanied by a factor of $\hbar^{-1/2}$. Second, as mentioned above, we must distinguish between the momentum of a massless particle p^μ and its wavenumber $\bar{p}^\mu$. They are related through

$$p^\mu = \hbar \bar{p}^\mu. \tag{345}$$

In the classical limit, the momenta and masses of the massive particles must be kept constant, whereas for massless particles it is the wavenumber that must be kept constant. While this result is achieved formally through the consideration of wavefunctions in Ref. [220], an intuitive way to see this is that massless particles are classically treated as waves whose propagation can be described by a wavenumber, whereas massive particles are treated as point particles whose motion is described by their momenta.

In this work, we are interested in the scattering of two massive particles, where the momentum q is transferred via massless particles (photons or gravitons). Letting the incoming momenta be p_1 and p_2, the amplitudes will thus take the form

$$i\mathcal{M}(p_1, p_2 \to p_1 - \hbar\bar{q}, p_2 + \hbar\bar{q}). \tag{346}$$

As the momentum transfer is carried by massless particles, the wavenumber $\bar{q}$ remains fixed in the classical limit, whereas the momentum q scales with $\hbar$, as indicated in Eq. (346). The classical limit of the kinematics is therefore associated with the limit $|q| \to 0$.

10.2.1 *Counting at one-loop*

With these rules for restoring powers of $\hbar$ in amplitudes, we can preemptively deduce which operators from the EFT expansion can contribute classically at one-loop level. First we must determine the $\hbar$-scaling that produces classical results.

The usual Newtonian potential can be obtained from the Fourier transform of the leading order non-relativistic contribution to the tree-level graviton exchange amplitude (see Fig. 10.5.1). Using a non-relativistic normalization of the external states,

$$\langle p_1 | p_2 \rangle = (2\pi)^3 \delta^3(\vec{p}_1 - \vec{p}_2), \tag{347}$$

this contribution to the amplitude is

$$\mathcal{M}^{(1)} \approx -\frac{\kappa^2 m_1 m_2}{8q^2}, \tag{348}$$

where $\kappa = \sqrt{32\pi G/\hbar}$ and G is Newton's constant. Here q is the four-momentum of the mediating graviton. Following the discussion above, we can thus make all factors of $\hbar$ explicit by writing q in terms of the graviton wavenumber. We find

$$\mathcal{M}^{(1)} \approx -\frac{4\pi G m_1 m_2}{\hbar^3 \bar{q}^2}. \tag{349}$$

We conclude that classical contributions to scattering amplitudes in momentum space with the current conventions scale as $\hbar^{-3}$. A quantum mechanical term is thus one that scales with a more positive power of $\hbar$ than this, as such a term will be less significant in the $\hbar \to 0$ limit.

Indeed, this must be the $\hbar$-scaling of any term in the amplitude contributing classically to the potential. At tree-level, the relation between the amplitude and the potential is simply

$$V = -\int \frac{d^3q}{(2\pi)^3} e^{-\frac{i}{\hbar}\vec{q}\cdot\vec{r}} \mathcal{M} = -\hbar^3 \int \frac{d^3\bar{q}}{(2\pi)^3} e^{-i\vec{\bar{q}}\cdot\vec{r}} \mathcal{M}, \tag{350}$$

where we have made factors of $\hbar$ explicit. The scaling of classical contributions from the amplitude must be such that they cancel the overall $\hbar^3$ in the Fourier transform.

Central to the applicability of the Feynman diagram expansion to the computation of classical corrections to the interaction potential is the counterintuitive fact that loop diagrams can contribute classically to scattering amplitudes [200, 216]. Which loop diagrams may give rise to classical terms can be determined by requiring the same $\hbar$-scaling as in Eq. (349).

Diagrams at one-loop level have four powers of the coupling constant, which are accompanied by a factor of $\hbar^{-2}$. This implies that classical contributions from one-loop need to carry exactly one more inverse power of $\hbar$, arising from the loop integral. The only kinematic parameter in the scattering that can bring the needed $\hbar$ is the transfer momentum q, and even then only in the non-analytic form $1/\sqrt{-q^2}$. Non-analytic terms at one-loop arise from one-loop integrals with two massless propagators [200, 216]. There are three topologies at one-loop that have two massless propagators per loop, and hence three topologies from which the requisite non-analytic form can arise: the bubble, triangle, and (crossed-)box topologies. We will determine the superficial $\hbar$-scaling of these topologies.

First we note that the loop momentum l can always be assigned to a massless propagator, and hence should scale with $\hbar$. The bubble integral is thus

$$\begin{aligned} i\mathcal{M}^{(2)}_{\text{bubble}} &\sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l} \frac{1}{\hbar^2\bar{l}^2} \frac{1}{\hbar^2(\bar{l}+\bar{q})^2} + \mathcal{O}(\hbar^{-1}) \\ &= \mathcal{O}(\hbar^{-2}). \end{aligned} \tag{351}$$

We conclude that the bubble contains no classical pieces.

Triangle integrals must have an extra HQET/HBET matter propagator, which, as will be seen below, is linear in the residual momentum. Therefore, triangle diagrams scale as

$$\begin{aligned} i\mathcal{M}^{(2)}_{\text{triangle}} &\sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l} \frac{1}{\hbar^2\bar{l}^2} \frac{1}{\hbar^2(\bar{l}+\bar{q})^2} \frac{1}{\hbar v\cdot(\bar{l}+\bar{k})} + \mathcal{O}(\hbar^{-2}) \\ &= \mathcal{O}(\hbar^{-3}). \end{aligned} \tag{352}$$

Here, v is the velocity of the heavy quark, and k is the residual HQET/HBET momentum. These quantities and their $\hbar$-scaling are discussed in Section 10.3. The scaling of the triangle integral suggests that triangles must contain classical pieces.

Finally, box and crossed-box integrals scale as

$$\begin{aligned} i\mathcal{M}^{(2)}_{\text{(crossed}-\text{)box}} &\sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l} \frac{1}{\hbar^2\bar{l}^2} \frac{1}{\hbar^2(\bar{l}+\bar{q})^2} \frac{1}{\hbar v\cdot(\bar{l}+\bar{k}_1)} \frac{1}{\hbar v\cdot(\bar{l}+\bar{k}_2)} + \mathcal{O}(\hbar^{-3}) \\ &= \mathcal{O}(\hbar^{-4}). \end{aligned} \tag{353}$$

There are potentially classical pieces in the subleading terms of the (crossed-)box – that is, in higher rank (crossed-)box loop integrals. However, the leading terms in the box and crossed-box diagrams look to be too classical, scaling as $1/\hbar^4$. In order for the amplitude to have a sensible classical limit, such contributions must cancel in physical classical quantities. Two types of cancellations occur at one-loop level: cancellations between the box and crossed-box, and cancellations due to the Born iteration of lower order terms when calculating the potential [203, 214, 221, 222].

In this paper we compute long-range effects arising from one-loop integrals, which are proportional to the non-analytic factors $S \equiv \pi^2/\sqrt{-\bar{q}^2}$ and $L \equiv \log(-q^2)$.[1] When considering only spinless terms at one-loop order, those proportional to S are classical, and those proportional to L are quantum. With the established $\hbar$ counting, classical terms at one-loop can arise from operators with at most one positive power of $\hbar$, and quantum terms arise from operators with at most two positive powers of $\hbar$. In the operator expansion of HQET/HBET, powers of $\hbar$ come from partial derivatives.

The inclusion of spin slightly complicates this counting. In order to identify spin multipoles with those of the classical angular momentum, we must allow the spin to be arbitrarily large while simultaneously taking the classical limit. More precisely, for a spin S_i the simultaneous limits $S_i \to \infty$, $\hbar \to 0$ must be taken while keeping $\hbar S_i$ constant [223, 224].[2] When considering spin-inclusive parts of the amplitude we must therefore neglect one positive power of $\hbar$ for each power of spin when identifying the classical and quantum contributions. To make the expansion in classical operators explicit, in this paper we keep track only of the factors of $\hbar$ that count towards the determination of the classicality of terms in the amplitudes. Practically, this amounts to rescaling the Dirac sigma matrices in the operators as $\sigma^{\mu\nu} \to \sigma^{\mu\nu}/\hbar$, or the spins in the amplitudes as $S_i \to S_i/\hbar$. At linear order in spin, this leads again to the interpretation at 2PM order of terms proportional to S as being classical, and those proportional to L as being quantum. At quadratic order in spin, however, terms such as $q^3 S$ and qL begin arising, which respectively have quantum and classical $\hbar$ scaling.

Altogether, operators contributing classically contain either up to one deriative, or up to two derivatives and a Dirac sigma matrix, which will be seen to be related to the spin vector.

10.2.2 *Counting at n-loops*

We can extend this analysis to determine which operators can produce classical terms at arbitrary loop order. First we consider two-loop diagrams, contributing 3PM corrections to the classical potential.

[1] In contrast to Refs. [203, 221], we define S in terms of the wavenumber $\bar{q}$ to make powers of $\hbar$ explicit in the amplitude.

[2] The universality of the multipole expansion in gravitational interactions ensures that the expansion remains unchanged in this limit [203, 224].

The highest order operator needed is determined by the most classical $\hbar$-scaling attainable at a given loop order, *i.e*, by the $\hbar$-scaling of the diagram that scales with the most inverse powers of $\hbar$. In Appendix 10.D we show that the leading order $\hbar$-scaling of a graviton-matter vertex is always $\hbar^0$. The Einstein-Hilbert action governing pure graviton vertices involves two derivatives of the graviton field, so that pure graviton vertices always scale as $\hbar^2$. It follows that the most classical diagrams at 3PM are the box and crossed-box with four massive and three massless propagators; we refer to these as ladder diagrams. The overall coupling is $G^3/\hbar^3$, and the two integrals over loop momenta contribute eight positive powers of $\hbar$. In total, the amplitude superficially scales as

$$\mathcal{M}^{(3)}_{\text{ladder}} \sim \frac{1}{\hbar^3}\hbar^8\frac{1}{\hbar^{10}} = \frac{1}{\hbar^5}. \tag{354}$$

At nPM — corresponding to $n-1$ loops — the dominant diagrams in the $\hbar \to 0$ limit are still the ladder diagrams, with n massless propagators and $2(n-1)$ massive propagators. The scaling is then

$$\begin{aligned}\mathcal{M}^{(n)}_{\text{ladder}} &\sim \frac{1}{\hbar^n}\hbar^{4(n-1)}\frac{1}{\hbar^{2n}}\frac{1}{\hbar^{2(n-1)}} \\ &\sim \frac{1}{\hbar^{2+n}}.\end{aligned} \tag{355}$$

From the HBET point of view, this means that we need to include operators that scale with one more power of $\hbar$ whenever we go from nPM to $(n+1)$PM order. Starting with the observation from the previous section of classical operators at 2PM, we will need operators with at most $n-1$ derivatives, or n derivatives and one Dirac sigma matrix, to obtain the full classical correction at nPM. Furthermore, to have a sensible classical limit, all superclassical contributions must cancel in physical quantities. We see that the order of cancellation scales with the number of loops.

Note that, according to this counting, starting at 3PM, spinless terms proportional to L can contribute classically. This is consistent with the classical 3PM scalar-scalar amplitude in Refs. [210, 212].

10.3 HEAVY QUARK EFFECTIVE THEORY

As the concepts and methods we will use to derive HBET are based on those of HQET, we give a brief review of the latter here.

HQET is used in calculations involving a bound state of a heavy quark Q with mass $m_Q \gg \Lambda_{\text{QCD}}$, and a light quark with mass smaller than Λ_{QCD}. The energy scale of the interactions between the light and heavy quark is on the order of the QCD scale, and is thus small compared to the mass of the heavy quark. The momentum p^μ of the system is therefore decomposed into a large part representing the energy of the heavy quark, $m_Q v^\mu$, which is approximately conserved in interactions between the two quarks, and a small residual momentum parameterizing the remaining momentum, k^μ, which is due to the motion of the light quark and interactions between the light and heavy quarks. Altogether,

$$p^\mu = m_Q v^\mu + k^\mu, \qquad |k^\mu| \sim \mathcal{O}(\Lambda_{\text{QCD}}) \text{ where } \Lambda_{\text{QCD}} \ll m_Q. \tag{356}$$

A hierarchy of scales is present, and we can organize an effective theory which expands in this hierarchy.

An interesting feature of HQET, as will be seen below, is that its propagating degrees of freedom are massless. The propagating degrees of freedom carry the residual momentum k^μ. Therefore, since we are interested in classical scattering, we can rewrite the residual momentum according to Eq. (345):

$$p^\mu = m_Q v^\mu + \hbar \bar{k}^\mu. \tag{357}$$

The procedure we will use to derive the HBET Lagrangian for spinors in the next section is identical to that used to derive the HQET Lagrangian. As such, we outline the derivation of the HQET Lagrangian for one quark coupled to a $U(1)$ gauge field.[3] Our starting point is the QED Lagrangian,

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\,(i\not{D} - m)\,\psi, \qquad \text{where} \quad D^\mu\psi \equiv (\partial^\mu + ieA^\mu)\psi. \tag{358}$$

Next, following the pedagogical derivation in Ref. [226], we introduce the projection operators

$$P_\pm \equiv \frac{1 \pm \not{v}}{2}, \tag{359a}$$

and two eigenfunctions of these operators

$$Q \equiv e^{imv\cdot x} P_+ \psi, \tag{359b}$$

$$\tilde{Q} \equiv e^{imv\cdot x} P_- \psi. \tag{359c}$$

[3] The non-abelian case is discussed in *e.g.* Ref. [225].

This allows us to decompose the spinor field as

$$\psi = \frac{1+\not{v}}{2}\psi + \frac{1-\not{v}}{2}\psi = e^{-imv\cdot x}\left(Q + \tilde{Q}\right). \tag{360}$$

The details pertaining to the external states of the fields Q and $\tilde{Q}$ are explained in Appendix 10.A.

Substituting Eq. (360) into Eq. (358), using some simple gamma matrix and projection operator identities, and integrating out $\tilde{Q}$ using its equation of motion, we arrive at the HQET Lagrangian,

$$\mathcal{L}_{\text{HQET}} = \bar{Q}\left(iv\cdot D - \frac{D_\perp^2}{2m} - \frac{e}{4m}\sigma^{\mu\nu}F_{\mu\nu}\right)Q + \frac{1}{2m}\bar{Q}i\not{D}\sum_{n=1}^{\infty}\left(-\frac{iv\cdot D}{2m}\right)^n P_- i\not{D}Q. \tag{361}$$

Here, $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ is the Dirac sigma matrix.

The redundant operators proportional to the leading order equation of motion can be removed by a field redefinition [225] leading to the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{HQET}} = \bar{Q}\Big(iv\cdot D - \frac{D^2}{2m} + \frac{D^4}{8m^3} - \frac{e}{4m}\sigma_{\mu\nu}F^{\mu\nu} - \frac{e}{8m^2}v^\mu[D^\nu F_{\mu\nu}] \\ + \frac{ie}{8m^2}v_\rho\sigma_{\mu\nu}\{D^\mu, F^{\rho\nu}\} + \frac{e}{16m^3}\{D^2, \sigma_{\mu\nu}F^{\mu\nu}\} + \frac{e^2}{16m^3}F_{\mu\nu}F^{\mu\nu}\Big)Q + \mathcal{O}(m^{-4}).\end{aligned} \tag{362}$$

Square brackets enclosing a derivative denote that the derivative acts only within the brackets.

Once Fourier transformed, partial derivatives produce the momentum of the differentiated field. In the specific case of HQET, the partial derivatives produce either a residual momentum (when acting on the spinor field) or a photon momentum (when acting on the vector field) in the Feynman rules. As both types of momenta correspond to massless modes, they both scale with $\hbar$, and hence partial derivatives always result in one positive power of $\hbar$.

We note the appearance of the covariant derivative D^μ in Eq. (362) instead of the orthogonal covariant derivative $D_\perp^\mu \equiv D^\mu - v^\mu(v\cdot D)$ that typically appears in HQET. The two types of the derivative can be swapped before integrating out the anti-field using

$$\bar{Q}\not{D}\tilde{Q} = \bar{Q}\not{D}_\perp\tilde{Q}, \tag{363a}$$

$$\bar{\tilde{Q}}\not{D}Q = \bar{\tilde{Q}}\not{D}_\perp Q. \tag{363b}$$

In Appendix 10.C we use the form with D^μ to more easily compare with previous calculations.

10.4 HEAVY BLACK HOLE EFFECTIVE THEORY

We now turn to the case of a heavy particle minimally coupled to gravity. The derivation of the Lagrangian for a heavy scalar coupled to gravity differs from the derivation of the spinor theory, because the scalar field whose heavy-mass limit we are interested in describing is real. The initial Lagrangian is that of a minimally coupled scalar matter field:

$$\mathcal{L}_{\text{sc-grav}} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{364a}$$

The metric is given by a small perturbation around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where the perturbation $h_{\mu\nu}$ is identified with the graviton.

The heavy-field limit of a real scalar field can be expressed in terms of a complex scalar field χ by employing a suitable field-redefinition. Motivated by earlier analyses in Refs. [227–229], we decompose

$$\phi \to \frac{1}{\sqrt{2m}}\left(e^{-imv\cdot x}\chi + e^{imv\cdot x}\chi^*\right). \tag{365}$$

Substituting this into Eq. (364a) and dropping quickly oscillating terms (those proportional to $e^{\pm 2imv\cdot x}$) gives the HBET Lagrangian for scalars:

$$\mathcal{L}^{s=0}_{\text{HBET}} = \sqrt{-g}\chi^*\left[g^{\mu\nu}iv_\mu\partial_\nu + \frac{1}{2}m(g^{\mu\nu}v_\mu v_\nu - 1) - \frac{1}{2m}g^{\mu\nu}\partial_\mu\partial_\nu\right]\chi + \mathcal{O}(1/m^2). \tag{366}$$

Comparing the Feynman rules for this theory in Appendix 10.D with the Feynman rules for the full theory in Ref. [203], we see that they are related by simply decomposing the momenta as in Eq. (356) and dividing by $2m$.

Next, we consider the case of a heavy spin-1/2 particle. We begin with the Lagrangian of a minimally coupled Dirac field ψ

$$\mathcal{L}_{\text{grav}} = \sqrt{-g}\bar{\psi}\left(ie^\mu{}_a\gamma^a D_\mu - m\right)\psi, \tag{367a}$$

where $e^\mu{}_a$ is a vierbein, connecting curved space (with Greek indices) and flat space (with Latin indices) tensors. The expansion of the vierbein in terms of the metric perturbation is given in Ref. [203]. The covariant derivative is [230]

$$D_\mu \psi \equiv \left(\partial_\mu + \frac{i}{2}\omega_\mu{}^{ab}\sigma_{ab}\right)\psi, \tag{367b}$$

where the spin connection $\omega_\mu{}^{ab}$ is given in terms of vierbeins in Eq. (41) of Ref. [230]. To quadratic order in the graviton field, the spin-connection is [203]

$$\omega_\mu{}^{ab} = -\frac{\kappa}{4}\partial^b h_\mu{}^a - \frac{\kappa^2}{16}h^{\rho b}\partial_\mu h^a{}_\rho + \frac{\kappa^2}{8}h^{\rho b}\partial_\rho h_\mu{}^a - \frac{\kappa^2}{8}h^{\rho b}\partial^a h_\mu{}^\rho - (a \leftrightarrow b). \tag{367c}$$

Eq. (367c) differs from that in Ref. [203] by a factor of $-1/2$. The spin connection of Ref. [230] differs from that of Ref. [203] by this same factor, and we use the connection of Ref. [230].

We make the same decomposition of the fermion field ψ as in HQET, Eq. (360), and integrate out the anti-field by substituting its equation of motion. As in the case of HQET, this gives a non-local form of the HBET Lagrangian, which we expand in $1/m$:

$$\begin{aligned}\mathcal{L}^{s=1/2}_{\text{HBET}} = {} & \sqrt{-g}\bar{Q}\left[ie^\mu{}_a\gamma^a D_\mu + m v_\mu v^a(e^\mu{}_a - \delta^\mu_a)\right]Q \\ & + \frac{\sqrt{-g}}{2m}\bar{Q}\left[ie^\mu{}_a\gamma^a D_\mu + m v_\mu \gamma^a(e^\mu{}_a - \delta^\mu_a)\right] \\ & \times \sum_{n=0}^{\infty}\frac{1}{[1+\frac{1}{2}v_\nu v^b(e^\nu{}_b - \delta^\nu{}_b)]^{n+1}}\frac{F[h]^n}{m^n} \times \left[ie^\rho{}_c\gamma^c D_\rho + m v_\rho \gamma^c(e^\rho{}_c - \delta^\rho_c)\right]Q,\end{aligned} \tag{368a}$$

where

$$F[h] \equiv \frac{i}{2}e^\mu{}_a\frac{1 - v_\nu\gamma^\nu}{2}\gamma^a D_\mu \frac{1 - v_\rho\gamma^\rho}{2} \equiv \frac{i}{2}e^\mu{}_a[\gamma^a D_\mu]_{-v}. \tag{368b}$$

As the operator $F[h]$ contains a covariant derivative, and the other factor in the sum in Eq. (368a) is covariantly conserved, the placement of $F[h]$ to the right of everything in the sum is unambiguous.

We can recover a local form of this Lagrangian by further expanding the denominator of the sum in κ. We will only need vertices involving two spinors and at most two gravitons, so we expand up to $\mathcal{O}(\kappa^2)$. The result is

$$\mathcal{L}_{\text{HBET}}^{s=1/2} = \sqrt{-g}\bar{Q}\left[ie^{\mu}{}_{a}\gamma^{a} D_{\mu} + mv_{\mu}v^{a}(e^{\mu}{}_{a} - \delta^{\mu}_{a})\right] Q \tag{369a}$$
$$+ \frac{\sqrt{-g}}{2m}\bar{Q}\left[ie^{\mu}{}_{a}\gamma^{a} D_{\mu} + mv_{\mu}\gamma^{a}(e^{\mu}{}_{a} - \delta^{\mu}_{a})\right] \sum_{n=0}^{\infty} G_n[h]\frac{F[h]^n}{m^n}\left[ie^{\rho}{}_{c}\gamma^{c} D_{\rho} + mv_{\rho}\gamma^{c}(e^{\rho}{}_{c} - \delta^{\rho}_{c})\right] Q,$$

where

$$G_n[h] \equiv \left\{1 + (n+1)H_1[h]\kappa + \left[\left(\frac{n^2}{2} + \frac{3n}{2} + 1\right) H_1[h]^2 - (n+1)H_2[h]\right]\kappa^2 + \dots\right\}, \tag{369b}$$
$$H_1[h] \equiv \frac{1}{4}v_{\mu}v_{\nu}h^{\mu\nu}, \qquad H_2[h] \equiv \frac{3}{16}v_{\mu}v^{a}h_{a\rho}h^{\mu\rho}. \tag{369c}$$

Though we started with massive matter fields, Eqs. (366), (368a), and (369a) contain no mass terms for the matter fields. The propagating modes of HBET are therefore massless, so their momenta scale with $\hbar$ in the classical limit. As in the case of HQET, this allows us to interpret the operator expansion of HBET as an expansion in $\hbar$.

The Feynman rules of both theories (Appendix 10.D) are suggestive of the universality of the multipole expansion from Ref. [203]; all terms present in the scalar Feynman rules also appear in the spinor Feynman rules. There are, of course, extra terms in the spinor Feynman rules which encode spin effects. Moreover, we find additional spin-independent terms in the spinor Feynman rules that do not appear in the scalar rules. This is not necessarily inconsistent with Ref. [203]: as will be discussed further below, we expect these additional terms to not contribute to the properly defined potential at one-loop level.

10.5 LONG RANGE $2 \to 2$ GRAVITATIONAL SCATTERING AMPLITUDES

We will demonstrate the utility of the above EFTs for systems of two heavy particles. We do so by calculating the amplitudes for the scattering of scalars and fermions mediated by gravitons up to the leading quantum order at one-loop level. To maximize the efficiency of the computation of the following amplitudes, one could obtain them as double copies of HQET amplitudes. Focusing on the validation of HBET, however, we compute them using standard Feynman diagram techniques applied

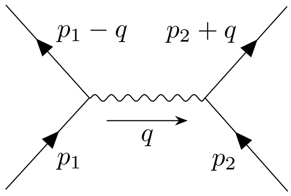


Figure 10.5.1: Classical scattering of two particles at tree-level.

directly to the HBET Lagrangians in Eqs. (366) and (369a), with graviton dynamics described by the usual Einstein-Hilbert action,

$$S_{\mathrm{GR}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R. \tag{370}$$

To obtain the classical portions of the amplitudes, we use only the HBET operators described in Section 10.2. The leading quantum terms arise by also including operators that scale with one more factor of $\hbar$.

In what follows we make use of the reparameterization invariance of HBET [231–233] to work in a frame in which the initial momenta are $p_i^\mu = m_i v_i^\mu$, where v_i^μ is the initial four-velocity of particle i. We then define $\omega \equiv v_{1\mu}v_2^\mu$, which, in such a frame, is related to the Mandelstam variable $s = (p_1 + p_2)^2$ via

$$s - s_0 = 2m_1 m_2(\omega - 1), \tag{371}$$

where $s_0 \equiv (m_1 + m_2)^2$. From Eq. (371) it is evident that the non-relativistic limit of the kinematics of both particles, $s - s_0 \to 0$, is equivalent to the limit $\omega \to 1$. As a check on the results, we reproduce the amplitudes in Ref. [203] in the non-relativistic limit.

Amplitudes for scalar-scalar scattering arise as a portion of the fermion-fermion scattering amplitude [203]. For this reason we present here the amplitudes for fermion-fermion scattering.

10.5.1 *First Post-Minkowskian Order*

At 1PM order, the relevant diagram is the tree-level graviton exchange diagram, shown in Fig. 10.5.1. Using the $\hbar$-counting in Section 10.2, we see that the coupling constants provide one inverse power of $\hbar$, while the graviton propagator scales as $1/\hbar^2$. The leading tree-level amplitude becomes

$$\mathcal{M}_t^{(1)} = -\frac{4\pi m_1 m_2 G}{\hbar^3 q^2}\left[(2\omega^2-1)\mathcal{U}_1\mathcal{U}_2 + \frac{2i\omega}{m_1^2 m_2}\mathcal{E}_1\mathcal{U}_2 + \frac{2i\omega}{m_1 m_2^2}\mathcal{E}_2\mathcal{U}_1 \right.$$
$$\left. - \frac{1}{m_1^3 m_2^3}\mathcal{E}_1\mathcal{E}_2 + \frac{\omega}{m_1^2 m_2^2}\mathcal{E}_1^\mu \mathcal{E}_{2\mu}\right]. \tag{372}$$

This is in agreement with Ref. [203] at leading order in $\mathcal{O}(|q|)$. We use the shorthand notation

$$\mathcal{U}_1 \equiv \bar{u}(p_1-q)u(p_1) \equiv \bar{u}_2 u_1, \tag{373a}$$

$$\mathcal{U}_2 \equiv \bar{u}(p_2+q)u(p_2) \equiv \bar{u}_4 u_3, \tag{373b}$$

$$\mathcal{E}_i \equiv \epsilon^{\mu\nu\alpha\beta} p_{1\mu} p_{2\nu} \bar{q}_\alpha S_{i\beta}, \tag{373c}$$

$$\mathcal{E}_i^\mu \equiv \epsilon^{\mu\nu\alpha\beta} p_{i\nu} \bar{q}_\alpha S_{i\beta}, \tag{373d}$$

with the relativistic normalization of the spinors, $\bar{u}(p)u(p) = 2m$. The Levi-Civita tensor is defined by $\epsilon^{0123} = 1$. The spin vector is defined as

$$S_i^\mu \equiv \frac{1}{2}\bar{u}_{2i}\gamma_5\gamma^\mu u_{2i-1}, \tag{373e}$$

where $\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$. The definition of the HQET spinor in Eq. (359b) automatically imposes the orthogonality of the spin vector and the momentum of the corresponding particle, since it implies the relation $\not{v}u = u$.

10.5.2 *Second Post-Minkowskian Order*

At 2PM order, eleven one-loop diagrams can contribute, shown in Fig. 10.5.2. Only triangles or box diagrams contribute to the classical amplitude, but as we also compute the leading quantum contributions, all eleven diagrams are needed.

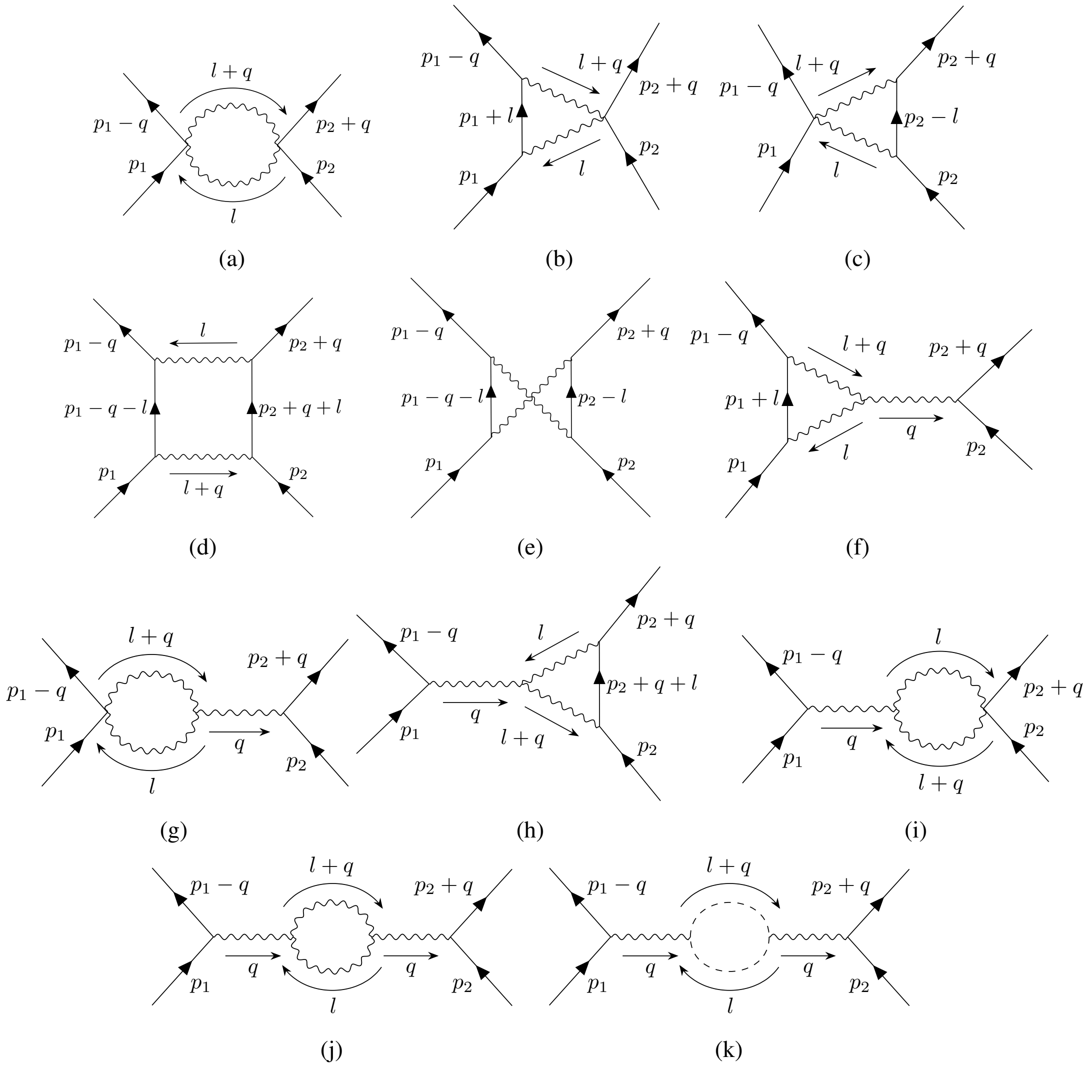


Figure 10.5.2: The one-loop Feynman diagrams containing non-analytic pieces that contribute to the classical scattering of two particles in GR. Solid lines represent fermions, wavy lines represent gravitons, and dashed lines represent the ghost field arising from working in the harmonic gauge [203].

For clarity, we split the 2PM amplitude into three parts: the spinless, spin-orbit, and spin-spin contributions. These are, respectively,

$$\begin{aligned}
&\mathcal{M}^{(2)}_{\text{spinless}} = \\
&\quad \frac{G^2}{\hbar^3} m_1 m_2 \mathcal{U}_1 \mathcal{U}_2 S \frac{3}{2} (5\omega^2 - 1)(m_1 + m_2) \\
&\quad + \frac{G^2 \mathcal{U}_1 \mathcal{U}_2 L}{30\hbar^2(\omega^2-1)^2} \Big[2m_1 m_2 (18\omega^6 - 67\omega^4 + 50\omega^2 - 1) \\
&\quad -60 m_1 m_2 \omega (12\omega^4 - 20\omega^2 + 7) L_\times(\omega) \sqrt{\omega^2-1} \; -15 i\pi (m_1^2 + m_2^2)(24\omega^4 - 37\omega^2 + 13)\sqrt{\omega^2-1} \\
&\quad - \frac{120}{\hbar^2 \bar{q}^2} i\pi m_1^2 m_2^2 (4\omega^6 - 8\omega^4 + 5\omega^2 - 1)\sqrt{\omega^2-1} \Bigg],
\end{aligned} \tag{374a}$$

$$\begin{aligned}
&\mathcal{M}^{(2)}_{\text{spin-orbit}} = \\
&\quad \frac{G^2 m_1 m_2 \omega (5\omega^2 - 3) S}{2\hbar^3 (\omega^2 - 1)} \left[(3m_1 + 4m_2) \frac{i\mathcal{U}_1 \mathcal{E}_2}{m_1 m_2^2} \right] \\
&\quad + \frac{G^2 L}{10\hbar^2 (\omega^2-1)^2} \Big\{ 2m_1 m_2 \omega (\omega^2 - 1)(46\omega^2 - 31) - 20 m_2^2 i\pi\omega(\omega^2 - 2)\sqrt{\omega^2-1} \\
&\quad - \frac{80}{\hbar^2 \bar{q}^2} i\pi m_1^2 m_2^2 \omega (\omega^2 - 1)(2\omega^2 - 1)\sqrt{\omega^2-1} - 5 m_1^2 i\pi\omega (12\omega^4 - 10\omega^2 - 5)\sqrt{\omega^2-1} \\
&\quad -5 m_1 m_2 [(40\omega^4 - 48\omega^2 + 7) L_\times(\omega) - (8\omega^4 - 1)(L_\square(\omega) + i\pi)]\sqrt{\omega^2-1} \Big\} \frac{i\mathcal{U}_1 \mathcal{E}_2}{m_1 m_2^2} + (1 \leftrightarrow 2),
\end{aligned} \tag{374b}$$

$$\begin{aligned}
&\mathcal{M}^{(2)}_{\text{spin-spin}} = \\
&\quad G^2 (m_1 + m_2) \frac{S}{\hbar^3} \left[\frac{(20\omega^4 - 21\omega^2 + 3)}{2(\omega^2-1)} (\bar{q} \cdot S_1 \bar{q} \cdot S_2 - \bar{q}^2 S_1 \cdot S_2) + \frac{2\bar{q}^2 \omega^3 (5\omega^2 - 4)}{m_1 m_2 (\omega^2-1)^2} p_2 \cdot S_1 p_1 \cdot S_2 \right] \\
&\quad + \frac{G^2 L}{\hbar^3 m_1 m_2} [m_1 C_1(m_1, m_2) p_2 \cdot S_1 \bar{q} \cdot S_2 - m_2 C_1(m_2, m_1) \bar{q} \cdot S_1 p_1 \cdot S_2] \\
&\quad + \frac{G^2 L}{60 m_1 m_2 \hbar^2 (\omega^2-1)^2} (2C_2 \bar{q} \cdot S_1 \bar{q} \cdot S_2 + C_3 \bar{q}^2 S_1 \cdot S_2) + \frac{G^2 \bar{q}^2 L}{20\hbar^4 m_1^2 m_2^2 (\omega^2-1)^{5/2}} C_4 p_2 \cdot S_1 p_1 \cdot S_2,
\end{aligned} \tag{374c}$$

where

$$L_{\square}(\omega) \equiv \log\left|\frac{\omega-1-\sqrt{\omega^2-1}}{\omega-1+\sqrt{\omega^2-1}}\right|, \tag{374d}$$

$$L_{\times}(\omega) \equiv \log\left|\frac{\omega+1+\sqrt{\omega^2-1}}{\omega+1-\sqrt{\omega^2-1}}\right|, \tag{374e}$$

$$C_1(m_i,m_j) \equiv \frac{(8\omega^4-8\omega^2+1)}{(\omega^2-1)^{3/2}} i\pi(m_i+\omega m_j), \tag{374f}$$

$$\begin{aligned} C_2 \equiv\, & 60m_1m_2\omega\Big((L_{\square}(\omega)+i\pi)(4\omega^4-2\omega^2-1)+L_{\times}(\omega)(-8\omega^4+14\omega^2-5)\Big)\sqrt{\omega^2-1} \\ & -\frac{120}{\hbar^2\bar{q}^2} i\pi m_1^2m_2^2(\omega^2-1)(1-2\omega^2)^2\sqrt{\omega^2-1} \\ & -30i\pi(m_1^2+m_2^2)(2\omega^6-4\omega^4-\omega^2+2)\sqrt{\omega^2-1} \\ & +2m_1m_2(\omega^2-1)\Big(258\omega^4-287\omega^2+29\Big), \end{aligned} \tag{374g}$$

$$\begin{aligned} C_3 \equiv\, & 60m_1m_2\omega\Big((L_{\square}(\omega)+i\pi)(3-4\omega^2)+12L_{\times}(\omega)(2\omega^4-3\omega^2+1)\Big)\sqrt{\omega^2-1} \\ & +\frac{120}{\hbar^2\bar{q}^2} i\pi m_1^2m_2^2(\omega^2-1)(8\omega^4-8\omega^2+1)\sqrt{\omega^2-1} \\ & +15i\pi(m_1^2+m_2^2)(8\omega^6-12\omega^4-3\omega^2+5)\sqrt{\omega^2-1} \\ & +4m_1m_2(\omega^2-1)\Big(-258\omega^4+287\omega^2-44\Big), \end{aligned} \tag{374h}$$

$$C_4 \equiv -\frac{40}{\bar{q}^2} i\pi m_1^2m_2^2\omega\Big(\omega^2-1\Big)\Big(8\omega^4-8\omega^2+1\Big). \tag{374i}$$

The classical contributions are in the first lines of each of Eqs. (374a)-(374b), and in the first and second lines of Eq. (374c). The classical spinless contribution is in agreement with Ref. [213]. The classical spin-orbit contribution is consistent with the spin holonomy map of Ref. [234]. The classical spin-spin contribution compliments the results in Ref. [209]. In particular, we find that the coefficient of $(-q \cdot S_1 q \cdot S_2)$ in Eq. (374c) agrees with $A_{1,1}^{2\,\mathrm{PM}}$ in Eq. (7.18) in Ref. [209], which is the corresponding coefficient in the Leading Singularity approach [205, 206], whereas the remainder of the terms are not presented therein. To the best of our knowledge, this is the first presentation of the leading quantum contributions to the spinless, spin-orbit and spin-spin amplitudes at 2PM order. There are additional spin-spin terms at the quantum level proportional to $p_i \cdot S_j$ for $i \neq j$ that we have not included in our calculation. We note that additional spin quadrupole terms are also present at the second order in spin, which can be calculated from vector-scalar scattering.

To obtain the results in this section we have made use of the identity

$$\bar{u}_{2i}\sigma^{\mu\nu}u_{2i-1} = -2\epsilon^{\mu\nu\alpha\beta}v_{i\alpha}S_{i\beta}, \tag{375}$$

which is valid for HQET spinors. This identity merits some discussion. Replacing the HQET spinors by Dirac spinors (denoted with a subscript D), the identity becomes

$$\bar{u}_{2i,\mathrm{D}}\sigma^{\mu\nu}u_{2i-1,\mathrm{D}} = -2ip^{[\nu}_{2i-1}\bar{u}_{2i,\mathrm{D}}\gamma^{\mu]}u_{2i-1,\mathrm{D}} - \frac{2}{m}\epsilon^{\mu\nu\alpha\beta}p_{2i-1,\alpha}S_{i,\mathrm{D}\beta}. \tag{376}$$

The second term above is the same as in Eq. (375). The first, by contrast, arises only with Dirac spinors, and through the Gordon decomposition contains both a spinless term involving only the spinor product $\mathcal{U}_i$, and a term like that on the left hand side of the equation. Eq. (376) thereby mixes spinless and spin-inclusive effects. This is an advantage of this EFT approach, at least at one-loop level. Eq. (375) allows one to target spinless or spin-inclusive terms in the amplitude simply by ignoring or including operators involving the Dirac sigma matrix. It is also consistent with the universality of the spin-multipole expansion observed in Refs. [203, 221], where spin effects were found to not mix with, and to be corrections to the universal spin-independent amplitude.

At face value, there is one complication to this interpretation of Eq. (375). Due to the heavy propagators, terms such as $\overline{u}\sigma^{\mu\nu}(1+\not{v})\sigma^{\alpha\beta}u$ begin to arise at one-loop level. Through some gamma matrix manipulations, it can be shown that these terms contain spinless (containing no sigma matrices) and spin-inclusive (containing one sigma matrix) components. At one-loop level the spinless components contribute to the classical and leading quantum portions of the spinless part of the amplitude only through the term proportional to $(m_1^2+m_2^2)L$. As this term is purely imaginary, we expect it to be subtracted by the Born iteration when extracting the potential. Thus, if one is interested only in non-imaginary terms at one-loop, spinless or spin-inclusive terms can be independently targeted by exploiting the separation of spin effects at the level of the Lagrangian.

While spinless and spin-inclusive effects are cleanly separated in spinor HBET, the presence of additional spin-independent operators in spinor HBET compared to scalar HBET makes it ostensibly possible that the spinless parts of its amplitudes differ from the amplitudes of scalar HBET. In fact, calculating scalar-scalar scattering explicitly with scalar HBET, we find that the term proportional to $(m_1^2+m_2^2)L$ in Eq. (374a) does not arise. In addition to receiving contributions from the $\overline{u}\sigma^{\mu\nu}(1+\not{v})\sigma^{\alpha\beta}u$ tensor structure in the loop amplitudes — a structure that certainly does not arise in scalar HBET — it is also the only term that is affected by the spin-independent operators in spinor HBET that are not present in scalar HBET. We therefore find that we preserve the universality of the multipole expansion from Ref. [203] in the one-loop relativistic regime as well, up to terms which are subtracted by the Born iteration.

As a check on the validity of our results, we compare their non-relativistic limits with what exists in the literature, simply by taking the limit $\omega - 1 \to 0$ in the PM amplitudes. At 1PM order we find that our results agree with those in Ref. [203]. At 2PM the amplitudes above contain those in Ref. [203], but there are two discrepancies:

1. We find an additional spinless term that we expect to be subtracted by Born iteration, arising from the imaginary term proportional to $(m_1^2 + m_2^2)L$.

2. The contraction $p_i \cdot S_j$ for $i \neq j$ vanishes in the non-relativistic limit. However, these terms in Eq. (374c) also have denominators that vanish in this limit. Without knowing explicitly how $p_i \cdot S_j \to 0$, we therefore cannot say that these terms will not remain in the limit.

We note that this limit only represents the non-relativistic limit of the kinematics; the non-relativistic limit of the spinors must also be taken in order to obtain the fully non-relativistic amplitude.

10.6 CONCLUSION

While significant progress has been made in understanding the relationship between gravitational scattering amplitudes and classical gravitational quantities, it remains uneconomical to extract the few classically contributing terms from the multitude of other terms that constitute the full amplitude. With an eye to addressing this inefficiency, we have introduced HBET, an EFT which describes the interactions of heavy scalars and heavy fermions with gravity. By restoring $\hbar$ at the level of the Feynman rules, we have been able to infer the $\hbar$-scaling of HBET operators, and exploit it to determine which operators can contribute to the classical amplitude at arbitrary loop order. One may see the present construction as a step towards isolating just those terms of the scattering amplitude that will contribute to the classical scattering of two massive objects, order by order in the loop expansion. Crucially, a method does not yet exist to convert fully relativistic amplitudes including spin to interaction potentials.

We used HBET to directly calculate the 2PM classical gravitational scattering amplitude for the scattering of two fermions, and checked that the spinless part of the amplitude matches the amplitude for scalar-scalar scattering, up to terms that we expect to be subtracted from the potential. To validate the EFT, we compared the fully relativistic amplitudes and their non-relativistic limits with what has been previously calculated, and found agreement. We presented the classical and leading quantum spinless, spin-orbit and spin-spin contributions at 2PM order, up to terms proportional to $p_i \cdot S_j$ where $i \neq j$ for the spin-spin contribution, complementing and extending the results in the literature.

While we derived HBET only for heavy particles of spin $s \leq 1/2$, we believe it is possible to also derive an HBET for heavy higher-spin particles: as long as a Lagrangian can be written for a massive particle of spin s, we can apply similar techniques to those herein to derive the HBET applicable for spin s. This would allow the computation of the classical amplitude for higher order terms in the multipole expansion.

For full efficiency, the HBET formalism should be used in combination with modern scattering amplitude techniques. First, the Feynman rules of scalar and spinor HBET, and the property in Eq. (375), are suggestive of the universality of the multipole expansion presented in Ref. [203]. An interesting next step is to express the degrees of freedom of HBET in terms of massive on-shell variables [235]. It would be interesting to study whether this universality can be made manifest in such variables, and how the observed separation of spinless and spin-inclusive effects arises. An on-shell formulation of HBET should also include an explicit $\hbar$ expansion, further elucidating the classical limit for amplitudes computed using on-shell variables. Moreover, the work in Refs. [209, 224, 235] suggests that massive on-shell variables may facilitate the extension of HBET to higher spins. Second, as HQET is derived from QCD, and HBET is derived from GR, we expect the double copy structure of the scattering amplitudes to still hold as a relation between the effective theories. While certainly not the only way to study such a relation, we expect it to be more readily apparent in on-shell versions of HQET and HBET. We leave the on-shell formulation of HBET for future work.

By combining the power counting (which includes the $\hbar$ counting) and multipole expansion of the effective field theories with the on-shell formalism, unitarity methods, and the double copy, we believe that higher order calculations are within reach.

APPENDIX

10.A HQET SPINORS

In this section, we make precise the external states of the HQET spinor field by expressing them in terms of the external states of the original Dirac spinor field ψ. To do so, we begin with the mode expansion of ψ:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s_{\mathrm{D}}(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} w^s_{\mathrm{D}}(p) e^{ip\cdot x} \right), \tag{377}$$

where $\mathbf{p}$ represents the three-momentum, $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, s is a spin index, and $a^s_{\mathbf{p}}$ and $b^{s\dagger}_{\mathbf{p}}$ are annihilation and creation operators for the particle and antiparticle respectively. We use the unconventional notation w_D for the antiparticle spinor to differentiate it from the four-velocity. The spinors $u^s_{\mathrm{D}}(p)$ and $w^s_{\mathrm{D}}(p)$ satisfy the Dirac equation,

$$(\not{p} - m) u^s_{\mathrm{D}}(p) = 0, \tag{378a}$$

$$(\not{p} + m) w^s_{\mathrm{D}}(p) = 0. \tag{378b}$$

Recall the definition of the HQET spinor field Q_v,

$$Q_v = e^{imv\cdot x} \frac{1+\not{v}}{2} \psi, \tag{379}$$

where v^μ is defined by the HQET momentum decomposition in Eq. (356). The mode expansion for Q_v is then

$$Q_v(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} \frac{1+\not{v}}{2} u^s_{\mathrm{D}}(p) e^{-ik\cdot x} + b^{s\dagger}_{\mathbf{p}} \frac{1+\not{v}}{2} w^s_{\mathrm{D}}(p) e^{i(2mv+k)\cdot x} \right). \tag{380}$$

After the decomposition in Eq. (356), the Dirac equation can be rewritten as

$$\not{v} u^s_{\mathrm{D}}(p) = \left(1 - \frac{\not{k}}{m}\right) u^s_{\mathrm{D}}(p), \tag{381a}$$

$$\not{v} w^s_{\mathrm{D}}(p) = -\left(1 - \frac{\not{k}}{m}\right) w^s_{\mathrm{D}}(p). \tag{381b}$$

Using this in the mode expansion for Q_v we find

$$Q_v(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s_v(p) e^{-ik\cdot x} + b^{s\dagger}_{\mathbf{p}} w^s_v(p) e^{i(2mv+k)\cdot x} \right). \tag{382a}$$

where

$$u^s_v(p) \equiv \left(1 - \frac{\not{k}}{2m}\right) u^s_{\mathrm{D}}(p), \tag{382b}$$

$$w^s_v(p) \equiv \frac{\not{k}}{2m} w^s_{\mathrm{D}}(p). \tag{382c}$$

Similarly, the mode expansion of $\tilde{Q}_v$ is

$$\tilde{Q}_v = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} \frac{\not{k}}{2m} u^s_D(p) e^{-ik\cdot x} + b^{s\dagger}_{\mathbf{p}} \left(1 - \frac{\not{k}}{2m}\right) w^s_D(p) e^{i(2mv+k)\cdot x} \right). \tag{383}$$

The mode expansion in Eq. (382a) makes it apparent that, when considering only particles and not antiparticles, the derivative of Q_v translates to a factor of the residual momentum k^μ in the Feynman rules.

10.B HEAVY SCALAR EFFECTIVE THEORY

For completeness we include the derivation of an effective theory for Scalar Quantum Electrodynamics (SQED). That is, we want the effective theory that arises when ϕ in

$$\mathcal{L}_{\mathrm{SQED}} = (D_\mu \phi)^* D^\mu \phi - m^2 \phi^2, \qquad D^\mu \phi = (\partial^\mu + ieA^\mu)\phi, \tag{384}$$

is very massive. To do so, we simply make the field redefinition [226]

$$\phi \to \frac{e^{-imv\cdot x}}{\sqrt{2m}} (\chi + \tilde{\chi}) . \tag{385}$$

The anti-field $\tilde{\chi}$ is to be integrated out. At leading order, we can drop this term. Inserting Eq. (385) into Eq. (384), and performing a field redefinition to eliminate redundant operators, we obtain Heavy Scalar Effective Theory (HSET):

$$\mathcal{L}_{\mathrm{HSET}} = \chi^* \left(iv \cdot D - \frac{D^2}{2m} \right) \chi + \mathcal{O}(1/m^3). \tag{386}$$

Higher order terms can be restored by keeping contributions coming from integrating out the anti-field.

10.C LONG RANGE $2 \to 2$ ELECTROMAGNETIC SCATTERING AMPLITUDES

In this section we demonstrate that HSET and HQET can be used to calculate the classical and leading quantum contributions to the $2 \to 2$ scattering amplitudes. We present here the results up to one-loop order. As in the gravity case, electromagnetic interactions also possess a universal spin-multipole expansion [221], so we present this calculation using HQET.

At tree level, the diagram in Fig. 10.5.1 is once again the only one that contributes. The amplitude is, up to leading order in $|q|$,

$$\mathcal{A}^{(0)} = \frac{4\pi\alpha}{\hbar^3 q^2} \left[\omega \mathcal{U}_1 \mathcal{U}_2 + \frac{i\mathcal{U}_1 \mathcal{E}_2}{m_1 m_2^2} + \frac{i\mathcal{E}_1 \mathcal{U}_2}{m_1^2 m_2} + \frac{\omega}{m_1 m_2} \mathcal{E}_1^{\mu} \mathcal{E}_{2\mu} \right] . \tag{387}$$

This amplitude is in agreement with Ref. [221] in the relativistic and non-relativistic regimes.

At one-loop level, the abelian nature of QED reduces the number of relevant diagrams compared to the gravity case. There are only five relevant diagrams in the electromagnetic case: they are diagrams (a) to (e) in Fig. 10.5.2. Of course, the wavy lines are reinterpreted as photons. We find the amplitude

$$\begin{aligned}
\mathcal{A}^{(1)}_{\text{spinless}} &= \frac{\alpha^2}{\hbar^3 m_1 m_2}\left[S(m_1+m_2) - \frac{\hbar L}{2m_1 m_2(\omega^2-1)^2}\Big(2m_1m_2(\omega^4-1)\right.\\
&\quad +4m_1m_2\omega(\omega^2-2)L_\times(\omega)\sqrt{\omega^2-1}\\
&\quad +(m_1^2+m_2^2)i\pi(\omega^2-1)^2\sqrt{\omega^2-1}\\
&\quad \left.+\frac{8i\pi}{\hbar^2\bar{q}^2}m_1^2m_2^2\omega^2(\omega^2-1)\sqrt{\omega^2-1}\Big)\right]\mathcal{U}_1\mathcal{U}_2, &\text{(388a)}\\
\mathcal{A}^{(1)}_{\text{spin-orbit}} &= \frac{\alpha^2}{4\hbar^3 m_1 m_2(\omega-1)}S(2m_2(\omega+1)-m_1\omega(\omega-3))\frac{i\mathcal{U}_1\mathcal{E}_2}{m_1m_2^2}\\
&\quad +\frac{\alpha^2 L}{4\hbar^2 m_1^2 m_2^2}\Big[4m_2^2(i\pi\omega\sqrt{\omega^2-1}+\omega^2-1)\\
&\quad +2m_1m_2((2\omega^2+1)(L_\square(\omega)+i\pi)-(\omega^2-2)L_\times(\omega))\sqrt{\omega^2-1}\\
&\quad +m_1^2\omega(-2(2\omega^2-3)(L_\square(\omega)+i\pi)-(5\omega^2-7)L_\times(\omega))\sqrt{\omega^2-1}\Big]\frac{i\mathcal{U}_1\mathcal{E}_2}{m_1m_2^2}\\
&\quad -\frac{4i\pi\alpha^2 L\omega}{\hbar^4\bar{q}^2\sqrt{\omega^2-1}}\frac{i\mathcal{U}_1\mathcal{E}_2}{m_1m_2^2}+(1\leftrightarrow 2), &\text{(388b)}\\
\mathcal{A}^{(1)}_{\text{spin-spin}} &= \frac{\alpha^2 S(m_1+m_2)}{\hbar^3 m_1^2 m_2^2(\omega^2-1)}\left[(2\omega^2-1)(q\cdot S_1 q\cdot S_2 - q^2 S_1\cdot S_2)+\frac{2q^2\omega^3}{m_1m_2(\omega^2-1)}p_2\cdot S_1 p_1\cdot S_2\right]\\
&\quad +\frac{\alpha^2}{\hbar^2 m_1 m_2}[m_1C_1'(m_1,m_2)p_2\cdot S_1 q\cdot S_2 - m_2 C_1'(m_2,m_1)q\cdot S_2 p_1\cdot S_2]\\
&\quad +\frac{\alpha^2 L}{2\hbar^2 m_1^3 m_2^3(\omega^2-1)^2}(C_2' q\cdot S_1 q\cdot S_2 + 2C_3' q^2 S_1\cdot S_2)\\
&\quad +\frac{\alpha^2 L}{2m_1^3m_2^3(\omega^2-1)^{5/2}}C_4' p_2\cdot S_1 p_1\cdot S_2, &\text{(388c)}
\end{aligned}$$

where

$$C_1'(m_i,m_j) \equiv -\frac{q^2 S}{m_i^3 m_j^3(\omega^2-1)^2}[m_i^2(3\omega^4+8\omega^2-3)+4m_im_j(\omega+1)^2(2\omega-1)+2m_j^2\omega(5\omega^2-1)] + \frac{L(2\omega^2-1)}{\hbar m_i^2 m_j^2} i\pi(m_i+\omega m_j), \tag{388d}$$

$$\begin{aligned} C_2' \equiv\ & 4m_1m_2\omega\left(L_\times(\omega)+L_\square(\omega)\omega^2+i\pi\omega^2\right)\sqrt{\omega^2-1} \\ & - i\pi(m_1^2+m_2^2)\left(2\omega^4-5\omega^2+1\right)\sqrt{\omega^2-1} \\ & + 6m_1m_2\left(\omega^2-1\right)^2 - \frac{8}{\hbar^2\bar{q}^2} i\pi m_1^2m_2^2\omega^2\left(\omega^2-1\right)\sqrt{\omega^2-1}, \end{aligned} \tag{388e}$$

$$\begin{aligned} C_3' \equiv\ & 2m_1m_2\omega\left(-L_\square(\omega)+2L_\times(\omega)\left(\omega^2-1\right)-i\pi\right)\sqrt{\omega^2-1} \\ & + i\pi(m_1^2+m_2^2)\left(2\omega^4-4\omega^2+1\right)\sqrt{\omega^2-1} \\ & + 2m_1m_2\left(\omega^2-1\right)\left(2-3\omega^2\right) + \frac{4}{\hbar^2\bar{q}^2} i\pi m_1^2m_2^2\left(\omega^2-1\right)\left(2\omega^2-1\right)\sqrt{\omega^2-1}, \end{aligned} \tag{388f}$$

$$\begin{aligned} C_4' \equiv\ & -\frac{4}{\hbar^2\bar{q}^2} i\pi m_1^2m_2^2\omega\left(\omega^2-1\right)\left(2\omega^2-1\right) + 6m_1m_2\omega^3\sqrt{\omega^2-1} \\ & + m_1m_2\left(\left(L_\square(\omega)+i\pi\right)\left(2\omega^4+5\omega^2-1\right)+L_\times(\omega)\left(-2\omega^4+3\omega^2-1\right)\right) \\ & - i\pi\omega\left(2\omega^4-6\omega^2+1\right)\left(m_1^2+m_2^2\right). \end{aligned} \tag{388g}$$

The non-relativistic limit of this amplitude is in agreement with Ref. [221], with discrepancy number 2 from the gravitational case applying here as well.

Calculating explicitly the amplitude for scalar-scalar scattering using HSET, we find the same amplitude as in Eq. (388a), but without the imaginary term proportional to $(m_1^2+m_2^2)L$. This term vanishes in the non-relativistic limit, thus preserving the non-relativistic universality of the multipole expansion in Ref. [221]. Furthermore, we expect it to be subtracted by the Born iteration when calculating the potential, thus extending the multipole univerality to the relativistic potential.

10.D FEYNMAN RULES

We list here the Feynman rules used to perform the calculations in this paper. Below we denote the matter wave vector entering the vertex by k_1 and the matter wave vector leaving by k_2. q_1 and q_2 are incoming photon (graviton) wave vectors with indices μ,ν $(\mu\nu,\alpha\beta)$, respectively.

We use the photon propagator in the Feynman gauge. The graviton propagator, three graviton vertex, as well as the ghost propagator and two-ghost-one-graviton vertex are given in the harmonic gauge in Ref. [203].

10.D.1 *Abelian HSET*

Starting with HSET, the one- and two-photon vertex Feynman rules are

$$\tau^{\mu}_{\chi\chi^*\gamma}(m,v,k_1,k_2) = -\frac{ie}{\sqrt{\hbar}}\left[v^{\mu} + \frac{\hbar}{2m}(k_1^{\mu}+k_2^{\mu}) + \mathcal{O}\left(\frac{\hbar^3}{m^3}\right)\right], \tag{389a}$$

$$\tau^{\mu\nu}_{\chi\chi^*\gamma\gamma}(m,v,k_1,k_2) = \frac{ie^2}{m\hbar}\left[\eta^{\mu\nu} + \mathcal{O}\left(\frac{\hbar^2}{m^2}\right)\right]. \tag{389b}$$

10.D.2 *Scalar HBET*

For HSBET the one- and two-graviton vertex Feynman rules are

$$\begin{aligned}\tau^{\mu\nu}_{\chi\chi^*h}(m,v,k_1,k_2) = -\frac{i\kappa}{2\sqrt{\hbar}}\Big\{ & mv^{\mu}v^{\nu} - \frac{\hbar}{2}[\eta^{\mu\nu}v_{\rho}(k_1^{\rho}+k_2^{\rho}) - v^{\mu}(k_1^{\nu}+k_2^{\nu}) - v^{\nu}(k_1^{\mu}+k_2^{\mu})] \\ & + \frac{\hbar^2}{2m}\left[(k_1^{\mu}k_2^{\nu}+k_2^{\mu}k_1^{\nu}) - \eta^{\mu\nu}k_{1\alpha}k_2^{\alpha}\right] + \mathcal{O}\left(\frac{\hbar^3}{m^2}\right)\Big\},\end{aligned} \tag{390a}$$

$$\begin{aligned}\tau^{\mu\nu,\alpha\beta}_{\chi\chi^*hh}(m,v,k_1,k_2) = \frac{i\kappa^2}{\hbar}\Big\{ & mv_{\tau}v_{\lambda}\left[I^{\mu\nu,\tau\gamma}I_{\gamma}{}^{\lambda,\alpha\beta} - \frac{1}{4}(\eta^{\mu\nu}I^{\alpha\beta,\tau\lambda} + \eta^{\alpha\beta}I^{\mu\nu,\tau\lambda})\right] \\ & + \frac{\hbar}{4}\{-P^{\mu\nu,\alpha\beta}v_{\rho}(k_1^{\rho}+k_2^{\rho}) - (\eta^{\mu\nu}I^{\alpha\beta,\tau\lambda} + \eta^{\alpha\beta}I^{\alpha\beta,\tau\lambda})v_{\tau}(k_{1\lambda}+k_{2\lambda}) \\ & + 2I^{\mu\alpha,\tau\gamma}I_{\gamma}{}^{\lambda,\nu\beta}[v_{\tau}(k_{1\lambda}+k_{2\lambda}) + v_{\lambda}(k_{1\tau}+k_{2\tau})]\} \\ & + \frac{\hbar^2}{4m}[-P^{\mu\nu,\alpha\sigma}k_{1\rho}k_2^{\rho} - (\eta^{\mu\nu}I^{\alpha\beta,\tau\lambda} + \eta^{\alpha\beta}I^{\mu\nu,\tau\lambda})k_{1\tau}k_{2\lambda} \\ & + 2I^{\mu\alpha,\tau\gamma}I_{\gamma}{}^{\lambda,\nu\beta}(k_{1\tau}k_{2\lambda} + k_{2\tau}k_{1\lambda})] + \mathcal{O}\left(\frac{\hbar^3}{m^2}\right)\Big\},\end{aligned} \tag{390b}$$

where

$$P_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}). \tag{390c}$$

The propagator in both scalar theories is

$$D_v^{s=0}(k) = \frac{i}{\hbar v \cdot k}. \tag{391}$$

10.D.3 *Abelian HQET*

The one- and two-photon Feynman rules in HQET are

$$\begin{aligned}\tau^{\mu}_{\bar{Q}Q\gamma}(m, v, k_1, k_2) = -\frac{ie}{\sqrt{\hbar}} \Big\{ & v^{\mu} + \frac{i}{2m}\sigma^{\mu\nu}(k_{2\nu} - k_{1\nu}) + \frac{\hbar}{2m}(k_1^{\mu} + k_2^{\mu}) \\ & + \frac{i\hbar}{8m^2} v_{\rho}\sigma_{\alpha\beta}(k_1 + k_2)^{\alpha}\left[(k_2^{\rho} - k_1^{\rho})\eta^{\mu\beta} - (k_2^{\beta} - k_1^{\beta})\eta^{\mu\rho}\right] \\ & + \frac{\hbar^2}{8m^2}[v^{\mu}(k_2 - k_1)^2 - v_{\rho}(k_2^{\rho} - k_1^{\rho})(k_2^{\mu} - k_1^{\mu})] \\ & + \frac{i\hbar^2}{8m^3}(k_1^2 + k_2^2)\sigma^{\mu\rho}(k_{2\rho} - k_{1\rho}) + \mathcal{O}\left(\frac{\hbar^3}{m^3}\right)\Big\}, \end{aligned} \tag{392a}$$

$$\begin{aligned}\tau^{\mu\nu}_{\bar{Q}Q\gamma\gamma}(m, v, k_1, k_2) = \frac{ie^2}{m\hbar} \Big\{ & \eta^{\mu\nu} - \frac{i}{4m}[\sigma^{\mu\rho}v^{\nu}q_{1\rho} + \sigma^{\nu\rho}v^{\mu}q_{2\rho} - \sigma^{\mu\nu}v_{\rho}(q_1^{\rho} - q_2^{\rho})] \\ & - \frac{i\hbar}{4m^2}(k_{2\rho} + k_{1\rho})\sigma_{\alpha\beta}(\eta^{\rho\mu}\eta^{\beta\nu}q_1^{\alpha} + \eta^{\rho\nu}\eta^{\beta\mu}q_2^{\alpha}) + \mathcal{O}\left(\frac{\hbar^2}{m^2}\right)\Big\}. \end{aligned} \tag{392b}$$

10.D.4 *Spinor HBET*

Finally, the one- and two-graviton Feynman rules in HBET are

$$\begin{aligned}
\tau^{\mu\nu}_{\bar{Q}Qh}(m,v,k_1,k_2) = \frac{i\kappa}{2\sqrt{\hbar}}\Bigg\{ & -mv^\mu v^\nu + \frac{i}{4}(v^\mu\sigma^{\rho\nu} + v^\nu\sigma^{\rho\mu})(k_{2\rho} - k_{1\rho}) \\
& + \frac{\hbar}{2}\left[v_\alpha(k_1^\alpha + k_2^\alpha)\eta^{\mu\nu} - v^\mu(k_1^\nu + k_2^\nu) - v^\nu(k_1^\mu + k_2^\mu)\right. \\
& \left. + 3v^\mu(k_2^\nu - k_1^\nu) + 3v^\nu(k_2^\mu - k_1^\mu) - 6\eta^{\mu\nu}v_\rho(k_2^\rho - k_1^\rho)\right] \\
& + \frac{i\hbar}{4m}\left[(k_{2\rho} - k_{1\rho})(k_1^\mu\sigma^{\rho\nu} + k_1^\nu\sigma^{\rho\mu}) - v^\mu v^\nu\sigma^{\rho\tau}k_{2\rho}k_{1\tau}\right. \\
& - \frac{1}{2}v_\rho(k_2^\rho k_{2\tau} - k_1^\rho k_{1\tau})(v^\mu\sigma^{\tau\nu} + v^\nu\sigma^{\tau\mu}) \\
& \left. + (k_{2\rho} - k_{1\rho})(k_2^\mu - k_1^\mu)\sigma^{\rho\nu} + (k_{2\rho} - k_{1\rho})(k_2^\nu - k_1^\nu)\sigma^{\rho\mu}\right] \\
& + \frac{\hbar^2}{2m}\left[-(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu) + \eta^{\mu\nu}k_1^\rho k_{2\rho} + \frac{1}{2}v^\mu v^\nu k_{1\rho}k_2^\rho\right. \\
& + \frac{1}{2}\eta^{\mu\nu}(k_{2\rho} - k_{1\rho})(k_2^\rho - k_1^\rho) - \frac{1}{2}(k_2^\mu - k_1^\mu)(k_2^\nu - k_1^\nu) \\
& \left. + \frac{1}{4}v_\rho v^\mu(k_2^\nu k_2^\rho + k_1^\nu k_1^\rho) + \frac{1}{4}v_\rho v^\nu(k_2^\mu k_2^\rho + k_1^\mu k_1^\rho)\right] + \mathcal{O}\left(\frac{\hbar^2}{m^2}\right)\Bigg\}, \qquad (393a)
\end{aligned}$$

$$\begin{aligned}
\tau^{\mu\nu,\alpha\beta}_{\bar{Q}Qhh}(m,v,k_1,k_2) = \frac{i\kappa^2}{\hbar}\Bigg\{ & mv_\kappa v_\lambda\left[I^{\mu\nu,\kappa\gamma}I_\gamma{}^{\lambda,\alpha\beta} - \frac{1}{4}(\eta^{\alpha\beta}I^{\mu\nu,\kappa\lambda} + \eta^{\mu\nu}I^{\alpha\beta,\kappa\lambda})\right] \\
& - \frac{i}{16}\epsilon^{\lambda\rho\tau\delta}\gamma_\delta\gamma_5(I^{\mu\nu,\kappa}{}_\lambda I^{\alpha\beta,}{}_{\tau\kappa}q_{2\rho} + I^{\alpha\beta,\kappa}{}_\lambda I^{\mu\nu,}{}_{\tau\kappa}q_{1\rho}) \\
& + \frac{i}{16}v_\kappa v_\sigma v_\rho\sigma_{\lambda\tau}[I^{\mu\nu,\kappa\lambda}I^{\alpha\beta,\sigma\tau}(q_{2\rho} + k_{1\rho}) + I^{\alpha\beta,\kappa\lambda}I^{\mu\nu,\sigma\tau}(q_{1\rho} + k_{1\rho})] \\
& - \frac{i}{8}v_\kappa\sigma_{\lambda\tau}(k_{1\sigma} - k_{2\sigma})(I^{\mu\nu,\kappa\lambda}I^{\alpha\beta,\sigma\tau} + I^{\alpha\beta,\kappa\lambda}I^{\mu\nu,\sigma\tau}) \\
& - \frac{3i}{16}v_\kappa\sigma_{\sigma\rho}(k_1^\rho - k_2^\rho)(I^{\mu\nu,\kappa\tau}I^{\alpha\beta,\sigma\tau} + I^{\alpha\beta,\kappa\tau}I^{\mu\nu,\sigma\tau}) \\
& + \frac{i}{8}v_\kappa\sigma_{\lambda\rho}(k_1^\rho - k_2^\rho)(\eta^{\mu\nu}I^{\alpha\beta,\kappa\lambda} + \eta^{\alpha\beta}I^{\mu\nu,\kappa\lambda}) \\
& + \frac{i}{16}v_\kappa\sigma_{\lambda\rho}(k_1^\rho - k_2^\rho)(v^\mu v^\nu I^{\alpha\beta,\kappa\lambda} + v^\alpha v^\beta I^{\mu\nu,\kappa\lambda}) \\
& + \frac{i}{8}v_\kappa\sigma_{\lambda\rho}(\eta^{\mu\nu}I^{\alpha\beta,\kappa\lambda}q_1^\rho + \eta^{\alpha\beta}I^{\mu\nu,\kappa\lambda}q_2^\rho) \\
& + \frac{i}{8}v_\rho\sigma_{\lambda\tau}(I^{\mu\nu,\kappa\lambda}I^{\alpha\beta,\rho\tau}q_{1\kappa} + I^{\alpha\beta,\kappa\lambda}I^{\mu\nu,\rho\tau}q_{2\kappa}) + \mathcal{O}(\hbar)\Bigg\}, \qquad (393b)
\end{aligned}$$

where

$$I^{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}). \tag{393c}$$

Based on the $\hbar$ counting, there are additional terms that could contribute to the amplitude, but we find that they contribute only at subleading quantum levels, and thus don't include them.

The propagator in both spinor theories is

$$D_v^{s=\frac{1}{2}}(k) = \frac{i}{\hbar v \cdot k}\frac{1+\not{v}}{2}. \tag{394}$$

10.E ONE-LOOP INTEGRAL BASIS

In this section, we point out some subtleties that arise from the linear matter propagators characteristic of HQET/HBET. We first address the appearance of non-analytical contributions to loop integrals when using linear matter propagators instead of quadratic ones. Then we discuss how we circumvent the infamous pinch singularity of HQET.

10.E.1 *Non-analytic portions of loop integrals*

Consider, for example, the box integral with quadratic massive propagators:

$$I_{\text{quad}} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2(l+q)^2\left[(p_1 - l - q)^2 - m_1^2 + i\epsilon\right]\left[(p_2 + l + q)^2 - m_2^2 + i\epsilon\right]}. \tag{395}$$

Letting the incoming momenta be $p_1^\mu = m_1 v_1^\mu$ and $p_2^\mu = m_2 v_2^\mu$, and making explicit the factors of $\hbar$ from the massless momenta,

$$I_{\text{quad}} = \tag{396}$$
$$\int \frac{d^4\bar{l}}{(2\pi)^4} \frac{1}{\bar{l}^2(\bar{l}+\bar{q})^2\left[-2m_1\hbar v_1 \cdot (\bar{l}+\bar{q}) + \hbar^2(\bar{l}+\bar{q})^2 + i\epsilon\right]\left[2m_2\hbar v_2 \cdot (\bar{l}+\bar{q}) + \hbar^2(\bar{l}+\bar{q})^2 + i\epsilon\right]}.$$

Note that the massive propagators remain quadratic in the loop momentum.

The box integral with the linear massive propagators of HQET/HBET takes the form

$$I_{\text{HQET}} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 (l+q^2) \left[-v_1 \cdot (l+q) + i\epsilon\right] \left[v_2 \cdot (l+q) + i\epsilon\right]}. \tag{397}$$

We are concerned with addressing how the non-analytic pieces of the integrals in Eqs. (395) and (397) are related.

We see from Eq. (397) that the HQET integral is, up to a factor of $1/4m_1m_2$, the leading term of the integral in Eq. (395) when it has been expanded in $\hbar$ or $1/m$ — the equivalence of the two expansions is once again manifest. However, when including subleading terms in the expansion of Eq. (395), additional factors of $(l+q)^2$ appear in the numerator, cancelling one of the massless propagators. We conclude that all non-analytic contributions to Eq. (395) must be produced by the leading term of its expansion in $\hbar$ $(1/m)$. The same argument holds in the cases of triangle and crossed-box integrals, so the non-analytic pieces of integrals with quadratic massive propagators are reproduced (up to a factor of $2m$ for each propagator of mass m) by the HQET integrals. Another way of seeing why this should be the case is to invoke generalized unitarity. Upon cutting two massless propagators l^2 and $(l+q)^2$, there is no distinction between I_{quad} and I_{HQET}. Consequently, the one-loop integrals needed to perform the calculations in this paper are those in Ref. [221] with $p^\mu \to mv^\mu + k^\mu$ and multiplied by $2m$ for each massive propagator of mass m.[4]

10.E.2 *Pinch singularity*

HQET box integrals suffer from the so-called pinch singularity, which causes it to be ill-defined and means that HQET cannot be used to describe a bound state of two heavy particles beyond tree level. The cause of this issue is that, in such a scenario, the two heavy particles would have the same velocity, $v_1^\mu = v_2^\mu = v^\mu$. The HQET box integral in Eq. (395) then becomes

$$I_{\text{HQET}} = -\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 (l+q^2) \left[v \cdot (l+q) - i\epsilon\right] \left[v \cdot (l+q) + i\epsilon\right]}. \tag{398}$$

Any contour one tries to use to evaluate this integral is then "pinched" in the $\epsilon \to 0$ limit by the singularities above and below the real axis at $v \cdot (l+q) = 0$ [236].

For bound systems, the resolution is to reorganize the power counting expansion in terms of v/c instead of q/m. The resulting effective theory is non-relativistic QCD (NRQCD), which restores the

[4] The integrals in Ref. [221] contain only IR and UV finite terms. It was shown in Ref. [212] that the interference of such terms does not contribute to the classical potential, so we have omitted them from our calculations.

quadratic pieces of the propagators. In the case at hand, however, we are considering the scattering of two unbound heavy particles, the crucial difference being that the velocities of the heavy particles are in general distinct, ($v_1^\mu \neq v_2^\mu$). Thus, the HQET integral remains well defined.[5] Note that the limit where the HQET box integral becomes ill-defined ($v_1^\mu \to v_2^\mu$) is precisely the limit in which the box integral with quadratic massive propagators obtains the singularity which is removed by the Born iteration [221].

[5] We thank Aneesh Manohar for discussions on this point.

Part III

SCATTERING AMPLITUDES

11

THE DOUBLE COPY FOR HEAVY PARTICLES

We show how to double-copy Heavy Quark Effective Theory (HQET) to Heavy Black Hole Effective Theory (HBET) for spin $s \leq 1$. In particular, the double copy of spin-s HQET with scalar QCD produces spin-s HBET, while the double copy of spin-1/2 HQET with itself gives spin-1 HBET. Finally, we present novel all-order-in-mass Lagrangians for spin-1 heavy particles.

11.1 INTRODUCTION

An expanding family of field theories has been observed to obey double-copy[1] relations [237–268]. In particular, scattering amplitudes of gravitational theories with massive matter can be calculated from the double copy of gauge theories with massive matter [212, 269–278].

As Heavy Quark Effective Theory (HQET) [217] is derived from QCD and Heavy Black Hole Effective Theory (HBET) [6] is derived from gravity coupled to massive particles, the amplitudes of HBET should be obtainable as double-copies of HQET amplitudes. Indeed, this is the main result of this paper. We show through direct computation that the three-point and Compton amplitudes of HQET and HBET satisfy the schematic relations

$$(\mathrm{QCD}_{s=0}) \times (\mathrm{HQET}_s) = \mathrm{HBET}_s, \tag{399a}$$

$$(\mathrm{HQET}_{s=1/2}) \times (\mathrm{HQET}_{s=1/2}) = \mathrm{HBET}_{s=1}, \tag{399b}$$

for $s \leq 1$, where the spin-s HQET and HBET matter states are equal in the free-field limit, and the spin-1 heavy polarization vectors are related to the heavy spinors through eq. (427). While we

[1] For a review of the double-copy program, see ref. [21].

only show here the double copy for three-point and Compton amplitudes, invariance of the S-matrix under field redefinitions implies that eq. (399) holds more generally whenever QCD double-copies to gravitationally interacting matter. Equation (399) expands the double copy in powers of $\hbar$ since the operator expansion for heavy particles can be interpreted as an expansion in $\hbar$ [6]. The $\hbar \to 0$ limit of the double copy is currently of particular relevance [212, 271, 274].

We will begin in Section 11.2 with a brief review of the color-kinematics duality, and we will also discuss double-copying with effective matter fields. In Sections 11.3 to 11.5 we demonstrate the double copy at tree level for three-point and Compton amplitudes for spins 0, 1/2, and 1, respectively. We conclude in Section 11.6. The Lagrangians used to produce the amplitudes in this paper are presented in Section 11.A. Among them are novel all-order-in-mass Lagrangians for spin-1 HQET and HBET given in eqs. (435) and (440).

11.2 COLOR-KINEMATICS DUALITY AND HEAVY FIELDS

An n-point gauge-theory amplitude, potentially with external matter, can be written as[2]

$$\mathcal{A}_n = \sum_{i\in\Gamma} \frac{c_i n_i}{d_i}, \tag{400}$$

where Γ is the set of all diagrams with only cubic vertices. Also, c_i are color factors, n_i encode the kinematic information, and d_i are propagator denominators. A subset of the color factors satisfies the identity

$$c_i + c_j + c_k = 0. \tag{401}$$

If the corresponding kinematic factors satisfy the analogous identity,

$$n_i + n_j + n_k = 0, \tag{402}$$

[2] We omit coupling constants for the sake of clarity. Reinstating them is straight-forward: after double-copying the gauge theory coupling undergoes the replacement $g \to \sqrt{\kappa}/2$.

and have the same anti-symmetry properties as the color factors, then the color and kinematic factors are dual. In this case, the color factors in eq. (400) can be replaced by kinematic factors to form the amplitude

$$\mathcal{M}_n = \sum_{i \in \Gamma} \frac{n_i' n_i}{d_i}, \tag{403}$$

which is a gravity amplitude with anti-symmetric tensor and dilaton contamination.[3] In general, n_i' and n_i need not come from the same gauge theory, and only one of the sets must satisfy the color-kinematics duality.

In this paper we are interested in applying the double-copy procedure to HQET. A complicating factor to double-copying effective field theories (EFTs) is that Lagrangian descriptions of EFTs are not unique, as the Lagrangian can be altered by redefining one or more of the fields. The LSZ procedure [81] guarantees the invariance of the S-matrix, and in particular eqs. (400) and (403), under such field redefinitions by accounting for wavefunction normalization factors (WNFs) $\mathcal{R}^{-1/2}$, which contribute to the on-shell residues of two-point functions.[4] Under the double copy the WNFs from each matter copy combine in a spin-dependent manner, which complicates the matching of the double-copied amplitude to one derived from a gravitational Lagrangian.

In order to ease the double-copying of HQET to HBET, we would like to avoid having to compensate for the WNFs. This can be achieved by ensuring that HQET and HBET have the same WNFs – i.e. that the asymptotic states for the spin-s particles in HQET and HBET are equal – and double-copying HQET with QCD, which has a trivial WNF.

[3] For an amplitude of arbitrary multiplicity containing massive external states with an arbitrary spectrum, eq. (403) may not represent a physical amplitude [279]. However, for the cases under consideration in this paper, the application of the double copy will yield a well-defined gravitational amplitude.

[4] Note that $\mathcal{R}^{-1/2} = 1$ for canonically normalized fields. The WNF for an effective state $\tilde{\varepsilon}$ can thus be determined by relating it to a canonically normalized state ε through

$$\varepsilon = \mathcal{R}^{-1/2} \cdot \tilde{\varepsilon}. \tag{404}$$

The asymptotic states – that is, the states in the free-field limit – of the canonically normalized theories (given by complex Klein-Gordon, Dirac, and symmetry-broken Proca actions) are related to their respective asymptotic heavy states (labelled by a velocity v) in position-space through

$$\varphi(x) = \frac{e^{-imv\cdot x}}{\sqrt{2m}}\left[1 - \frac{1}{2m + iv\cdot\partial + \frac{\partial_\perp^2}{2m}}\frac{\partial_\perp^2}{2m}\right]\phi_v(x), \tag{405a}$$

$$\psi(x) = e^{-imv\cdot x}\left[1 + \frac{i}{2m + iv\cdot\partial}(\not\partial - v\cdot\partial)\right]Q_v(x), \tag{405b}$$

$$A^\mu(x) = \frac{e^{-imv\cdot x}}{\sqrt{2m}}\left[\delta^\mu_\nu - \frac{iv^\mu\partial_\nu - \partial^\mu\partial_\nu/2m}{m + iv\cdot\partial/2}\right]B^\nu_v(x), \tag{405c}$$

where $a_\perp^\mu = a^\mu - v^\mu(v\cdot a)$ for a vector a^μ. Here, the momentum is decomposed as $p^\mu = mv^\mu + k^\mu$ in the usual heavy-particle fashion. The Lagrangians for the heavy fields in eq. (405) are given in Section 11.A. Converting to momentum space, eq. (405) gives the WNFs

$$\mathcal{R}^{-1/2}_{s=0}(p) = \frac{1}{\sqrt{2m}}\left[1 + \frac{k_\perp^2}{4m^2 + 2mv\cdot k - k_\perp^2}\right], \tag{406a}$$

$$\mathcal{R}^{-1/2}_{s=1/2}(p) = 1 + \frac{1}{2m + v\cdot k}(\not k - v\cdot k), \tag{406b}$$

$$\left(\mathcal{R}^{-1/2}_{s=1}(p)\right)_\mu^{\ \nu} = \frac{1}{\sqrt{2m}}\left[\delta^\nu_\mu - \frac{v_\mu k^\nu + k_\mu k^\nu/2m}{m + v\cdot k/2}\right]. \tag{406c}$$

We will demonstrate that spin-s HBET amplitudes can directly be obtained by double-copying spin-s HQET amplitudes with scalar QCD for spins $s \le 1$. At $s = 1$ there is also the possibility to double-copy using two spin-1/2 amplitudes. We will discuss this point further below.

11.3 SPIN-0 GRAVITATIONAL AMPLITUDES

We begin with the simplest case of spinless amplitudes. Consider first the three-point amplitude. For scalar HQET we have that

$$\mathcal{A}_3^{\mathrm{H},s=0} = -\mathbf{T}^a_{ij}\epsilon_q^{*\mu}\phi_v^*\left(1 + \frac{k_1^2 + k_2^2}{4m^2}\right)\phi_v \times \left[v_\mu + \frac{(k_1+k_2)_\mu}{2m}\right] + \mathcal{O}\left(m^{-4}\right), \tag{407}$$

where $k_2 = k_1 - q$. For scalar QCD the amplitude is

$$\mathcal{A}_3^{s=0} = -\mathbf{T}^a_{ij}\epsilon_q^{*\mu}\left[2mv_\mu + (k_1+k_2)_\mu\right]. \tag{408}$$

Note that we have left the external heavy scalar factor ϕ_v explicit in the HQET amplitude. This is because, in contrast to the canonically normalized scalar fields, the heavy scalar factors are not equal to 1 in momentum space. Indeed, for the HQET amplitude to be equal to the QCD amplitude, the heavy scalar factor in momentum space must be equal to the inverse of eq. (406a). This will cancel the extra factor in round brackets in eq. (407).

The double copy at three-points is simply given by a product of amplitudes:

$$\begin{aligned}\mathcal{A}_3^{s=0}\mathcal{A}_3^{\mathrm{H},s=0} =& \epsilon_q^{*\mu}\epsilon_q^{*\nu}\phi_v^*\left(1+\frac{k_1^2+k_2^2}{4m^2}\right)\phi_v \times 2m\left[v_\mu v_\nu + v_\mu\frac{k_{1\nu}+k_{2\nu}}{m} + \frac{(k_1+k_2)_\mu(k_1+k_2)_\nu}{4m^2}\right]\\ &+\mathcal{O}(m^{-3}).\end{aligned} \tag{409}$$

As the only massless particle in this process is external, we can easily eliminate the massless non-graviton degrees of freedom by identifying the outer product of gluon polarization vectors with the graviton polarization tensor. After doing so, eq. (409) agrees with the three-point amplitude derived from eq. (438).

As another example, consider the Compton amplitude. The color decomposition for Compton scattering[5] is

$$\mathcal{A}_4^s = \frac{c_s n_s}{d_s} + \frac{c_t n_t}{d_t} + \frac{c_u n_u}{d_u}, \tag{410a}$$

where

$$c_s = \mathbf{T}^a_{ik}\mathbf{T}^b_{kj}, \quad c_t = if^{abc}\mathbf{T}^c_{ij}, \quad c_u = \mathbf{T}^b_{ik}\mathbf{T}^a_{kj}. \tag{410b}$$

[5] We have computed all Compton amplitudes using NRQCD propagators. It is also possible to perform the computations using HQET propagators: in that case, a comparison to the Compton amplitude for the emission of bi-adjoint scalars from heavy particles (described by the Lagrangians in eqs. (430) to (432)) – analogous to the treatment in ref. [280] – is necessary to identify kinematic numerators. Both methods produce the same results.

The kinematic numerators for scalar HQET are

$$n_s^{\mathrm{H},s=0} = -2m\phi_v^* \epsilon_{q_1}^{*\mu} \epsilon_{q_2}^{*\nu} v_\mu v_\nu \left(1 + \frac{k_1^2 + k_2^2}{4m^2}\right) \phi_v, \tag{411a}$$

$$n_t^{\mathrm{H},s=0} = 0, \tag{411b}$$

$$n_u^{\mathrm{H},s=0} = n_s^{\mathrm{H},s=0}|_{q_1 \leftrightarrow q_2}, \tag{411c}$$

where $k_2 = k_1 - q_1 - q_2$. Those for scalar QCD are

$$n_s^{s=0} = -4m^2 \epsilon_{q_1}^{*\mu} \epsilon_{q_2}^{*\nu} v_\mu v_\nu, \tag{412a}$$

$$n_t^{s=0} = 0, \tag{412b}$$

$$n_u^{s=0} = n_s^{s=0}|_{q_1 \leftrightarrow q_2}. \tag{412c}$$

For brevity we have written the numerators under the conditions $k_1 = q_i \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0$; the initial residual momentum can always be set to 0 by reparameterizing v, and such a gauge exists for opposite helicity gluons. We have checked explicitly up to and including $\mathcal{O}(m^{-2})$ that the following results hold when relaxing all of these conditions.

Both the HQET and QCD numerators satisfy the color-kinematics duality in the form

$$c_s - c_u = c_t \;\Leftrightarrow\; n_s - n_u = n_t. \tag{413}$$

We can therefore replace the color factors in the HQET amplitude with the QCD kinematic numerators,

$$\mathcal{M}_4^{\mathrm{H},s=0} = \frac{n_s^{s=0} n_s^{\mathrm{H},s=0}}{d_s} + \frac{n_t^{s=0} n_t^{\mathrm{H},s=0}}{d_t} + \frac{n_u^{s=0} n_u^{\mathrm{H},s=0}}{d_u}. \tag{414}$$

Identifying once again the outer products of gluon polarization vectors with graviton polarization tensors, we find that the Compton amplitude derived from eq. (438) agrees with eq. (414).

To summarize, we have explicitly verified that

$$(\mathrm{QCD}_{s=0}) \times (\mathrm{HQET}_{s=0}) = \mathrm{HBET}_{s=0} \tag{415}$$

for three-point and Compton amplitudes.

11.4 SPIN-1/2 GRAVITATIONAL AMPLITUDES

We now move on to the double copy of spin-1/2 HQET with scalar QCD to obtain spin-1/2 HBET. The three-point spin-1/2 HQET amplitude is

$$\mathcal{A}_3^{\mathrm{H},s=\frac{1}{2}} = -\mathbf{T}^a_{ij}\bar{u}_v u_v \epsilon_q^{*\mu}\left(v_\mu + \frac{k_{1\mu}}{m} + \frac{k_1^2 - k_1\cdot q}{4m^2}v_\mu\right) - \frac{i\mathbf{T}^a_{ij}}{2m}\bar{u}_v\sigma^{\alpha\beta}u_v\epsilon_q^{*\mu}\left[q_\alpha\eta_{\beta\mu} - \frac{1}{2m}q_\alpha k_{1\beta}v_\mu\right] + \mathcal{O}(m^{-3}) \tag{416}$$

Double-copying with scalar QCD, we find

$$\mathcal{M}_3^{\mathrm{H},s=\frac{1}{2}} = \mathcal{A}_3^{s=0}\mathcal{A}_3^{\mathrm{H},s=\frac{1}{2}}, \tag{417}$$

where $\mathcal{M}_3^{\mathrm{H},s=\frac{1}{2}}$ is the amplitude derived from eq. (439).

We turn now to Compton scattering. For brevity we write here the amplitudes in the case $k_1 = q_i\cdot\epsilon_j = \epsilon_i\cdot\epsilon_j = 0$. We have checked explicitly that the results hold when these conditions are relaxed. Also, we have performed the calculation up to $\mathcal{O}(m^{-2})$ but only present the kinematic numerators up to $\mathcal{O}(m^{-1})$. They are

$$n_s^{\mathrm{H},s=\frac{1}{2}} = -2m\bar{u}_v\left[v\cdot\epsilon_{q_1}^* v\cdot\epsilon_{q_2}^* - \frac{iv_\rho}{2m}\sigma_{\mu\nu}(\epsilon_{q_1}^{*\mu}q_1^\nu\epsilon_{q_2}^{*\rho} + \epsilon_{q_2}^{*\mu}q_2^\nu\epsilon_{q_1}^{*\rho} - q_2^\rho\epsilon_{q_2}^{*\mu}\epsilon_{q_1}^{*\nu})\right]u_v, \tag{418a}$$

$$n_t^{\mathrm{H},s=\frac{1}{2}} = 0, \tag{418b}$$

$$n_u^{\mathrm{H},s=\frac{1}{2}} = n_s^{\mathrm{H},s=\frac{1}{2}}|_{q_1\leftrightarrow q_2}. \tag{418c}$$

In this case, the color-kinematic duality eq. (413) is violated at $\mathcal{O}(m^{-2})$. Nevertheless, since the scalar QCD kinematic numerators satisfy the duality we can use them to double copy the spin-1/2 Compton amplitude. Doing so we find

$$\mathcal{M}_4^{\mathrm{H},s=\frac{1}{2}} = \frac{n_s^{s=0}n_s^{\mathrm{H},s=\frac{1}{2}}}{d_s} + \frac{n_t^{s=0}n_t^{\mathrm{H},s=\frac{1}{2}}}{d_t} + \frac{n_u^{s=0}n_u^{\mathrm{H},s=\frac{1}{2}}}{d_u}, \tag{419}$$

where $\mathcal{M}_4^{\mathrm{H},s=\frac{1}{2}}$ is the spin-1/2 HBET Compton amplitude derived from eq. (439).

We have seen that

$$(\mathrm{QCD}_{s=0}) \times (\mathrm{HQET}_{s=1/2}) = \mathrm{HBET}_{s=1/2} \tag{420}$$

for the three-point and Compton amplitudes.

11.5 SPIN-1 GRAVITATIONAL AMPLITUDES

Gravitational amplitudes with spin-1 matter can be obtained by double-copying two gauge theories with matter in two ways: spin-0 $\times$ spin-1 or spin-1/2 $\times$ spin-1/2 [275–277]. This fact also holds for heavy particles. We now show this in two examples by deriving the spin-1 gravitational three-point and Compton amplitudes using both double-copy procedures.

11.5.1 0×1 *Double Copy*

The three-point spin-1 HQET amplitude is

$$\begin{aligned}\mathcal{A}_3^{\mathrm{H},s=1} = \mathbf{T}^a_{ij}\varepsilon_v^{*\beta}\varepsilon_v^{\alpha}\epsilon_q^{*\mu}\Big[&\eta_{\alpha\beta}v_\mu + \frac{1}{2m}(\eta_{\alpha\beta}(k_1+k_2)_\mu - 2q_\beta\eta_{\alpha\mu} + 2q_\alpha\eta_{\beta\mu}) \\ &+\frac{1}{2m^2}v_\mu(-k_{1\beta}q_\alpha + q_\alpha q_\beta + q_\beta k_{1\alpha})\Big],\end{aligned} \tag{421}$$

where $k_2^\mu = k_1^\mu - q^\mu$. Double-copying with scalar QCD we find

$$\mathcal{M}_3^{\mathrm{H},s=1} = \mathcal{A}_3^{s=0}\mathcal{A}_3^{\mathrm{H},s=1}, \tag{422}$$

where $\mathcal{M}_3^{\mathrm{H},s=1}$ is the amplitude derived from eq. (440) after applying the field redefinition in eq. (441).

Compton scattering for spin-1 HQET is given by the kinematic numerators

$$n_s^{\mathrm{H},s=1} = 2m\varepsilon_v^{*\beta}\varepsilon_v^{\alpha}\Big[v\cdot\epsilon_{q_1}^{*}v\cdot\epsilon_{q_2}^{*}\eta_{\alpha\beta} + \frac{v_\rho}{m}(\eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\mu}\eta_{\beta\nu})(\epsilon_{q1}^{*\mu}q_1^{\nu}\epsilon_{q2}^{*\rho} + \epsilon_2^{*\mu}q_2^{\nu}\epsilon_{q1}^{*\rho})$$
$$-\frac{v\cdot q_2}{2m}(\epsilon_{q1\alpha}^{*}\epsilon_{q2\beta}^{*} - \epsilon_{q2\alpha}^{*}\epsilon_{q1\beta}^{*})\Big], \tag{423a}$$

$$n_t^{\mathrm{H},s=1} = 0, \tag{423b}$$

$$n_u^{\mathrm{H},s=1} = n_s^{\mathrm{H},s=1}|_{q_1\leftrightarrow q_2}, \tag{423c}$$

where, for brevity, we again write the numerators up to $\mathcal{O}(m^{-1})$ and in the case where $k_1 = \epsilon_i\cdot\epsilon_j = q_i\cdot\epsilon_j = 0$. We have performed the calculation up to $\mathcal{O}(m^{-2})$ and checked the general case explicitly. The double copy becomes

$$\mathcal{M}_4^{\mathrm{H},s=1} = \frac{n_s^{s=0}n_s^{\mathrm{H},s=1}}{d_s} + \frac{n_t^{s=0}n_t^{\mathrm{H},s=1}}{d_t} + \frac{n_u^{s=0}n_u^{\mathrm{H},s=1}}{d_u}, \tag{424}$$

where $\mathcal{M}_4^{\mathrm{H},s=1}$ is derived from eq. (440) after applying the field redefinition in eq. (441).

Thus, we find that

$$(\mathrm{QCD}_{s=0})\times(\mathrm{HQET}_{s=1}) = \mathrm{HBET}_{s=1} \tag{425}$$

for three-point and Compton amplitudes.

11.5.2 $\frac{1}{2}\times\frac{1}{2}$ *Double Copy*

The spin-1 gravitational amplitudes can also be obtained by double-copying the spin-1/2 HQET amplitudes. To do so, we use the on-shell heavy particle effective theory (HPET) variables of ref. [11] to modify eq. (2.11) of ref. [277] for the case of heavy particles. Using the fact that the on-shell HPET variables correspond to momenta $p_v^\mu = m_k v^\mu$ with mass $m_k = m(1 - k^2/4m^2)$, following the derivation of ref. [277] leads to

$$\mathcal{M}_n^{\mathrm{H},\frac{1}{2}\times\frac{1}{2}} = \frac{m_{k_1}m_{k_2}}{m}\sum_{\alpha\beta}K_{\alpha\beta}\mathrm{Tr}[\mathcal{A}_{n,\alpha}^{\mathrm{H},\frac{1}{2}}P_+\not{\epsilon}_v\bar{\mathcal{A}}_{n,\beta}^{\mathrm{H},\frac{1}{2}}P_-\not{\epsilon}_v^{*}], \tag{426}$$

where $P_\pm = (1 \pm \not{v})/2$, $K_{\alpha\beta}$ is the KLT kernel, and α, β represent color orderings. Here $\mathcal{A}^{\rm H}$ and $\bar{\mathcal{A}}^{\rm H}$ are amplitudes with the external states stripped, and $\bar{\mathcal{A}}^{\rm H} = -\gamma_5(\mathcal{A}^{\rm H})^\dagger\gamma_5$. We have also adopted the convention that only the initial matter momentum is incoming. Converting to the on-shell HPET variables, it can be easily seen that

$$\varepsilon^{IJ}_{v\mu}(p) = \frac{1}{2\sqrt{2}m_k}\bar{u}^I_v(p)\gamma_5\gamma_\mu u^J_v(p), \tag{427}$$

with I, J being massive little group indices. Given the WNF for the heavy spinors, the WNF for the polarization vector can easily be computed by comparing eq. (427) to its canonical polarization vector analog. We find that it is indeed given by eq. (406c).

Applying eq. (426) to eq. (416) with the three-point KLT kernel $K_3 = 1$, we immediately recover the left-hand side of eq. (422). For Compton scattering the KLT kernel is

$$K_4 = \frac{(s-m^2)(u-m^2)}{2q_1 \cdot q_2}. \tag{428}$$

Then, applying eq. (426) to the spin-1/2 HQET Compton amplitude with $k_1, q_i \cdot \epsilon_j, \epsilon_i \cdot \epsilon_j \neq 0$ up to and including terms of order $\mathcal{O}(m^{-2})$, we find eq. (424) up to $\mathcal{O}(m^{-1})$. When imposing $k_1 = q_i \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0$, cancellations make the double copy valid up to $\mathcal{O}(m^{-2})$. The extension to higher inverse powers of the mass amounts to simply including the contributions of higher-order operators in the HQET and HBET amplitudes.

Therefore, by using eq. (426) to convert heavy spinors in amplitudes to heavy polarization vectors, we have shown that

$$(\text{HQET}_{s=1/2}) \times (\text{HQET}_{s=1/2}) = \text{HBET}_{s=1} \tag{429}$$

for three-point and Compton amplitudes.

11.6 CONCLUSION

We have shown that the three-point and Compton amplitudes derived from HQET can be double-copied to those of HBET for spins $s \leq 1$. As long as the matter states of HQET and HBET are related through the double copy, in the sense described in Section 11.2, and as long as higher-point amplitudes obey the spectral condition of ref. [279], we see no obstacles to extending the double copy to higher-point amplitudes.

As mentioned in the introduction, due to the operator expansion of HPETs, the double-copy relation between HQET and HBET can be studied at each order in the $\hbar$ expansion, with the classical limit being of special interest. Studying the double copy of HPETs through this lens may provide some insight into the connection between the double copy with matter at the quantum and classical levels. We leave this study for future work.

APPENDIX

11.A LAGRANGIANS FOR HEAVY PARTICLES

We present Lagrangians for heavy particles coupled to bi-adjoint scalars, gluons, and gravitons. The heavy-particle Lagrangians were used to derive the scattering amplitudes in the paper. For clarity, we omit the subscript v for the heavy spin-1 fields.

Bi-adjoint scalars and heavy particles

We couple the bi-adjoint scalars Φ to heavy particles with spins $s \leq 1$. The spin-0 Lagrangian is

$$\mathcal{L}^{s=0}_{\mathrm{bi-adjoint}} = \phi^*_v \left[iv\cdot\partial - \frac{\partial^2_\perp - y_s\Phi}{2m} + \left(\frac{\partial^2_\perp - y_s\Phi}{2m}\right) \frac{1}{2m + iv\cdot\partial + \frac{\partial^2_\perp - y_s\Phi}{2m}} \left(\frac{\partial^2_\perp - y_s\Phi}{2m}\right)\right] \phi_v. \tag{430}$$

The spin-1/2 Lagrangian is

$$\mathcal{L}^{s=1/2}_{\mathrm{bi-adjoint}} = \overline{Q}_v \left[iv\cdot\partial + y_f\Phi + (i\not{\partial}_\perp) \frac{1}{2m + iv\cdot\partial - y_f\Phi} (i\not{\partial}_\perp)\right] Q_v. \tag{431}$$

The spin-1 Lagrangian is

$$\mathcal{L}^{s=1}_{\mathrm{bi-adjoint}} = -B^*_\mu (iv\cdot\partial) B^\mu - \frac{1}{4m} B^*_{\mu\nu} B^{\mu\nu} + \frac{y_v}{2m} B^*_\mu \Phi B^\mu - \left(\mathcal{F}^\lambda_- B^*_\lambda\right) \frac{2}{m + \frac{1}{m}\partial^2_\perp} \left(\mathcal{F}^\lambda_+ B^*_\lambda\right) \tag{432a}$$

where

$$\mathcal{F}^\mu_\pm = \left(\pm\frac{i}{2}\partial^\mu - \frac{1}{2m}\partial^\mu (v\cdot\partial) + \frac{y_v\Phi}{2m}\right). \tag{432b}$$

The coupling constants between the bi-adjoint scalars and the heavy scalars, fermions, and vectors are y_s, y_f, and y_v, respectively.

Gluons and heavy particles

We couple gluons to heavy particles. The covariant derivative in this case is given by $D_\mu = \partial_\mu + ig_s \mathbf{T}^a A^a_\mu$. The scalar Lagrangian is

$$\mathcal{L}^{s=0}_{\text{gluon}} = \phi^*_v \left[iv \cdot D - \frac{D^2_\perp}{2m} + \left(\frac{D^2_\perp}{2m}\right) \frac{1}{2m + iv\cdot D + \frac{D^2_\perp}{2m}} \left(\frac{D^2_\perp}{2m}\right) \right] \phi_v. \tag{433}$$

The spin-1/2 Lagrangian is

$$\mathcal{L}^{s=1/2}_{\text{gluon}} = \overline{Q}_v \left[iv \cdot D + (i\not{D}_\perp) \frac{1}{2m + iv\cdot D} (i\not{D}_\perp) \right] Q_v. \tag{434}$$

The spin-1 Lagrangian [281] with gyromagnetic ratio $g = 2$ can be written as

$$\mathcal{L}^{s=1}_{\text{gluon}} = -B^*_\mu (iv\cdot D) B^\mu - \frac{1}{4m} B^*_{\mu\nu} B^{\mu\nu} + \frac{ig}{2m} F^{\mu\nu} B^*_\mu B_\nu - \left(\mathcal{E}^\lambda_- B^*_\lambda\right) \frac{2}{m + \frac{1}{m} D^2_\perp} \left(\mathcal{E}^\mu_+ B_\mu\right) \tag{435a}$$

where

$$\mathcal{E}^\mu_\pm = \left(\pm \frac{i}{2} D^\mu - \frac{1}{2m} D^\mu (v\cdot D) \pm \frac{igv_\nu F^{\nu\mu}}{2m} \right). \tag{435b}$$

The heavy spin-1 states described by this Lagrangian are related to the canonical massive spin-1 states through

$$A^\mu(x) = \frac{e^{-imv\cdot x}}{\sqrt{2m}} \left[\delta^\mu_\nu - \frac{1}{1 + iv\cdot\partial/m} \frac{iv^\mu \partial_\nu}{m} \right] B^\nu(x). \tag{436}$$

To obtain the desired heavy spin-1 states we apply the field redefinition

$$B_\mu \to \left[\delta^\nu_\mu + \frac{1}{2m^2} \left(-v_\mu v\cdot D + D_\mu\right) D^\nu \right] B_\nu + \mathcal{O}(m^{-3}). \tag{437}$$

Gravitons and heavy particles

We couple gravitons to heavy particles. The spin-0 Lagrangian is

$$\sqrt{-g}\mathcal{L}^{s=0}_{\text{graviton}} = \sqrt{-g}\phi_v^* \left[\mathcal{A}_1 + (\mathcal{A}_{2-})\,\frac{1}{2m + i(v^\mu \nabla_\mu + \nabla_\mu v^\mu) - \mathcal{A}_1}\,(\mathcal{A}_{2+})\right]\phi_v, \tag{438a}$$

where

$$\mathcal{A}_1 = \frac{1}{2} i g^{\mu\nu}(v_\mu \nabla_\nu + \nabla_\mu v_\nu) + \frac{1}{2} m (g^{\mu\nu} - \eta^{\mu\nu}) v_\mu v_\nu - \frac{1}{2m}\nabla_\mu \left((g^{\mu\nu} - \eta^{\mu\nu})\nabla_\nu + \eta^{\mu\nu}\nabla_{\perp\nu}\right), \tag{438b}$$

$$\mathcal{A}_{2\pm} = \frac{1}{2m}(imv_\mu - \nabla_\mu)\left((g^{\mu\nu} - \eta^{\mu\nu})(-imv_\nu + \nabla_\nu)\right) - \frac{1}{2m}\nabla_\mu(\eta^{\mu\nu}\nabla_{\perp\mu}) \pm \frac{1}{2} i \left[\nabla_\mu v^\mu\right], \tag{438c}$$

with $v^\mu \equiv \eta^{\mu\nu} v_\nu$ and $\nabla_{\perp\mu} \equiv \nabla_\mu - v_\mu(v^\nu \nabla_\nu)$. The spin-1/2 Lagrangian is

$$\sqrt{-g}\mathcal{L}^{s=1/2}_{\text{graviton}} = \sqrt{-g}\ \overline{Q}_v \left[i\not{\nabla} + \mathcal{B} + (i\not{\nabla} + \mathcal{B})\,P_- \frac{1}{2m - (i\not{\nabla} + \mathcal{B})P_-}\,(i\not{\nabla} + \mathcal{B})\right] Q_v, \tag{439a}$$

where $\not{\nabla} \equiv \delta^\mu_a \gamma^a \nabla_\mu$ and

$$\mathcal{B} = (e^\mu_a - \delta^\mu_a)(i\gamma^a \nabla_\mu + m\gamma^a v_\mu). \tag{439b}$$

The spin-1 Lagrangian can be written as

$$\begin{aligned}\sqrt{-g}\mathcal{L}^{s=1}_{\text{graviton}} = &\sqrt{-g}\left[-\frac{m}{2}(v_\mu B^*_\nu)(v_\rho B_\sigma)\left((g^{\mu\rho} - \eta^{\mu\rho})g^{\nu\sigma} - (g^{\mu\sigma} - \eta^{\mu\sigma})(g^{\nu\rho} - \eta^{\nu\rho})\right)\right.\\ &+ \frac{i}{2}\left[\left(\nabla_\mu B^*_\nu\right)(v_\rho B_\sigma) - \left(v_\mu B^*_\nu\right)(\nabla_\rho B_\sigma)\right](g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})\\ &\left.- \frac{1}{4m} B^*_{\mu\nu} B_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} - (\mathcal{C}^\alpha_- B^*_\alpha)\,\frac{1}{\mathcal{D}}\left(\mathcal{C}^\beta_+ B_\beta\right)\right],\end{aligned} \tag{440a}$$

where

$$\mathcal{C}^{\alpha}_{\pm} = -\frac{m}{2}(g^{\alpha\nu} - \eta^{\alpha\nu})v_\nu \pm \frac{i}{2} v_\nu \left[g^{\mu\rho}g^{\alpha\nu} - g^{\alpha\mu}g^{\nu\rho}\right] \nabla_\mu \left(v_\rho \pm \frac{i}{m}\nabla_\rho\right), \tag{440b}$$

$$\mathcal{D} = \frac{m}{2}(v_\nu v_\sigma g^{\nu\sigma}) + \frac{1}{2m} v_\nu \left[g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\rho\nu}\right] \nabla_\mu \nabla_\rho v_\sigma. \tag{440c}$$

Note that though the velocity four-vector is constant its covariant derivative does not vanish because of the metric connection. The heavy spin-1 states described by this Lagrangian are related to the canonical massive spin-1 states through eq. (436). To obtain the desired heavy spin-1 states we apply the field redefinition

$$\begin{aligned} B_\mu \to & \left[\delta^\nu_\mu + \frac{1}{2m^2}\left(-g^{\alpha\beta} v_\alpha D_\beta v_\mu + D_\mu\right) g^{\nu\lambda} D_\lambda\right] B_\nu \\ & + \mathcal{O}(m^{-3}). \end{aligned} \tag{441}$$

Extending this redefinition to higher orders in $1/m$ is straight-forward.

12

ON-SHELL HEAVY PARTICLE EFFECTIVE THEORIES

We introduce on-shell variables for Heavy Particle Effective Theories (HPETs) with the goal of extending Heavy Black Hole Effective Theory to higher spins and of facilitating its application to higher post-Minkowskian orders. These variables inherit the separation of spinless and spin-inclusive effects from the HPET fields, resulting in an explicit spin-multipole expansion of the three-point amplitude for any spin. By matching amplitudes expressed using the on-shell HPET variables to those derived from the one-particle effective action, we find that the spin-multipole expansion of a heavy spin-s particle corresponds exactly to the multipole expansion (up to order $2s$) of a Kerr black hole, that is, without needing to take the infinite spin limit. Finally, we show that tree-level radiative processes with same-helicity bosons emitted from a heavy spin-s particle exhibit a spin-multipole universality.

12.1 INTRODUCTION

The relationship between quantum scattering amplitudes and classical physics has enjoyed a surge of attention in recent years, in large part due to the observation of gravitational waves by the LIGO and Virgo collaborations as of 2015 [189]. Motivating studies in this direction has been the realization that perturbative techniques from quantum field theory are well suited to the computation of the complementary post-Newtonian (PN) and post-Minkowskian (PM) expansions of the binary inspiral problem in General Relativity (GR). Indeed, the effective field theory (EFT) of GR [200, 201] has been used extensively to compute classical corrections to the gravitational potential [201–206, 209, 222, 282]. Furthermore, effective-field-theoretic methods have been used to develop EFTs for gravitationally interacting objects whose operator expansions are tailored to computing terms in the PN approximation [196–199, 283]. In fact, using EFT methods, the entire 4PN spinless conservative dynamics were derived in refs. [284, 285], and the computation of the 5PN spinless conservative dynamics was approached in refs. [286, 287]. Including spin, the current state-of-the-art computations from the PN approach were performed in refs. [288, 289] using the EFT of ref. [283].

On the PM side, it has also recently been shown that quantum scattering amplitudes can be used to extract fully relativistic information about the classical scattering process [208, 210–214, 220, 223, 290–293]. Moreover, a direct relationship between the scattering amplitude and the scattering angle has been uncovered in refs. [191, 192, 215, 294].[1] All of these developments suggest that the $2 \to 2$ gravitational scattering amplitude encodes information that is crucial for the understanding of classical gravitational binary systems, to all loop orders [200, 216].

Various methods exist for identifying the classical component of a scattering amplitude [205, 206, 220, 223]. Towards this same end, Heavy Black Hole Effective Theory (HBET) was recently formulated by Damgaard and two of the present authors in ref. [6] with the aim of streamlining the extraction of classical terms from gravitational scattering amplitudes. It was shown there that the operator expansion of HBET is equivalent to an expansion in $\hbar$. Exploiting this fact, the authors were able to identify which HBET operators can induce classical effects at arbitrary loop order, and the classical portion of the $2 \to 2$ amplitude was computed up to one-loop order for spins $s \leq 1/2$. These results were obtained using Lagrangians and Feynman diagram techniques which, while tractable at the perturbative orders and spins considered, become non-trivial and computationally unwieldy to extend to higher spins or loop orders. Nevertheless, the separation of classical and quantum effects and the observed separation of spinless and spin-inclusive effects are desirable features of the EFT that will prove quite convenient when cast as part of a more user-friendly formalism.

We aim in this paper to present such a formalism that will allow the extension of HBET to higher spins and to facilitate its application to higher loop orders. A means to do so comes in the formalism presented in ref. [235]. Spinor-helicity variables were presented there that describe the scattering of massive matter with arbitrary spin. Based solely on kinematic considerations, these variables were used to construct the most general three-point amplitude for a massive spin-s particle emitting a massless boson with a given helicity.[2] In this most general amplitude, the term that is best behaved in the UV limit is termed the minimal coupling amplitude. When $s \leq 1/2$ it reduces to the three-point amplitude arising from the relevant Lagrangian that is minimally coupled in the sense of covariantized derivatives. This terminology is preserved for higher spins; the minimal coupling amplitude for a general spin-s particle is a tensor product of $2s$ factors of spin-1/2 minimal coupling amplitudes. Note that this definition of minimal coupling generally differs from the typical definition from the

[1] We thank Andrea Cristofoli for bringing earlier work on this relationship to our attention.

[2] For alternative approaches to the application of spinor-helicity variables to massive spinning particles see e.g. refs. [295, 296].

Lagrangian perspective. Phenomenologically, these minimal coupling amplitudes are those that produce a gyromagnetic ratio of $g = 2$ for all spins [209, 297, 298].

This minimal coupling amplitude has proven to be quite useful in the study of classical Kerr black holes, which have been shown to couple minimally to gravity [206, 208, 209, 211, 224, 299]. Such a description of Kerr black holes is in fact not immediately exact when using the variables of ref. [235] due to the difference between the momenta of the initial and final states, leading to an ill-defined matrix element of the spin-operator. This gap has been overcome using various methods in the above references. However we will show that expressing the degrees of freedom of HBET in on-shell variables reduces the discrepancy to a mere choice of the kinematics. The appropriate kinematics can sometimes be imposed (when a process is described by diagrams with no internal matter lines), but are always recovered in the classical limit; $\hbar \to 0$.

In this paper, we express the asymptotic states of Heavy Particle Effective Theories (HPETs) — the collection of effective field theories treating large mass particles — using the massive on-shell spinor-helicity variables of ref. [235]. An explicit $\hbar$ expansion will arise from these variables, which makes simple the task of taking classical limits of amplitudes. Such an expression of the asymptotic states of HPET will also lead to an explicit separation of spinning and spinless effects in the three-point minimal coupling amplitude. From the lens of the classical gravitational scattering of two spinning black holes, this results in the finding that the asymptotic states of HPET are naturally identified with a Kerr black hole with truncated spin-multipole expansion.

Our construction will also allow us to gain insight into this class of effective field theories. We will derive a conjecture for the three-point amplitude arising from an arbitrary HPET, and posit a form for this same amplitude for heavy matter of any spin. Then, in the appendices, we comment on the link between reparameterization invariance of a momentum and its little group, and finally compute the operator projecting onto a heavy particle of spin $s \leq 2$, the derivation of which can be extended to general spin.

The layout of this paper is as follows. We begin with a very brief review of HPETs in Section 12.2. Also, we introduce on-shell variables that describe the heavy field. The three-point amplitudes of HPETs are analyzed in Section 12.3. In particular, we construct the three-point amplitude of HPET resummed to all orders in the expansion parameter. Furthermore, the construction of ref. [235] provides a method of extending HPET amplitudes to arbitrary spin. In Section 12.4, we interpret the on-shell HPET variables as Kerr black holes with truncated spin-multipole expansions, and show that heavy spin-s particles possess the same spin-multipole expansion as a Kerr black hole, up to the $2s^{\text{th}}$ multipole. This is in contrast to previous work [209, 224], which found that minimally coupled particles possess the same spin multipoles as Kerr black holes only in the infinite spin limit.

Section 12.5 is dedicated to the computation of on-shell amplitudes, and we show the simplicity of taking the classical limit of an amplitude when it is expressed in on-shell HPET variables. The main body of the paper is concluded in Section 12.6. Our conventions are summarized in Appendix 12.A. The question of the uniqueness of the constructed variables is addressed in Appendix 12.B. We then relate the little group of a momentum p to its invariance under the HPET reparameterization (see Section 12.2) in Section 12.C. In Appendix 12.D we use spin-s polarization tensors for heavy particles to explicitly construct propagators and projection operators for heavy particles with spins $s \leq 2$. We then use these results to conjecture the forms of the projection operators for arbitrary spin. Finally, we describe in Appendix 12.E the forms of the spin-1/2 HPET Lagrangians that must be used to match to the on-shell minimal coupling amplitudes. We also show there that the three-point amplitude derived from a Lagrangian for a heavy spin-1 particle is reproduced by the extension of the variables to arbitrary spin in Section 12.3.

12.2 EFFECTIVE THEORIES WITH HEAVY PARTICLES

When describing a scattering process in which the transfer momentum, q^μ, is small compared to the mass of one of the scattered particles, m, we can exploit the separation of scales by expanding in the small parameter $|q|/m$. Heavy Quark Effective Theory (HQET) [217–219] is the effective field theory that employs this expansion in the context of QCD, with HBET being its gravitational analog. Central to the separation of scales is the decomposition of the momentum of the heavy particle as

$$p^\mu = mv^\mu + k^\mu, \tag{442}$$

where v^μ is the (approximately constant) four-velocity ($v^2 = 1$) of the heavy particle, and k^μ is a residual momentum that parameterizes the energy of the interaction; it is therefore comparable in magnitude to the momentum transfer, $|k^\mu| \sim |q^\mu|$. When decomposed in this way, the on-shell condition, $p^2 = m^2$, is equivalent to

$$v \cdot k = -\frac{k^2}{2m}. \tag{443}$$

As was argued in ref. [6], using results from ref. [220], the residual momentum scales with $\hbar$ in the limit $\hbar \to 0$. We discuss the counting of $\hbar$ in Section 12.5.1.

With some background about the construction and motivation behind HPETs, we introduce in this section on-shell variables that describe spin-1/2 HPET states. Then, the transformation of these variables under a reparameterization of the momentum eq. (442) is given. We end the section by defining the spin operator for heavy particles.

12.2.1 *On-shell HPET variables*

The spinors $u_v^I(p)$ that describe the particle states of HPET are related to the Dirac spinors $u^I(p)$ via [6]

$$u_v^I(p) = \left(\frac{\mathbb{I} + \not{v}}{2}\right) u^I(p) = \left(\mathbb{I} - \frac{\not{k}}{2m}\right) u^I(p), \tag{444}$$

where I is an $SU(2)$ little group index, and v^μ and k^μ are defined in eq. (442). The operator $P_+ \equiv \frac{1+\not{v}}{2}$ is the projection operator that projects on to the heavy particle states. Writing the Dirac spinor in terms of massive on-shell spinors $|\mathbf{p}\rangle_\alpha$ and $|\mathbf{p}]^{\dot\alpha}$, we define on-shell variables for the HPET spinor field:

$$\begin{pmatrix} |\mathbf{p}_v\rangle \\ \\ |\mathbf{p}_v] \end{pmatrix} = \left(\mathbb{I} - \frac{\not{k}}{2m}\right) \begin{pmatrix} |\mathbf{p}\rangle \\ \\ |\mathbf{p}] \end{pmatrix}. \tag{445}$$

The bold notation for the massive on-shell spinors was introduced in ref. [235], and represents symmetrization over the little group indices. We refer to the on-shell variables of ref. [235] as the traditional on-shell variables, and those introduced here as the on-shell HPET variables. The on-shell HPET variables are labelled by their four-velocity v. We emphasize that the relation between the traditional and HPET on-shell variables is exact in k/m. See Section 12.A for conventions.

When working with heavy particles, the Dirac equation is replaced by the relation $\not{v} u_v^I = u_v^I$, which can be seen by multiplying the first equation in eq. (444) by $\not{v}$. This relates the on-shell HPET variables in different bases through

$$v_{\alpha\dot\beta}|\mathbf{p}_v]^{\dot\beta} = |\mathbf{p}_v\rangle_\alpha, \quad v^{\dot\alpha\beta}|\mathbf{p}_v\rangle_\beta = |\mathbf{p}_v]^{\dot\alpha}, \tag{446a}$$

$$[\mathbf{p}_v|_{\dot\alpha} v^{\dot\alpha\beta} = -\langle\mathbf{p}_v|^\beta, \quad \langle\mathbf{p}_v|^\alpha v_{\alpha\dot\beta} = -[\mathbf{p}_v|_{\dot\beta}. \tag{446b}$$

We associate the momentum p_v^μ with the on-shell HPET spinors, where

$$\not{p}_v = \begin{pmatrix} 0 & |p_v\rangle^I{}_I[p_v| \\ |p_v]_I{}^I\langle p_v| & 0 \end{pmatrix} = m_k \not{v}, \tag{447}$$

and

$$m_k \equiv \left(1 - \frac{k^2}{4m^2}\right) m. \tag{448}$$

We see that the momentum p_v^μ is proportional to v^μ, regardless of the residual momentum. The momentum p_v^μ is related to the momentum p^μ through

$$P_+ \not{p}_v = P_+ \not{p} P_+. \tag{449}$$

The on-shell HPET variables naturally describe heavy particles in a context with no anti-particles. To see this, note that the relation between the HPET spinor and the Dirac spinor in eq. (445) can be inverted [233]

$$\begin{aligned} u^I(p) &= \left(\mathbb{I} - \frac{\not{k}}{2m}\right)^{-1} u_v^I(p) \\ &= \left[1 + \frac{1}{2m}\left(1 + \frac{k \cdot v}{2m}\right)^{-1} (\not{k} - k \cdot v)\right] u_v^I(p). \end{aligned} \tag{450}$$

In the free theory, this corresponds to the relation between the fields in the full and effective theories once the heavy anti-field has been integrated out by means of its equation of motion. Thus, eq. (444) is equivalent to integrating out heavy anti-particle states.

12.2.2 *Reparameterization*

There is an ambiguity in the choice of v and k in the decompositon of the momentum in eq. (442). The momentum is invariant under reparameterizations of v and k of the forms

$$(v, k) \to (w, k') \equiv \left(v + \frac{\delta k}{m}, k - \delta k\right), \tag{451}$$

where $|\delta k|/m \ll 1$ and $(v + \delta k/m)^2 = 1$. Given that observables can only depend on the total momentum, observables computed in heavy particle effective theories must be invariant under this reparameterization [231–233]. In particular, the S-matrix is reparameterization invariant.

The on-shell HPET variables transform under the reparameterization of the momentum in eq. (451). The HPET spinors $u_v^I(p)$ and $u_w^I(p)$ are related through

$$\begin{aligned} u_v^I(p) &= \frac{1+\not{v}}{2} u^I(p) \\ &= \frac{1+\not{v}}{2}\left[1 + \frac{1}{2m}\left(1 + \frac{k'\cdot w}{2m}\right)^{-1}(\not{k}' - k'\cdot w)\right] u_w^I(p), \end{aligned} \tag{452}$$

where the second line is simply eq. (450) with $(v, k) \to (w, k')$. Rewriting this in terms of the on-shell HPET variables, we find

$$|\mathbf{p}_v\rangle = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[\left(1 - \frac{k^2}{4m^2} + \frac{\not{k}\delta\not{k}}{4m^2}\right)|\mathbf{p}_w\rangle - \frac{\delta\not{k}}{2m}|\mathbf{p}_w]\right], \tag{453a}$$

$$|\mathbf{p}_v] = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[\left(1 - \frac{k^2}{4m^2} + \frac{\not{k}\delta\not{k}}{4m^2}\right)|\mathbf{p}_w] - \frac{\delta\not{k}}{2m}|\mathbf{p}_w\rangle\right]. \tag{453b}$$

Similarly,

$$\langle\mathbf{p}_v| = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[\langle\mathbf{p}_w|\left(1 - \frac{k^2}{4m^2} + \frac{\delta\not{k}\not{k}}{4m^2}\right) + [\mathbf{p}_w|\frac{\delta\not{k}}{2m}\right], \tag{453c}$$

$$[\mathbf{p}_v| = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[[\mathbf{p}_w|\left(1 - \frac{k^2}{4m^2} + \frac{\delta\not{k}\not{k}}{4m^2}\right) + \langle\mathbf{p}_w|\frac{\delta\not{k}}{2m}\right]. \tag{453d}$$

The transformed spinors $|\mathbf{p}_w\rangle$ and $|\mathbf{p}_w]$ are related to the traditional on-shell variables via eq. (445), with the replacement $k \to k'$. This transformation is singular at the point where the new residual momentum has magnitude squared $k'^2 = 4m^2$. This pole is ubiquitous when using these variables, and signals the point where fluctuations of the matter field are energetic enough to allow for pair-creation. As we have integrated out the anti-particle through eq. (444), such energies are outside the region of validity of this formalism. In fact, the working assumption of the formalism is that the residual momentum is small compared to the mass, so one would expect the formalism to lose predictive power well before this point.

12.2.3 *Spin operator*

We identify the spin operator with the Pauli-Lubanski pseudovector,

$$S^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}, \tag{454}$$

where $J^{\mu\nu}$ is the generator of rotations, p^μ is the momentum with respect to which the operator is defined, and $m^2 = p^2$. For our purposes, it will be convenient to choose $p^\mu = p_v^\mu$: this ensures that, irrespective of the value of the residual momentum, the momentum $p_v^\mu = m_k v^\mu$ will always be orthogonal to the spin operator. Thus, S^μ is the spin vector of a particle with velocity v^μ and any value of residual momentum. With this choice for the reference momentum, the spin-operator is

$$S^\mu = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}v_\nu J_{\alpha\beta}. \tag{455}$$

Its action on irreducible representations of $SL(2,\mathbb{C})$ is

$$(S^\mu)_\alpha{}^\beta = \frac{1}{4}\left[(\sigma^\mu)_{\alpha\dot{\alpha}}v^{\dot{\alpha}\beta} - v_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\right], \tag{456a}$$

$$(S^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{1}{4}\left[(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}v_{\alpha\dot{\beta}} - v^{\dot{\alpha}\alpha}(\sigma^\mu)_{\alpha\dot{\beta}}\right]. \tag{456b}$$

These two representations of the spin-vector are related via

$$(S^\mu)_\alpha{}^\beta = v_{\alpha\dot{\alpha}}(S^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}v^{\dot{\beta}\beta}, \quad (S^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = v^{\dot{\alpha}\alpha}(S^\mu)_\alpha{}^\beta v_{\beta\dot{\beta}}. \tag{457}$$

On three-particle kinematics, the spin-vector can be written more compactly by introducing the x factor for a massless momentum q [235],

$$mx\langle q| \equiv [q|p_1, \tag{458a}$$

$$\Rightarrow mx^{-1}[q| = \langle q|p_1. \tag{458b}$$

Using this, when the initial residual momentum is $k = 0$, we can re-express the contraction $q \cdot S$ as

$$(q \cdot S)_\alpha{}^\beta = \frac{x}{2} |q\rangle\langle q|, \tag{459a}$$

$$(q \cdot S)^{\dot\alpha}{}_{\dot\beta} = -\frac{x^{-1}}{2} |q][q|. \tag{459b}$$

For general initial residual momentum, we find an additional term:

$$(q \cdot S)_\alpha{}^\beta = \frac{1}{4}\left(2x|q\rangle\langle q| + \frac{1}{m}[k,q]_\alpha{}^\beta\right), \tag{460a}$$

$$(q \cdot S)^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{4}\left(2x^{-1}|q][q| + \frac{1}{m}[k,q]^{\dot\alpha}{}_{\dot\beta}\right). \tag{460b}$$

Note that eq. (460) reduces to eq. (459) when $k = 0$.

When choosing the reference momentum to be p_v^μ, we can identify the spin-vector with the classical spin-vector of a Kerr black-hole with classical momentum $p^\mu_{\text{Kerr}} = \frac{m}{m_k} p_v^\mu$. This is because the Lorentz generator in eq. (454) can be replaced with the black hole spin-tensor $S^{\mu\nu} = J_\perp^{\mu\nu}$ which satisfies the condition [283, 300]

$$p^\mu_{\text{Kerr}} S_{\mu\nu} = 0, \tag{461}$$

known as the spin supplementary condition.

In ref. [6], the spin vector was defined as

$$S_v^\mu \equiv \frac{1}{2} \bar{u}_v(p_2)\gamma_5\gamma^\mu u_v(p_1), \tag{462}$$

and it was found that this spin vector satisfied the relation

$$\bar{u}_v(p_2)\sigma^{\mu\nu}u_v(p_1) = -2\epsilon^{\mu\nu\alpha\beta} v_\alpha S_{v\beta}. \tag{463}$$

We can therefore relate these two definitions of the spin vector:

$$S_v^\mu = \bar{u}_v(p_2) S^\mu u_v(p_1) = -2\langle \mathbf{2}_v | S^\mu | \mathbf{1}_v \rangle = 2[\mathbf{2}_v | S^\mu | \mathbf{1}_v]. \tag{464}$$

Thus the two definitions are consistent, with one being the one-particle matrix element of the other.

12.3 THREE-POINT AMPLITUDE

We study in this section the on-shell three-point amplitudes of HPET. The main goal here will be to express the most general three-point on-shell amplitude for two massive particles (mass m, spin s) and one massless boson (helicity h) in terms of on-shell HPET variables. Focusing on the minimal coupling portion of such an expression, we will be left with a resummed form of the HPET three-point amplitude, valid for any spin. Moreover, we will find that a certain choice of the residual momentum results in the exponentiation of the minimally coupled three-point amplitude.

In the traditional on-shell variables, the most general three-point amplitude for two massive particles of mass m and spin s, and one massless particle with momentum q and helicity h is [235]

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h}\frac{x^{|h|}}{m^{2s}}\left[g_0\langle\mathbf{21}\rangle^{2s} + g_1\langle\mathbf{21}\rangle^{2s-1}\frac{x\langle\mathbf{2}q\rangle\langle q\mathbf{1}\rangle}{m} + \cdots + g_{2s}\frac{\left(x\langle\mathbf{2}q\rangle\langle q\mathbf{1}\rangle\right)^{2s}}{m^{2s}}\right], \quad (465)$$

$$\mathcal{M}^{-|h|,s} = (-1)^{h}\frac{x^{-|h|}}{m^{2s}}\left[\tilde{g}_0[\mathbf{21}]^{2s} + \tilde{g}_1[\mathbf{21}]^{2s-1}\frac{x^{-1}[\mathbf{2}q][q\mathbf{1}]}{m} + \cdots + \tilde{g}_{2s}\frac{\left(x^{-1}[\mathbf{2}q][q\mathbf{1}]\right)^{2s}}{m^{2s}}\right]. \quad (466)$$

The overall sign differs from the expression in ref. [235], due to our convention that p_1 is incoming. The positive helicity amplitude is expressed in the chiral basis, and the negative helicity amplitude in the anti-chiral basis. The minimal coupling portion of this is the amplitude with all couplings except g_0 and $\tilde{g}_0$ set to zero:

$$\mathcal{M}^{+|h|,s}_{\text{min}} = (-1)^{2s+h}\frac{g_0 x^{+|h|}}{m^{2s}}\langle\mathbf{21}\rangle^{2s}, \quad (467)$$

$$\mathcal{M}^{-|h|,s}_{\text{min}} = (-1)^{h}\frac{\tilde{g}_0 x^{-|h|}}{m^{2s}}[\mathbf{21}]^{2s}. \quad (468)$$

Thus we see that expressing this in terms of on-shell HPET variables requires that we convert the spinor products $\langle\mathbf{21}\rangle$, $x\langle\mathbf{2}q\rangle\langle q\mathbf{1}\rangle$ (and their anti-chiral basis counterparts) to the on-shell HQET variables.

In the remainder of this section we take $p_1^\mu = mv^\mu + k_1^\mu$ incoming, and q^μ and $p_2^\mu = mv^\mu + k_2^\mu$ outgoing. With this choice of kinematics, the initial and final residual momenta are related by $k_2^\mu = k_1^\mu - q^\mu$. We can relate a spinor with incoming momentum to the spinor with outgoing momentum using analytical continuation, eq. (551). Also, the x factor picks up a negative sign when the directions of p_1 or q are flipped, $x \to -x$.

12.3.1 *General residual momentum*

We start by converting the $s = 1/2$ amplitude to on-shell HPET variables. Inverting eq. (445) and simply taking the appropriate spinor products, we can relate the traditional and HPET spinor products:

$$\langle \mathbf{21} \rangle = \frac{m^2}{m_{k_2} m_{k_1}} \left[\frac{m_{k_1}}{m} \langle \mathbf{2}_v \mathbf{1}_v \rangle + \frac{1}{4m} [\mathbf{2}_v q] \langle q \mathbf{1}_v \rangle + \frac{x^{-1}}{4m} [\mathbf{2}_v q][q \mathbf{1}_v] \right], \tag{469a}$$

$$\langle \mathbf{2} q \rangle \langle q \mathbf{1} \rangle = \frac{m^2}{4 m_{k_2} m_{k_1}} \Big(\langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle + x^{-1} \langle \mathbf{2}_v q \rangle [q \mathbf{1}_v] + x^{-1} [\mathbf{2}_v q] \langle q \mathbf{1}_v \rangle + x^{-2} [\mathbf{2}_v q][q \mathbf{1}_v] \Big). \tag{469b}$$

Similarly, the spinor products in the anti-chiral basis become

$$[\mathbf{21}] = \frac{m^2}{m_{k_2} m_{k_1}} \left[\frac{m_{k_1}}{m} [\mathbf{2}_v \mathbf{1}_v] + \frac{1}{4m} \langle \mathbf{2}_v q \rangle [q \mathbf{1}_v] + \frac{x}{4m} \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \right], \tag{470a}$$

$$[\mathbf{2} q][q \mathbf{1}] = \frac{m^2}{4 m_{k_2} m_{k_1}} \left([\mathbf{2}_v q][q \mathbf{1}_v] + x [\mathbf{2}_v q] \langle q \mathbf{1}_v \rangle + x \langle \mathbf{2}_v q \rangle [q \mathbf{1}_v] + x^2 \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \right). \tag{470b}$$

By substituting eqs. (469) and (470) in eqs. (467) and (468) for $s = 1/2$, the minimally coupled amplitudes for positive and negative helicity become

$$\mathcal{M}^{+|h|, s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{1+h} g_0 x^{|h|} \frac{m}{m_{k_2} m_{k_1}} \left[\frac{m_{k_1}}{m} \langle \mathbf{2}_v \mathbf{1}_v \rangle + \frac{1}{4m} [\mathbf{2}_v q] \langle q \mathbf{1}_v \rangle + \frac{x^{-1}}{4m} [\mathbf{2}_v q][q \mathbf{1}_v] \right], \tag{471a}$$

$$\mathcal{M}^{-|h|, s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{h} \tilde{g}_0 x^{-|h|} \frac{m}{m_{k_2} m_{k_1}} \left[\frac{m_{k_1}}{m} [\mathbf{2}_v \mathbf{1}_v] + \frac{1}{4m} \langle \mathbf{2}_v q \rangle [q \mathbf{1}_v] + \frac{x}{4m} \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \right], \tag{471b}$$

One can expand the m_{k_i} in powers of $|k|/m$, which is the characteristic expansion of HPETs. These three-point amplitudes therefore provide a conjecture for the resummed spin-1/2 HPET amplitude. Comparing the expansions of eq. (471) with that computed directly from the spin-1/2 HPET Lagrangians, we have confirmed that they agree at least up to $\mathcal{O}(m^{-2})$ for HQET, and $\mathcal{O}(m^{-1})$ for HBET.[3] Some subtleties of the matching to the Lagrangian calculation are discussed in Section 12.E.

[3] Note that the power counting of the HBET operators starts one power of m higher than HQET, at $\mathcal{O}(m)$. Thus both of these checks account for the operators up to and including NNLO.

The spin-dependence of these amplitudes can be made explicit by using the on-shell form of $q \cdot S$ in eq. (460):

$$\mathcal{M}^{+|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{1+h} g_0 x^{|h|} \frac{m}{m_{k_2} m_{k_1}} \langle \mathbf{2}_v| \left[1 - \frac{\not{v} \not{k}_1 \not{k}_2 \not{v}}{4m^2} + \frac{q \cdot S}{m} \right] |\mathbf{1}_v\rangle, \tag{472a}$$

$$\mathcal{M}^{-|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{h} \tilde{g}_0 x^{-|h|} \frac{m}{m_{k_2} m_{k_1}} [\mathbf{2}_v| \left[1 - \frac{\not{v} \not{k}_1 \not{k}_2 \not{v}}{4m^2} - \frac{q \cdot S}{m} \right] |\mathbf{1}_v]. \tag{472b}$$

Written in this way, it is immediately apparent how the $k_1 = 0$ parameterization can be obtained from the general case. We turn now to this scenario.

12.3.2 *Zero initial residual momentum*

We now consider the parameterization where $k_1^\mu = 0$ and $k_2^\mu = -q^\mu$ With zero initial residual momentum, we can switch between the chiral and anti-chiral bases using eq. (446):

$$\langle \mathbf{2}_v \mathbf{1}_v \rangle = -[\mathbf{2}_v \mathbf{1}_v], \tag{473a}$$

$$\langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle = x^{-2} [\mathbf{2}_v q][q \mathbf{1}_v]. \tag{473b}$$

Recognizing eqs. (473a) and (473b) as directly relating spinless effects and the spin-vector respectively in different bases, we see that, for this parameterization, spin effects are never obscured by working in any particular basis. This is in contrast to the traditional on-shell variables, where the analog to eq. (473a) includes a spin term, thus hiding or exposing spin dependence when working in a certain basis. Thus we have gained a basis-independent interpretation of spinless and spin-inclusive terms.

Either setting $k_1 = 0$ in eq. (472), or applying eqs. (473a) and (473b) to eqs. (469) and (470), the minimally coupled three-point amplitude with zero residual momentum is obtained:

$$\mathcal{M}^{+|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{1+h} \frac{g_0 x^{|h|}}{m} \left[\langle \mathbf{2}_v \mathbf{1}_v \rangle + \frac{x}{2m} \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \right], \tag{474a}$$

$$\mathcal{M}^{-|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{h} \frac{\tilde{g}_0 x^{-|h|}}{m} \left[[\mathbf{2}_v \mathbf{1}_v] + \frac{x^{-1}}{2m} [\mathbf{2}_v q][q \mathbf{1}_v] \right], \tag{474b}$$

Note the negative signs which come from treating p_1 as incoming.

Three-point kinematics are restrictive enough when $k_1 = 0$ that we can derive the three-point amplitude in eq. (474) in an entirely different fashion. The full three-point amplitude for a heavy spin-$1/2$ particle coupled to a photon can be written as[4]

$$\mathcal{A}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, q^h) = f(m,v,q) e v_\mu \epsilon_q^{h,\mu} \bar{u}_v(p_2) u_v(p_1) + g(m,v,q) e q^\mu \epsilon_q^{h,\nu} \bar{u}_v(p_2) \sigma_{\mu\nu} u_v(p_1). \quad (475)$$

The negative in the argument of the amplitude signifies an incoming momentum. The three-point operators in the HQET Lagrangian, as well as any non-minimal couplings, modify the functions f and g, but there are no other spinor structures that can arise. We therefore have two spinor contractions in terms of which we would like to express the spinor brackets of interest. We proceed by writing the two contractions in terms of the traditional on-shell variables, and equating this to the contractions expressed in terms of the on-shell HPET variables. Working with, say, a positive helicity photon, this yields

$$v_\mu \epsilon_q^{+,\mu} \bar{u}_v(p_2) u_v(p_1) = -\sqrt{2} x \langle \mathbf{2}_v \mathbf{1}_v \rangle = -\frac{x}{\sqrt{2}} \left(-\frac{x}{m} \langle \mathbf{2} q \rangle \langle q \mathbf{1} \rangle + 2 \langle \mathbf{2} \mathbf{1} \rangle \right), \quad (476a)$$

$$\bar{u}_v(p_2) \sigma_{\mu\nu} u_v(p_1) q^\mu \epsilon_q^{+,\nu} = \sqrt{2} i x^2 \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle = \sqrt{2} i x^2 \langle \mathbf{2} q \rangle \langle q \mathbf{1} \rangle. \quad (476b)$$

Solving for the traditional spinor products, we find

$$\langle \mathbf{2} \mathbf{1} \rangle = \langle \mathbf{2}_v \mathbf{1}_v \rangle + \frac{x}{2m} \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle, \quad (477a)$$

$$\langle \mathbf{2} q \rangle \langle q \mathbf{1} \rangle = \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle. \quad (477b)$$

Similarly,

$$[\mathbf{2} \mathbf{1}] = [\mathbf{2}_v \mathbf{1}_v] + \frac{x^{-1}}{2m} [\mathbf{2}_v q][q \mathbf{1}_v], \quad (478a)$$

$$[\mathbf{2} q][q \mathbf{1}] = [\mathbf{2}_v q][q \mathbf{1}_v]. \quad (478b)$$

Note that eqs. (477) and (478) decompose the spinor brackets into spinless and spin-inclusive terms. Applying eq. (473), it is easy to check that this separation of different spin multipoles is independent of the basis used to express the traditional spinor brackets.

[4] We use $\mathcal{A}$ to denote a Yang-Mills amplitude.

With eqs. (476a) and (476b), we can rewrite eq. (475) as

$$\mathcal{A}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, q^+) = \sqrt{2}xe\left(-f(m,v,q)\langle \mathbf{2}_v \mathbf{1}_v\rangle + g(m,v,q)ix\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right). \tag{479}$$

The three-point amplitude in QED — with interaction term $\mathcal{L}_{\text{int}} = e\bar{\psi}\not{A}\psi$ — for a positive helicity photon is

$$\begin{aligned}\mathcal{A}_{\text{QED}}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, q^+) &= e\bar{u}(p_2)\gamma_\mu u(p_1)\epsilon_q^{+,\mu} \\ &= \sqrt{2}ex\langle \mathbf{21}\rangle,\end{aligned} \tag{480}$$

where in the first line we use Dirac spinors instead of HQET spinors. Substituting eq. (477a) into the above equation gives

$$\mathcal{A}_{\text{QED}}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, q^+) = \sqrt{2}ex\left(\langle \mathbf{2}_v \mathbf{1}_v\rangle + \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right). \tag{481}$$

As abelian HQET is an effective theory derived from QED, it must reproduce the on-shell QED amplitudes when all operators are accounted for. This means that eqs. (479) and (481) are equal, so we can solve for the functions f and g:

$$f(m,v,q) = -1, \tag{482a}$$

$$g(m,v,q) = \frac{i}{2m}. \tag{482b}$$

As a consequence of eqs. (482a) and (482b), we conclude that only the leading spin and leading spinless three-point operators of HQET are non-vanishing on-shell when $k_1 = 0$. Indeed, in this case the transfer momentum q^μ is the only parameter that can appear in the HQET operator expansion. In the three-point amplitude, it can only appear in the scalar combinations $q^2 = 0$ by on-shellness of the photon, $v \cdot q \sim q^2 = 0$ by on-shellness of the quarks, or $q \cdot \epsilon(q) = 0$ by transversality of the polarization.

To sum up, we list the three-point amplitude for two equal mass spin-1/2 particles and an outgoing photon for both helicities, and in both the chiral and anti-chiral bases:[5]

$$\mathcal{A}^{+1,s=\frac{1}{2}} = \sqrt{2}ex\left(\langle \mathbf{2}_v \mathbf{1}_v\rangle + \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right) = -\sqrt{2}ex\left([\mathbf{2}_v\mathbf{1}_v] - \frac{x^{-1}}{2m}[\mathbf{2}_v q][q\mathbf{1}_v]\right), \tag{483a}$$

$$\mathcal{A}^{-1,s=\frac{1}{2}} = \sqrt{2}ex^{-1}\left(\langle \mathbf{2}_v \mathbf{1}_v\rangle - \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right) = -\sqrt{2}ex^{-1}\left([\mathbf{2}_v\mathbf{1}_v] + \frac{x^{-1}}{2m}[\mathbf{2}_v q][q\mathbf{1}_v]\right), \tag{483b}$$

so $g_0 = \tilde{g}_0 = \sqrt{2}em$. When a graviton is emitted instead of a photon, we simply make the replacement $e \to -\frac{\kappa m}{2\sqrt{2}}$ and square the overall factors of x.

We can obtain the amplitude with general initial residual momentum by reparameterizing the states by means of eq. (453).

12.3.3 *Most general three-point amplitude*

Recall the most general three-point amplitude for two massive particles of spin s and mass m and a massless boson with helicity h in the chiral basis, eq. (465):

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h}\frac{x^{|h|}}{m^{2s}}\left[g_0\langle \mathbf{21}\rangle^{2s} + g_1\langle \mathbf{21}\rangle^{2s-1}\frac{x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle}{m} + \cdots + g_{2s}\frac{(x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle)^{2s}}{m^{2s}}\right]. \tag{484}$$

When expressing eq. (465) in terms of the on-shell HPET variables, setting the initial residual momentum to zero, and applying the binomial expansion, we find that

$$\mathcal{M}_3^{+|h|,s} = (-1)^{2s+h}\frac{x^{|h|}}{m^{2s}}\sum_{k=0}^{2s} g^{\mathrm{H}}_{s,k}\langle \mathbf{2}_v\mathbf{1}_v\rangle^{2s-k}\left(\frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right)^k, \quad g^{\mathrm{H}}_{s,k} = \sum_{i=0}^{k} g_i\binom{2s-i}{2s-k}. \tag{485a}$$

We can express this in the anti-chiral basis using eq. (473):

$$\mathcal{M}_3^{+|h|,s} = \frac{x^{|h|}}{m^{2s}}\sum_{k=0}^{2s} g^{\mathrm{H}}_{s,k}(-1)^{k+h}[\mathbf{2}_v\mathbf{1}_v]^{2s-k}\left(\frac{x^{-1}}{2m}[\mathbf{2}_v q][q\mathbf{1}_v]\right)^k. \tag{485b}$$

[5] We abbreviate the arguments of the amplitude here, but still use p_1 incoming.

The k^{th} spin-multipole can be isolated by choosing the k^{th} term in the sum. There are $2s+1$ combinations of the spinor brackets in this sum, consistent with the fact that a spin s particle can only probe up to the $2s^{\text{th}}$ spin order term of the spin-multipole expansion. Note also that the coefficient of the spin monopole term is always equal to its value for minimal coupling, making the monopole term universal in any theory.[6]

The minimal coupling amplitudes are those in eqs. (467) and (468), which correspond to setting $g_{i>0}=0$. Translating to the on-shell HPET variables, minimal coupling in eqs. (485a) and (485b) corresponds to $g^{\mathrm{H}}_{s,k}=g_0\binom{2s}{k}$.

We can write the analogous expressions to eqs. (485a) and (485b) for a negative helicity massless particle. Expressing eq. (466) using eqs. (478a) and (478b),

$$\mathcal{M}_3^{-|h|,s} = (-1)^h \frac{x^{-|h|}}{m^{2s-1}} \sum_{k=0}^{2s} \tilde{g}^{\mathrm{H}}_{s,k} [\mathbf{2}_v \mathbf{1}_v]^{2s-k} \left(\frac{x^{-1}}{2m} [\mathbf{2}_v q][q \mathbf{1}_v] \right)^k, \quad \tilde{g}^{\mathrm{H}}_{s,k} = \sum_{i=0}^{k} \tilde{g}_i \binom{2s-i}{2s-k}. \tag{486a}$$

Converting to the chiral basis,

$$\mathcal{M}_3^{-|h|,s} = \frac{x^{-|h|}}{m^{2s}} \sum_{k=0}^{2s} \tilde{g}^{\mathrm{H}}_{s,k} (-1)^{2s+h+k} \langle \mathbf{2}_v \mathbf{1}_v \rangle^{2s-k} \left(\frac{x}{2m} \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \right)^k. \tag{486b}$$

Minimal coupling in this case corresponds to $\tilde{g}_{i>0}=0$, and thus $\tilde{g}^{\mathrm{H}}_{s,k}=\tilde{g}_0\binom{2s}{k}$.

12.3.4 *Infinite spin limit*

Various methods have been used to show that the minimal coupling three-point amplitude in traditional on-shell variables exponentiates in the infinite spin limit [208, 211, 299]. All of them require a slight manipulation of the minimal coupling to do so, with refs. [208, 211] employing a change of basis between the chiral and anti-chiral bases, ref. [208] applying a generalized expectation value, and refs. [211, 299] using a Lorentz boost – analogous to the gauge-fixing of the spin operator in ref. [283] – to rewrite the minimal coupling amplitude. As the on-shell HPET variables inherently make the spin-dependence of the minimal coupling manifest, the exponentiation of the three-point amplitude is immediate.

[6] This is consistent with the reparameterization invariance of HQET, which fixes the Wilson coefficients of the spinless operators in the HQET Lagrangian up to order $1/m$ [231]. As argued above, when the initial residual momentum is set to 0, these are the only operators contributing to the spin monopole.

Consider the minimal coupling three-point amplitude for two massive spin s particles and one massless particle:

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h}\frac{g_0 x^{|h|}}{m^{2s}}\langle \mathbf{2}_v|^{2s}\sum_{k=0}^{2s}\frac{(2s)!}{(2s-k)!}\frac{\left(\frac{x}{2m}|q\rangle\langle q|\right)^k}{k!}|\mathbf{1}_v\rangle^{2s}. \tag{487}$$

The quantity in the sum is the rescaled spin-operator $q\cdot S/m$ for a spin s particle, raised to the power of k and divided by $k!$ [209],

$$\left(\frac{q\cdot S}{m}\right)^n = \frac{(2s)!}{(2s-n)!}\left(\frac{x}{2m}|q\rangle\langle q|\right)^n, \tag{488}$$

where we have suppressed the spinor indices. The amplitude is therefore

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h}\frac{g_0 x^{|h|}}{m^{2s}}\langle \mathbf{2}_v|^{2s}\sum_{k=0}^{2s}\frac{\left(\frac{q\cdot S}{m}\right)^k}{k!}|\mathbf{1}_v\rangle^{2s}. \tag{489}$$

We identify the sum with an exponential, with the understanding that the series truncates at the $2s^{\text{th}}$ term for a spin $2s$ particle:

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h}\frac{g_0 x^{|h|}}{m^{2s}}\langle \mathbf{2}_v|^{2s}e^{q\cdot S/m}|\mathbf{1}_v\rangle^{2s}. \tag{490}$$

Taking the infinite spin limit, the exponential is exact as its Taylor series does not truncate. We treat the exponential as a number in this limit and remove it from between the spinors [211]:

$$\lim_{s\to\infty}\mathcal{M}^{+|h|,s} = \lim_{s\to\infty}(-1)^{2s+h}\frac{g_0 x^{|h|}}{m^{2s}}e^{q\cdot S/m}\langle \mathbf{2}_v\mathbf{1}_v\rangle^{2s}. \tag{491}$$

Note that since the initial residual momentum is 0, both spinors are associated with the same momentum. Then, using the on-shell conditions for these variables,[7]

$$\lim_{s\to\infty}\mathcal{M}^{+|h|,s} = (-1)^h g_0 x^{|h|}e^{q\cdot S/m}. \tag{492}$$

This amplitude immediately agrees with the three-point amplitude in refs. [208, 211]: it is the scalar three-point amplitude multiplied by an exponential containing the classical spin-multipole moments.

[7] The validity of using the on-shell conditions can be checked explicitly by rewriting the bracket in terms of traditional on-shell variables, then boosting one of the momenta into the other as in ref. [299].

Also notable is that the generalized expectation value (GEV) of ref. [208] or the Lorentz boosts of refs. [211, 299] are not necessary here to interpret the spin dependence classically.

For the emission of a negative helicity boson, the n^{th} power of the spin-operator projected along the direction of the boson's momentum is

$$\left(\frac{q\cdot S}{m}\right)^n = \frac{(2s)!}{(2s-n)!}\left(-\frac{x^{-1}}{2m}|q][q|\right)^n . \tag{493}$$

Starting with eq. (486a), the three-point amplitude exponentiates as

$$\mathcal{M}^{-|h|,s} = (-1)^h \frac{\tilde{g}_0 x^{-|h|}}{m^{2s}} [\mathbf{2}_v|^{2s} e^{-q\cdot S/m}|\mathbf{1}_v]^{2s}, \tag{494}$$

with the exponential being truncated at the $2s^{\text{th}}$ term. Taking the infinite spin limit, we find

$$\lim_{s\to\infty} \mathcal{M}^{-|h|,s} = \lim_{s\to\infty} (-1)^h \frac{\tilde{g}_0 x^{-|h|}}{m^{2s}} e^{-q\cdot S/m}[\mathbf{2}_v \mathbf{1}_v]^{2s}. \tag{495}$$

Applying the on-shell conditions for these variables, we get

$$\lim_{s\to\infty} \mathcal{M}^{-|h|,s} = (-1)^h \tilde{g}_0 x^{-|h|} e^{-q\cdot S/m}. \tag{496}$$

Once again we find the scalar three-point amplitude multiplied by an exponential containing the classical spin dependence.

That the exponentials in this section are functions of $q\cdot S$ instead of $2q\cdot S$, as is the case when the traditional on-shell variables are naïvely exponentiated — that is, without normalizing by the GEV, or Lorentz boosting one of the spinors — is significant. We discuss the implications of this in the next section.

12.4 KERR BLACK HOLES AS HEAVY PARTICLES

In this section, we apply the on-shell HPET variables to the classical gravitational scattering of two spinning black holes. We show that, with the correct momentum parameterization, a heavy spin-s particle minimally coupled to gravity possesses precisely the same spin-multipole expansion as a Kerr black hole, up to the order $2s$ multipole. The reason for this is that on-shell HPET variables for a given velocity v^μ, residual momentum k^μ, and mass m always correspond to momenta $m_k v^\mu$, where m_k is defined in eq. (448).

We begin with a brief review of the effective field theory for spinning gravitating bodies. The action of a particle interacting with gravitational radiation of wavelength much larger than its spatial extent (approximately a point particle) was formulated in ref. [196]. The generalization to the case of spinning particles was first approached in ref. [197]. The effective action formulated in ref. [283] takes the form

$$S = \int d\sigma \left\{ -m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{\mathrm{SI}}[u^\mu, S_{\mu\nu}, g_{\mu\nu}(x^\mu)] \right\}, \tag{497}$$

where σ parameterizes the wordline of the particle, $u^\mu = \frac{dx^\mu}{d\sigma}$ is the coordinate velocity, $S_{\mu\nu}$ is the spin operator, $\Omega^{\mu\nu}$ is the angular velocity, and L_{SI} contains higher spin-multipoles that are dependent on the inner structure of the particle through non-minimal couplings.

The first two terms in eq. (497) are the spin monopole and dipole terms, and are universal for spinning bodies with any internal configuration. We assign to them respectively the coefficients $C_{S^0} = C_{S^1} = 1$. From an amplitudes perspective, the universality of the spin-monopole coefficient can be seen from the on-shell HPET variables since the coefficient of the spin-monopole term in eqs. (485a) and (486a) is always equal to its minimal coupling value. The universality of the spin-dipole coefficient was argued in refs. [209, 224] from general covariance, and by requiring the correct factorization of the Compton scattering amplitude. Explicitly, the higher spin-multipole terms L_{SI} are

$$\begin{aligned} L_{\mathrm{SI}} = &\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{S^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \dots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \dots S^{\mu_{2n}} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{S^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \dots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \dots S^{\mu_{2n+1}}. \end{aligned} \tag{498}$$

See ref. [283] for the derivation and formulation of this action. The Wilson coefficients C_{S^k} contain the information about the internal structure of the object, with a Kerr black hole being described by $C_{S^k}^{\mathrm{Kerr}} = 1$ for all k.

The three-point amplitude derived from this action was expressed in traditional spinor-helicity variables in refs. [209, 224], where it was shown that the spin-multipole expansion is necessarily truncated at order $2s$ when the polarization tensors of spin s particles are used. By matching this three-point amplitude with the most general form of a three-point amplitude, it was found there that in the case of minimal coupling one obtains the Wilson coefficients of a Kerr black hole in the infinite spin limit. Following their derivation, but using on-shell HPET variables instead, we find (with all momenta incoming)

$$\mathcal{M}^{+2,s} = \sum_{a+b\leq s} \frac{\kappa m x^2}{2m^{2s}} C_{S^{a+b}} n^s_{a,b} \langle \mathbf{2}_{-v} \mathbf{1}_v \rangle^{s-a} \left(-x \frac{\langle \mathbf{2}_{-v} q \rangle \langle q \mathbf{1}_v \rangle}{2m} \right)^a [\mathbf{2}_{-v} \mathbf{1}_v]^{s-b} \left(x^{-1} \frac{[\mathbf{2}_{-v} q][q \mathbf{1}_v]}{2m} \right)^b,$$
$$n^s_{a,b} \equiv \binom{s}{a}\binom{s}{b}. \tag{499}$$

As in refs. [209, 224], we refer to this representation of the amplitude in a form symmetric in the chiral and anti-chiral bases as the polarization basis. Flipping the directions of p_2 and q (to allow us to directly compare with eq. (485a)), then converting the polarization basis to the chiral basis:

$$\mathcal{M}^{+2,s} = \frac{x^2}{m^{2s}} (-1)^{2s} \sum_{a+b\leq 2s} \frac{\kappa m}{2} C_{S^{a+b}} n^s_{a,b} \langle \mathbf{2}_v \mathbf{1}_v \rangle^{2s-a-b} \left(\frac{x}{2m} \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \right)^{a+b}. \tag{500}$$

Comparing with eq. (485a), we obtain a one-to-one relation between the coupling constants of both expansions:

$$g^{\mathrm{H}}_{s,k} = \frac{\kappa m}{2} C_{S^k} \sum_{j=0}^{k} n^s_{k-j,j}. \tag{501}$$

Such a one-to-one relation is consistent with the interpretation of eq. (485a) as being a spin-multipole expansion. Focusing on the minimal coupling case, we set $g_{i>0} = 0$, which means $g^{\mathrm{H}}_{s,k} = g_0 \binom{2s}{k}$. Normalizing $g_0 = \kappa m/2$, the coefficients of the one-particle effective action for finite spin take the form

$$C^{\min}_{S^k} = \binom{2s}{k} \left[\sum_{j=0}^{k} \binom{s}{k-j} \binom{s}{j} \right]^{-1} = 1. \tag{502}$$

The final equality is the Chu-Vandermonde identity, valid for all k. This suggests that the minimal coupling expressed in the on-shell HPET variables produces precisely the multipole moments of a Kerr black hole, even before taking the infinite spin limit.

Using the same matching technique, refs. [209, 224] showed that, when using traditional on-shell variables, the minimal coupling three-point amplitude for finite spin s corresponded to Wilson coefficients that deviated from those of a Kerr black hole by terms of order $\mathcal{O}(1/s)$. Why is it then that the polarization tensors of finite spin HPET possess the same spin-multipole expansion as a Kerr black hole? Analyzing the matching performed in refs. [209, 224], the s dependence there arises from the conversion of the polarization basis to the chiral basis. The reason for this is that new spin contributions arise from this conversion since the chiral and anti-chiral bases are mixed by two times the spin-operator:

$$\langle \mathbf{12} \rangle = -[\mathbf{12}] + \frac{1}{xm}[\mathbf{1}q][q\mathbf{2}], \tag{503a}$$

$$[\mathbf{12}] = -\langle \mathbf{12} \rangle + \frac{x}{m}\langle \mathbf{1}q \rangle \langle q\mathbf{2} \rangle. \tag{503b}$$

The second terms on the right hand sides of these equations encode spin effects, while the first terms were interpreted to be purely spinless. However, the left hand sides of these equations contradict the latter interpretation; the spinor brackets $\langle \mathbf{12} \rangle$ and $[\mathbf{12}]$ themselves contain spin effects. This is the origin of the observed deviation from $C^{\mathrm{Kerr}}_{S^k}$: eq. (503), while exposing some spin-dependence, does not entirely separate the spinless and spin-inclusive effects encoded in the traditional minimal coupling amplitude. The result is the matching of an exact spin-multipole expansion on the one-particle effective action side, to a rough separation of different spin-multipoles on the amplitude side.

A similar mismatch to Kerr black holes was seen in ref. [211], where the minimal coupling amplitude was shown to produce the spin dependence[8]

$$\langle \mathbf{21} \rangle = -[\mathbf{2}|e^{2q\cdot S/m}|\mathbf{1}], \tag{504}$$

where S^μ is the Pauli-Lubanski pseudovector defined with respect to p_1. Expanding the exponential and noting that the series terminates after the spin-dipole term in this case, it's easy to see the equivalence between this and eq. (503). The spin-dependence here differs from that of a Kerr black hole by a factor of two in the exponential [208, 301]. Motivated by arguments in ref. [283], an exact match to the Kerr black hole spin multipole expansion was obtained in ref. [211] by noting that additional spin contributions are hidden in the fact that the polarization vectors $[\mathbf{2}|$ and $|\mathbf{1}]$ represent different momenta. Writing $[\mathbf{2}|$ as a Lorentz boost of $[\mathbf{1}|$, the true spin-dependence of the minimal coupling bracket was manifested:

$$\langle \mathbf{21} \rangle \sim -[\mathbf{1}|e^{q\cdot S/m}|\mathbf{1}], \tag{505}$$

up to an operator acting on the little group index of $[\mathbf{1}|$. The spin-dependence here matches that of a Kerr black hole, and also matches what has been made explicit in Section 12.3.4. Using a similar

[8] Ref. [211] worked exclusively with integer spin. However the only adaptation that must be made to the results therein when working with half integer spins is the inclusion of a factor of $(-1)^{2s} = -1$.

Lorentz boost, the authors of ref. [299] also showed that the minimal coupling bracket indeed contains the spin-dependence of a Kerr black hole. We see that in the absence of a momentum mismatch between the polarization states used, the full spin-dependence is manifest, and the multipole expansion of a finite spin s particle corresponds exactly to that of a Kerr black hole up to $2s^{\text{th}}$ order.

This mismatch of momenta is avoided entirely when using on-shell HPET variables. Recall that in general the momentum p_v represented by on-shell HPET variables is

$$p_v^\mu = m_k v^\mu. \tag{506}$$

Working in the case where the initial residual momentum is zero, as in the rest of this section, this reduces to simply $mv_{\alpha\dot{\alpha}}$ for the case of $p_{v,1}$. For $p_{v,2}$, where $p_2 = p_1 - q$ and q is the null transfer momentum,

$$p_{v,2} = \left(1 - \frac{q^2}{4m^2}\right) mv^\mu = mv^\mu. \tag{507}$$

Consequently, although the initial and final momenta of the massive particle differ by q, the degrees of freedom are arranged in such a way that the external states $|\mathbf{1}_v\rangle$ and $|\mathbf{2}_v\rangle$ are associated with the same momentum. This explains why we have recovered precisely the Wilson coefficients of a Kerr black hole. We identify this common momentum with that of the Kerr black hole $p^\mu_{\text{Kerr}} = mv^\mu$. From the point of view of spinor products, eq. (473) shows that on-shell HPET variables provide an unambiguous and basis-independent interpretation of spinless and spin-inclusive spinor brackets. Thus, the entire spin dependence of the minimal coupling amplitude is automatically made explicit, and is isolated from spinless terms.

In the case of $k_1 \neq 0$, the three-term structure of the minimal coupling amplitude spoils its exponentiation. The matching to the Kerr black hole spin-multipole moments is therefore obscured, but is recovered in the reparameterization where k_1 is set to 0. This mismatching of the spin-multipole moments can be attributed to the fact that the polarization tensors for the initial and final states no longer correspond to the same momentum, since generally $m_{k_1 - q} \neq m_{k_1}$.

A similar matching analysis has recently been performed in ref. [302] for the case of Kerr-Newman black holes. It was also found there that minimal coupling to electromagnetism reproduces the classical spin multipoles of a Kerr-Newman black hole in the infinite spin limit, when the matching is performed using traditional on-shell variables. Repeating their analysis, but using on-shell HPET variables instead, we find again that the classical multipoles are reproduced exaclty, even for finite spin.

12.5 ON-SHELL AMPLITUDES

In this section, we compute electromagnetic and gravitational amplitudes for the scattering of minimally coupled spin-s particles in on-shell HPET variables using eqs. (469), (470), (477) and (478). Our goal in this section is two-fold: first, we will show how spin effects remain separated from spinless effects, at the order considered in this work, when using on-shell HPET variables. Second, we will exploit the explicit $\hbar$ dependence of eqs. (469) and (470) to isolate the classical portions of the computed amplitudes. Given that the momenta of the on-shell HPET variables always reduce to the momentum of a Kerr black hole in the classical limit, we expect to recover the spin-multipoles of a Kerr black hole in this limit. We show that, at tree-level, the spin dependence of the leading $\hbar$ portions factorizes into a product of the classical spin-dependence at three-points. This is simply a consequence of factorization for boson exchange amplitudes (a result that has already been noted in ref. [211]). For same-helicity tree-level radiation processes this results from a spin-multipole universality that we will uncover, and for the opposite helicity Compton amplitude there will be an additional factor accounting for its non-uniqueness at higher spins.[9]

12.5.1 *Counting $\hbar$*

Given that we will be interested in isolating classical effects, we summarize here the rules for restoring the $\hbar$ dependence in the amplitude [220], and adapt these rules to the on-shell variables.

Powers of $\hbar$ are restored in such a way so as to preserve the dimensionality of amplitudes and coupling constants. To do so, the coupling constants of electromagnetism and gravity are rescaled as $e \to e/\sqrt{\hbar}$ and $\kappa \to \kappa/\sqrt{\hbar}$. Furthermore, when taking the classical limit $\hbar \to 0$ of momenta, massive momenta and masses are to be kept constant, whereas massless momenta vanish in this limit — for a massless momentum q, it is the associated wave number $\bar{q} = q/\hbar$ that is kept constant in the classical limit. Thus each massless momentum in amplitudes is associated with one power of $\hbar$. Translating this to on-shell variables, we assign a power of $\hbar^{\alpha}$ to each $|q\rangle$, and a power of $\hbar^{1-\alpha}$ to each $|q]$.[10] Momenta that are treated with the massless $\hbar$ scaling are

- photon and graviton momenta, whether they correspond to external or virtual particles;
- loop momenta, which can always be assigned to an internal massless boson;

[9] We contrast the factorization for radiation processes here with that in ref. [208] by noting that the entire quantum amplitude was factorized there, whereas we show that the factorization holds also for the leading $\hbar$ contribution.

[10] The value of α can be determined by fixing the $\hbar$ scaling of massless polarization tensors for each helicity. Requiring that the dimensions of polarization vectors remain unchanged when $\hbar$ is restored results in the democratic choice $\alpha = 1/2$.

- residual momenta [6].

Finally, we come to the case of spin-inclusive terms. When taking the classical limit $\hbar \to 0$, we simultaneously take the limit $s \to \infty$ where s is the magnitude of the spin. These limits are to be taken in such a way so as to keep the combination $\hbar s$ constant. This means that for every power of spin in a term, there is one factor of $\hbar$ that we can neglect when taking the classical limit. Effectively, we can simply scale all powers of spin with one inverse power of $\hbar$, and understand that $\hbar$ is to be taken to 0 wherever it appears in the amplitude.

As in ref. [6], we identify the components of an amplitude contributing classically to the interaction potential as those with the $\hbar$ scaling

$$\mathcal{M} \sim \hbar^{-3}. \tag{508}$$

Terms with more positive powers of $\hbar$ contribute quantum mechanically to the interaction potential. Also, we use $\mathcal{M}^{\text{cl.}}$ to denote the leading $\hbar$ portion of an amplitude.

12.5.2 *Boson exchange*

We begin with the tree-level amplitudes for photon/graviton[11] exchange between two massive spinning particles. We consider first spin-1/2 – spin-1/2 scattering, to show that the spin-multipole expansion remains explicit in these variables at four points. The classical part of the amplitude can be computed by factorizing it into two three-point amplitudes. To simplify the calculation, we are free to set the initial residual momentum of each massive leg to 0, so we will need only eqs. (477) and (478). Letting

[11] We will denote an amplitude involving photons by $\mathcal{A}$, and one involving gravitons by $\mathcal{M}$.

particle a have mass m_a and incoming/outgoing momenta p_1/p_2, and particle b have mass m_b and incoming/outgoing momenta p_3/p_4, we find for an exchanged photon

$$\begin{aligned}
i\mathcal{A}_{\text{tree}}(-\mathbf{1}_a^{\frac{1}{2}},\mathbf{2}_a^{\frac{1}{2}},-\mathbf{3}_b^{\frac{1}{2}},\mathbf{4}_b^{\frac{1}{2}}) &= \sum_h \mathcal{A}_{\text{tree}}(-\mathbf{1}^{\frac{1}{2}},\mathbf{2}^{\frac{1}{2}},-q^h)\frac{i}{q^2}\mathcal{A}_{\text{tree}}(q^{-h},-\mathbf{3}^{\frac{1}{2}},\mathbf{4}^{\frac{1}{2}}) \\
&= -\frac{ie^2}{q^2}\Big[4\omega\langle\mathbf{2}_{v_a}\mathbf{1}_{v_a}\rangle\langle\mathbf{4}_{v_b}\mathbf{3}_{v_b}\rangle \\
&\quad -\frac{2}{m_b}\sqrt{\omega^2-1}\,\langle\mathbf{2}_{v_a}\mathbf{1}_{v_a}\rangle x_b\langle\mathbf{4}_{v_b}q\rangle\langle q\mathbf{3}_{v_b}\rangle \\
&\quad +\frac{2}{m_a}\sqrt{\omega^2-1}\,x_a\langle\mathbf{2}_{v_a}q\rangle\langle q\mathbf{1}_{v_a}\rangle\langle\mathbf{4}_{v_b}\mathbf{3}_{v_b}\rangle \\
&\quad -\frac{\omega}{m_a m_b}x_a\langle\mathbf{2}_{v_a}q\rangle\langle q\mathbf{1}_{v_a}\rangle x_b\langle\mathbf{4}_{v_b}q\rangle\langle q\mathbf{3}_{v_b}\rangle\Big],
\end{aligned} \tag{509}$$

where $\omega \equiv p_1 \cdot p_3/m_a m_b = (x_a x_b^{-1} + x_a^{-1} x_b)/2$, $v_a = p_1/m_a$, $v_b = p_3/m_b$, and negative momenta are incoming. The x variables are defined as

$$x_a = -\frac{[q|p_1|\xi\rangle}{m_a\langle q\xi\rangle}, \quad x_a^{-1} = -\frac{\langle q|p_1|\xi]}{m_a[q\xi]}, \tag{510a}$$

$$x_b = \frac{[q|p_3|\xi\rangle}{m_b\langle q\xi\rangle}, \quad x_b^{-1} = \frac{\langle q|p_3|\xi]}{m_b[q\xi]}. \tag{510b}$$

The negative sign in the definitions of x_a and x_a^{-1} account for the fact that the massless boson is incoming to particle a.

The gravitational amplitude is computed analogously:

$$\begin{aligned}
i\mathcal{M}_{\text{tree}}(-\mathbf{1}_a^{\frac{1}{2}},\mathbf{2}_a^{\frac{1}{2}},-\mathbf{3}_b^{\frac{1}{2}},\mathbf{4}_b^{\frac{1}{2}}) &= -\frac{im_a m_b\kappa^2}{8q^2}\Big[4\left(2\omega^2-1\right)\langle\mathbf{2}_{v_a}\mathbf{1}_{v_a}\rangle\langle\mathbf{4}_{v_b}\mathbf{3}_{v_b}\rangle \\
&\quad -\frac{4\omega}{m_a}\sqrt{\omega^2-1}\,x_a\langle\mathbf{2}_{v_a}q\rangle\langle q\mathbf{1}_{v_a}\rangle\langle\mathbf{4}_{v_b}\mathbf{3}_{v_b}\rangle \\
&\quad +\frac{4\omega}{m_b}\sqrt{\omega^2-1}\,\langle\mathbf{2}_{v_a}\mathbf{1}_{v_a}\rangle x_b\langle\mathbf{4}_{v_b}q\rangle\langle q\mathbf{3}_{v_b}\rangle \\
&\quad -\frac{(2\omega^2-1)}{m_a m_b}x_a\langle\mathbf{2}_{v_a}q\rangle\langle q\mathbf{1}_{v_a}\rangle x_b\langle\mathbf{4}_{v_b}q\rangle\langle q\mathbf{3}_{v_b}\rangle\Big].
\end{aligned} \tag{511}$$

Both amplitudes agree with known results [6, 203, 221]. Furthermore, the amplitudes as written are composed of terms which each individually correspond to a single order in the spin-multipole expansion. All terms in these amplitudes scale as $\hbar^{-3}$, so these amplitudes are classical in the sense mentioned in the previous section.

Using the exponential forms of the three-point amplitudes in Section 12.3.4, we can write down the boson-exchange amplitudes in the infinite spin case. We find the same result in the gravitational case as ref. [211]. However we have obtained this result immediately simply by gluing together the three-point amplitudes; we had no need to boost the external states such they represent the same momentum. Omitting the momentum arguments, the amplitudes are

$$\lim_{s_a,s_b\to\infty} \mathcal{A}^{s_a,s_b}_{\text{tree}} = -\frac{2e^2}{q^2}\sum_{\pm}(\omega \pm \sqrt{\omega^2-1})\,\exp\left[\pm q\cdot\left(\frac{S_a}{m_a}+\frac{S_b}{m_b}\right)\right], \tag{512a}$$

$$\lim_{s_a,s_b\to\infty} \mathcal{M}^{s_a,s_b}_{\text{tree}} = -\frac{\kappa^2 m_a m_b}{4q^2}\sum_{\pm}(\omega \pm \sqrt{\omega^2-1})^2\,\exp\left[\pm q\cdot\left(\frac{S_a}{m_a}+\frac{S_b}{m_b}\right)\right]. \tag{512b}$$

The gravitational result corresponds to the first post-Minkowskian (1PM) order amplitude.

12.5.3 *Compton scattering*

Our focus shifts now to the electromagnetic and gravitational Compton amplitudes. These computations will enable the exploitation of the explicit $\hbar$ and spin-multipole expansions to relate the classical limit $\hbar\to 0$ and the classical spin-multipole expansion. Concretely, we will show that the spin-multipole expansion of the leading-in-$\hbar$ terms factorizes into a product of factors of the classical spin-dependence at three-points.

First, consider the spin-s electromagnetic Compton amplitude with two opposite helicity photons, $\mathcal{A}(-\mathbf{1}^s,\mathbf{2}^s,q_3^{-1},q_4^{+1})$. To simplify calculations, we can set the initial residual momentum to 0, so that $p_1^\mu = mv^\mu$. Note that it is impossible to set both initial and final residual momenta to 0 simultaneously, so we will need eqs. (469) and (470). We perform the computation by means of Britto-Cachazo-Feng-Witten (BCFW) recursion [303, 304], using the $[3,4\rangle$-shift

$$|\hat{4}\rangle = |4\rangle - z|3\rangle, \quad |\hat{3}] = |3] + z|4]. \tag{513a}$$

Under this shift, two factorization channels contribute to this amplitude:

$$\begin{aligned}\mathcal{A}(-\mathbf{1}^s,\mathbf{2}^s,q_3^{-1},q_4^{+1}) = {}& \left.\frac{\mathcal{A}(-\mathbf{1}^s,\hat{q}_3^{-1},\hat{P}_{13}^s)\mathcal{A}(\mathbf{2}^s,\hat{q}_4^{+1},-\hat{P}_{13}^s)}{\langle 3|p_1|3]}\right|_{\hat{P}_{13}^2=m^2} \\ &+ \left.\frac{\mathcal{A}(-\mathbf{1}^s,\hat{q}_4^{+1},\hat{P}_{14}^s)\mathcal{A}(\mathbf{2}^s,\hat{q}_3^{-1},-\hat{P}_{14}^s)}{\langle 4|p_1|4]}\right|_{\hat{P}_{14}^2=m^2}.\end{aligned} \tag{514}$$

This shift avoids boundary terms for $s \leq 1$ as $z \to \infty$. When expressing the factorization channels in terms of on-shell HPET variables, there is a question about whether new boundary terms arise relative to the traditional on-shell variables for $z \to \infty$, as would generally be expected because of higher-dimensional operators present in EFTs. This is not the case here, since eq. (450) shows that the definition of the on-shell HPET variables accounts for the contributions from all higher order HPET operators. Another way to see this is that, since the relation between the traditional and on-shell HPET variables is exact, an amplitude must always have the same large z scaling for any shift when expressed using the on-shell HPET variables as when expressed with the traditional on-shell variables. Consider for example the spinor contraction part of the P_{13} factorization channel. In the traditional variables, this is

$$\langle \mathbf{2} P_{13}\rangle^{I}{}_{I}[\hat{P}_{13}\mathbf{1}], \tag{515}$$

which scales as z when $z \to \infty$. In the on-shell HPET variables:

$$\frac{m}{m_{q_3+q_4}}\left(\langle \mathbf{2}_v P_{13v}\rangle_I + \frac{1}{4m}[\mathbf{2}_v 4]\langle \hat{4} P_{13v}\rangle_I + \frac{1}{4m\hat{x}_4}[\mathbf{2}_v 4][4\hat{P}_{13v}]_I\right)\langle P_{13v}|^I\left(\mathbb{I} - \frac{1}{2m\hat{x}_3^{-1}}|3\rangle\langle 3|\right)|\mathbf{1}_v\rangle. \tag{516}$$

Choosing appropriate reference vectors for $\hat{x}_3^{-1}$ and $\hat{x}_4$ ($|4]$ and $|3\rangle$ respectively), we recover the unshifted x_3^{-1} and x_4. Thus this also scales as z when $z \to \infty$. All other factors involved in the factorization channel are common to both sets of variables.

Adding the P_{13} and P_{14} factorization channels, we find the spin-s Compton amplitude

$$\begin{aligned}\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-1}, q_4^{+1}) &= (-1)^{2s}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{+1})[4|p_1|3\rangle^{-2s}\left(1 - \frac{q_3\cdot q_4}{2m^2}\right)^{-2s} \\ &\quad\times\left(\langle 3\mathbf{1}_v\rangle[4\mathbf{2}_v] - \langle 3\mathbf{2}_v\rangle[4\mathbf{1}_v] + \frac{[43]}{2m}\langle \mathbf{2}_v 3\rangle\langle 3\mathbf{1}_v\rangle - \frac{\langle 34\rangle}{2m}[\mathbf{2}_v 4][4\mathbf{1}_v]\right)^{2s}, \\ \mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{+1}) &= -\frac{e^2[4|p_1|3\rangle^2}{\langle 4|p_1|4]\langle 3|p_1|3]},\end{aligned} \tag{517}$$

which is in agreement with the result in ref. [209] for QED when the massive spinors are replaced with on-shell HPET spinors. In the gravitational case, we find

$$\begin{aligned}
\mathcal{M}(-\mathbf{1}^s,\mathbf{2}^s,q_3^{-2},q_4^{+2}) &= (-1)^{2s}\mathcal{M}(-\mathbf{1}^0,\mathbf{2}^0,q_3^{-2},q_4^{+2})[4|p_1|3\rangle^{-2s}\left(1-\frac{q_3\cdot q_4}{2m^2}\right)^{-2s} \\
&\quad\times\left(\langle 3\mathbf{1}_v\rangle[4\mathbf{2}_v]-\langle 3\mathbf{2}_v\rangle[4\mathbf{1}_v]+\frac{[43]}{2m}\langle\mathbf{2}_v3\rangle\langle 3\mathbf{1}_v\rangle-\frac{\langle 34\rangle}{2m}[\mathbf{2}_v4][4\mathbf{1}_v]\right)^{2s}, \\
\mathcal{M}(-\mathbf{1}^0,\mathbf{2}^0,q_3^{-2},q_4^{+2}) &= -\frac{\kappa^2[4|p_1|3\rangle^4}{8q_3\cdot q_4\langle 4|p_1|4]\langle 3|p_1|3]}.
\end{aligned} \tag{518}$$

Note the appearance of spurious poles for $s>1$ in the electromagnetic case, and for $s>2$ in the gravitational case, consistent with the necessarily composite nature of higher spin particles [235].

Spin effects are isolated in the last two terms in parentheses. This can be seen in two ways. The first is to rewrite these last two terms in the language of ref. [208]:

$$\begin{aligned}
\mathcal{M}(-\mathbf{1}^s,\mathbf{2}^s,q_3^{-2},q_4^{+2}) &= \frac{(-1)^{2s}}{m^{2s}}\mathcal{M}(-\mathbf{1}^0,\mathbf{2}^0,q_3^{-2},q_4^{+2})\left(1-\frac{q_3\cdot q_4}{2m^2}\right)^{-2s} \\
&\quad\times\langle\mathbf{2}_v|^{2s}\left(\mathbb{I}+\frac{1}{2}i\frac{q_{3,\mu}\varepsilon^-_{3,\nu}J^{\mu\nu}}{p_1\cdot\varepsilon_3^-}+\frac{1}{2}i\not{v}\frac{q_{4,\mu}\varepsilon^+_{4,\nu}J^{\mu\nu}}{p_1\cdot\varepsilon_4^+}\not{v}\right)^{2s}|\mathbf{1}_v\rangle^{2s}.
\end{aligned} \tag{519}$$

Alternatively, as is more convenient for our purposes, the factorization into classical three-point amplitudes can be made more visible by application of the Schouten identity to these terms:

$$\mathcal{M}(-\mathbf{1}^s,\mathbf{2}^s,q_3^{-2},q_4^{+2}) = \frac{(-1)^{2s}}{m^{2s}}\mathcal{M}(-\mathbf{1}^0,\mathbf{2}^0,q_3^{-2},q_4^{+2})(\mathcal{N}_1+\mathcal{N}_2)^{2s}, \tag{520a}$$

where

$$\mathcal{N}_1 \equiv \langle\mathbf{2}_v|\left[\mathbb{I}+\frac{(q_4-q_3)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v\rangle, \tag{520b}$$

$$\begin{aligned}
\mathcal{N}_2 &\equiv \langle\mathbf{2}_v|\left[v|4]\langle 3|\frac{p_1\cdot q_4}{m_{q_3+q_4}[4|p_1|3\rangle}+|3\rangle[4|v\frac{p_1\cdot q_3}{m_{q_3+q_4}[4|p_1|3\rangle}\right]|\mathbf{1}_v\rangle \\
&= \langle\mathbf{2}_v|\left[v|4]\langle 3|\frac{q_3\cdot q_4}{m_{q_3+q_4}[4|p_1|3\rangle}+\frac{w\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v\rangle,
\end{aligned} \tag{520c}$$

and

$$w_{\alpha\dot\alpha} \equiv 2p_1\cdot q_3\frac{|3\rangle_\alpha[4|_{\dot\alpha}}{[4|p_1|3\rangle}, \quad w^{\dot\alpha\alpha} = 2p_1\cdot q_3\frac{|4]^{\dot\alpha}\langle 3|^\alpha}{[4|p_1|3\rangle}. \tag{520d}$$

$\mathcal{N}_2$ is the term that contributes spurious poles for high enough spins. The contraction $w \cdot S$ has been defined through eq. (456a). The momentum w^μ scales linearly with $\hbar$, so the contraction $w \cdot S$ does not scale with $\hbar$. Compared to this term, the first term in $\mathcal{N}_2$ is subleading in $\hbar$. Ignoring it in the classical limit, and noting that binomial combinatoric factors must be absorbed into the spin-vector when it is raised to some power, the remaining terms imply an exponential spin structure:

$$\mathcal{M}^{\text{cl.}}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-2}, q_4^{+2}) = \frac{(-1)^{2s}}{m^{2s}} \mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{+2}) \langle \mathbf{2}_v|^{2s} \exp\left[\frac{(q_4 - q_3 + w) \cdot S}{m}\right] |\mathbf{1}_v\rangle^{2s}. \tag{521}$$

The same exponentiation holds in the electromagnetic case, with the spinless amplitude above replaced by the corresponding spinless amplitude for QED.

The leading $\hbar$ scaling for these amplitudes is $\hbar^{-1}$ whereas naïve counting of the vertices and propagators says that the scaling should be $\hbar^{-2}$. The source of this discrepancy is interference between the two factorization channels, yielding a factor in the numerator of $p_1 \cdot (\hbar\bar{q}_3 + \hbar\bar{q}_4) = \hbar^2 \bar{q}_3 \cdot \bar{q}_4$. It is thus possible for the naïve $\hbar$ counting to over-count inverse powers of $\hbar$, and hence overestimate the classicality of an amplitude. This has consequences for the extension of these results to the emission of n bosons: factorization channels with a cut graviton line are naïvely suppressed by one factor of $\hbar$ relative to those with cut matter lines. The interference described here means that both factorizations may actually have the same leading $\hbar$ behavior.

Consider now the same-helicity amplitudes. The two-negative-helicity amplitude for spin-1 has been computed by one of the present authors in ref. [305] by shifting one massive and one massless leg. Extending the amplitude found there to spin s,

$$\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-1}, q_4^{-1}) = \frac{1}{m^{2s}} \mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1}) [\mathbf{21}]^{2s}, \tag{522a}$$

$$\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1}) = \frac{e^2 m^2 \langle 34\rangle^2}{\langle 3|p_1|3]\langle 4|p_1|4]} \tag{522b}$$

We have replaced the coupling in ref. [305] with e^2, as is appropriate for QED. Expressing this in terms of on-shell HPET variables, we find

$$\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-1}, q_4^{-1}) = \frac{1}{m^{2s}} \mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1}) [\mathbf{2}_v|^{2s} \left(\mathbb{I} - \frac{(q_3 + q_4) \cdot S}{m_{q_3+q_4}}\right)^{2s} |\mathbf{1}_v]^{2s}. \tag{523}$$

The spin-dependence immediately becomes explicit after the change of variables. The exponential spin structure is obvious:

$$\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-1}, q_4^{-1}) = \frac{1}{m^{2s}}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1})[\mathbf{2}_v|^{2s}\exp\left[-\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v]^{2s}. \tag{524}$$

When the gyromagnetic ratio $g = 2$, the arbitrary spin $s = s_1 + s_2$ gravitational Compton amplitude is proportional to the product between the spin s_1 and s_2 electromagnetic amplitudes [275, 306, 307]. As we have constructed the electromagnetic Compton amplitude using the minimal coupling three-point amplitude, this condition is satisfied. The same-helicity gravitational Compton amplitude is thus

$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-2}, q_4^{-2}) = \frac{1}{m^{2s}}\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{-2})[\mathbf{2}_v|^{2s}\exp\left[-\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v]^{2s}, \tag{525a}$$

$$\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{-2}) = \frac{\kappa^2}{8e^4}\frac{\langle 3|p_1|3]\langle 4|p_1|4]}{q_3\cdot q_4}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1})^2. \tag{525b}$$

Analogous results hold for the emission of two positive helicity bosons:

$$\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{+1}, q_4^{+1}) = \frac{1}{m^{2s}}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+1}, q_4^{+1})\langle\mathbf{2}_v|^{2s}\exp\left[\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v\rangle^{2s}, \tag{526a}$$

$$\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+1}, q_4^{+1}) = \frac{e^2[34]^2}{\langle 3|p_1|3]\langle 4|p_1|4]}, \tag{526b}$$

$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{+2}, q_4^{+2}) = \frac{1}{m^{2s}}\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+2}, q_4^{+2})\langle\mathbf{2}_v|^{2s}\exp\left[\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v\rangle^{2s}, \tag{526c}$$

$$\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+2}, q_4^{+2}) = \frac{\kappa^2}{8e^4}\frac{\langle 3|p_1|3]\langle 4|p_1|4]}{q_3\cdot q_4}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+1}, q_4^{+1})^2. \tag{526d}$$

Taking the classical limit, we can simply replace $m_{q_3+q_4} \to m$ to obtain the leading $\hbar$ behavior of these amplitudes.

To see that the spin-dependence of the leading $\hbar$ portions of the amplitudes in this section factorize into a product of the three-point amplitudes, note that

$$[q_i\cdot S, q_j\cdot S]_\alpha{}^\beta = -\left(v\cdot q_{[i}q_{j]}\cdot S - iq_i^\mu q_j^\nu J_{\mu\nu}\right)_\alpha{}^\beta = \mathcal{O}(\hbar), \tag{527}$$

where square brackets around indices represent normalized anti-symmeterization of the indices. We can thus combine exponentials and split exponentials of sums only at the cost of subleading-in-$\hbar$ corrections.

The on-shell HPET variables have made it immediate that the spin exponentiates in the same-helicity Compton amplitudes, and this exponentiation is preserved in the $\hbar \to 0$ limit. In the opposite helicity case, the composite nature of higher spin particles can be seen to influence dynamics already at the leading $\hbar$ level. It does so through the contraction $w \cdot S$ for the unphysical momentum w^μ, which appears in a spin exponential in the leading $\hbar$ term. The focus in this section has been on the emission of two bosons, but we will now show that the exponentiation in the same-helicity case extends to the n bosons scenario.

12.5.4 *Emission of n bosons*

We can generalize the exponentiation of the spin observed in the same-helicity Compton amplitudes. In particular, focusing on integer spins for simplicity, we show that for the tree-level emission of n same-helicity bosons with a common helicity h from a heavy spin-s particle, the amplitude satisfies

$$\begin{aligned} M^s_{n+2} &= \frac{(-1)^{nh}}{m^{2s}} M^{s=0}_{n+2} \langle \mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^{n} q_i \cdot S\right] |\mathbf{1}_v\rangle^{2s} \\ &= \frac{(-1)^{nh}}{m^{2s}} M^{s=0}_{n+2} [\mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^{n} q_i \cdot S\right] |\mathbf{1}_v]^{2s}. \end{aligned} \tag{528}$$

We use $q \equiv \sum_{i=1}^n q_i$ throughout this section. Once we have proven the first line, the second follows from the fact that the velocity commutes with the spin-vector. The easiest way to proceed is inductively, constructing the $n+2$ point amplitude using BCFW recursion. The cases $n = 1, 2$ were the focus of previous sections. Note that the result holds for $n = 1$ even when $k_1 \neq 0$, since a non-zero k_1 results in an additional subleading $\mathcal{O}(\hbar^2)$ term.

First, note that when expressed in terms of traditional on-shell variables, the spin dependence in eq. (528) is simply

$$\langle \mathbf{21}\rangle^{2s} = \langle \mathbf{2}_v|^{2s} \exp\left(\frac{q\cdot S}{m_q}\right)|\mathbf{1}_v\rangle^{2s} = [\mathbf{2}_v|^{2s} \exp\left(\frac{q\cdot S}{m_q}\right)|\mathbf{1}_v]^{2s}, \quad \text{for } h > 0, \tag{529a}$$

$$[\mathbf{21}]^{2s} = [\mathbf{2}_v|^{2s} \exp\left(-\frac{q\cdot S}{m_q}\right)|\mathbf{1}_v]^{2s} = \langle \mathbf{2}_v|^{2s} \exp\left(-\frac{q\cdot S}{m_q}\right)|\mathbf{1}_v\rangle^{2s}, \quad \text{for } h < 0. \tag{529b}$$

Thus the problem becomes to prove that the spin dependence is isolated in these spinor contractions. Having already proven this for the base cases, let us now assume it holds up to the emission of

$n-1$ bosons and show that this implies the relations for the emission of n bosons. Constructing the $n+2$-point amplitude using BCFW, the amplitude takes the general form

$$M^{s}_{n+2}=\sum_{k=1}^{n-1}\sum_{\sigma(k)}\left[\hat{M}^{s,I}_{\sigma(k),k+2}\frac{i\epsilon_{IJ}}{P^2_{1,\sigma(k)}}\hat{M}^{s,J}_{\sigma(n-k),n-k+2}+\sum_{h=\pm}\hat{M}^{s,h}_{\sigma(k),k+3}\frac{i}{P^2_{0,\sigma(k)}}\hat{M}^{-h}_{\sigma(n-k),n-k+1}\right], \tag{530}$$

where $P_{1,\sigma(k)}\equiv p_1+\sum_{i=1}^{k}q_{\rho(i,\sigma(k))}\equiv p_1+P_{0,\sigma(k)}$. The permutations $\sigma(k)$ and $\sigma(n-k)$ account for all the ways of organizing the boson legs into $k+2$ and $n-k+2$ point amplitudes, in which shifted legs are never in the same sub-amplitude. $\rho(i,\sigma(k))$ denotes the i^{th} index in the permutation $\sigma(k)$. The notation $\hat{M}$ reminds us that the sub-amplitudes are functions of shifted momenta. The first term in eq. (530) represents factorizations where a massive propagator is on-shell, whereas the second accounts for a massless propagator going on-shell — h in this second term is the helicity of the cut boson.

We will treat each term in eq. (530) separately. We begin with the first term, which is the only contribution for QED. For the case of n positive-helicity bosons, we shift $|\mathbf{1}]$ and, say, $|q_1\rangle$ as in ref. [305]. Then, applying the induction hypothesis, this term is

$$\begin{aligned}\frac{(-1)^{nh}}{m^{4s}}&\sum_{k=1}^{n-1}\sum_{\sigma(k)}\hat{M}^{s=0,I}_{\sigma(k),k+2}\frac{i}{P^2_{1,\sigma(k)}}\hat{M}^{s=0,J}_{\sigma(n-k),n-k+2}\langle\mathbf{2}\hat{P}^{I}_{1,\sigma(k)}\rangle^{2s}\langle\hat{P}_{1,\sigma(k)I}\mathbf{1}\rangle^{2s}\\&=\frac{(-1)^{nh}}{m^{2s}}\langle\mathbf{21}\rangle^{2s}\sum_{k=1}^{n-1}\sum_{\sigma(k)}\hat{M}^{s=0,I}_{\sigma(k),k+2}\frac{i}{P^2_{1,\sigma(k)}}\hat{M}^{s=0,J}_{\sigma(n-k),n-k+2}\end{aligned} \tag{531}$$

The case of n negative-helicity bosons can be shown similarly by shifting $|\mathbf{1}\rangle$ and, say, $|q_1]$. In particular, choosing an appropriate shift of one massive and one massless leg results in no massive shift appearing in the sub-amplitudes. Applying eq. (529) to this, the form of the first term in eq. (530) is therefore

$$\frac{(-1)^{nh}}{m^{2s}}\langle\mathbf{2}_v|^{2s}\exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^{n}q_i\cdot S\right]|\mathbf{1}_v\rangle^{2s}\sum_{k=1}^{n-1}\sum_{\sigma(k)}\hat{M}^{s=0}_{\sigma(k),k+2}\frac{i}{P^2_{1,\sigma(k)}}\hat{M}^{s=0}_{\sigma(n-k),n-k+2} \tag{532}$$

The remaining sum here is the BCFW form of the amplitude for n-photon emission from a massive scalar. Thus we have proven eq. (528) for the photon case.

The non-linear nature of gravity allows contributions from the second term in eq. (530). The contribution of this term to the amplitude is predictable for unique-helicity configurations. The only non-vanishing factorization channels will involve the product of $(n-1)+2$ point amplitudes with

$n-1$ same-helicity gravitons, and a three-graviton amplitude with one distinct helicity graviton, which is the cut graviton. For example, consider the all-plus helicity amplitude. Applying the induction hypothesis,

$$\begin{aligned}\sum_{k=1}^{n-1}\sum_{\sigma(k)}\sum_{h=\pm}\hat{\mathcal{M}}^{s,h}_{\sigma(k),k+3}\frac{i}{P^2_{0,\sigma(k)}}\hat{\mathcal{M}}^{-h}_{\sigma(n-k),n-k+1}\Bigg|_{\text{cl.}} &= \sum_{\sigma(n-2)}\hat{\mathcal{M}}^{s,+}_{\sigma(n-2),n+1}\frac{i}{P^2_{0,\sigma(n-2)}}\hat{\mathcal{M}}^{-}_{\sigma(2),3}\Bigg|_{\text{cl.}} \\ &= \frac{1}{m^{2s}}\langle\mathbf{2}_v|^{2s}\exp\left[\frac{1}{m_q}\sum_{i=1}^{n}q_i\cdot S\right]|\mathbf{1}_v\rangle^{2s}\sum_{\sigma(n-2)}\hat{\mathcal{M}}^{s=0,+}_{\sigma(n-2),n+1}\frac{i}{P^2_{0,\sigma(n-2)}}\hat{\mathcal{M}}^{-}_{\sigma(2),3}. \end{aligned} \tag{533}$$

We have used momentum conservation to write the cut momentum in terms of the sum of the momenta of the gravitons in the all-graviton subamplitude. The argument is identical in the all-negative case. Adding eqs. (532) and (533) and identifying the remaining sums of sub-amplitudes as the scalar amplitude for the emission of $n+2$ gravitons, we find

$$\begin{aligned}\mathcal{M}^s_{n+2} &= \frac{1}{m^{2s}}\mathcal{M}^{s=0}_{n+2}\langle\mathbf{2}_v|^{2s}\exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^{n}q_i\cdot S\right]|\mathbf{1}_v\rangle^{2s} \\ &= \frac{1}{m^{2s}}\mathcal{M}^{s=0}_{n+2}[\mathbf{2}_v|^{2s}\exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^{n}q_i\cdot S\right]|\mathbf{1}_v]^{2s}.\end{aligned} \tag{534}$$

In amplitudes where this spin universality is manifest, we can eliminate the dependence on the specific states used by taking the infinite spin and classical limits of the result,

$$\lim_{\substack{s\to\infty\\ \hbar\to 0}} M^s_{n+2} = M^{s=0}_{n+2}\exp\left[\frac{1}{m}\frac{h}{|h|}\sum_{i=1}^{n}q_i\cdot S\right], \tag{535}$$

where we have used that $\lim_{\hbar\to 0}p^\mu_{v,2} = \lim_{\hbar\to 0}p^\mu_{v,1} = mv^\mu$ to apply on-shell conditions. This makes contact between the classical limit of the kinematics, and the classical spin limit: for tree-level same-helicity boson emission processes, the spin dependence of the leading-in-$\hbar$ term factorizes into factors of the classical three-point spin-dependence.

12.6 SUMMARY AND OUTLOOK

We have presented an on-shell formulation of HPETs by expressing their asymptotic states as a linear combination of the chiral and anti-chiral massive on-shell helicity variables of ref. [235]. This expression automatically takes into account the infinite tower of higher-dimensional operators present in HPETs, which result from the integrating out of the anti-field. The variables defined in this manner possess manifest spin multipole and $\hbar$ expansions. Consequently, using the most general three-point amplitude of ref. [235], we have been able to derive a closed form for the amplitude arising from the sum of all three-point operators in an arbitrary spin HPET. This form of the amplitude has been checked explicitly up to NNLO in the operator expansion of spin-1/2 HQET and HBET. We will also show in Section 12.E that the extension to higher spins is suitable for describing the three-point amplitude for zero initial residual momentum for a heavy spin-1 particle coupled to electromagnetism.

We have shown that the spin-multipole expansion of minimally coupled heavy particles corresponds exactly to a truncated Kerr black hole expansion when the initial residual momentum is set to zero. This has been done in two ways. First, we exponentiated the spin dependence of the minimally coupled three-point amplitude in Section 12.3.4. Doing so directly produced the same spin exponential as that in refs. [208, 301] for a Kerr black hole coupled to a graviton. Unlike previous approaches, no further manipulation of the three-point amplitude was needed to match to refs. [208, 301]. An exact match to all spin orders was achieved in the infinite spin limit. An alternative approach to matching the Kerr black hole multipole moments was carried out in refs. [209, 224], by matching to the EFT of ref. [283]. Following this matching procedure but using on-shell HPET variables, an exact match to the Kerr black hole Wilson coefficients was achieved without the need to take an infinite spin limit. The reason that the three-point amplitude in on-shell HPET variables immediately matches the Kerr black hole multipole expansion is that the heavy spinors representing the initial and final states are both associated with the same momentum, which is identified with that of the black hole.

We set out to provide a framework that would enable the extension of HPETs to higher spins, and to enable the application of HPETs to the computation of higher order classical amplitudes. As a step in this direction, we applied recursion relations to the minimal coupling amplitude for heavy particles to build arbitrary-spin higher-point tree amplitudes. Doing so, we showed that the explicit $\hbar$ and spin multipole expansions at three points remained manifest in all amplitudes considered. We also easily constructed the tree-level boson exchange amplitude to all orders in spin for QED and GR, without having to further manipulate the states to produce the correct classical black hole spin multipole expansion.

Moving on to radiative processes, we showed that the same-helicity electromagnetic and gravitational Compton amplitudes exhibit a spin universality: they can be written as

$$M_4^s = M_4^{s=0} \langle \mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_{q_1+q_2}}\frac{h}{|h|}\sum_{i=1}^{2} q_i \cdot S\right] |\mathbf{1}_v\rangle^{2s}. \tag{536}$$

This universality extends to the emission of n same-helicity bosons (eq. (528)). In the four-point opposite-helicity case, a similar exponential was obtained only in the classical limit. However the sum in the exponential also included an unphysical momentum contracted with the spin, representing the non-uniqueness of the amplitude for large enough spins. It would be interesting to examine whether the opposite-helicity amplitude possesses an n-boson extension analogous to eq. (528). Another natural extension is to study how the leading $\hbar$ behaviour changes when a second matter line is included in radiation processes; this is relevant to the understanding of non-conservative effects in spinning binaries. The understanding of radiative processes is paramount to the PM amplitude program, as the construction of higher PM amplitudes using unitarity methods requires knowledge of tree-level radiative amplitudes. Combining radiative amplitudes with the $\hbar$ counting of the on-shell HPET variables in a unitarity-based approach, the classical limits of amplitudes can be easily identified and taken before integration to simplify computations of classical loop amplitudes including spin.

Because of the topicality of the subject, we have focused in the main body of this paper on the application of these variables to their interpretation as spinning black holes and the construction of classical tree-level amplitudes. Nevertheless, they are equally applicable to the QCD systems which HQET was formulated to describe. Moreover an on-shell perspective is useful for the understanding of HPETs as a whole. Indeed, we take an on-shell approach in the appendices to make further statements about HPETs.

APPENDIX

12.A CONVENTIONS

We list here our conventions for reference. In the Weyl basis, the Dirac gamma matrices take the explicit form

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\alpha}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} & 0 \end{pmatrix}, \tag{537}$$

where $\sigma^\mu = (1, \sigma^i)$, $\bar{\sigma}^\mu = (1, -\sigma^i)$, and σ^i are the Pauli matrices. The gamma matrices obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We use the mostly minus metric convention, $\eta^{\mu\nu} = \mathrm{diag}\{+,-,-,-\}$. The fifth gamma matrix is defined as

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{538}$$

The generator of Lorentz transforms is

$$J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{539}$$

We express massless momenta in terms of on-shell variables:

$$q_{\alpha\dot{\alpha}} \equiv q^\mu(\sigma_\mu)_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \equiv |\lambda\rangle_\alpha [\lambda|_{\dot{\alpha}}, \tag{540a}$$

$$q^{\dot{\alpha}\alpha} \equiv q^\mu(\sigma_\mu)^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}} \lambda^\alpha \equiv |\lambda]^{\dot{\alpha}} \langle\lambda|^\alpha. \tag{540b}$$

Here α, $\dot{\alpha}$ are $SL(2,\mathbb{C})$ spinor indices. Spinor brackets are formed by contracting the spinor indices,

$$\langle\lambda_1\lambda_2\rangle \equiv \langle\lambda_1|^{\alpha}|\lambda_2\rangle_{\alpha}, \tag{541}$$

$$[\lambda_1\lambda_2] \equiv [\lambda_1|_{\dot{\alpha}}|\lambda_2]^{\dot{\alpha}}. \tag{542}$$

For massive momenta, we have that

$$p_{\alpha\dot{\alpha}} = \lambda_{\alpha}{}^{I}\tilde{\lambda}_{\dot{\alpha}I} \equiv |\lambda\rangle_{\alpha}^{I}[\lambda|_{\dot{\alpha}I}, \tag{543a}$$

$$p^{\dot{\alpha}\alpha} = \tilde{\lambda}_{I}^{\dot{\alpha}}\lambda^{\alpha I} \equiv |\lambda]_{I}^{\dot{\alpha}}\langle\lambda|^{\alpha I}, \tag{543b}$$

where I is an $SU(2)$ little group index. Spinor brackets for massive momenta are also formed by contracting spinor indices, identically to the massless case. We also use the bold notation introduced in ref. [235] to suppress the symmetrization over $SU(2)$ indices in amplitudes:

$$\langle\mathbf{2}q_1\rangle\langle\mathbf{2}q_2\rangle \equiv \begin{cases} \langle 2^I q_1\rangle\langle 2^J q_2\rangle & I = J, \\ \langle 2^I q_1\rangle\langle 2^J q_2\rangle + \langle 2^J q_1\rangle\langle 2^I q_2\rangle & I \neq J. \end{cases} \tag{544}$$

The Levi-Civita symbol, used to raise and lower spinor and $SU(2)$ little group indices, is defined by

$$\epsilon^{12} = -\epsilon_{12} = 1. \tag{545}$$

Spinor and $SU(2)$ indices are raised and lowered by contracting with the second index on the Levi-Civita symbol. For example,

$$\lambda^{I} = \epsilon^{IJ}\lambda_{J}, \quad \lambda_{I} = \epsilon_{IJ}\lambda^{J}. \tag{546}$$

The on-shell conditions for the massive helicity variables are

$$\lambda^{\alpha I}\lambda_{\alpha J} = m\delta^I{}_J, \quad \lambda^{\alpha I}\lambda_\alpha{}^J = -m\epsilon^{IJ}, \quad \lambda^\alpha{}_I\lambda_{\alpha J} = m\epsilon_{IJ}, \tag{547a}$$

$$\tilde{\lambda}^I_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_J = -m\delta^I{}_J, \quad \tilde{\lambda}^I_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}J} = m\epsilon^{IJ}, \quad \tilde{\lambda}_{\dot{\alpha}I}\tilde{\lambda}^{\dot{\alpha}}_J = -m\epsilon_{IJ}. \tag{547b}$$

Given eq. (445), we can derive the on-shell conditions of the HPET variables, analogous to eq. (547). We find

$$\lambda^{\alpha I}_v\lambda_{v\alpha J} = m_k\delta^I{}_J, \quad \lambda^{\alpha I}_v\lambda_{v\alpha}{}^J = -m_k\epsilon^{IJ}, \quad \lambda^\alpha_{vI}\lambda_{v\alpha J} = m_k\epsilon_{IJ}, \tag{548a}$$

$$\tilde{\lambda}^I_{v\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{vJ} = -m_k\delta^I{}_J, \quad \tilde{\lambda}^I_{v\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}J}_v = m_k\epsilon^{IJ}, \quad \tilde{\lambda}_{v\dot{\alpha}I}\tilde{\lambda}^{\dot{\alpha}}_{vJ} = -m_k\epsilon_{IJ}, \tag{548b}$$

where

$$m_k \equiv \left(1 - \frac{k^2}{4m^2}\right)m. \tag{548c}$$

In Section 12.C we will decompose massive momenta into two massless momenta, as in eq. (559). When identifying

$$\lambda_\alpha{}^1 = |a\rangle_\alpha, \quad \lambda_\alpha{}^2 = |b\rangle_\alpha, \tag{549a}$$

$$\tilde{\lambda}_{\dot{\alpha}1} = [a|_{\dot{\alpha}}, \quad \tilde{\lambda}_{\dot{\alpha}2} = [b|_{\dot{\alpha}}, \tag{549b}$$

we use $\langle ba\rangle = [ab] = m$.

On-shell variables can be assigned to the upper and lower Weyl components of a Dirac spinor so that the spinors satisfy the Dirac equation [209],

$$u^I(p) = \begin{pmatrix} \lambda_\alpha{}^I \\ \tilde{\lambda}^{\dot{\alpha}I} \end{pmatrix}, \quad \bar{u}_I(p) = \begin{pmatrix} -\lambda^\alpha{}_I & \tilde{\lambda}_{\dot{\alpha}I} \end{pmatrix}, \tag{550}$$

where p is expressed in terms of λ and $\tilde{\lambda}$ as in eq. (543).

Using analytic continuation, under a sign flip of the momentum, the on-shell variables transform as

$$|-\mathbf{p}\rangle = -|\mathbf{p}\rangle, \quad |-\mathbf{p}] = |\mathbf{p}], \tag{551a}$$

which means

$$|-\mathbf{p}_v\rangle = |\mathbf{p}_{-v}\rangle = -|\mathbf{p}_v\rangle, \quad |-\mathbf{p}_v] = |\mathbf{p}_{-v}] = |\mathbf{p}_v]. \tag{551b}$$

12.B UNIQUENESS OF ON-SHELL HPET VARIABLES

In this section, we address the question of uniqueness of the on-shell HPET variables as defined in eq. (445). In particular, we relate the on-shell HPET variables $|\mathbf{p}_v\rangle$ and $|\mathbf{p}_v]$ to the traditional on-shell variables under two conditions:

1. The new variables describe a very massive spin-1/2 state that acts as a source for mediating bosons, meaning that the velocity of the state is approximately constant. Since the motion of the particle is always very closely approximated by its velocity, we demand that the new variables satisfy the Dirac equation for a velocity v^μ and mass $v^2 = 1$:

   $$\not{v}|\mathbf{p}_v\rangle = |\mathbf{p}_v], \quad \not{v}|\mathbf{p}_v] = |\mathbf{p}_v\rangle. \tag{552}$$

   Clearly these relations can be scaled to give the state an arbitrary mass.

2. When describing a heavy particle with mass m and velocity v^μ, the new variables must reduce to the traditional on-shell variables with $p^\mu = mv^\mu$ when $k = 0$.

We express the on-shell HPET variables in the basis of traditional on-shell variables:

$$|\mathbf{p}_v\rangle = a(k)|\mathbf{p}\rangle + \not{\Gamma}_1(k)|\mathbf{p}], \tag{553a}$$

$$|\mathbf{p}_v] = b(k)|\mathbf{p}] + \not{\Gamma}_2(k)|\mathbf{p}\rangle. \tag{553b}$$

The fact that the functions a, b, Γ_1, Γ_2 can, without loss of generality, be assumed to be functions of only k^μ (and m) follows from on-shellness and the Dirac equation. Any dependence on v^μ must be either in a scalar form, $v \cdot v = 1$ or $v \cdot k = -k^2/2m$, or in matrix form $\not{v}$, which can be eliminated for $\not{k}/m$ using the Dirac equation for $\not{p}$. This also means that we can rewrite $\Gamma_{1,2}^\mu = c_{1,2}(k)k^\mu$, where the $c_i(k)$ are scalars and potentially functions of k^2. Moreover, given that a and b are functions only of k, they must also be scalars; the only possible matrix combinations they can contain to preserve the correct spinor indices are even powers of $\not{k}$, which would reduce to some power of k^2. Condition 2 provides a final constraint on these four functions:

$$a(0) = b(0) = 1, \tag{554a}$$

$$\Gamma_1(0) = \Gamma_2(0) = 0. \tag{554b}$$

Since $\Gamma_i^\mu = c_i(k)k^\mu$, the second line imposes that the $c_i(k)$ are regular at $k = 0$. From now on we drop the arguments of these functions for brevity.

Applying condition 1 to eqs. (553), we derive relations among the four functions $a,\ b,\ c_1,\ c_2$:

$$b = a, \tag{555a}$$

$$c_2 = -\frac{a}{m} - c_1. \tag{555b}$$

The most general on-shell HPET variables are thus

$$|\mathbf{p}_v\rangle = a|\mathbf{p}\rangle + c_1\not{k}|\mathbf{p}], \tag{556a}$$

$$|\mathbf{p}_v] = a|\mathbf{p}] - \left(\frac{a}{m} + c_1\right)\not{k}|\mathbf{p}\rangle. \tag{556b}$$

The momentum associated with these states is

$$\not{p}_v = \begin{pmatrix} 0 & |p_v\rangle^I{}_I[p_v| \\ |p_v]_I{}^I\langle p_v| & 0 \end{pmatrix} = m\left[a^2 + c_1\left(\frac{a}{m} + c_1\right)k^2\right]\not{v}. \tag{557}$$

The functions a and c_1 cannot be constrained further by conditions 1 and 2. However we can choose $c_1 = -a/2m$ to describe non-chiral interactions. Then, from an off-shell point of view, the function a

simply corresponds to the (potentially non-local) field redefinition $Q \to Q/a$ in the spin-1/2 HPET Lagrangian. We are free to redefine our fields such that $a = 1$. The final result is

$$|\mathbf{p}_v\rangle = |\mathbf{p}\rangle - \frac{\not{k}}{2m}|\mathbf{p}], \tag{558a}$$

$$|\mathbf{p}_v] = |\mathbf{p}] - \frac{\not{k}}{2m}|\mathbf{p}\rangle. \tag{558b}$$

Thus we recover the on-shell HPET variables in eq. (445). We conclude that, up to scaling by an overall function of k^2, eq. (445) is the unique decomposition in terms of traditional variables of non-chiral heavy particle states. The overall scalings correspond to field redefinitions in the Lagrangian formulation.

12.C REPARAMETERIZATION AND THE LITTLE GROUP

As is apparent from eq. (442), reparameterization transformations leave p^μ unchanged. It is therefore reasonable to expect that there exists a relation between reparameterizations and the little group of p^μ. There is indeed a relationship between infinitesimal little group transformations of $\lambda_\alpha{}^I$ and $\tilde{\lambda}_I^{\dot{\alpha}}$ and reparameterizations of the total momentum. The focus of this section is the derivation of such a connection, which is easy to explore by employing the so-called Light Cone Decomposition (LCD) [308, 309] of massive momenta.

The LCD allows any massive momentum to be written as a sum of two massless momenta. That is, for a momentum p^μ of mass m, there exist two massless momenta a^μ and b^μ such that

$$p^\mu = a^\mu + b^\mu. \tag{559}$$

When p^μ is real, we can assume wihtout loss of generality that a^μ and b^μ are real as well, since any imaginary components must cancel anyway. The condition $p^2 = m^2$ then implies $a \cdot b = m^2/2$. Expressing this in on-shell variables,

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha{}^I \tilde{\lambda}_{\dot{\alpha}I} = |a\rangle_\alpha [a|_{\dot{\alpha}} + |b\rangle_\alpha [b|_{\dot{\alpha}}, \tag{560a}$$

$$p^{\dot{\alpha}\alpha} = \tilde{\lambda}_I^{\dot{\alpha}} \lambda^{\alpha I} = |a]^{\dot{\alpha}} \langle a|^\alpha + |b]^{\dot{\alpha}} \langle b|^\alpha. \tag{560b}$$

This allows us to make the identifications

$$\lambda_\alpha{}^1 = |a\rangle_\alpha, \quad \lambda_\alpha{}^2 = |b\rangle_\alpha, \quad \tilde{\lambda}_{\dot{\alpha}1} = [a|_{\dot{\alpha}}, \quad \tilde{\lambda}_{\dot{\alpha}2} = [b|_{\dot{\alpha}}. \tag{560c}$$

In the spirit of the momentum decomposition in eq. (442) we can break this up into a large and a small part

$$p^\mu = \alpha a^\mu + \beta b^\mu + (1-\alpha)a^\mu + (1-\beta)b^\mu, \tag{561}$$

where $|\alpha|, |\beta| \sim 1$. We identify

$$mv^\mu \equiv \alpha a^\mu + \beta b^\mu, \quad k^\mu \equiv (1-\alpha)a^\mu + (1-\beta)b^\mu. \tag{562}$$

Since v^μ is a four-velocity, it must satify $v^2 = 1$, which constrains α and β to obey $\alpha\beta = 1$. Once we require this, the on-shell condition that $2mv \cdot k = -k^2$ is automatically imposed.

Now, consider a reparameterization of the momentum as in eq. (451). We can use the LCD to rewrite the shift momentum as

$$\delta k^\mu = c^\mu + d^\mu, \tag{563}$$

where $|c+d|/m \ll 1$. For this to be a reparameterization, the new velocity $v^\mu + \delta k^\mu/m$ must have magnitude 1, which means c^μ and d^μ must be such that

$$(\alpha a + \beta b) \cdot (c + d) = -c \cdot d. \tag{564}$$

Contracting the shift momentum with the gamma matrices and using the Schouten identity,

$$\begin{aligned}\delta k_{\alpha\dot{\alpha}} = \frac{2}{m^2} b \cdot (c+d) |a\rangle_\alpha [a|_{\dot{\alpha}} + \frac{2}{m^2} a \cdot (c+d) |b\rangle_\alpha [b|_{\dot{\alpha}} \\ - \frac{[a|(\not{c} + \not{d})|b\rangle}{m^2} |a\rangle_\alpha [b|_{\dot{\alpha}} - \frac{[b|(\not{c} + \not{d})|a\rangle}{m^2} |b\rangle_\alpha [a|_{\dot{\alpha}}.\end{aligned} \tag{565}$$

Note that setting $k = 0$ is always allowed for an on-shell momentum by reparameterization: indeed, choosing $c^\mu = (1-\alpha)a^\mu$ and $d^\mu = (1-\beta)b^\mu$ trivially satisfies eq. (564).

Consider an infinitesimal little group transformation of the on-shell variables $W^I{}_J$ where $W \in SU(2)$. Then we can write

$$W^I{}_J = \mathbb{I}^I{}_J + i\epsilon^j U^{jI}{}_J, \tag{566}$$

where ϵ^j are real and infinitesimal parameters, and $U^{jI}{}_J$ is traceless and Hermitian. We suppress the color index j below. Under this transformation, the on-shell variables transform as [235]

$$\lambda_\alpha{}^I \to W^I{}_J \lambda_\alpha{}^J, \tag{567a}$$

$$\tilde{\lambda}_{\dot{\alpha}I} \to (W^{-1})^J{}_I \tilde{\lambda}_{\dot{\alpha}J}. \tag{567b}$$

Up to linear order in the infinitesimal parameter, the momentum transforms as

$$\begin{aligned} p_{\alpha\dot{\alpha}} = \lambda_\alpha{}^I \tilde{\lambda}_{\dot{\alpha}I} \to\ & (1 + i\epsilon U^1{}_1)\lambda_\alpha{}^1 \tilde{\lambda}_{\dot{\alpha}1} + (1 + i\epsilon U^2{}_2)\lambda_\alpha{}^2 \tilde{\lambda}_{\dot{\alpha}2} + i\epsilon U^2{}_1 \lambda^1_\alpha \tilde{\lambda}_{\dot{\alpha}2} + i\epsilon U^1{}_2 \lambda^2_\alpha \tilde{\lambda}_{\dot{\alpha}1} \\ & - i\epsilon U^2{}_1 \lambda^1_\alpha \tilde{\lambda}_{\dot{\alpha}2} - i\epsilon U^1{}_2 \lambda^2_\alpha \tilde{\lambda}_{\dot{\alpha}1} - i\epsilon U^1{}_1 \lambda_\alpha{}^1 \tilde{\lambda}_{\dot{\alpha}1} - i\epsilon U^2{}_2 \lambda_\alpha{}^2 \tilde{\lambda}_{\dot{\alpha}2}. \end{aligned} \tag{568}$$

Comparing with eq. (565), we would like to identitfy the following map to the reparameterization in eq. (451):

$$i\epsilon U^I{}_J \to R^I{}_J \equiv \frac{1}{m} \begin{pmatrix} 2b \cdot \frac{\delta k}{m} & -[b|\frac{\delta k}{m}|a\rangle \\ -[a|\frac{\delta k}{m}|b\rangle & 2a \cdot \frac{\delta k}{m} \end{pmatrix}. \tag{569}$$

The reparameterization matrix $R^I{}_J$ is infintesimal because of the appearance of $\delta k^\mu/m$ in each entry. Moreover, $R^I{}_J$ is traceless up to corrections of order $\mathcal{O}(\delta k^2/m^2)$ because of eq. (564). However, we cannot equate it to $i\epsilon U^I{}_J$ because the latter is always anti-Hermitian, whereas $R^I{}_J$ need not be. Indeed, when δk^μ is real $R^I{}_J$ is Hermitian, and when δk^μ is imaginary it is anti-Hermitian. It can thus be seen that the condition for equality is that δk^μ is imaginary:

$$\delta k^\mu \in i\mathbb{R} \Rightarrow \mathbb{I}^I{}_J + R^I{}_J \in SU(2), \tag{570}$$

where $\mathbb{I}^I{}_J + R^I{}_J$ induces the reparameterization in eq. (451). It is straightforward to check that this quantity also has determinant 1, up to infinitesimal corrections of order $\mathcal{O}(\delta k^2/m^2)$.

12.D PROPAGATORS

In ref. [123], massive on-shell variables were used to construct propagators for massive spin-1/2 and spin-1 states. In this section, we use the on-shell HPET variables to do the same for a spin $s \leq 2$ state. We find that the propagator for a heavy particle with spin $s \leq 2$ is

$$D_v^s(p_v) = P^s \frac{N^s(p_v)}{p^2 - m^2} P^s, \tag{571}$$

where P^s is the spin-s projection operator whose eigenstate is the HPET state, and $N^s(p_v)$ is the numerator of the propagator for a massive particle of that spin. By recognizing the form of the numerator, this will allow us to extract the higher spin projection operators. The methods used in this section can be applied to arbitrary spin, but become quite cumbersome as the number of little group invariant objects that must be computed grows as $s + 1/2$ for half-integer spins, and as s for integer spins. Nevertheless, we are able to use our results to conjecture projection operators for any spin.

Spin-1/2

We begin with the spin-1/2 propagator, which can be constructed as

$$\frac{1}{p^2 - m^2}\left[\begin{pmatrix} |p_v^I\rangle \\ |p_v^I] \end{pmatrix} \epsilon_{IJ} \begin{pmatrix} \langle -p_v^J| & [-p_v^J| \end{pmatrix}\right] = P_+ \frac{2m_k}{p^2 - m^2} P_+ = P_+ \frac{1}{\not{p} - m} P_+. \tag{572}$$

We do indeed recover the projection operator for a heavy spin-1/2 field.

Spin-1

We can do the same for a massive spin-1 field. In this case, we posit that the polarization vector is obtained by replacing $p \to p_v$ and $m \to m_k$ in the usual polarization vector:

$$\varepsilon_{v,\mu}^{IJ}(p) = \frac{1}{2\sqrt{2}m_k}(\langle p_v^I|\gamma_\mu|p_v^J] + \langle p_v^J|\gamma_\mu|p_v^I]). \tag{573}$$

It is straightforward to see that the polarization vector satisfies the requisite condition on the heavy spin-1 particle, $v \cdot \varepsilon_v^{IJ} = 0$ for $p^\mu = mv^\mu + k^\mu$, as well as the orthonormality condition

$$\varepsilon_v^{IJ} \cdot \varepsilon_v^{LK} = -\frac{1}{2}(\epsilon^{IL}\epsilon^{JK} + \epsilon^{IK}\epsilon^{JL}). \tag{574}$$

The heavy spin-1 propagator is

$$\frac{1}{p^2 - m^2}\left[\varepsilon_{v,\mu}^{IJ}(p)\epsilon_{IK}\epsilon_{JL}\varepsilon_{v,\nu}^{LK}(-p)\right] = (g_\mu{}^\lambda - v_\mu v^\lambda)\frac{-g_{\lambda\sigma} + v_\lambda v_\sigma}{p^2 - m^2}(g^\sigma{}_\nu - v^\sigma v_\nu). \tag{575}$$

From this we can read off that the operator projecting onto the heavy spin-1 particle is $P_-^{\mu\nu}$ in Section 12.E.

Spin-3/2

The spin-3/2 polarization tensor is

$$\varepsilon_{v,\mu}^{IJK}(p) = \varepsilon_{v,\mu}^{(IJ}u_v^{K)} = \frac{1}{\sqrt{2}m_k}\langle p_v^{(I}|\gamma_\mu|p_v^J]\begin{pmatrix}|p_v^{K)}\rangle \\ |p_v^{K)}]\end{pmatrix}, \tag{576}$$

where the round brackets around sets of indices denote normalized symmetrization over the indices. Using the symmetry of the spin-1 polarization vector in its little group indices, we have that

$$\varepsilon_{v,\mu}^{IJK}(p) = \frac{1}{3}\left(\varepsilon_{v,\mu}^{IJ}u_v^K + \varepsilon_{v,\mu}^{JK}u_v^I + \varepsilon_{v,\mu}^{IK}u_v^J\right). \tag{577}$$

The propagator is

$$\begin{aligned}
&\frac{1}{p^2 - m^2}\left[\varepsilon_{v,\mu}^{IJK}(p)\epsilon_{IA}\epsilon_{JB}\epsilon_{KC}\varepsilon_{v,\nu}^{ABC}(-p)\right] \\
&\quad = \frac{1}{p^2 - m^2}\frac{1}{3}\left(\varepsilon_{v,\mu}^{IJ}\varepsilon_{v,\nu IJ}u_v^K\bar{u}_{v,K} + 2\varepsilon_{v,\mu}^{IJ}\varepsilon_{v,\nu IK}u_v^K\bar{u}_{v,J}\right) \\
&\quad = -P_+P_{-,\mu\alpha}\frac{2m_k}{p^2 - m^2}\left[g^{\alpha\beta} - \frac{1}{3}\gamma^\alpha\gamma^\beta - \frac{1}{3}(\not{v}\gamma^\alpha v^\beta + v^\alpha\gamma^\beta\not{v})\right]P_{-,\beta\nu}P_+.
\end{aligned} \tag{578}$$

We recognize the quantity between the projection operators as the propagator for a massive spin-3/2 particle with momentum $m_k v^\mu$ [310, 311]. The heavy spin-3/2 projection operator can thus be identified as

$$P^{\mu\nu}_{\frac{1}{2},-} \equiv P_+ P^{\mu\nu}_-. \tag{579}$$

Spin-2

The spin-2 polarization tensor is

$$\varepsilon^{I_1 J_1 I_2 J_2}_{v,\mu_1\mu_2}(p) = \varepsilon^{(I_1 J_1}_{v,\mu_1}\varepsilon^{I_2 J_2)}_{v,\mu_2} = \frac{1}{2m_k^2}\langle p_v^{(I_1}|\gamma_{\mu_1}|p_v^{J_1}]\langle p_v^{I_2}|\gamma_{\mu_2}|p_v^{J_2)}]. \tag{580}$$

Using the symmetry of each spin-1 polarization vector in its little group indices, we find that

$$\varepsilon^{I_1 J_1 I_2 J_2}_{v,\mu_1\mu_2}(p) = \frac{1}{3}\left(\varepsilon^{I_1 J_1}_{v,(\mu_1}\varepsilon^{I_2 J_2}_{v,\mu_2)} + \varepsilon^{I_1 I_2}_{v,(\mu_1}\varepsilon^{J_1 J_2}_{v,\mu_2)} + \varepsilon^{I_1 J_2}_{v,(\mu_1}\varepsilon^{I_2 J_1}_{v,\mu_2)}\right). \tag{581}$$

The propagator is

$$\begin{aligned}
&\frac{1}{p^2-m^2}\left[\varepsilon^{I_1 J_1 I_2 J_2}_{v,\mu\nu}(p)\epsilon_{I_1 K_1}\epsilon_{J_1 L_1}\epsilon_{I_2 K_2}\epsilon_{J_2 L_2}\varepsilon^{K_1 L_1 K_2 L_2}_{v,\alpha\beta}(-p)\right] \\
&\quad = \frac{1}{p^2-m^2}\frac{1}{3}\left(\varepsilon^{I_1 J_1}_{v,(\mu}\varepsilon^{I_2 J_2}_{v,\nu)}\varepsilon_{v,\alpha I_1 J_1}\varepsilon_{v,\beta I_2 J_2} + 2\varepsilon^{I_1 J_1}_{v,(\mu}\varepsilon^{I_2 J_2}_{v,\nu)}\varepsilon_{v,\alpha I_1 J_2}\varepsilon_{v,\beta I_2 J_1}\right) \\
&\quad = \frac{1}{p^2-m^2}P_{-,\mu\mu'}P_{-,\nu\nu'}\left[-\frac{1}{2}(P_-^{\mu'\alpha'}P_-^{\nu'\beta'} + P_-^{\mu'\beta'}P_-^{\nu'\alpha'}) + \frac{1}{3}P_-^{\mu'\nu'}P_-^{\alpha'\beta'}\right]P_{-,\alpha'\alpha}P_{-,\beta'\beta}.
\end{aligned} \tag{582}$$

The quantity in square brackets is the numerator of the massive spin-2 propagator with momentum $m_k v^\mu$ [312]. We therefore identify the heavy spin-2 projection operator:

$$P^{\mu\nu,\alpha\beta}_- \equiv P^{\mu\nu}_- P^{\alpha\beta}_-. \tag{583}$$

12.D.1 *Spin-s Projection Operator*

Based on the above discussion, as well as the properties of a general spin heavy field, we conjecture the projection operator for a spin-s field. An integer spin-s field $Z^{\mu_1\cdots\mu_s}$ must be symmetric and traceless [313]. When the mass of the particle is very large, the particle component $\mathcal{Z}$ must satisfy [314]

$$v_{\mu_1}\mathcal{Z}^{\mu_1\cdots\mu_s} = 0. \tag{584}$$

By symmetry, this condition holds regardless of the index with which the velocity is contracted. The general spin-s projection operator for a field satisfying eq. (584), and which reduces to the above cases for $s = 1$ and $s = 2$ is

$$P_{-}^{\mu_1\nu_1,\ldots,\mu_s\nu_s} = \prod_{i=1}^{s} P_{-}^{\mu_i\nu_i}. \tag{585}$$

The integer spin projection operator is simply a product of spin-1 projection operators.

A half-integer spin-$(s+1/2)$ field $\Psi^{\mu_1\cdots\mu_s}$ must be symmetric and γ-traceless [315],

$$\gamma_{\mu_1}\Psi^{\mu_1\cdots\mu_s} = 0. \tag{586}$$

Symmetry ensures that the condition holds for any index the γ matrix is contracted with. When the mass of the field becomes very large, its particle component $\mathcal{Q}$ must satisfy [314]

$$\not{v}\mathcal{Q}^{\mu_1\cdots\mu_s} = \mathcal{Q}^{\mu_1\cdots\mu_s}. \tag{587}$$

These constraints also imply, among other things, the v-tracelessness of the heavy field. The general spin-$(s+1/2)$ projection operator that results in a field satisfying these conditions, and that reduces to the above cases for spin-1/2 and spin-3/2, is

$$P_{\frac{1}{2},-}^{\mu_1\nu_1,\ldots,\mu_s\nu_s} \equiv P_{+}P_{-}^{\mu_1\nu_1,\ldots,\mu_s\nu_s}. \tag{588}$$

From this we see that knowledge of the spin-1/2 heavy particle states is enough to construct the polarization tensors and projection operators for higher spin states. In this sense, HPETs are unified in terms of the basic building blocks in eq. (445).

12.E MATCHING TO HPET LAGRANGIANS

In this section, we address the matching of on-shell amplitudes to those derived from HPET Lagrangians. First, there is a subtlety that must be accounted for when matching the minimal coupling in eqs. (469) and (470) to an HPET Lagrangian. We focus the discussion of this to the case of spin-1/2 HPET. Next, we confirm explicitly that the general spin three-point amplitude derived from the Zeeman coupling in ref. [209] reproduces the amplitude derived from spin-1 abelian HQET when expressed using on-shell HPET variables.

12.E.1 *Matching spin-1/2 minimal coupling*

For any quantum field theory, the form of the Lagrangian that produces a given S-matrix is not unique: indeed the S-matrix is invariant under appropriate redefinitions of the fields composing the Lagrangian [16]. Generally, a field redefinition will alter the Green's function for a given process. To relate the Green's functions of two forms of a Lagrangian, the relation between both sets of external states must be specified. The same holds for HQET, which has been presented in various forms in the literature.

Fortunately, the definition of the heavy spinors in eq. (444) specifies for us the form of the spin-1/2 HPET Lagrangian whose external spinors are expressible as such. By inverting eq. (444), we see that the field redefinition converting the full theory to its HPET form must reduce to

$$\psi(x) = e^{-imv\cdot x}\left[\frac{1+\not{v}}{2} + \frac{1-\not{v}}{2}\frac{1}{iv\cdot\partial + 2m} i\not{\partial}\right] Q_v(x). \tag{589}$$

in the free-field limit. For spin-1/2 HQET, this means we must match the minimal coupling to the Lagrangian in the form

$$\mathcal{L}^{s=\frac{1}{2}}_{\text{HQET}} = \bar{Q} iv\cdot DQ + \bar{Q} i\not{D} P_{-}\frac{1}{2m + iv\cdot D} i\not{D} Q. \tag{590}$$

This form of the Lagrangian appears in e.g. ref. [232], and differs from the forms in refs. [6, 316] by the presence of a projection operator in the non-local term. The Lagrangian of HBET presented in ref. [6] must similarly be modified to compare to the minimal coupling amplitude. The suitable form for spin-1/2 HBET is

$$\mathcal{L}_{\mathrm{HBET}}^{s=\frac{1}{2}} = \sqrt{-g}\bar{Q}i\mathcal{D}Q + \frac{\sqrt{-g}}{2m}\bar{Q}i\mathcal{D}P_{-}\sum_{n=0}^{\infty} G_n[h]\frac{F[h]^n}{m^n}i\mathcal{D}Q, \tag{591a}$$

where

$$i\mathcal{D} \equiv ie^{\mu}{}_{a}\gamma^{a}D_{\mu} + mv_{\mu}\gamma^{a}(e^{\mu}{}_{a} - \delta^{\mu}_{a}), \tag{591b}$$

and all other notation is described in ref. [6].

12.E.2 *Matching spin-1 Zeeman coupling*

We demonstrate explicitly the applicability of the on-shell HPET variables to spin-1 heavy particle systems. To do so, we will show that the same variables are suitable for describing the three-point amplitude arising from the Proca action. First, we note that a massive spin-1 particle described by the Proca action has a gyromagnetic ratio $g = 1$ [297]. As such, it should not be expected that the corresponding three-point amplitude matches with the minimal coupling amplitude for $s = 1$. To understand which three-point amplitude we should match with, we recast the three-point amplitude derived from the Zeeman coupling in ref. [209] into on-shell HPET variables (with $k_1 = 0$):

$$\mathcal{A}^{+,s} = \frac{g_0 x}{m^{2s}}\left[\langle \mathbf{2}_v \mathbf{1}_v\rangle^{2s} + x\frac{sg}{2m}\langle \mathbf{2}_v \mathbf{1}_v\rangle^{2s-1}\langle \mathbf{2}_v 3\rangle\langle 3\mathbf{1}_v\rangle + \dots\right], \tag{592}$$

where the dots represent higher spin multipoles. When $g = 2$ we recover the spin-dipole term from $2s$ factors of the spin-1/2 minimal coupling amplitude. Setting $s = g = 1$ for the Proca action,

$$\mathcal{A}^{+,1} = \frac{g_0 x}{m^{2}}\left[\langle \mathbf{2}_v \mathbf{1}_v\rangle^{2} + \frac{x}{2m}\langle \mathbf{2}_v \mathbf{1}_v\rangle\langle \mathbf{2}_v 3\rangle\langle 3\mathbf{1}_v\rangle + \dots\right]. \tag{593}$$

This is the three-point amplitude that we expect from a very heavy spin-1 Proca particle.

Consider now the Proca Lagrangian for a massive vector field B^{μ} coupled to electromagnetism:

$$\mathcal{L} = -\frac{1}{4}F^{*}_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 B^{*}_{\mu}B^{\mu}, \tag{594a}$$

where

$$F^{\mu\nu} = D^{\mu}B^{\nu} - D^{\nu}B^{\mu}, \quad D^{\mu}B^{\nu} = (\partial^{\mu} + ieA^{\mu})B^{\nu}, \tag{594b}$$

and A^{μ} is the $U(1)$ gauge field. We now need a condition that splits the light component $\mathcal{B}^{\mu}$ from the heavy (anti-field) component $\tilde{\mathcal{B}}^{\mu}$. Furthermore, the light component has to satisfy $v_{\mu}\mathcal{B}^{\mu} = 0$ [314]. The appropriate decomposition of the massive vector field is

$$\mathcal{B}^{\mu} = e^{imv\cdot x}P_{-}^{\mu\nu}B_{\nu}, \tag{595a}$$

$$\tilde{\mathcal{B}}^{\mu} = e^{imv\cdot x}P_{+}^{\mu\nu}B_{\nu}, \tag{595b}$$

where $P_{-}^{\mu\nu} \equiv g^{\mu\nu} - v^{\mu}v^{\nu}$ — this is the projection operator that has been derived explicitly in Appendix 12.D — and $P_{+}^{\mu\nu} \equiv v^{\mu}v^{\nu}$. Next, we substitute eq. (595) into the Proca Lagrangian, and integrate out $\tilde{\mathcal{B}}^{\mu}$ using its equation of motion to find

$$\mathcal{L}_{\text{HQET}}^{s=1} = -m\mathcal{B}_{\mu}^{*}(iv\cdot D)\mathcal{B}^{\mu} - \frac{1}{4}\mathcal{B}_{\mu\nu}^{*}\mathcal{B}^{\mu\nu} + \frac{1}{2}\mathcal{B}_{\nu}^{*}D^{\nu}D_{\mu}\mathcal{B}^{\mu} + \mathcal{O}(m^{-1}), \tag{596}$$

where $\mathcal{B}^{\mu\nu} = D^{\mu}\mathcal{B}^{\nu} - D^{\nu}\mathcal{B}^{\mu}$. Computing the three-point amplitude with this Lagrangian for $k_1 = 0$ and expressing it using on-shell HPET variables, we find agreement with eq. (593) for $g_0 = -em/\sqrt{2}$. This supports the hypothesis that the on-shell information of spin-1/2 HPET is sufficient to extend HPETs to higher spins.

13

SOFT THEOREMS AND THE KLT-RELATION

We find new relations for the non-universal part of the Yang-Mills amplitudes by combining the KLT-relation and the soft behavior of gauge and gravity amplitudes. We also extend the relations to include contributions from effective operators.

13.1 INTRODUCTION

The study of scattering amplitudes when the momentum of one or more particles becomes soft has a long history [317–328]. Weinberg showed that the scattering amplitudes factorize when a photon or graviton becomes soft, and that this factorization is universal [325, 326]. The universality of the soft photon theorem is due to charge conservation, while for a soft graviton it follows from the equivalence principle. A subleading soft theorem for photons at tree-level was proven by Low [321]. Similar subleading soft theorems for gravitons and gluons have more recently been discussed using eikonal methods [329–331]. Also, Cachazo and Strominger showed that the sub-subleading soft graviton correction at tree-level is also universal [332].

The soft graviton theorem was shown to be connected to the Bondi, van der Burg, Metzner and Sachs (BMS) symmetry [333, 334], as a Ward identity [335–337]. This has sparked an interest in the connection between asymptotic symmetries and soft theorems (see Ref. [338] for a list of references). The subleading soft theorem is known to be related to the supertranslations and superrotations for asymptotic symmetries and the sub-subleading soft theorem related symmetries was recently analysed in Ref. [339, 340]. The authors found a new class of vector fields, which hints in the direction of a BMS algebra extension.

At loop level, the leading soft theorems for photons and gravitons remain unchanged. However, the loop corrections to the subleading soft theorems for gluons and gravitons were discussed in Refs. [341, 342].

The soft behavior of scattering amplitudes when more than one particle is taken soft has also been studied (see e.g. Ref. [343] and references therein). The soft theorems have also been discussed outside four dimensions [344–347] using the scattering equation framework by Cachazo, He, and Yuan (CHY) [348–350].

Much of the recent progress in calculating gravitational scattering amplitudes relies the connection between gauge amplitudes and gravity amplitudes [21, 239]. One manifestation of this connection is the Kawai-Lewellen-Tye (KLT) relations [237], which relates open and closed string amplitudes at tree level. In the field theory limit, the KLT-relation states that a sum of products of two color-ordered Yang-Mills amplitudes is a gravity amplitude.

In this paper, we study the connection between gravity and gauge soft theorems via the KLT-relations. We compare both sides of the formula at sub-subleading order in the soft-momentum expansion and obtain relations for the *non-universal* part of the Yang-Mills amplitude. To the best of our knowledge, no other relations have been obtained for the non-universal piece of the Yang-Mills amplitudes. We further study the insertion of effective operators, which start contributing at sub-leading order. We also obtain relations for the non-universal effective amplitude at sub- and sub-subleading order.

The paper is organized as follows: Section 13.2 reviews the derivation of the soft theorems using Britto-Cachazo-Feng-Witten (BCFW) recursion relations. Section 13.3 presents the soft theorems of Yang-Mills and gravity amplitudes, while section 13.4 introduces the KLT-relations. Two of the new results of the paper are presented in section 13.5, where the soft limit of the amplitudes and the KLT-relation are used to find non-trivial relations which the Yang-Mills amplitudes must satisfy. An extension of these results is presented in section 13.6, when effective operators are included. We conclude in section 13.7.

13.2 BFCW

We will review the derivation of the soft theorems for Yang-Mills and gravity amplitudes using the BCFW recursion relations [303, 304]. This follows closely the derivation of the new soft theorems by Cachazo and Strominger [332]. We use the spinor-helicity formalism, with the convension

$s_{ab} = \langle a,b\rangle[b,a]$. For an $(n+1)$-point amplitude with a soft particle s with positive helicity,[1] $\mathcal{A}_{n+1}(s,1,\ldots,n)$, we perform the BCFW shift

$$\lambda_s(z) = \lambda_s + z\lambda_n, \qquad \tilde{\lambda}_n(z) = \tilde{\lambda}_n - z\tilde{\lambda}_s. \tag{597}$$

The original amplitude can be recovered from the complex deformed amplitude as the residue at $z = 0$,

$$\mathcal{A}_{n+1} = \frac{1}{2\pi i}\int \frac{dz}{z}\mathcal{A}_{n+1}(z). \tag{598}$$

Using Cauchy's residue theorem, we find the following relation

$$\mathcal{A}_{n+1} = \sum_{\text{diagrams } I} \mathcal{A}_L(z_I)\frac{1}{P_I^2}\mathcal{A}_R(z_I), \tag{599}$$

where the amplitude factorizes into a sum of two lower-point amplitudes, assuming the contribution from $|z| \to \infty$ vanishes. The product of the two amplitudes is the residue in Cauchy's theorem, therefore they are evaluated at the pole z_I. The pole is found for each diagram by solving the equation $P_I^2(z_I) = 0$. From this relation we can build up higher-point amplitudes recursively.

The sum in eq. (599) can be split into two parts as

$$\mathcal{A}_{n+1} = \mathcal{A}_3(s(z_I),1,I)\frac{1}{P_I^2}\mathcal{A}_n(-I,2,\ldots,n(z_I)) + \mathcal{R}_{n+1}. \tag{600}$$

The remainder term is written as

$$\mathcal{R}_{n+1} = \sum_{\text{diagrams I}} \mathcal{A}_L(s(z_I),1,2,\ldots,j)\frac{1}{P_I^2}\mathcal{A}_R(j+1,\ldots,n(z_I)) \tag{601}$$

[1] The negative helicity case can be derived analogously.

where $\mathcal{A}_L$ in eq. (601) is a four- or higher-point amplitude. Under the holomorphic scaling $\lambda_s \to \epsilon\lambda_s$, $\tilde{\lambda}_s \to \tilde{\lambda}_s$, the amplitude[2] for a soft gluon takes the form

$$\begin{aligned}\mathcal{A}_{n+1} &= \frac{1}{\epsilon^2}\frac{\langle n1\rangle}{\langle ns\rangle\langle s1\rangle}\mathcal{A}_n\left(\{\lambda_1, \tilde{\lambda}_1 + \epsilon\frac{\langle ns\rangle}{\langle n1\rangle}\tilde{\lambda}_s\}, \ldots, \{\lambda_n, \tilde{\lambda}_s + \epsilon\frac{\langle s1\rangle}{\langle n1\rangle}\tilde{\lambda}_s\}\right) + \mathcal{R}_{n+1} \\ &= \frac{1}{\epsilon^2}\frac{\langle n1\rangle}{\langle ns\rangle\langle s1\rangle}e^{\epsilon\left(\frac{\langle ns\rangle}{\langle n1\rangle}\tilde{\lambda}_s\frac{d}{d\tilde{\lambda}_1} + \frac{\langle s1\rangle}{\langle n1\rangle}\tilde{\lambda}_s\frac{d}{d\tilde{\lambda}_n}\right)}\mathcal{A}_n + \mathcal{R}_{n+1}.\end{aligned} \tag{602}$$

This form was written down in Ref. [351]. Also, the authors of Ref. [332] showed that $\mathcal{R}_{n+1}$ is of order $\mathcal{O}(\epsilon^0)$. A similar expression can be written down for gravity, with the leading pole being $\mathcal{O}(\epsilon^{-3})$. By expanding the exponential we can find the universal leading and (at tree-level) subleading soft factors, $S^{(0)}_{\mathrm{YM}}$ and $S^{(1)}_{\mathrm{YM}}$. The exponential contains derivative terms, for which we use the notation $\tilde{\nabla}_{a,b} = \tilde{\lambda}^{\dot{\alpha}}_a \frac{d}{d\tilde{\lambda}^{\dot{\alpha}}_b}$. The soft factors can be written as

$$S^{(k)}_{\mathrm{YM}} = \frac{\langle n1\rangle}{\langle ns\rangle\langle s1\rangle}\frac{1}{k!}\left(\frac{\langle ns\rangle}{\langle n1\rangle}\tilde{\nabla}_{s,1} + \frac{\langle s1\rangle}{\langle n1\rangle}\tilde{\nabla}_{s,n}\right)^k, \tag{603}$$

where the soft factors with $k \geq 2$ give only a part of the amplitude. Again, a similar partial infinite soft factor for gravity can be found analogously.

A partial infinite soft theorem for effective operators can also be found using the same method. For Yang-Mills, we find that

$$\overline{S}^{(k+1)}_{\mathrm{YM}} = -\left(\frac{[ns]}{\langle ns\rangle} + \frac{[1s]}{\langle 1s\rangle}\right)\frac{1}{k!}\left(\frac{\langle ns\rangle}{\langle n1\rangle}\tilde{\nabla}_{s,1} + \frac{\langle s1\rangle}{\langle n1\rangle}\tilde{\nabla}_{s,n}\right)^k. \tag{604}$$

The $k = 0$ term reproduces the leading soft theorem for effective operators with a spin-1 particle discussed in Ref. [352]. The first contribution from effective operators for a Yang-Mills amplitude appears at subleading order. For gravity, the first contribution from effective operators only enters at sub-subleading order.

13.3 SOFT THEOREM

We will show the connection between the soft factors of gauge theory and gravity using the KLT-formula. In particular, at sub-subleading order in the soft expansion we find new relations between

[2] We will denote a Yang-Mills amplitude by $\mathcal{A}$, and a gravity amplitude by $\mathcal{M}$.

tree-level amplitudes. We start with an amplitude with one external soft particle. The soft limits of an amplitude with an external soft spin-1/spin-2 particle are

$$\mathcal{A}_{n+1} = \left(\frac{S_{\text{YM}}^{(0)}}{\epsilon^2} + \frac{S_{\text{YM}}^{(1)}}{\epsilon} + S_{\text{YM}}^{(2)} \right) \mathcal{A}_n + \mathcal{R}_{n+1} + \mathcal{O}(\epsilon^1), \tag{605}$$

$$\mathcal{M}_{n+1} = \left(\frac{S_{\text{GR}}^{(0)}}{\epsilon^3} + \frac{S_{\text{GR}}^{(1)}}{\epsilon^2} + \frac{S_{\text{GR}}^{(2)}}{\epsilon} \right) \mathcal{M}_n + \mathcal{O}(\epsilon^0), \tag{606}$$

respectively. The $S_{\text{YM}}^{(i)}$ is the ith subleading soft factor of an amplitude with a soft spin-1 particle, and similarly for gravity. The *non-universal* part of the Yang-Mills amplitude enters at sub-subleading order. In the KLT-formula, two Yang-Mills amplitudes with different color-ordering are required: $\mathcal{A}_n(t, \sigma, n-1, n)$ and $\tilde{\mathcal{A}}_n(n-1, \rho, n, t)$. From now on we assume that whenever $\mathcal{A}_n$ and $\tilde{\mathcal{A}}_n$ is written, we have this particular ordering. The same holds for the $(n+1)$-point amplitudes; the relevant color-orderings are $\mathcal{A}_{n+1}(t, \sigma, n-1, n, n+1)$, $\tilde{\mathcal{A}}_{n+1}(n-1, \rho, n, t, n+1)$, $\mathcal{R}_{n+1}(t, \sigma, n-1, n, n+1)$ and $\tilde{\mathcal{R}}_{n+1}(n-1, \rho, n, t, n+1)$. We leave the ordering implicit from now on.

The soft limit of the Yang-Mills amplitude is given by the soft factors in eq. (603). The soft limit of the gravity amplitude is

$$\mathcal{M}_{n+1}(1,2,\ldots,n+1) = -\frac{1}{\epsilon^3} \sum_{k=1}^{n} \frac{[n+1,k]}{\langle n+1,k\rangle} \tag{607}$$
$$\left[\frac{\langle x,k\rangle\langle y,k\rangle}{\langle x,n+1\rangle\langle y,n+1\rangle} \frac{\epsilon}{2} \left(\frac{\langle x,k\rangle}{\langle x,n+1\rangle} + \frac{\langle y,k\rangle}{\langle y,n+1\rangle} \right) \tilde{\nabla}_{n+1,k} + \frac{\epsilon^2}{2} \left(\tilde{\nabla}_{n+1,k}\right)^2 \right] \mathcal{M}_n(1,2,\ldots,n),$$

where x, y are reference spinors which specifies a gauge. The amplitudes are independent of the choice of x, y. Some care is needed when implementing the momentum conservation. We use $(n+1)$-point and n-point momentum conservation, given by

$$\tilde{\lambda}_i = -\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \frac{\langle j,k\rangle}{\langle j,i\rangle}\tilde{\lambda}_k - \epsilon \frac{\langle j,n+1\rangle}{\langle j,i\rangle}\tilde{\lambda}_{n+1}, \qquad \tilde{\lambda}_j = -\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \frac{\langle i,k\rangle}{\langle i,j\rangle}\tilde{\lambda}_k - \epsilon \frac{\langle i,n+1\rangle}{\langle i,j\rangle}\tilde{\lambda}_{n+1}. \tag{608}$$

and

$$\tilde{\lambda}_i = -\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \frac{\langle j,k\rangle}{\langle j,i\rangle}\tilde{\lambda}_k, \qquad \tilde{\lambda}_j = -\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \frac{\langle i,k\rangle}{\langle i,j\rangle}\tilde{\lambda}_k. \tag{609}$$

We use eq. (608) for all $(n+1)$-point amplitudes $\mathcal{A}_{n+1}$ and $\mathcal{M}_{n+1}$, and eq. (609) for all n-point amplitudes $\mathcal{A}_n$ and $\mathcal{M}_n$, with $i, j = n-1, n$. Also, the total derivatives in the soft factors are

$$\tilde{\nabla}_{s,k} = \tilde{\lambda}_s^{\dot{\alpha}} \frac{d}{d\tilde{\lambda}_k^{\dot{\alpha}}} = \tilde{\lambda}_s^{\dot{\alpha}} \left[\frac{\partial}{\partial \tilde{\lambda}_k^{\dot{\alpha}}} + \left(-\frac{\langle j, k\rangle}{\langle j, i\rangle} \right) \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} + \left(-\frac{\langle i, k\rangle}{\langle i, j\rangle} \right) \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{\alpha}}} \right]. \tag{610}$$

When first using momentum conservation before applying the soft factors, the total derivatives reduce to partial derivatives.

13.4 KLT-RELATION

In string theory, the KLT-relation provides a connection between open and closed string amplitudes. In the limit of infinite string tension, field theory is recoved and a relation between gravity and gauge amplitudes is obtained. Once all the proper permutations are taken into account, the KLT-relation gives the gravity amplitude as the "square" of the gauge amplitudes. For low-point amplitudes, the formulas are relatively simple, which helps streamlining gravitational scattering-amplitude calculations.

The most general form of the KLT-relation is [353, 354]

$$\begin{aligned} \mathcal{M}_n(1,2,\dots,n) =&(-1)^{n+1} \sum_{\sigma\in S_{n-3}} \sum_{\alpha\in S_{j-1}} \sum_{\beta\in S_{n-2-j}} \mathcal{A}_n(1, \sigma_{2,j}, \sigma_{j+1,n-2}, n-1, n) \\ &\times \mathcal{S}[\alpha_{\sigma(2),\sigma(j)}|\sigma_{2,j}]_{p_1} \mathcal{S}[\sigma_{j+1,n-2}|\beta_{\sigma(j+1),\sigma(n-2)}]_{p_{n-1}} \tilde{\mathcal{A}}_n(\alpha_{\sigma(2),\sigma(j)}, 1, n-1, \beta_{\sigma(j+1),\sigma(n-2)}, n), \end{aligned} \tag{611}$$

where $\alpha, \beta, \sigma, \rho$ are particular orderings of the color-ordered Yang-Mills amplitudes. The KLT-kernel $\mathcal{S}$ is defined as

$$\mathcal{S}[i_1,\dots,i_k|j_1,\dots,j_k]_{p_1} = \prod_{t=1}^{k} (s_{i_t 1} + \sum_{q>t}^{k} \theta(i_t, i_q) s_{i_t i_q}), \tag{612}$$

where $\theta(i_a, i_b)$ is 0 if i_a sequentially comes before i_b in the set $\{j_1, \dots, j_k\}$, and otherwise it takes the value 1. One of the properties of the kernel is to take into account the fact that Yang-Mills amplitudes are color-order while gravity amplitudes are not. It was also proven in Ref. [354] that the KLT-relation, as written in eq. (611), is independent of the choice of j. Therefore, with $j = 2$, we have for an $(n+1)$-point amplitude that

$$\mathcal{M}_{n+1}(1,2,\ldots,n,n+1) = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \mathcal{A}_{n+1}(t,\sigma,n-1,n,n+1) \mathcal{S}[t|t]_{p_{n+1}} \mathcal{S}[\sigma|\rho]_{p_{n-1}} \times \tilde{\mathcal{A}}_{n+1}(n-1,\rho,n,t,n+1). \tag{613}$$

In the next section, we are going to apply the soft theorems for each amplitude and collect terms at different orders in $1/\epsilon$. Thus, we also need the soft limit of the KLT-kernel, which is [351]

$$\mathcal{S}[t|t]_{p_{n+1}} \mathcal{S}[\sigma|\rho]_{p_{n-1}} \to \epsilon s_{t,n+1} e^{\epsilon \frac{\langle n,n+1\rangle}{\langle n,t\rangle} \tilde{\nabla}_{n+1,t}} \mathcal{S}[\sigma|\rho]_{p_{n-1}}. \tag{614}$$

We also have the $S_{n-3}-$symmetric form of the KLT-relation for n-point amplitudes

$$\mathcal{M}_n(1,2,\ldots,n) = (-1)^{n+1} \sum_{\sigma,\rho \in S_{n-3}} \mathcal{A}_n(1,\sigma,n-1,n) \mathcal{S}[\sigma|\rho]_{p_{n-1}} \tilde{\mathcal{A}}_n(1,n-1,\rho,n). \tag{615}$$

The different forms of the KLT-relation will be useful shortly.

13.5 NON-UNIVERSAL RELATIONS

The usual procedure when using the KLT-relation is to obtain gravity amplitudes from Yang-Mills amplitudes, since usually the Yang-Mills amplitudes are easier to calculate. Here, we go in the opposite direction. We use information about the gravity amplitudes to obtain relations on the Yang-Mills side. As we noted before, the *non-universal* part of the Yang-Mills amplitude enters at sub-subleading order in the soft-momentum expansion. At this order, we also have a universal part which comes from an exponential of the associated soft factor. Both terms contribute in the KLT-formula. On the other hand, gravity contains only *universal* pieces at $\mathcal{O}(1/\epsilon)$. We equate the soft limit of the gravity amplitude with the soft limit of the Yang-Mills side in the KLT-relation. This immediately gives constraints for the *non-universal* part of the Yang-Mills amplitudes. We describe the procedure in the following and give a detailed derivation in Section 13.A.

We use the KLT-relation in eq. (613), which we write as

$$\mathcal{M}_{n+1} = \sum_{\sigma,\rho} \mathcal{A}_{n+1}(\sigma)\mathcal{S}_{n+1}[\sigma|\rho]\tilde{\mathcal{A}}_{n+1}(\rho). \tag{616}$$

For the left-hand side of eq. (616), we apply the soft-graviton theorem in eq. (607), and then apply the KLT-relation in eq. (615) for each k in the sum for the soft factor,

$$\begin{aligned}\mathcal{M}_{n+1} &= \left(\frac{S^{(0)}_{\text{GR}}}{\epsilon^3} + \frac{S^{(1)}_{\text{GR}}}{\epsilon^2} + \frac{S^{(2)}_{\text{GR}}}{\epsilon}\right)\mathcal{M}_n \\ &= \left(\frac{S^{(0)}_{\text{GR}}}{\epsilon^3} + \frac{S^{(1)}_{\text{GR}}}{\epsilon^2} + \frac{S^{(2)}_{\text{GR}}}{\epsilon}\right)\sum_{\alpha,\beta}\mathcal{A}_n(\alpha)\mathcal{S}_n[\alpha|\beta]\tilde{\mathcal{A}}_n(\beta).\end{aligned} \tag{617}$$

The right-hand side of eq. (616) becomes

$$\begin{aligned}\sum_{\sigma,\rho}\mathcal{A}_{n+1}\mathcal{S}_{n+1}[\sigma|\rho]\tilde{\mathcal{A}}_{n+1} = \sum_{\sigma,\rho}&\left[\left(\frac{S^{(0)}_{\text{YM}}}{\epsilon^2} + \frac{S^{(1)}_{\text{YM}}}{\epsilon} + S^{(2)}_{\text{YM}}\right)\mathcal{A}_n + \mathcal{R}_{n+1}\right]\mathcal{S}_{n+1}[\sigma,\rho] \\ &\times\left[\left(\frac{S^{(0)}_{\text{YM}}}{\epsilon^2} + \frac{S^{(1)}_{\text{YM}}}{\epsilon} + S^{(2)}_{\text{YM}}\right)\tilde{\mathcal{A}}_n + \tilde{\mathcal{R}}_{n+1}\right],\end{aligned} \tag{618}$$

when we use the soft limit of the Yang-Mills amplitudes in eq. (603). We can match the left-hand side and the right-hand side of eq. (616) at each order in $1/\epsilon$. A detailed analysis of the relation at order $1/\epsilon^3$ and $1/\epsilon^2$ was performed in Ref. [351], resulting in new relations for the KLT-kernel.

Focusing on $\mathcal{O}(1/\epsilon)$, we find simple relations between the universal and non-universal piece of the Yang-Mills amplitudes. A more detailed derivation can be found in Section 13.A. The non-universal pieces are defined as

$$R_1 = (-1)^{n+1}\sum_{t=1}^{n-2}\sum_{\sigma,\rho\in S_{n-3}}\frac{1}{\epsilon}\frac{[t,n+1]\langle t,n-1\rangle}{\langle n+1,n-1\rangle}\mathcal{R}^{\epsilon\to 0}_{n+1}\,\mathcal{S}[\sigma|\rho]_{p_{n-1}}\,\tilde{\mathcal{A}}_n, \tag{619}$$

$$R_2 = (-1)^{n+1}\sum_{t=1}^{n-2}\sum_{\sigma,\rho\in S_{n-3}}\frac{1}{\epsilon}\frac{[t,n+1]\langle n,t\rangle}{\langle n+1,n\rangle}\mathcal{A}_n\,\mathcal{S}[\sigma|\rho]_{p_{n-1}}\,\tilde{\mathcal{R}}^{\epsilon\to 0}_{n+1}, \tag{620}$$

while the universal pieces come from second derivatives,

$$T_1 = (-1)^{n}\sum_{t=1}^{n-2}\sum_{\sigma,\rho\in S_{n-3}}\frac{1}{\epsilon}\frac{[t,n+1]\langle n,n-1\rangle}{\langle t,n\rangle\langle n+1,n-1\rangle}\left[\frac{1}{2}\tilde{\nabla}^2_{n+1,t}\left(\mathcal{A}_n\,\mathcal{S}[\sigma|\rho]_{p_{n-1}}\right)\right]\tilde{\mathcal{A}}_n, \tag{621}$$

$$T_2 = (-1)^{n}\sum_{t=1}^{n-2}\sum_{\sigma,\rho\in S_{n-3}}\frac{1}{\epsilon}\frac{[t,n+1]\langle n-1,n\rangle}{\langle t,n-1\rangle\langle n+1,n\rangle}\mathcal{A}_n\,\mathcal{S}[\sigma|\rho]_{p_{n-1}}\left[\frac{1}{2}\tilde{\nabla}^2_{n+1,t}\tilde{\mathcal{A}}_n\right]. \tag{622}$$

The relation between the universal and non-universal pieces of the Yang-Mills amplitudes is

$$T_1 + T_2 = R_1 + R_2. \tag{623}$$

Equation (623) is a new, non-trival relation, illustrating that the non-universal part of the Yang-Mills amplitude possesses some hidden structure. Remarkably, find that the relation simplifies into two parts,[3] given by

$$T_1 = R_1, \tag{624}$$

$$T_2 = R_2. \tag{625}$$

Explicitly, for e.g. eq. (625), this means that

$$\sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\frac{[t,n+1]\langle n,n-1\rangle}{\langle t,n\rangle\langle n+1,n-1\rangle} \left[\frac{1}{2} \tilde{\nabla}^2_{n+1,t} \, \tilde{\mathcal{A}}_n \right] + \frac{[t,n+1]\langle n,t\rangle}{\langle n+1,n\rangle} \tilde{\mathcal{R}}^{\epsilon\to 0}_{n+1} \right] = 0. \tag{626}$$

This is a non-trivial relation for $\mathcal{R}_{n+1}$, which previously have not been discussed. It relates the second derivative of the universal part of a n-point amplitude to the *non-universal* piece of the $(n+1)$-point amplitude. Taking as an example the $n+1 = 6$ NMHV amplitude. There are usually three BCFW-diagrams contributing to the amplitude. One of them contains all the universal soft behavior, as in eq. (600), while the other two diagrams belongs to $\mathcal{R}_{n+1}$ (see eq. (601)). In the soft momenta limit, $\epsilon \to 0$, we have a relation between these two diagrams and the second derivative of the 5-point amplitude $\mathcal{A}_n$.

13.6 EFFECTIVE OPERATORS

The inclusion of effective operators in soft theorems were first studied in Ref. [355]. The authors considered the operators F^3, R^3, and $R^2\phi$, and found that the soft theorems hold for the two first operators while the soft graviton theorem receives a contribution from the last operator at sub-sub-leading order. More general operators were considered in Ref. [352], where was shown that the soft theorem for a Yang-Mills particle is corrected at the subleading and sub-subleading order, while

[3] We have explicitly verified this through seven-points.

gravity amplitudes get corrections at the sub-subleading order. The modified soft theorems take the form

$$\mathcal{A}_{n+1} = \left(\frac{S^{(0)}_{\rm YM}}{\epsilon^2} + \frac{S^{(1)}_{\rm YM}}{\epsilon} + S^{(2)}_{\rm YM}\right)\mathcal{A}_n + \left(\frac{\overline{S}^{(1)}_{\rm YM}}{\epsilon} + \overline{S}^{(2)}_{\rm YM}\right)\overline{\mathcal{A}}_n + \mathcal{R}_{n+1} + \mathcal{O}(\epsilon), \tag{627}$$

$$\mathcal{M}_{n+1} = \left(\frac{S^{(0)}_{\rm GR}}{\epsilon^3} + \frac{S^{(1)}_{\rm GR}}{\epsilon^2} + \frac{S^{(2)}_{\rm GR}}{\epsilon}\right)\mathcal{M}_n + \frac{\overline{S}^{(2)}_{\rm GR}}{\epsilon}\overline{\mathcal{M}}_n + \mathcal{O}(\epsilon). \tag{628}$$

All amplitudes, including the remainder terms $\mathcal{R}_{n+1}$, can contain contributions from effective operators. The bar and superscript (k) denote that, when corrected by effective operators, the particle k of the $(n+1)$- and n-point amplitudes may be of different particle type. The soft theorems for the effective operator corrections are

$$\overline{S}^{(1)}_{\rm YM}\,\overline{\mathcal{A}}_n = -\sum_k \frac{[s,k]}{\langle s,k\rangle}\overline{\mathcal{A}}^{(k)}_n, \qquad \overline{S}^{(2)}_{\rm GR}\,\overline{\mathcal{M}}_n = -\sum_k \frac{[s,k]^3}{\langle s,k\rangle}\overline{\mathcal{M}}^{(k)}_n, \tag{629}$$

where s is the soft particle and k is adjacent to the soft particle. We have absorbed the couplings into the amplitudes. The sum in eq. (629) for a Yang-Mills particle goes over the two adjacent legs, while for gravity it sums over all other particles. The sub-subleading soft term for Yang-Mills particles can be found in eq. (604).

The second ingredient we need is the KLT-relation. The open-closed string KLT-relations are similar to the field-theory KLT-relations, with a different kernel. For instance, the 4-point string KLT-relation turns into

$$\mathcal{M}^{\rm closed}_4(1,2,3,4) = \mathcal{A}^{\rm open}_4(1,2,3,4)\left[\frac{\kappa^2}{4\pi\alpha'}\sin(\pi x)\right]\tilde{\mathcal{A}}^{\rm open}_4(1,2,4,3) \tag{630}$$

where $x = -\alpha' s_{12}$ and α' is the inverse string tension. A generalized prescription for the KLT-relations for effective amplitudes was analysed in Refs. [356–358]. The new, generalized kernel used in Refs. [356–358] was organized as a Taylor expansion in powers of $\alpha' s_{12}$,

$$\frac{\sin(\pi x)}{\pi} \to x(1 + c_1 x + c_2 x^2 + ...) \tag{631}$$

The first order in α' recovers the usual KLT-kernel.

To consider the KLT-relation for effective amplitudes, we need to make some assumptions. First, we assume that a general n-point KLT-relation for effective amplitudes follows the structure found in

Refs. [356–358], i.e. the kernel is generalized, where the leading order reproduces the original kernel, and the kernel can be expanded as a Taylor expansion in powers of s_{ij}/Λ^2, where Λ is some energy scale. In string theory, α' takes the role of $1/\Lambda^2$, as can be seen from eq. (630). Second, we assume that the soft limit of the kernel is similar to eq. (614), with possibly more powers of $s_{t,n+1}$. Therefore, we assume that the soft limit of the kernel is

$$\mathcal{S}[t|t]_{p_{n+1}}\mathcal{S}[\sigma|\rho]_{p_{n-1}} \to \epsilon s_{t,n+1}\left(\sum_{\ell=0}^{\infty} c_\ell(\epsilon s_{t,n+1})^\ell\right) e^{\epsilon\frac{\langle n,n+1\rangle}{\langle n,t\rangle}\tilde{\nabla}_{n+1,t}}\,\mathcal{S}[\sigma|\rho]_{p_{n-1}}, \tag{632}$$

where $c_0 = 1$. We absorb any mass scale into the unknown coefficiencts c_ℓ, such that the mass dimension of c_ℓ is -2ℓ.

The left-hand side of the KLT-relation in eq. (616) now becomes

$$\begin{aligned}\mathcal{M}_{n+1} &= \left(\frac{S^{(0)}_{\text{GR}}}{\epsilon^3}+\frac{S^{(1)}_{\text{GR}}}{\epsilon^2}+\frac{S^{(2)}_{\text{GR}}}{\epsilon}\right)\mathcal{M}_n + \left(\frac{\overline{S}^{(2)}_{\text{GR}}}{\epsilon}\right)\overline{\mathcal{M}}_n\\ &= \left(\frac{S^{(0)}_{\text{GR}}}{\epsilon^3}+\frac{S^{(1)}_{\text{GR}}}{\epsilon^2}+\frac{S^{(2)}_{\text{GR}}}{\epsilon}\right)\sum_{\alpha,\beta}\mathcal{A}_n(\alpha)\mathcal{S}_n[\alpha|\beta]\tilde{\mathcal{A}}_n(\beta) + \left(\frac{\overline{S}^{(2)}_{\text{GR}}}{\epsilon}\right)\overline{\mathcal{M}}_n.\end{aligned} \tag{633}$$

Note that the only difference between eq. (617) and eq. (633) is the additional term coming from the effective-operator extension of soft-graviton theorem. The right-hand side of eq. (616) is

$$\begin{aligned}&\sum_{\sigma,\rho}\left[\left(\frac{S^{(0)}_{\text{YM}}}{\epsilon^2}+\frac{S^{(1)}_{\text{YM}}}{\epsilon}+S^{(2)}_{\text{YM}}\right)\mathcal{A}_n+\left(\frac{\overline{S}^{(1)}_{\text{YM}}}{\epsilon}+\overline{S}^{(2)}_{\text{YM}}\right)\overline{\mathcal{A}}_n+\mathcal{R}_{n+1}\right]\mathcal{S}_{n+1}[\sigma,\rho]\\ &\times\left[\left(\frac{S^{(0)}_{\text{YM}}}{\epsilon^2}+\frac{S^{(1)}_{\text{YM}}}{\epsilon}+S^{(2)}_{\text{YM}}\right)\tilde{\mathcal{A}}_n+\left(\frac{\overline{S}^{(1)}_{\text{YM}}}{\epsilon}+\overline{S}^{(2)}_{\text{YM}}\right)\tilde{\overline{\mathcal{A}}}_n+\tilde{\mathcal{R}}_{n+1}\right].\end{aligned} \tag{634}$$

By equating eqs. (633) and (634) and comparing order-by-order in $1/\epsilon$, we see that the first correction from the effective operators appears at subleading order for the right-hand side and at sub-subleading order for the left-hand side. Therefore, at order $\mathcal{O}(1/\epsilon^2)$, we find the relation

$$0 = \overline{U}_1+\overline{U}_2+U_3+U_4, \tag{635}$$

where the first two terms are given by the modifications of the soft theorem for effective operators,

$$\overline{U}_1 = \frac{(-1)^{n+1}}{\epsilon^2} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]\langle n,t\rangle}{\langle n,n+1\rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\frac{[n+1,n-1]}{\langle n+1,n-1\rangle} \overline{\tilde{\mathcal{A}}}_n^{(n-1)} + \frac{[n+1,t]}{\langle n+1,t\rangle} \overline{\tilde{\mathcal{A}}}_n^{(t)} \right] \tag{636}$$

$$\overline{U}_2 = \frac{(-1)^{n+1}}{\epsilon^2} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \left[\frac{[n+1,n]}{\langle n+1,n\rangle} \overline{\mathcal{A}}_n^{(n)} + \frac{[n+1,t]}{\langle n+1,t\rangle} \overline{\mathcal{A}}_n^{(t)} \right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \frac{[t,n+1]\langle t,n-1\rangle}{\langle n-1,n+1\rangle} \tilde{\mathcal{A}}_n. \tag{637}$$

The third term comes from the expansion of the kernel, which has an unknown parameter c_1,

$$U_3 = \frac{(-1)^n}{\epsilon^2} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} c_1 \frac{[t,n+1]^2\langle t,n\rangle\langle t,n-1\rangle}{\langle n,n+1\rangle\langle n+1,n-1\rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \tilde{\mathcal{A}}_n.$$

The last term is

$$U_4 = \frac{(-1)^n}{\epsilon^2} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]\langle n,n-1\rangle}{\langle n-1,n+1\rangle\langle n,n+1\rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t} \, \tilde{\mathcal{A}}_n \right].$$

which was also found in the analysis in Ref. [351], and was shown to vanish as long as the original kernel, defined in eq. (612), satisfies two non-trivial identities. As we have not specified the generalized kernel fully, we keep this term for generality.

At $\mathcal{O}(1/\epsilon)$, we proceed analogously. For the left-hand side of eq. (616) we obtain the same terms P_{1-6} in eqs. (639) to (644) in addition to the gravity effective operator term. For the right-hand side, we find the same Q_{1-6} in eqs. (645) to (650) and $R_{1,2}$ in eqs. (619) and (620) as in section 13.5. The new contributions appear when we have used $\overline{S}^{(1)}_{\mathrm{YM}}$, $\overline{S}^{(2)}_{\mathrm{YM}}$, or higher terms in the expansion of the kernel. All the new contributions are given in Section 13.B.

After some manipulations, we obtain the following relation

$$\frac{\overline{S}^{(2)}_{\mathrm{GR}}}{\epsilon} \overline{\mathcal{M}}_n + T_1 + T_2 = \overline{Q} + R_1 + R_2. \tag{638}$$

where $\overline{Q}$ is defined in the appendix. As expected, when the effective operators are turned off we recover eq. (623). Although the relation eq. (638) is more complicated than the relation for the original amplitudes (with no effective operators), we can still systematically organize the contributions from the effective operators into a simple formula. We found here the most general relations for the effective terms. However, further assumptions can be made on the type of effective operators that contribute to the amplitude and the form of the kernel, which can further simplify the relations. We leave this for a future investigation.

13.7 CONCLUSION

Using the KLT-relation and the soft limit of Yang-Mills and gravity amplitudes, we have found new, non-trivial relations for the sub-subleading part of the Yang-Mills amplitudes. Previous analysis has only considered the subleading terms, and the sub-subleading part has not previously been fully discussed. The new relations provide non-trivial constraints for the behavior of the Yang-Mills amplitudes under the soft limit. We also studied the analogous relations when contributions from effective operators are included.

The new relations give information about the *non-universal* part of the Yang-Mills amplitude. In obtaining the relations, we went in the oppositve direction of most of uses of the KLT-formula, where we made use of the behavior of the gravity amplitude to extract information for the Yang-Mills amplitude.

As we have used the spinor-helicity formalism, our results are restricted to four dimensions. Extending the analysis to arbitrary dimensions would provide insight into the generality of the result. A natural framework for studying the relations in arbitrary dimensions is the CHY-formalism.

Recently, infinite partial soft theorems were discussed in Refs. [359, 360]. Understanding the connection between our results and the infinite partial soft theorems would be illuminating. Also, non-linear relations for Yang-Mills amplitudes were presented in Refs. [361, 362]. We leave the study of the connection between these non-linear relations and the relations presented in this paper as a future project.

APPENDIX

13.A SUB-SUBLEADING TERMS

We outline the derivation of the relation for the non-universal terms at $\mathcal{O}(1/\epsilon)$. Recall that the color-ordered amplitudes are given by $\mathcal{A}_n \equiv \mathcal{A}_n(t,\sigma,n-1,n)$ and $\tilde{\mathcal{A}}_n \equiv \tilde{\mathcal{A}}_{n+1}(n-1,\rho,n,t)$ and the definition of the derivative operator is $\tilde{\nabla}_{a,b} = \tilde{\lambda}_a^{\dot{\alpha}} \frac{d}{d\tilde{\lambda}_b^{\dot{\alpha}}}$. The left-hand side of eq. (616), after applying the soft-graviton theorem and the KLT-relation, is given in eq. (617). At $\mathcal{O}(1/\epsilon)$, the terms in eq. (617) are given by P_{1-6}, where

$$P_1 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t,n+1]}{\langle t,n+1\rangle} \mathcal{A}_n \left[\frac{1}{2}\tilde{\nabla}^2_{n+1,t}\, \mathcal{S}[\sigma|\rho]_{p_{n-1}}\right] \tilde{\mathcal{A}}_n, \tag{639}$$

$$P_2 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t,n+1]}{\langle t,n+1\rangle} \left[\frac{1}{2}\tilde{\nabla}^2_{n+1,t}\, \mathcal{A}_n\right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \tilde{\mathcal{A}}_n, \tag{640}$$

$$P_3 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t,n+1]}{\langle t,n+1\rangle} \mathcal{A}_n \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\frac{1}{2}\tilde{\nabla}^2_{n+1,t}\, \tilde{\mathcal{A}}_n\right], \tag{641}$$

$$P_4 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t,n+1]}{\langle t,n+1\rangle} \left[\tilde{\nabla}_{n+1,t}\, \mathcal{A}_n\right] \left[\tilde{\nabla}_{n+1,t}\, \mathcal{S}[\sigma|\rho]_{p_{n-1}}\right] \tilde{\mathcal{A}}_n, \tag{642}$$

$$P_5 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t,n+1]}{\langle t,n+1\rangle} \left[\tilde{\nabla}_{n+1,t}\, \mathcal{A}_n\right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t}\, \tilde{\mathcal{A}}_n\right], \tag{643}$$

$$P_6 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t, n+1]}{\langle t, n+1 \rangle} \mathcal{A}_n \left[\tilde{\nabla}_{n+1,t}\, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \right] \left[\tilde{\nabla}_{n+1,t}\, \tilde{\mathcal{A}}_n \right]. \tag{644}$$

Similarly, the right-hand side of eq. (616) is given in eq. (618), consisting of Q_{1-6} and $R_{1,2}$. The non-universal terms $R_{1,2}$ are given in eqs. (619) and (620), with the residual terms being

$$Q_1 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t, n+1]\langle t, n-1 \rangle \langle n, n+1 \rangle}{\langle t, n+1 \rangle \langle n+1, n-1 \rangle \langle n, t \rangle} \mathcal{A}_n \left[\frac{1}{2} \tilde{\nabla}^2_{n+1,t}\, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \right] \tilde{\mathcal{A}}_n, \tag{645}$$

$$Q_2 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t, n+1]\langle t, n-1 \rangle \langle n, n+1 \rangle}{\langle t, n+1 \rangle \langle n+1, n-1 \rangle \langle n, t \rangle} \left[\frac{1}{2} \tilde{\nabla}^2_{n+1,t}\, \mathcal{A}_n \right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \tilde{\mathcal{A}}_n, \tag{646}$$

$$Q_3 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t, n+1]\langle n, t \rangle \langle n-1, n+1 \rangle}{\langle t, n+1 \rangle \langle n, n+1 \rangle \langle n-1, t \rangle} \mathcal{A}_n \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\frac{1}{2} \tilde{\nabla}^2_{n+1,t}\, \tilde{\mathcal{A}}_n \right], \tag{647}$$

$$Q_4 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t, n+1]\langle t, n-1 \rangle \langle n, n+1 \rangle}{\langle t, n+1 \rangle \langle n+1, n-1 \rangle \langle n, t \rangle} \left[\tilde{\nabla}_{n+1,t}\, \mathcal{A}_n \right] \left[\tilde{\nabla}_{n+1,t}\, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \right] \tilde{\mathcal{A}}_n, \tag{648}$$

$$Q_5 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t, n+1]}{\langle t, n+1 \rangle} \left[\tilde{\nabla}_{n+1,t}, \mathcal{A}_n \right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t}, \tilde{\mathcal{A}}_n \right], \tag{649}$$

$$Q_6 = (-1)^n \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{1}{\epsilon} \frac{[t, n+1]}{\langle t, n+1 \rangle} \mathcal{A}_n \left[\tilde{\nabla}_{n+1,t}, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \right] \left[\tilde{\nabla}_{n+1,t}\, \tilde{\mathcal{A}}_n \right]. \tag{650}$$

Note that

$$P_5 = Q_5, \tag{651}$$

$$P_6 = Q_6, \tag{652}$$

which reduces eq. (616) to

$$P_1 + P_2 + P_3 + P_4 = Q_1 + Q_2 + Q_3 + Q_4 + R_1 + R_2. \tag{653}$$

We can simplify this further by using that

$$T_1 = P_1 + P_2 + P_4 - Q_1 - Q_2 - Q_4 \quad \text{and} \quad T_2 = P_3 - Q_3, \tag{654}$$

which are the terms used in the main text, given in eqs. (621) and (622). With these manipulations, we find new, non-trivial relations for the non-universal part of the Yang-Mills amplitudes. The new relations, eqs. (624) and (625), have surprisingly simple forms.

13.B SUB-SUBLEADING TERMS FROM EFFECTIVE OPERATOR

The new contributions to eq. (616) coming from the soft theorems for effective operators involve $\overline{S}^{(1)}_{\mathrm{YM}}$ and $\overline{S}^{(2)}_{\mathrm{YM}}$. We will denote the terms by $\overline{Q}_{2-6}$ as they resemble Q_{2-6} in eqs. (646) to (650). In general, as $\overline{S}^{(1,2)}_{\mathrm{YM}}$ contain two different terms, we express $\overline{Q}_{2-6}$ as two terms, e.g. $\overline{Q}_2 = \overline{Q}_2^{(t)} + \overline{Q}_2^{(n)}$. Note that $\overline{Q}_5$ is split into eight terms,

$$\overline{Q}_5 = \overline{Q}_5^{(n,n-1)} + \overline{Q}_5^{(n,t)} + \overline{Q}_5^{(t,n-1)} + \overline{Q}_5^{(t,t)} + \overline{Q}_5^{(L,n)} + \overline{Q}_5^{(L,t)} + \overline{Q}_5^{(R,n-1)} + \overline{Q}_5^{(R,t)}. \tag{655}$$

The terms are

$$\overline{Q}_2^{(t)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t, n+1]^2 \langle t, n-1\rangle \langle n, n+1\rangle}{\langle n+1, n-1\rangle \langle n+1, t\rangle \langle n, t\rangle} \left[\tilde{\nabla}_{n+1,t}\, \overline{\mathcal{A}}_n^{(t)}\right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \tilde{\mathcal{A}}_n,$$

$$\overline{Q}_2^{(n)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t, n+1] \langle t, n-1\rangle [n+1, n]}{\langle n+1, n-1\rangle \langle n, t\rangle} \left[\tilde{\nabla}_{n+1,t}\, \overline{\mathcal{A}}_n^{(n)}\right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \tilde{\mathcal{A}}_n,$$

$$\overline{Q}_3^{(t)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]^2 \langle n,t\rangle \langle n-1,n+1\rangle}{\langle n+1,n\rangle \langle n+1,t\rangle \langle n-1,t\rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t} \overline{\tilde{\mathcal{A}}}_n^{(t)}\right],$$

$$\overline{Q}_3^{(n-1)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1][n+1,n-1]\langle n,t\rangle}{\langle n+1,n\rangle \langle n-1,t\rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t} \overline{\tilde{\mathcal{A}}}_n^{(n-1)}\right].$$

$$\overline{Q}_4^{(n)} = \frac{(-1)^n}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]\langle t,n-1\rangle \langle n,n+1\rangle [n+1,n]}{\langle n+1,n-1\rangle \langle n,t\rangle \langle n+1,n\rangle} \overline{\mathcal{A}}_n^{(n)} \left[\tilde{\nabla}_{n+1,t} \mathcal{S}[\sigma|\rho]_{p_{n-1}}\right] \tilde{\mathcal{A}}_n,$$

$$\overline{Q}_4^{(t)} = \frac{(-1)^n}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]\langle t,n-1\rangle \langle n,n+1\rangle [n+1,t]}{\langle n+1,n-1\rangle \langle n,t\rangle \langle n+1,t\rangle} \overline{\mathcal{A}}_n^{(t)} \left[\tilde{\nabla}_{n+1,t} \mathcal{S}[\sigma|\rho]_{p_{n-1}}\right] \tilde{\mathcal{A}}_n,$$

$$\overline{Q}_5^{(n,n-1)} = \frac{(-1)^n}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]\langle n+1,t\rangle [n+1,n][n+1,n-1]}{\langle n+1,n\rangle \langle n+1,n-1\rangle} \overline{\mathcal{A}}_n^{(n)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \overline{\tilde{\mathcal{A}}}_n^{(n-1)},$$

$$\overline{Q}_5^{(n,t)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]^2 [n+1,n]}{\langle n+1,n\rangle} \overline{\mathcal{A}}_n^{(n)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \overline{\tilde{\mathcal{A}}}_n^{(t)},$$

$$\overline{Q}_5^{(t,n-1)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]^2 [n+1,n-1]}{\langle n+1,n-1\rangle} \overline{\mathcal{A}}_n^{(n)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \overline{\tilde{\mathcal{A}}}_n^{(n-1)},$$

$$\overline{Q}_5^{(t,t)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[n+1,t]^3}{\langle n+1,t\rangle} \overline{\mathcal{A}}_n^{(t)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \overline{\tilde{\mathcal{A}}}_n^{(t)},$$

$$\overline{Q}_5^{(L,n)} = \frac{(-1)^n}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1][n+1,n]}{\langle n+1,n\rangle} \overline{\mathcal{A}}_n^{(n)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t} \tilde{\mathcal{A}}_n\right],$$

$$\overline{Q}_5^{(L,t)} = \frac{(-1)^n}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1][n+1,t]}{\langle n+1,t\rangle} \overline{\mathcal{A}}_n^{(t)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \, \left[\tilde{\nabla}_{n+1,t}\tilde{\mathcal{A}}_n\right],$$

$$\overline{Q}_5^{(R,n-1)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1][n+1,n-1]}{\langle n+1,n-1\rangle} \left[\tilde{\nabla}_{n+1,t}\mathcal{A}_n\right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \overline{\tilde{\mathcal{A}}}_n^{(n-1)},$$

$$\overline{Q}_5^{(R,t)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1][n+1,t]}{\langle n+1,t\rangle} \left[\tilde{\nabla}_{n+1,t}\mathcal{A}_n\right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \overline{\tilde{\mathcal{A}}}_n^{(t)}.$$

$$\overline{Q}_6^{(n-1)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1][n+1,n-1]}{\langle n+1,n-1\rangle} \mathcal{A}_n \left[\tilde{\nabla}_{n+1,t}\mathcal{S}[\sigma|\rho]_{p_{n-1}}\right] \overline{\tilde{\mathcal{A}}}_n^{(n-1)},$$

$$\overline{Q}_6^{(t)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1][n+1,t]}{\langle n+1,t\rangle} \mathcal{A}_n \left[\tilde{\nabla}_{n+1,t}\mathcal{S}[\sigma|\rho]_{p_{n-1}}\right] \overline{\tilde{\mathcal{A}}}_n^{(t)}.$$

The strategy now is to group $\overline{Q}_2^{(t)}$ and $\overline{Q}_4^{(t)}$ with $\overline{Q}_5^{(L,t)}$. Notice that each term has a derivative and comes with the superscript (t). We apply the Schouten identity on the spinor brackets in the first two terms to match the last one. This produces a 'total' derivative called $\overline{Q}_L^{(t)}$ and an extra term, which we call $\overline{Q}_S^{(t1)}$. We do the same for others operators, finding the following rearragements

$$\overline{Q}_2^{(t)} + \overline{Q}_4^{(t)} + \overline{Q}_5^{(L,t)} = \overline{Q}_L^{(t)} + \overline{Q}_S^{(t1)}, \tag{656}$$

$$\overline{Q}_2^{(n)} + \overline{Q}_4^{(n)} + \overline{Q}_5^{(L,n)} = \overline{Q}_L^{(n)} + \overline{Q}_S^{(n)}, \tag{657}$$

$$\overline{Q}_3^{(t)} + \overline{Q}_6^{(t)} + \overline{Q}_5^{(R,t)} = \overline{Q}_R^{(t)} + \overline{Q}_S^{(t2)}, \tag{658}$$

$$\overline{Q}_3^{(n-1)} + \overline{Q}_6^{(n-1)} + \overline{Q}_5^{(R,n-1)} = \overline{Q}_R^{(n-1)} + \overline{Q}_S^{(n-1)}, \tag{659}$$

where

$$\overline{Q}_L^{(k)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} (-1)\,[t, n+1] \frac{[n+1, k]}{\langle n+1, k\rangle} \tilde{\nabla}_{n+1,t} \left[\overline{\mathcal{A}}_n^{(k)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \, \tilde{\mathcal{A}}_n \right], \quad (660)$$

$$\overline{Q}_R^{(k)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} (-1)\,[t, n+1] \frac{[n+1, k]}{\langle n+1, k\rangle} \tilde{\nabla}_{n+1,t} \left[\mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \, \tilde{\overline{\mathcal{A}}}_n^{(k)} \right]. \quad (661)$$

The extra terms are

$$\overline{Q}_S^{(t1)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t, n+1]^2 \langle n-1, n\rangle}{\langle n+1, n-1\rangle \langle n, t\rangle} \tilde{\nabla}_{n+1,t} \left[\overline{\mathcal{A}}_n^{(t)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \right] \tilde{\mathcal{A}}_n, \quad (662)$$

$$\overline{Q}_S^{(n)} = \frac{(-1)^{n}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t, n+1][n, n-1]\langle t, n+1\rangle \langle n, n-1\rangle}{\langle n+1, n-1\rangle \langle n, t\rangle \langle n+1, n\rangle} \tilde{\nabla}_{n+1,t} \left[\overline{\mathcal{A}}_n^{(n)} \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \right] \tilde{\mathcal{A}}_n, \quad (663)$$

$$\overline{Q}_S^{(t2)} = \frac{(-1)^{n+1}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t, n+1]^2 \langle n, n-1\rangle}{\langle n+1, n\rangle \langle n-1, t\rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t} \tilde{\overline{\mathcal{A}}}_n^{(t)} \right], \quad (664)$$

$$\overline{Q}_S^{(n-1)} = \frac{(-1)^{n}}{\epsilon} \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t, n+1][n+1, n-1]\langle n, n-1\rangle \langle t, n+1\rangle}{\langle n+1, n\rangle \langle n-1, t\rangle \langle n+1, n-1\rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t} \tilde{\overline{\mathcal{A}}}_n^{(n-1)} \right]. \quad (665)$$

We group the terms together,

$$\overline{Q}_S = \overline{Q}_S^{(n)} + \overline{Q}_S^{(t1)} + \overline{Q}_S^{(t2)} + \overline{Q}_S^{(n-1)}, \quad (666)$$

$$\overline{Q}_{5'} = \overline{Q}_5^{(n,n-1)} + \overline{Q}_5^{(n,t)} + \overline{Q}_5^{(t,n-1)} + \overline{Q}_5^{(t,t)}, \quad (667)$$

$$\overline{Q}_B = \overline{Q}_L^{(t)} + \overline{Q}_L^{(n)} + \overline{Q}_R^{(t)} + \overline{Q}_R^{(n-1)}. \quad (668)$$

We also have contributions from the higher-order terms in the expansion of the kernel. We group them as

$$\overline{Q}_K = T_1' + T_2' + T_3' + T_4' + \overline{U}_1' + \overline{U}_2' \tag{669}$$

The two last terms are related to U_1 and U_2 in eqs. (636) and (637) as

$$\overline{U}_1' = \epsilon c_1 s_{t,n+1} U_1, \tag{670}$$

$$\overline{U}_2' = \epsilon c_1 s_{t,n+1} U_2, \tag{671}$$

where the factor $s_{t,n+1}$ is understood to be inside the sum over t.

The other terms are

$$T_1' = \frac{(-1)^{n+1}}{\epsilon} c_1 \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]^2 \langle n,t \rangle}{\langle n,n+1 \rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t} \tilde{\mathcal{A}}_n\right], \tag{672}$$

$$T_2' = \frac{(-1)^{n+1}}{\epsilon} c_1 \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]^2 \langle n-1,t \rangle}{\langle n-1,n+1 \rangle} \left[\tilde{\nabla}_{n+1,t} \mathcal{A}_n\right] \mathcal{S}[\sigma|\rho]_{p_{n-1}} \tilde{\mathcal{A}}_n, \tag{673}$$

$$T_3' = \frac{(-1)^{n+1}}{\epsilon} c_1 \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]^2 \langle n-1,t \rangle}{\langle n-1,n+1 \rangle} \mathcal{A}_n \left[\tilde{\nabla}_{n+1,t} \mathcal{S}[\sigma|\rho]_{p_{n-1}}\right] \tilde{\mathcal{A}}_n, \tag{674}$$

$$T_4' = \frac{(-1)^{n}}{\epsilon} c_2 \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t,n+1]^3 \langle n,t \rangle \langle t,n-1 \rangle \langle t,n+1 \rangle}{\langle n,n+1 \rangle \langle n+1,n-1 \rangle} \left[\mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \tilde{\mathcal{A}}_n\right]. \tag{675}$$

We can rewrite T_{1-3}' using the Schouten identity to

$$T_1' + T_2' + T_3' = T' + T_s', \tag{676}$$

where

$$T' = \frac{(-1)^{n+1}}{\epsilon} c_1 \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t, n+1]^2 \langle n-1, t\rangle}{\langle n-1, n+1\rangle} \tilde{\nabla}_{n+1,t} \left[\mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \, \tilde{\mathcal{A}}_n\right], \tag{677}$$

$$T'_s = \frac{(-1)^{n}}{\epsilon} c_1 \sum_{t=1}^{n-2} \sum_{\sigma,\rho \in S_{n-3}} \frac{[t, n+1]^2 \langle n, n-1\rangle \langle t, n+1\rangle}{\langle n, n+1\rangle \langle n+1, n-1\rangle} \mathcal{A}_n \, \mathcal{S}[\sigma|\rho]_{p_{n-1}} \left[\tilde{\nabla}_{n+1,t} \tilde{\mathcal{A}}_n\right]. \tag{678}$$

In total, we have that

$$\overline{Q} = \overline{Q}_{5'} + \overline{Q}_B + \overline{Q}_S + \overline{Q}_K. \tag{679}$$

14

NEW FACTORIZATION RELATIONS FOR NON-LINEAR SIGMA MODEL AMPLITUDES

We obtain novel factorization identities for non-linear sigma model amplitudes using a new integrand in the CHY double-cover prescription. We find that it is possible to write very compact relations using only longitudinal degrees of freedom. We discuss implications for on-shell recursion.

14.1 INTRODUCTION

Cachazo, He and Yuan invented in ref. [348] a new method for calculating S-matrix elements. This formalism has numerous applications and many interesting connections, see for instance refs. [363–365]. The CHY construction was formally proven by Dolan and Goddard in ref. [366].

The main ingredients are the n-point scattering equations

$$0 = S_a \equiv \sum_{b=1,b\neq a}^{n} \frac{s_{ab}}{z_{ab}}, \quad z_{ab} \equiv z_a - z_b, \quad s_{ab} \equiv 2k_a \cdot k_b, \tag{680}$$

where z_a are auxiliary variables on the Riemann sphere and k_a are momenta. In the CHY formalism one has to integrate over a contour containing the $(n-3)!$ independent solutions of the scattering equations.

As computations in the CHY formalism grow factorially in complexity with n, integration rules have been developed at tree [367–370] and loop level [371], so that analytical results for amplitudes can be derived without solving the scattering equations explicitly.

Recently, the CHY formalism was reformulated by one of us in the context of a double cover [372] (called the 'Λ-formalism' in refs. [372, 373]). Here, the basic variables are elements of $\mathbb{CP}^2$, and not $\mathbb{CP}^1$ as in the original CHY formalism. One advantage of the extra machinery is that amplitudes in the double-cover formulation naturally factorize into smaller $\mathbb{CP}^1$ pieces, and this is a useful laboratory for deriving new amplitude identities.

We will start by reviewing the CHY formalism for the non-linear sigma model (NLSM) and provide an alternative formulation that employs a new integrand. Next, we will show how the double-cover formalism naturally factorizes this new CHY formulation in a surprising way.

14.2 A NEW CHY INTEGRAND

As explained in ref. [246], the flavor-ordered partial $U(N)$ non-linear sigma model amplitude in the scattering equation framework is given by the contour integral

$$A_n(\alpha) = \int d\mu_n H_n(\alpha)\,, \tag{681}$$
$$d\mu_n \equiv (z_{ij}\, z_{jk}\, z_{ki})^2 \prod_{\substack{a=1 \\ a\neq\{i,j,k\}}}^{n} \frac{dz_a}{S_a},$$

where $(\alpha) = (\alpha(1), ..., \alpha(n))$ denotes a partial ordering. The integrand is given by

$$H_n(\alpha) = \mathrm{PT}(\alpha) \times (\mathrm{Pf}'\mathsf{A})^2, \tag{682}$$

$$\mathrm{PT}(\alpha) \equiv \frac{1}{z_{\alpha(1)\alpha(2)} z_{\alpha(2)\alpha(3)} \cdots z_{\alpha(n)\alpha(1)}}, \tag{683}$$
$$\mathrm{Pf}'\mathsf{A} \equiv \frac{(-1)^{i+j}}{z_{ij}}\, \mathrm{Pf}[(\mathsf{A})^{ij}_{ij}]. \tag{684}$$

Here $\mathrm{PT}(\alpha)$ and $\mathrm{Pf}'\mathsf{A}$ are the Parke-Taylor (PT) factor and the reduced Pfaffian of matrix A, respectively. The $n \times n$ anti-symmetric matrix, A, is defined as,

$$\mathsf{A}_{ab} \equiv \frac{s_{ab}}{z_{ab}} \quad \text{for } a \neq b, \quad \text{and} \quad \mathsf{A}_{ab} \equiv 0 \qquad\qquad \text{for } a = b. \tag{685}$$

In general, $(\mathsf{A})^{i_1\cdots i_p}_{j_1\cdots j_p}$ denotes the reduced matrix obtained by removing the rows, $\{i_1,...,i_p\}$, and columns, $\{j_1,...,j_p\}$, from A. Note that when the number of external particles n is odd, $\text{Pf}'\mathsf{A} = 0$, and $A_n(\alpha)$ vanishes.

Using eq. (684), we have

$$(\text{Pf}'\mathsf{A})^2 = \frac{(-1)^{i+j+m+p}}{z_{ij}\, z_{mp}} \text{Pf}[(\mathsf{A})^{ij}_{ij}] \times \text{Pf}[(\mathsf{A})^{mp}_{mp}]. \tag{686}$$

With the choice $\{i,j\} = \{m,p\}$, this product of Pfaffians becomes a determinant,

$$(\text{Pf}'\mathsf{A})^2 = -\text{PT}(m,p)\, \det[(\mathsf{A})^{mp}_{mp}]. \tag{687}$$

We will now discuss the following new matrix identities. On the support of the scattering equations and the massless condition, $\{S_a = 0, k_a^2 = 0\}$, we find when $m \neq p \neq q$

$$\text{Pf}[(\mathsf{A})^{mp}_{mp}] \times \text{Pf}[(\mathsf{A})^{pq}_{pq}] = \det[(\mathsf{A})^{mp}_{pq}], \tag{688}$$

$$\det[(\mathsf{A})^{mp}_{pq}] = 0 \quad \text{if}\, n \text{ odd}. \tag{689}$$

A proof of these identities will be provided in ref. [5]. Using the non-antisymmetric matrix, $(\mathsf{A})^{ij}_{jk}$, we define the objects (with $i<j<k$)

$$A'_n(\alpha) = \int d\mu_n\ \text{PT}(\alpha)\, \frac{(-1)^{i+k}}{z_{ij}\, z_{jk}} \det[(\mathsf{A})^{ij}_{jk}], \tag{690}$$

$$A^{(ij)}_n(\alpha) = \int d\mu_n\ \text{PT}(\alpha)\, \frac{(-1)^{i+j}}{z_{ij}} \det[(\mathsf{A})^{i}_{j}]. \tag{691}$$

Note that in eqs. (690) and (691) we have reduced the A matrix with the indices $\{i,j,k\}$ associated with the Faddeev-Popov determinant. This gauge choice will be convenient later. We now have the following equality,

$$A'_n(\alpha) = A_n(\alpha), \tag{692}$$

when all particles are on-shell. When there are off-shell particles, the identity is true only if the number of particles is even. When the number of particles is odd and there are off-shell particles, one has $A_n(\alpha) = 0$ while $A'_n(\alpha) \neq 0$. Since the A matrix has co-rank 2 on the support of the scattering equations and the massless condition, $\{S_a = 0, k_a^2 = 0\}$, $A_n^{(ij)}(\alpha)$ vanishes trivially. However, when there are off-shell particles the amplitude $A_n^{(ij)}(\alpha)$ is no longer zero.

These observations will be crucial in obtaining the new factorization relations.

14.3 THE DOUBLE-COVER REPRESENTATION

In the double-cover version of the CHY construction, the n-point amplitude is given as a contour integral on the double-covered Riemann sphere with n punctures. The pairs $(\sigma_1, y_1), (\sigma_2, y_2), \ldots, (\sigma_n, y_n)$ provide the new set of doubled variables restricted to the curves

$$0 = C_a \equiv y_a^2 - \sigma_a^2 + \Lambda^2 \qquad \text{for } a = 1, \ldots, n. \tag{693}$$

A translation table has been worked out in detail in ref. [372]. The double-cover formulation of the NLSM is given by the integral

$$\begin{aligned} A_n(\alpha) &= \int_\Gamma d\mu_n^\Lambda \, \frac{(-1)\,\Delta(ijk)\,\Delta(ijk|r)}{S_r^\tau} \times \mathcal{I}_n(\alpha), \\ d\mu_n^\Lambda &\equiv \frac{1}{2^2}\frac{d\Lambda}{\Lambda}\prod_{a=1}^{n}\frac{y_a dy_a}{C_a} \times \prod_{\substack{d=1 \\ d\neq i,j,k,r}}^{n} \frac{d\sigma_d}{S_d^\tau}, \end{aligned} \tag{694}$$

$$\begin{aligned} \tau(a,b) &\equiv \frac{1}{2\,\sigma_{ab}}\left(\frac{y_a + y_b + \sigma_{ab}}{y_a}\right), \quad S_a^\tau \equiv \sum_{\substack{b=1 \\ b\neq a}}^{n} s_{ab}\,\tau(a,b)\,, \\ \Delta(ijk) &\equiv \big(\tau(i,j)\,\tau(j,k)\,\tau(k,i)\big)^{-1}, \\ \Delta(ijk|r) &\equiv \sigma_i\Delta(jkr) - \sigma_r\Delta(ijk) + \sigma_k\Delta(rij) - \sigma_j\Delta(kri). \end{aligned} \tag{695}$$

The Γ contour is defined by the $2n-3$ equations

$$\Lambda = 0, \quad S_d^\tau(\sigma, y) = 0, \quad C_a = 0, \tag{696}$$

for $d \neq \{i, j, k, r\}$ and $a = 1, ..., n$.

The integrand is given by

$$\mathcal{I}_n(\alpha) = -\mathrm{PT}^{\tau}(\alpha) \prod_{a=1}^{n} \frac{(y\sigma)_a}{y_a} \, \mathrm{PT}^{T}(m,p) \, \det[(\mathsf{A}^{\Lambda})^{mp}_{mp}], \tag{697}$$

where $(y\sigma)_a \equiv y_a + \sigma_a$. To obtain the kinematic matrix and the Parke-Taylor factors we need to do the following replacements

$$\mathsf{A} \to \mathsf{A}^{\Lambda}, \text{ and } \mathrm{PT} \to \mathrm{PT}^{T} \quad \text{for} \quad z_{ab} \to T_{ab}^{-1}, \tag{698}$$

$$\mathrm{PT} \to \mathrm{PT}^{\tau} \qquad\qquad \text{for} \quad z_{ab} \to \tau(a,b)^{-1}, \tag{699}$$

with $T_{ab} \equiv \frac{1}{(y\sigma)_a - (y\sigma)_b}$. Analogous to eq. (690), we can now write down a new form for the integrand

$$\mathcal{I}'_n(\alpha) = \mathrm{PT}^{\tau}(\alpha) \prod_{a=1}^{n} \frac{(y\sigma)_a}{y_a} (-1)^{i+k} T_{ij} T_{jk} \det[(\mathsf{A}^{\Lambda})^{ij}_{jk}], \tag{700}$$

where $\{i, j, k\}$ are the same labels as in $\Delta(ijk)\,\Delta(ijk|r)$. For more details on the double-cover prescription, see refs. [5, 372, 374].

14.4 FACTORIZATION

Let us start by considering the four-point amplitude, $A'_4(1,2,3,4)$, with the gauge fixing $(ijk|r) = (123|4)$. We will denote sums of cyclically-consecutive external momenta (modulo the total number of external momenta) by $P_{i:j} \equiv k_i + k_{i+1} + \ldots + k_{j-1} + k_j$,. For expressions involving only two (not necessarily consecutive) momenta, we are using the shorthand notation $P_{ij} \equiv k_i + k_j$. We focus on the configuration where the sets of punctures (σ_1, σ_2) and (σ_3, σ_4) are respectably on the upper and the lower sheet of the curves

$$\begin{aligned} &\left(y_1 = +\sqrt{\sigma_1^2 - \Lambda^2}, \sigma_1\right), \quad \left(y_2 = +\sqrt{\sigma_2^2 - \Lambda^2}, \sigma_2\right), \\ &\left(y_3 = -\sqrt{\sigma_3^2 - \Lambda^2}, \sigma_3\right), \quad \left(y_4 = -\sqrt{\sigma_4^2 - \Lambda^2}, \sigma_4\right). \end{aligned} \tag{701}$$

Expanding all elements in $A'_4(1,2,3,4)$ around $\Lambda = 0$, we obtain (to leading order)

$$\mathrm{PT}^{\tau}(1,2,3,4)\Big|^{1,2}_{3,4} = \frac{\Lambda^2}{2^2}\frac{1}{(\sigma_{12}\sigma_{2P_{34}}\sigma_{P_{34}1})}\frac{1}{(\sigma_{P_{12}3}\sigma_{34}\sigma_{4P_{12}})},$$

$$\frac{\Delta(123)\Delta(123|4)}{S_4^{\tau}}\Big|^{1,2}_{3,4} = \frac{2^5}{\Lambda^4}(\sigma_{12}\,\sigma_{2P_{34}}\,\sigma_{P_{34}1})^2\left(\frac{1}{s_{34}}\right)\times(\sigma_{P_{12}3}\,\sigma_{34}\,\sigma_{4P_{12}})^2, \tag{702}$$

$$\prod_{a=1}^{4}\frac{(y\sigma)_a}{y_a}\,T_{12}T_{23}\,\det[(\mathsf{A}^{\Lambda})^{12|}_{23|}]\Big|^{1,2}_{3,4} = \frac{\Lambda^2}{2^2}\frac{1}{\sigma_{12}\,\sigma_{2P_{34}}}\frac{s_{14}}{\sigma_{P_{34}1}}\frac{1}{\sigma_{P_{12}3}}\frac{(-1)\,s_{34}}{\sigma_{34}\,\sigma_{4P_{12}}}, \tag{703}$$

where we have introduced the new fixed punctures $\sigma_{P_{34}} = \sigma_{P_{12}} = 0$. Since we want to arrive at factorization identity for non-linear sigma model amplitudes (inspired by previous work for Yang-Mills theories, see ref. [374]) we are now going to introduce polarizations associated with the punctures, $\sigma_{P_{34}} = \sigma_{P_{12}} = 0$, *i.e.* ϵ^M_{34} and ϵ^M_{12}. Thus,

$$\begin{aligned} s_{14} &= 2(k_1\cdot k_4) = 2\,(k_{1\,\mu}\times\eta^{\mu\nu}\times k_{4\,\nu}) \\ &= \sum_M(\sqrt{2}\,k_1\cdot\epsilon^M_{34})\times(\sqrt{2}\,k_4\cdot\epsilon^M_{12}), \end{aligned} \tag{704}$$

employing,

$$\sum_M \epsilon_i^{M\,\mu}\epsilon_j^{M\,\nu} = \eta^{\mu\nu}. \tag{705}$$

After separating the labels $\{1,2\}$ and $\{3,4\}$, it is simple to rearrange the eq. (702) as a product of two reduced determinants,

$$\frac{1}{\sigma_{12}\,\sigma_{2P_{34}}}\frac{(\sqrt{2}\,k_1\cdot\epsilon^M_{34})}{\sigma_{P_{34}1}} = \frac{1}{\sigma_{12}\,\sigma_{2P_{34}}}\det\left[\frac{\sqrt{2}\,k_1\cdot\epsilon^M_{34}}{\sigma_{P_{34}1}}\right],$$

$$\frac{(-1)}{\sigma_{P_{12}3}}\frac{s_{34}\,(\sqrt{2}\,k_4\cdot\epsilon^M_{12})}{\sigma_{34}\,\sigma_{4P_{12}}} = \frac{(-1)}{\sigma_{P_{12}3}}\det\begin{bmatrix}\frac{\sqrt{2}\,k_3\cdot\epsilon^M_{12}}{\sigma_{3P_{12}}} & \frac{s_{34}}{\sigma_{34}} \\ \frac{\sqrt{2}\,k_4\cdot\epsilon^M_{12}}{\sigma_{4P_{12}}} & 0\end{bmatrix},$$

therefore

$$\prod_{a=1}^{4}\frac{(y\sigma)_a}{y_a}\; T_{12}T_{23}\,\det[(\mathsf{A}^{\Lambda})^{12}_{23}]\Big|^{1,2}_{3,4} \quad = -\frac{\Lambda^2}{2^2}\times\sum_{M}\frac{1}{\sigma_{12}\,\sigma_{2P_{34}}}\times$$
$$\det\left[\frac{\sqrt{2}\,k_1\cdot\epsilon^M_{34}}{\sigma_{P_{34}1}}\right]\times\frac{(-1)}{\sigma_{P_{12}3}}\det\begin{bmatrix}\frac{\sqrt{2}\,k_3\cdot\epsilon^M_{12}}{\sigma_{3P_{12}}} & \frac{s_{34}}{\sigma_{34}}\\ \frac{\sqrt{2}\,k_4\cdot\epsilon^M_{12}}{\sigma_{4P_{12}}} & 0\end{bmatrix}. \tag{706}$$

The new matrices in eq. (706) can be obtained from the A matrix by replacing the off-shell momenta, P_{34} and P_{12}, by their corresponding off-shell polarization vectors,

$$\det[(\mathsf{A})^{12}_{2P_{34}}]\to\det\left[\frac{\sqrt{2}\,k_1\cdot\epsilon^M_{34}}{\sigma_{P_{34}1}}\right]\ \text{ for } P_{34}\to\frac{1}{\sqrt{2}}\epsilon^M_{34}, \tag{707}$$

$$\det[(\mathsf{A})^{P_{12}}_{3}]\to\det\begin{bmatrix}\frac{\sqrt{2}\,k_3\cdot\epsilon^M_{12}}{\sigma_{3P_{12}}} & \frac{s_{34}}{\sigma_{34}}\\ \frac{\sqrt{2}\,k_4\cdot\epsilon^M_{12}}{\sigma_{4P_{12}}} & 0\end{bmatrix}\ \text{ for } P_{12}\to\frac{1}{\sqrt{2}}\epsilon^M_{12}, \tag{708}$$

where the A matrix in eq. (707) is the 3×3 matrix related with the punctures $(\sigma_1,\sigma_2,\sigma_{P_{34}})$, while the matrix in eq. (708) corresponds to the punctures $(\sigma_{P_{12}},\sigma_3,\sigma_4)$.

Using the measure, $d\mu_4^{\Lambda}=\frac{1}{2^2}\frac{d\Lambda}{\Lambda}$, we now perform the Λ integral and the amplitude becomes

$$A'_4(1,2,3,4)\Big|^{1,2}_{3,4}= \tag{709}$$
$$\frac{1}{2}\sum_{M}\frac{A'_3(1,2,P^{\epsilon^M}_{34})\times A_3^{(P_{12}3)}(P^{\epsilon^M}_{12},3,4)}{s_{12}}=\frac{s_{14}}{2},$$

where the notation, $P_i^{\epsilon^M}$, means one must make the replacement, $P_i\to\frac{1}{\sqrt{2}}\epsilon_i^M$, and use eq. (705). The overall factor $1/2$ cancels out after summing over mirrored configurations, *i.e.*, $A'_4(1,2,3,4)\Big|^{1,2}_{3,4}+A'_4(1,2,3,4)\Big|^{3,4}_{1,2}=s_{14}$.

Following the integration rules in ref. [374], we also have the contribution (up to summing over mirrored configurations)

$$A'_4(1,2,3,4)\Big|_{2,3}^{4,1} = \tag{710}$$
$$\frac{1}{2}\sum_M \frac{A_3^{(1P_{23})}(1,P_{23}^{\epsilon^M},4)\times A'_3(P_{41}^{\epsilon^M},2,3)}{s_{14}} = \frac{s_{12}}{2}.$$

Thus, the final result is

$$A'_4(1,2,3,4) = \tag{711}$$
$$\sum_M \Bigg[\frac{A'_3(1,2,P_{34}^{\epsilon^M})\times A_3^{(P_{12}3)}(P_{12}^{\epsilon^M},3,4)}{s_{12}}$$
$$+\frac{A_3^{(1P_{23})}(1,P_{23}^{\epsilon^M},4)\times A'_3(P_{41}^{\epsilon^M},2,3)}{s_{14}}\Bigg] = -s_{13}.$$

The four-point amplitude is factorized in terms of three-point functions. The general three-point functions where some or all particles can be off-shell, are

$$A'_3(P_a,P_b,P_c) = s_{P_cP_a} = -(P_a^2-P_b^2+P_c^2), \tag{712}$$

$$\begin{aligned} A_3^{(P_a\,P_b)}(P_a,P_b,P_c) &= s_{P_bP_c}s_{P_cP_a} \\ &= (P_c^2-P_a^2+P_b^2)(P_a^2-P_b^2+P_c^2). \end{aligned} \tag{713}$$

Since the non-linear sigma model is a scalar theory it is an interesting proposition to consider longitudinal degrees of freedom only

$$\sum_L \epsilon_i^{L\,\mu}\epsilon_j^{L\,\nu} = \frac{k_i^\mu k_j^\nu}{k_i\cdot k_j}. \tag{714}$$

Doing so we arrive at the equation

$$
\begin{aligned}
&A_4'(1,2,3,4) = \\
&2\sum_L \Bigg[(-1)^3 \frac{A_3'(1,2,P_{34}^{\epsilon^L}) \times A_3^{(P_{12}3)}(P_{12}^{\epsilon^L},3,4)}{s_{12}} \\
&+(-1)^3 \frac{A_3'(P_{41}^{\epsilon^L},2,3) \times A_3^{(1P_{23})}(1,P_{23}^{\epsilon^L},4)}{s_{14}} \Bigg] = -s_{13}.
\end{aligned} \tag{715}
$$

Surprisingly, it is possible to generalize this equation to higher point amplitudes. Here the overall sign of each contribution depends of the number of points of the sub-amplitudes. In ref. [5], we will give more details on this phenomenon.

14.5 NEW RELATIONS

As will be shown in great detail elsewhere [5], using the double-cover prescription for a partial non-linear sigma model amplitude one is led to the following general formula where an n-point amplitude is factorized into a product of two (single-cover) lower-point amplitudes:

$$
\begin{aligned}
&A_n'(1,2,3,4,\ldots,n) = \\
&\sum_{i=4,\,M}^{n} \frac{A_{n-i+3}'\big(1,2,P_{3:i}^{\epsilon^M},i{+}1,...n\big)\, A_{i-1}^{(P_{i+1:2}3)}\big(P_{i+1:2}^{\epsilon^M},3,...i\big)}{P_{i+1:2}^2} \\
&+\sum_M \frac{A_3'\big(P_{4:1}^{\epsilon^M},2,3\big)\times A_{n-1}^{(1P_{23})}\big(1,P_{23}^{\epsilon^M},4,\ldots,n\big)}{P_{23}^2}.
\end{aligned} \tag{716}
$$

Here n is an even integer and we have used eq. (705). The above expression is valid using the Möbius and scale-invariance gauge choice $(ijk|r) = (123|4)$.

From the decomposition obtained by the double-cover method in eq. (716), we are able to write down a new factorization relation, where only longitudinal degrees of freedom contribute,

$$A'_n(1,2,3,4,\ldots,n) = 2\left[\sum_{i=4,\,L}^{n}(-1)^{i-1}\times \frac{A'_{n-i+3}(1,2,P^{\epsilon^L}_{3:i},i{+}1,\ldots n)\times A^{(P_{i+1:2}3)}_{i-1}(P^{\epsilon^L}_{i+1:2},3,\ldots i)}{P^2_{i+1:2}} + \sum_{L}(-1)^3\frac{A'_3(P^{\epsilon^L}_{4:1},2,3)\times A^{(1P_{23})}_{n-1}(1,P^{\epsilon^L}_{23},4,\ldots,n)}{P^2_{23}}\right], \tag{717}$$

where eq. (714) was used. We checked this formula up to ten points.

Since the above factorization relation includes only longitudinal contributions, we can rewrite it in a more elegant form, involving only the A'_q amplitudes. Using the definitions given in eqs. (690) and (691) and under the gauge fixing (ijk), with $i<j<k$, we have the following two identities [5]

$$\begin{aligned} A^{(ij)}_q(\ldots,P_i,\ldots) &= P_i^2\,A'_q(\ldots,P_i,\ldots), \;\; q=2m+1 \\ A^{(ij)}_q(\ldots,P_i,\ldots) &= -P_i^2\,A'_q(\ldots,P_i,\ldots), \;\;\; q=2m, \end{aligned} \tag{718}$$

where $P_i^2\neq 0$. In addition, $A^{(ij)}_q$ satisfies the useful identities

$$\begin{aligned} A^{(ij)}_q(1,\ldots,i,\ldots P_j,\ldots k\ldots q) &= A^{(jk)}_q(1,\ldots,i,\ldots P_j,\ldots k\ldots q), \\ A^{(ij)}_q(1\ldots,i,\ldots P_j,\ldots k\ldots q) &= A^{(ij)}_q(2\ldots i\ldots P_j\ldots k\ldots q,1) = \\ \cdots = A^{(jk)}_q(\ldots P_j\ldots k\ldots q\ldots i) &= A^{(ij)}_q(\ldots k\ldots q,1,\ldots i\ldots P_j\ldots). \end{aligned} \tag{719}$$

Applying the identities eqs. (718) and (719), it is straightforward to obtain

$$A'_n(1,2,3,4,\ldots,n) = \sum_{i=4}^{n}\frac{A'_{n-i+3}(1,2,P_{3:i},i{+}1,\ldots n)\times A'_{i-1}(P_{i+1:2},3,\ldots i)}{P^2_{i+1:2}} + \frac{A'_3(P_{4:1},2,3)\times A'_{n-1}(P_{23},4,\ldots,n,1)}{P^2_{23}}, \tag{720}$$

where the factorization formula has been written in terms of the generalized amplitude A'_q. Other gauge choices will naturally with lead to alternative factorization formulas.

14.5.1 *BCFW recursion*

It is interesting to analyse the new factorization identities in comparison with expressions originating from Britto-Cachazo-Feng-Witten (BCFW) recursion [304]. We introduce the momentum deformation

$$k_2^\mu(z) = k_2^\mu + z\, q^\mu, \quad k_3^\mu(z) = k_3^\mu - z\, q^\mu, \quad z \in \mathbb{C}, \tag{721}$$

where q^μ satisfies $k_2 \cdot q = k_3 \cdot q = q \cdot q = 0$. Deformed momenta are conserved and on-shell: $k_1 + k_2(z) + k_3(z) + k_4 + \cdots + k_n = 0$ and $k_2^2(z) = k_3^2(z) = 0$. We consider the general amplitude, $A_n(1, \ldots, n)$, where n is an even integer. From eq. (720) using Cauchy's theorem we have

$$\begin{aligned} &A_n(1,2,...,n) = \\ &-\sum_{i=3}^{n/2} \mathrm{Res}_{P^2_{2i:2}(z)=0}\Big[A'_{n-2i+4}(1,2,P_{3:2i-1},2i,...,n)\times \\ &\frac{A'_{2i-2}(P_{2i:2},3,...,2i-1)}{z\,P^2_{2i:2}(z)}\Big] - \mathrm{Res}_{z=\infty}\left[\frac{A'_n(1,2,...,n)(z)}{z}\right]. \end{aligned} \tag{722}$$

Only the even amplitudes, namely A'_{2q}, contribute to the physical residues. This is simple to understand as we have the identity, $A_{2q}(1, \ldots, 2q) = A'_{2q}(1, \ldots, 2q)$, so only sub-amplitudes with an even number of particles produce physical factorization channels. On the other hand, when the number of particles is odd, the off-shell ($P_i^2 \neq 0$) amplitude, $A'_{2q+1}(..., P_i, ...)$, is proportional to P_i^2, since it must vanish when all particles are on-shell. So, the poles, $P^2_{2i-1:2}$, $i = 3, ..., \frac{n}{2} + 1$ and P_{23}, are all spurious and the sub-amplitudes with an odd number of particles only contribute the boundary term at $z = \infty$.

Finally, it is important to remark that after evaluating the residues, $P^2_{2i:2}(z) = 0$, in eq. (722), one obtains extra non-physical contributions, which cancel out combining with terms associated with the residue at $z = \infty$. Therefore, the effective boundary contribution is just given by the sub-amplitudes with an odd number of particles

$$
\begin{aligned}
&\mathrm{Res}_{z=\infty}\left[\frac{A'_n(1,2,...,n)(z)}{z}\right]^{\text{Effective}} = \\
&\partial_{\frac{1}{z}}\left[\sum_{i=3}^{n/2+1} A'_{n-2i+5}(1,2,P_{3:2i-2},2i-1,...,n)\right. \\
&\times \frac{A'_{2i-3}(P_{2i-1:2},3,...,2i-2)}{z\,P^2_{2i-1:2}(z)} + \\
&\left.\frac{A'_3(P_{4:1},2,3)\times A'_{n-1}(P_{23},4,\ldots,n,1)}{z\,P^2_{23}}\right]_{z=\infty}.
\end{aligned}
\tag{723}
$$

14.6 CONCLUSIONS

We have proposed a new CHY integrand for the $U(N)$ non-linear sigma model. For this new integrand, the kinematic matrix, $(\mathsf{A})^{ij}_{jk}$, is no longer anti-symmetric. We have found two new factorization identities, eq. (716) and eq. (717). We have written the second factorization formula in an elegant way, which only involves the generalized amplitude, A'_q. This formula turns out to be surprisingly compact (we have checked agreement of the soft-limit of this formula with ref.[259]).

This has implications for BCFW recursion since the two new factorization formulas can be split among even and odd sub-amplitudes, for example $A'_{2q}\times A'_{2m}$ and $A'_{2q+1}\times A'_{2m+1}$ respectively. Using this we are able to give a physical meaning to the odd sub-amplitudes as boundary contributions under such recursions.

Work in progress [5] is going to present a new recurrence relation and investigate its connection to Berends-Giele [243, 375–378] currents and Bern-Carrasco-Johansson (BCJ) numerators [238, 379, 380]. Similar relations for others effective field theories [246, 259, 375] are expected and will be another focus.

Despite similarities between the three-point amplitudes with the Feynman vertices obtained in ref. [262], the construction presented here is different. For example, the numerators found in eq. (711) are not reproduced by the Feynman rules found in ref. [262]. Understanding the relationship between the formalisms would be interesting.

15

SCATTERING EQUATIONS AND FACTORIZATION OF AMPLITUDES II: EFFECTIVE FIELD THEORIES

We continue the program of extending the scattering equation framework by Cachazo, He and Yuan to a double-cover prescription. We discuss how to apply the double-cover formalism to effective field theories, with a special focus on the non-linear sigma model. A defining characteristic of the double-cover formulation is the emergence of new factorization relations. We present several factorization relations, along with a novel recursion relation. Using the recursion relation and a new prescription for the integrand, any non-linear sigma model amplitude can be expressed in terms of off-shell three-point amplitudes. The resulting expression is purely algebraic, and we do not have to solve any scattering equation. We also discuss soft limits, boundary terms in BCFW recursion, and application of the double cover prescription to other effective field theories, like the special Galileon theory.

15.1 INTRODUCTION

The S-matrix elements of gravity, gauge theories and various scalar theories can be calculated using the novel scattering equation framework by Cachazo, He and Yuan (CHY) [348–350]. The n-point scattering amplitude in the CHY-formalism is expressed as contour integrals localized to the solutions of the scattering equations

$$S_a = 0, \qquad \text{where} \quad S_a = \sum_{b \neq a} \frac{s_{ab}}{z_{ab}}, \tag{724}$$

with $z_{ab} = z_a - z_b$ and z_a are auxiliary variables on the Riemann sphere. Unless otherwise specified, we let $a, b \in \{1, \ldots, n\}$. The momentum of the a^{th} external particle is k_a^μ and $s_{ab} = 2k_a \cdot k_b$ are the usual Mandelstam variables. The scattering equations are invariant under $\mathrm{PSL}(2,\mathbb{C})$ transformations of the variables,

$$z_a \to z_a' = \frac{Az_a + B}{Cz_a + D}, \qquad \text{where} \quad AD - BC = 1, \tag{725}$$

using overall momentum conservation, $\sum k_a = 0$, and the massless condition, $k_a^2 = 0$. This means that if z_a is a solution to eq. (724), then so is z_a'. Thus, only $(n-3)$ of the scattering equations are independent, which can be seen from the fact that

$$\sum_a S_a = \sum_a z_a S_a = \sum_a z_a^2 S_a = 0. \tag{726}$$

There is a redundancy in the integration variables which needs to be fixed, similar to how gauge redundancy is fixed. We choose three of the integration variables to be fixed, leaving $(n-3)$ unfixed variables, which are integrated over. Thus, the number of integration variables and the number of constraints from the scattering equations are equal, which fully localizes the integral to the solutions of the scattering equations. However, the number of independent solutions to the scattering equations is $(n-3)!$, and it becomes impractical to deal with them when n is not small. The computational cost becomes huge when the number of external particles increases. Integration rules have been developed to circumvent this problem, both at tree [367–370, 381, 382] and loop level [371], where no scattering equation has to be explicitly solved. A formal proof of the CHY-formalism was provided in Ref. [366]. See also Ref. [383].

Recently, one of us extended the scattering equation formalism to a double cover of the Riemann sphere (called the Λ-algorithm in Refs. [372, 373, 384, 385]). The auxiliary double-cover variables live in $\mathbb{CP}^2$, contrasted with the original auxiliary variables z_a, which live in $\mathbb{CP}^1$ in the standard CHY formulation. More precisely, we consider curves in $\mathbb{CP}^2$ defined by

$$C_a \equiv y_a^2 - \sigma_a^2 + \Lambda^2 = 0, \tag{727}$$

where Λ is a non-zero constant. This curve is invariant under a simultaneous scaling of the parameters y, σ, Λ. In the new double-cover formulation, the punctures on the Riemann sphere are given by the pair (σ_a, y_a). As eq. (727) is a quadratic equation, two branches develop. The value of y_a specifies which branch the solution is on. To make sure we pick up the puncture on the correct branch, the scattering equations have to be modified

$$\tilde{S}_a^\tau(\sigma, y) = \sum_{b \neq a} \frac{1}{2}\left(\frac{y_b}{y_a} + 1\right)\frac{s_{ab}}{\sigma_{ab}}, \tag{728}$$

where $\sigma_{ab} = \sigma_a - \sigma_b$. The factor $\frac{1}{2}\left(\frac{y_b}{y_a} + 1\right)$ projects out the solution where y_b approaches $-y_a$, and gives 1 when y_b approaches y_a. Another (equivalent) way of defining the double cover scattering equations is to postulate the map

$$S_a(z) = \sum_{a \neq b} \frac{s_{ab}}{z_{ab}} \to S_a^\tau(\sigma, y) = \sum_{a \neq b} s_{ab}\tau_{(a,b)}, \qquad \text{where} \quad \tau_{(a,b)} = \frac{1}{2\sigma_{ab}}\left(\frac{y_a + y_b + \sigma_{ab}}{y_a}\right). \tag{729}$$

It is easy to check that the two prescriptions for the double cover scattering equations are equivalent by using overall momentum conservation and the on-shell condition. The map $z_{ij} \to \tau_{(i,j)}^{-1}$ will be useful later when we define the double cover integrand. For a full formulation of the double-cover prescription, see Ref. [372].

In the double cover prescription, three variables need to be fixed due to Möbius invariance. In addition, the integrand is invariant under a scale transformation. This gives an additional redundancy which needs to be fixed (as the integrand is $\mathrm{PSL}(2, \mathbb{C})$ and scale invariant, *i.e.* $\mathrm{GL}(2, \mathbb{C})$ invariant). Using the scale symmetry, we fix an extra puncture, and promote Λ to a variable and include a scale invariant measure $\frac{d\Lambda}{\Lambda}$. Using the global residue theorem, we can deform the integration contour to go around $\Lambda = 0$ instead of the solution to the scattering equation for the puncture fixed by the scale symmetry. This scattering equation is left free. Thus, in the double-cover prescription we gauge fix four points, three from the usual gauge fixing procedure, and one from the scale transformation.

The two sheets of the Riemann sphere are separated by a branch cut, and by integrating over Λ, lead to the factorization into two regular lower-point CHY amplitudes. This is the origin of the new factorization relations which we will discuss in the main part of this paper. By iteratively promoting the scattering amplitudes to the double-cover formulation, and using certain matrix identities, any n-point scattering amplitude for the non-linear sigma model can be fully factorized into off-shell three-point amplitudes.

This paper is organized as follows. In Section 15.2 we formulate the non-linear sigma model amplitudes in the usual CHY formalism. In Section 15.3 we introduce the double-cover prescription for effective field theories. In Section 15.4 we describe the graphical representations for the scattering amplitudes in the double-cover formalism. In Section 15.5 we list the double-cover integration rules. In Section 15.6 we define the three-point functions which will serve as the building blocks for higher-point amplitudes. In Sections 15.7 and 15.8 we present the new factorization formulas for the non-linear sigma model. In Section 15.9 we present a novel recursion relation, which fully factorizes the non-linear sigma model amplitudes in terms of off-shell three-point amplitudes. This is one of the main results of the paper. Section 15.10 takes the soft limit of the non-linear sigma model amplitudes, and presents a new relation for NLSM $\oplus\, \phi^3$ amplitudes. In Section 15.11 we apply the double-cover prescription to the special Galileon theory. We end with conclusions and outlook in Section 15.12. The Sections 15.A and 15.B contain matrix identities and details of the six-point calculation.

15.2 CHY FORMALISM

We briefly review the construction of non-linear sigma model (NLSM) scattering amplitudes in the CHY formalism to fix notation. The flavor-ordered partial $\mathrm{U}(N)$ amplitude for the non-linear sigma model in the scattering equation framework is defined by the integral

$$\mathcal{A}_n(\alpha) = \int \mathrm{d}\mu_n^{\mathrm{CHY}} (z_{pq} z_{qr} z_{rp})^2 H_n(\alpha), \tag{730}$$

$$\mathrm{d}\mu_n^{\mathrm{CHY}} = \prod_{a=1, a\neq p,q,r}^{n} \frac{\mathrm{d}z_a}{S_a}, \tag{731}$$

where a partial ordering is denoted by $(\alpha) = (\alpha_1, \ldots, \alpha_n)$. We have fixed the punctures $\{z_p, z_q, z_r\}$. The integrand is given by the Parke-Taylor factor $\mathrm{PT}(\alpha)$ and the reduced Pfaffian of the matrix A_n, $\mathrm{Pf}'\mathsf{A}_n$,

$$H_n(\alpha) = \mathrm{PT}(\alpha)\ \left(\mathrm{Pf}'\mathsf{A}_n\right)^2, \tag{732}$$

$$\mathrm{PT}(\alpha) = \frac{1}{z_{\alpha_1\alpha_2} z_{\alpha_2\alpha_3} \ldots z_{\alpha_n\alpha_1}}, \tag{733}$$

$$\left(\mathrm{Pf}'\mathsf{A}_n\right)^2 = \frac{(-1)^{i+j+l+m}}{z_{ij} z_{lm}}\ \mathrm{Pf}\left[(\mathsf{A}_n)^{ij}_{ij}\right] \times \mathrm{Pf}\left[(\mathsf{A}_n)^{lm}_{lm}\right]. \tag{734}$$

The matrix A_n is $n \times n$ and antisymmetric,

$$(\mathsf{A}_n)_{ab} = \begin{cases} \frac{s_{ab}}{z_{ab}} & \text{for} \quad a \neq b \\ 0 & \text{for} \quad a = b. \end{cases} \tag{735}$$

We will in general denote a reduced matrix by $(\mathsf{A}_n)^{i_1 \cdots i_p}_{j_1 \cdots j_p}$, where we have removed rows $\{i_1, \ldots, i_p\}$ and columns $\{j_1, \ldots, j_p\}$ from the matrix A_n. As an example, we can remove rows $\{i, j\}$ and columns $\{j, k\}$ from A_n in eq. (735), denoted by $(\mathsf{A}_n)^{ij}_{jk}$.

With the conventional choice $\{l, m\} = \{i, j\}$, the product of Pfaffians turns into a determinant

$$\left(\mathrm{Pf}'\mathsf{A}_n\right)^2 = -\mathrm{PT}(i,j) \det\left[(\mathsf{A}_n)^{ij}_{ij}\right]. \tag{736}$$

We will denote the amplitude with this choice by

$$A_n(\alpha) = -\int \mathrm{d}\mu_n^{\mathrm{CHY}} (z_{pq} z_{qr} z_{rp})^2 \, \mathrm{PT}(\alpha) \, \mathrm{PT}(i,j) \det\left[(\mathsf{A}_n)^{ij}_{ij}\right]. \tag{737}$$

We can make a different choice, specifically $\{l, m\} = \{j, k\}$. We will make use of the matrix identities

$$\mathrm{Pf}\left[(\mathsf{A}_n)^{ij}_{ij}\right] \times \mathrm{Pf}\left[(\mathsf{A}_n)^{jk}_{jk}\right] = \det\left[(\mathsf{A}_n)^{ij}_{jk}\right], \tag{738}$$

$$\det\left[(\mathsf{A}_n)^{ij}_{jk}\right] = 0 \qquad \text{if } n \text{ is odd.} \tag{739}$$

Equation (739) depends on momentum conservation and the massless condition. A proof of the matrix identities in eqs. (738) and (739) is found in section 15.A. The amplitude with this new choice is denoted by

$$A'_n(\alpha) = \int \mathrm{d}\mu_n^{\mathrm{CHY}} (z_{pq} z_{qr} z_{rp})^2 \, \mathrm{PT}(\alpha) \, \frac{(-1)^{i+k}}{z_{ij} z_{jk}} \det\left[(\mathsf{A}_n)^{ij}_{jk}\right]. \tag{740}$$

This definition differs from the conventional one, and will be of great practical use in the following [386]. It will often be useful to remove columns and rows from the set of fixed punctures. For the objects in eqs. (737) and (740), we will encode the information of which rows and columns are removed in the labeling of the partial ordering α. When removing columns and rows (i, j), we bold the corresponding elements in the partial ordering, *i.e.* $A_n(\ldots, \boldsymbol{i}, \ldots, \boldsymbol{j}, \ldots)$. For the new prescription, the choice (ijk) is labeled by $A'_n(\ldots,\boldsymbol{i},\ldots,\boldsymbol{j},\ldots,\boldsymbol{k},\ldots)$, where the set is chosen to be ordered as $i < j < k$. Unless otherwise specified, we assume the set of removed rows and columns are in the two or three first positions, *i.e.* $A_n = A_n(\boldsymbol{i},\boldsymbol{j},\ldots)$ and $A'_n = A'_n(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k},\ldots)$. In this case, we will suppress the bold notation. For an odd number of external particles n, $\det\left[(\mathsf{A}_n)^{ij}_{ij}\right] = \det\left[(\mathsf{A}_n)^{ij}_{jk}\right] = 0$, and the amplitudes vanish.

When evaluating the double cover amplitudes, it will be necessary to relax the requirement of masslessness, as the full amplitude is splits into off-shell lower-point amplitudes. The off-shell punctures are part of the set of fixed punctures. We will also use the object

$$A_n^{(ij)}(\alpha) = \int \mathrm{d}\mu_n^{\mathrm{CHY}} (z_{pq}z_{qr}z_{rp})^2\, \mathrm{PT}(\alpha)\, \frac{(-1)^{i+j}}{z_{ij}} \det\left[(\mathsf{A}_n)^i_j\right]. \tag{741}$$

As the matrix A_n has co-rank 2 on the support of the massless condition and the scattering equations, $\{k_a^2 = 0, S_a = 0\}$, $A_n^{(ij)}(\alpha)$ vanishes trivially. However, when some of the particles are off-shell, $A_n^{(ij)}(\alpha)$ is non-zero in general. Similarly, the object $A'_n(\alpha)$ is non-zero for odd number of particles, if and only if some of the particles are off-shell.

15.3 EFFECTIVE FIELD THEORIES IN THE DOUBLE-COVER PRESCRIPTION

In Ref. [386], it was argued that the n-point NLSM scattering amplitude in the double-cover language is given by the integral

$$\mathcal{A}_n^{\mathrm{NLSM}}(\alpha) = \int_\Gamma \mathrm{d}\mu_n^\Lambda\, \frac{(-1)\Delta(pqr)\Delta(pqr|m)}{S_m^\tau}\, \mathcal{I}_n^{\mathrm{NLSM}}(\alpha), \tag{742}$$

$$\mathrm{d}\mu_n^\Lambda = \frac{1}{2^2}\frac{\mathrm{d}\Lambda}{\Lambda}\prod_{a=1}^n \frac{y_a \mathrm{d}y_a}{C_a} \prod_{d=1,d\neq p,q,r,m}^n \frac{\mathrm{d}\sigma_d}{S_d^\tau}, \tag{743}$$

$$\Delta(pqr) = \frac{1}{\tau_{(p,q)}\tau_{(q,r)}\tau_{(r,p)}}, \tag{744}$$

$$\Delta(pqr|m) = \sigma_p\Delta(qrm) - \sigma_q\Delta(rmp) + \sigma_r\Delta(mpq) - \sigma_m\Delta(pqr). \tag{745}$$

In this section we will include a superscript to denote the amplitudes. In the rest of the paper we keep this superscript implicit. When not otherwise specified, an amplitude without a superscript refers to an NLSM amplitude. The integration contour Γ is constrained by the $(2n-3)$ equations

$$\Lambda = 0, \qquad S_d^\tau(\sigma, y) = 0, \qquad C_a = 0, \tag{746}$$

for $d \neq \{p, q, r, m\}$ and $a = 1, \ldots, n$.

In a similar fashion, one can obtain the expressions for the NLSM $\oplus\, \phi^3$ and special Galileon amplitudes, *i.e.* for $A_n^{\text{NLSM}\oplus\phi^3}(\alpha|\beta)$ and A_n^{sGal}, by specifying the integrand. The integrands in the double-cover scattering equation framework for the NLSM, NLSM $\oplus\, \phi^3$ and special Galileon theory are given by the expressions

$$\mathcal{I}_n^{\text{NLSM}}(\alpha) = \text{PT}^{\tau}(\alpha) \times \mathbf{det}' \mathsf{A}_n^{\Lambda}\,, \tag{747}$$

$$\mathcal{I}_n^{\text{NLSM}\oplus\phi^3}(\alpha||\beta) = \text{PT}^{\tau}(\alpha) \left(\left[\prod_{a=1}^{n} \frac{(y\sigma)_a}{y_a} \right] \text{PT}^{T}(\beta)\, \text{det} \left[\mathsf{A}_n^{\Lambda} \right]_{\beta_1 \ldots \beta_p}^{\beta_1 \ldots \beta_p} \right), \tag{748}$$

$$\mathcal{I}_n^{\text{sGal}} = \mathbf{det}' \mathsf{A}_n^{\Lambda} \times \mathbf{det}' \mathsf{A}_n^{\Lambda}\,, \tag{749}$$

where $(y\sigma)_a \equiv y_a + \sigma_a$. The bold reduced determinant is defined as

$$\mathbf{det}' \mathsf{A}_n^{\Lambda} = - \left[\prod_{a=1}^{n} \frac{(y\sigma)_a}{y_a} \right] \text{PT}(i,j) \text{det} \left[\mathsf{A}_n^{\Lambda} \right]_{ij}^{ij} \tag{750}$$

$$= \left[\prod_{a=1}^{n} \frac{(y\sigma)_a}{y_a} \right] (-1)^{i+k} T_{ij} T_{jk} \text{det} \left[\mathsf{A}_n^{\Lambda} \right]_{jk}^{ij}, \tag{751}$$

where the second equality is used to define the A' amplitude in the double cover language, similar to eq. (740). The Parke-Taylor factors and the kinematic matrix are defined by the following replacement

$$\mathsf{A}_n \to \mathsf{A}_n^{\Lambda} \qquad \text{for} \quad z_{ab} \to T_{ab}^{-1}, \tag{752}$$

$$\text{PT} \to \text{PT}^{T} \qquad \text{for} \quad z_{ab} \to T_{ab}^{-1}, \tag{753}$$

$$\text{PT} \to \text{PT}^{\tau} \qquad \text{for} \quad z_{ab} \to \tau_{(a,b)}^{-1}, \tag{754}$$

where $T_{ab}^{-1} = (y\sigma)_a - (y\sigma)_b$.

Notice that the generalization to theories such as sGal $\oplus$ NLSM^2 $\oplus\, \phi^3$ or Born-Infeld theory, among others, is straightforward [246, 259, 375].

15.3.1 *The* Π *Matrix*

Most integrands in the CHY approach depend on the auxiliary variable z_i through the combination $z_{ij} = z_i - z_j$. As shown in eqs. (752) to (754), we can construct the double cover integrand by replacing z_{ij} with T_{ij}^{-1} or $\tau_{(i,j)}^{-1}$.[1] This makes for an easy map between the traditional CHY approach and the new double cover method for most integrands.

However, the Π matrix, defined in Refs. [246, 259, 375], has elements such as, $\frac{z_a\, k_a\cdot k_b}{z_{ab}}$, which so far have not been studied in the double cover framework. Explicitly, the $\Pi_{\beta_1,\beta_2,\ldots,\beta_m}$ matrix, defined in Ref. [375], is

$$\Pi_{\beta_1,\ldots,\beta_m} = \begin{array}{cc} \begin{array}{cccc} j\in\overline{\beta} & b\in\{\beta_1,\ldots,\beta_m\} & j\in\overline{\beta} & b'\in\{\beta_1,\ldots,\beta_m\} \end{array} & \\ \left(\begin{array}{c|c|c|c} \mathsf{A}_{ij} & \Pi_{ib} & \mathsf{A}_{ij} & \Pi_{ib'} \\ \hline \Pi_{aj} & \Pi_{ab} & \Pi_{aj} & \Pi_{ab'} \\ \hline \mathsf{A}_{ij} & \Pi_{ib} & 0 & \Pi_{ib'} \\ \hline \Pi_{a'j} & \Pi_{a'b} & \Pi_{a'j} & \Pi_{a'b'} \end{array}\right) & \begin{array}{c} i\in\overline{\beta} \\ a\in\{\beta_1,\ldots,\beta_m\} \\ i\in\overline{\beta} \\ a'\in\{\beta_1,\ldots,\beta_m\} \end{array} \end{array} .$$

Here, the β_a's sets are such that $\beta_a\cap\beta_b=\emptyset$, $a\neq b$, and $\overline{\beta}$ is the complement, namely, $\overline{\beta}=\{1,2,\ldots,n\}\setminus\beta_1\cup\beta_2\cup\cdots\cup\beta_m$, where n is the total number of particles. The Π submatrices are given by the expressions

$$\Pi_{ib}=\sum_{c\in\beta_b}\frac{k_i\cdot k_c}{z_{ic}},\quad \Pi_{ib'}=\sum_{c\in\beta_b}\frac{z_c\, k_i\cdot k_c}{z_{ic}},\quad \Pi_{ab}=\sum_{\substack{c\in\beta_a\\ d\in\beta_b}}\frac{k_c\cdot k_d}{z_{cd}},$$
$$\Pi_{ab'}=\sum_{\substack{c\in\beta_a\\ d\in\beta_b}}\frac{z_d\, k_c\cdot k_d}{z_{cd}},\quad \Pi_{a'b'}=\sum_{\substack{c\in\beta_a\\ d\in\beta_b}}\frac{z_c\, z_d\, k_c\cdot k_d}{z_{cd}}. \tag{755}$$

As shown in Refs. [374, 386], to obtain the usual CHY matrices in the double-cover prescription we use the identification $\frac{1}{z_{ab}}\to T_{ab}=\frac{1}{(y_a+\sigma_a)-(y_b+\sigma_b)}$ (see the above section), which gives us the naive identification $z_a\to(y_a+\sigma_a)$. However, we need all elements of $\Pi_{\beta_1,\ldots,\beta_m}$ to transform in the same way under a global scaling $(y_1,\sigma_1,\ldots,y_n,\sigma_n,\Lambda)\to\rho\,(y_1,\sigma_1,\ldots,y_n,\sigma_n,\Lambda)$, $\rho\in\mathbb{C}^*$. We

[1] Of course, the measure is also redefined in the double cover prescription.

use the map[2] $z_a \to \frac{(y_a+\sigma_a)}{\Lambda}$. Thus, the Π matrix in the double-cover representation is given by the replacement,

$$\Pi^{\Lambda}_{\beta_1,\beta_2,\ldots,\beta_m} \equiv \Pi_{\beta_1,\beta_2,\ldots,\beta_m} \qquad \text{for} \quad \frac{1}{z_{ab}} \to T_{ab}, \quad z_a \to \frac{(y\sigma)_a}{\Lambda}. \tag{756}$$

The multi-trace amplitude for interactions among NLSM pions and bi-adjoint scalars is given by the integrand [375]

$$\mathcal{I}_n^{\text{NLSM}\oplus BA}(\alpha||\beta_1|\cdots|\beta_m) = \text{PT}^{\tau}(\alpha) \times \left(\left[\prod_{a=1}^{n} \frac{(y\sigma)_a}{y_a}\right] \times \text{PT}^{T}(\beta_1)\ldots\text{PT}^{T}(\beta_m) \times \text{Pf}'\left[\Pi^{\Lambda}_{\beta_1\ldots\beta_p}\right] \right).$$

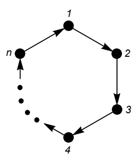
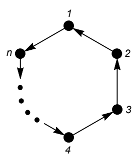
The integrand is the defined using eqs. (753), (754) and (756). The reduced Pfaffian is defined as

$$\text{Pf}'\left[\Pi^{\Lambda}_{\beta_1\ldots\beta_p}\right] = \text{Pf}\left[(\Pi^{\Lambda}_{\beta_1\ldots\beta_p})^{ab'}_{ab'}\right]. \tag{757}$$

15.4 GRAPHICAL REPRESENTATION

The graphical representation for effective field theory amplitudes in the double-cover prescription is analogous to one presented in Ref. [374]. The only difference is that we are going to work with determinants instead of Pfaffians. We will briefly review the graphical notation used in this paper.

First, the Parke-Taylor factor is drawn by a sequence of arrows joining vertices. The orientation of the arrow represents the ordering,

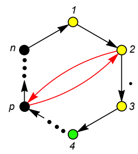
$\text{PT}^{\tau}(1,\ldots,n) =$  $= (-1)^n \times$  $= (-1)^n \times \text{PT}^{\tau}(n,\ldots,1)\,.$ (758)

To describe the half-integrand $(-1)\left[\prod_{a=1}^{n}\frac{(y\sigma)_a}{y_a}\right](T_{ij}T_{ji})\det[(\text{A}^{\Lambda}_n)^{ij}_{ij}]$, we recall how the Pfaffian in Yang-Mills theory was represented [374]. In YM, the half-integrand $(-1)^{i+j}\left[\prod_{a=1}^{n}\frac{(y\sigma)_a}{y_a}\right](T_{ij})\text{Pf}[(\Psi^{\Lambda}_n)^{ij}_{ij}]$ was represented by a red arrow from $i \to j$. We associate this red arrow with the factor T_{ij} of the reduced Pfaffian. In the case of NLSM, we draw two red arrows, $i \rightleftarrows j$, for the factor $T_{ij}T_{ji}$ of the reduced determinant. With the new definition of the NLSM integrand, $(-1)^{i+k}\left[\prod_{a=1}^{n}\frac{(y\sigma)_a}{y_a}\right] T_{ij}T_{jk}\det[(\text{A}^{\Lambda}_n)^{ij}_{jk}]$, we draw two red arrows, $i \to j \to k$.

[2]This is in agreement with the single and double-cover equivalence given in Ref. [372].

If we choose to fix the punctures $(pqr|m) = (123|4)$ and reduce the determinant with $(i,j) = (2,p)$, we can graphically represent the NLSM amplitude $A_n(\alpha)$ by an *NLSM-graph*,

$$A_n(1,\mathbf{2},3,4,...,\mathbf{p},...,n) = \int d\mu_n^\Lambda$$



Recall that the removed columns and rows (i,j) are written in bold in the partial ordering. The notation for the fixed punctures by yellow, green and red vertices is the same as in Ref. [374]. When all particles are on-shell, the expression is independent of the choice of fixed punctures and reduced determinant. However, as we shall see later, when we have off-shell particles, the expression depends on the choices.

Lastly, the following two properties

$$\begin{aligned}
A_n(1,\mathbf{2},3,4,...,\mathbf{p},...,n) &= A_n(\text{cyc}(1,\mathbf{2},3,4,...,\mathbf{p},...,n)),\\
A_n(1,\mathbf{2},3,4,...,\mathbf{p},...,n) &= (-1)^n\, A_n(n,...,\mathbf{p},...,4,3,\mathbf{2},1)\,,
\end{aligned} \tag{759}$$

are satisfied even if some of the particles are off-shell. The graphical representation for other effective field theories are similar. Also, the double-cover representation reduces to the usual CHY representation when the green vertex is replaced by a black vertex.

15.5 THE DOUBLE-COVER INTEGRATION RULES

We will formulate the double-cover integration rules, applicable for the effective field theory amplitudes for the NLSM and special Galileon theory (sGal). Generalizing the integration rules to other effective field theories is straightforward. The integration rules share a strong resemblance to the Yang-Mills integration rules given in Ref. [374].

The integration of the double-cover variables y_a localizes the integrand to the curves $C_a = 0$, with the solutions $y_a = \pm\sqrt{\sigma_a^2 - \Lambda^2}$, $\forall\, a$. The double cover splits into an upper and a lower Riemann sheet, connected by a branch-cut, defined by the branch-points $-\Lambda$ and Λ. The punctures are distributed among the two sheets in all 2^n possible combinations.[3] When performing the integration of Λ, the two sheets factorize into two single covers connected by an off-shell propagator (the scattering equation S_m^τ in eq. (742) reduces to the off-shell propagator under the Λ integration). On each of the two lower-point single covers three punctures need to be fixed due to the PSL$(2,\mathbb{C})$ redundancy. The

[3] Only 2^{n-1} configurations are distinct, due to a $\mathbb{Z}_2$ symmetry.

branch-cut closes to a point when $\Lambda \to 0$, which becomes an off-shell particle. The corresponding puncture is fixed. In addition, two more punctures need to be fixed on each of the sheets. These fixed punctures must come from the fixed punctures in the original double cover (graphically represented by colored vertices, yellow or green). If there is not exactly two colored vertices on each of the new single covers, the configuration vanishes. We summarize this in the first integration rule [372, 374];

- **Rule-I.** *All configurations (or cuts) with fewer (or more) than two colored vertices (yellow or green) vanish trivially.*

The first integration rule, **Rule-I**, is general for any theory formulated in a double-cover language. In addition, we need to formulate supplementary integration rules specific to the NLSM and special Galileon amplitudes.

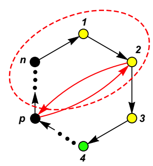
We start by determining how different parts of the integrand (and the measure) scale with Λ. Without loss of generality, consider a configuration where the punctures $\{\sigma_{p+1}, \ldots, \sigma_n, \sigma_1, \sigma_2\}$ are located on the upper sheet, and the punctures $\{\sigma_3, \sigma_4, \ldots, \sigma_p\}$ are located on the lower sheet. This configuration (or cut) will be graphically represented by a dashed red line, which separates the two sets. **Rule-I** forces two of the fixed punctures to be on the upper sheet, and the other two to be on the lower sheet. By expanding around $\Lambda = 0$, the measure and the Faddeev-Popov determinants become

$$
\begin{aligned}
d\mu_n^{\Lambda}\Big|_{3,4,\ldots,p}^{p+1,\ldots,1,2} &= \frac{d\Lambda}{\Lambda} \times \left[\frac{d\sigma_{p+1}}{S_{p+1}} \cdots \frac{d\sigma_n}{S_n}\right] \times \left[\frac{d\sigma_5}{S_5} \cdots \frac{d\sigma_p}{S_p}\right] + \mathcal{O}(\Lambda) \\
&= \frac{d\Lambda}{\Lambda} \times d\mu^{\text{CHY}}_{n-(p-2)+1} \times d\mu^{\text{CHY}}_{(p-2)+1} + \mathcal{O}(\Lambda),
\end{aligned} \tag{760}
$$

$$
\frac{\Delta(123)\Delta(123|4)}{S_4^{\tau}}\Big|_{3,4,\ldots,p}^{p+1,\ldots,1,2} = \frac{2^5}{\Lambda^4}(\sigma_{12}\,\sigma_{2P_{3:p}}\,\sigma_{P_{3:p}1})^2 \left[\frac{1}{s_{34\ldots p}}\right] (\sigma_{P_{p+1:2}3}\,\sigma_{34}\,\sigma_{4P_{p+1:2}})^2 + \mathcal{O}\left(\Lambda^{-2}\right), \tag{761}
$$

where $P_{3:p}$ and $P_{p+1:2}$ denote the momentum of the off-shell punctures on the upper and lower sheets, respectively. Here, $P_{3:p} = k_3 + \cdots + k_p$, $P_{p+1:2} = k_{p+1} + \cdots + k_2$ and $s_{34\ldots p} = 2\sum_{i<j=3}^{p} k_i \cdot k_j$. For concreteness, we have fixed the punctures $(pqr|m) = (123|4)$. Graphically, this configuration is represented by

$$
A_n(1, \mathbf{2}, 3, 4, \ldots, \mathbf{p}, \ldots, n)\Big|_{3,4,\ldots,p}^{p+1,\ldots,1,2} = \tag{762}
$$



.

Notice how the measure and the Faddeev-Popov determinants scale with Λ at leading order,

$$d\mu_n^\Lambda \sim \frac{d\Lambda}{\Lambda}, \tag{763}$$

$$\frac{\Delta(123)\Delta(123|4)}{S_4^\tau} \sim \frac{1}{\Lambda^4}. \tag{764}$$

We also need to know how the Parke-Taylor factor and the reduced determinant scale with Λ. Table 15.5.1 shows how the integrand factors depend on Λ when expanded around $\Lambda = 0$. We see

No. of cut arrows	Factor: $\mathrm{PT}^\tau(\alpha)$	Factor: $\mathbf{det}'(\mathsf{A}_n^\Lambda)$
0	Λ^0	Λ^0
1	-	Λ^2
2	Λ^2	Λ^2
3	-	-
4	Λ^4	-

Table 15.5.1: The table displays the dependence of Λ in the integrand factors when expanding around $\Lambda = 0$. Some entries are empty, meaning that they are impossible to achieve. E.g. the Parke-Taylor factor only appears when an even number of arrows are cut. This is because the PT factor forms a closed ring. Similarly, the reduced determinant enters with two arrows, so at most two arrows can be cut.

that how the integrand scales with Λ is very dependent on the number of cut arrows. For an NLSM amplitude, for each possible non-zero cut, we find that

$\mathrm{PT}^\tau(1,\ldots,n) \times \mathbf{det}'\mathsf{A}_n^\Lambda \sim \mathcal{O}(\Lambda^6)$, *The dashed red line cuts more than four arrows.*

$\mathrm{PT}^\tau(1,\ldots,n) \times \mathbf{det}'\mathsf{A}_n^\Lambda \sim \Lambda^4 + \mathcal{O}(\Lambda^2)$, *The dashed red line cuts three or four arrows.*

$\mathrm{PT}^\tau(1,\ldots,n) \times \mathbf{det}'\mathsf{A}_n^\Lambda \sim \Lambda^2 + \mathcal{O}(\Lambda^0)$, *The dashed red line cuts two arrows (singular cut).*

Similarly, for an sGal-graph, we find that

$$
\begin{aligned}
&\mathbf{det}'\mathsf{A}_n^{\Lambda} \times \mathbf{det}'\mathsf{A}_n^{\Lambda} \sim \Lambda^4 + \mathcal{O}(\Lambda^2), && \textit{The dashed red line cuts one or two arrows} \\
& && \textit{from each of the determinants.} \\
&\mathbf{det}'\mathsf{A}_n^{\Lambda} \times \mathbf{det}'\mathsf{A}_n^{\Lambda} \sim \Lambda^2 + \mathcal{O}(\Lambda^2), && \textit{The dashed red line cuts one or two arrows} \\
& && \textit{from a single the determinant (singular cut).} \\
&\mathbf{det}'\mathsf{A}_n^{\Lambda} \times \mathbf{det}'\mathsf{A}_n^{\Lambda} \sim \Lambda^0 + \mathcal{O}(\Lambda^2), && \textit{The dashed red line cuts no arrows (singular cut).}
\end{aligned}
$$

We combine this with eqs. (763) and (764). For an NLSM-graph, there is no residue when more than four arrows are cut, and the configuration vanishes. When three or four arrows are cut, the factor of $1/\Lambda^4$ from the Faddeev-Popov determinants is canceled by the integrand, and we have a simple pole in Λ. We can evaulate the contribution directly. However, when only two arrows are cut, we do not have a simple pole, and we need to expand beyond leading order. We call this configuration a *singular cut*. We summarize this in the second integration rule for an NLSM-graph;

- **Rule-II (***NLSM-graph***).** *If the dashed red line cuts fewer than three arrows over the NLSM-graph, the integrand must be expanded to next to leading order (singular cut). If the dashed red line cuts three or four arrows, the leading order expansion is sufficient. Otherwise, the cut is zero.*

We can perform a similar analysis for an sGal-graph. If one or two arrows from each of the determinants are cut, we have a simple pole and the contribution can be evaluated directly. Otherwise, the cut is singular and we need to expand beyond leading order. This produces the second integraion rule for an sGal-graph;

- **Rule-II (***sGal-graph***).** *If the dashed red line cuts at least one arrow from each of the determinants, the leading order expansion is sufficient. Otherwise, the integrand must be expanded to next to leading order.*

In Ref. [372], this rule was called the Λ-theorem. In general, we want to avoid singular cuts. If the graph in question is regular (not singular), the following rule apply

- **Rule-IIIa (***NLSM- and sGal-graphs***).** *When the dashed red line cuts four arrows, the graph breaks into two smaller graphs (times a propagator). The off-shell puncture corresponds to a scalar particle.*

- **Rule-IIIb** (*NLSM- and sGal-graphs*). *If the dashed red line cuts three arrows in a graph, there is an off-shell vector field (gluon) propagating among the two resulting graphs. The two resulting graphs must be glued by the identity,* $\sum_M \epsilon^{M\,\mu}\,\epsilon^{M\,\nu} = \eta^{\mu\nu}$.

- **Rule-IIIc** (*sGal-graph*). *If the dashed red line cuts two arrows, there is an off-shell spin-2 field (graviton) propagating between the two resulting smaller graphs. The two sub-graphs are glued together by the identity* $\sum_M \epsilon^{M\,\mu\alpha}\epsilon^{M\,\nu\beta} = \eta^{\mu\nu}\eta^{\alpha\beta}$.

When there are off-shell gluons or gravitons connecting the sub-graphs, we must replace the corresponding off-shell momentum by a polarization vector, $P_i^\mu \to P_i^{M\,\mu} = \frac{1}{\sqrt{2}}\epsilon_i^{M\,\mu}$, in the reduced determinants [3].

Finally, we note that the integration rules are independent of the embedding,

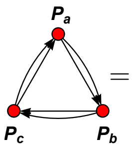
- **Rule-IV.** *The number of intersection points among the dashed red-line and the arrows is given mod 2.*

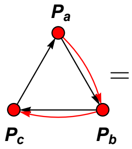
We can always find an embedding where the dashed red line cuts any arrow zero or one time.

15.6 THREE-POINT FUNCTIONS

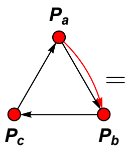
Before we look at examples, it is useful to compute the three-point amplitudes that will work as building blocks for higher-point amplitudes.

We are using the objects defined in eqs. (740) and (741). For the non-linear sigma model, the fundamental three-point functions are given by the expressions

$$A^{\phi^3}(P_a, P_b, P_c) =$$  $$= \int d\mu_3^{\text{CHY}} (\sigma_{P_aP_b}\sigma_{P_bP_c}\sigma_{P_cP_a})^2 \, \text{PT}(P_a, P_b, P_c)^2 = 1\,, \tag{765}$$

$$A'_3(P_a, P_b, P_c) =$$  $$= \int d\mu_3^{\text{CHY}} (\sigma_{P_aP_b}\sigma_{P_bP_c}\sigma_{P_cP_a})^2 \, \text{PT}(P_a, P_b, P_c) \, \frac{1}{\sigma_{P_aP_b}\,\sigma_{P_bP_c}} \frac{s_{P_cP_a}}{\sigma_{P_cP_a}}$$

$$= s_{P_cP_a}, \tag{766}$$

$$A_3^{(P_aP_b)}(P_a, P_b, P_c) =$$  $$= \int d\mu_3^{\text{CHY}} (\sigma_{P_aP_b}\sigma_{P_bP_c}\sigma_{P_cP_a})^2 \, \text{PT}(P_a, P_b, P_c)$$

$$\times \frac{(-1)}{\sigma_{P_aP_b}} \det \begin{bmatrix} \frac{s_{P_bP_a}}{\sigma_{P_bP_a}} & \frac{s_{P_bP_c}}{\sigma_{P_bP_c}} \\ \frac{s_{P_cP_a}}{\sigma_{P_cP_a}} & 0 \end{bmatrix} = s_{P_bP_c}\, s_{P_cP_a}, \tag{767}$$

where $P_a^\mu + P_b^\mu + P_c^\mu = 0$ and all particles could be off-shell, *i.e.* $P_i^2 \neq 0$. Using momentum conservation, we reformulate the expressions as

$$A'_3(P_a, P_b, P_c) = s_{P_cP_a} = -(P_a^2 - P_b^2 + P_c^2), \tag{768}$$

$$A_3^{(P_aP_b)}(P_a, P_b, P_c) = s_{P_bP_c}\, s_{P_cP_a} = (P_c^2 - P_a^2 + P_b^2) \times (P_a^2 - P_b^2 + P_c^2)$$

$$= A'_3(P_c, P_a, P_b) \times A'_3(P_a, P_b, P_c). \tag{769}$$

We see that the three-point functions in eqs. (768) and (769) vanish when the particles are on-shell.

15.7 FACTORIZATION RELATIONS

We will presents three different prescriptions for computing NLSM amplitudes. As we will see, they lead to three different factorization relations.

First, we start with the conventional NLSM prescription given in eq. (737) (in the double-cover language). It is useful to remember that for an odd number of external particles, the amplitude vanishes,

$$A_{2n+1}(1, \ldots, \boldsymbol{P_i}, \ldots, \boldsymbol{P_j}, \ldots, n) = 0. \tag{770}$$

This relation holds even when the particles removed from the determinant by the choice (i, j) are off-shell, *i.e.* when $P_i^2 \neq 0$ and/or $P_j^2 \neq 0$.

Secondly, we will use the alternative prescription given in eq. (740) with two different gauge fixing choices, resulting in two new factorization formulas. Parts of the results were reported by us in Ref. [3].

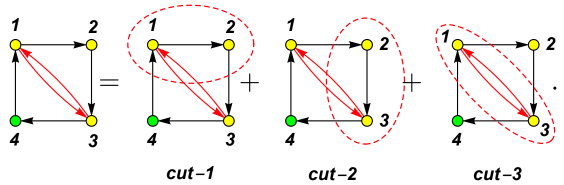
In general, we denote the sum of cyclically-consecutive external momenta (modulo the total number of particles) by $P_{i:j} \equiv k_i + \cdots + k_j$. We also use the shorthand notation $P_{i,j} \equiv k_i + k_j$ for two (not necessarily consecutive) momenta. We also define the generalized Mandelstam variables $s_{i:i+j} \equiv s_{ii+1\ldots i+j}$ and $s_{i:i+j,L} \equiv s_{ii+1\ldots i+jL}$, with $s_{i_1\ldots i_p} \equiv \sum_{a\neq b,a,b=1}^{p} k_{i_a} \cdot k_{i_b}$.

15.7.1 *Four-Point*

The Usual Integrand Prescription

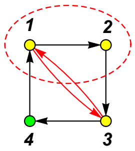
Let us start by considering the four-point amplitude, $A_4(1, 2, 3, 4)$. Without loss of generality, we choose the gauge fixing $(pqr|m) = (123|4)$. In order to avoid singular cuts (see Section 15.5), we remove the columns and rows $(i, j) = (1, 3)$ for the determinant in eq. (737). For notational simplicity, we define $\mathbb{I}_n = (1, \ldots, n)$, $\mathbb{I}_n^{(ij)} = (1, \ldots, \boldsymbol{i}, \ldots, \boldsymbol{j}, \ldots, n)$, and $\mathbb{I}_n^{(ijk)} = (1, \ldots, \boldsymbol{i}, \ldots, \boldsymbol{j}, \ldots, \boldsymbol{k}, \ldots, n)$. Graphically, the amplitude factorizes into

$$A_4(\mathbb{I}_4^{(13)}) = \int d\mu_4^{\Lambda}$$



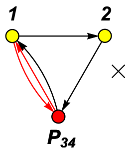
(771)

By applying **rule-III**, we can evaluate *cut-1*, finding

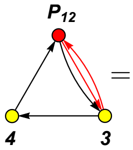


cut−1

$$=$$

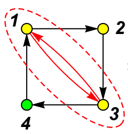


$$\times \left(\frac{1}{s_{34}}\right) \times$$



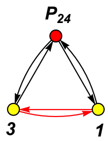
$$= \frac{A_3(P_{34}, 1, 2) \times A_3(P_{12}, 3, 4)}{s_{34}} = 0, \tag{772}$$

where we have used eq. (770). *Cut-2* can be evaluated in a similar manner. Finally, it is straightforward to see that the last cut (*cut-3*) is broken into

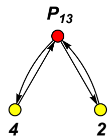


cut−3

$$=$$

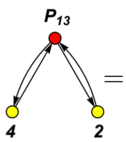


$$\times \left(\frac{1}{s_{24}}\right) \times$$



$$. \tag{773}$$

From the normalization of the three-point function in eq. (765), the first graph evaluates to (-1), while the second is (using **rule-III**)



$$= \frac{(\sigma_{P_{13}2}\,\sigma_{24}\,\sigma_{4P_{13}})^2}{(\sigma_{P_{13}2}\,\sigma_{2P_{13}}) \times (\sigma_{P_{13}4}\sigma_{4P_{13}})} \times \det \begin{bmatrix} 0 & \frac{s_{24}}{\sigma_{24}} \\ \frac{s_{24}}{\sigma_{42}} & 0 \end{bmatrix} = s_{24}^2\,. \tag{774}$$

We can also rewrite the cut using matrix relations defined in section 15.A.2,

$$\textit{cut-3} = -\frac{A_3'(P_{13}, 2, 4)A_3'(1, 3, P_{24})}{s_{24}}. \tag{775}$$

By evaluating the cuts, we have that

$$\begin{aligned} A_4(\mathbb{I}_4^{(13)}) =& \frac{A_3(P_{34}, 1, 2)A_3(P_{12}, 3, 4)}{s_{34}} + \frac{A_3(P_{23}, 1, 4)A_3(3, P_{14}, 2)}{s_{23}} \\ &- \frac{A_3'(P_{13}, 2, 4)A_3'(1, 3, P_{24})}{s_{24}} \\ =& - \frac{A_3'(P_{13}, 2, 4)A_3'(1, 3, P_{24})}{s_{24}} = -\frac{(-s_{13})\,(-s_{24})}{s_{24}} = -s_{13}. \end{aligned} \tag{776}$$

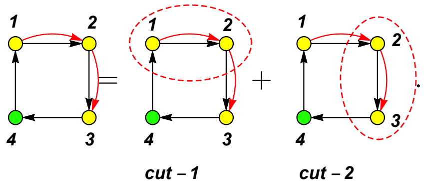
Here we have used eqs. (768) and (770) when evaluating the amplitude. Notice that the factorization channels with poles s_{34} and s_{23} vanish because they factorize into an odd NLSM amplitude, see eq. (770). The last contribution does not vanish, as it is not the usual NLSM prescription, but rather an off-shell amplitude with the new prescription given in eq. (740). Of course, the subamplitudes would vanish if all particles, including intermediate particles, were on-shell. In particular if P_{24} was on-shell (collinear limit). We can see this reflected by the answer, which would vanish in that case.

The New Integrand Prescription

In the previous section, we expressed the factorized non-linear sigma model amplitude with the usual prescription in terms of lower-point amplitudes with the new prescription. In this section we are going to do the calculations using the new prescription.

Let us consider the four-point amplitude, with gauge fixing $(pqr|m) = (123|4)$. In order to get a better understanding of the method, we are going to choose two different reduced determinants, *i.e.* we consider removing columns and rows such that $(ijk) = (123)$ in the first example, and $(ijk) = (134)$ in the second example. In the first example, we have the graphical representation

$$A'_4(\mathbb{I}_4) = \int d\mu_4^\Lambda \tag{777}$$



The graphs can be evaluated as

$$A'_4\left(\mathbb{I}_4\right) = \sum_M \left[\frac{A'_3(1,2,P_{34}^M) A_3^{(P_{12}3)}(P_{12}^M,3,4)}{s_{34}} + \frac{A_3^{(1P_{23})}(1,P_{23}^M,4) A'_3(P_{41}^M,2,3)}{s_{41}} \right]. \tag{778}$$

We see that all factorization contributions are glued together by an off-shell vector field (off-shell gluon). The notation P_i^M means the replacement $P_i^\mu \to \frac{1}{\sqrt{2}} \epsilon_i^{M\,\mu}$ in the A_n matrix. Also, the gluing relation is

$$\sum_M \epsilon_i^{M\,\mu} \epsilon_j^{M\,\nu} = \eta^{\mu\nu}. \tag{779}$$

Explicitly, the two factorization contributions become

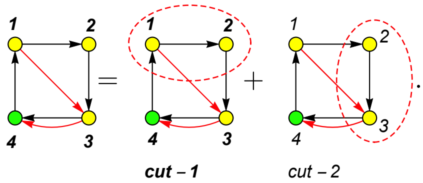
$$\sum_M \frac{A'_3(1,2,P_{34}^M) A_3^{(P_{12}3)}(P_{12}^M,3,4)}{s_{34}} = \sum_M \frac{\left(\sqrt{2}\epsilon_{34}^M \cdot k_1\right) \times s_{34} \left(\sqrt{2}\epsilon_{12}^M \cdot k_4\right)}{s_{34}} = \frac{s_{14}s_{34}}{s_{34}} = s_{14}, \tag{780}$$

and

$$\sum_M \frac{A_3^{(1P_{23})}(1, P_{23}^M, 4) A_3'(P_{41}^M, 2, 3)}{s_{23}} = \sum_M \frac{\left(\sqrt{2}\epsilon_{23}^M \cdot k_4\right) s_{41} \times \left(\sqrt{2}\epsilon_{41}^M \cdot k_3\right)}{s_{23}} = \frac{s_{14}s_{34}}{s_{23}} = s_{12}. \tag{781}$$

As a second example, consider

$$A_4'(\mathbb{I}_4^{(134)}) = \int d\mu_4^\Lambda \tag{782}$$



The graphs evaluate to

$$A_4'(\mathbb{I}_4^{(134)}) = \sum_M \frac{A_3^{(1P_{34})}(1, 2, P_{34}^M) A_3'(P_{12}^M, 3, 4)}{s_{34}} + \frac{A_3'(1, P_{23}, 4) A_3(3, P_{41}, 2)}{s_{23}} \tag{783}$$

Notice that only one of the factorization contributions (*cut-1*) is glued together by an off-shell gluon, while the second contribution (*cut-2*) is a purely scalar contribution. Evaluating the contributions, we find that

$$\sum_M \frac{A_3^{(1P_{34})}(1, 2, P_{34}^M) A_3'(P_{12}^M, 3, 4)}{s_{34}} = \sum_M \frac{-\left(\sqrt{2}\epsilon_{34}^M \cdot k_2\right) s_{12} \times \left(\sqrt{2}\epsilon_{12}^M \cdot k_4\right)}{s_{34}} = -\frac{s_{12}s_{24}}{s_{34}} = -s_{13}, \tag{784}$$

and

$$\frac{A_3'(1, P_{23}, 4) A_3(3, P_{41}, 2)}{s_{23}} = \frac{P_{23}^2 \times 0}{s_{23}} = 0. \tag{785}$$

The scalar contribution vanishes, as an odd amplitude in the usual prescription vanishes, see eq. (770).

Summing the contributions, we obtain

$$A_4'(\mathbb{I}_4^{(123)}) = s_{14} + s_{12} = -s_{13}, \tag{786}$$

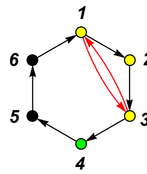
$$A_4'(\mathbb{I}_4^{(134)}) = -s_{13} + 0 = -s_{13}. \tag{787}$$

This agrees with eq. (776).

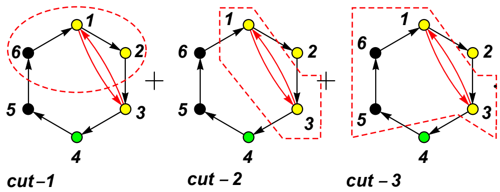
15.7.2 *Six-Point*

Next, we compute the six-point amplitude using the double-cover formalism. We stick to the gauge fixing $(pqr|m) = (123|4)$, and to removing the columns and rows $(i,j) = (1,3)$. Graphically, the amplitude factorizes into

$$A_6(\mathbb{I}_6^{(13)}) = \int d\mu_6^{\Lambda}$$

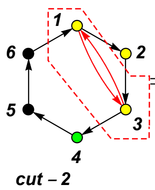


$$=$$

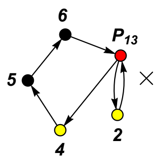


(788)

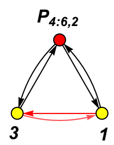
We have omitted some factorizations, which evaluate to zero by analogy to the four-point case. The full calculation is presented in section 15.B.3. *Cut-1* is straightforward to evaluate, as it factorizes into lower-point NLSM amplitudes. However, *cut-2* and *cut-3* do not have straightforward interpretations yet (which is why they sometimes are referred to as *strange-cuts*). Take *cut-2* as an example, it graphically takes the form



$$= \int d\mu_5^{\text{CHY}}$$



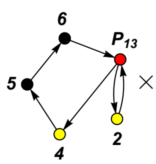
$$\times \left(\frac{1}{s_{4:6,2}}\right) \times$$



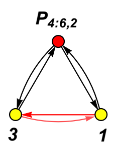
. (789)

The first graph looks non-simple to be computed since there is no way to avoid the *singular cuts*. Nevertheless, such as in Yang-Mills theory, Ref. [374], this strange-cut can be rewritten in the following way

$$\int d\mu_5^{\text{CHY}}$$



$$\times$$



$$= \quad (-1)\, A_5'(P_{13}, 2, 4, 5, 6) \times A_3'(1, 3, P_{4:6,2}), \tag{790}$$

which comes from the matrix identities given in section 15.A.2. We can do a similar rewriting for *cut-3*. Putting it all together, the six-point amplitude factorizes as

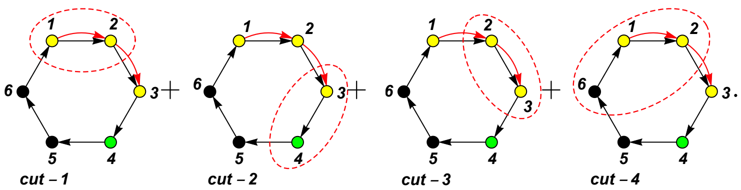
$$\begin{aligned} A_6(\mathbb{I}_6^{(13)}) =& \frac{A_4(\mathbf{1},2,\boldsymbol{P}_{3:5},6)A_4(P_{6:2},3,4,5)}{s_{3:5}} - \frac{A_5'(P_{13},2,4,5,6)A_3'(1,3,P_{4:6,2})}{s_{13}} \\ & - \frac{A_3'(P_{5:1,3},2,4)A_5'(1,3,P_{24},5,6)}{s_{24}} \\ =& \frac{s_{26}s_{35}}{s_{3:5}} + s_{13}\left[\frac{s_{46}}{s_{4:6}} + \frac{s_{26}+s_{46}}{s_{56P_{13}}}\right] + s_{24}\left[\frac{s_{26}+s_{46}}{s_{56P_{24}}} + \frac{s_{26}+s_{36}+s_{46}}{s_{5:1}}\right]. \end{aligned} \tag{791}$$

By using momentum conservation, all unphysical poles cancel, and we match with the known result

$$\begin{aligned} A_6\left(\mathbb{I}_6\right) =& \frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} + \frac{(s_{23}+s_{34})(s_{56}+s_{61})}{s_{234}} + \frac{(s_{34}+s_{45})(s_{56}+s_{61})}{s_{345}} \\ & - (s_{12}+s_{23}+s_{34}+s_{45}+s_{56}+s_{61}). \end{aligned} \tag{792}$$

The six-point amplitude can also be computed using the new prescription. The first example with the choice $(ijk) = (123)$ gives, graphically,

$$A_6'(\mathbb{I}_6^{(123)}) = $$



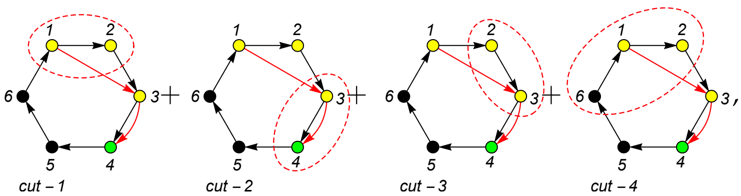
(793)

The full calculation is found in section 15.B.1. The contributions unambiguously evaluate to

$$\begin{aligned} & A_6'(\mathbb{I}_6^{(123)}) = \\ & \sum_M \left[\frac{A_3'\left(1,2,P_{3:6}^M\right) A_5^{(P_{123})}(P_{12}^M,3,4,5,6)}{s_{3:6}} + \frac{A_5'(1,2,P_{34}^M,5,6)A_3^{(P_{5:2}3)}(P_{5:2}^M,3,4)}{s_{34}} \right. \\ & \left. + \frac{A_3'\left(P_{4:1}^M,2,3\right) A_5^{(1P_{23})}(1,P_{23}^M,4,5,6)}{s_{4:1}} + \frac{A_4'\left(1,2,P_{3:5}^M,6\right) A_4^{(P_{6:2}3)}\left(P_{6:2}^M,3,4,5\right)}{s_{3:5}} \right]. \end{aligned} \tag{794}$$

Graphically, the second example, with the choice $(ijk) = (134)$, is

$$A_6'(\mathbb{I}_6^{(134)}) = $$



(795)

which becomes (see section 15.B.2)

$$
\begin{aligned}
&A_6'(\mathbb{I}_6^{(134)}) = \qquad (796)\\
&\sum_M \left[\frac{A_3^{(1P_{3:6})}\left(1,2,P_{3:6}^M\right) A_5'(P_{12}^M,3,4,5,6)}{s_{3:6}} + \frac{A_5^{(1P_{34})}(1,2,P_{34}^M,5,6) A_3'(P_{5:2}^M,3,4)}{s_{34}} \right.\\
&\left. + \frac{A_4^{(1P_{3:5})}\left(1,2,P_{3:5}^M,6\right) A_4'\left(P_{6:2}^M,3,4,5\right)}{s_{3:5}} \right] + \frac{A_3\left(3,P_{4:1},2\right) A_5'(1,P_{23},4,5,6)}{s_{4:1}} .
\end{aligned}
$$

Notice that the last contribution (*cut-3*) evaluates to zero. We can check that both examples with the new integrand prescription reproduce the correct result. The full six-point calculation for both choices of gauge fixing is presented in section 15.B. Notice that in the first example, all factorization contributions are glued together with off-shell gluons, while in the second example, three contributions involve off-shell gluons, and one contribution is purely in terms of scalar particles.

So far we have seen three different kinds of factorization relations. The first kind, presented in eqs. (776) and (791), all particles were scalar. In the second case, given by eqs. (778) and (794), the intermediate particles were vector fields (off-shell gluons). Finally, in the last case, eqs. (783) and (796), the factorization relation involved both intermediate scalar and vector fields.[4]

15.7.3 *Longitudinal Contribution*

As the non-linear sigma model is a scalar theory, it is an interesting proposition to only consider longitudinal contributions. An off-shell vector field can be decomposed in terms of transverse and longitudinal degrees of freedom. Let us consider only including the longitudinal degrees of freedom.

Practically, this means that instead of using the relation in eq. (779), we keep only the longitudinal sector,

$$
\sum_L \epsilon_i^{L\,\mu} \epsilon_j^{L\,\nu} = \frac{k_i^\mu k_j^\nu}{k_i \cdot k_j} = \overline{k}_i^\mu k_j^\nu, \;\; \text{with, } k_i^\mu = -k_j^\mu, \;\; \overline{k}_i^\mu = -\left(\frac{k_i^\mu}{k_i^2}\right). \qquad (797)
$$

Here we label the polarization vectors by a superscript L instead of M when keeping only longitudinal degrees of freedom.

[4]Although in this case, the factorization contribution where the propagated particle is a scalar field vanishes, it is simple to find an example where this does not happen. For instance, let us choose the gauge, $(pqr|m) = (134|6)$, and the reduced A_n matrix with $(ijk) = (146)$. It is not hard to check that for this gauge fixing the amplitude, $A_6'(\mathbb{I}_6^{(146)})$, has the two types of factorization contributions which are non-zero.

In the four-point example, we have that

$$\sum_L \left[\frac{A_3'(1,2,P_{34}^L) A_3^{(P_{12}3)}(P_{12}^L,3,4)}{s_{34}} + \frac{A_3^{(1P_{23})}(1,P_{23}^L,4) A_3'(P_{41}^L,2,3)}{s_{23}} \right]$$
$$= -\frac{1}{2}\left[\frac{s_{12}^2}{s_{12}} + \frac{s_{14}^2}{s_{14}}\right] = \frac{s_{13}}{2} = -\frac{1}{2}A_4(\mathbb{I}_4) \tag{798}$$

and

$$\sum_L \frac{A_3^{(1P_{34})}(1,2,P_{34}^L) A_3'(P_{12}^L,3,4)}{s_{34}} + \frac{A_3'(1,P_{23},4) A_3(3,P_{41},2)}{s_{23}}$$
$$= \frac{1}{2}\left[\frac{s_{12}^2}{s_{12}} + \frac{0}{s_{14}}\right] = \frac{s_{12}}{2} \neq \rho A_4(\mathbb{I}_4) \tag{799}$$

where is ρ is a real constant. The sum of longitudinal contributions in eq. (798) is proportional to the correct answer, while the sum of longitudinal contributions in eq. (799) is not.

Applying the same ideas to the six-point amplitude in eq. (794), we have that

$$\sum_L \left[\frac{A_3'\left(1,2,P_{3:6}^L\right) A_5^{(P_{12}3)}(P_{12}^L,3,4,5,6)}{s_{3:6}} + \frac{A_5'(1,2,P_{34}^L,5,6) A_3^{(P_{5:2}3)}(P_{5:2}^L,3,4)}{s_{34}} \right.$$
$$\left. + \frac{A_3'\left(P_{4:1}^L,2,3\right) A_5^{(1P_{23})}(1,P_{23}^L,4,5,6)}{s_{4:1}} + (-1)\frac{A_4'\left(1,2,P_{3:5}^L,6\right) A_4^{(P_{6:2}3)}\left(P_{6:2}^L,3,4,5\right)}{s_{3:5}} \right]$$
$$= -\frac{1}{2}A_6(\mathbb{I}_6). \tag{800}$$

Notice that the relative sign of the contribution from even subamplitudes (physical pole) was flipped in order to reproduce the correct amplitude.[5] In the four-point example, all subamplitudes are odd, and no relative sign flip is needed. All the longitudinal contributions are computed in section 15.B.4.

Now, let us focus on the factorization relation given in eq. (796) and its longitudinal contributions

$$\sum_L \left[(-1)^{i_1} \frac{A_3^{(1P_{3:6})}\left(1,2,P_{3:6}^L\right) A_5'(P_{12}^L,3,4,5,6)}{s_{3:6}} + (-1)^{i_2} \frac{A_5^{(1P_{34})}(1,2,P_{34}^L,5,6) A_3'(P_{5:2}^L,3,4)}{s_{34}} \right.$$
$$\left. + (-1)^{i_3} \frac{A_4^{(1P_{3:5})}\left(1,2,P_{3:5}^L,6\right) A_4'\left(P_{6:2}^L,3,4,5\right)}{s_{3:5}} \right] + \frac{A_3\left(3,P_{4:1},2\right) A_5'(1,P_{23},4,5,6)}{s_{4:1}}$$
$$\neq \rho A_6(\mathbb{I}_6), \tag{801}$$

[5] We have tested all possible sign combinations, and this is the only one which is proportional to the correct amplitude.

where the non-equality is preserved for all $2^3 = 8$ possible combinations of relative signs, *i.e.* $(i_1, i_2, i_3) \in \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}$. Thus, like the four-point example, the amplitude with both off-shell gluons and scalars does not reproduce the full answer when only longitudinal contributions are kept. Again, the longitudinal contributions are presented in section 15.B.4.

In summary, we have obtained examples of three different factorization relations, involving only intermediate scalars, off-shell gluons, or both scalars and off-shell gluons, respectively. In the case where we have only off-shell gluons, we are able to reproduce the full answer by only keeping the longitudinal degrees of freedom (with a relative sign flip between even and odd factorization contributions).

15.8 GENERAL FACTORIZATION RELATIONS

The factorization relations from the previous section can be generalized. In this section, we present three different factorization formulas. One formula is given in terms of exchange of off-shell vector fields, while the other two formulas are given in terms of purely scalar fields.

First, let us consider the case, $A_{2n}(\mathbb{I}_{2n}^{(13)})$. Thus, as in the section 15.7.1, we choose the gauge fixing $(pqr|m) = (123|4)$ and the reduced matrix with $(ij) = (13)$, namely $[\mathsf{A}_{2n}]^{13}_{31}$. Applying the integration rules, the amplitude becomes

$$A_{2n}(\mathbb{I}_{2n}^{(13)}) = \sum_{i=3}^{n} \frac{A'_{2(n-i+2)}\,(1,2,P_{3:2i-1},2i,...,2n) \times A'_{2(i-1)}\,(P_{2i:2},3,4,...,2i-1)}{s_{3:2i-1}} +$$
$$(-1)\sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}\,(1,3,P_{4:2i-2,2},2i-1,...,2n) \times A'_{2(i-1)-1}\,(P_{2i-1:1,3},2,4,...,2i-2)}{s_{4:2i-2,2}}\,. \quad (802)$$

This formula has been check up to ten points. In order to obtain the this relation, we used the matrix identities formulated in section 15.A.2. In the first line, we used that

$$A_{2i}\left(..., \boldsymbol{P_p}, ..., \boldsymbol{P_q}, ..., P_r, ...\right) = A_{2i}\left(..., \boldsymbol{P_p}, ..., P_q, ..., \boldsymbol{P_r}, ...\right)$$
$$= A_{2i}\left(..., P_p, ..., \boldsymbol{P_q}, ..., \boldsymbol{P_r}, ...\right) = A'_{2i}\left(..., \boldsymbol{P_p}, ..., \boldsymbol{P_q}, ..., \boldsymbol{P_r}, ...\right). \quad (803)$$

For the second line, we used properties **I** and **III** in section 15.A.2.

Thus, as the formula obtained in eq. (802), our second factorization relation, that was already presented in Ref. [3], is supported on the double-cover formalism. In order to generalize the eqs. (778)

and (794), we choose the same gauge fixing, $(pqr|m) = (123|4)$, and the reduced matrix with, $(ijk) = (123)$, (i.e. $[\mathsf{A}_{2n}]^{12}_{23}$). By the integration rules formulated in section 15.5, it is straightforward to see the amplitude turns into

$$
\begin{aligned}
A'_{2n}\left(\mathbb{I}_{2n}\right) = \sum_{M} \Bigg[& \sum_{i=3}^{n} \frac{A'_{2(n-i+2)}\left(1,2,P^{M}_{3:2i-1},2i,...,2n\right) \times A^{(P_{2i:2}3)}_{2(i-1)}\left(P^{M}_{2i:2},3,4,...,2i-1\right)}{s_{3:2i-1}} \\
& + \sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}\left(1,2,P^{M}_{3:2i-2},2i-1,...,2n\right) \times A^{(P_{2i-1:2}3)}_{2(i-1)-1}\left(P^{M}_{2i-1:2},3,4,...,2i-2\right)}{s_{3:2i-2}} \\
& + \frac{A'_{3}\left(P^{M}_{4:1},2,3\right) \times A^{(1P_{23})}_{2n-1}\left(1,P^{M}_{23},4,\ldots,2n\right)}{s_{4:1}} \Bigg],
\end{aligned} \tag{804}
$$

where we use eq. (779). This second general formula has been verified up to ten points.

On the other hand, from the results obtained in the eqs. (798) and (800) for four and six points, respectively, we can generalize the idea presented in section 15.7.3 to higher number of points. Therefore, by considering just the longitudinal degrees of freedom in eq. (804), we conjecture the following factorization formula [3],

$$
\begin{aligned}
A'_{2n}(\mathbb{I}_{2n}) = 2\sum_{L} \Bigg[& \sum_{i=3}^{n} \frac{A'_{2(n-i+2)}\left(1,2,P^{L}_{3:2i-1},2i,...,2n\right) \times A^{(P_{2i:2}3)}_{2(i-1)}\left(P^{L}_{2i:2},3,4,...,2i-1\right)}{s_{3:2i-1}} \\
& + (-1)\sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}\left(1,2,P^{L}_{3:2i-2},2i-1,...,2n\right) \times A^{(P_{2i-1:2}3)}_{2(i-1)-1}\left(P^{L}_{2i-1:2},3,4,...,2i-2\right)}{s_{3:2i-2}} \\
& + (-1)\,\frac{A'_{3}\left(P^{L}_{4:1},2,3\right) \times A^{(1P_{23})}_{2n-1}\left(1,P^{L}_{23},4,\ldots,2n\right)}{s_{4:1}} \Bigg],
\end{aligned} \tag{805}
$$

where we use eq. (797). Finally, by applying the identities

$$
\begin{aligned}
A^{(P_pP_q)}_{2i}(...,P_p,...,P_q,...,P_r,...) &= A^{(P_qP_r)}_{2i}(...,P_p,...,P_q,...,P_r,...) \\
&= -(P_p^2+P_q^2+P_r^2)\times A'_{2i}(...,\boldsymbol{P}_p,...,\boldsymbol{P}_q,...,\boldsymbol{P}_r,...), \\
A^{(P_pP_q)}_{2i+1}(...,P_p,...,P_q,...,P_r,...) &= A^{(P_qP_r)}_{2i+1}(...,P_p,...,P_q,...,P_r,...) \\
&= (P_p^2-P_q^2-P_r^2)\times A'_{2i+1}(...,\boldsymbol{P}_p,...,\boldsymbol{P}_q,...,\boldsymbol{P}_r,...),
\end{aligned} \tag{806}
$$

which are a consequence from the properties in appendix 15.A.2, it is straightforward to see the eq. (805) becomes

$$
\begin{aligned}
A_{2n}(\mathbb{I}_{2n}) &= \sum_{i=3}^{n} \frac{A'_{2(n-i+2)}\left(1,2,P_{3:2i-1},2i,...,2n\right) \times A'_{2(i-1)}\left(P_{2i:2},3,4,...,2i-1\right)}{s_{3:2i-1}} \\
&+ \sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}\left(1,2,P_{3:2i-2},2i-1,...,2n\right) \times A'_{2(i-1)-1}\left(P_{2i-1:2},3,4,...,2i-2\right)}{s_{3:2i-2}} \\
&+ (-1)\,\frac{A'_3\left(P_{4:1},2,3\right) \times A'_{2n-1}\left(1,P_{23},4,\ldots,2n\right)}{s_{4:1}}.
\end{aligned} \tag{807}
$$

This is our third general factorization formula.

15.8.1 *A New Relationship for the Boundary Terms*

As we argued in Ref. [3], the amplitudes with an odd number of particles, i.e. amplitudes of the form $A'_{2m+1}(...,\boldsymbol{P}_a,...)$ (odd amplitude), are proportional to P_a^2 since that them must vanish when all particles are on-shell. Thus, the poles given by the odd contributions, namely expressions of the form $\frac{A'_{2m+1}(...,\boldsymbol{P}_a,...)\times A'_{2k+1}(...,\boldsymbol{P}_b,...)}{2\,P_a\cdot P_b}$, are spurious and, therefore, those terms are on the boundary of any usual BCFW deformation [304]. In particular, under the BCFW deformation,

$$
k_2^\mu(z) = k_2^\mu + z\,q^\mu\,, \qquad k_3^\mu(z) = k_3^\mu - z\,q^\mu\,, \quad \text{with } \ q^2 = 0, \tag{808}
$$

all even contributions (physical poles), which are given by the sum

$$
\sum_{i=3}^{n} \frac{A'_{2(n-i+2)}\left(1,2,P_{3:2i-1},2i,...,2n\right) \times A'_{2(i-1)}\left(P_{2i:2},3,4,...,2i-1\right)}{P^2_{3:2i-1}(z)} \tag{809}
$$

in eqs. (802) and (807), are localized over the z-plane at, $P^2_{3:2i-1}(z) = 0$. Thus, by the above discussion, all odd contributions in eqs. (802) and (807) are localized at the point $z = \infty$ on the z-plane and, hence, we call those odd amplitudes the boundary terms.

Now, clearly, by comparing the factorization relations obtained in eqs. (802) and (807), this is straightforward to see that one arrives to the identity

$$\sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}(1,2,P_{3:2i-2},2i-1,...,2n)\times A'_{2(i-1)-1}(P_{2i-1:2},3,4,...,2i-2)}{s_{3:2i-2}} + (2\leftrightarrow 3)$$
$$= \frac{A'_3(P_{4:1},2,3)\times A'_{2n-1}(1,P_{23},4,\ldots,2n)}{s_{4:1}}, \tag{810}$$

which lies on the boundary of any usual BCFW deformation. We have checked this identity up to $n=10$.

15.9 A NOVEL RECURSION RELATION

In this section, we are going to present a new recursion relationship, which can be used to write down any NLSM amplitude in terms of the three-point building-block, $A'_3(P_a,P_b,P_c)=-(P_a^2-P_b^2+P_c^3)$, given in eq. (766).

Previously, in eq. (805), we arrived at an unexpected factorization expansion, which, although it emerged accidentally from the integration rules, a formal proof is yet unknown.[6] Thus, since applying the integration rules is an iterative process, we would like to know if the relationship in eq. (805) could be extended to off-shell amplitudes (both for an even and odd number of particles). Here, we are going to show how to do that.

First, consider the four-point computation, $A'_4(P_1,P_2,P_3,4)$, where the particles, $\{P_1,P_2,P_3\}$, can be off-shell. By the integration rules, we obtain the same decomposition as in eq. (778),

$$A'_4(P_1,P_2,P_3,4) = \tag{811}$$
$$\sum_M\left[\frac{A'_3(P_1,P_2,P^M_{34})A_3^{(P_{12}P_3)}(P^M_{12},P_3,4)}{s_{P_3P_4}}+\frac{A_3^{(P_1P_{23})}(P_1,P^M_{23},4)A'_3(P^M_{41},P_2,P_3)}{s_{P_4P_1}}\right]=-s_{4P_2}.$$

[6] It is important to remind ourselves that the longitudinal contributions give the right answer only when, after applying the integration rules, all factorization channels are mediated by an off-shell vector field. This was exemplified in section 15.7.3.

Now, by using the longitudinal gluing relation given in eq. (797), *i.e.* $\sum_L \epsilon_{34}^{\mu\,L}\epsilon_{12}^{\nu\,L} = \overline{P}_{34}^{\mu}P_{12}^{\nu}$ and $\sum_L \epsilon_{23}^{\mu\,L}\epsilon_{41}^{\nu\,L} = P_{23}^{\mu}\overline{P}_{41}^{\nu}$, over the above factorized amplitude, one arrives at

$$
\begin{aligned}
&(-2)\sum_L \left[\frac{A_3'(P_1,P_2,P_{34}^L)A_3^{(P_{12}P_3)}(P_{12}^L,P_3,4)}{s_{P_3P_4}} + \frac{A_3^{(P_1P_{23})}(P_1,P_{23}^L,4)A_3'(P_{41}^L,P_2,P_3)}{s_{P_4P_1}}\right]\\
&= \frac{-(P_1^2-P_2^2+P_{34}^2)\,s_{4P_{12}}}{P_{34}^2} + \frac{-(P_{41}^2-P_2^2+P_3^2)\,s_{4P_{23}}}{P_{41}^2}.
\end{aligned}
\tag{812}
$$

Clearly, since $\{P_1,P_2,P_3\}$ are off-shell, the results found in eqs. (811) and (812) do not match. However, there is a simple way to make them coincide. Instead of using the usual longitudinal identity, we employ a generalized version where $\overline{P}_a^{\mu}$ is redefined as

$$
\overline{P}_{34}^{\mu} = -\left(\frac{P_{34}^{\mu}}{P_{34}^2}\right) \to \overline{P}_{34}^{\mu} = -\left(\frac{P_{34}^{\mu}}{P_1^2-P_2^2+P_{34}^2}\right),\ \overline{P}_{41}^{\mu} = -\left(\frac{P_{41}^{\mu}}{P_{41}^2}\right) \to \overline{P}_{41}^{\mu} = -\left(\frac{P_{41}^{\mu}}{P_{41}^2-P_2^2+P_3^2}\right).
$$

It is straightforward to check that under this redefinition, the factored expression in eq. (812) reproduces the same result as in eq. (811). The generalization to a higher number of points is straightforward, so, when the particles $\{P_1,P_2,P_3\}$ are off-shell, the longitudinal gluing relations that must be used in eq. (805) are given by

$$
\begin{aligned}
&\sum_L A_{2m+1}'(\boldsymbol{P_r^L},\ldots,\boldsymbol{P_2},\ldots,\boldsymbol{P_3},\ldots)\times A_{2q+1}^{(P_1P_k)}(P_1,\ldots,P_k^L,\ldots,) \;\to\; \sum_L \epsilon_r^{\mu\,L}\epsilon_k^{\nu\,L} = \overline{P}_r^{\mu}P_k^{\nu},\\
&\sum_L A_{2j}^{(P_1P_k)}(P_1,\ldots,P_k^L,\ldots,)\times A_{2i}'(\boldsymbol{P_r^L},\ldots,\boldsymbol{P_2},\ldots,\boldsymbol{P_3},\ldots) \;\to\; \sum_L \epsilon_k^{\mu\,L}\epsilon_r^{\nu\,L} = \overline{P}_k^{\mu}P_r^{\nu},
\end{aligned}
$$

where, $P_r^{\mu} = -P_k^{\mu}$, and

$$
\overline{P}_r^{\mu} = -\left(\frac{P_r^{\mu}}{P_r^2-P_2^2+P_3^2}\right), \qquad \overline{P}_k^{\mu} = -\left(\frac{P_k^{\mu}}{P_1^2+P_k^2}\right). \tag{813}
$$

Thus, by applying the identities in eq. (806), we obtain the following simple and compact expression

$$
\begin{aligned}
& A'_{2n}(P_1, P_2, P_3, 4, ..., 2n) = \\
& \quad \sum_{i=3}^{n} \frac{A'_{2(n-i+2)}(P_1, P_2, P_{3:2i-1}, 2i, ..., 2n) \times A'_{2(i-1)}(P_{2i:2}, P_3, 4, ..., 2i-1)}{s_{3:2i-1}} \\
& \quad + \sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}(P_1, P_2, P_{3:2i-2}, 2i-1, ..., 2n) \times A'_{2(i-1)-1}(P_{2i-1:2}, P_3, 4, ..., 2i-2)}{P_1^2 - P_2^2 + P_{3:2i-2}^2} \\
& \quad + (-1)\, \frac{A'_3(P_{4:1}, P_2, P_3) \times A'_{2n-1}(P_1, P_{23}, 4, \ldots, 2n)}{P_{4:1}^2 - P_2^2 + P_3^2}. \qquad (814)
\end{aligned}
$$

Obviously, when $\{P_1, P_2, P_3\}$ become on-shell, we rediscover eq. (807).

In order to achieve a completed recursion-relationship, it is needed to get a closed formula for the odd amplitude, $A'_{2n+1}(P_1, P_2, P_3, 4, ..., 2n+1)$. Therefore, applying the integration rules over this amplitude, one obtains the following two types of combinations

$$
\begin{aligned}
&\textbf{I.} \qquad \sum_{M} A'_{2m+1}(\boldsymbol{P_r^M}, \ldots, \boldsymbol{P_2}, \ldots, \boldsymbol{P_3}, \ldots) \times A_{2j}^{(P_1 P_k)}(P_1, \ldots, P_k^M, \ldots,), \\
&\textbf{II.} \qquad \sum_{M} A_{2q+1}^{(P_1 P_k)}(P_1, \ldots, P_k^M, \ldots,) \times A'_{2i}(\boldsymbol{P_r^M}, \ldots, \boldsymbol{P_2}, \ldots, \boldsymbol{P_3}, \ldots).
\end{aligned}
$$

We found that, to land on the right result by using just longitudinal degrees of freedom, the combination **I** must be glued by the relation

$$
\textbf{I.} \qquad \sum_{L} \epsilon_r^{\mu\, L} \epsilon_k^{\nu\, L} = (-1)(P_1^2 - P_2^2 + P_3^3) \times \overline{P}_r^{\mu} \overline{P}_k^{\nu}, \qquad (815)
$$

where $\overline{P}_r^{\mu}$ and $\overline{P}_k^{\nu}$ are defined in eq. (813), while the combination **II** has to be discarded. Note that the overall factor, $(P_1^2 - P_2^2 + P_3^3)$, implies that when the off-shell external particles become on-shell, the amplitude A'_{2n+1} vanishes trivially, such as it is required.

To summarize, after applying the integration rules over an even or odd amplitude, such that the factorized subamplitudes are glued only by virtual vector fields, then, we can compute this process just by considering the longitudinal degrees of freedom and the rules given in the following box

$$
\begin{tikzcd}
A'_{2m+1}(\boldsymbol{P_r^\epsilon},\ldots,\boldsymbol{P_2},\ldots,\boldsymbol{P_3},\ldots)\Big|_{\epsilon_r^\mu\to\overline{P}_r^\mu} \arrow[r, Leftrightarrow, "\text{Product Allowed}"] \arrow[d, Leftrightarrow, "\text{Product Allowed}"', "\times(-1)\left(P_1^2-P_2^2+P_3^2\right)"] & A_{2q+1}^{(P_1P_k)}(P_1,\ldots,P_k^\epsilon,\ldots,)\Big|_{\epsilon_k^\mu\to P_k^\mu} \arrow[d, Leftrightarrow, "\text{Product Forbidden}"] \\
A_{2j}^{(P_1P_k)}(P_1,\ldots,P_k^\epsilon,\ldots,)\Big|_{\epsilon_k^\mu\to\overline{P}_k^\mu} \arrow[r, Leftrightarrow, "\text{Product Allowed}"] & A'_{2i}(\boldsymbol{P_r^\epsilon},\ldots,\boldsymbol{P_2},\ldots,\boldsymbol{P_3},\ldots)\Big|_{\epsilon_r^\mu\to P_r^\mu}
\end{tikzcd}
$$

where $\overline{P}_r^\mu$ and $\overline{P}_k^\nu$ are given in eq. (813). Notice that the horizontal rules on the box work over the even amplitudes, *i.e.* $A'_{2n}(P_1,P_2,P_3,4,...,2n)$, while the vertical rules work over the odd ones, $A'_{2n+1}(P_1,P_2,P_3,4,...,2n+1)$.

Finally, by employing the identities in eq. (806) and the above box, we are able to write down a compact formula for $A'_{2n+1}(P_1,P_2,P_3,4,...,2n+1)$,

$$
\begin{aligned}
A'_{2n+1}(P_1,P_2,P_3,4,...,2n+1) = \left(P_1^2-P_2^2+P_3^2\right)\times\Bigg[\sum_{i=3}^{n+1}\left(\frac{1}{P_1^2-P_2^2+P_{3:2i-1}^2}\right) \\
\times\frac{A'_{2(n-i+2)+1}\left(P_1,P_2,P_{3:2i-1},2i,...,2n+1\right)\times A'_{2(i-1)}\left(P_{2i:2},P_3,4,...,2i-1\right)}{s_{3:2i-1}} \\
+\left(\frac{1}{P_{4:1}^2-P_2^2+P_3^2}\right)\times\frac{A'_3\left(P_{4:1},P_2,P_3\right)\times A'_{2n}\left(P_1,P_{23},4,\ldots,2n+1\right)}{s_{4:1}}\Bigg].
\end{aligned}
\tag{816}
$$

Evidently, the formulas, eqs. (814) and (816), give us a novel recursion relation, which we have checked against known results for up to ten points. The big advantage with this relation is that it is purely algebraic, as any non-linear sigma model amplitude can be decomposed to off-shell three-point amplitudes (without solving any scattering equations).

15.10 THE SOFT LIMIT AND A NEW RELATION FOR $A_n^{\mathrm{NLSM}\oplus\phi^3}$

The soft limit for the $\mathrm{U}(N)$ non-linear sigma model in its CHY representation was already studied by Cachazo, Cha and Mizera (CCM) in Ref. [259]. One of the main results is given by the expression (at leading order)

$$A_n(1,\ldots,n) = \epsilon \sum_{a=2}^{n-2} 2\,\tilde{k}_n \cdot k_a\, A_{n-1}^{\mathrm{NLSM}\oplus\phi^3}(1,\ldots,n-1||n-1,a,1) + \mathcal{O}(\epsilon^2), \tag{817}$$

where $k_n^\mu = \epsilon\,\tilde{k}_n^\mu$ and $\epsilon \to 0$.

In this section we carry out, in detail, the soft limit behaviour at six-point, but using the new recursion relation proposed in section 15.9. Although the generalization to a higher number of points is not straightforward, it is not complicated. We will not take into account the general case in this work.

Let us consider the amplitude, $A_6(1,2,3,4,5,6) = A_6'(5,6,1,2,3,4)$, where the soft particle is, $k_6^\mu = \epsilon\,\tilde{k}_6^\mu$, with $\epsilon \to 0$. From eq. (814), we have

$$\begin{aligned} A_6'(5,6,1,2,3,4) \;&=\; \frac{A_3'(5,6,P_{1:4}) \times A_5'(P_{56},1,2,3,4)}{P_{56}^2} - \frac{A_3'(P_{2:5},6,1) \times A_5'(5,P_{61},2,3,4)}{P_{61}^2} \\ &\quad + \frac{A_3'(P_{3:6},1,2) \times A_5'(5,6,P_{12},3,4)}{P_{12}^2} + \frac{A_4'(5,6,P_{1:3},4) \times A_4'(P_{4:6},1,2,3)}{P_{1:3}^2} \\ &= -A_5'(P_{56},1,2,3,4) + A_5'(5,P_{61},2,3,4) - A_5'(5,6,P_{12},3,4) - \frac{2\,\epsilon\,\tilde{k}_6 \cdot k_4 \; \times A_4'(P_{456},1,2,3)}{s_{45} + 2\epsilon\,\tilde{k}_6 \cdot P_{45}}, \end{aligned} \tag{818}$$

where the three-point building-blocks in eq. (767) have been used. Applying the off-shell formula proposed in eq. (816), it is not hard to check that, at leading order, the above five-point amplitudes become

$$-A_5'(P_{56},1,2,3,4) = (2\,\epsilon\,\tilde{k}_6 \cdot k_5)\left[\frac{A_4'(P_{51},2,3,4)}{s_{51}} + \frac{A_4'(5,P_{12},3,4)}{s_{12}}\right], \tag{819}$$

$$A_5'(5,P_{61},2,3,4) = (2\,\epsilon\,\tilde{k}_6 \cdot k_1)\left[\frac{A_4'(P_{51},2,3,4)}{s_{51}} + \frac{A_4'(5,P_{12},3,4)}{s_{12}}\right], \tag{820}$$

$$-A_5'(5,6,P_{12},3,4) = -(2\,\epsilon\,\tilde{k}_6 \cdot P_{125}) \times \frac{A_4'(5,P_{12},3,4)}{s_{12}} - 2\,\epsilon\,\tilde{k}_6 \cdot k_4. \tag{821}$$

Therefore, the six-point amplitude at leading order in ϵ is given by

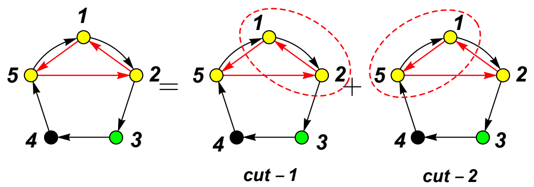
$$\begin{aligned} A_6(1,2,3,4,5,6) &= (2\,\epsilon\,\tilde{k}_6 \cdot k_2)\left[-\frac{A_4'(P_{51},2,3,4)}{s_{15}} - \frac{A_4'(5,P_{12},3,4)}{s_{12}}\right] \\ &\quad +(2\,\epsilon\,\tilde{k}_6 \cdot k_3)\left[-\frac{A_4'(P_{51},2,3,4)}{s_{15}}\right] \\ &\quad +(2\,\epsilon\,\tilde{k}_6 \cdot k_4)\left[-\frac{A_4'(P_{51},2,3,4)}{s_{15}} - \frac{A_4'(P_{45},1,2,3)}{s_{45}} - 1\right]. \end{aligned} \tag{822}$$

Now, from the CCM formula in eq. (817) one has

$$\begin{aligned} A_6(1,2,3,4,5,6) &= (2\,\epsilon\,\tilde{k}_6 \cdot k_2) \times A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,2,1) \\ &\quad +(2\,\epsilon\,\tilde{k}_6 \cdot k_3) \times A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,3,1) \\ &\quad +(2\,\epsilon\,\tilde{k}_6 \cdot k_4) \times A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,4,1). \end{aligned} \tag{823}$$

Although at first glance, the eqs. (822) and (823) do not seem to be the same, notice that by choosing the gauge, $(pqr|m) = (512|3)$, the amplitude $A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,2,1)$ turns into

$$A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,2,1) = \int d\mu_5^{\Lambda}$$



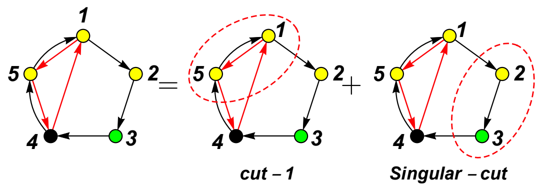
$$\begin{aligned} &= -\frac{A_3^{\phi^3}(1,2,P_{3:5}) \times A_4'(5,P_{12},3,4)}{s_{12}} - \frac{A_3^{\phi^3}(1,P_{2:4},5) \times A_4'(P_{51},2,3,4)}{s_{15}} \\ &= -\frac{A_4'(5,P_{12},3,4)}{s_{12}} - \frac{A_4'(P_{51},2,3,4)}{s_{15}}, \end{aligned} \tag{824}$$

where we employed the integration rules, the building-block, $A_3^{\phi^3}(P_1,P_2,P_3) = 1$, and the second property from the appendix 15.A.2. Following the same procedure, it is straightforward to see

$$A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,3,1) = -\frac{A_4'(P_{51},2,3,4)}{s_{15}}. \tag{825}$$

Clearly, the first two lines in eqs. (822) and (823) match perfectly, however, to compare the last lines we must take care. By direct computation, it is not hard to show that, in fact, the third lines in eqs. (822) and (823) produce the same result, but, we can extract more information from them. For example, under the gauge fixing, $(pqr|m) = (512|3)$, the amplitude $A_5^{\text{NLSM}\oplus\phi^3}(1,2,3,4,5||5,4,1)$ is given by the cuts

$$A_5^{\text{NLSM}\oplus\phi^3}(1,2,3,4,5||5,4,1) = \int d\mu_5^{\Lambda}$$



$$= -\frac{A'_4(P_{51},2,3,4)}{s_{15}} + \text{Singular-cut}. \tag{826}$$

Clearly, by comparing the above expression with the last line in eq. (822), we arrive at

$$\text{Singular-cut} = -\frac{A'_4(P_{45},1,2,3)}{s_{45}} - 1, \tag{827}$$

which is a simple but a strong result. As it has been argued several times [372, 374] (see section 15.5), the integration rules, which were obtained by expanding at leading order the Λ parameter of the double cover representation, can not be applied over singular cuts. In order to achieve an extension of these rules to singular cuts, one must expand beyond leading order the Λ parameter and find a pattern, which is a highly non-trivial task. Nevertheless, eq. (827) tells us that the soft limit behaviour could help us to figure out this issue. This is an interesting subject to be studied in a future project.

15.10.1 *A New Relation for $A_n^{\text{NLSM}\oplus\phi^3}$*

In the previous section, we observe that, using the recursion relation proposed in section 15.9, the soft limit behaviour of the six-point amplitude, $A_6(1,2,3,4,5,6)$, gives a factorized formula for $A_5^{\text{NLSM}\oplus\phi^3}(1,2,3,4,5||5,a,1)$ in terms of off-shell NLSM amplitudes. In this section, we are going to show a new factorization formula for the general amplitude, $A_n^{\text{NLSM}\oplus\phi^3}(1,\ldots,n||n,a,1)$.

First, let us consider the gauge fixing $(pqr|m) = (1an|2)$, so, we can suppose that the set of particles, $\{P_1, P_a, P_n\}$, are off-shell (here a is a label between $2 < a < n$). Since the $A_n^{\text{NLSM}\oplus\phi^3}(1,\ldots,n||n,a,1)$ amplitude vanishes trivially when n is even, then, it is enough to define, $n = 2m+1$. Thus, applying the integration rules with the previous setup the amplitude is factorized into

$$A_n^{\text{NLSM}\oplus\phi^3}(1,\ldots,a-1,a,a+1,\ldots,n||n,a,1) = \tag{828}$$
$$\sum_{i=2}^{\lfloor\frac{a}{2}\rfloor}\frac{A'_{2i}(P_{2i:n},1,2,\ldots,2i-1)\times A^{\text{NLSM}\oplus\phi^3}_{2(m-i)+3}(P_{1:2i-1},2i,\ldots,a,\ldots,n||n,a,P_{1:2i-1})}{s_{1:2i-1}}+$$
$$\sum_{i=\lceil\frac{a}{2}\rceil}^{m}\frac{A'_{2i}(\boldsymbol{P_{2i+1:1},2,..,a,...,}2i)\times A^{\text{NLSM}\oplus\phi^3}_{2(m-i)+3}(1,P_{2:2i},2i+1,\ldots,n||n,P_{2:2i},1)}{s_{2:2i}},$$

where $\lfloor x\rfloor$ and $\lceil x\rceil$ are the Floor and Ceiling functions, respectively. Notice that when $a=3$, the first line doesn't contribute because of the properties of the Floor function.

In the particular case when $a=2$, we choose the gauge fixing $(pqr|m)=(12n|3)$, and the factorization relation becomes

$$A_n^{\text{NLSM}\oplus\phi^3}(1,2,\ldots,n||n,2,1) = \tag{829}$$
$$\frac{A'_{2m}(n,P_{12},3,\ldots,n-1)\times A_3^{\text{NLSM}\oplus\phi^3}(P_{3:n},1,2||P_{3:n},2,1)}{s_{3:n}}+$$
$$\sum_{i=2}^{m}\frac{A'_{2i}(P_{2i+1:1},2,3,\ldots,2i)\times A^{\text{NLSM}\oplus\phi^3}_{2(m-i)+3}(1,P_{2:2i},2i+1,\ldots,n||n,P_{2:2i},1)}{s_{2:2i}}.$$

Clearly, when $n=2m+1=5$, the relations obtained above are in agreement with the ones in eqs. (824) and (825).

15.11 SPECIAL GALILEON THEORY

In Ref. [246], Cachazo, He and Yuan proposed the CHY prescription to compute the S-Matrix of a special Galileon theory (sGal). The Galileon theories arise as effective field theories in the decoupling limit of massive gravity [312, 387, 388]. The special Galileon theory was discovered in Refs. [246, 389] as a special class of theory with soft limits that vanish particularly fast.

As discussed previously (for more details, see Ref. [246]), the CHY prescription of the sGal is given by the integral

$$A_n^{\text{sGal}} = \int d\mu_n^{\text{CHY}}\,(z_{pq}z_{qr}z_{rp})^2\times\left[\det{}'\text{A}_n\times\det{}'\text{A}_n\right]. \tag{830}$$

From this expression, it is straightforward to see the sGal is the square of the NLSM, where the product is by means of the field theory Kawai-Lewellen-Tye (KLT) kernel [237]. Schematically, one has

$$A_n^{\rm sGal} = A_n \overset{\rm KLT}{\otimes} A_n, \tag{831}$$

where the KLT matrix, usually denoted as $S[\alpha|\beta]$, is the inverse matrix of the double-color partial amplitude for the bi-adjoint ϕ^3 scalar theory [348, 350]. Notice that, from this double copy formula, we can use whole technology developed for NLSM and apply it in sGal. Nevertheless, since our main aim is to show how the integration rules work, we will not use eq. (831).

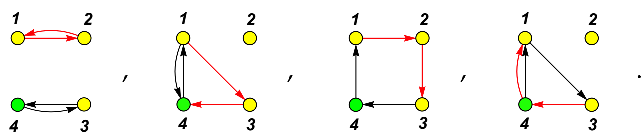
15.11.1 *A Simple Example*

In this section, we will show how the integration rules work in a theory without partial ordering. As a simple example, we will calculate the four-point amplitude for sGal.

From eq. (749), the sGal in the double cover representation is given by the integral

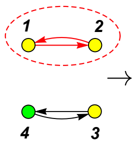
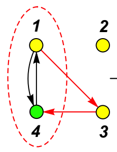
$$A_n^{\rm sGal} = \int d\mu_n^\Lambda \frac{(-1)\Delta(pqr)\Delta(pqr|m)}{S_m^\tau} \times \left[\mathbf{det}'\mathsf{A}_n^\Lambda \times \mathbf{det}'\mathsf{A}_n^\Lambda\right]. \tag{832}$$

where we have defined, $\mathbf{det}'\mathsf{A}_n^\Lambda = \prod_{a=1}^n \frac{(y\sigma)_a}{y_a} \times \det'\mathsf{A}_n^\Lambda$. After choosing a gauge fixing, by the **rule-I** in section 15.5 we know that the Faddeev-Popov factor goes as, $\frac{(-1)\Delta(pqr)\Delta(pqr|m)}{S_m^\tau} \sim \Lambda^{-4} + \mathcal{O}(\Lambda^{-2})$, (eq. (763)). Thus, in order to cancel this Λ^{-4} factor, at leading order, a cut-contribution in the special Galileon theory must cut at least one arrow of each reduced determinant, this fact comes from table 15.5.1. This is summarized in **Rule-II**. For example, for the four-point amplitude, $A_4^{\rm sGal}(1,2,3,4)$, let us consider the following four different setups



(833)

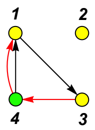
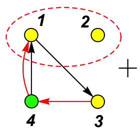
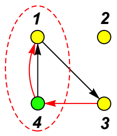
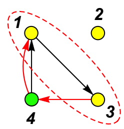
where the red/black arrows denote a given reduced determinant. Clearly, the first two graphs with reduced matrices, $(\mathsf{A}_4^{\Lambda})^{12}_{12} \times (\mathsf{A}_4^{\Lambda})^{34}_{34}$ and $(\mathsf{A}_4^{\Lambda})^{13}_{34} \times (\mathsf{A}_4^{\Lambda})^{14}_{14}$, respectively, have the following singular cuts


$$\rightarrow \mathbf{det}'\mathsf{A}_4^{\Lambda} \times \mathbf{det}'\mathsf{A}_4^{\Lambda}\Big|^{12}_{34} \sim \Lambda^0\,,$$

$$\rightarrow \mathbf{det}'\mathsf{A}_4^{\Lambda} \times \mathbf{det}'\mathsf{A}_4^{\Lambda}\Big|^{41}_{23} \sim \Lambda^2\,.$$

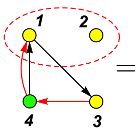
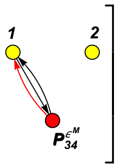
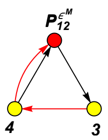
On the other hand, the third and fourth graphs do not have any singular cuts, therefore, we can apply the integration rules over them.

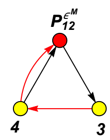
The Four-Point Computation

To carry out the four-point sGal amplitude, we choose the fourth setup in eq. (833). Thus, from the integration rules, we have three cut contributions given by

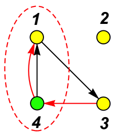
$$A_4^{\text{sGal}}(1,2,3,4) = \int d\mu_4^{\Lambda}$$

$$=$$

cut − 1
$$+$$

cut − 2
$$+$$

cut − 3
$$. \qquad (834)$$

It is straightforward to see that the first contribution vanishes trivially,

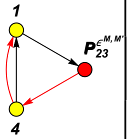

cut − 1
$$=$$

$$\times \left(\frac{1}{s_{34}}\right) \times$$

$$= \sum_M (\sigma_{12}\sigma_{2P_{34}}\sigma_{P_{34}1})^2 \times$$

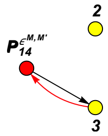
$$\mathrm{PT}(1,P_{34})\,\det\Big[(\mathsf{A}_3)^{1P_{34}}_{1P_{34}}\Big] \times \frac{1}{\sigma_{P_{34}1}}\det\Big[(\mathsf{A}_3)^{P_{34}}_{1}\Big]\Big|_{P_{34}\to\frac{\epsilon_{34}^M}{\sqrt{2}}} \times \left(\frac{1}{s_{34}}\right) \times$$

$$= 0,$$

where we used the identity, $\det\Big[(\mathsf{A}_3)^{1P_{34}}_{1P_{34}}\Big] = 0$. The first and second reduced determinants correspond to the black and red arrows, respectively. In the following, we associate the first reduced determinant with the black arrows, and the second reduced determinant with the red arrows. By a similar computation, the *cut-3* also vanishes, then, the only non-zero contribution comes from the *cut-2*.



cut − 2

$=$ $\Big[$ 

$\Big] \times \left(\frac{1}{s_{14}}\right) \times \Big[$ 

$\Big] = \sum_{M,M'} (\sigma_{41}\sigma_{1P_{23}}\sigma_{P_{23}4})^2 \times$

$$\left[\frac{1}{\sigma_{41}\sigma_{1P_{23}}} \det\left[(\mathsf{A}_3)^{41}_{1P_{23}}\right]\Big|_{P_{23}\to\frac{\epsilon_{23}^{M}}{\sqrt{2}}} \times \frac{1}{\sigma_{P_{23}4}\sigma_{41}} \det\left[(\mathsf{A}_3)^{P_{23}4}_{41}\right]\Big|_{P_{23}\to\frac{\epsilon_{23}^{M'}}{\sqrt{2}}}\right] \times \left(\frac{1}{s_{14}}\right) \times$$

$$(\sigma_{23}\sigma_{3P_{14}}\sigma_{P_{14}2})^2 \times \left[\frac{1}{\sigma_{P_{14}3}} \det\left[(\mathsf{A}_3)^{P_{14}}_{3}\right]\Big|_{P_{14}\to\frac{\epsilon_{14}^{M}}{\sqrt{2}}} \times \frac{1}{\sigma_{3P_{14}}} \det\left[(\mathsf{A}_3)^{3}_{P_{14}}\right]\Big|_{P_{14}\to\frac{\epsilon_{14}^{M'}}{\sqrt{2}}}\right]$$

$$= -s_{12}\, s_{13}\, s_{14}\ ,$$

where the completeness identities, $\sum_M \epsilon_{23}^{\mu\,M}\epsilon_{14}^{\nu\,M} = \eta^{\mu\nu}$ and $\sum_{M'} \epsilon_{23}^{\mu\,M'}\epsilon_{14}^{\nu\,M'} = \eta^{\mu\nu}$, have been used. Therefore, we obtain

$$A_4^{\rm sGal}(1,2,3,4) = -s_{12}\, s_{13}\, s_{14}, \tag{835}$$

which is the right answer.

Finally, it is straightforward to generalize this simple example to a higher number of points. Additionally, it would be interesting to understand the properties of the special Galileon theory similar to ones obtained for NLSM in sections 15.7.3, 15.8.1 and 15.9.

15.12 CONCLUSIONS

The double-cover version of the CHY formalism is an intriguing extension that sheds new light on how scattering amplitudes can emerge as factorized pieces. Focusing on the non-linear sigma model, we have illustrated how unphysical channels appear at intermediate steps, always canceling in the end, and thus producing the right answer. The origin of factorizations is the appearance of one 'free' scattering equation. By fixing four σ-variables rather than three as in the ordinary CHY formalism, there is no longer a one-to-one match between the σ-variables and the number of independent scattering equations. This is the origin of the off-shell channel through which the amplitudes factorize.

We have analyzed the factorizations obtained in the non-linear sigma model because they perfectly illustrate the mechanism, and the cancellations that eventually render the full result free of unphysical poles. For this theory, we have obtained three different factorization relationships, two of them emerged naturally from the double-cover framework (by using the A_{2n} and A'_{2n} prescriptions), while the other one was obtained fortuitously by considering the longitudinal degrees of freedom of the cut-contributions from the new A'_{2n} prescription. By comparing to BCFW on-shell recursion relations we have found a perfect correspondence between the unphysical terms of the double-cover formalism and terms that arise from poles at infinity in the BCFW formalism. In that sense, the double-cover version of CHY succeeds in evaluating what appears as poles at infinity in BCFW recursion as simple CHY-type integrals of the double cover. It would be interesting if this correspondence could be made more explicit. Certainly, it hints at the possibility that an alternative formulation of the problem of poles at infinity in BCFW recursion exists, without recourse to the particular double-cover formalism.

Using the new prescription for the reduced determinant in the integrand, we found a factorization relation where all the intermediate off-shell particles are spin-1 (gluons). The corresponding momenta in the reduced determinants are replaced by polarization vectors. We would like to investigate further the connection between this new object and the integrand for generalized Yang-Mills-Scalar theory [246]. At first sight, we thought that this new matrix could be related to the novel model proposed by Cheung, Remmen, Shen, and Wen in [254, 262], nevertheless, after comparing the numerators at the four-point computation, the relation among these two approaches is unclear.

However, when we replaced the off-shell gluons with only the longitudinal degrees of freedom, we were able to rewrite the factorized pieces in terms of lower-point NLSM amplitudes in the new prescription, with up to three off-shell punctures. This is a very surprising result, and understanding the origin of this connection is left for future work. The big advantage of being able to rewrite the factorized pieces is that we can iteratively promote the lower-point NLSM amplitudes to the double cover, which would lead to further factorization. Thus, any NLSM amplitude can be factorized entirely in terms of off-shell three-point amplitudes. This is a novel off-shell recursion relation. The resulting expression is algebraic, and no scattering equation needs to be solved. We have checked the validity of the recursion relation up to ten points (17 points for odd amplitudes). We would like to find the connection between the recursion relation and Berends-Giele currents [375, 376].

The novel recursion relation can also be used to investigate singular cuts and NLSM $\oplus\,\phi^3$ amplitudes through the soft limit. CCM showed how the soft limit of an NLSM amplitude can be expressed in terms of NLSM $\oplus\,\phi^3$ amplitudes [259]. We calculated the soft limit of a six-point NLSM amplitude in two ways, using the CCM formula and using the novel recursion relation. This gives a relation for a specific singular cut. Further investigations into the nature of the soft limits might reveal insight into the singular cuts in general. Also, we were able to find a factorization relation for the NLSM $\oplus\,\phi^3$ amplitudes.

Lastly, we showed how the special Galileon amplitudes can be calculated in a double cover language. One intriguing feature is that for some configurations, the off-shell particle propagating between the lower-point pieces is spin-2 (graviton). So, we have observed that for the NLSM, off-shell gluons appear, while for the special Galileon theory, both off-shell gluons and gravitons appear. This might be connected to the fact that the NLSM originated as an effective theory of pion scattering, while the

Galileon theories arise as effective field theories in the decoupling limit of massive gravity. This also seems natural, as the special Galileon theory is the square of the NLSM, using the KLT relation.

It seems evident that there are numerous aspects of CHY on a double cover that need to be investigated.

APPENDIX

15.A SOME MATRIX IDENTITIES

In this section, we are going to provide some useful properties of the determinant of the A_n matrix. Although we lack formal proofs for many of the relations, we have performed numerous checks, up to ten points.

15.A.1 *A New NLSM Prescription from CHY*

In this appendix, we formulate two propositions which have been employed to redefine the n-point NLSM amplitude from the CHY framework.

Proposition 1: Let M be a $2n \times 2n$ antisymmetric matrix. Then M satisfy the identity

$$\mathrm{Pf}\Big[(M)^{ik}_{ik}\Big] \times \mathrm{Pf}\Big[(M)^{kj}_{kj}\Big] = \det\Big[(M)^{ik}_{kj}\Big], \tag{836}$$

up to an overall sign.

Proof: We start with the Desnanot-Jacobi identity [390], given by

$$\det\left[M\right]\ \det\left[(M)^{ij}_{ij}\right] = \det\left[(M)^{i}_{i}\right]\ \det\left[(M)^{j}_{j}\right] - \det\left[(M)^{i}_{j}\right]\ \det\left[(M)^{j}_{j}\right]. \tag{837}$$

Now, let M be a $2n \times 2n$ antisymmetric matrix, therefore, $(M)^k_k$ is a $(2n-1)\times(2n-1)$ antisymmetric matrix. Thus, from the identity in eq. (837), it is straightforward to see that

$$0 = \det\left[(M)^{ki}_{ki}\right]\ \det\left[(M)^{kj}_{kj}\right] - \det\left[(M)^{ki}_{kj}\right]\ \det\left[(M)^{kj}_{ki}\right], \tag{838}$$

where we used the fact, $\det\left[(M)^k_k\right] = \det\left[(M)^{kij}_{kij}\right] = 0$. Since, $[(M)^{kj}_{ki}] = [(M)^{ki}_{kj}]^{\mathrm{t}} = -[(M)^{ki}_{kj}]$, then

$$\Big\{\mathrm{Pf}\Big[(M)^{ik}_{ik}\Big] \times \mathrm{Pf}\Big[(M)^{kj}_{kj}\Big]\Big\}^2 = \Big\{\det\Big[(M)^{ik}_{kj}\Big]\Big\}^2, \tag{839}$$

and **proposition 1** has been proved.

Proposition 2: Let A be the antisymmetric matrix defined in eq. (735). When its size is $(2n+1)\times(2n+1)$, then

$$\det\left[(\mathsf{A})^{ik}_{kj}\right] = 0. \tag{840}$$

Proof: Let us consider the $2n\times 2n$ antisymmetric matrix given by $(\mathsf{A})^{k}_{k}$. Thus, from the Desnanot-Jacobi identity in eq. (837), one has

$$\det\left[(\mathsf{A})^{k}_{k}\right]\times\det\left[(\mathsf{A})^{kij}_{kij}\right] = -\left\{\det\left[(\mathsf{A})^{ik}_{kj}\right]\right\}^2, \tag{841}$$

where we used, $\det\left[(\mathsf{A})^{ki}_{ki}\right] = \det\left[(\mathsf{A})^{kj}_{kj}\right] = 0$. Under the support of the scattering equations, $S_a = 0$, and the on-shell conditions, $k_a^2 = 0$, it is simple to show that the A matrix has co-rank 2, therefore, $\det\left[(\mathsf{A})^{k}_{k}\right] = 0$. This implies that, $\det\left[(\mathsf{A})^{ik}_{kj}\right] = 0$, and the proof is completed.

15.A.2 *Off-shell Determinant Properties*

In this appendix we give some properties of the determinant when there is an off-shell particle. These properties involve the matrices, A_n and $\mathsf{A}_n\Big|_{P_i\to\frac{1}{\sqrt{2}}\epsilon_i}$.

This is very important to remark that those properties are supported on the solution of the scattering equations, and, although we do not have a formal proof, they have been checked up to ten points.

Let us consider n-particles with momenta, $(P_1, P_2, P_3, k_4, ..., k_n)$, where the first three are off-shell, *i.e.* $P_i^2 \neq 0$, and the momentum conservation condition is satisfied, $P_1 + P_2 + P_3 + k_4 \cdots + k_n = 0$. Additionally, the three off-shell punctures are fixed, $\sigma_{P_1} = \mathrm{c}_1$, $\sigma_{P_2} = \mathrm{c}_2$, $\sigma_{P_3} = \mathrm{c}_3$, $\mathrm{c}_i \in \mathbb{C}$, where $\mathrm{c}_1 \neq \mathrm{c}_2 \neq \mathrm{c}_3$. Thus, the "$n-3$" scattering equations are given by

$$S_a = \frac{2\,k_a\cdot P_1}{\sigma_{aP_1}} + \frac{2\,k_a\cdot P_2}{\sigma_{aP_2}} + \frac{2\,k_a\cdot P_3}{\sigma_{aP_3}} + \sum_{\substack{b=4\\ a\neq b}}^{n}\frac{2\,k_a\cdot k_b}{\sigma_{ab}} = 0, \quad a = 4,\ldots,n. \tag{842}$$

Properties:

Under the support of the scattering equations and using the above setup, we have the following properties

I. Let n an odd number, $n = 2m+1$, then

$$\det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right] = (P_1^2 - P_2^2 - P_3^2)\times\frac{(-1)}{\sigma_{P_2\,P_3}}\det\left[(\mathsf{A}_n)^{P_1P_2}_{P_2P_3}\right]. \tag{843}$$

Notice that if all particles are on-shell, $P_i^2 = 0$, the right hand side vanishes trivially by the overall factor, $(P_1^2 - P_2^2 - P_3^2)$.

When the momentum P_1^μ is replaced by an off-shell polarization vector, $P_1^\mu \to \frac{1}{\sqrt{2}}\epsilon_1^\mu$, $(\epsilon_1 \cdot P_1 \neq 0)$, the identity keeps the same form, namely

$$\det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right]\Big|_{P_1^\mu \to \frac{1}{\sqrt{2}}\epsilon_1^\mu} = (P_1^2 - P_2^2 - P_3^2) \times \frac{(-1)}{\sigma_{P_2P_3}} \det\left[(\mathsf{A}_n)^{P_1P_2}_{P_2P_3}\right]\Big|_{P_1^\mu \to \frac{1}{\sqrt{2}}\epsilon_1^\mu}. \tag{844}$$

This identity is no longer satisfied if there are two off-shell polarization vectors.

II. Let n an even number, $n = 2m$, then

$$\begin{aligned} \frac{(-1)}{\sigma_{P_1P_2}} \det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right] &= -(P_1^2 + P_2^2 + P_3^2) \times \frac{1}{\sigma_{P_1P_2}\,\sigma_{P_2P_3}} \det\left[(\mathsf{A}_n)^{P_1P_2}_{P_2P_3}\right] \\ &= -(P_1^2 + P_2^2 + P_3^2) \times \frac{(-1)}{\sigma_{P_1P_2}\,\sigma_{P_2P_1}} \det\left[(\mathsf{A}_n)^{P_1P_2}_{P_2P_1}\right]. \end{aligned} \tag{845}$$

If all particles are on-shell, $P_i^2 = 0$, the right hand side vanishes trivially by the overall factor, $(P_1^2 + P_2^2 + P_3^2)$.

When the momentum P_1^μ is replaced by an off-shell polarization vector, $P_1^\mu \to \frac{1}{\sqrt{2}}\epsilon_1^\mu$, $(\epsilon_1 \cdot P_1 \neq 0)$, then, eq. (845) is no longer an identity. Instead, we have a new identity given by

$$\frac{(-1)}{\sigma_{P_1\,P_2}} \det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right]\Big|_{P_1^\mu \to \frac{1}{\sqrt{2}}\epsilon_1^\mu} = \frac{1}{\sigma_{P_1\,P_3}} \det\left[(\mathsf{A}_n)^{P_1}_{P_3}\right]\Big|_{P_1^\mu \to \frac{1}{\sqrt{2}}\epsilon_1^\mu}. \tag{846}$$

If there are two off-shell polarization vectors, then, this equality is no longer true.

III. Let n an odd number, $n = 2m + 1$, and let us consider the particles P_1 and P_2 on-shell $(P_1^2 = P_2^2 = 0)$. Then, we have the following identities

$$\frac{1}{\sigma_{P_1\,P_3}} \det\left[(\mathsf{A}_n)^{P_1}_{P_1}\right] = \frac{(-1)}{\sigma_{P_2\,P_3}} \det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right], \tag{847}$$

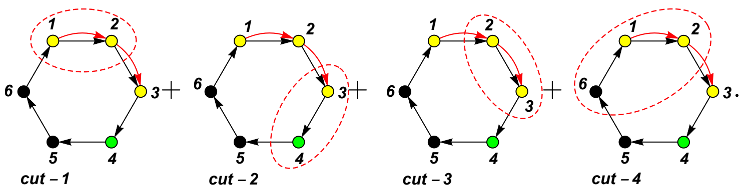
$$\det\left[(\mathsf{A}_n)^{P_1}_{P_1}\right] = \left[P_1^2 \times \frac{1}{\sigma_{P_2\,P_3}}\right]^2 \det\left[(\mathsf{A}_n)^{P_1P_2P_3}_{P_1P_3P_3}\right]. \tag{848}$$

15.B SIX-POINT COMPUTATIONS

In this section we are going to explicitly calculate the six-point NLSM amplitudes $A'_6(\mathbb{I}^{(123)})$, $A'_6(\mathbb{I}^{(134)})$ and $A_6(\mathbb{I}^{(13)})$, where the two first are defined with the new integrand prescription, while the third is defined with the standard integrand. We will calculate some of the cut-contributions in detail, with the hope that the reader becomes more familiar with the double cover formalism. The rest of the cut-contributions can be computed in a similar way.

15.B.1 $A'_6(\mathbb{I}^{(123)})$

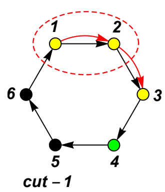
Let us consider the six-point NLSM amplitude, $\mathcal{A}_6(1,2,3,4,5,6)$, with the gauge fixing, $(pqr|m) = (123|4)$, and the reduced matrix $[\mathsf{A}_n]^{12}_{23}$ (*i.e.* $(ijk) = (123)$). Applying **rule-I**, this amplitude has the following contributions

$A'_6(\mathbb{I}^{(123)}) =$ 

(849)

We will compute in detail the first contribution, which we call *cut-1*. The other cuts can be evaluated using the same techniques.

From the **integration rules**, *cut-1* is evaluated as



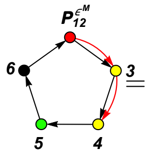
$$= \sum_M \frac{A'_3(1,2,P^{\epsilon^M}_{3:6}) \times A_5^{(P_{12}\,3)}(P^{\epsilon^M}_{12},3,4,5,6)}{s_{3:6}}. \qquad (850)$$

The three-point amplitude was already computed in eq. (766). We remind ourselves that the notation $P^{\epsilon^M}_{3:6}$ means that the off-shell momentum, $P^\mu_{3:6}$, must be replacement by the polarization vector, $P^\mu_{3:6} \to \frac{1}{\sqrt{2}}\,\epsilon^{M\,\mu}_{3:6}$. More precisely, the three-point amplitude becomes

$$A'_3(1,2,P^{\epsilon^M}_{3:6}) = \sqrt{2}\,(\epsilon^M_{3:6}\cdot k_1)\,. \qquad (851)$$

Before computing the five-point amplitude in eq. (850), it is useful to use the identity, $A_5^{(P_{12}\,3)}(P_{12}^{\epsilon^M},3,4,5,6) = P_{12}^2 \times A_5'(P_{12}^{\epsilon^M},3,4,5,6)$. Thus, by applying the integration rules for $A_5'(P_{12}^{\epsilon^M},3,4,5,6)$ one has

$$A_5'(P_{12}^{\epsilon^M},3,4,5,6) = \int d\mu_5^{\Lambda}$$



$$= \sum_N \left\{ \frac{A_3'(P_{12}^{\epsilon^M},3,P_{4:6}^{\epsilon^N}) \times A_4^{(P_{1:3}\,4)}(P_{1:3}^{\epsilon^N},4,5,6)}{s_{4:6}} + \frac{A_4^{(P_{12}P_{34})}(P_{12}^{\epsilon^M},P_{34}^{\epsilon^N},5,6) \times A_3'(P_{5:2}^{\epsilon^N},3,4)}{s_{56P_{12}}} + \frac{A_4'(P_{12}^{\epsilon^M},3,P_{45}^{\epsilon^N},6) \times A_3^{(P_{6:3}\,4)}(P_{6:3}^{\epsilon^N},4,5)}{s_{45}} \right\}, \tag{852}$$

with $\sum_N \epsilon_i^{N\,\mu} \epsilon_j^{N\,\nu} = \eta^{\mu\nu}$. From the building blocks in eqs. (766) and (767), the above three-point amplitudes are straightforward to compute. We find that

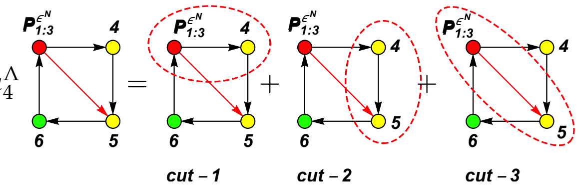
$$A_3'(P_{12}^{\epsilon^M},3,P_{4:6}^{\epsilon^N}) = \epsilon_{12}^M \cdot \epsilon_{4:6}^N, \quad A_3'(P_{5:2}^{\epsilon^N},3,4) = \sqrt{2}\,\epsilon_{5:2}^N \cdot k_4, \quad A_3^{(P_{6:3}\,4)}(P_{6:3}^{\epsilon^N},4,5) = \sqrt{2}\,s_{45}\,(\epsilon_{6:3}^N \cdot k_5). \tag{853}$$

Next, using the same procedure as in eq. (777), we evaluate the four-point graph, $A_4'(P_{12}^{\epsilon^M},3,P_{45}^{\epsilon^N},6)$, arriving at

$$A_4'(P_{12}^{\epsilon^M},3,P_{45}^{\epsilon^N},6) = 2(\epsilon_{12}^M \cdot k_6)\,(\epsilon_{45}^N \cdot k_6) \left(\frac{1}{s_{6P_{45}}} + \frac{1}{s_{6P_{12}}} \right). \tag{854}$$

On the other hand, in order to avoid singular cuts when applying the integration rules over $A_4^{(P_{1:3}\,4)}(P_{1:3}^{\epsilon^N},4,5,6)$, we employ the identity, $A_4^{(P_{1:3}\,4)}(P_{1:3}^{\epsilon^N},4,5,6) = A_4^{(P_{1:3}\,5)}(P_{1:3}^{\epsilon^N},4,5,6)$. Thus,

$$A_4^{(P_{1:3}\,5)}(P_{1:3}^{\epsilon^N},4,5,6) = \int d\mu_4^{\Lambda}$$



$$= -\sqrt{2}\,s_{46}(\epsilon_{1:3}^N \cdot k_4) - \frac{\sqrt{2}\,s_{46}\,s_{45}\,(\epsilon_{1:3}^N \cdot k_6)}{s_{6P_{1:3}}} - \sqrt{2}\,s_{46}\,(\epsilon_{1:3}^N \cdot k_5), \tag{855}$$

where we again have used the three-point building blocks in eqs. (766) and (767). Lastly, since for the amplitude, $A_4^{(P_{12}\,P_{34})}(P_{12}^{\epsilon^M}, P_{34}^{\epsilon^N}, 5, 6)$, the above identity is no longer valid, namely[7]

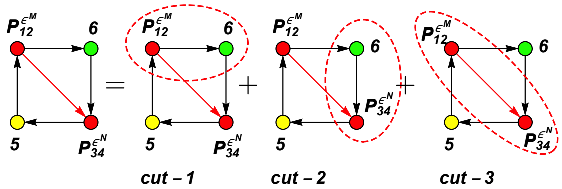
$$A_4^{(P_{12}\,P_{34})}(P_{12}^{\epsilon^M}, P_{34}^{\epsilon^N}, 5, 6) \neq A_4^{(P_{12}\,5)}(P_{12}^{\epsilon^M}, P_{34}^{\epsilon^N}, 5, 6), \tag{856}$$

we make use of the BCJ relation [369, 370],

$$s_{65}\,\mathrm{PT}(5, 6, P_{12}, P_{34}) + s_{6P_{125}}\,\mathrm{PT}(5, P_{12}, 6, P_{34}) = 0. \tag{857}$$

From this we obtain the equality $A_4^{(P_{12}\,P_{34})}(P_{12}^{\epsilon^M}, P_{34}^{\epsilon^N}, 5, 6) = \left(\frac{s_{6P_{34}}}{s_{56}}\right) \times A_4^{(P_{12}\,P_{34})}(P_{12}^{\epsilon^M}, 6, P_{34}^{\epsilon^N}, 5)$. Now, applying the integration rules, one has

$$A_4^{(P_{12}\,P_{34})}(P_{12}^{\epsilon^M}, P_{34}^{\epsilon^N}, 5, 6) = \left(\frac{s_{6P_{34}}}{s_{56}}\right) \times \int d\mu_4^{\Lambda}$$



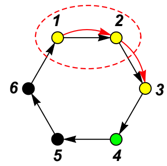
$$= -\frac{2\,s_{6P_{34}}\,(\epsilon_{12}^M \cdot k_6)\,(\epsilon_{34}^N \cdot k_5)}{s_{6P_{12}}} - 2\,(\epsilon_{12}^M \cdot k_5)\,(\epsilon_{34}^N \cdot k_6) - s_{6P_{34}}(\epsilon_{12}^M \cdot \epsilon_{34}^N). \tag{858}$$

Utilizing the results obtained in eqs. (853) to (855) and (858), it is straightforward to check the five-point amplitude, $A_5^{(P_{12}\,3)}(P_{12}^{\epsilon^M}, 3, 4, 5, 6)$, is given by

$$\begin{aligned} A_5^{(P_{12}\,3)}(P_{12}^{\epsilon^M}, 3, 4, 5, 6) = -s_{12}\,\sqrt{2}\,\Bigg\{ &\frac{s_{46}}{s_{4:6}}\left[(\epsilon_{12}^M \cdot k_4) + \frac{s_{45}(\epsilon_{12}^M \cdot k_6)}{s_{6P_{1:3}}} + (\epsilon_{12}^M \cdot k_5)\right] \\ + \frac{s_{6P_{34}}}{s_{56P_{12}}}\left[\frac{s_{45}\,(\epsilon_{12}^M \cdot k_6)}{s_{6P_{12}}} + \frac{s_{46}\,(\epsilon_{12}^M \cdot k_5)}{s_{6P_{34}}} + (\epsilon_{12}^M \cdot k_4)\right] &- s_{56}\,(\epsilon_{12}^M \cdot k_6)\left[\frac{1}{s_{6P_{45}}} + \frac{1}{s_{6P_{12}}}\right]\Bigg\}, \end{aligned} \tag{859}$$

[7] This is because there is more than one off-shell polarization vector.

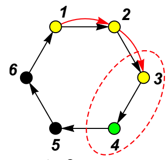
and therefore *cut-1* in eq. (850) is given by



cut − 1

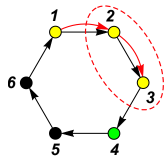
$$
= \sum_M \frac{A'_3(1,2,P^{\epsilon M}_{3:6}) \times A_5^{(P_{12}\,3)}(P^{\epsilon M}_{12},3,4,5,6)}{s_{3:6}} = -\left\{ \frac{s_{46}}{s_{4:6}} \left[s_{14} + \frac{s_{45}s_{16}}{s_{6P_{1:3}}} + s_{15} \right] + \frac{s_{6P_{34}}}{s_{56P_{12}}} \left[\frac{s_{45}\,s_{16}}{s_{6P_{12}}} + \frac{s_{46}\,s_{15}}{s_{6P_{34}}} + s_{14} \right] - s_{56}\,s_{16} \left[\frac{1}{s_{6P_{45}}} + \frac{1}{s_{6P_{12}}} \right] \right\}. \qquad (860)
$$

The other contributions, *cut-2,3,4*, are calculated in a similar fashion. We find that



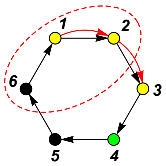
cut − 2

$$
= \sum_M \frac{A'_5(1,2,P^{\epsilon M}_{34},5,6) \times A_3^{(P_{5:2}\,3)}(P^{\epsilon M}_{5:2},3,4)}{s_{34}} = -\left\{ \frac{s_{15}}{s_{5:1}} \left[s_{14} + \frac{s_{45}s_{16}}{s_{5P_{2:4}}} + s_{46} \right] + \frac{s_{5P_{12}}}{s_{56P_{34}}} \left[\frac{s_{45}\,s_{16}}{s_{5P_{34}}} + \frac{s_{46}\,s_{15}}{s_{5P_{12}}} + s_{14} \right] - s_{56}\,s_{45} \left[\frac{1}{s_{5P_{16}}} + \frac{1}{s_{5P_{34}}} \right] \right\}, \qquad (861)
$$



cut − 3

$$
= \sum_M \frac{A'_3(P^{\epsilon M}_{4:1},2,3) \times A_5^{(1\,P_{23})}(1,P^{\epsilon M}_{23},4,5,6)}{s_{4:1}} = -\left\{ \frac{s_{46}}{s_{4:6}} \left[s_{34} + \frac{s_{45}s_{36}}{s_{6P_{1:3}}} + s_{35} \right] - \frac{s_{56}\,s_{36}}{s_{6P_{45}}} + \frac{s_{6P_{2:4}}\,s_{34}}{s_{5:1}} \right\}, \qquad (862)
$$

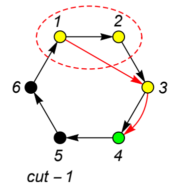


cut − 4

$$
= \sum_M \frac{A'_4(1,2,P^{\epsilon M}_{3:5},6) \times A_4^{(P_{6:2}\,3)}(P^{\epsilon M}_{6:2},3,4,5)}{s_{3:5}} = -\frac{s_{16}\,s_{35}}{s_{3:5}} \times \left(\frac{1}{s_{16}} + \frac{1}{s_{6P_{3:5}}} \right) \times \left(s_{36} + \frac{s_{34}\,s_{56}}{s_{5P_{6:2}}} + s_{46} \right). \qquad (863)
$$

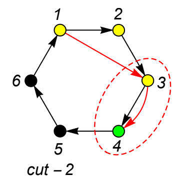
15.B.2 $A'_6(\mathbb{I}^{(134)})$

In this section, we just write down the results found for the cut-contributions obtained in eq. (795). Using the same method presented above, it is straightforward to arrive



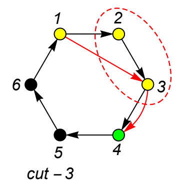
$$= \sum_M \frac{A_3^{(1\,P_{3:6})}(1,2,P_{3:6}^{\epsilon^M})\times A_5'(P_{12}^{\epsilon^M},3,4,5,6)}{s_{3:6}} = \frac{s_{46}}{s_{4:6}}\left[s_{24}+\frac{s_{45}s_{26}}{s_{6P_{1:3}}}+s_{25}\right]$$

$$+\frac{s_{6P_{34}}}{s_{56P_{12}}}\left[\frac{s_{45}\,s_{26}}{s_{6P_{12}}}+\frac{s_{46}\,s_{25}}{s_{6P_{34}}}+s_{24}\right]-s_{56}\,s_{26}\left[\frac{1}{s_{6P_{45}}}+\frac{1}{s_{6P_{12}}}\right], \tag{864}$$

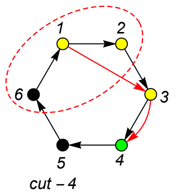


$$= \sum_M \frac{A_5^{(1\,P_{34})}(1,2,P_{34}^{\epsilon^M},5,6)\times A_3'(P_{5:2}^{\epsilon^M},3,4)}{s_{34}} = -\frac{s_{26}\,s_{56}\,s_{45}}{s_{5P_{34}}\,s_{6P_{3:5}}}+\frac{s_{24}\,s_{6P_{2:4}}}{s_{5:1}}$$

$$+\frac{1}{s_{P_{34}56}}\left[s_{25}\,s_{46}+\frac{s_{26}\,s_{6P_{34}}\,s_{45}}{s_{6P_{12}}}+s_{24}\,s_{6P_{34}}\right], \tag{865}$$



$$= \frac{A_3(3,P_{4:1},2)\times A_5'(1,P_{23},4,5,6)}{s_{4:1}} = 0, \tag{866}$$



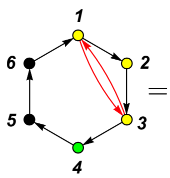
$$= \sum_M \frac{A_4^{(1\,P_{3:5})}(1,2,P_{3:5}^{\epsilon^M},6)\times A_4'(P_{6:2}^{\epsilon^M},3,4,5)}{s_{3:5}} = -\frac{s_{26}\,s_{45}}{s_{3:5}}\times$$

$$\left(\frac{1}{s_{45}}+\frac{1}{s_{5P_{6:2}}}\right)\times\left(s_{15}+\frac{s_{12}\,s_{56}}{s_{6P_{3:5}}}+s_{25}\right). \tag{867}$$

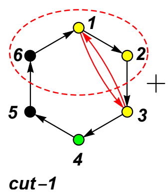
15.B.3 $A_6(\mathbb{I}^{(13)})$

Now, we focus to apply the **integration rules** for $A_6(\mathbb{I}^{(13)})$. We recall that this notation means that the reduced Pfaffian is given by $-\mathrm{PT}^T(1,3)\times\det[(\mathrm{A}_6^\Lambda)^{13}_{13}]$. In addition, such as in the previous examples, we fix the gauge by $(pqr|m) = (123|4)$. Thus, from the eq. (788), we have that

$$A_6(\mathbb{I}^{(13)}) = \int d\mu_6^\Lambda$$



$=$



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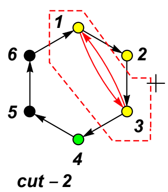

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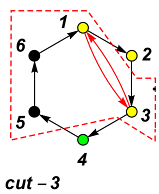
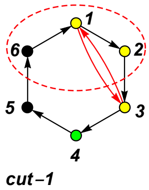
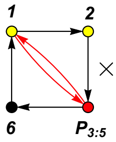


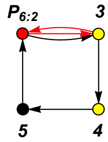
Applying the **integration rules**, *cut-1* is split into



$$= \int d\mu_4^{\text{CHY}}$$



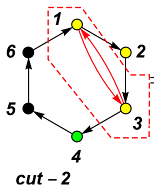
$$\times \left(\frac{1}{s_{345}}\right) \times \int d\mu_4^{\text{CHY}}$$



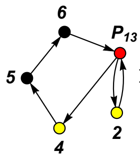
$$= \frac{A_4(\mathbf{1},2,\boldsymbol{P_{3:5}},6) \times A_4(\boldsymbol{P_{6:2}},\mathbf{3},4,5)}{s_{3:5}} = \frac{s_{26}\, s_{35}}{s_{3:5}}. \tag{868}$$

On the last equality we used the identity, $A_4(\boldsymbol{P_{6:2}},\mathbf{3},4,5) = A_4(\boldsymbol{P_{6:2}},3,\mathbf{4},5)$ (in order to avoid singular cuts), and the same procedure as in eq. (771). This identity is supported over the off-shell Pfaffian properties given in appendix 15.A.2.

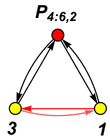
The following contribution is the *cut-2* (*strange-cut*), which, by the **integration rules**, is broken as



$$= \int d\mu_5^{\text{CHY}}$$



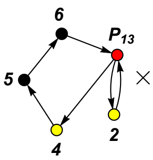
$$\times \left(\frac{1}{s_{4:6,2}}\right) \times$$



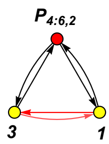
$$. \tag{869}$$

Notice that on the first graph the our method can not be employed. Nevertheless, similar to Yang-Mills theory [374], this strange-cut can be rewritten in the following way

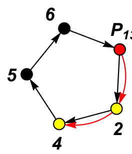
$$\int d\mu_5^{\text{CHY}}$$



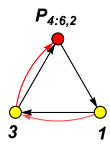
$$\times$$



$$= (-1) \int d\mu_5^{\text{CHY}}$$

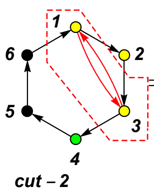


$$\times$$



$$= (-1)\, A_5'(P_{13},2,4,5,6) \times A_3'(1,3,P_{4:6,2}). \tag{870}$$

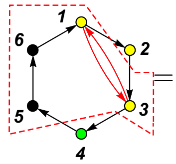
where we used the identities formulated in appendix 15.A.2. Therefore, this cut turns into



$$= (-1)\, \frac{A_5'(P_{13},2,4,5,6) \times A_3'(1,3,P_{4:6,2})}{s_{4:6,2}} = s_{13} \left[\frac{s_{46}}{s_{456}} + \frac{s_{26}+s_{46}}{s_{56P_{13}}} \right], \tag{871}$$

The five-point amplitude, $A_5'(P_{13},2,4,5,6)$, was already calculated in eq. (852) and the three-point function is given in eq. (766).

Lastly, the strange *cut-3* is



cut – 3

$$= (-1)\,\frac{A_3'(P_{5:1,3},2,4)\times A_5'(1,3,P_{24},5,6)}{s_{24}} = s_{24}\left[\frac{s_{26}+s_{46}}{s_{56P_{24}}}+\frac{s_{26}+s_{36}+s_{46}}{s_{561}}\right]. \tag{872}$$

15.B.4 *Longitudinal Contributions*

In this section, we consider just the longitudinal degrees of freedom of all cut-contributions obtained from $A_6'(\mathbb{I}^{(123)})$ and $A_6'(\mathbb{I}^{(134)})$. Those results are used in section 15.7.3.

First, we begin with the cut-structure given in eq. (849) for $A_6'(\mathbb{I}^{(123)})$. We replace $\epsilon^M \to \epsilon^L$, and use eq. (797). The longitudinal contributions become

$$\sum_L \frac{A_3'(1,2,P^{\epsilon^L}_{3:6})\times A_5^{(P_{12}3)}(P^{\epsilon^L}_{12},3,4,5,6)}{s_{3:6}} = \frac{s_{1P_{3:6}}}{2\,s_{12}}\times\left\{\frac{s_{46}}{s_{4:6}}\left[s_{P_{12}P_{45}}+\frac{s_{45}s_{P_{12}6}}{s_{6P_{1:3}}}\right]\right.$$
$$\left.+\frac{s_{6P_{34}}}{s_{56P_{12}}}\left[\frac{s_{45}\,s_{P_{12}6}}{s_{6P_{12}}}+\frac{s_{46}\,s_{P_{12}5}}{s_{6P_{34}}}+s_{P_{12}4}\right]-s_{56}\,s_{P_{12}6}\left[\frac{1}{s_{6P_{45}}}+\frac{1}{s_{6P_{12}}}\right]\right\}. \tag{873}$$

$$\sum_L \frac{A_5'(1,2,P^{\epsilon^L}_{34},5,6)\times A_3^{(P_{5:2}3)}(P^{\epsilon^L}_{5:2},3,4)}{s_{34}} = \frac{s_{4P_{5:2}}}{2\,s_{34}}\times\left\{\frac{s_{15}}{s_{5:1}}\left[s_{P_{34}P_{16}}+\frac{s_{P_{34}5}s_{16}}{s_{5P_{2:4}}}\right]\right.$$
$$\left.+\frac{s_{5P_{12}}}{s_{56P_{34}}}\left[\frac{s_{P_{34}5}\,s_{16}}{s_{5P_{34}}}+\frac{s_{P_{34}6}\,s_{15}}{s_{5P_{12}}}+s_{1P_{34}}\right]-s_{56}\,s_{5P_{34}}\left[\frac{1}{s_{5P_{16}}}+\frac{1}{s_{5P_{34}}}\right]\right\}, \tag{874}$$

$$\sum_L \frac{A_3'(P^{\epsilon^L}_{4:1},2,3)\times A_5^{(1P_{23})}(1,P^{\epsilon^L}_{23},4,5,6)}{s_{4:1}} = \frac{s_{3P_{4:1}}}{2\,s_{23}}\times\left\{\frac{s_{46}}{s_{4:6}}\left[s_{P_{23}P_{45}}+\frac{s_{45}s_{P_{23}6}}{s_{6P_{1:3}}}\right]\right.$$
$$\left.-\frac{s_{56}\,s_{P_{23}6}}{s_{6P_{45}}}+\frac{s_{6P_{2:4}}\,s_{P_{23}4}}{s_{5:1}}\right\}, \tag{875}$$

$$\sum_L \frac{A_4'(1,2,P_{3:5}^{\epsilon^L},6) \times A_4^{(P_{6:2}3)}(P_{6:2}^{\epsilon^L},3,4,5)}{s_{3:5}} = \frac{s_{16}\, s_{6P_{3:5}}}{2\, s_{3:5}} \times \left(\frac{1}{s_{16}} + \frac{1}{s_{6P_{3:5}}}\right) \times$$
$$\frac{s_{35}}{s_{3:5}} \times \left(s_{P_{6:2}P_{34}} + \frac{s_{34}\, s_{5P_{6:2}}}{s_{5P_{6:2}}}\right). \tag{876}$$

To end, we carry out the longitudinal contributions for all cut-contributions of $A_6'(\mathbb{I}^{(134)})$,

$$\sum_L \frac{A_3^{(1P_{3:6})}(1,2,P_{3:6}^{\epsilon^L}) \times A_5'(P_{12}^{\epsilon^L},3,4,5,6)}{s_{3:6}} = -\frac{s_{2P_{3:6}}}{2\, s_{12}} \times \left\{ \frac{s_{46}}{s_{4:6}} \left[s_{P_{12}P_{45}} + \frac{s_{45} s_{P_{12}6}}{s_{6P_{1:3}}} \right]\right.$$
$$\left. + \frac{s_{6P_{34}}}{s_{56P_{12}}} \left[\frac{s_{45}\, s_{P_{12}6}}{s_{6P_{12}}} + \frac{s_{46}\, s_{P_{12}5}}{s_{6P_{34}}} + s_{P_{12}4} \right] - s_{56}\, s_{P_{12}6} \left[\frac{1}{s_{6P_{45}}} + \frac{1}{s_{6P_{12}}} \right] \right\}. \tag{877}$$

$$\sum_L \frac{A_5^{(1P_{34})}(1,2,P_{34}^{\epsilon^L},5,6) \times A_3'(P_{5:2}^{\epsilon^L},3,4)}{s_{34}} = -\frac{s_{4P_{5:2}}}{2\, s_{34}} \times \left\{ \frac{s_{26}\, s_{56}\, s_{P_{34}5}}{s_{5P_{34}}\, s_{6P_{3:5}}} + \frac{s_{2P_{34}}\, s_{6P_{2:4}}}{s_{5:1}} \right.$$
$$\left. + \frac{1}{s_{P_{34}56}} \left[s_{25}\, s_{P_{34}6} + \frac{s_{26}\, s_{6P_{34}}\, s_{5P_{34}}}{s_{6P_{12}}} + s_{2P_{34}}\, s_{6P_{34}} \right] \right\}, \tag{878}$$

$$\sum_L \frac{A_4^{(1P_{3:5})}(1,2,P_{3:5}^{\epsilon^L},6) \times A_4'(P_{6:2}^{\epsilon^L},3,4,5)}{s_{3:5}} = \frac{s_{5P_{6:2}}\, s_{45}}{s_{3:5}} \times \left(\frac{1}{s_{45}} + \frac{1}{s_{5P_{6:2}}}\right) \times$$
$$\frac{s_{26}}{s_{3:5}} \times \left(s_{1P_{3:5}} + \frac{s_{12}\, s_{P_{3:5}6}}{s_{6P_{3:5}}} + s_{2P_{3:5}}\right). \tag{879}$$

$$\frac{A_3(3,P_{4:1},2) \times A_5'(1,P_{23},4,5,6)}{s_{4:1}} = 0, \tag{880}$$

Part IV

CONCLUSION AND FUTURE WORK

16

CONCLUSION

Many interesting quantum field theories relevant for describing phenomena in Nature are effective field theories. In this thesis we have discussed several different effective field theories, ranging from the effective-field-theory extension of the Standard Model to an effective field theory describing the classical (and quantum) behavior of a black hole. In order to extract information from the effective theories, we must perform calculations. Tree- and loop-level calculations are crucial in order to connect the theories to experimentally relevant quantities. In particular, one object of great theoretical and experimental interest is the scattering amplitude. Traditional methods for calculating scattering amplitudes are well-established, and have their use. However, the traditional methods also have several limitations. This is particularly true when dealing with effective field theories. Therefore, modern methods for calculating scattering amplitudes have received a surge of interest in recent years.

One outstanding problem is to fully connect the realm of effective field theory with the modern methods for scattering amplitudes. The work presented in this thesis connects several effective field theories with these modern methods. In particular, we have developed the gravitational theory for a heavy particle, both from an effective-field-theory perspective, with a Lagrangian and associated Feynman rules, as well as from an on-shell methodology. This illustrates the interplay between the two approaches. We believe that this connection will be a fruitful object of study in years to come.

We also presented many novel results which sit more comfortably in one of the two subjects. For the Standard Model Effective Field Theory (SMEFT), we introduced the notion of a curved field space for the gauge sector. Using the curved field space and the background field method, we derived a gauge-fixing term which breaks quantum field gauge invariance while keeping background field gauge invariance. A direct consequence of the background field gauge invariance is background field Ward identities, which we also derived. The background field Ward identities are valid to all orders in the perturbative expansion. Another consequence of the curved field space of the theory is a geometric description of the SMEFT. In particular, the canonically normalized gauge couplings, mixing angles, and masses have geometric definitions. Since we can write down a closed form for the metric defining the curved field space to all orders in the $\sqrt{2H^\dagger H}/\Lambda$ expansion, the geometric definitions of the couplings, masses, and mixing angles are also defined to all orders in the power-counting expansion.

Other results for the SMEFT include a detailed discussion on interference and non-interference effects, as well as a group-theoretic discussion of operators violating baryon and/or lepton number when flavor symmetries are imposed.

When describing phenomena below the electroweak scale, a new effective field theory can be applied, namely the Low-Energy Effective Field Theory (LEFT). We discussed the equations of motion and symmetry currents of the LEFT.

On the scattering-amplitudes side, we investigated several effective field theories using the scattering equation framework extended to a double cover. The theory in focus was the non-linear sigma model, but similar techniques were also applied to the special Galilean theory and the combination of the non-linear sigma model and ϕ^3 theory. We derived novel recursion relations, where any tree-level scattering amplitude in the non-linear sigma model can be recursively calculated from (off-shell) three-point amplitudes.

We also discussed the connection between the soft limit of gauge and gravitational amplitudes using the KLT-relation. In particular, we derived novel relations for the gauge theory amplitudes, which are constrained by the soft behavior of the gravitational amplitudes. These novel relations where generalized to include contributions from effective operators.

Lastly, we developed the Heavy Black Hole Effective Theory (HBET), the gravitational analog of Heavy Quark Effective Theory (HQET), both from a Lagrangian and an on-shell perspective. We also demonstrated the double-copy relation between HQET and HBET.

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FUTURE WORK

Based on the work presented in this thesis, there are many avenues of further research to consider. On the effective-field-theory side, the geometric framework for the Standard Model Effective Field Theory should be developed further. In particular, applying it to phenomenological calculations would be of great value. A separate effective field theory is also relevant for Higgs physics beyond the Standard Model, namely Higgs Effective Field Theory (HEFT). There is a close connection between the SMEFT and the HEFT, but also some crucial differences. In particular, in the HEFT the Higgs field transforms non-linearly under the electroweak gauge symmetry. The geometric framework could be applied to the HEFT. Work has been performed for the scalar sector of the theory. However, analogous to the SMEFT, the geometric framework can be applied for the gauge sector as well. Similar definitions of the gauge couplings, masses and mixing angles can then be defined.

The on-shell description of heavy particles deserves more investigation. The heavy limit is closely related to the non-relativistic limit, which hints at an on-shell description of non-relativistic systems. A non-relativistic version of the on-shell program would be useful for calculations in e.g. post-Newtonian corrections to the binary inspiral problem, relevant for LIGO/LISA observations of gravitational waves.

One of the groundbreaking new results of the amplitude program is the double-copy relation between gauge and gravitational theories, through the KLT-relation, color-kinematics duality, or the scattering-equation framework. One natural extension of this is to investigate general effective field theories in the light of the double-copy relations. This would highlight the underlying structure of the effective theories, as well as to recast complicated calculations involving gravitational interactions in terms of simpler gauge-theory calculations.

Hopefully, several of these research directions will be pursued in the near future.

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