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Quantum gravity in two dimensions

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Abstract

The topic of this thesis is quantum gravity in $1 + 1$ dimensions. We will focus on two formalisms, namely Causal Dynamical Triangulations (CDT) and Dynamical Triangulations (DT). Both theories regularize the gravity path integral as a sum over triangulations. The difference lies in the class of triangulations considered. While the CDT triangulations have a natural Lorentzian structure, in DT the triangulations are Euclidean.

The thesis is built up around three papers, reproduced as Chapter 3, 4 and 6. The outline is as follows: The first two chapters provides background material on path integral quantization and the CDT formalism. In Chapter 3 we consider a generalization of CDT (introduced in Ref. [43]) and show that the continuum limit is the same as for plain CDT. This provides evidence for the robustness of the CDT universality class. Chapter 4 provides an analysis of CDT coupled to Yang-Mills theory. In Chapter 5 we review the DT formalism and some basic aspects of Liouville Theory. We put special emphasis on some subtleties of the continuum limit. Finally, Chapter 6 contains a discussion on mixing between geometrical and matter degrees of freedom, when DT is coupled to non-unitary CFTs.

Most of the material in Chapter 1, 2 and 5 is not new, and we attempt to provide relevant references. Chapter 3 and 4 is co-authored with J. Ambjørn, while Chapter 6 is co-authored with J. Ambjørn, A. Görlich and H.-G. Zhang.

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Contents

Chapter 1

Path integral quantization

In this chapter we will review some aspects of path integrals in quantum mechanics that we will need mainly in the CDT part of the thesis. Let us start by considering a particle moving in one dimensional space subject to the (time independent) Hamiltonian $\hat{H}$. We assume that $\hat{H}$ is bounded from below. A basic object of interest is the propagator

$$G(x', x; T) := \langle x' | e^{-T\hat{H}} | x \rangle, \qquad T \geq 0. \tag{1.1}$$

As usual in the context of path integrals, we prefer to work with the Euclidean propagator (1.1), rather than the physically more natural Lorentzian version

$$G_{\mathrm{Lo}}(x', x; T) := \langle x' | e^{-iT\hat{H}} | x \rangle. \tag{1.2}$$

Inserting $N-1$ copies of the resolution of the identity,

$$\int_{-\infty}^{\infty} \mathrm{d}x \, |x\rangle\langle x| = I, \tag{1.3}$$

we get

$$\begin{aligned} &G(x', x; T) \\ &\quad = \int \left(\prod_{j=1}^{N-1} \mathrm{d}x_j \right) \langle x' | e^{-\epsilon\hat{H}} | x_{N-1} \rangle \langle x_{N-1} | e^{-\epsilon\hat{H}} | x_{N-2} \rangle \cdots \langle x_1 | e^{-\epsilon\hat{H}} | x \rangle, \end{aligned} \tag{1.4}$$

with

$$\epsilon := \frac{T}{N}. \tag{1.5}$$

If we introduce $\tilde{\mathcal{N}}_\epsilon$ and $S_\epsilon(x', x)$ by the definition

$$\tilde{\mathcal{N}}_\epsilon e^{-\epsilon \tilde{S}_\epsilon(x', x)} := G(x', x; T = \epsilon), \tag{1.6}$$

we can write (1.4) in the more suggestive way

$$G(x', x; T) = \int \left(\prod_{j=1}^{N-1} \tilde{\mathcal{N}}_\epsilon \mathrm{d}x_j \right) e^{-\sum_{j=0}^{N-1} \epsilon \tilde{S}_\epsilon(x_{j+1}, x_j)}, \tag{1.7}$$

where we set $x_0 = x$, and $x_N = x'$. Note that the definition (1.6) is ambiguous, since one can shift constant factors around between $\tilde{\mathcal{N}}_\epsilon$ and $\tilde{S}_\epsilon(x', x)$. We will refer to the exponent

$$\sum_{j=0}^{N-1} \epsilon \tilde{S}_\epsilon(x_{j+1}, x_j) \tag{1.8}$$

as the action.

The identity (1.7) is not terribly useful, since one needs to compute the exact propagator to determine $\tilde{S}_\epsilon(x', x)$. The point is that, as long as one is only concerned with the $\epsilon \to 0$ limit, there is a large freedom in the choice of the action.

For example, consider the standard Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.9}$$

It is then well known that, with the choice

$$\mathcal{N}_\epsilon = \sqrt{\frac{m}{2\pi\epsilon}}, \qquad S_\epsilon(x', x) = \frac{m(x'-x)^2}{2\epsilon^2} + V(x'), \tag{1.10}$$

we have

$$\lim_{\epsilon \to 0^+} \int \left(\prod_{j=1}^{N-1} \mathcal{N}_\epsilon \mathrm{d}x_j \right) e^{-\sum_{j=0}^{N-1} \epsilon S_\epsilon(x_{j+1}, x_j)} = G(x', x; T). \tag{1.11}$$

This basically follows from the Trotter product formula.

When $\epsilon \to 0$ the discrete path $j \mapsto x_j$ formally goes to a continuum path $t \mapsto X(t)$, with

$$X(t) = x_{j=t/\epsilon}. \tag{1.12}$$

The action (1.10) then formally becomes

$$\sum_{j=0}^{N-1} \epsilon S_\epsilon(x_{j+1}, x_j) \to S[X] = \int \mathrm{d}t \left(\frac{m}{2} \left[\frac{\mathrm{d}X(t)}{\mathrm{d}t} \right]^2 + V\big(X(t)\big) \right). \tag{1.13}$$

The path integral can now be written as[1]

$$G(x', x; T) = \int_{\mathcal{P}(x', x; T)} \mathcal{D}[X] e^{-S[X]}, \tag{1.14}$$

where the integration domain $\mathcal{P}(x', x; T)$ is the set of continuous paths $t \mapsto X(t)$, with $X(0) = x$ and $X(T) = x'$. A very nice property of the continuum formulation is that the action $S[X]$ is just the (Wick rotated) classical action corresponding to the classical Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \tag{1.15}$$

But, as we will discuss now, the situation is more complicated when the Hamiltonian is not of the form (1.9).

[1] The integral $\int_{\mathcal{P}(x', x; T)} \mathcal{D}[X]$ can be made rigorous using the Wiener measure (at least for certain choices of the action), but we will not need this formalism here.

1.1 Operator ordering and saddle point integration

In the CDT part of this thesis we will need to deal with path integrals corresponding to Hamiltonians of more general form than (1.9). Let us therefore consider the most general Hamiltonian which is quadratic in the momentum. We can parametrize it as

$$\hat{H} = \frac{1}{2m(\hat{x})}\hat{p}^2 + A(\hat{x})\hat{p} + V(\hat{x}), \tag{1.16}$$

where $m(\hat{x})$, $A(\hat{x})$ and $V(\hat{x})$ are arbitrary functions[2]. In analogy with the simple case considered above, we might expect the action for the path integral to be the classical (Wick rotated) action

$$S[X] \stackrel{?}{=} \int \mathrm{d}t \left(\frac{m\big(X(t)\big)}{2} \left[\frac{\mathrm{d}X(t)}{\mathrm{d}t}\right]^2 + im\big(X(t)\big)A\big(X(t)\big)\frac{\mathrm{d}X(t)}{\mathrm{d}t} \right. \\ \left. + V\big(X(t)\big) - \frac{m\big(X(t)\big)}{2}A\big(X(t)\big)^2 \right). \tag{1.17}$$

In a certain sense this is true, but it is not the hole story.

The problem is that in (1.16) we had chosen a particular operator ordering, but this choice is not reflected in the action (1.17). We will see that the resolution to this 'paradox' is related to the fact that the time derivative

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} \tag{1.18}$$

does not behave as one would expect classically.

To understand the situation better, we will derive the correct quantum action in a (semi) careful way, and then check that this action leads back to our initial Hamiltonian (1.16) using saddle point integration. The saddle point technique will also be useful for dealing with the CDT path integrals.

In order to determine the action, we need to compute the small time propagator

$$G(x', x; \epsilon) = \langle x'|e^{-\epsilon \hat{H}}|x\rangle. \tag{1.19}$$

The limit $\epsilon \to 0$ is clearly singular, and simply expanding the exponential in powers of ϵ will not lead to a nice action. A trick is to introduce the usual $\hat{p}$ eigenstates $|p\rangle$, normalized as

$$\langle x|p\rangle := e^{ipx}, \qquad \int \frac{\mathrm{d}p}{2\pi}|p\rangle\langle p| = I. \tag{1.20}$$

Inserting the resolution of the identity into $G(x', x; \epsilon)$ we find

$$\langle x'|e^{-\epsilon\hat{H}}|x\rangle = \int \frac{\mathrm{d}p}{2\pi}\langle x'|e^{-\epsilon\hat{H}}|p\rangle\langle p|x\rangle \tag{1.21}$$

$$\simeq \int \frac{\mathrm{d}p}{2\pi} e^{-ipx}[e^{ipx'} - \epsilon H(x', p)] \tag{1.22}$$

$$\simeq \int \frac{\mathrm{d}p}{2\pi} e^{ip(x'-x) - \epsilon H(x', p)}, \tag{1.23}$$

[2] If we want $\hat{H}$ to be Hermitian we clearly get some constraints on $m(\hat{x})$, $A(\hat{x})$ and $V(\hat{x})$.

where the function $H(x,p)$ is

$$H(x,p) := \langle x|\hat{H}|p\rangle = \frac{p^2}{2m(x)} + A(x)p + V(x). \tag{1.24}$$

Note that this works particularly well due to the specific operator ordering in the parametrization of $\hat{H}$. Carrying out the Gaussian integration over p, we obtain

$$G(x',x;\epsilon) \simeq \mathcal{N}_\epsilon e^{-\epsilon S_\epsilon(x',x)}, \tag{1.25}$$

with

$$S_\epsilon(x',x) = \frac{m(x')(x'-x)^2}{2\epsilon^2} + im(x')A(x')\frac{(x'-x)}{\epsilon} + V(x') - \frac{m(x')A(x')^2}{2}, \tag{1.26}$$

and

$$\mathcal{N}_\epsilon = \sqrt{\frac{m}{2\pi\epsilon}}. \tag{1.27}$$

The formal continuum limit of this expression leads back to the classical action (1.17), but, as we will see, the information about operator ordering is also contained in (1.26).

A useful way for dealing with distributional objects like $G(x',x;\epsilon)$ is to integrate them against smooth test functions. In our case we can introduce an arbitrary smooth wave function $\psi(x)$ to form

$$\int \mathrm{d}x\, G(x',x;\epsilon)\psi(x) = (e^{-\epsilon\hat{H}}\psi)(x'), \tag{1.28}$$

which is now well behaved in ϵ. In particular, we can expand the RHS to find

$$\int \mathrm{d}x\, G(x',x;\epsilon)\psi(x) = ([I - \epsilon\hat{H}]\psi)(x') + O(\epsilon^2). \tag{1.29}$$

We can now explain what Eq. (1.25) really means, namely that

$$\mathcal{N}_\epsilon \int \mathrm{d}x\, e^{-\epsilon S_\epsilon(x',x)}\psi(x) = \int \mathrm{d}x\, G(x',x;\epsilon)\psi(x) + O(\epsilon^2). \tag{1.30}$$

If this equation is satisfied, it follows that the path integral with action $S_\epsilon(x',x)$ yields the propagator $G(x',x;T)$ in the continuum limit. To check that (1.26) is a correct action for the path integral, we thus have to compute the LHS of (1.30) to order ϵ.

Let us write the integrand as

$$e^{-\epsilon S(x',x)}\psi(x) = \left[e^{-\frac{m(x')(x'-x)^2}{2\epsilon}}\right]\left[e^{-im(x')A(x')(x'-x)-\epsilon V(x')+\epsilon\frac{m(x')A(x')^2}{2}}\psi(x)\right]. \tag{1.31}$$

We see that, for small ϵ, the first bracket is sharply peaked around $x = x'$, while the second bracket is smooth. This means that we can use saddle point integration to evaluate the integral. In expanding the second bracket around $x = x'$ we need to know how $x' - x$ scales. By dimensional considerations, we have

$$\frac{1}{\sqrt{\epsilon}}\int \mathrm{d}x\, e^{-\frac{m(x')(x'-x)^2}{2\epsilon}}(x'-x)^{2n} \propto \left(\frac{\epsilon}{m(x')}\right)^n. \tag{1.32}$$

We can loosely express this as

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} \sim \frac{x_{j+1}-x_j}{\epsilon} \propto \frac{1}{\sqrt{m\epsilon}}. \tag{1.33}$$

This is a reflection of the fractal nature of the paths dominating the path integral.

Expanding the second bracket of (1.31) we find

$$\begin{aligned}
&e^{-im(x')A(x')(x'-x)-\epsilon V(x')+\epsilon\frac{m(x')A(x')^2}{2}}\psi(x) && (1.34)\\
&\simeq \Big[1 - im(x')A(x')(x'-x) - \epsilon V(x') \\
&\qquad + \epsilon\frac{m(x')A(x')^2}{2} - \frac{1}{2}m(x')^2A(x')^2(x'-x)^2\Big] \\
&\quad \times \Big[\psi(x') - (x'-x)\psi'(x') + \frac{1}{2}(x'-x)^2\psi''(x')\Big] && (1.35)\\
&\simeq 1 + im(x')A(x')(x'-x)^2\psi'(x') - \epsilon V(x') \\
&\quad + \epsilon\frac{m(x')A(x')^2}{2} - \frac{1}{2}m(x')^2A(x')^2(x'-x)^2 + \frac{1}{2}(x'-x)^2\psi''(x'). && (1.36)
\end{aligned}$$

Here we have dropped terms that are linear in $(x'-x)$, since they will integrate to zero. It is now straight forward to perform the integral, with the result

$$\begin{aligned}
&\mathcal{N}_\epsilon \int \mathrm{d}x\, e^{-\epsilon S(x',x)}\psi(x) \\
&\quad = \left(1 - \epsilon\left[-\frac{1}{2m(x')}\frac{\mathrm{d}^2}{\mathrm{d}x'^2} - iA(x')\frac{\mathrm{d}}{\mathrm{d}x'} + V(x')\right]\right)\psi(x') + O(\epsilon^2). && (1.37)
\end{aligned}$$

This matches exactly (1.29) with the Hamiltonian (1.16), showing that the action $S_\epsilon(x',x)$ correctly reproduces the propagator $G(x',x;T)$ in the continuum limit.

1.2 Lattice gauge theory

In this section we will solve lattice gauge theory in two dimensions as an example of path integral quantization[68, 87, 86]. This will also serve as useful background for Ch. 4 where the theory is coupled to CDT. A summary of the group theory results and definitions we will need is given in the appendix following this section. In Sec. 2.5 we discuss 2d Yang-Mills from a canonical point of view.

Consider a square lattice, n links high and l links wide, made into a cylinder by identifying the left and right hand boundary, see Fig. 1.1. Each oriented link ℓ is assigned a group element U_ℓ of some fixed Lie group G. We define a propagator by

$$G(\{U'_\ell\},\{U_\ell\};n) := \int \prod_\ell \mathrm{d}U_\ell\, e^{-S}, \tag{1.38}$$

where the integration is over all group elements, except those on the lower and upper boundary, which are fixed to be $\{U_\ell\}$ and $\{U'_\ell\}$, respectively. The action

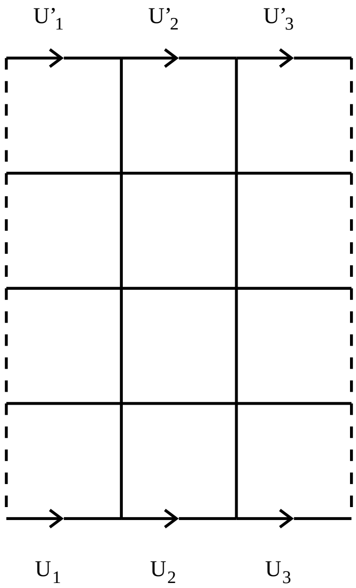

Figure 1.1: A 3×4 lattice. The dashed edges are identified.

is a product of all plaquettes P of the lattice

$$e^{-S} = \prod_P Z_P(U_P), \tag{1.39}$$

where $Z_P(U_P)$ is a class function, and U_P is the product of the group elements on the links surrounding P. In U_P the links are oriented anti-clockwise around the plaquette. The orientation of the boundary links is indicated in Fig. 1.1. By the completeness (1.91), we can write $Z_P(U_P)$ as

$$Z_P(U_P) := \sum_R d_R \chi_R(U_P) e^{-\frac{1}{2}c_R}, \tag{1.40}$$

for some set of coefficients c_R, and where the sum is over all irreducible representations of G.

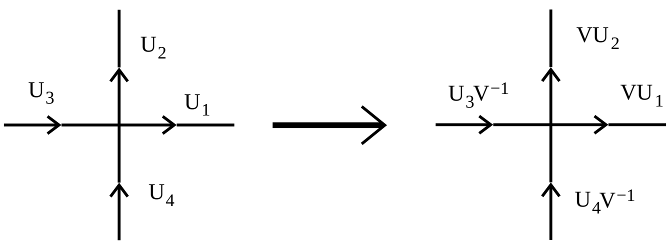

Figure 1.2: A gauge transformation.

The propagator is invariant under gauge transformations, which acts as illustrated in Fig. 1.2. In particular, this means that

$$G(\{U'_\ell\},\{U_1,\dots,U_j,U_{j+1},\dots,U_l\},n) \\ = G(\{U'_\ell\},\{U_1,\dots,U_jV^{-1},VU_{j+1},\dots,U_l\},n), \quad (1.41)$$

and similarly for the outgoing boundary. It follows that $G(\{U'_\ell\},\{U_\ell\},n)$ only depends on the holonomies,

$$G(\{U'_\ell\},\{U_\ell\};n) = G(U'_1U'_2\cdots U'_l, U_1U_2\cdots U_l;n). \quad (1.42)$$

We will abbreviate $U = U_1U_2\cdots U_l$, and $U' = U'_1U'_2\cdots U'_l$. Note that if we had not made that spatial direction compact, the propagator would have been completely independent of the boundary group elements, by gauge invariance.

By construction, the propagator satisfies the gluing relation

$$G(U',U;n_2+n_1) = \int \prod_{\ell=1}^{l} \mathrm{d}U''_\ell\, G(U',U'';n_2)G(U'',U;n_1). \quad (1.43)$$

Using the translation invariance of the Haar measure (and normalization), we can collapse the l integral to a single integral over the holonomy U'',

$$G(U',U;n_2+n_1) = \int \mathrm{d}U'', G(U',U'';n_2)G(U'',U;n_1). \quad (1.44)$$

By repeated use of this equation, we can write the propagator as

$$G(U',U,n) = \int \prod_{j=1}^{n-1} \mathrm{d}V_j\, G(U',V_{n-1};n=1) \\ \times G(V_{n-1},V_{n-2};n=1)\cdots G(V_1,U;n=1). \quad (1.45)$$

We thus see that 2d Yang-Mills reduces to ordinary quantum mechanics.

It remains to compute the one-step propagator $G(U',U;n=1)$. Using the identity (1.93), we find

$$\int \mathrm{d}V\, Z_P(U_1V;\alpha_1)Z_P(V^{-1}U_2;\alpha_2) = Z_P(U_1U_2;\alpha_1+\alpha_2), \quad (1.46)$$

where

$$Z_P(U_P;\alpha) := \sum_R d_R\chi_R(U_P)e^{-\frac{\alpha}{2}c_R}. \quad (1.47)$$

It follows that we can integrate out all the time-like links, except one, to get

$$G(U',U;n=1) = \int \mathrm{d}V\, Z_P(UVU'^{-1}V^{-1};\alpha=l). \quad (1.48)$$

Inserting (1.47) and using (1.96), we finally have

$$G(U',U;n=1) = \sum_R \overline{\chi_R(U')}\chi_R(U)e^{-\frac{l}{2}c_R}. \quad (1.49)$$

With this representation of the one-step propagator, we can immediately perform the integrals of Eq. (1.45) (using (1.93) again), and we find the simple result[68, 87]

$$G(U', U; n) = \sum_R \overline{\chi_R(U')}\chi_R(U)e^{-\frac{nl}{2}c_R}. \tag{1.50}$$

This is the solution to 2d lattice Yang-Mills (with topology $[0,1] \times S^1$). The solution for more complicated topologies can be found in Ref. [87].

Let us make some remarks on this result. First, let us write (1.50) as

$$G(U', U; n) = \langle U'| \sum_R |\chi_R\rangle\langle\chi_R| e^{-\frac{nl}{2}c_R}|U\rangle. \tag{1.51}$$

If we let

$$\hat{P} := \sum_R |\chi_R\rangle\langle\chi_R| \tag{1.52}$$

be the projection onto the space of class functions, we have

$$G(U', U; n) = \langle U'|\hat{P}e^{-nl\hat{H}}\hat{P}|U\rangle, \tag{1.53}$$

with

$$\hat{H} := \frac{1}{2}\sum_R c_R|\chi_R\rangle\langle\chi_R|. \tag{1.54}$$

We see that $\hat{H}$ is Hermitian when the c_R are real, and bounded from below when

$$\inf_R c_R > -\infty. \tag{1.55}$$

We can take the continuum limit of the propagator by introducing a lattice constant a, and keeping $T = an$ and $L = al$ fixed. If we then rescale $c_R \to a^2 c_R$, we obtain

$$G(U', U; T) = \langle U'|\hat{P}e^{-TL\hat{H}}\hat{P}|U\rangle, \tag{1.56}$$

with $\hat{H}$ still given by (1.54). Note that we do not actually need to take the $a \to 0$ limit.

From the lattice point of view, there is no preferred choice for the constants c_R, but there is a particular choice of Z_P which is natural from the Hamiltonian perspective, namely[38, 67]

$$Z_P(U_P) = \langle U_P|e^{-\frac{1}{2}g^2\Delta}|I\rangle, \tag{1.57}$$

where Δ is defined as

$$\Delta_G := \sum_a E^a E^a, \tag{1.58}$$

and g is a coupling constant. Here E^a is the first order differential operator generating left translation, see Eq. 1.80. The operator Δ_G is analogous to the kinetic term $\hat{p}^2$ of the free particle. In fact, one can understand Δ_G as the kinetic energy of a particle constrained to move on the (curved) manifold of the group G (see e.g. [30, 37]). Eq. (1.57) is known as the heat kernel action.

Comparing (1.58) with (1.72) we see that Δ_G is the Casimir operator of the infinite dimensional representation of G on $L^2(G)$. Using (1.86) we see that

$$\Delta_G|\chi_R\rangle = C_2(R)|\chi_R\rangle, \tag{1.59}$$

where $C_2(R)$ is just a number, since R is irreducible. The completeness relation for characters (1.91) gives

$$|I\rangle = \sum_R |\chi_R\rangle\langle\chi_R|I\rangle. \tag{1.60}$$

Here we use that $U \mapsto \langle U|I\rangle$ is a class function (or rather, a class distribution). Inserting this into (1.57) we have[38, 67]

$$Z_P(U_P) = \sum_R e^{-\frac{1}{2}g^2C_2(R)}\langle U_P|\chi_R\rangle\langle\chi_R|I\rangle \tag{1.61}$$

$$= \sum_R d_R\chi_R(U_P)e^{-\frac{1}{2}g^2C_2(R)}. \tag{1.62}$$

This explicitly exhibits (1.57) as a special case of (1.40), with $c_R = g^2C_2(R)$. Incidentally, this computation also verifies that the heat kernel action is a class function.

1.3 Appendix: Some group theory

Here we collect some basic results on the representation of Lie groups, following Refs. [44, 79, 85]. We will, for simplicity, consider a compact subgroup $G \subset U(N)$. The Lie algebra, $\mathfrak{g}$, is the subspace of $n \times n$ matrices t such that

$$e^{it} \in G. \tag{1.63}$$

Let t^a be a basis for $\mathfrak{g}$. The structure constants f^{abc} are defined by

$$[t^a, t^b] = f^{abc}t^c. \tag{1.64}$$

Since the generators t^a are Hermitian (this follows from (1.63) and the fact that $G \subset U(n)$) and linearly independent, the matrix

$$K^{ab} := \mathrm{tr}[t^at^b] \tag{1.65}$$

is positive-definite. We can thus choose the basis for $\mathfrak{g}$ such that

$$\mathrm{tr}[t^at^b] = \frac{1}{2}\delta^{ab}, \tag{1.66}$$

and in the following we will assume that this has been done.

With the convention (1.66) we see that the structure constants are given by the formula

$$f^{abc} = 2\,\mathrm{tr}([t^a, t^b]t^c). \tag{1.67}$$

In particular f^{abc} is anti-symmetric in all three indices.

A finite-dimensional unitary representation of G is a map

$$\rho_R : G \to U(d_R), \tag{1.68}$$

where

$$\rho_R(U_1U_2) = \rho_R(U_1)\rho_R(U_2) \tag{1.69}$$

for all $U_1, U_2 \in G$. Here d_R is the dimension of the representation. The map ρ_R induces the Lie algebra counterpart ($H(d_R)$ is the Hermitian matrices of dimension d_R)

$$\tilde{\rho}_R : \mathfrak{g} \to H(d_R), \tag{1.70}$$

with the defining property

$$\rho_R\left(e^{ix^a t^a}\right) = e^{ix^a \tilde{\rho}_R(t^a)}, \qquad x^a \in \mathbb{R}. \tag{1.71}$$

For any representation R, we can construct the Casimir operator

$$C_2(R) := \sum_a \rho_R(t^a)\rho_R(t^a). \tag{1.72}$$

Using the anti-symmetry of f^{abc} (see (1.67)), we find (summation over b implied)

$$\begin{aligned}
[\rho_R(t^a), C_2(R)] &= [\rho_R(t^a), \rho_R(t^b)]\rho_R(t^b) + \rho_R(t^b)[\rho_R(t^a), \rho_R(t^b)] \\
&= f^{abc}\rho_R(t^c)\rho_R(t^b) + f^{abc}\rho_R(t^b)\rho_R(t^c) \\
&= 0.
\end{aligned} \tag{1.73}$$

By exponentiating this result, it (almost) follows that $C_2(R)$ commutes with all group elements,

$$[\rho_R(U), C_2(R)] = 0. \tag{1.74}$$

If R is a irreducible representation, Schur's Lemma then tells us that $C_2(R)$ is proportional to the identity.

Given G we can construct the Hilbert space $L^2(G)$ of square-integrable functions on G. To define the inner product, we note that there exists a unique measure, the Haar measure, on G, which is left and right invariant,

$$\int_G \mathrm{d}U\, f(V_1 U V_2) = \int_G \mathrm{d}U f(U), \tag{1.75}$$

and normalized according to

$$\int_G \mathrm{d}U = 1. \tag{1.76}$$

The inner product on $L^2(G)$ is then

$$\langle \phi | \psi \rangle := \int \mathrm{d}U\, \overline{\phi(U)}\psi(U), \tag{1.77}$$

with the measure understood to be the Haar measure. There is natural infinite-dimensional unitary representation of G on $L^2(G)$, sending a group element U to the unitary translation operator L_U defined by

$$L_U|V\rangle = |UV\rangle. \tag{1.78}$$

Here $|V\rangle$ denote the 'position' eigenstate,

$$|\psi\rangle = \int \mathrm{d}U\, \psi(U)|U\rangle, \qquad \psi(U) = \langle U|\psi\rangle. \tag{1.79}$$

The unitarity of L_U follows from the invariance (1.75) of the measure. We denote the generators of left translation E^a, meaning that

$$L_U = e^{ix^a E^a}, \qquad \text{when } U = e^{ix^a t^a}. \tag{1.80}$$

It is clear that E^a is a first order differential operator.

From now on R, R' will denote finite dimensional *irreducible* representations. Given a representation R we can construct a states in $L^2(G)$ by

$$|R, m, n\rangle := \int \mathrm{d}U \, \rho_R(U)_{mn} |U\rangle, \qquad m, n = 1, \ldots, d_R. \tag{1.81}$$

By (part of) the Peter-Weyl Theorem, these states are orthogonal,

$$\langle R, m, n | R', m', n' \rangle = \int \mathrm{d}U \, \overline{\rho_R(U)_{mn}} \rho_{R'}(U)_{m'n'} = \frac{1}{d_R} \delta_{RR'} \delta_{mm'} \delta_{nn'}, \tag{1.82}$$

and complete

$$\sum_R \sum_{m,n=1}^{d_R} d_R |R, m, n\rangle \langle R, m, n| = I_{L^2(G)}. \tag{1.83}$$

Using the invariance of the measure, we have

$$L_U |R, m, n\rangle = \int \mathrm{d}V \, \rho_R(U^{-1}V) |V\rangle \tag{1.84}$$

hence, by the fundamental identity (1.69), we conclude that

$$L_U |R, m, n\rangle = \sum_l \rho_R(U^{-1})_{ml} |R, l, n\rangle. \tag{1.85}$$

This shows that (1.83) is a decomposition of the representation $U \mapsto L_U$ into a direct sum of irreducible representations. Inserting (1.80) into (1.85) and expanding, we obtain

$$E^a |R, m, n\rangle = -\sum_l \rho_R(t^a)_{ml} |R, l, n\rangle. \tag{1.86}$$

The last concept we will need is that of a *class function*. We say that $\psi : G \to \mathbb{C}$ is a class function if

$$\psi(VUV^{-1}) = \psi(U) \tag{1.87}$$

for all $V \in G$. Let $L^2_{\mathcal{C}}(G) \subset L^2(G)$ denote the subspace of class functions. For $\psi \in L^2_{\mathcal{C}}(G)$ we trivially have

$$\psi(U) = \int \mathrm{d}V \, \psi(VUV^{-1}), \tag{1.88}$$

and it follows that

$$\begin{aligned} \langle R, m, n | \psi \rangle &= \int \mathrm{d}U \, \psi(U) \int \mathrm{d}V \, \overline{\rho_R(V^{-1}UV)_{mn}} \\ &= \int \mathrm{d}U \, \psi(U) \sum_{l,k} \overline{\rho_R(U)_{lk}} \int \mathrm{d}V \, \rho_R(V)_{lm} \overline{\rho_R(V)_{kn}} \\ &= \frac{\delta_{mn}}{d_R} \langle \chi_R | \psi \rangle, \end{aligned} \tag{1.89}$$

where we define

$$|\chi_R\rangle := \sum_m |R, m, m\rangle. \tag{1.90}$$

Combining (1.83) and (1.89) we deduce that $|\chi_R\rangle$ form a orthonormal basis for the class functions,

$$\sum_R |\chi_R\rangle\langle\chi_R| = I_{L^2_{\mathbb{C}}(G)}. \tag{1.91}$$

The wavefunction of $|\chi_R\rangle$ is the character $\chi_R(U)$ of the representation R,

$$\chi_R(U) := \langle U|\chi_R\rangle = \mathrm{tr}[\rho_R(U)]. \tag{1.92}$$

Using the cyclic invariance of the trace we explicitly see that $\chi_R(U)$ is a class function.

Finally, let us mention two identities involving characters. First we have

$$\begin{aligned}\int \mathrm{d}U\, \chi_R(V_1 U)\chi_{R'}(U^{-1}V_2) &= \int \mathrm{d}U\, \chi_R(V_1 V_2 U)\chi_{R'}(U^{-1}) \\ &= \sum_{l,m,n} \int \mathrm{d}U\, \rho_R(V_1V_2)_{lm}\rho_R(U)_{ml}\overline{\rho_{R'}(U)_{nn}} \\ &= \frac{\delta_{RR'}}{d_R}\chi_R(V_1V_2).\end{aligned} \tag{1.93}$$

Next, consider the function

$$f(V_1) := \int \mathrm{d}U \chi_R(V_1 U V_2 U^{-1}). \tag{1.94}$$

It is clear that f is a class function. Using (1.91) and (1.93), we can thus write

$$\begin{aligned}f(V_1) &= \sum_{R'} \chi_{R'}(V_1) \int \mathrm{d}V\, \chi_{R'}(V^{-1})f(V) \\ &= \sum_{R'} \chi_{R'}(V_1)\frac{\delta_{RR'}}{d_R} \int \mathrm{d}U \chi_R(UV_2U^{-1}).\end{aligned} \tag{1.95}$$

We conclude that

$$\int \mathrm{d}U \chi_R(V_1UV_2U^{-1}) = \frac{1}{d_R}\chi_R(V_1)\chi_R(V_2). \tag{1.96}$$

Chapter 2

Causal Dynamical Triangulations

In analogy with the path integral construction of quantum field theories, one might attempt to define quantum gravity by a path integral of the form

$$\int_{\mathcal{M}} \mathcal{D}[g] e^{iS[g]}. \tag{2.1}$$

Here the integration domain $\mathcal{M}$ is supposed to be some class of Lorentzian manifolds. If the topology of space-time is kept fixed, which we will always assume, the manifolds can be parametrized by the metric $g_{\mu\nu}$. The classical Einstein-Hilbert action is

$$S[g] := \frac{1}{16\pi G_N} \int \mathrm{d}\xi \, \sqrt{-g}(R - 2\Lambda), \tag{2.2}$$

where

$$\sqrt{-g} := \sqrt{-\det g_{\mu\nu}}, \tag{2.3}$$

R is the Ricci scalar, Λ is the cosmological constant and G_N is Newton's constant.

In this thesis we will only consider the case of two-dimensional space-times. In this case the term

$$\int \mathrm{d}\xi \, \sqrt{-g} R \tag{2.4}$$

only depends on the topology of space-time, by the Gauss-Bonnet Theorem. This means that it will only contribute an overall constant to the path integral (since we keep the topology fixed), and we might as well drop it from the action. We then simply have

$$S[g] = -\Lambda A[g], \tag{2.5}$$

where

$$A[g] := \int \mathrm{d}\xi \, \sqrt{-g} \tag{2.6}$$

is the area of spacetime and we have absorbed some constants in Λ.

The integral (2.1) is not a well-defined mathematical object. One could try to make sense of it either as some perturbative expansion around a solvable limit[1] or make some kind of non-perturbative regularization. Here we will pursue the latter option.

Causal Dynamical Triangulations[12, 8] (CDT) is a specific regularization of (2.1). An important feature of CDT is that there is a preferred notion of time. This means that we can define a propagator of the form

$$G_{\mathrm{Lo}}(L', L; \Lambda; T) := \int_{\mathcal{M}(L',L;T)} \mathcal{D}[g] e^{iS[g]}. \tag{2.7}$$

Here $\mathcal{M}(L', L; T)$ is a class of Lorentzian manifolds with incoming (outgoing) boundary of length L (L'), and where the time separation between the boundaries is T. We will find that there is a well defined self-adjoint Hamiltonian $\hat{H}$ such that $G_{\mathrm{Lo}}(L', L)$ can be written as

$$G_{\mathrm{Lo}}(L', L; \Lambda; T) = \langle L'|e^{-iT\hat{H}}|L\rangle. \tag{2.8}$$

Before we define the CDT model, let us make some remarks on Wick rotations. By Wick rotation we simply mean that the path integral we will write down will compute the 'Euclidean' propagator

$$G_{\mathrm{Eu}}(L', L; \Lambda; T) := \langle L'|e^{-T\hat{H}}|L\rangle = G_{\mathrm{Lo}}(L', L; \Lambda; -iT). \tag{2.9}$$

Here we will not interpret the Wick rotation as a change of the signature of the underlying manifolds to the Euclidean signature, but just as an analytical continuation in the parameter T, which is well defined as long as the Hamiltonian $\hat{H}$ is bounded from below. One motivation for performing the Wick rotation is that it makes it easier to write down a path integral with the right properties, as we will now argue.

The basic observation is simply that $e^{-iT\hat{H}}$ is unitary, while $e^{-T\hat{H}}$ is Hermitian. For the Euclidean propagator the condition of Hermiticity is

$$G_{\mathrm{Eu}}(L', L; T) = \overline{G_{\mathrm{Eu}}(L, L'; T)}, \tag{2.10}$$

while the analogous statement for the Lorentzian propagator is[2]

$$G_{\mathrm{Lo}}(L', L; T) = \overline{G_{\mathrm{Lo}}(L, L'; -T)}. \tag{2.11}$$

It is much easier to construct a path integral manifestly satisfying (2.10), than (2.11).

If we assume time reversal symmetry, then we can replace (2.10) with an even simpler condition. This is a natural assumption since classical gravity is time reversal invariant. In quantum mechanics time reversal symmetry means that there is an anti-unitary[3] operator $\mathcal{T}$,

$$(\mathcal{T}|\psi\rangle, \mathcal{T}|\phi\rangle) = (|\phi\rangle, |\psi\rangle) = \langle\phi|\psi\rangle, \tag{2.12}$$

[1]For *pure* quantum gravity in two space-time dimensions we see that there is no candidate for a small parameter to expand in. One can, however, define a semi-classical limit for QG coupled to matter with large negative central charge, see also Sec. 5.1.1.

[2]In a more careful approach to constructing well defined unitary theories using Euclidean path integrals, one uses the concept of reflection positivity[11, 47, 70].

[3]When working with anti-linear operators we will write the inner product on the Hilbert space explicitly as $(\cdot, \cdot)$, since the bra-ket notation becomes ambiguous.

that commutes with the Hamiltonian,

$$\mathcal{T}\hat{H}\mathcal{T}^{-1} = \hat{H}. \tag{2.13}$$

Since $\mathcal{T}i\mathcal{T}^{-1} = -i$, one then has

$$\mathcal{T}e^{-iT\hat{H}}\mathcal{T}^{-1} = e^{-i(-T)\hat{H}}, \tag{2.14}$$

as expected for the time reversal operator. It seems natural to define $\mathcal{T}$ to act trivially in the length basis,[4]

$$\mathcal{T}|L\rangle = |L\rangle. \tag{2.16}$$

From (2.13) and (2.16) we derive

$$\Big(|L'\rangle, e^{-T\hat{H}}|L\rangle\Big) = \Big(\mathcal{T}e^{-T\hat{H}}\mathcal{T}^{-1}|L\rangle, \mathcal{T}|L'\rangle\Big) = \Big(e^{-T\hat{H}}|L\rangle, |L'\rangle\Big), \tag{2.17}$$

or

$$G_{\mathrm{Eu}}(L', L; T) = \overline{G_{\mathrm{Eu}}(L', L; T)}. \tag{2.18}$$

The conditions (2.10) and (2.18) together are equivalent to demanding that $G_{\mathrm{Eu}}(L', L; T)$ is real and symmetric,

$$\operatorname{Im} G_{\mathrm{Eu}}(L', L; T) = 0, \qquad G_{\mathrm{Eu}}(L', L; T) = G_{\mathrm{Eu}}(L, L'; T). \tag{2.19}$$

We will begin by defining 2d CDT and deriving the continuum Hamiltonian in Sec. 2.1. In Sec. 2.2 we will briefly analyze the Hamiltonian by relating it to the 2d isotropic harmonic oscillator. The continuum limit of CDT can also be obtained by canonical quantization of Hořava-Lifshitz gravity[5], as we will outline in Sec. 2.3. In Sec. 2.4 we will make some speculations about the universality of CDT, which is also the subject of Ch. 3. Finally, we will canonically quantize CDT coupled to Yang-Mills in Sec. 2.5. The lattice quantization of the same model is given in Ch. 4.

2.1 The CDT model

The main idea of CDT is to replace the integral over Lorentzian manifolds in (2.7) with a sum over triangulated manifolds. The basic building block is a triangle with two time-like edges and one space-like. It comes in two variants, depending on whether the space-like edge is in the future or the past. One can then build discretized space-time manifolds by gluing these triangles together, subject to the constraint that one may only glue space-like edges to space-like edges and time-like to time-like (respecting also the direction of time).

Let $\tau^o(l', l; n)$ be the set of triangulations with topology $[0,1] \times [0,1]$, with the left and right edges time-like and the top and bottom edges space-like, with the top and bottom edges of length l' and l, respectively, and with n layers of triangles between the top and bottom edge. See Fig. 2.1 for an example of a triangulation. Note that the gluing rules forces the triangulation to have

[4]If $\mathcal{T}$ is diagonal in the length basis, we can eliminate phases by a change of basis

$$|L\rangle \to e^{i\phi}|L\rangle. \tag{2.15}$$

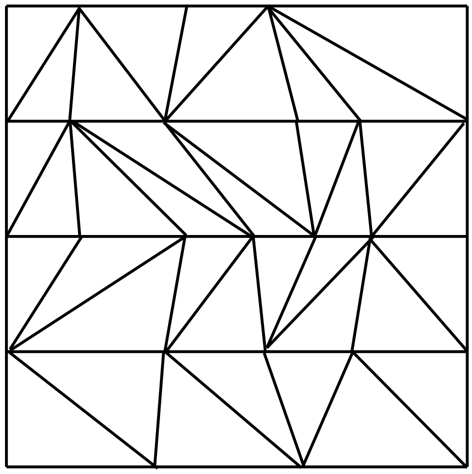

Figure 2.1: A triangulation in $\tau^o(l' = 4, l = 3; n = 4)$. Time flows upwards.

a layered structure, which can be interpreted as a preferred notion of time. Recently, a modification of CDT has been proposed, where there is no strict foliation structure.[59, 60]

The discretized propagator is defined to be

$$G^o(l', l; g; n) := \sum_{t \in \tau^o(l', l; n)} g^{|t|}, \tag{2.20}$$

where $|t|$ is the number of triangles in the triangulation t. The factor $g^{|t|}$ acts like a Wick rotated version of the continuum weight $e^{i\Lambda A[g]}$. We clearly have

$$G^o(l', l; g; n) = G^o(l, l'; g; n), \tag{2.21}$$

and, if g is real, then so is $G^o(l', l; g; n)$. This gives us hope, by the arguments of the previous section, that we might obtain a Hermitian Hamiltonian in the continuum limit.

By inserting a resolution of the identity into (we drop the Eu subscript now)

$$G(L', L; \Lambda; T) = \langle L' | e^{-T\hat{H}} | L \rangle, \tag{2.22}$$

we obtain

$$G(L', L; \Lambda; T = T_1 + T_2) = \int_0^\infty \mathrm{d}L \, G(L', L''; \Lambda; T_2) G(L'', L; \Lambda; T_1). \tag{2.23}$$

It is easy to convince oneself that the regularized path integral satisfies the analogous property

$$G^o(l', l; g; n = n_1 + n_2) = \sum_{l''=0}^{\infty} G^o(l', l''; g; n_2) G^o(l'', l; g; n_1). \tag{2.24}$$

This means that we can write (2.20) as

$$G^o(l', l; g; n) = \left(\prod_{j=1}^{n-1} \sum_{l_j=0}^{\infty} \right) G^o(l', l_{n-1}; g; n = 1)$$
$$\times G^o(l_{n-1}, l_{n-2}; g; n = 1) \cdots G^o(l_1, l; g; n = 1), \quad (2.25)$$

which is very close to the ordinary quantum mechanics path integral (1.7), except that we have sums instead of integrals.

The one-step propagator $G^o(l', l; g; n = 1)$ can be computed by elementary combinatorics. The relevant triangulations consist of a single row of $l' + l$ triangles, with l pointing upwards and l' pointing downwards. The number of such configurations is just $\binom{l'+l}{l}$, hence

$$G^o(l', l; g; n = 1) = g^{l'+l} \binom{l' + l}{l} = g^{l'+l} \frac{(l' + l)!}{l'!l!}. \quad (2.26)$$

It is also possible to formulate CDT with spacetime topology $S^1 \times [0, 1]$ (in fact, this was the case when CDT was originally introduced[12]), i.e. with each time slice having the topology of a circle. The propagator is

$$G^c(l', l; g; n) = \sum_{t \in \tau^c(l', l; n)} g^{|t|} \quad (2.27)$$

as before, only the class of triangulations is different. Let us first consider how to count the number of single step geometries $G^c(l', l; g; n = 1)$. Let us fix a closed triangulation in $\tau^c(l', l; n = 1)$. By cutting this geometry along one of its $l' + l$ time-like edges we get an open geometry in $\tau^o(l', l; n = 1)$. Assuming that there is no accidental rotational symmetry, these $l' + l$ open triangulations will be different. We conclude that there are $l' + l$ times as many open triangulations as closed triangulations,[5]

$$G^c(l', l; g; n = 1) := \frac{1}{l' + l} G^o(l', l; g; n = 1) = g^{l'+l} \frac{(l' + l - 1)!}{l'!l!}, \qquad l' + l > 0. \quad (2.29)$$

To make this equation compatible with Eq. (2.27), it is necessary to include a symmetry factor in (2.27). The analogue of Eq. (2.24) is

$$G^c(l', l; g, n) := \sum_{l''=0}^{\infty} l'' G^c(l', l''; g; n_2) G^c(l'', l; g; n_1). \quad (2.30)$$

The factor l'' accounts for the fact that there are l'' different ways to orient the two triangulations relative to each other when gluing. Again, this is only strictly true in the generic case where there is no rotational symmetry. We will take (2.29) and (2.30) as the definition of $G^c(l', l; g; n)$, which means that (2.27) is only true up to symmetry factors. In any case, these symmetry factors should

[5] Eq. (2.29) does not define $G^c(l' = 0, l = 0; g; n = 1)$. In Ref. [12] the definition

$$G^c(l' = 0, l; g; n = 1) = G^c(l', l = 0; g; n = 1) = 0 \quad (2.28)$$

was used.

not affect the continuum limit, since only a small fraction of large triangulations are symmetric.

We will turn to the computation of the continuum limit in the next two sections. We introduce the lattice spacing a, which relate the discrete and continuum quantities as

$$L = al, \qquad L' = al', \qquad T = an. \tag{2.31}$$

The task is then to determine

$$\lim_{a\to 0} G(l' = a^{-1}L', l = a^{-1}L; g; n = a^{-1}T). \tag{2.32}$$

2.1.1 Hamiltonian in real space

In this section we compute the Hamiltonian of 2d CDT with open boundaries following the saddle point approach introduced in Section 1.1. A generalization of this computation can be found in Ref. [90]. The basic idea was to integrate the propagator against a smooth wave function, and then read off the Hamiltonian from the short-time behavior. One complication is that the Hilbert space for the discrete path integral is the sequence space ℓ^2, while the continuum Hilbert space is $L^2(\mathbb{R}^+)$.

We deal with this by fixing a smooth wavefunction $\psi(L)$ on $\mathbb{R}^+$, and defining the discrete wavefunction to be

$$\psi_l := \psi(L = al). \tag{2.33}$$

The continuum Hamiltonian is then defined by the ansatz

$$\sum_{l=0}^{\infty} G^o(l', l; g; n = 1)\psi_l = ([1 - a\hat{H}]\psi)(L' = al') + O(a^2). \tag{2.34}$$

The first step is to use Stirling's formula

$$n! = \sqrt{2\pi}e^{-n}n^{n+1/2}\left(1 + \frac{1}{12n} + O(n^{-2})\right) \tag{2.35}$$

to simplify the propagator for large l, l'. We find

$$\begin{aligned} & g^{l'+l}\frac{(l'+l)!}{l'!l!} && (2.36)\\ &= \frac{g^{l'+l}}{\sqrt{2\pi}}\frac{(l'+l+\frac{1}{2})^{l'+l+1/2}}{l'^{l'+1/2}l^{l+1/2}}\left(1 + \frac{1}{12(l+l')} - \frac{1}{12l} - \frac{1}{12l'} + O(l^{-2})\right) && (2.37)\\ &= \frac{(2g)^{l'+l}}{\sqrt{2\pi l'}}\sqrt{1+\frac{l'}{l}}e^{l'f(\frac{l}{l'})}\left(1 + \frac{1}{12(l+l')} - \frac{1}{12l} - \frac{1}{12l'} + O(l^{-2})\right), && (2.38) \end{aligned}$$

where

$$f(r) := (1+r)\log\left(\frac{1}{2} + \frac{r}{2}\right) - r\log r, \qquad 0 < r. \tag{2.39}$$

For the ansatz (2.34) to make sense, it is necessary that $G^o(l, l'; n = 1)$ is strongly peaked about $l = l'$. It must clearly be the factor $e^{l'f(\frac{l}{l'})}$ which is responsible for the peak, and we indeed find that $f(r)$ has a global maximum at $r = 1$.

So far we have not discussed how to introduce the continuum cosmological constant. The factor $(2g)^{l'+l}$ will not have a nice limit unless g is close to $\frac{1}{2}$, so we are motivated to introduce the continuum cosmological constant Λ by setting

$$g = \frac{1}{2} - a^2\Lambda. \tag{2.40}$$

With this we find that the short-time propagator is given by

$$\begin{aligned} G^o(l', l; g; n = 1) &= \sqrt{\frac{a}{2\pi L'}}\sqrt{1 + \frac{L'}{L}} e^{a^{-1}L' f(\frac{L}{L'})} \\ &\times \Big(1 + \frac{a}{12}\big((L' + L)^{-1} - L'^{-1} - L^{-1}\big) - 2a(L' + L)\Lambda + O(a^2)\Big). \end{aligned} \tag{2.41}$$

Returning to the LHS of (2.34), we thus have

$$\sum_{l=0}^{\infty} G^o(l', l; g; n = 1)\psi(L) = \sum_{l=0}^{\infty} K_{a,\Lambda,L'}(L = al) + O(a^2) \tag{2.42}$$

with

$$\begin{aligned} K_{a,\Lambda,L'}(L) &:= \sqrt{\frac{a}{2\pi L'}}\sqrt{1 + \frac{L'}{L}} e^{a^{-1}L' f(\frac{L}{L'})} \\ &\times \Big(1 + \frac{a}{12}\big((L' + L)^{-1} - L'^{-1} - L^{-1}\big) - 2a(L' + L)\Lambda\Big)\,\psi(L). \end{aligned} \tag{2.43}$$

Note that $K_{a,\Lambda,L'}(L)$ is defined for general real values of L, not just at the lattice points $L = al$.

In order to proceed, we need to replace the sum over l with an integral over L. The Euler-Maclaurin formula (see e.g. [14]) tells us that (here we drop the subscripts on $K_{a,\Lambda,L'}(L)$)

$$\begin{aligned} \sum_{l=0}^{N} K(L = al) &= a^{-1}\int_0^{aN} \mathrm{d}L\, K(L) + \frac{1}{2}\big(K(0) + K(aN)\big) \\ &+ \sum_{r=1}^{2} \frac{a^{2r-1}B_{2r}}{(2r)!}\big(K^{(2r-1)}(aN) - K^{(2r-1)}(0)\big) + R, \end{aligned} \tag{2.44}$$

where the remainder is given by

$$R := -\frac{a^3}{4!}\int_0^{aN} \mathrm{d}L\, P_4(L)K^{(4)}(L), \qquad K^{(4)}(L) := \frac{\partial^4 K(L)}{\partial L^4}, \tag{2.45}$$

and we have (temporarily) introduced an upper cut-off N on the sum. Here $B_2 := \frac{1}{6}$, $B_4 := -\frac{1}{30}$ and $B_4(x)$ is the polynomial

$$B_4(x) := x^4 - 2x^3 + x^2 - \frac{1}{30}. \tag{2.46}$$

Finally, the Bernoulli function $P_4(x)$ is defined by

$$P_4(x) := B_4(x - \lfloor x \rfloor), \tag{2.47}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Let us first note that, since $K(L)$ is exponentially peaked around $L = L'$, we can safely take the limit $N \to \infty$. For the same reason, we can also throw the boundary terms away, with the result

$$\sum_{l=0}^{\infty} K(L = al) = a^{-1} \int_0^\infty \mathrm{d}L \, K(L) + R + \text{exponentially small.} \tag{2.48}$$

It now remains to argue that R is also small. The function $P_4(x)$ is clearly bounded, so we can estimate

$$|R| \leq \frac{a^3}{4!} \left(\sup_x |P_4(x)| \right) \int_0^\infty \mathrm{d}L \, |K^{(4)}(L)|. \tag{2.49}$$

The behavior of $K^{(4)}(L)$ is dominated by the factor $e^{a^{-1}L' f(\frac{L}{L'})}$. Near the maximum of the exponent we have

$$a^{-1} L' f\left(\frac{L}{L'}\right) = -\frac{(L'-L)^2}{4aL'} + O\big((L'-L)^3\big). \tag{2.50}$$

It follows that $K^{(4)}(L)$ is only significantly different from zero when $L' - L$ is of the order $\sqrt{aL'}$. In this small interval the remaining factors of $K^{(4)}(L)$ can be considered constant, and we thus have

$$\int_0^\infty \mathrm{d}L \, |K^{(4)}(L)| \sim \sqrt{\frac{a}{L'}} \psi(L') \int_0^\infty \mathrm{d}L \left| \partial_L^4 e^{-\frac{(L'-L)^2}{4aL'}} \right| \sim \frac{\psi(L')}{aL'^2}. \tag{2.51}$$

Together with (2.49) we thus have

$$|R| \lesssim \frac{a^2}{L'^2} \psi(L'), \tag{2.52}$$

which is sufficient to show that we can ignore R in our computation.

We are now ready to perform the actual saddle point integration. First we expand $K(L)$ around the saddle point,

$$\begin{aligned} K(L) = \sqrt{\frac{a}{\pi L'}} e^{-\frac{(L'-L)^2}{4aL'}} \bigg(1 + \frac{(L'-L)^6}{128a^2 L'^4} - \frac{5(L'-L)^4}{48aL'^3} + \frac{7(L'-L)^2}{32L'^2} - \frac{1}{8L'} \\ + \frac{(L'-L)^4 \partial_{L'}}{8aL'^2} - \frac{(L'-L)^2 \partial_{L'}}{4L'} + \frac{(L'-L)^2 \partial_{L'}^2}{2} - 4a\Lambda L' + \dots \bigg) \psi(L'). \end{aligned} \tag{2.53}$$

We only show the terms that will contribute to the Hamiltonian. Performing the Gaussian integration we obtain

$$\begin{aligned} a^{-1} \int_0^\infty \mathrm{d}L \, K(L) &= 2\big(1 + a[L'\partial_{L'}^2 + \partial_{L'} - 4\Lambda L']\big)\psi(L') + O(a^2) & (2.54) \\ &= e^{-a\hat{H}^o} \psi(L') + O(a^2), & (2.55) \end{aligned}$$

with the continuum Hamiltonian

$$\hat{H}^o(L, \partial_L) = -L\partial_L^2 - \partial_L + 4\Lambda L - a^{-1} \log 2. \tag{2.56}$$

This is the main result of this section.

Let us make a couple of remarks on the result. The divergent shift of the ground state energy can be absorbed by redefining the propagator as

$$G^o(l', l; g; n) \to \frac{1}{2^n} G^o(l', l; g; n). \tag{2.57}$$

In any case, the shift does not affect the physics (in particular, it is *not* a cosmological term), so it is usually not displayed. It is also conventional to rescale Λ by a factor of four, resulting in the Hamiltonian

$$\hat{H}^o(L, \partial_L) = -L\partial_L^2 - \partial_L + \Lambda L. \tag{2.58}$$

A curious feature of the above calculation is that four of the terms in Eq. (2.53) would have resulted in a $\frac{1}{L}$ term in the Hamiltonian, but the overall coefficient turns out to be exactly zero! This is somewhat puzzling, since the term does not seem to violate any symmetries. We will return to this question in Sec. 2.4.

One can repeat the above exercise in the case of circular boundary conditions and find the Hamiltonian (we have again rescaled $\Lambda \to \frac{1}{4}\Lambda$)

$$\hat{H}^c(L, \partial_L) = -L\partial_L^2 - 2\partial_L + \Lambda L. \tag{2.59}$$

Here we find that the divergent shift is absent. An important difference is that the inner product is not the usual one, but instead

$$\langle\phi|\psi\rangle := \int_0^\infty L\mathrm{d}L\, \overline{\phi(L)}\psi(L). \qquad \text{(circular boundary conditions)} \tag{2.60}$$

This is due to the factor of l'' in the gluing relation (2.30).

2.1.2 Hamiltonian in Laplace space

In the preceding calculation, we assumed l', l to be large throughout. This means that the contribution to the propagator from microscopic universes (i.e. universes with a finite number of triangles) is not accounted for. In this section we determine the Hamiltonian by an alternative method[12], where the finite triangulations are visible.

We start by introducing a generating function for the propagator,

$$G(y, x; g; n) := \sum_{l', l} y^{l'} x^l G(l', l; g; n), \tag{2.61}$$

and the wave function,

$$\psi(\omega) := \sum_{l=0}^{\infty} \psi_l \omega^l. \tag{2.62}$$

The key observation is now that the action of the propagator on a wavefunction can be written as a contour integral of the generating functions. To see this, let

$\mathcal{C}$ be a contour encircling the origin once in the positive direction, we then have

$$\int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi i \omega} G(y, x = \omega^{-1}; g; n)\psi(\omega) = \sum_{l',l,k=0}^{\infty} y^{l'} G(l', l; g; n)\psi_k \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi i} \omega^{k-l-1} \tag{2.63}$$

$$= \sum_{l',l}^{\infty} y^{l'} G(l', l; g; n)\psi_l, \tag{2.64}$$

assuming that $\mathcal{C}$ is within the combined region of convergence of $G(x, \omega^{-1})$ and $\psi(\omega)$.

The generating function for $G^o(l', l; g; n = 1)$ is particularly simple. Plugging (2.26) into (2.61) we obtain

$$G^o(y, x; g; n = 1) = \sum_{l',l=0}^{\infty} (gy)^{l'} (gx)^l \binom{l'+l}{l} = \sum_{k=0}^{\infty} (gx + gy)^k = \frac{1}{1 - gy - gx}. \tag{2.65}$$

We see that $G^o(y, \omega^{-1}; g; n = 1)$ is analytic as a function of ω, except for a simple pole at

$$\omega_* = \frac{g}{1 - gy}. \tag{2.66}$$

To perform the integral of (2.63) we need to decide where to place the contour. For normalizable wave functions $\psi(L)$ it is clear that $\psi(\omega)$ is convergent for

$$|\omega| < 1. \tag{2.67}$$

On the other hand, $G^o(y, \omega^{-1})$ is convergent when (here we assume that $|gy| < 1$)

$$|\omega| > |\omega_*|. \tag{2.68}$$

Our contour $\mathcal{C}$ should thus lie between the limits (2.67) and (2.68). When $|g| \leq \frac{1}{2}$ and $|y| < 1$, which is the physically relevant region, this is possible, since we then have $|\omega_*| < 1$. Using the residue theorem we find[6]

$$\int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi i \omega} G^o(y, \omega^{-1}; g; n = 1)\psi(\omega) = \frac{1}{1 - gy} \psi\left(\frac{g}{1 - gy}\right). \tag{2.69}$$

Note that this result is exact, i.e. there is no expansion in a.

It is straight forward to take the continuum limit of the above result. First we introduce the variable $Z \geq 0$ by

$$\omega = 1 - aZ. \tag{2.70}$$

For small a we then have

$$\psi(\omega = 1 - aZ) \simeq a^{-1} \int_0^{\infty} \mathrm{d}L\, \psi(L)(1 - aZ)^{a^{-1}L} \simeq a^{-1} \int_0^{\infty} \mathrm{d}L\, \psi(L) e^{-LZ}. \tag{2.71}$$

Up to the factor of a^{-1}, this is just the Laplace transform of $\psi(L)$, which we will denote

$$\psi(Z) := \int_0^{\infty} \mathrm{d}L\, \psi(L) e^{-LZ}. \tag{2.72}$$

[6]Note that the apparent singularity at $\omega = 0$ is removable.

With $g = 1 - a^2\Lambda$ and $y = 1 - aY$, the RHS of (2.69) becomes

$$2\big(1 - aY + O(a^2)\big)\psi\left(\omega = 1 - a\big[Y - aY^2 + 4a\Lambda + O(a^2)\big]\right)$$
$$= 2\big(1 - a[Y^2\partial_Y + Y - 4\Lambda\partial_Y]\big)\psi(Y) + O(a^2). \quad (2.73)$$

From this expression we directly read of the Laplace space Hamiltonian

$$\hat{H}^o(Z, \partial_Z) = (Z^2 - 4\Lambda)\partial_Z + Z + a^{-1}\log 2. \quad (2.74)$$

One can check that the real space Hamiltonian (2.56) is recovered by an inverse Laplace transform of $\hat{H}^o(Z, \partial_Z)$.

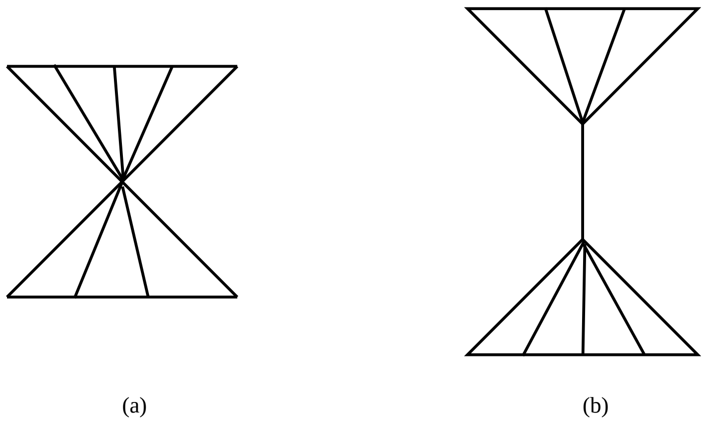

Figure 2.2: Both configuration (a) and (b) is allowed when computing $G^o(l', l; g; n)$, but in the definition of $G^o_r(l', l; g; n)$ we exclude configuration (b).

The method of this section is in some sense more sensitive to the microscopic details of the model. This can be demonstrated by a simple modification of the CDT propagator. Imagine that we decided that having time-slices of space-time volume zero (i.e. time-slices where both the incoming and outgoing boundary is of length zero) in the path integral is too pathological, see Fig 2.2. A natural way to eliminate them is to define a new restricted propagator by

$$G^o_r(l', l; g; n = 1) := G^o(l', l; g; n = 1) - \delta_{l',0}\delta_{l,0}. \quad (2.75)$$

The new and old propagator coincide, except that $G^o_r(l' = 0, l = 0; g; n = 1) = 0$. The corresponding generating function is simply

$$G^o_r(y, x; g; n = 1) = G^o(y, x; g; n = 1) - 1. \quad (2.76)$$

Repeating the above calculation we find

$$\int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi i\omega} G^o_r(y, \omega^{-1}; g; n = 1)\psi(\omega) = \frac{1}{1 - gy}\psi\left(\frac{g}{1 - gy}\right) - \psi(\omega = 0) \quad (2.77)$$

The extra term leads to the non-local Hamiltonian (we drop the divergent constant)

$$(\hat{H}^o_r\psi)(Z) = [(Z^2 - 4\Lambda)\partial_Z + Z]\psi(Z) + \lim_{Y\to\infty} Y\psi(Y) \quad (2.78)$$

in Laplace space, but if we transform to real space we find

$$\hat{H}_r^o(L, \partial_L) = -L\partial_L^2 - \partial_L + \delta(L) + 4\Lambda L, \tag{2.79}$$

where the operator $\delta(L)$ is defined by

$$\langle\phi|\delta(L)|\psi\rangle = \overline{\phi(L=0)}\psi(L=0). \tag{2.80}$$

Looking back at (2.75) it is not too surprising that we get the delta potential in the Hamiltonian. One should note, however, that the analysis of the previous section would have missed the term, since there we assumed that $l, l' \gg 1$.

It is clear that modifying the propagator for microscopic universes will in general lead to a Hamiltonian of the form

$$\hat{H}(L, \partial_L) = -L\partial_L^2 - \partial_L + \alpha\delta(L) + 4\Lambda L, \tag{2.81}$$

for some real constant α. We will not attempt a full analysis of the Hamiltonian (2.81) here, but see the comments at the end of the next section.

One can adopt different attitudes towards the interpretation of the $\delta(L)$ term. One possibility is too regard it as a lattice artifact that should be excluded from the continuum model. In this case the relation between the discrete and continuum propagator can be more complicated than a simple scaling limit. This is the view we take in Ch. 3.[7]

On the other hand, one could take the term seriously. One would the need to do a more careful study of the continuum limit (to determine e.g. which self-adjoint extension of (2.81) a given model corresponds to).

2.2 Relation between CDT and the harmonic oscillator

To gain a better understanding of the CDT Hamiltonian, it is useful to relate it to the harmonic oscillator[43]. In particular we will use the identification to derive the spectrum of $\hat{H}^o$ and express its eigenstates in terms of harmonic oscillator eigenstates.

Let us first perform the change of variables

$$L = \frac{\rho^2}{2} \tag{2.82}$$

to obtain

$$\hat{H}^o(\rho, \partial_\rho) = -\frac{1}{2}\partial_\rho^2 - \frac{1}{2\rho}\partial_\rho + \frac{\Lambda}{2}\rho^2. \tag{2.83}$$

This Hamiltonian is reminiscent of radial Hamiltonian for rotationally invariant problems. To make this precise, let us consider the isotropic harmonic oscillator in two dimensions,

$$\hat{H}_{\mathrm{h.o.}}(x, y, \partial_x, \partial_y) := -\frac{1}{2}\partial_x^2 - \frac{1}{2}\partial_y^2 + \frac{\omega^2}{2}(x^2 + y^2). \tag{2.84}$$

[7] In the model considered in Ref. [43] and Ch. 3 the generating function for the propagator has a cut in the ω plane, instead of the simple pole at $\omega = 0$ we encountered in (2.77).

In polar coordinates,

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta, \tag{2.85}$$

the Hamiltonian reads

$$\hat{H}_{\text{h.o.}}(\rho,\theta,\partial_\rho,\partial_\theta) = -\frac{1}{2}\partial_\rho^2 - \frac{1}{2\rho}\partial_\rho - \frac{1}{2\rho^2}\partial_\theta^2 + \frac{1}{2}\omega^2\rho^2. \tag{2.86}$$

As usual, the rotation symmetry allows us to diagonalize the system with wave-functions of the form

$$\psi(\rho,\theta) = e^{im\theta}\chi(\rho), \qquad m \in \mathbb{Z}. \tag{2.87}$$

Acting on this ansatz with $\hat{H}_{\text{h.o.}}(\rho,\theta,\partial_\rho,\partial_\theta)$ we obtain the eigenvalue problem

$$\left(-\frac{1}{2}\partial_\rho^2 - \frac{1}{2\rho}\partial_\rho + \frac{m^2}{2\rho^2} + \frac{1}{2}\omega^2\rho^2\right)\chi(\rho) = E\chi(\rho). \tag{2.88}$$

We see that the operator on the LHS coincides exactly with $\hat{H}^o(\rho,\partial_\rho)$ when $m=0$ and with the identification

$$\omega^2 = \Lambda. \tag{2.89}$$

The harmonic oscillator can be diagonalized in the usual way. We introduce annihilation operators

$$a_x := \frac{1}{\sqrt{2}}\left(\sqrt{\omega}x + \frac{1}{\sqrt{\omega}}\partial_x\right), \qquad a_y := \frac{1}{\sqrt{2}}\left(\sqrt{\omega}y + \frac{1}{\sqrt{\omega}}\partial_y\right), \tag{2.90}$$

and find

$$\hat{H}_{\text{h.o.}} = \omega\big(a_x^\dagger a_x + a_y^\dagger a_y + 1\big). \tag{2.91}$$

It follows that the spectrum is

$$E_{n_x,n_y} = \omega(n_x + n_y + 1), \qquad n_x, n_y \geq 0, \tag{2.92}$$

and the eigenstates are

$$|n_x, n_y\rangle \propto (a_x^\dagger)^{n_x}(a_y^\dagger)^{n_y}|0\rangle, \tag{2.93}$$

where $|0\rangle$ is the ground state of the oscillator. These states are, however, not of the form (2.87), and thus not the most practical for relating to CDT. This problem is easily solved by introducing a different pair of annihilation operators (see e.g. Ref. [25]),

$$a_\pm := \frac{1}{\sqrt{2}}\big(a_x \mp i a_y\big). \tag{2.94}$$

These operators mutually commute, and satisfy the usual anti-commutation relations. The Hamiltonian looks like

$$\hat{H}_{\text{h.o.}} = \omega\big(a_+^\dagger a_+ + a_-^\dagger a_- + 1\big), \tag{2.95}$$

but now the angular momentum operator is also diagonalized,

$$\hat{L}_z := x(-i\partial_y) - y(-i\partial_x) = -i\partial_\theta = a_+^\dagger a_+ - a_-^\dagger a_-. \tag{2.96}$$

We see that the $m = 0$ states are exactly those of the form

$$(a_+^\dagger a_-^\dagger)^n|0\rangle, \qquad n \geq 0. \tag{2.97}$$

The CDT eigenstates are then

$$\psi_n(L) \propto \langle x = \sqrt{2L}, y = 0 \big| (a_+^\dagger a_-^\dagger)^n \big| 0 \rangle, \tag{2.98}$$

with energies

$$E_n = \sqrt{\Lambda}(2n+1), \qquad n \geq 0. \tag{2.99}$$

More explicit formulae for the eigenstates can be found in Ref. [90].

The Hamiltonian for circular boundary conditions can also be related to the 2d harmonic oscillator. In that case, one has to perform a similarity transformation in addition to the change of variable (2.82). One then finds that CDT corresponds to the $m = 1$ (or, equivalently, $m = -1$) sector of the harmonic oscillator.

Let us finally comment on the $\delta(L)$ term considered in the previous section. In the new coordinates the term is simply

$$\delta(L) \to 2\delta(\rho^2). \tag{2.100}$$

This is formally equivalent to a delta function potential at the origin of the 2d oscillator, since we have

$$\delta(x)\delta(y) \to \frac{1}{\pi}\delta(\rho^2) \tag{2.101}$$

in polar coordinates. Delta function potentials in 2d quantum systems have been studied in detail[58, 48, 53, 24, 69, 73, 45, 35]. Note in particular that the combination of a 2d harmonic potential with a delta potential was considered in Ref. [35]. It would be interesting to translate the results of that reference to the CDT model.

2.3 CDT and Hořava-Lifshitz gravity

Two dimensional CDT can be seen as a regularization of Hořava-Lifshitz (HL) gravity[55], as was shown in Ref. [5]. Here we will sketch the relation between 2d HL gravity and 2d CDT. Note that the two theories are also expected to be related in higher dimensions[56, 7].

The dynamical degree of freedom in HL gravity is the metric $g_{\mu\nu}$ as for ordinary gravity, but full diffeomorphism symmetry is not assumed. Instead one only demands invariance under foliation preserving diffeomorphisms,

$$t \to t'(t), \qquad x \to x'(t, x). \tag{2.102}$$

By a foliation we mean a preferred slicing of the space-time manifold into disjoint space-like surfaces. Here the foliation is defined by the constant-t surfaces.

To express the action of HL gravity it is useful to parametrize the metric as (we denote $x^0 = t$ and $x^1 = x$)

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N(x,t)^2\mathrm{d}t^2 + g_{11}(x,t)(\mathrm{d}x + N^1(t)\mathrm{d}t)^2, \tag{2.103}$$

where $N(x,t)$ is the lapse function, and $N^1(t)$ is the shift vector. Here we are working with the projectable version of HL gravity, which means that $N^1(t)$ only depends on t. The action for 2d HL gravity is[55, 5]

$$S_{\mathrm{HL}} := \int \mathrm{d}t\mathrm{d}x\sqrt{-g}\,[(1-\lambda)K^2 - 2\Lambda], \tag{2.104}$$

where λ is a dimensionless parameter, and K is the extrinsic curvature

$$K := \frac{g^{11}}{2N}(\partial_0 g_{11} - 2\nabla_1 N_1). \tag{2.105}$$

When $\lambda = 1$ the action coincides with the Einstein-Hilbert action (2.5) and it is trivially invariant under diffeomorphisms, but in the $\lambda \neq 1$ case it is only (2.102) that survives as a symmetry.

The main result of Ref. [5] is that, after dealing with constraints, and in the proper time gauge ($N(t) = 1$), the dynamics of the theory (2.104) is described by the Hamiltonian

$$H_{\mathrm{HL}} = L\Pi^2 + \Lambda L, \tag{2.106}$$

where the length of the spatial universe, L, is defined by

$$L(t) := \int \mathrm{d}x\,\sqrt{g_{11}(x,t)}, \tag{2.107}$$

and Π is the momentum canonically conjugate to L. Upon quantization of H_{HL} we recover the CDT Hamiltonian.

2.4 Universality of CDT

Whenever a theory is defined as the continuum limit of some lattice theory, it it natural to ask how the continuum theory depends on the microscopic details of the lattice system. For ordinary renormalizable QFTs the answer is basically that the continuum theory only depends on the microscopic theory through the coefficients of the (finite number of) relevant (and possibly marginal) operators.

One way to approach the question of universality for CDT is to work out what the restrictions on the Hamiltonian are. Let us go back to the starting point of this chapter. We want to make sense of the integral

$$\int_{\mathcal{M}(L',L;T)} \mathcal{D}[g]e^{-i\Lambda A[g]}, \tag{2.108}$$

and we assume that there is some Hermitian Hamiltonian $\hat{H}$ such that

$$\int_{\mathcal{M}(L',L;T)} \mathcal{D}[g]e^{-i\Lambda A[g]} = \langle L'|e^{-iT\hat{H}}|L\rangle. \tag{2.109}$$

From the form of the action it is natural to assume that Λ should enter the Hamiltonian as

$$\hat{H} = \hat{H}_{\mathrm{kin}} + \Lambda L. \tag{2.110}$$

We now have to determine what the possibilities for the kinetic part of the Hamiltonian could be.

Demanding Hermiticity, time reversal invariance and correct dimension (we assume that there are no dimensionful coupling constants besides Λ), the only possible terms for the kinetic part of the Hamiltonian (which are, at most, quadratic in the momentum[8]) are[9]

$$\mathcal{O}_1 := -L\partial_L^2 - \partial_L \quad \text{and} \quad \mathcal{O}_2 := \frac{1}{L}. \tag{2.111}$$

By a rescaling of L and T we must then have

$$\hat{H} = -L\partial_L^2 - \partial_L + \frac{\kappa}{L} + \Lambda L, \tag{2.112}$$

with κ some real number. As discussed in Ch. 3, in all the explicitly solved CDT like models, the coefficient κ is zero (see Refs. [12, 42, 43, 39, 5, 10] and also Sec. 2.3). Usually, this would indicate that $\mathcal{O}_2$ violated some symmetry, but in this case there does not seem to be any relevant symmetry.

One could speculate that locality is behind the universality of CDT, and thus is the mechanism that suppresses the $1/L$ term. When the CDT path integral is written in terms of L, manifest locality is lost (since L is a global degree of freedom), so it is not clear how to make this hypothesis into a precise statement. Still, it might be possible to derive some consistency relation for the path integral from locality, and show that this relation excludes the $1/L$ term. We leave this to future work.

2.5 CDT coupled to Yang-Mills

In Ref. [9] (Ch. 4) a model of CDT coupled to Yang-Mills theory is defined and solved. There the focus is on a lattice formulation of the theory. Here we will give some details of the solution of the model starting from a Hořava-Lifshitz formulation. We follow the flat-space calculation of Ref. [77].

We consider Yang-Mills theory coupled to two-dimensional projectable HL gravity with circular spatial topology. We take the spatial coordinate to run from $x = 0$ to $x = 1$, with the two points identified. We start from a classical theory

$$S = S_{\text{HL}} + S_{\text{YM}} = \int \mathrm{d}t\mathrm{d}x\sqrt{-g}\,[(1-\lambda)K^2 - 2\Lambda] + \frac{1}{2}\int \mathrm{d}t\mathrm{d}x\sqrt{-g}\;\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}], \tag{2.113}$$

with

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu - ig_{\text{YM}}[A_\mu, A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{\text{YM}}[A_\mu, A_\nu]. \tag{2.114}$$

The field A_μ takes values in the Lie algebra of some group G. By gauge invariance, the gauge field has no local degrees of freedom. It is thus natural to rewrite the theory in terms of the holonomy of the gauge field along the constant-t loops. Let us define

$$S^{-1}(x,t) := \mathcal{P}e^{-ig_{\text{YM}}\int_0^x \mathrm{d}y\, A_1(y,t)} \in G, \tag{2.115}$$

[8]One could, in principle, consider Hamiltonians that are of higher order in the momentum, but then special care is needed (see e.g. Refs. [15, 13]). It is clear that any regularized path integral that allows for a saddle point treatment as in Sec. 2.1.1 will be quadratic in the momentum.

[9]Here we are excluding delta function potentials like $\delta(L)$.

where $\mathcal{P}e^{\int \cdots}$ is the path-ordered exponential, that is, $S(x,t)$ solves the equations

$$\partial_1 S^{-1}(x,t) = -ig_{\mathrm{YM}} S^{-1}(x,t) A_1(x,t), \qquad S^{-1}(x=0,t) = I. \tag{2.116}$$

The holonomy is then

$$q(t) := S(x=1,t). \tag{2.117}$$

We now want to express S_{YM} in terms of q. This is possible after enforcing the Gauss constraint.

The field strength only has one independent component, $E := F^{01}$, but it is more convenient to work with

$$\tilde{E} := (\sqrt{-g})^{-1} E. \tag{2.118}$$

This quantity transforms as a scalar under diffeomorphisms, as can be seen from the relation

$$F_{\mu\nu} F^{\mu\nu} = 2(\tilde{E})^2. \tag{2.119}$$

The equations of motion following from S_{YM} are

$$\nabla_\mu F^{\mu\nu} - ig_{\mathrm{YM}} [A_\mu, F^{\mu\nu}] = 0, \tag{2.120}$$

and the Gauss constraint is the $\nu = 0$ component. The presence of the covariant derivative in (2.120) is inconvenient, but we can use the relation

$$(\sqrt{-g})^{-1} \partial_\mu \sqrt{-g} = \Gamma^\nu_{\mu\nu} \tag{2.121}$$

to deduce that

$$\nabla_\mu F^{\mu 0} = -(\sqrt{-g})^{-1} \partial_1 \tilde{E}. \tag{2.122}$$

The constraint can thus be written as

$$\partial_1 \tilde{E} - ig_{\mathrm{YM}} [A_1, \tilde{E}] = 0. \tag{2.123}$$

From this last equation, it follows immediately that $\operatorname{tr} F_{\mu\nu} F^{\mu\nu}$ is constant as a function of x. The Yang-Mills action can then be simplified to (since we are working with projectable HL, the determinant of the metric takes the form $\sqrt{-g} = \sqrt{g_{11}} N(t)$)

$$S_{\mathrm{YM}} = \frac{1}{2} \int \mathrm{d}t\, N(t) L(t) \operatorname{tr} p(t)^2, \tag{2.124}$$

where

$$p(t) := \tilde{E}(x=0,t), \tag{2.125}$$

and

$$L(t) := \int_0^1 \mathrm{d}x \sqrt{g_{11}(x,t)} \tag{2.126}$$

is the spatial size of the universe. We note that we can explicitly solve (2.123) for $\tilde{E}$ to get

$$\tilde{E}(x,t) = S(x,t) p(t) S(x,t)^{-1}. \tag{2.127}$$

We must now relate $p(t)$ and $q(t)$. Since $p(t)$ is in the Lie algebra, a natural object to consider would be $q^{-1}\partial_0 q$. Let us more generally define

$$P(x,t) := S^{-1} \partial_0 S. \tag{2.128}$$

At this point we choose the temporal gauge, $A_0 = 0$, for simplicity, such that $E = \partial_0 A_1$. The x derivative of P is

$$\partial_1 P = igS^{-1}ES = ig_{\rm YM}\sqrt{-g}S^{-1}\tilde{E}S. \tag{2.129}$$

Using (2.127), we can integrate the equation and find

$$P(x=1,t) = q^{-1}\partial_0 q = ig_{\rm YM}NLp. \tag{2.130}$$

We conclude that the action for the holonomy variable is

$$S_{YM} = -\frac{1}{2g_{\rm YM}^2}\int \mathrm{d}t \frac{1}{NL}\,\mathrm{tr}(q^{-1}\partial_0 q)^2. \tag{2.131}$$

This action is, up to the (time dependent) factor $(NL)^{-1}$, just the action for a free particle on a Lie group. The corresponding quantum Hamiltonian is well known(see e.g. [30, 37])

$$\hat{H}_{YM} = \frac{1}{2}g_{\rm YM}^2 NL\Delta_G, \tag{2.132}$$

where Δ is the group Laplacian defined by Eq. (1.58). Gauge fixing to proper time gauge ($N(t) = 1$), and using the result mentioned in the previous section, we thus obtain the total Hamiltonian

$$\hat{H} = -L\partial_L^2 - \partial_L + \frac{1}{2}g_{\rm YM}L\Delta_G + \Lambda L. \tag{2.133}$$

This is in complete agreement with the result obtained by the lattice approach in Ch. 4.

Chapter 3

Paper: "Universality of 2d causal dynamical triangulations"

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Universality of 2d causal dynamical triangulations

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Abstract

The formalism of Causal Dynamical Triangulations (CDT) attempts to provide a non-perturbative regularization of quantum gravity, viewed as an ordinary quantum field theory. In two dimensions one can solve the lattice theory analytically and the continuum limit is universal, not depending on the details of the lattice regularization.

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1 Introduction

Two-dimensional quantum gravity has been a fruitful laboratory for studying aspects of string theory as well as quantum gravity. One somewhat surprising aspect of Euclidean two-dimension quantum gravity coupled to matter in the form of a conformal field theory, is that the regularized lattice theory, using the so-called dynamical triangulations (DT), can be solved analytically. The details of the DT regularization are unimportant for the continuum limit. In fact it has been a wonderful example of universality in the Wilsonian sense, the critical surface where the continuum limit can be taken being of finite co-dimension in an infinite dimensional coupling constant space (see e.g. [1] for a review). The lattice regularization known as causal dynamical triangulations (CDT) uses a subset of the triangulations used in DT [2, 3]. The original idea was to consider a path integral where spacetime histories before rotating to Euclidean signature were locally causal, i.e. had non-degenerate light cones (see [4] for a review of the CDT approach also in higher dimensions than two). In two dimensions, which is the only case we will consider here, the precise relation between the CDT triangulations and the DT triangulations was described in [5].

There is good evidence of universality of the CDT scaling limit, although one does not have the same comprehensive evidence as for the DT case. First, a related model, in a certain way more general, the so-called string-bit model [6], led to the same scaling limit. Further it was shown in [7] that one could add dimers on the "spatial" CDT links without changing the universality class. Thus it was somewhat surprising that adding further "dressing", but only along the spatial links, seemingly led to new continuum models, depending on a continuous parameter β (to be defined below) [8]. The purpose of this letter is to show that also for this general set of models one obtains indeed the standard CDT scaling limit.

2 Defining the model

The modified CDT model (not to be mistaken for what has later been called "generalized CDT" [9]) is most easily defined using a lattice dual to the triangulation, i.e. a ϕ^3 graph with a "time" foliation . Fig. 1 shows the dual CDT lattice and its generalization. In this dual picture each vertex represents a triangle in the "original" triangulation and each polygon represents a vertex, the order of which is equal the number of sides in the polygon.

In the modified model one allows a dressing of the horizontal links between two vertical links by rainbow diagrams.

Three coupling constants are assigned to the model: to each vertex one associates a coupling constant g, to a vertex with an incident vertical link an additional

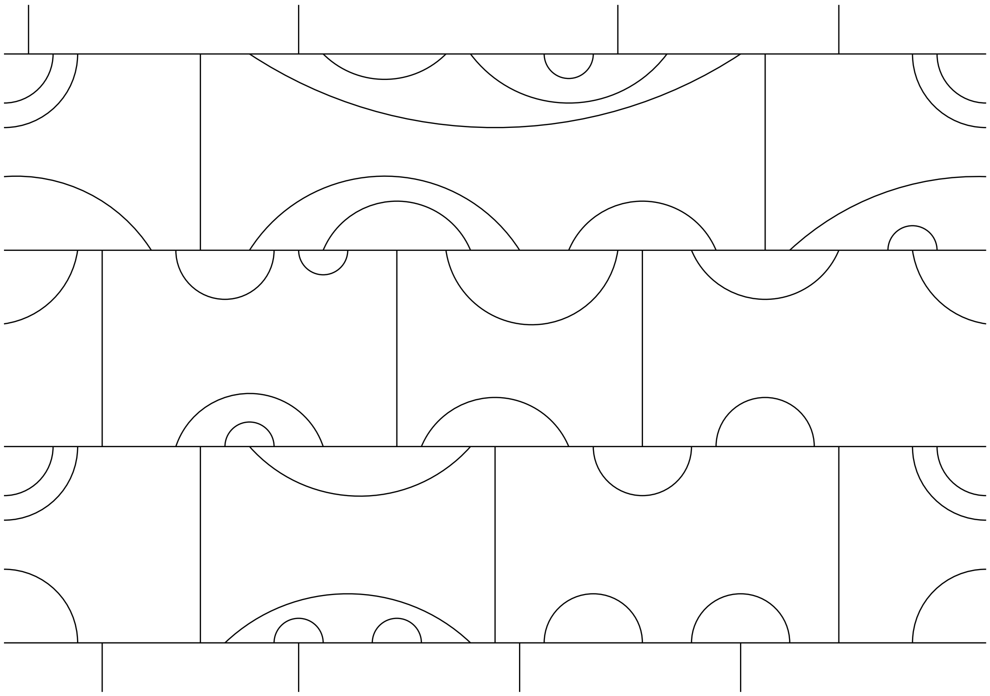

Figure 1: Modified CDT configuration, dual graph.

coupling constant h, and finally to each vertex with an incident rainbow link a coupling constant θ. The parameter

$$\beta = \frac{\theta}{h} \tag{1}$$

governs the density of rainbow links compared to the number of vertical links, i.e. "time-like" links in the original CDT-like ϕ^3-graph. In this article we will only consider $0 \leq \beta < 1$, which is the range leading to CDT-like theories [8].

As shown in [8] one can define and calculate a transfer matrix for this model. The result is

$$\Theta_{ij} = \sum_k \Theta^{(2)}_{ik} \Theta^{(1)}_{kj} \tag{2}$$

where the index j refers to the number of incoming half-lines which is incident from below on the horizontal line at time t and index k refers to the number of half-lines leaving the horizontal line at time t. Index k plays the same role as index j, only at time-slice $t+1$. In this way $\Theta^{(2)}_{ik}$ connects outgoing vertical half-lines at t to incoming half-lines at $t+1$ and Θ_{ij} incoming half-lines at t to incoming half-lines at $t+1$.

$\Theta^{(1)}$ is the CDT transfer matrix, already discussed in [2] and analyzed in detail in [7]. If $\theta = 0$ and $h = 1$ there are no rainbow lines and $\Theta^{(2)}$ becomes the identity matrix and Θ also the CDT transfer matrix.

It is convenient to work with the discrete Laplace transforms of Θ, $\Theta^{(1)}$ and $\Theta^{(2)}$. To simplify the expressions somewhat we make the following redefinitions compared to [8]:

$$\Theta^{(1)}_{ij} \to (2g)^{-i-j}\Theta^{(1)}_{ij}, \qquad \Theta^{(2)}_{ij} \to (2g)^{i+j}\Theta^{(2)}_{ij}. \tag{3}$$

The explicit expressions are then:

$$\Theta^{(1)}(x,y) = \sum_{ij} x^i y^j \Theta^{(1)}_{ij} = \frac{1}{1-\frac{1}{2}x-\frac{1}{2}y} \tag{4}$$

$$\Theta^{(2)}(x,y) = \frac{C(\hat{x}^2)C(\hat{y}^2)}{(1-\hat{x}^2C(\hat{x}^2))(1-\hat{y}^2C(\hat{y}^2))(1-\beta^{-2}\hat{x}\hat{y}\,C(\hat{x}^2)C(\hat{y}^2))} \tag{5}$$

$$\Theta(x,y) = \oint_{\mathcal{C}} \frac{d\omega}{2\pi i\omega}\,\Theta^{(1)}(x,\omega^{-1})\Theta^{(2)}(\omega,y), \tag{6}$$

where the contour encloses cuts and poles and where

$$\hat{x} = 2g\theta x, \quad \hat{y} = 2g\theta y, \qquad C(z) = \frac{1-\sqrt{1-4z}}{2z}. \tag{7}$$

Integrating over the simple pole of $\Theta^{(1)}$ one obtains

$$\Theta(x,y) = \frac{1}{1-\frac{1}{2}x}\,\frac{C(\bar{x}^2)C(\hat{y}^2)}{(1-\bar{x}^2C(\bar{x}^2))(1-\hat{y}^2C(\hat{y}^2))}\,\frac{1}{1-\beta^{-2}\bar{x}\hat{y}C(\bar{x}^2)C(\hat{y}^2)}, \tag{8}$$

where

$$\bar{x} = \frac{2g\theta}{2-x} \tag{9}$$

The partition function with open horizontal boundaries after t time steps is[1]

$$Z(l,k;t) = \Big((\Theta^{(1)}(\Theta^{(2)}\Theta^{(1)})^t\Big)_{kl}, \tag{10}$$

and the (discrete) Laplace transformed function is denoted $Z(x,y)$

$$Z(x,y;t) = \sum_{l,k} x^l y^k Z(l,k;t). \tag{11}$$

The partition function after t time steps with periodic boundary conditions in the time direction is

$$Z(t) = \mathrm{tr}\,(\Theta^t). \tag{12}$$

[1] The same continuum limit is obtained by setting $Z(l,k;t) = \big(\Theta^t\big)_{kl}$.

3 The continuum limit using the transfer matrix

As shown in [8] the partition function $Z(t)$ has a singularity at

$$\xi_c = 2g\theta \left(\beta + \frac{1}{\beta} \right) = 1. \tag{13}$$

We want to take to continuum limit by approaching this singularity. This is done in the following way [8]:

$$\xi \equiv 2g\theta \left(\beta + \frac{1}{\beta} \right) = 1 - \frac{1}{2} a^2 \Lambda \left(\frac{1-\beta^2}{1+\beta^2} \right)^2. \tag{14}$$

The interpretation is that a is the lattice spacing, i.e. the link length in the triangulation, and Λ the cosmological constant, such that the average number of triangles is proportional to $1/(\Lambda a^2)$. Thus the average "continuum" area is proportional to $1/\Lambda$.

Until now t has denoted the integer number of time steps in the triangulation. We are interested in a limit where we have a finite continuum time T scaling as

$$T = ta, \tag{15}$$

where a is the lattice spacing defined by (14). We can then write

$$Z(T) = \mathrm{tr}\, \Theta^t = \mathrm{tr}\, e^{-TH}, \quad \Theta = \mathrm{e}^{-aH}. \tag{16}$$

Thus an expansion of Θ to lowest order in a should allow us to determine H.

If the continuum area is proportional to $1/\Lambda$ we expect the continuum length of a time slice to be proportional to $1/(\Lambda T)$. Thus we expect a scaling $L \propto l\, a$ where l is the number of space-like links. We can also enforce this on the boundaries:

$$Z(l, k; t) \to Z(L_0, L_T; T). \tag{17}$$

The discrete Laplace transform of $Z(x, y; t)$ has poles in x, y and it is at these poles one extracts the continuum function $Z(L_0, L_T; T)$. These poles are at $x_c = y_c = 1$ for $a \to 0$. The terms x^l and y^k in (11) can then be given an interpretation as the part of the action coming from a continuum boundary cosmological term proportional to X if we scale:

$$x = 1 - aX \left(\frac{1-\beta^2}{1+\beta^2} \right)^2, \qquad L = a\, l \left(\frac{1-\beta^2}{1+\beta^2} \right)^2, \tag{18}$$

and thus

$$x^l \to \mathrm{e}^{-LX} \quad \text{for} \quad a \to 0. \tag{19}$$

With this scaling we obtain a relation similar to (17), going from the discretized expression to the continuum expression:

$$Z(x,y,t) \to Z(X,Y;T), \tag{20}$$

where the continuum analogue of (11) reads

$$Z(X,Y;T) = \int_0^\infty \mathrm{d}L_0 \mathrm{d}L_T \; \mathrm{e}^{-L_0 X - L_T Y} Z(L_0, L_T; T). \tag{21}$$

We will return to (17) and (20) in the next section.

We now extract H from $\Theta = e^{-aH}$. It is convenient to use the Laplace transform (6) of Θ. Expanding in a we obtain [8]:

$$\Big((1 - aH + O(a^2))\psi\Big)(x) = \frac{1}{2}\frac{1-\beta^2}{1+\beta^2} \oint \frac{d\omega}{2\pi i\omega} \; \Theta\Big(x, \frac{1}{\omega}\Big) \, \psi(\omega). \tag{22}$$

Here $\psi(\omega)$ is the discrete Laplace transform of a function $\psi(l)$:

$$\psi(\omega) = \sum_l \omega^l \psi(l). \tag{23}$$

The function $\Theta(x, 1/\omega)$ has a pole in ω at 1 for $a \to 0$ and it has a branch cut located at $\omega \in [-\omega_*, \omega_*]$, where

$$\omega_* = 2\left(\beta + \frac{1}{\beta}\right)^{-1} + O(a) < 1 \quad \text{for } a \text{ sufficiently small.} \tag{24}$$

We can deform the contour to be a small circle around one and an integration along the branch cut. The integration around $\omega = 1$ allows us to use the expansion (18) for x and ω, and we obtain

$$\oint \frac{dZ}{2\pi i} \left[\frac{1}{Z-X} + \frac{a}{(Z-X)^2} \left(\Lambda + \frac{\beta^2 X^2 - (1+3\beta^2) XZ + \beta^2 Z^2}{1+\beta^2} \right) \right] \psi(Z) + O(a^2), \tag{25}$$

Performing the integration (and ignoring the contribution from the cut) we can identify H as

$$H(X) = (X^2 - \Lambda)\frac{\partial}{\partial X} + X, \tag{26}$$

and by an inverse Laplace transformation

$$H(L) = -L\frac{\partial^2}{\partial L^2} - \frac{\partial}{\partial L} + \Lambda L. \tag{27}$$

This is precisely the ordinary CDT Hamiltonian, the only difference is that in order to obtain it in this form we had to perform a dressing (or renormalization) of the continuum boundary cosmological constant from a value X, corresponding to $\beta = 0$ to the β dependent value given in (18). This renormalization of X and a similar renormalization of the coupling cosmological coupling constant Λ in (14) is all that is needed to include the effects of the rainbow diagrams.

The contribution from the cut can be written as

$$\tilde{\psi}(x) = \int_{-\omega_*}^{\omega_*} d\omega f(x,\omega)\psi(\omega), \tag{28}$$

where $f(x,\omega)$ is integrable in $[-\omega_*, \omega_*]$ and $\tilde{\psi}(x)$ analytic in the neighborhood of 1 and finite when $a \to 0$. We cannot view such a function as the Laplace transform of any function $\psi(\sqrt{\Lambda}L)$ depending on the continuum length $L > 0$, the reason being that the inverse Laplace transformation from (26) to (27) gives

$$\int_{i\infty+c}^{i\infty+c} \frac{dX}{2\pi i}\, \mathrm{e}^{XL}\tilde{\psi}(1-aX) = \delta(L)\tilde{\psi}(1) - a\delta'(L)\tilde{\psi}'(1) + \cdots + O(a^n). \tag{29}$$

Thus we do not associate any continuum physics with the analytic function $\tilde{\psi}(x)$ defined by (28) [2].

4 The Schwinger representation and the continuum

In [8] the modified CDT Hamiltonian was not derived using the transfer matrix as described above, but rather a so-called Schwinger representation of $Z(x,y;t)$. We now show that this method also leads to (27), i.e. the ordinary CDT Hamiltonian.

The starting point is the following representation of $Z(x,y;t)$ ([8], formula (5.19)):

$$Z(x,y;t) = \prod_{s=0}^{t}\left(\int_0^\infty d\alpha_s e^{-\alpha_s}\right) e^{\frac{1}{2}(\alpha_0 x + \alpha_t y)} \prod_{r=0}^{t-1} \phi_\beta(g\theta\alpha_r, g\theta\alpha_{r+1}) \tag{30}$$

where

$$\phi_\beta(x,y) = \sum_{k\geq 0} I_k(2x) I_k(2y)/\beta^{2k}. \tag{31}$$

[2] Of course a function like $\tilde{\psi}(\omega)$ would also not contribute to continuum physics if inserted in (25). The part of a function $\psi(\omega)$ defined as in (23) which *does* contribute to continuum physics in (25) is the part which has a continuum Laplace transform, i.e. the part where $\psi(l)$ in (23) has the form $\psi(\sqrt{\xi-\xi_c}\, l) \to \psi(\sqrt{\Lambda}L)$. Since $\sqrt{\xi-\xi_c} \propto a\sqrt{\Lambda}$ it can at most be the tail at infinite l which contributes to continuum physics for a given $\psi(\omega) = \sum_l \omega^l \psi(l)$.

x and y only appears in the exponential function and we can write

$$Z(x,y;t) = \int_0^\infty d\alpha_0 \int_0^\infty d\alpha_t \, \mathrm{e}^{-\frac{1}{2}(1-x)\alpha_0 - \frac{1}{2}(1-y)\alpha_t} F(\alpha_0, \alpha_t; t), \tag{32}$$

where

$$F(\alpha_0,\alpha_t;t) = \left(\prod_{s=1}^{t-1} \int_0^\infty d\alpha_s \right) \prod_{r=0}^{t-1} \mathrm{e}^{-(\alpha_r+\alpha_{r+1})/2} \phi_\beta(g\theta\alpha_r, g\theta\alpha_{r+1}). \tag{33}$$

Since $1-x \propto aX$ and $1-y \propto aY$, (32) states that in the limit where $a \to 0$ and thus $Z(x,y;t) \to Z(X,Y;T)$, $Z(X,Y;T)$ is the Laplace transform of $F(\alpha_0,\alpha_t;t)$, $t = T/a$. Thus, in accordance with (21) we have

$$F(\alpha_0,\alpha_t;t) \propto Z(L_0, L_T; T), \tag{34}$$

where

$$L_0 = \frac{1}{2} a\alpha_0 \left(\frac{1-\beta^2}{1+\beta^2} \right)^2, \quad L_T = \frac{1}{2} a\alpha_t \left(\frac{1-\beta^2}{1+\beta^2} \right)^2, \quad a\, t = T. \tag{35}$$

If we change variables from α_s to φ_s,

$$\alpha_s = \frac{\varphi_s^2}{a} \left(\frac{1+\beta^2}{1-\beta^2} \right)^2, \tag{36}$$

we obtain

$$Z(L_0,L_T;T) \propto \frac{1}{\sqrt{\varphi_0\varphi_t}} \int_0^\infty \prod_{s=1}^{t-1} \mathrm{d}\varphi_s \prod_{r=0}^{t-1} \frac{\sqrt{\varphi_r\varphi_{r+1}}}{a} \frac{1+\beta^2}{1-\beta^2} \mathrm{e}^{-\frac{\alpha_r+\alpha_{r+1}}{2}} \phi_\beta(g\theta\alpha_r, g\theta\alpha_{r+1}). \tag{37}$$

The right hand side can be interpreted as a (quantum mechanical) path integral, i.e.

$$\sqrt{\varphi_0\varphi_t}\, Z(L_0,L_T;T) \propto \langle \varphi_0 | e^{-TH} | \varphi_t \rangle \tag{38}$$

for some Hamiltonian H. We will now proceed to determine H.

Following [8] we use the notation

$$\mathrm{e}^{-\frac{\alpha_0+\alpha_1}{2}} \phi_\beta(g\theta\alpha_0, g\theta\alpha_1) \sim U_\beta(\alpha_0,\alpha_1)\, \mathrm{e}^{-S_\beta(\alpha_0,\alpha_1)}. \tag{39}$$

According to [8]

$$S_\beta(\alpha_0,\alpha_1) = \frac{1}{2}(\alpha_0+\alpha_1) - 2g\theta\sqrt{(\alpha_0+\beta^2\alpha_1)(\alpha_0+\beta^{-2}\alpha_1)} \tag{40}$$

and

$$U_\beta(\alpha_0,\alpha_1) = \frac{1}{\sqrt{4\pi g\theta}}\frac{1}{((\alpha_0+\beta^2\alpha_1)(\alpha_0+\beta^{-2}\alpha_1))^{1/4}} \times\left(1+\frac{1}{16g\theta\sqrt{(\alpha_0+\beta^2\alpha_1)(\alpha_0+\beta^{-2}\alpha_1)}}+\cdots\right). \quad (41)$$

We now expand in a, with

$$\Delta\varphi = \varphi_1 - \varphi_0 \quad (42)$$

counted as being of order $\sqrt{a}$ as one has to do in a path integral (here we differ from [8]):

$$S_\beta(\alpha_0,\alpha_1) = \frac{\Delta\varphi^2}{2a} - \frac{\beta^2}{(1+\beta^2)^2}\frac{\Delta\varphi^4}{2a\varphi_0^2} + \frac{a\Lambda}{2}\varphi_0^2 + O(a^{3/2}). \quad (43)$$

We see that we get a standard kinetic term, justifying $\Delta\varphi \propto \sqrt{a}$. (Note that the $\Delta\varphi^4$ term is not present in [8]).

Similarly, we find

$$\frac{\sqrt{\varphi_r\varphi_{r+1}}}{a}\frac{1+\beta^2}{1-\beta^2}U_\beta(\alpha_0,\alpha_1) = \frac{1}{\sqrt{2\pi a}}\left(1+\frac{a}{8\varphi_0^2} - \frac{\beta^2}{(1+\beta^2)^2}\frac{\Delta\varphi^2}{\varphi_0^2} + O(a^{3/2})\right). \quad (44)$$

(We note that the $\Delta\varphi^2$ term is not present in [8].)

The Hamilton is finally determined by integrating against a trial state:

$$((1-aH)\psi)(\varphi_0) = \int_0^\infty \frac{\mathrm{d}\varphi_1}{\sqrt{2\pi a}}e^{-\frac{\Delta\varphi^2}{2a}}\left[1+\left(\frac{1-\beta^2}{1+\beta^2}\right)^2\frac{a}{8\varphi_0^2} - \frac{\beta^2}{(1+\beta^2)^2}\frac{\Delta\varphi^2}{\varphi_0^2} + \frac{\beta^2}{(1+\beta^2)^2}\frac{\Delta\varphi^4}{2a\varphi_0^2} - \frac{a\Lambda}{2}\varphi_0^2\right]\left[1+\Delta\varphi\frac{\partial}{\partial\varphi}+\frac{\Delta\varphi^2}{2}\frac{\partial^2}{\partial\varphi^2}\right]\psi(\varphi_1). \quad (45)$$

Carrying out the Gaussian integral, we obtain

$$H = -\frac{1}{2}\frac{\partial^2}{\partial\varphi^2} + \frac{\Lambda}{2}\varphi^2 - \frac{1}{8\varphi^2}. \quad (46)$$

This is precisely the CDT Hamiltonian when changing back to the L variable.

5 Critical arches

In principle a new behavior could be possible for $\beta \to 1$ from below, since in this case the rescaling of lengths and boundary cosmological constants, as defined by eqs. (18), diverges and it is precisely the limit where the cut will merge with the pole in the expression (8) for Θ. Let us investigate this case by assuming

$$\beta = 1 - a^\eta B, \tag{47}$$

where B is a new physical constant with mass dimension η. To understand the analytic structure of Θ for $a \to 0$, i.e. $\beta \to 1$ from below, we expand the argument of the square root related to the Catalan number in the expression for Θ:

$$\sqrt{1-4\hat{x}^2} = a^\eta B(1 + aX + \frac{1}{2}a^2(\Lambda - X^2) + O(a^\eta B) + O(a^3)) \tag{48}$$

From this expression it is clear that that the cut has disappeared from the expression even though it hits the pole when expressed in terms of unrenormalized variables. To find the Hamiltonian we use the same approach as in Sec. 3, eqs. (22) and (25) and write

$$\tilde{\psi}(x) = (1 - a^\nu H + \cdots)\psi(x) := \frac{a^\eta B}{2} \oint \frac{\mathrm{d}\omega}{2\pi i\omega}\, \Theta(x, \frac{1}{\omega})\psi(\omega), \tag{49}$$

where ν is determined by the expansion, We find:

$$\tilde{\psi}(X) = \oint \frac{\mathrm{d}Z}{2\pi i} \left[\frac{1 - a^\eta B/2}{Z - X} + a\, \frac{\Lambda + \frac{1}{2}(X^2 - 4XZ + Z^2)}{(Z-X)^2} \right] \psi(Z). \tag{50}$$

Thus, if $\eta > 1$ we obtain the same results as before (eq. (25) with $\beta = 1$) and if $\eta < 1$ we obtain a trivial Hamiltonian. $\eta = 1$ just adds the positive constant $B/2$ to the CDT Hamiltonian (26). So far we have ignored the contributions from the cut. However, arguments like the ones used in Sec. 3 show that the cut will not contribute in the scaling limit.

6 Discussion

We have shown that the CDT scaling limit is quite universal and independent of details of the lattice regularization, as long as we maintain a reasonable "memory" of the underlying assumed time foliation. Dressing the spatial slices with a few outgrowths should not alter the scaling limit and this is indeed what we have proven to be the case. Potentially there could have been a different behavior in the limit $\beta \to 1$ where the rainbow diagrams become critical, but explicit calculations showed that it was not the case. The CDT model provides us with a regularized of a theory of fluctuating spacetime which is invariant under spatial diffeomorphisms and which allows for a time foliation. The simplest such continuum model is a Hořava-Lifshitz gravity model in two-dimensions where we only keep terms with at most second order derivatives of the metric, and one can indeed show that such a model has a classical CDT Hamiltonian which when quantized is compatible with the $H(L)$ considered in this paper [10].

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Chapter 4

Paper: "Two-dimensional causal dynamical triangulations with gauge fields"

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2d CDT with gauge fields

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Abstract

Two-dimensional Causal Dynamical Triangulations provides a definition of the path integral for projectable two-dimensional Horava-Lifshitz quantum gravity. We solve the theory coupled to gauge fields.

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Keywords: quantum gravity, lower dimensional models, lattice models.

1 Introduction

Two-dimensional causal dynamical triangulations (CDT) [1] provides a well defined path integral representation of two-dimensional projectable Hořava-Lifshitz quantum gravity (HL [2]), as was recently shown in [3]. 2d CDT coupled to conformal field theories with central charges $c = 1/2$ and $c = 4/5$ as well as $c \geq 1$ have been investigate numerically [4, 5, 6]. However, it has not yet been possible to provide exact solutions of the gravity theory coupled to a well defined continuum matter theory despite the existence of a matrix formulation [9][1]. Here we will provide a first such step and solve CDT coupled to gauge theories.

Gauge theories are simple in two dimensions since there are no propagating field degrees of freedom. However, if the geometry is non-trivial there can still be non-trivial dynamics, involving a finite number of degrees of freedom. In the CDT case we consider space-time with the topology of a cylinder, space being compactified to S^1, and we thus have non-trivial dynamics associated with the holonomies of S^1. This has been studied in great detail in flat space-time (see [13] and references therein). We will use the results from these studies to solve CDT coupled to gauge theory. The rest of this article is organized in the following way. In Sec. 2 we review the part of [13] that we need for the construction the CDT quantum Hamiltonian. In Sec. 3 we find the lattice transfer matrix and the corresponding continuum Hamiltonian and finally in Sec. 4 we discuss "cosmological" applications.

2 2d gauge theories on a cylinder

Let us first heuristically understand the Hamiltonian for gauge theory on the cylinder, the gauge group G being a simple Lie group (we can think of $G = SU(N)$ if needed). The action is

$$S_{YM} = \frac{1}{4} \int dtdx \; (F^a_{\mu\nu})^2, \qquad \mu, \nu = 0, 1, \tag{1}$$

where $F^a_{01} = E^a_1$ is the chromo-electric field. Quantizing in the temporal gauge, $A^a_0 = 0$, say, one obtains the Hamiltonian

$$\hat{H} = \frac{1}{2} \int dx \; (\hat{E}^a_1)^2, \quad \hat{E}^a_1 \equiv -i \frac{\delta}{\delta A^a_1(x)}, \tag{2}$$

[1] To be precise, CDT has been solved when coupled to some "non-standard" hard dimer models [10, 11], but it is unknown if these dimer models have an interesting continuum limit. Also, " generalized CDT" models coupled to ordinary hard dimer models have been solved [10, 12], using matrix models.

and this Hamiltonian acts on physical states, i.e. wavefunctions which satisfy Gauss law

$$(D_1\hat{E}^1)^a\Psi(A) = 0, \tag{3}$$

where D_1 denotes the covariant derivative. Since D_1E^1 are the generators of gauge transformations (3) just tells us that $\Psi(A)$ is gauge invariant. But on S^1 the only gauge invariant functions are class functions of the holonomies and any class function can be expanded in characters of irreducible unitary representations of the group. Let T^a_R denote the Lie algebra generators of the representation R, where $\mathrm{tr}\,_R T^a_R T^a_R = C_2(R)$, the value of quadratic Casimir for the representation R. For a holonomy

$$U_R(A) = P\mathrm{e}^{ig\oint dx A^a_1(x) T^a_R}, \quad \chi_R(U) \equiv \mathrm{tr}\,_R U_R, \tag{4}$$

where g is the gauge coupling, one easily finds that the action of $\hat{H}$ on the wavefunction $\chi_R(U(A))$ is

$$\hat{H}\chi_R(U(A)) = \frac{1}{2}g^2 L C_2(R)\chi_R(U(A)). \tag{5}$$

From this we take along that on the gauge invariant wave functions we can write[2]

$$\hat{H} = \frac{1}{2}g^2 L\Delta_G \tag{6}$$

where Δ_G is the Laplace-Beltrami operator on the group G (here $SU(N)$), and further that the gauge invariant eigenfunctions are the irreducible characters of G.

Let us now quantize the theory using a lattice, i.e. using a (regularized) path integral. The lattice partition function is defined as

$$Z(g) = \int \prod_{\ell} dU_\ell \prod_{\text{plaquettes}} Z_P[U_P], \tag{7}$$

where we to each link ℓ associate a $U_\ell \in G$, and U_P is the product of the U_ℓ's around the plaquette (since we always take the trace of U_P it does not matter which link is first in the product as long as we keep the orientation). One writes $U_\ell = e^{agiA^b_\ell t^b}$ where ℓ signifies a link in direction $\mu = 0$ or $\mu = 1$, a is the length

[2] We are clearly not very precise here when discussing the quantization (that is why we started this section with the word "heuristic"). We have still available a time independent gauge transformation which we can use to gauge the holonomy $U(A)$ to a Cartan subalgebra of G, i.e. to diagonalize $U(A)$, and further to permute the diagonal elements. Strictly speaking $\hat{H}$ should be defined on this subspace which is the orbifold T^{N-1}/S_N for $G = SU(N)$. We refer the reader to [13] for details.

of a lattice link and we choose t^a to be generators of the Lie algebra of G in the fundamental representation, normalized to $\mathrm{tr}\, t^b t^c = 1/2$. This establishes a formal relation between the gauge fields A_ℓ and the group variables U_ℓ. One has a large choice for $Z_P[U_p]$, the only requirement being that $Z(g)$ in (7) should formally become the continuum path integral when the lattice spacing is taken to zero. Often the so-called Wilson action is used where

$$Z_P[U_P] = \mathrm{e}^{\beta \mathrm{tr}\,(U_p + U_P^{-1})}, \qquad \beta = \frac{1}{4g^2 a^2}. \tag{8}$$

In the limit where $a \to 0$ one has $\mathrm{tr}\,(U_p + U_P^{-1}) \to 1 - a^4 g^2 (F^b_{\mu\nu})^2 + 0(a^6)$, thus leading to the correct naive continuum limit in (7) if β scales as in (8). For the purpose of extracting the Hamiltonian it is convenient for us to use a different $Z_P[U_p]$, the so-called heat kernel action

$$Z_P[U_P] = \langle U_P | e^{-\frac{1}{2} g^2 A_P \Delta_G} | I \rangle = \sum_R d_R \chi_R(U_P)\, \mathrm{e}^{-\frac{1}{2} g^2 A_P C_2(R)}, \tag{9}$$

where $A_P = a_t a_s$ denotes the area of the plaquette with spatial lattice link length a_s and time-like link length a_t (we will usually think of $a_s = a_t$), I denotes the identity element in G and, as above Δ_G the Laplace-Beltrami operator on G. Using $U_\ell = (I + agA^a_\ell t^a \cdots)$ in the limit $a \to 0$, and $\sum_R d_R \chi_R(U_P) = \delta(U_P - I)$, one can show that the continuum Yang-Mills action is formally reproduced. The convenient property of the heat kernel action in 2d is that it is additive, i.e. if we integrate over a link in (7) the action is unchanged: write $U_{P_1} = U_4 U_3 U_2 U_1$ and $U_{P_2} = U_4^{-1} U_7 U_6 U_5$, then

$$\int dU_4 Z_{P_1}[U_{P_1}] Z_{P_2}[U_{P_2}] = Z_{P_1+P_2}[U_{P_1+P_2}], \tag{10}$$

where $U_{P_1+P_2} = U_7 U_6 U_5 U_3 U_3 U_1$, see Fig. 1.

The relation follows from the orthogonality of the group characters:

$$\int dU\ \chi_R(XU) \chi_{R'}(U^{-1}Y) = \frac{1}{d_R} \delta_{RR'} \chi_R(XY). \tag{11}$$

Let us now consider a lattice with t links in the time direction and l links in the spatial direction. We have two boundaries, with gauge field configurations $\{U_\ell\}$ and $\{U'_\ell\}$, which we can choose to keep fixed (Dirichlet-like boundary conditions (BC)), integrate over (free BC), or identify and integrate over (periodic BC). We write, using Dirichlet BCs

$$Z(g, \{U'_\ell\}, \{U_\ell\}) = \langle \{U'_\ell\} |\, \hat{T}^t \,| \{U_\ell\} \rangle, \qquad \hat{T} = \mathrm{e}^{-a_t \hat{H}} \tag{12}$$

where $\hat{T}$ is the transfer matrix, giving us the transition amplitude between link configurations at neighboring time slices. However in 2d we can restrict $\hat{T}$ to be

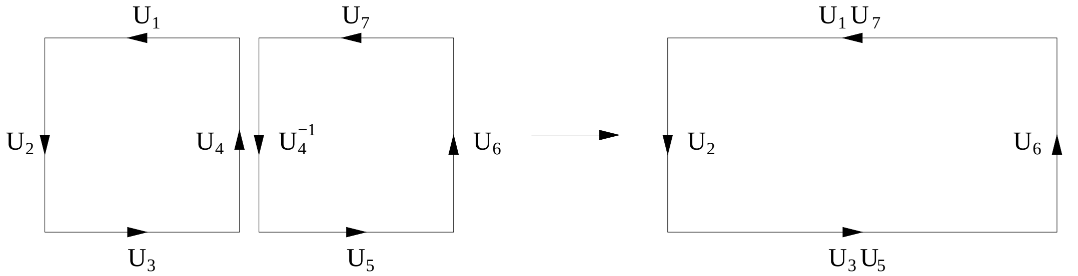

Figure 1: Integrating out the link U_4 using the heat kernel action. The graphic notation is such one has cyclic matrix multiplication on loops and if an arrow is reversed (oriented link $\ell \to -\ell$) then $U_{-\ell} = U_\ell^{-1}$.

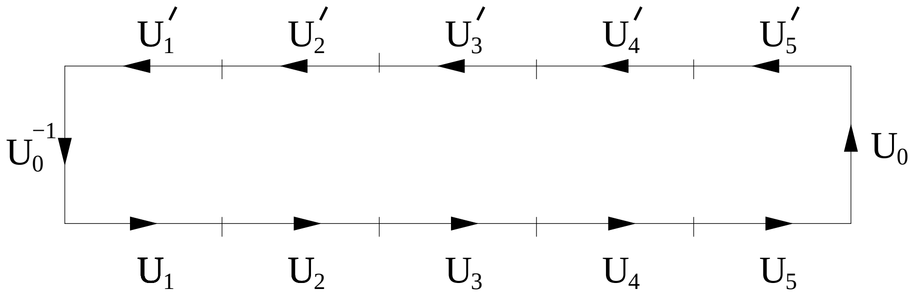

Figure 2: Integrating out the temporal links in a time-slab, except a last link U_0. The temporal links U_0^{-1} and U_0 are identified on the cylinder, and the result is $Z_P[U_P]$, $U_P = U_0(U_5U_4U_3U_2U_1)U_0^{-1}(U_1'U_2'U_3'U_4'U_5')$ using the heat kernel action.

an operator only acting on the holonomies since we can use (10) to integrate out the temporal links U_ℓ which connect two time slices. We obtain

$$\langle U'|\hat{T}|U\rangle = \int dU_0 \; \langle U'U_0U^{-1}U_0^{-1}|\, \mathrm{e}^{-la_sa_t\frac{1}{2}g^2\Delta_G}|e\rangle, \tag{13}$$

where we have not integrated over the last temporal link U_0 and U is the holonomy for the links at the time slice t', say, and U' the holonomy for the links at the neighbor time slice $t'+1$ (see Fig. 2).

Using $\langle U'U^{-1}|\mathrm{e}^{-\Delta_G}|e\rangle = \langle U'|\mathrm{e}^{-\Delta_G}|U\rangle$ we can write the transfer matrix elements as

$$\langle U'|\, \mathrm{e}^{-a_t(la_s\frac{1}{2}g^2\Delta_G)}\hat{P}|U\rangle = \langle U'|\hat{P}\; \mathrm{e}^{-a_t(la_s\frac{1}{2}g^2\Delta_G)}\hat{P}|U\rangle, \tag{14}$$

where the projection operator $\hat{P}$ is defined by

$$\hat{P}|U\rangle = \int dG\,|GUG^{-1}\rangle. \tag{15}$$

$\hat{P}$ commutes with Δ_G, a fact which allows us to write the right hand side of eq. (14) and thus ensures that we can restrict the action of the transfer matrix even further, namely to the class functions on G. To make this explicit consider an arbitrary state

$$|\Psi\rangle = \int dU\, |U\rangle \Psi(U), \qquad \Psi(U) = \langle U|\Psi\rangle, \tag{16}$$

i.e. $(\hat{P}\Psi)(U) = \int dG \Psi(G^{-1}UG)$ which is clearly a class function.

Denote the length of the lattice $L = a_s l$. From (12) and (14) it follows that

$$\hat{H} = \frac{1}{2} g^2 L \Delta_G. \tag{17}$$

The expression agrees with the continuum expression. We have reviewed how the lattice theory, even if no gauge fixing is imposed, nevertheless makes it possible and natural to restrict the transfer matrix and the corresponding Hamiltonian to class functions of the holonomies. Finally, it is of course only for the heat kernel action that one derives an $\hat{H}$ formally identical to the continuum Hamiltonian even before the lattice spacings are taken to zero. The above arguments could be repeated for any reasonable action, e.g. the Wilson action mentioned above, and in the limit where $a_s, a_t \to 0$ one would obtain (17). Finally, the derivation can be repeated also for Abelian groups or discrete groups like Z_N groups, resulting in an expression like (17) with an appropriate group Laplacian Δ_G, in the Abelian case without the issue of reduction of domain of Δ_G.

3 Coupling to geometry

The covariant version of the Yang-Mills theory (1) is

$$S_{YM} = \frac{1}{4} \int d^2x \sqrt{g(x)}\; F^a_{\mu\nu} (F^{\mu\nu})^a. \tag{18}$$

We want a path integral formulation which includes also the integration over geometries. Here the CDT formulation is natural: one is summing over geometries which have cylindrical geometry and a time foliation, each geometry being defined by a triangulation and the sum over geometries in the path integral being performed by summing over all triangulations with topology of the cylinder and a time foliation. The coupling of gauge fields to a geometry via dynamical triangulations (where the length of a link is a) is well known [14]: One uses as plaquettes the triangles. Thus the 2d partition function becomes

$$Z(\Lambda, g, l', l, \{U'_\ell\}, \{U_\ell\}) = \sum_{\mathcal{T}} \mathrm{e}^{-\frac{1}{2} N_{\mathcal{T}} \Lambda \frac{\sqrt{3}}{4} a^2} Z^G_{\mathcal{T}}(\beta), \tag{19}$$

where the summation is over CDT triangulations $\mathcal{T}$, with an "entrance" boundary consisting of l links and an "exit" boundary consisting of l' links, Λ is the lattice cosmological constant, $N_{\mathcal{T}}$ the number of triangles in $\mathcal{T}$, and the gauge partition function for a given triangulation $\mathcal{T}$ is defined as

$$Z^G_{\mathcal{T}}(g,\{U'_\ell\},\{U_\ell\}) = \int \prod_\ell dU_\ell \prod_P Z_P[U_P]. \tag{20}$$

The integration is over links and $\prod_P$ is the product over plaquettes (here triangles) in $\mathcal{T}$. For the plaquette action defining $Z_P[U_P]$ we have again many choices, and for convenience we will use the heat kernel action (9).

We can introduce a transfer matrix $\hat{T}$, which connects geometry and fields at time label t' to geometry and fields at time label $t'+1$, and if the (discretized) universe has $t+1$ time labels we can write

$$Z(\Lambda, g, l', l, \{U'_\ell\},\{U_\ell\}) = \langle \{U'_\ell\}, l'|T^t|\{U_\ell\}, l\rangle, \qquad T = \mathrm{e}^{-a\hat{H}}. \tag{21}$$

The one-dimensional geometry at t' is characterized by the number l of links (each of length a), and on these links we have field configurations $\{U_\ell\}$. Similarly the geometry at $t'+1$ has l' links and field configurations $\{U'_\ell\}$. For fixed l and l' the number of plaquettes (triangles) in the spacetime cylinder "slab" between t' and $t'+1$ is $l+l'$ and the number of temporal links $l+l'$. There is a number of possible triangulations of the slab for fixed l and l', namely

$$N(l',l) = \frac{1}{l+l'}\binom{l+l'}{l}. \tag{22}$$

For each of these triangulations we can integrate over the $l+l'$ temporal link variables U_0, as we did for a fixed lattice and we obtain as in that case

$$\langle U'|\hat{P}\,\mathrm{e}^{-a(a(l+l')\frac{\sqrt{3}}{8}g^2\Delta_G)}\hat{P}|U\rangle, \tag{23}$$

where U' and U are the holonomies corresponding to $\{U'_\ell\}$ and $\{U_\ell\}$, respectively, and $\hat{P}$ is the projection operator (15) to class functions coming from the last integration over a temporal link U_0. The factor $\sqrt{3}/8$ rather than the factor $1/2$ appears because we are using equilateral triangles rather than squares as in Sec. 2. In order to have unified formulas we make a redefinition $g^2\sqrt{3}/4 \to g^2$ and thus we have the matrix element

$$\langle U'|\hat{P}\,\mathrm{e}^{-a(a(l+l')\frac{1}{2}g^2\Delta_G}\hat{P}|U\rangle, \tag{24}$$

If we did not have the matter fields the transfer matrix would be

$$\langle l'|\hat{T}_{\mathrm{geometry}}|l\rangle = N(l',l)\mathrm{e}^{-a((l+l')a\frac{1}{2}\Lambda)}, \tag{25}$$

where we have made a redefinition $\Lambda\sqrt{3}/4 \to \Lambda$, similar to the one made for g^2, in order to be in accordance with notations in other articles. The limit where $a \to 0$ and $L' = a\,l'$ and $L = a\,l$ are kept fixed has been studied [15] and one finds

$$\hat{T}_{\rm geometry} = \mathrm{e}^{-a(\hat{H}_{\rm cdt}+O(a))}, \qquad \hat{H}_{\rm cdt} = -\frac{d^2}{dL^2}L + \Lambda L. \tag{26}$$

From the definition (21) of $\hat{H}$ and (24) it follows that

$$\hat{H} = \hat{H}_{\rm cdt} + \frac{1}{2}g^2 L \Delta_G, \tag{27}$$

acting on the Hilbert space which is the tensor product of the Hilbert space of square integrable class functions on G and the Hilbert space of the square integrable functions on R_+ with measure $d\mu(L) = LdL$.

We have obtained the Hamiltonian (27) using the path integral, starting out with a lattice regularization. Alternatively one can use that the classical 2d YM action (1) on the (flat) cylinder can be formulated in terms of the holonomies $U(t)$ defined in eq. (4) (see [16] for details):

$$S_{YM} = \frac{1}{2}\int dxdt \;\mathrm{tr}\, E_1^2 = \frac{1}{2g^2 L}\int dt \;\mathrm{tr}\,(iU^{-1}\partial_0 U)^2. \tag{28}$$

Let us now couple the YM theory to geometry as in (18). One observes that $\tilde{E} = E^1/\sqrt{g} = E_1\sqrt{g}$ behaves as a scalar under diffeomorphisms. Thus $D_1\tilde{E} = 0$, where D_1 is the ordinary gauge covariant derivative as in (3). This implies that the derivation in [16] which led to (28) for flat spacetime is essentially unchanged. As we are interested in HL projectable 2d quantum geometries we assume the geometry is defined by a laps function $N(t)$, a shift function $N_1(x,t)$ and a spatial metric $\gamma(x,t)$. $\sqrt{g(x,t)} = N(t)\sqrt{\gamma(x,t)}$, and introducing the spatial length $L(t) = \int dx \sqrt{\gamma(x,t)}$ one obtains instead of (28)

$$S_{YM} = \frac{1}{2}\int dtdx\; \sqrt{g(x,t)}\,\mathrm{tr}\,\tilde{E}^2 = \frac{1}{2g^2}\int dt\; \frac{\mathrm{tr}\,(iU^{-1}\partial_0 U)^2}{N(t)L(t)}. \tag{29}$$

Combined with the results from [3] for the HL-action one can write the total action as

$$S_{TOT} = \int dt\; \left[\frac{1}{2N(t)L(t)}\left(\frac{1}{2}(\partial_0 L)^2 + \frac{1}{g^2}\mathrm{tr}\,(iU^{-1}\partial_0 U)^2\right) + \Lambda N(t)L(t)\right]. \tag{30}$$

This classical action leads to the quantum Hamiltonian (27).

Let us return to the quantum Hamiltonian (27). Since the eigenfunctions of Δ_G after projection with $\hat{P}$ are just the characters $\chi_R(U)$ on G and they have eigenvalues $C_2(R)$, we can solve the eigenvalue equation for $\hat{H}$ by writing $\Psi(L,U) = \psi_R(L)\chi_R(U)$. For $\hat{H}_{\rm cdt}$ we have [15, 8]

$$\hat{H}_{\rm cdt}\psi_n(L,\Lambda) = \varepsilon_n \psi_n(L,\Lambda), \qquad \varepsilon_n = 2n\sqrt{\Lambda}, \quad n > 0, \tag{31}$$

where the eigenfunctions are of the form $\Lambda\, p_n(L\sqrt{\Lambda})\mathrm{e}^{-\sqrt{\Lambda}L}$, $p_n(x)$ being a polynomial of degree $n-1$. The corresponding solution for $\psi_R(L)$ is obtained by the substitution

$$\Lambda \to \Lambda_R = \Lambda + \frac{1}{2} g^2 C_2(R), \tag{32}$$

i.e.

$$\hat{H}\Psi_{n,R} = E(n,R)\Psi_{n,R}, \quad E(n,R) = 2n\sqrt{\Lambda_R}, \quad n > 0 \tag{33}$$

$$\Psi_{n,R}(L,U) = \Lambda_R\, p_n(L\sqrt{\Lambda_R})\, \mathrm{e}^{-L\sqrt{\Lambda_R}}\, \chi_R(U), \tag{34}$$

with the reservation that the correct variable is not really the group variable U but rather the conjugacy class corresponding to U. In the simplest case of $SU(2)$ the group manifold can be identified with S^3 and Δ_G is the Laplace-Beltrami operator on S^3. The conjugacy classes are labeled by the geodesic distance θ to the north pole and the representations are labeled by $R = j$ and we have[3]

$$C_j = j(j+1), \qquad \chi_j(\theta) = \frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \quad j = 0, \frac{1}{2}, 1, \ldots \tag{35}$$

As already mentioned the above results are also valid in simpler cases. If $G = U(1)$ where one has

$$U(\theta) = \mathrm{e}^{i\theta}, \qquad \Delta_G = -\frac{d^2}{d\theta^2}, \tag{36}$$

$$C_n = n^2, \qquad \chi_n(\theta) = \mathrm{e}^{in\theta}, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{37}$$

and if $G = Z_N$, the discrete cyclic group of order N,

$$U(k) = \mathrm{e}^{\frac{2\pi}{N}k}, \qquad (\Delta_G)_{k,k'} = \delta_{k,k'+1} + \delta_{k,k'-1} - 2\delta_{k,k'}, \quad k = 0, \ldots, N-1, \tag{38}$$

$$C_n = 2\left(1 - \cos\left(\frac{2\pi}{N}n\right)\right) \qquad \chi_n(k) = \mathrm{e}^{i\frac{2\pi n}{N}k}, \quad n = 0, 1, \ldots, N-1. \tag{39}$$

[3] Using the lattice we have effectively performed a quantization using the fact that $SU(2)$ is a compact group. However, as already mentioned in footnote 2, there are subtleties associated with the quantization, more precisely whether one chooses first to project to the algebra and quantized there, or first to quantized using the group variables and then project to the holonomies. We refer to [13] for a detailed discussion.

4 The ground state of the universe

In CDT the disk amplitude is defined as

$$W_\Lambda(L) = \int_0^\infty dt \; \langle L|\, \mathrm{e}^{-t\hat{H}_{\mathrm{cdt}}} |L' \to 0\rangle. \tag{40}$$

It is a version of the Hartle-Hawking wave function. One can calculate $W_\Lambda(L)$ [1]:

$$W_\Lambda(L) = \frac{\mathrm{e}^{-\sqrt{\Lambda}L}}{L}. \tag{41}$$

This function satisfy

$$\hat{H}_{\mathrm{cdt}} W_\Lambda(L) = 0, \tag{42}$$

and one can view (42) as the Wheeler-deWitt equation. Formally $W_\Lambda(L) \propto \psi_0(L)$ in the notation used in eq. (31), but it was not included as an eigenfunction in the listing in (31) since it does not belong to the Hilbert space $L^2(R_+)$ with measure LdL.

If we couple the theory of fluctuating geometries to gauge fields as above we have to decide what kind of boundary condition to impose in the limit $L' \to 0$ in (40). A possible interpretation of this "singularity" in the discrete setting is that all the vertices of the first time slice at time $t' = 1$ have additional temporal links joining a single vertex at time $t' = 0$ (see Fig. 3). We can view this as an explicit, discretized, realization of the matter part of the Hartle-Hawking boundary condition.

Denote by $\{U^{(0)}_\ell\}$, $\ell = 1, \ldots, l$ the gauge fields on these temporal links and by $\{U_\ell\}$, $\ell = 1, \ldots, l$ the gauge fields on the spatial links constituting the first loop at time $t' = 1$ and denote by $U(1)$ the corresponding holonomy at time $t' = 1$. The contribution to the matter partition function coming from this first "big bang" part of the universe is then

$$\int \prod_{k=1}^{l} dU^{(0)}_k \prod_{k'=1}^{l} Z_{P_{k'}}[U_{P_{k'}}] = Z_{\mathrm{disk}}[U(1)] = \langle U(1)|\, \mathrm{e}^{-\frac{1}{2}g^2 l a^2 \Delta_G} |I\rangle, \tag{43}$$

where we have integrated out the temporal links $\{U^{(0)}_\ell\}$. The matter partition function can now be written (after integrating out the temporal links in the rest of the lattice too, as the integral over t holonomies $U(1), U(2), \ldots, U(t)$

$$\int \prod_{i=1}^{t} \Big(dU(i) \langle U(i)|\, \mathrm{e}^{-\frac{1}{2}g^2 (l_i + l_{i-1}) a^2 \Delta_G} |U(i-1)\rangle \Big), \tag{44}$$

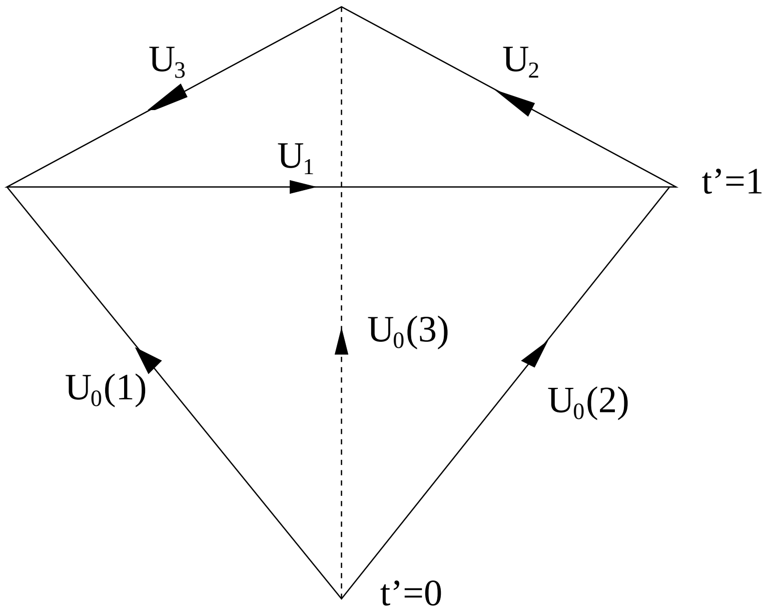

Figure 3: The "beginning of the universe" at $t' = 0$ and the connection to the first loop at $t' = 1$.

where $U(0) \equiv I$ and $l_0 = 0$. From this expression it is natural to say that the universe starts out in the matter state $|I\rangle$, or expanded in charaters:

$$\langle U|I\rangle = \delta(U - I) = \sum_R d_R \chi_R(U). \tag{45}$$

This wave function is not normalizable if the group has infinitely many representations, but neither is $W_\Lambda(L)$ as we just saw. Combining the two we might define the Hartle-Hawking wavefunction for 2d CDT coupled to gauge fields as

$$W(L, U) = \int_0^\infty dT \, \langle L, U| \, \mathrm{e}^{-T\hat{H}} |L = 0, U = I\rangle = \sum_R d_R \chi_R(U) \, W_{\Lambda_R}(L), \tag{46}$$

where Λ_R is defined in eq. (32). We have explicitly:

$$W(L, k) = \sum_r \mathrm{e}^{\frac{i2\pi rk}{N}} \, \frac{\exp\Big(-L(\sqrt{\Lambda + g^2[1 - \cos(2\pi r/n)]})\Big)}{L}, \tag{47}$$

for the Z_N theory,

$$W(L, \theta) = \sum_{r=-\infty}^{\infty} \mathrm{e}^{ir\theta} \, \frac{\exp\Big(-L\sqrt{\Lambda + \frac{1}{2}r^2 g^2}\Big)}{L}. \tag{48}$$

for the $U(1)$ theory, and

$$W(L,\theta) = \sum_{k=0}^{\infty} \frac{\sin\left(\frac{(k+1)\theta}{2}\right)}{\sin\frac{\theta}{2}} \; \frac{\exp\left(-L\sqrt{\Lambda + \frac{1}{8}g^2 k(k+2)}\right)}{L}. \tag{49}$$

for the $SU(2)$ theory.

We have tried to define the initial matter state $|I\rangle$ in the Hartle-Hawking spirit as coming from "no boundary" conditions by closing the universe into a disk. Even if the "initial" (Big Bang) state is then a simple tensor product $|L = 0\rangle \otimes |I\rangle$, the corresponding Hartle-Hawking wave function is the result of a non-trivial interaction between matter and geometry. However, we cannot claim that the model points to such a "no boundary" condition in a really *compelling* way. From a continuum point of view it should not make a difference if we, rather than implementing the continuum statement $L' \to 0$ by adding a little cap, had implemented it by insisting that the first time slice had $l = 2$ or $l = 3$, say. The calculation of $W_\Lambda(L)$ is insensitive to such details. However, if our universe really started with such a microscopic loop, there is no reason that we should not choose the matter ground state, i.e. the trivial, constant, character as the initial state. In this case absolutely nothing happens with matter during the time evolution of the universe. It just stays in this state and the state does not influence the geometry. Clearly the state $|I\rangle$ is much more interesting and more in accordance with the picture we have of the Big Bang of the real 4d world where matter and geometry have interacted. Even if the argument for the state $|I\rangle$ are not compelling, as just mentioned, it is nevertheless encouraging that the "natural" Hartle-Hawking like boundary condition leads to a non-trivial interaction between geometry and matter.

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Chapter 5

Dynamical Triangulations

The formalism of dynamical triangulation (DT) provides a regularization of the continuum theory[29, 23, 76]

$$\mathcal{Z} = \int_{\mathcal{M}} \mathcal{D}_g[g]\, e^{-\mu_0 \int d^2\xi \sqrt{g}} \int \mathcal{D}_g[X]\, e^{-S_m[X;g]}, \tag{5.1}$$

known as the Polyakov path integral. The difference between this and the CDT path integral (2.1) is that the theory (5.1) is completely Euclidean. By this we mean both that the manifolds we intend to integrate over are Euclidean (i.e. the metric $g_{\mu\nu}$ now has signature $(+,+)$), but also that the action is 'Wick rotated'. In Eq. (5.1) we have explicitly included matter fields X coupled to the fluctuating geometry, and we will see that it is actually possible to solve DT with certain types of matter theories.

It does not seem to be possible to Wick rotate the DT path integral back to a Lorentzian theory by a simple analytical continuation. The problem is basically that there is no obvious way to map Euclidean manifolds to Lorentzian manifolds (see e.g. [12, 54]).[1] On the other hand, the DT formalism provides a rich class of statistical mechanics models that can be solved non-perturbatively. In some cases it might actually be easier to solve a given model on a dynamical triangulation, and then extrapolate back to a fixed lattice setting (see e.g. [22]).

In Sec. 5.1 we will attack the continuum integral (5.1) directly. The main outcome will be the KPZ-DDK relations,[62, 28, 33] which determine the scaling properties of certain operators. These scaling dimensions will then be an input to Sec. 5.2 where we introduce some general machinery to solve DT models (orthogonal polynomials). We finally consider in detail DT coupled to the critical Ising model and to the Yang-Lee edge singularity in Secs. 5.3 and 5.4.

[1] There is a map, which uses the geodesic distance on the Euclidean manifold to introduce a preferred Lorentzian time. The resulting Lorentzian geometry has singularities where the lightcone degenerates and the universe splits up into disconnected branches. See e.g. [12, 2]

5.1 The Polyakov path integral

As a first step towards understanding Eq. (5.1), let us consider the partition function of the matter theory

$$\int \mathcal{D}_g[X]\, e^{-S_m[X;g]}. \tag{5.2}$$

In general, the problem of defining interacting QFTs on curved spacetimes is still a topic of research (see e.g. [54] for a recent review). However, there is a special class of QFTs where the situation is much better understood, namely when the QFT is a conformal field theory (CFT). A defining feature is that they transform 'covariantly' under Weyl rescalings of the metric, i.e. transformations of the form

$$g_{\mu\nu} \to e^{\psi} g_{\mu\nu}, \tag{5.3}$$

where ψ is some real function on the manifold.

For any two dimensional Riemannian manifold with spherical topology, one can choose coordinates such that the metric is proportional to the flat metric,

$$g_{\mu\nu} = e^{\psi}\delta_{\mu\nu}. \tag{5.4}$$

A metric of this form is said to be conformally flat. It then follows that in order to understand a QFT on a general curved surface, it is sufficient to understand the behavior under Weyl transformations. For CFTs, the partition function is given by

$$\int \mathcal{D}_g X\, e^{-S_m[X;g]} = e^{\frac{c_m}{48\pi}\int \mathrm{d}^2\xi \frac{1}{2}\partial_\mu\psi\,\partial_\mu\psi - \mu_m \int \mathrm{d}^2\xi\, e^{\psi}}, \tag{5.5}$$

where c_m is the central charge. The induced cosmological constant μ_m is formally divergent (note that there is no dimensionful parameter in the CFT that could set the scale of μ_m), and it will be absorbed into the bare μ_0 of (5.1). For this reason we will pretend that $\mu_m = 0$ in the following.

It is not obvious that the RHS of (5.5) is well-defined as a function of $g_{\mu\nu}$, since we had to choose a particular coordinate system in order to define ψ. However, we can express the exponent in the coordinate independent form

$$\int \mathrm{d}^2\xi\,\partial_\mu\psi\,\partial_\mu\psi = -\int \mathrm{d}^2\xi\,\sqrt{g(\xi)}\mathrm{d}^2\xi'\,\sqrt{g(\xi')}R(\xi)K(\xi,\xi')R(\xi'), \tag{5.6}$$

where R is the Ricci scalar corresponding to $g_{\mu\nu}$, and $K(\xi,\xi')$ is the Green's function defined by

$$\nabla^2_\xi K(\xi,\xi') = \frac{\delta(\xi-\xi')}{\sqrt{g}}. \tag{5.7}$$

To check (5.6) one needs to use that

$$R = -e^{-\psi}\partial_\mu\partial_\mu\psi, \qquad \nabla^2 = -e^{-\psi}\partial_\mu\partial_\mu \tag{5.8}$$

when the metric is given by (5.4).

In the following it will be very useful to write the metric as a Weyl rescaling of some arbitrary reference metric $\hat{g}$. We thus set

$$g_{\mu\nu} = e^{\phi}\hat{g}_{\mu\nu}, \tag{5.9}$$

with

$$\hat{g}_{\mu\nu} = e^{\hat{\phi}}\delta_{\mu\nu}. \tag{5.10}$$

The reason this is helpful is that all physical results must be independent of the choice of $\hat{g}$, and this independence will provide a crucial consistency condition.

From (5.5) it now follows that

$$\int \mathcal{D}_g X\, e^{-S_m[X;g]} = e^{\frac{c_m}{48\pi}S_{L,I}[\phi;\hat{g}]} \int \mathcal{D}_{\hat{g}} X\, e^{-S_m[X;\hat{g}]}, \tag{5.11}$$

where the so-called induced Liouville action $S_{L,I}[\phi;\hat{g}]$ is given by

$$S_{L,I}[\phi;\hat{g}] := \int \mathrm{d}^2\xi\, \frac{1}{2}(\partial_\mu(\phi+\hat{\phi})\partial_\mu(\phi+\hat{\phi}) - \partial_\mu\hat{\phi}\partial_\mu\hat{\phi}). \tag{5.12}$$

We can write this in a more conventional form

$$S_{L,I}[\phi;\hat{g}] = \int \mathrm{d}^2\xi\, \sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \hat{R}\phi\right), \tag{5.13}$$

by using that the Ricci scalar corresponding to $\hat{g}_{\mu\nu}$ is given

$$\hat{R} = -e^{-\hat{\phi}}\partial_\mu\partial_\mu\hat{\phi}. \tag{5.14}$$

Above we took $\hat{g}_{\mu\nu}$ to be conformally flat, but clearly Eq. (5.13) is diffeomorphism invariant if ϕ is defined to be a scalar.

Since the entire metric dependence of the matter partition function is given by the Liouville action, we can factorize our Polyakov path integral as

$$\mathcal{Z} = \left(\int \mathcal{D}_g[g]\, e^{\frac{c_m}{48\pi}S_{L,I}[\phi;\hat{g}] - \mu\int d^2\xi\sqrt{\hat{g}}e^{\phi}}\right)\left(\int \mathcal{D}_{\hat{g}}X\, e^{-S_m[X;\hat{g}]}\right). \tag{5.15}$$

The remaining problem is now to interpret the integral over the metric $g_{\mu\nu}$. It seems like a good idea to re-express this as a integral over ϕ, since $S_{L,I}[\phi;\hat{g}]$ already looks like an action for ϕ. We thus need to perform a change of variables from $g_{\mu\nu}$ to φ. In order to calculate the Jacobian of transformation, we should first construct a norm on infinitesimal variations of the metric (i.e. on the tangent space of metrics). A natural choice is

$$\|\delta g\|^2 := \int \mathrm{d}^2\xi\, \sqrt{g}(g^{\mu\nu}g^{\rho\sigma} + C g^{\mu\rho}g^{\nu\sigma})\delta g_{\mu\rho}\delta g_{\nu\sigma}, \tag{5.16}$$

where C is some constant. This is the most general thing you can write down which is diffeomorphism invariant, ultra local (i.e. it does not involve derivatives) and quadratic in the variation $\delta g_{\mu\nu}$.

To make the change of variables, we parametrize the neighborhood of the metric $g_{\mu\nu} = e^{\phi}\hat{g}_{\mu\nu}$ in terms of (infinitesimal) Weyl transformations and diffeomorphisms. Under an infinitesimal change of coordinates

$$\xi^\mu \to \tilde{\xi}^\mu, \tag{5.17}$$

with

$$\xi^\mu = \tilde{\xi}^\mu + v^\mu(\xi), \tag{5.18}$$

the metric transforms as (to first order in v^μ)

$$\tilde{g}_{\mu\nu}(\tilde{\xi}) = \frac{\partial \xi^\rho}{\partial \tilde{\xi}^\mu}\frac{\partial \xi^\sigma}{\partial \tilde{\xi}^\nu} g_{\rho\sigma}(\xi) \tag{5.19}$$

$$= g_{\mu\nu}(\xi) + g_{\nu\lambda}(\xi)\partial_\mu v^\lambda + g_{\mu\lambda}(\xi)\partial_\nu v^\lambda \tag{5.20}$$

$$= g_{\mu\nu}(\tilde{\xi}) + v^\lambda \partial_\lambda g_{\mu\nu}(\tilde{\xi}) + g_{\nu\lambda}(\tilde{\xi})\partial_\mu v^\lambda + g_{\mu\lambda}(\tilde{\xi})\partial_\nu v^\lambda. \tag{5.21}$$

In the second line we have evaluated the old metric at the new coordinate $\tilde{\xi}$, that is

$$g_{\mu\nu}(\tilde{\xi}) := g_{\mu\nu}(\xi = \tilde{\xi}). \tag{5.22}$$

We thus find that the change in the metric is

$$\delta g_{\mu\nu} = v^\lambda \nabla_\lambda g_{\mu\nu} + g_{\nu\lambda}\nabla_\mu v^\lambda + g_{\mu\lambda}\nabla_\nu v^\lambda = \nabla_\mu v_\nu + \nabla_\nu v_\mu, \tag{5.23}$$

where we use that the Levi-Civita connection is torsion free to change the partial derivatives to covariant derivatives (it is easy to check that the Christoffel symbols cancel), and use that the metric is covariantly constant to drop the first term.

Under a combined diffeomorphism and Weyl transformation $\phi \to \phi + \delta\phi$, the change of the metric is

$$\delta g_{\mu\nu} = \delta\phi g_{\mu\nu} + \nabla_\mu v_\nu + \nabla_\nu v_\mu = (\delta\phi + \nabla_\lambda v^\lambda) g_{\mu\nu} + (P_1 v)_{\mu\nu}, \tag{5.24}$$

where P_1 is defined as

$$(P_1 v)_{\mu\nu} := \nabla_\mu v_\nu + \nabla_\nu v_\mu - g_{\mu\nu}\nabla_\lambda v^\lambda. \tag{5.25}$$

We see that P_1 is a linear operator sending vectors to anti-symmetric tensors. Inserting (5.24) into (5.16) we find that the norm of the variation is

$$\|\delta g_{\mu\nu}\|^2 = (2+4C)\int \mathrm{d}^2\xi\, \sqrt{g}(\delta\phi + \nabla_\lambda v^\lambda)^2 + \int \mathrm{d}^2\xi\, \sqrt{g}(P_1 v)_{\mu\nu}(P_1 v)^{\mu\nu} \tag{5.26}$$

$$= (2+4C)\int \mathrm{d}^2\xi\, \sqrt{g}(\delta\phi + \nabla_\lambda v^\lambda)^2 + \int \mathrm{d}^2\xi\, \sqrt{g} v_\mu (P_1^T P_1 v)^\mu \tag{5.27}$$

Here the transpose P_1^T is defined by the second equality. From this expression we can directly read off that the Jacobian is $\sqrt{\det(P_1^T P_1)}$, i.e. that

$$\int \mathcal{D}_g[g] = \int \mathcal{D}_g[\phi]\mathcal{D}_g[v]\, \sqrt{\det(P_1^T P_1)}, \tag{5.28}$$

where the measures on ϕ and v are induced by the usual norms

$$\|\delta\phi\|^2 = \int \mathrm{d}^2\xi\, \sqrt{g}\,(\delta\phi)^2, \qquad \|v\|^2 = \int \mathrm{d}^2\xi\, \sqrt{g}\, v_\mu v^\mu. \tag{5.29}$$

Note that the linear operator $P_1^T P_1$ maps vector fields to vector fields, so the determinant makes sense, at least formally. A subtlety that we will not discuss here is that $P_1^T P_1$ has zero modes that have to be excluded when computing the determinant (see e.g. [1, 33, 51]).

The integral over v^μ is easy to take care of. First of all, nothing in the rest of the path integral can depend on v^μ, since that would mean that diffeomorphism invariance was not respected. In Eq. (5.29) it looks like the norm of v^μ depends on ϕ, but this not the case. In fact, we have

$$\|v\|^2 = \int \mathrm{d}^2\xi \, \sqrt{g} \, g_{\mu\nu} v^\mu v^\mu = \int \mathrm{d}^2\xi \, \sqrt{\hat{g}} \, \hat{g}_{\mu\nu} v^\mu v^\mu. \tag{5.30}$$

We conclude that

$$\int \mathcal{D}_g[v] = \int \mathcal{D}_{\hat{g}}[v] \tag{5.31}$$

is just some (ill-defined) overall constant, which we will simply drop in the following.

As usual, one can express the Jacobian of (5.28) as a path integral over ghosts. We claim that

$$\sqrt{\det(P_1^T P_1)} = \int \mathcal{D}_g[b]\mathcal{D}_g[c] \, e^{-S_{gh}[b,c;g]}, \tag{5.32}$$

where the ghost action

$$S_{gh}[b,c;g] := \frac{1}{2\pi} \int \mathrm{d}^2\xi \, b_{\mu\nu} (P_1 c)^{\mu\nu}. \tag{5.33}$$

The field $b^{\mu\nu}$ is an anti-symmetric tensor, while c^μ is a vector. Both are fermionic, so they violate the spin-statistics theorem.

To verify[2] (5.32) let us introduce an eigenbasis, $C_{j\mu}$, of (bosonic) vector fields such that

$$(P_1^T P_1 C_j)_\mu = \lambda_j^2 C_{j\mu}, \tag{5.34}$$

and

$$\int \mathrm{d}^2\xi \, \sqrt{g} C_{j\mu} C_k^\mu = \delta_{jk}. \tag{5.35}$$

We can now expand the field c^μ as

$$c_\mu = \sum_j c_j C_{j\mu}, \tag{5.36}$$

where c_j are Grassmann variables. For the measure we simply find

$$\mathcal{D}_g[c] = \prod_j \mathrm{d}c_j \tag{5.37}$$

because of the orthonormality (5.35). Similarly, we note that $\lambda_j^{-1}(P_1 C_j)_{\mu\nu}$ are orthonormal anti-symmetric tensors,

$$\int \mathrm{d}^2\xi \, \sqrt{g} \lambda_j^{-1} (P_1 C_j)_{\mu\nu} \lambda_k^{-1} (P_1 C_k)^{\mu\nu} = \int \mathrm{d}^2\xi \, \sqrt{g} \lambda_j^{-1} \lambda_k^{-1} C_{j\mu} (P_1^T P_1 C_k)^\mu = \delta_{jk}, \tag{5.38}$$

so we can expand $b^{\mu\nu}$ as

$$b_{\mu\nu} = \sum_j b_j \lambda_j^{-1} (P_1 C_j)_{\mu\nu}, \tag{5.39}$$

[2] Again, in these manipulations we ignore the issue of zero-modes.

with measure

$$\mathcal{D}_g[b] = \prod_j \mathrm{d}b_j. \tag{5.40}$$

Inserting (5.36) and (5.39) into the ghost action we find

$$S_{gh}[b, c; g] = \frac{1}{2\pi} \int \mathrm{d}^2\xi \, c_\mu (P_1^T b)^\mu = \frac{1}{2\pi} \sum_j \lambda_j b_j c_j. \tag{5.41}$$

It follows that the ghost partition function is simply

$$\int \mathcal{D}_g[b]\mathcal{D}_g[c] \, e^{-S_{gh}[b,c;g]} = \int \prod_j \left(\mathrm{d}b_j \mathrm{d}c_j \, e^{-\frac{1}{2\pi}\lambda_j b_j c_j} \right) = \prod_j \lambda_j. \tag{5.42}$$

On the other hand, from (5.34), it follows that the eigenvalues of $P_1^T P_1$ are λ_j^2, hence

$$\sqrt{\det(P_1^T P_1)} = \prod_j \lambda_j. \tag{5.43}$$

We conclude that the path integral over the ghost fields generates the correct Jacobian.

The ghost theory described by (5.33) can be quantized as a conformal field theory (it is free) with central charge $c_{gh} = -13$, hence

$$\int \mathcal{D}_g[b]\mathcal{D}_g[c] \, e^{-S_{gh}[b,c;g]} = e^{-\frac{13}{48\pi} S_{L,I}} \int \mathcal{D}_{\hat{g}}[b]\mathcal{D}_{\hat{g}}[c] \, e^{-S_{gh}[b,c;\hat{g}]}. \tag{5.44}$$

The full partition function thus takes the form

$$\mathcal{Z} = \left(\int \mathcal{D}_g[\phi] \, e^{\frac{c_m - 13}{48\pi} S_{L,I}[\phi;g] - \mu \int d^2\xi \sqrt{g}} \right) \\ \times \left(\int \mathcal{D}_{\hat{g}}[X]\mathcal{D}_{\hat{g}}[b]\mathcal{D}_{\hat{g}}[c] \, e^{-S_m[X;\hat{g}] - S_{gh}[b,c;\hat{g}]} \right). \tag{5.45}$$

We are now faced with the problem that the measure for ϕ (induced by the norm (5.29)) depends on ϕ itself. It is now tempting to postulate that the effect of this measure can be absorbed in the action for ϕ.[28, 33] We thus propose to replace

$$\mathcal{D}_g[\phi] \, e^{\frac{c_m - 13}{48\pi} S_{L,I}[\phi;g]} \to \mathcal{D}_{\hat{g}}[\phi] \, e^{-S_L[\phi;\hat{g}]}, \tag{5.46}$$

where $S_L[\varphi; \hat{g}]$ is to be determined. We will not attempt a direct calculation of $S_L[\phi; \hat{g}]$ (however, see [65, 32, 31]), but instead use consistency arguments.

After the replacement (5.46) the partition function looks like

$$\mathcal{Z} = \int \mathcal{D}_{\hat{g}}[\varphi, X, b, c] \, e^{-S_L[\varphi;\hat{g}] - S_m[X;\hat{g}] - S_{gh}[b,c;\hat{g}]}. \tag{5.47}$$

Since $\hat{g}_{\mu\nu}$ is an unphysical reference metric, we must demand that $\mathcal{Z}$ does not depend on it. We can ensure this by making the field theory described by $S_L[\phi; \hat{g}]$ conformal, with a central charge that exactly cancels the central charge from the matter and ghost systems. We thus set

$$S_L[\varphi; \hat{g}] := \frac{1}{8\pi} \int \mathrm{d}^2\xi \, \sqrt{\hat{g}} (\hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - Q \hat{R} \varphi), \tag{5.48}$$

where Q is a parameter, and we have rescaled $\phi \to \varphi$ in order to get the standard kinetic term. We note that S_L is of the same form as the induced action $S_{L,I}$. The action (5.48) is free, and can be quantized as a CFT with central charge $c_\varphi = 1+3Q^2$ (this is the Feigin-Fuchs construction). The consistency condition is then

$$c_{\text{tot}} := c_\varphi + c_m + c_{gh} = 1 + 3Q^2 + c_m - 26 \stackrel{!}{=} 0, \tag{5.49}$$

with solution

$$Q = \frac{1}{\sqrt{3}}\sqrt{25 - c_m}. \tag{5.50}$$

Note that the sign of Q is irrelevant, since it can be absorbed by a change of sign of φ.

In order to introduce the cosmological term, we need to make sense of the area $\int \mathrm{d}^2\xi\,\sqrt{\hat{g}}e^{\phi}$ as an operator on the quantum geometry. Let us first understand how CFT operators transform under Weyl-transformations and diffeomorphisms (see e.g. [80, 40, 75]). On flat space a primary operator $\mathcal{O}$ with dimensions $\Delta_0, \bar{\Delta}_0$ transforms as

$$\mathcal{O}(z) \to \mathcal{O}(z') = \left(\frac{\partial z'}{\partial z}\right)^{-\Delta_0} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{\Delta}_0} \mathcal{O}(z) \tag{5.51}$$

under a conformal transformation $z \to z' = f(z)$. For a spinless operator, i.e. when $\Delta_0 = \bar{\Delta}_0$, we in particular have

$$\mathcal{O}(z') = \left|\frac{\partial z'}{\partial z}\right|^{-2\Delta_0} \mathcal{O}(z). \tag{5.52}$$

The conformal transformation can be understood as a diffeomorphism, accompanied by the usual change of metric

$$\delta_{\mu\nu} \to \left|\frac{\partial z'}{\partial z}\right|^{-2} \delta_{\mu'\nu'}, \tag{5.53}$$

followed by a Weyl-transformation

$$\left|\frac{\partial z'}{\partial z}\right|^{-2} \delta_{\mu'\nu'} \to e^{\sigma} \left|\frac{\partial z'}{\partial z}\right|^{-2} \delta_{\mu'\nu'} = \delta_{\mu'\nu'} \tag{5.54}$$

restoring the flat metric $\delta_{\mu\nu}$. Here the conformal factor is clearly given by

$$e^{\sigma} = \left|\frac{\partial z'}{\partial z}\right|^{2}. \tag{5.55}$$

When defining a field $\mathcal{O}$ on a curved background, one has to specify how the field transforms under Weyl-transformation and diffeomorphisms separately, while (5.52) only tells us how it behaves under a combined transformation. One choice, that will be most convenient for us, is to say that the factor $\left|\frac{\partial z'}{\partial z}\right|^{-2\Delta_0}$ in (5.52) is due to the Weyl transformation (5.54), while $\mathcal{O}$ is defined to be a scalar under diffeomorphisms. For a spinless primary operator on the background geometry $\hat{g}$ we thus demand that

$$\mathcal{O}(\xi) \to e^{-\Delta_0 \sigma}\mathcal{O}(\xi) \tag{5.56}$$

under a general Weyl-transformation

$$\hat{g}_{\mu\nu} \to e^{\sigma}\hat{g}_{\mu\nu}. \tag{5.57}$$

After this digression, we now return to the problem of determining the area operator. For any $\alpha \in \mathbb{R}$ we have a spinless primary field $e^{\alpha\varphi}$ with dimension

$$\Delta_0(e^{\alpha\varphi}) = \bar{\Delta}_0(e^{\alpha\varphi}) = -\frac{1}{2}\alpha(Q+\alpha). \tag{5.58}$$

Following the above discussion we define $e^{\alpha\varphi}$ to be invariant under diffeomorphisms. A natural ansatz for the area operator is then[28, 33]

$$\hat{A} := \int \mathrm{d}^2\xi\sqrt{\hat{g}}\, e^{\alpha\varphi}. \tag{5.59}$$

We determine α in the same way we determined Q, that is, we demand invariance under Weyl-transformations of $\hat{g}$. Using (5.56) and (5.58) we find

$$\int \mathrm{d}^2\xi\sqrt{\hat{g}}\, e^{\alpha\varphi} \to \int \mathrm{d}^2\xi\sqrt{\hat{g}} e^{\sigma(1+\frac{1}{2}\alpha(Q+\alpha))}\, e^{\alpha\varphi} \tag{5.60}$$

under

$$\hat{g}_{\mu\nu} \to e^{\sigma}\hat{g}_{\mu\nu}, \tag{5.61}$$

which leads to the equation

$$1 + \frac{1}{2}\alpha(Q+\alpha) = 0. \tag{5.62}$$

The solutions are

$$\alpha_{\pm} := -\frac{1}{2\sqrt{3}}\left(\sqrt{25-c_m} \mp \sqrt{1-c_m}\right). \tag{5.63}$$

The CFT formally defined by the path integral

$$\int \mathcal{D}_{\hat{g}}[\varphi] e^{-S_L[\varphi;\hat{g}]-\mu\hat{A}}, \tag{5.64}$$

with $\hat{A}$ given by (5.59), is known as Liouville theory. See e.g. Refs. [84, 72] for reviews.

In the next section we will determine which sign gives the physical value of α by comparing with the classical limit. Note that $\alpha_{\pm}$ is only real for $c_m \leq 1$. This indicates that it might be problematic to make sense of the theory beyond $c_m = 1$. In any case we will assume that the matter sector is a minimal CFT, so we will have $c_m < 1$.

5.1.1 KPZ-DDK scaling dimensions

To get something interesting out of the Polyakov integral, we want to construct some operators that we can insert. A natural candidate for a diffeomorphism invariant operator is

$$\int \mathrm{d}^2\xi\sqrt{g}\, \mathcal{O}(\xi), \tag{5.65}$$

where $\mathcal{O}(\xi)$ is some spinless primary operator of the matter theory. To construct the analog of this operator in Liouville theory, we follow the logic used above for $\hat{A}$ and consider operators of the form

$$\textstyle\int \mathcal{O} := \int \mathrm{d}^2\xi \sqrt{\hat{g}} e^{\beta\varphi}\, \mathcal{O}(\xi) \tag{5.66}$$

Requiring this to be invariant under Weyl transformations of $\hat{g}$ we obtain the equation

$$1 - \Delta_0 + \frac{1}{2}\beta(Q+\beta) = 0, \tag{5.67}$$

where Δ_0 is the flat-space scaling dimension of $\mathcal{O}(\xi)$. The solutions are

$$\beta_\pm := -\frac{1}{2\sqrt{3}}\left(\sqrt{25-c_m} \mp \sqrt{1-c_m+24\Delta_0}\right). \tag{5.68}$$

We will determine which solution is the physical one at the end of the section. The objective of this section is to compute the *gravitational dimensions* of the operators of the form (5.66), following Refs. [28, 33].

First we need to define what we mean by the gravitational dimension. We start by introducing the fixed area expectation value,

$$\langle \bullet \rangle_A := \int \mathcal{D}_{\hat{g}}[\varphi, X, b, c]\, e^{-S_0} \delta(\hat{A} - A)\bullet, \tag{5.69}$$

where we abbreviate

$$S_0 := S_L[\varphi; \hat{g}] + S_m[X; \hat{g}] + S_{gh}[b, c; \hat{g}]. \tag{5.70}$$

A special case is the fixed area partition function

$$\mathcal{Z}_A := \langle 1 \rangle_A \tag{5.71}$$

We denote the normalized expectation value by

$$\langle\langle \bullet \rangle\rangle_A := \frac{\langle \bullet \rangle_A}{\mathcal{Z}_A}. \tag{5.72}$$

The gravitational dimension Δ of the operator $\mathcal{O}$ is then defined by

$$\textstyle\langle\langle \int \mathcal{O} \rangle\rangle_{e^\rho A} = e^{(1-\Delta)\rho} \langle\langle \int \mathcal{O} \rangle\rangle_A. \tag{5.73}$$

Note that it might well happen that the expectation value $\langle\langle \int \mathcal{O} \rangle\rangle_A$ vanishes. In that case one could consider $\langle\langle \int \mathcal{O} \int \mathcal{O} \rangle\rangle_A$ instead. This will not make any difference for the scaling argument we sketch bellow, so we will just pretend that the one point function is non-zero.

By definition we have

$$\textstyle\langle \int \mathcal{O} \rangle_{e^\sigma A} = e^{-\sigma} \int \mathcal{D}_{\hat{g}}[\varphi, X, b, c]\, e^{-S_0} \delta(e^{-\sigma}\hat{A} - A) \int \mathcal{O}. \tag{5.74}$$

We can relate this to $\langle \int \mathcal{O} \rangle_A$ by performing the change of variables $\varphi \to \varphi + \sigma/\alpha$. This leaves the measure invariant, and changes the operator as

$$\textstyle\hat{A} \to e^\sigma \hat{A}, \qquad \int \mathcal{O} \to e^{\frac{\beta}{\alpha}\sigma} \int \mathcal{O}. \tag{5.75}$$

The shift of the action is

$$S_L[\varphi, \hat{g}] \to S_L[\varphi, \hat{g}] - \frac{\sigma Q}{8\pi\alpha} \int \mathrm{d}^2\xi \sqrt{\hat{g}}\, \hat{R} = S_L[\varphi, \hat{g}] - \frac{\sigma Q \chi}{2\alpha}, \tag{5.76}$$

where we have used the Gauss-Bonnet Theorem (h is the genus of the surface)

$$\frac{1}{4\pi} \int \mathrm{d}^2\xi \sqrt{\hat{g}}\, \hat{R} = \chi = 2 - 2h. \tag{5.77}$$

Combining these results we obtain

$$\langle \textstyle\int \mathcal{O} \rangle_{e^\sigma A} = e^{(\frac{2\beta + Q\chi}{2\alpha} - 1)\sigma} \langle \textstyle\int \mathcal{O} \rangle_A. \tag{5.78}$$

and similarly we find

$$\mathcal{Z}_{e^\rho A} = e^{(\frac{Q\chi}{2\alpha} - 1)\sigma} \mathcal{Z}_A. \tag{5.79}$$

We conclude that

$$\langle\langle \textstyle\int \mathcal{O} \rangle\rangle_{e^\rho A} = e^{\frac{\beta}{\alpha}\sigma} \langle\langle \textstyle\int \mathcal{O} \rangle\rangle_A \tag{5.80}$$

and finally, from the definition (5.73), that[62, 28, 33]

$$\Delta := 1 - \frac{\beta}{\alpha}. \tag{5.81}$$

This is the main result of this section. It was first obtained in Ref. [62], using a light cone formulation of the Polyakov integral.

Eq. (5.79) is often written as

$$\mathcal{Z}_A \propto A^{\gamma_s - 3}, \tag{5.82}$$

where

$$\gamma_s := \frac{Q\chi}{2\alpha} + 2 \tag{5.83}$$

is known as the string susceptibility. Since we focus on spherical topology, the Euler characteristic is $\chi = 2$, and

$$\gamma_s := \frac{Q}{\alpha} + 2. \tag{5.84}$$

By a Laplace transform we can formally re-introduce the cosmological constant. In particular, consider the partition function

$$\mathcal{Z}_\mu := \int \mathcal{D}_{\hat{g}}[\varphi, X, b, c]\, e^{-S_0 - \mu \hat{A}}. \tag{5.85}$$

By the above scaling results we have

$$\begin{aligned} \mathcal{Z}_\mu &= \int_0^\infty \mathrm{d}A\, e^{-\mu A} \mathcal{Z}_A \propto \int_0^\infty \mathrm{d}A\, e^{-\mu A} A^{\gamma_s - 3} \\ &= \mu^{2-\gamma_s} \int_0^\infty \mathrm{d}x\, e^{-x} x^{\gamma_s - 3} \\ `&=' \Gamma(\gamma_s - 2) \mu^{2-\gamma_s} \end{aligned} \tag{5.86}$$

Note that the small area region of the integral is actually divergent[3]. The DT formalism regularizes this divergence, making the triangulated analogue of $\mathcal{Z}_\mu$ well defined. More generally, we can consider expectation values with several operators, and find

$$\langle \textstyle\int \mathcal{O}_j \int \mathcal{O}_k \cdots \rangle_\mu \propto \mu^{2-\gamma_s-(1-\Delta_j)-(1-\Delta_k)-\cdots}, \qquad \Delta_j := \Delta(\mathcal{O}_j), \tag{5.87}$$

where

$$\langle \bullet \rangle_\mu := \int \mathcal{D}_{\hat{g}}[\varphi, X, b, c]\, e^{-S_0-\mu\hat{A}}\, \bullet\, . \tag{5.88}$$

Let us now return to the problem of determining the correct solution for α and β. By a rescaling $\varphi \to Q\tilde{\varphi}$ we have

$$\begin{aligned} S_L[\varphi; \hat{g}] = \frac{1}{8\pi} \int \mathrm{d}^2\xi\, \sqrt{\hat{g}}(\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - Q\hat{R}\varphi) \\ = \frac{Q^2}{8\pi} \int \mathrm{d}^2\xi\, \sqrt{\hat{g}}(\hat{g}^{\mu\nu}\partial_\mu\tilde{\varphi}\partial_\nu\tilde{\varphi} - \hat{R}\tilde{\varphi}) \end{aligned} \tag{5.89}$$

This shows that for large Q (i.e. large negative c_m) the semi-classical approximation (i.e. saddle point integration) of the Liouville path integral becomes exact. In this limit we then expect that the theory behaves approximately as if the geometry is just a classical background. In particular we expect that

$$\Delta = \Delta_0 + O\left(\frac{1}{-c_m}\right). \tag{5.90}$$

We can now check that this is the case if and only if we pick the solutions

$$\alpha = \alpha_+, \qquad \beta = \beta_+. \tag{5.91}$$

5.1.2 Minimal CFTs

Among the 2d CFTs a particularly well understood class is the so called minimal CFTs. Each theory is labeled by a pair of mutually prime natural numbers (p, q), with (p, q) and (q, p) denoting the same theory. The central charge of a theory (p, q) is

$$c_m(p, q) := 1 - 6\frac{(p-q)^2}{pq}. \tag{5.92}$$

The spinless primary operators $\mathcal{O}_{r,s}$ are named by a pairs of integers (r, s),

$$0 < r < p, \qquad 0 < s < q, \tag{5.93}$$

and their scaling dimensions are given by

$$\Delta_0(r, s) := \frac{(qr - ps)^2 - (p-q)^2}{4pq}. \tag{5.94}$$

Note that the operator $\mathcal{O}_{r,s}$ is the same as the operator $\mathcal{O}_{p-r,q-s}$, and that $\mathcal{O}_{1,1} = \mathcal{O}_{p-1,q-1}$ is the identity operator.

[3]From (5.84) we have that $\gamma_s \leq 2$, since Q is positive and α is negative.

From $c_m(p,q)$ and $\Delta_0(r,s)$ we can now compute

$$Q = \frac{\sqrt{2}}{\sqrt{pq}}(p+q), \qquad \alpha = -\frac{\sqrt{2}}{\sqrt{pq}}\min(p,q), \tag{5.95}$$

and

$$\beta = -\frac{\sqrt{2}}{\sqrt{pq}}\frac{p+q-|qr-ps|}{2}. \tag{5.96}$$

This leads to the gravitational scaling dimension

$$1-\Delta_{r,s} = \frac{\beta}{\alpha} = \frac{p+q-|qr-ps|}{2\min(p,q)}, \tag{5.97}$$

and the string susceptibility

$$\gamma_s = \frac{Q}{\alpha}+2 = 2-\frac{p+q}{\min(p,q)} = 1-\frac{\max(p,q)}{\min(p,q)}. \tag{5.98}$$

Looking at the general formulae for α and β, Eqs. (5.63) and (5.68), it is remarkable that $\Delta_{r,s}$ and γ_s turn out to be rational numbers. One can take this as a hint that Liouville theory coupled to minimal CFTs is especially well behaved.

5.2 Dynamical triangulations and the two-matrix model

The idea of dynamical triangulations (DT) is to regularize the Polyakov path integral

$$\mathcal{Z} = \int \mathcal{D}_g[g]\, e^{-\mu_0\int d^2\xi\sqrt{g}} \int \mathcal{D}_g[X]\, e^{-S_m[X;g]} \tag{5.99}$$

as a sum over triangulations (we will use the word 'triangulation' to mean a combinatorial surface build by gluing together polygons). Schematically, the DT partition function is

$$\sum_\tau g^{|\tau|} Z_m[\tau], \tag{5.100}$$

where $|\tau|$ is the size of the triangulation (we will specify what 'size' means in a moment) and $Z_m[\tau]$ is the matter partition function on τ. The DT formalism is reviewed in e.g. [46, 41, 3].

A very useful way of representing the DT partition function is as a matrix model. Here we will focus on the two-matrix model, which is defined by

$$Z_{\mathrm{MM}} := \int \mathrm{d}^{N^2}X\,\mathrm{d}^{N^2}Y\, e^{-N\,\mathrm{tr}[g^{-1}U(\sqrt{g}X)+g^{-1}V(\sqrt{g}Y)-XY]}. \tag{5.101}$$

The integration variables X and Y are $N\times N$ Hermitian matrices, and the measure is normalized such that $Z_{\mathrm{MM}} = 1$ when $g=0$. The action depends on the polynomials

$$U(x) := \sum_{j=2}^{p}\frac{c_j}{j}x^j, \qquad V(y) := \sum_{j=2}^{q}\frac{\tilde{c}_j}{j}y^j. \tag{5.102}$$

Note that there are no zeroth or first order terms in $U(x)$ and $V(y)$.

It has been shown, at least formally,[4] that the two-matrix model has critical points corresponding to Liouville theory coupled to all of the minimal CFTs.[27]

When $g = 0$ the action of (5.101) is quadratic, so we can do a perturbative expansion of the integral in g. To do this, we need to know the Feynman rules. First we note that for a single matrix integral, the 'propagator' is simply

$$\int \mathrm{d}^{N^2} X\, e^{-\frac{1}{2}\operatorname{tr}[X^2]} X_{jk} X_{lm} = \delta_{jm}\delta_{kl}. \tag{5.103}$$

In the two-matrix model the quadratic part of the action is

$$\frac{N}{2}\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} c_2 & -1 \\ -1 & \tilde{c}_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \tag{5.104}$$

Inverting this matrix and using (5.103), we find the propagator

$$\begin{pmatrix} \langle X_{jk} X_{lm} \rangle & \langle X_{jk} Y_{lm} \rangle \\ \langle Y_{jk} X_{lm} \rangle & \langle Y_{jk} Y_{lm} \rangle \end{pmatrix} = \frac{1}{N}\frac{\delta_{jm}\delta_{kl}}{c_2\tilde{c}_2 - 1}\begin{pmatrix} \tilde{c}_2 & 1 \\ 1 & c_2 \end{pmatrix}, \tag{5.105}$$

where

$$\langle X_{jk} Y_{lm} \rangle := \int \mathrm{d}X\, \mathrm{d}Y\, e^{-N\operatorname{tr}[\frac{c_2}{2}X^2 + \frac{c_2}{2}X^2 + \frac{\tilde{c}_2}{2}Y^2 - XY]} X_{jk} Y_{lm}, \tag{5.106}$$

etc.

An example of a typical term in the expansion of Z_{MM} is

$$N^3 \int \mathrm{d}X\, \mathrm{d}Y\, e^{-N\operatorname{tr}[\frac{c_2}{2}X^2 + \frac{c_2}{2}X^2 + \frac{\tilde{c}_2}{2}Y^2 - XY]} \frac{gc_4}{4}\operatorname{tr}[X^4]\frac{g^2 c_6}{6}\operatorname{tr}[X^6]\frac{g\tilde{c}_4}{4}\operatorname{tr}[Y^4]. \tag{5.107}$$

Such a term can be evaluated by performing the usual Wick contractions of the matrices, with the propagator (5.105). The key observation is now that the different contractions can be interpreted as triangulations.[82, 57, 20] A trace of the form $\operatorname{tr}[X^n]$ (or $\operatorname{tr}[Y^n]$) corresponds to a n-sided polygon, with each factor of X being an edge. A wick contraction between two matrices is then interpreted as gluing the corresponding edges.

In general, the triangulations obtained in this way will have a complicated topology, but one can take advantage of the parameter N to pick out surfaces with a specific topology.[82, 57, 20] To see how this works, let us count the factors of N associated to each triangulation. For each face (polygon) we have one factor, since the action has an overall factor of N. For each edge we have a factor N^{-1} from the propagator (5.105). Finally we have a number of free matrix indices, that have not been killed by the deltas in the propagator. It is easy to see that we have one free index for each vertex of the triangulation. The total power of N is thus

$$\chi := \text{faces} - \text{edges} + \text{vertices}. \tag{5.108}$$

The number χ is known as the Euler characteristic of the surface. For a closed surface (which is what we are considering) its value is

$$\chi = 2 - 2h, \tag{5.109}$$

[4] There are subtleties in deciding whether these critical points are actually the ones controlling the critical behavior of the DT partition function.[27]

where h is the genus of the surface.

We will focus on surfaces with spherical topology (i.e. genus $h = 0$). From the above discussion it follows that the partition function of connected spherical triangulation is

$$Z_s := \lim_{N\to\infty} N^{-2} \ln Z_{\mathrm{MM}}. \tag{5.110}$$

Here the logarithm is used to extract only the connected diagrams.

5.2.1 The method of orthogonal polynomials

In order to compute the integral (5.101) we first change variables[5] to the eigenvalues of X and Y. It can be shown that[50, 57, 66, 61]

$$Z_{\mathrm{MM}} = \mathcal{N} \int \mathrm{d}^N\mathbf{x}\,\mathrm{d}^N\mathbf{y}\,\Delta(\mathbf{x})\Delta(\mathbf{y}) e^{-N\sum_{j=1}^{N}(g^{-1}U(\sqrt{g}x_j)+g^{-1}V(\sqrt{g}y_j)-x_jy_j)}, \tag{5.111}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ correspond to the eigenvalues, and $\mathcal{N}$ is some N-dependent normalization factor. We refer to the references given above for a derivation of this formula. The term $\Delta(\mathbf{x})\Delta(\mathbf{y})$ is loosely the Jacobian of the change of variables, where $\Delta(\cdot)$ is the Vandermonde determinant defined by

$$\Delta(\mathbf{x}) := \det \begin{pmatrix} 1 & x_1^1 & \cdots & x_1^{N-1} \\ 1 & x_2^1 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N^1 & \cdots & x_N^{N-1} \end{pmatrix}. \tag{5.112}$$

We now note, following[57, 20, 66], that using column operations we can rewrite the Vandermonde determinant as

$$\Delta(\mathbf{x}) = \det \begin{pmatrix} \pi_0(x_1) & \pi_1(x_1) & \cdots & \pi_{N-1}(x_1) \\ \pi_0(x_2) & \pi_1(x_2) & \cdots & \pi_{N-1}(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ \pi_0(x_N) & \pi_1(x_N) & \cdots & \pi_{N-1}(x_N) \end{pmatrix}, \tag{5.113}$$

where π_j is a monic polynomial of degree j. If we expand the two Vandermonde determinants in (5.111) we see that each term is the product of integrals of the form

$$\int \mathrm{d}x\,\mathrm{d}y\,\pi_j(x) e^{-N(g^{-1}U(\sqrt{g}x)+g^{-1}V(\sqrt{g}y)-xy)}\tilde{\pi}_k(y). \tag{5.114}$$

This suggests choosing the polynomials to kill as many of these integrals as possible. Equation (5.114) defines an inner product,

$$\langle \pi(x), \tilde{\pi}(y)\rangle := \int \mathrm{d}x\,\mathrm{d}y\,\pi(x) e^{-N(g^{-1}U(\sqrt{g}x)+g^{-1}V(\sqrt{g}y)-xy)}\tilde{\pi}(y). \tag{5.115}$$

Note that $\langle\cdot,\cdot\rangle$ is not in general symmetric. Using a variant of the Gram-Schmidt procedure we can construct two sets of polynomials, $\pi_j(x)$ and $\tilde{\pi}_j$, orthogonal with respect to this inner product, i.e. such that

$$\langle \pi_j(x), \tilde{\pi}_k(y)\rangle = \delta_{jk} h_j. \tag{5.116}$$

[5]Strictly speaking it is not just a change of variables, since one have to perform a non-trivial unitary integral[50, 57, 66, 61].

Explicitly, we find

$$
\begin{aligned}
\pi_0(x) &= 1 \\
\pi_1(x) &= x - \frac{\langle x, \tilde{\pi}_0(y) \rangle}{h_0} \pi_0(x) \\
\pi_2(x) &= x^2 - \frac{\langle x^2, \tilde{\pi}_1(y) \rangle}{h_1} \pi_1(x) - \frac{\langle x^2, \tilde{\pi}_0(y) \rangle}{h_0} \pi_0(x) \\
&\vdots
\end{aligned}
\tag{5.117}
$$

and similar expressions for the $\tilde{\pi}_j$ polynomials.

The expansion of $\Delta(\mathbf{x})$ is

$$
\Delta(\mathbf{x}) = \sum_{\sigma \in S(\{0,\ldots,N-1\})} (-)^\sigma \pi_{\sigma 0}(x_1) \pi_{\sigma 1}(x_2) \cdots \pi_{\sigma(N-1)}(x_N),
\tag{5.118}
$$

with a corresponding expansion for $\Delta(\mathbf{y})$. Because of the orthogonality, only terms where the two permutations are identical are non-zero. Since there are $N!$ such terms, and the sign factor cancels, we find

$$
Z_{\mathrm{MM}} = \frac{N! \prod_{j=0}^{N-1} h_j}{N! \prod_{j=0}^{N-1} h_j^{(0)}} = \prod_{j=0}^{N-1} \frac{h_j}{h_j^{(0)}},
\tag{5.119}
$$

where $h_j^{(0)}$ is h_j evaluated at $g = 0$. This is the first important result in this section: We can compute the partition function from the norms of the orthogonal polynomials.

In the following it will be more convenient to work with the rescaled polynomials

$$
\sigma_j(x) := g^{j/2} \pi_j\left(\frac{x}{\sqrt{g}}\right), \qquad \tilde{\sigma}_j(y) := g^{j/2} \tilde{\pi}_j\left(\frac{y}{\sqrt{g}}\right).
\tag{5.120}
$$

Let us also define a rescaled inner product

$$
\langle \sigma_j, \tilde{\sigma}_k \rangle := \int \mathrm{d}x \, \mathrm{d}y \, e^{-\frac{N}{g}(U(x)+V(y)-xy)} \sigma_j(x) \tilde{\sigma}_k(y)
\tag{5.121}
$$

We find

$$
\langle \sigma_j, \tilde{\sigma}_k \rangle = \delta_{jk} h'_j
\tag{5.122}
$$

with

$$
h'_j = g^{1+j} h_j.
\tag{5.123}
$$

The next step will be to write down some equations that can be solved for the norms h'_j. It turns out to be a good idea to define four linear operators on the vector space of polynomials by

$$
(Q\sigma_j)(x) := x\sigma_j(x) \qquad (\tilde{Q}\tilde{\sigma}_j)(y) := y\tilde{\sigma}_j(y),
\tag{5.124}
$$

and

$$
(P\sigma_j)(x) := \frac{1}{N}\sigma'_j(x) \qquad (\tilde{P}\tilde{\sigma}_j)(y) := \frac{1}{N}\tilde{\sigma}'_j(y).
\tag{5.125}
$$

Here the prime denotes differentiation with respects to x or y. By a simple integration by parts we get the identities

$$\langle P\sigma, \tilde{\sigma}\rangle = \frac{1}{g}\langle U'(Q)\sigma, \tilde{\sigma}\rangle - \frac{1}{g}\langle \sigma, \tilde{Q}\tilde{\sigma}\rangle, \tag{5.126}$$

and

$$\langle \sigma, \tilde{P}\tilde{\sigma}\rangle = \frac{1}{g}\langle \sigma, V'(\tilde{Q})\tilde{\sigma}\rangle - \frac{1}{g}\langle Q\sigma, \tilde{\sigma}\rangle. \tag{5.127}$$

Remarkably, the partition function is determined by these equations (with some simple boundary conditions).

Ultimately we are interested in the large N limit of the partition function, and in this limit there is a better way of writing Eqs. (5.126) and (5.127). Let us first define the matrix elements of Q and P:

$$(Q\sigma_j)(x) = \sum_{k=0}^{j+1} \sigma_k(x) Q_{kj} \qquad (P\sigma_j)(x) = \sum_{k=0}^{j-1} \sigma_k(x) P_{kj}. \tag{5.128}$$

Note that most statements have a dual obtained by adding or removing tildes and replacing $U \leftrightarrow V$. For brevity we will sometimes leave the dual statement implicit. In component form Eq. (5.126) looks like

$$gP_{jk} = [U'(Q)]_{jk} - \frac{h'_j}{h'_k}\tilde{Q}_{kj}. \tag{5.129}$$

This equation becomes easier to manipulate if we introduce generating (Laurant) polynomials for the entries of the matrices. Thus, for any matrix B_{jk}, define

$$B(t,z) := \sum_j B_j(t) z^j = \sum_j B_{Nt+j,Nt} z^j, \tag{5.130}$$

with

$$B_j(t) := B_{Nt+j,Nt}. \tag{5.131}$$

Let us now assume two things about B_{jk}: (1) The functions $B_j(t)$ will become smooth functions of t in the large-N limit. (2) The functions $B(t,z)$ are Laurant polynomials in z (i.e. $B_j(t)$ is only non-zero for a finite number of values of j). We then have the important property (with the same assumptions about C)

$$[BC](t,z) = B(t,z)C(t,z) + O\left(\frac{1}{N}\right). \tag{5.132}$$

In words: To leading order in $1/N$, we can replace matrix multiplication with multiplication of the corresponding polynomials. The result follows from the identity

$$[BC]_{jk} = \sum_l B_{j,k+l} C_{k+l,k} = \sum_l B_{j-l,k} C_{k+l,k} + O\left(\frac{1}{N}\right). \tag{5.133}$$

To express (5.129) in terms of the polynomials, we first compute

$$\sum_j \frac{h'_{Nt}}{h'_{Nt+j}} \tilde{Q}_{Nt,Nt+j} z^j = \sum_j v(t)^{-j} \tilde{Q}_{Nt-j,Nt} z^j + O\left(\frac{1}{N}\right) \tag{5.134}$$

$$= \tilde{Q}\left(t, \frac{v(t)}{z}\right) + O\left(\frac{1}{N}\right), \tag{5.135}$$

where $v(t)$ is defined to be the ratio

$$v(t) := \frac{h'_{Nt}}{h'_{Nt-1}}. \tag{5.136}$$

The expression $[U'(Q)]_{jk}$ can be simplified using (5.132), and we thus get the equation

$$gP(t,z) = U'(Q(t,z)) - \tilde{Q}\left(t, \frac{v(t)}{z}\right) + O\left(\frac{1}{N}\right). \tag{5.137}$$

Similarly, Eq. (5.127) becomes

$$g\tilde{P}(t,z) = V'(\tilde{Q}(t,z)) - Q\left(t, \frac{v(t)}{z}\right) + O\left(\frac{1}{N}\right). \tag{5.138}$$

These are the master equations for computing the large-N limit of $v(t)$, and hence the partition function.

Let us briefly discuss condition (2) mentioned above, namely that $Q(t,z)$ and $P(t,z)$ are Laurant polynomials in z. First we note that $Q\sigma_j$ is a polynomial of degree $j+1$. It follows that

$$Q_j(t) = \tilde{Q}_j(t) = 0, \qquad \text{for } j > 1. \tag{5.139}$$

Similarly $P\sigma_j$ is of degree $j-1$, hence

$$P_j(t) = \tilde{P}_j(t) = 0, \qquad \text{for } j > -1. \tag{5.140}$$

Combining these observations with (5.137) and (5.138) we infer[6]

$$Q_j(t) = 0, \qquad \text{for } j \notin \{-(q-1), \ldots, 1\}, \tag{5.141}$$

$$\tilde{Q}_j(t) = 0, \qquad \text{for } j \notin \{-(p-1), \ldots, 1\}, \tag{5.142}$$

and finally

$$P_j(t) = \tilde{P}_j(t) = 0, \qquad \text{for } j \notin \{-(p-1)(q-1), \ldots, -1\}. \tag{5.143}$$

We conclude that $Q(t,z)$ and $P(t,z)$ are indeed polynomials. This means that (5.137) and (5.138) represent a finite number of coupled algebraic equations. From the fact that σ_j are monic, we obtain the 'boundary conditions'

$$Q_1(t) = \tilde{Q}_1(t) = 1, \tag{5.144}$$

and

$$P_{-1}(t) = \tilde{P}_{-1}(t) = t. \tag{5.145}$$

Let us mention one more useful fact about $Q_j(t)$ and $P_j(t)$. We recall that a function is even iff $f(-x) = f(x)$ and odd iff $f(-x) = -f(x)$. When both $U(x)$ and $V(y)$ are even we have the following useful result: The polynomials $\sigma_j, \tilde{\sigma}_j$ are even (odd) when j is even (odd). To see this, we note that since U, V are even, the measure

$$\int \mathrm{d}x\, \mathrm{d}y\, e^{-\frac{N}{g}(U(x)+V(y)-xy)} \tag{5.146}$$

[6] Actually, it follows from (5.126) and (5.127) that the conclusion also holds for finite N.

is invariant under the simultaneous change of variables $x \to -x$, $y \to -y$. We then conclude that $\langle \sigma, \tilde{\sigma} \rangle$ is zero when σ and $\tilde{\sigma}$ have opposite parity. The claim follows by induction on the explicit expressions (5.117). We now remark that the operators Q and P reverse the parity of the polynomial they act on. Finally, we have that, when $U(x)$ and $V(y)$ are even, $Q_j(t)$ and $P_j(t)$ can only be non-zero for odd j.

In principle, we now know how to compute $v(t)$ to leading order $1/N$, so let us discuss how to compute the actual partition function. From (5.119) we have

$$\ln Z_{\mathrm{MM}} = \sum_{j=0}^{N-1} \ln \frac{h_j}{h_j^{(0)}} = N \ln \frac{h_0}{h_0^{(0)}} + \sum_{j=1}^{N-1} (N-j) \ln \frac{h_j h_{j-1}^{(0)}}{h_{j-1} h_j^{(0)}}. \tag{5.147}$$

The large-N (spherical) limit is then

$$Z_s := \lim_{N\to\infty} N^{-2} \ln Z_{\mathrm{MM}} = \int_0^1 \mathrm{d}t\, (1-t) \ln \frac{\nu(t)}{\nu^{(0)}(t)} \tag{5.148}$$

where we assume that

$$\nu(t) := \frac{h_{Nt}}{h_{Nt-1}} \tag{5.149}$$

becomes a smooth function, and

$$\nu^{(0)}(t) := \nu(t)|_{g=0}. \tag{5.150}$$

We note that $v^{(0)}(t)$ does not exist, which is why we bother to define both $v(t)$ and $\nu(t)$. Comparing (5.136) and (5.149) we find

$$v(t) = g\nu(t). \tag{5.151}$$

It is convenient to change integration variable to

$$\tau := gt \tag{5.152}$$

which leads to

$$g^2 Z_s = \int_0^g \mathrm{d}\tau\, (g-\tau) \ln \frac{v(\tau)}{g\nu^{(0)}(t)}. \tag{5.153}$$

It is not difficult to see (we will also check this in the concrete examples of Secs. 5.3 and 5.4) that

$$\nu^{(0)}(t) = \alpha t, \tag{5.154}$$

where α is some constant depending on the quadratic part of the polynomials U and V (but *not* on g). Our final expression for the spherical partition function is then

$$g^2 Z_s = \int_0^g \mathrm{d}\tau\, (g-\tau) \ln \frac{v(\tau)}{\alpha\tau}. \tag{5.155}$$

In practice Eq. (5.155) is not the most useful expression for Z_s, because one has to compute $v(\tau)$ for all values of $\tau \in [0, g]$. Luckily, there is a simple solution. First we note that the RHS of the equation only depends on g through the integration limit and the explicit occurrence in the integrand. This is the motivation for the particular way g was introduced in the matrix model (5.101). Differentiating (5.155) twice with respect to g we obtain

$$\frac{\partial^2 (g^2 Z_s)}{\partial g^2} = \ln \frac{v(\tau = g)}{\alpha g}, \tag{5.156}$$

which only depends on $v(\tau)$ at $\tau = g$. We note that the identity

$$\frac{\partial^2}{\partial g^2} g^3 \frac{\partial}{\partial g} Z_s = \frac{g}{v(\tau = g)} \frac{\partial v(\tau = g)}{\partial g} - 1, \tag{5.157}$$

which is also straight forward to derive starting from (5.155), is sometimes used instead of (5.156) in the literature (e.g. [41, 22]).

5.2.2 The continuum limit

We will now discuss how to connect the discrete and continuum formalisms. The precise relation is somewhat subtle, so we will start out in a heuristic manner. The basic intuition is that, when the coupling constants $(g, \{c_j\}, \{\tilde{c}_j\})$ are close to a critical point $(g^*, \{c_j^*\}, \{\tilde{c}_j^*\})$, then (the meaning of $\sim$ in this equation will become clear latter)

$$Z_s(g, \{c_j\}, \{\tilde{c}_j\}) \sim \mathcal{Z}(\mu, \{\lambda_j\}), \tag{5.158}$$

where $\mathcal{Z}(\mu, \{\lambda_j\})$ is a perturbed version of the continuum path integral (5.47), formally defined by

$$\mathcal{Z}(\mu, \{\lambda_j\}) := \int \mathcal{D}_{\hat{g}}[\varphi, X, b, c]\, e^{-S_0 - \mu \hat{A} + \sum_j \lambda_j \int \mathcal{O}_j}. \tag{5.159}$$

The matter theory will be a minimal CFT, and $\int \mathcal{O}_j$ are the dressed primary operators constructed in Section 5.1.1. Note that the correspondence (5.158) is seemingly not completely understood when one includes irrelevant operators[7].

So far it has not been possible to compute $\mathcal{Z}(\lambda, \mu)$ in the general case, but the first terms in the expansion (here we switch to a fixed area formulation for later convenience)

$$\mathcal{Z}(A, \{\lambda_j\}) = \mathcal{Z}_A + \sum_j \lambda_j \langle \textstyle\int \mathcal{O}_j \rangle_A + \frac{1}{2} \sum_{j,k} \lambda_j \lambda_k \langle \textstyle\int \mathcal{O}_j \int \mathcal{O}_k \rangle_A + \cdots \tag{5.160}$$

are known[49, 36, 83, 89, 18]. By the scaling arguments of Sec. 5.1.1 we can write the expansion as

$$\begin{aligned} \mathcal{Z}(A, \{\lambda_j\}) = K A^{\gamma_s - 3} &+ \sum_j K_j \lambda_j A^{\gamma_s - 3 + (1-\Delta_j)} \\ &+ \frac{1}{2} \sum_{j,k} K_{jk} \lambda_j \lambda_k A^{\gamma_s - 3 + (1-\Delta_j) + (1-\Delta_k)} + \cdots, \end{aligned} \tag{5.161}$$

where the Ks are constants, known as *correlation numbers*. These correlation numbers can be compared with DT calculations, and are found to match in

[7] In particular, the four-point correlation numbers involving irrelevant operators derived from the discrete approach are found not to satisfy the selection rules of the matter CFT[16].

all cases the author is aware of.[17, 16, 19] (Actually, to avoid ambiguities in the normalization of operators, one should compare ratios like (no sum over repeated indices)

$$\frac{K_{jkl}}{\sqrt{K_{jj}}\sqrt{K_{kk}}\sqrt{K_{ll}}} \tag{5.162}$$

where the normalization cancels.)

An important point is that the theory (5.159) should be conformal with $c_{\rm tot} = 0$, even for non-zero couplings λ_j, in order that the dependence on the fiducial metric $\hat{g}$ drops out. It has been shown that to maintain this condition it is necessary to include terms like

$$\delta S = \sum_{j,k,l} k_{j,k,l}\lambda_j\lambda_k \textstyle\int \mathcal{O}_j + \cdots \tag{5.163}$$

that are of higher order in the coupling constants[78, 4]. This means that the definition (5.159) should not be taken too literally. Even the expansion (5.160) has ambiguities due to contact terms[71].

For (5.158) to have any content, we need to specify the relation between the DT coupling constants $(g, \{c_j\}, \{\tilde{c}_j\})$ and the Liouville coupling constants $(\mu, \{\lambda_j\})$. It seems that the following prescription works: First we introduce a 'lattice length' a to set the overall scale as we approach the critical point. We assign a^2 the dimension of area (i.e. the inverse of the dimension of μ). Next we take $\{g, c_j, \tilde{c}_j\}$ to be regular functions (we will discuss later how to choose these functions) of the dimensionless quantities $\{a^2\hat{\mu}, a^{2(1-\Delta_j)}\hat{\lambda}_j\}$. On might then hope that the DT partition function behaves as (the power of a follows by dimensional analysis)

$$Z_s(g, \{c_j\}, \{\tilde{c}_j\}) \stackrel{?}{=} a^{2(2-\gamma_s)}\mathcal{Z}(\mu = \hat{\mu}, \{\lambda_j = \hat{\lambda}_j\}) + o(a^{2(2-\gamma_s)}). \tag{5.164}$$

This is, however, not quite true.

The basic problem with (5.164) is that there are contributions to Z_s from microscopic surfaces, and that these contributions do not scale with appropriate power of a. In more detail, let us write

$$Z_s(g, \{c_j\}, \{\tilde{c}_j\}) = \sum_n e^{-[\hat{\mu}a^2 + o(a^2)]n} Z_s(n, \{a^{2(1-\Delta_j)}\hat{\lambda}_j\}), \tag{5.165}$$

where n is the discrete area of the surface (think of n as the number of triangles). The fixed area partition function is similar to an ordinary lattice partition function, so we expect

$$Z_s(n = a^{-2}A, \{a^{2(1-\Delta_j)}\hat{\lambda}_j\}) = a^{2(3-\gamma_s)}\mathcal{Z}(A, \{\lambda_j = \hat{\lambda}_j\}) + o(a^{2(3-\gamma_s)}), \tag{5.166}$$

as the lattice spacing a is taken to zero. We see that $Z_s(n = a^{-2}A, \{c_j\}, \{\tilde{c}_j\})$ goes to zero in the continuum limit (remember that $\gamma_s \leq 2$). In contrast, for finite n, there is no reason for $Z_s(n, \{c_j\}, \{\tilde{c}_j\})$ to go to zero as the coupling constants approach the critical point.

The solution to the problem is to note that the singularities of Z_s in $\{\hat{\mu}, \hat{\lambda}_j\}$ can only come from the large n part of the sum (5.165)[8]. Let us start by

[8]If we cut the sum off at some finite n, the result is a polynomial in $\{g, c_j, \tilde{c}_j\}$, and these are, by assumption, regular functions of $\{\hat{\mu}, \hat{\lambda}_j\}$.

considering the case where all the $\hat{\lambda}_j$ are zero, but $\hat{\mu}$ is non-zero. We then have

$$Z_s(n = a^{-2}A, \{a^{2(1-\Delta_j)}\hat{\lambda}_j = 0\}) = a^{2(3-\gamma_s)}KA^{3-\gamma_s} + o(a^{2(3-\gamma_s)}). \tag{5.167}$$

Let us now separate the sum in (5.165) as

$$\sum_n = \sum_{n=1}^{A_0/a^2} + \sum_{n=A_0/a^2}^{\infty}, \tag{5.168}$$

where A_0 is some arbitrary area cut-off. It is clear that first sum will be regular in $\hat{\mu}$, so we have[9]

$$\begin{aligned}
Z_s(\hat{\mu}) &= \sum_{n=A_0/a^2}^{\infty} e^{-\hat{\mu}a^2}\left[Ka^{2(3-\gamma_s)}(a^2n)^{3-\gamma_s} + o(a^{2(3-\gamma_s)})\right] + \text{reg.} \\
&= Ka^{2(2-\gamma_s)}\int_{A_0}^{\infty} \mathrm{d}A\, A^{\gamma_s-3}e^{-\hat{\mu}A} + o(a^{2(3-\gamma_s)}) + \text{reg.} \\
&= Ka^{2(2-\gamma_s)}\hat{\mu}^{2-\gamma_s}\int_{\hat{\mu}A_0}^{\infty} \mathrm{d}x\, x^{\gamma_s-3}e^{-x} + o(a^{2(3-\gamma_s)}) + \text{reg.} \\
&= \Gamma(\gamma_s-2)Ka^{2(2-\gamma_s)}\hat{\mu}^{2-\gamma_s} - Ka^{2(2-\gamma_s)}A_0^{\gamma_s-2}\sum_{k=0}^{\infty}\frac{(-1)^k(\hat{\mu}A_0)^k}{k!(\gamma_s-2+k)} \\
&\qquad + o(a^{2(3-\gamma_s)}) + \text{reg.}
\end{aligned} \tag{5.170}$$

The sum over k produces a regular function of $\hat{\mu}$, so we conclude that

$$Z_s(\hat{\mu}) = a^{2(2-\gamma_s)}\Gamma(\gamma_s-2)K\hat{\mu}^{2-\gamma_s} + o(a^{2(3-\gamma_s)}) + \text{reg.} \tag{5.171}$$

We can repeat the above exercise for each term of the Taylor expansion of $Z_s(n = a^{-2}A, \{a^{2(1-\Delta_j)}\hat{\lambda}_j\})$ in $\hat{\lambda}_j$, and find

$$\begin{aligned}
Z_s(n = a^{-2}A, \{a^{2(1-\Delta_j)}\hat{\lambda}_j\}) &= a^{2(2-\gamma_s)}\Gamma(\gamma_s-2)K\hat{\mu}^{2-\gamma_s} \\
&\quad + a^{2(2-\gamma_s)}\sum_j \hat{\lambda}_j\Gamma(\gamma_s+\Delta_j-1)K_j\hat{\mu}^{1-\gamma_s-\Delta_j} \\
&\quad + a^{2(2-\gamma_s)}\sum_{j,k}\hat{\lambda}_j\hat{\lambda}_k\Gamma(\gamma_s+\Delta_j+\Delta_k)K_{jk}\hat{\mu}^{-\gamma_s-\Delta_j-\Delta_k} \\
&\quad + \cdots + o(a^{2(3-\gamma_s)}) + \text{reg.}
\end{aligned} \tag{5.172}$$

Terms where the total dimension $2-\gamma_s-(1-\Delta_j)-(1-\Delta_k)-\cdots$ is a non-negative integer should be excluded from the above sums.[10] In this case the term is not singular as a function of $\hat{\mu}$, and there is no way to distinguish continuum contributions from non-universal lattice artifacts. We can summarize (5.172) as

$$Z_s|_{a^{2(2-\gamma_s)}} = \mathcal{Z}(\mu = \hat{\mu}, \{\lambda_j = \hat{\lambda}_j\}) + \text{reg.}, \tag{5.173}$$

[9] The last equality uses the expansion

$$\Gamma(a,z) := \int_z^{\infty} \mathrm{d}x x^{a-1}e^{-x} = \Gamma(a) - \sum_{k=0}^{\infty}\frac{(-1)^k z^{a+k}}{k!(a+k)}, \qquad a \notin \{0,-1,-2,\ldots\} \tag{5.169}$$

of the upper incomplete gamma function $\Gamma(a,z)$. See e.g. Sec. 8.7 of Ref. [34].

[10] The manipulation corresponding to the last equality in (5.170) would clearly be wrong for these terms.

where $|_{a^{2(2-\gamma_s)}}$ denotes the term of order $a^{2(2-\gamma_s)}$.

Eq. (5.173) is still not quite what we want, because we will actually calculate (see Eq. (5.156))

$$\frac{\partial^2(g^2 Z_s)}{\partial g^2} \tag{5.174}$$

instead of Z_s. Let us assume that we can solve for $(\hat{\mu}, \{\hat{\lambda}_j\})$ in terms of $(a, g, \{c_j\}, \{\tilde{c}_j\})$. We then have

$$\frac{\partial}{\partial g} = \frac{\partial \hat{\mu}}{\partial g}\frac{\partial}{\partial \hat{\mu}} + \sum_j \frac{\partial \hat{\lambda}_j}{\partial g}\frac{\partial}{\partial \hat{\lambda}_j}. \tag{5.175}$$

By dimensional analysis, the a dependence is

$$\frac{\partial \hat{\mu}}{\partial g} \propto a^{-2} + o(a^{-2}), \qquad \frac{\partial \hat{\lambda}_j}{\partial g} \propto a^{-2(1-\Delta_j)} + o(a^{-2(1-\Delta_j)}). \tag{5.176}$$

Let $\hat{\Lambda}$ denote the most relevant coupling constant[11]. The leading term of $\partial/\partial g$ is thus

$$\frac{\partial}{\partial g} \propto a^{-2(1-\Delta_\Lambda)}\frac{\partial}{\partial \hat{\Lambda}} + o(a^{-2(1-\Delta_\Lambda)}). \tag{5.177}$$

We can now act with $\partial^2/\partial g^2$ on (5.172) to obtain (the factor of g^2 plays no role since it is not singular)

$$\left.\frac{\partial^2(g^2 Z_s)}{\partial g^2}\right|_{a^{-2\gamma_s-4\Delta_\Lambda}} \propto \frac{\partial^2 \mathcal{Z}(\mu = \hat{\mu}, \{\lambda_j = \hat{\lambda}_j\})}{\partial \Lambda^2} + \text{reg.} \tag{5.178}$$

Using (5.156) we finally have

$$v(t=1)|_{a^{-2\gamma_s-4\Delta_\Lambda}} \propto \frac{\partial^2 \mathcal{Z}(\mu = \hat{\mu}, \{\lambda_j = \hat{\lambda}_j\})}{\partial \Lambda^2} + \text{reg.} \tag{5.179}$$

So far, we have not explained how to choose the functional dependence of $(g, \{c_j\}, \{\tilde{c}_j\})$ on $(\hat{\mu}, \{\hat{\lambda}_j\})$. One can try to fix this dependence by simply demanding that the leading singularity of $v(t=1)$ in $\hat{\mu}$ is of order $a^{-2\gamma_s+4\Delta_\Lambda}$ (for all terms in the Taylor expansion in $\hat{\lambda}_j$). We will use this approach in Secs. 5.3 and 5.4.

The above consistency condition is almost sufficient to determine the correct scaling limit. However, a complication occurs when there are *resonances*, that is, when the dimension of some product of coupling constants coincide with the dimension of another coupling constant. As an example, consider three coupling constants $\hat{\lambda}_a$, $\hat{\lambda}_b$ and $\hat{\lambda}_c$ such that the product $\hat{\lambda}_a\hat{\lambda}_b$ has the same dimension as $\hat{\lambda}_c$. The procedure based purely on dimensions now becomes ambiguous, because one could end up with e.g.

$$Z_s|_{a^{2(2-\gamma_s)}} = \mathcal{Z}(\mu = \hat{\mu}, \{\ldots, \lambda_c = \hat{\lambda}_c + \kappa\hat{\lambda}_a\hat{\lambda}_b, \ldots\}) + \text{reg.}, \tag{5.180}$$

instead of (5.179). This ambiguity has been connected with the problem of contact terms of the continuum formalism[71].

[11] That is, the coupling constant for which Δ_j is smallest. For unitary matter theories we have $\hat{\Lambda} = \hat{\mu}$, but non-unitary theories can have operators with negative scaling dimension.

5.3 The Ising model

As a first example of the DT formalism, we will consider the Ising model on a random triangulation. This model was first solved in Refs. [61, 21].

Consider the potentials

$$U(x) = \frac{1}{4}x^4 + \frac{c_2}{2}x^2, \qquad V(y) = \frac{1}{4}y^4 + \frac{\tilde{c}_2}{2}y^2, \tag{5.181}$$

with

$$c_2, \tilde{c}_2 > 0, \qquad c_2\tilde{c}_2 > 1. \tag{5.182}$$

The resulting two-matrix has a natural interpretation as a Ising model coupled to random triangulation. Here, the 'triangulation' is constructed from squares, with a spin on each square (it is known that the same continuum limit is obtained if one uses triangles instead of squares[21]). The direction of the spin on a square is determined by whether it is of type X or type Y. With the identification $X \leftrightarrow\uparrow$, $Y \leftrightarrow\downarrow$, the propagators (5.105) look like

$$\langle\uparrow\uparrow\rangle = \frac{\tilde{c}_2}{c_2\tilde{c}_2 - 1}, \quad \langle\uparrow\downarrow\rangle = \frac{1}{c_2\tilde{c}_2 - 1}, \quad \langle\downarrow\downarrow\rangle = \frac{c_2}{c_2\tilde{c}_2 - 1}. \tag{5.183}$$

A little algebra shows that the overall weight of a triangulation is

$$\left[-g\frac{c_2\tilde{c}_2}{(c_2\tilde{c}_2 - 1)^2}\right]^{\#_4} \left(\frac{\tilde{c}_2}{c_2}\right)^{\#_\uparrow - \#_\downarrow} \left(\frac{1}{c_2\tilde{c}_2}\right)^{\frac{1}{2}\#_{\uparrow\downarrow}}, \tag{5.184}$$

where $\#_4$ is the number of squares, $\#_\uparrow$ ($\#_\downarrow$) is the number of spins pointing up (down), and $\#_{\uparrow\downarrow}$ is the number of neighbouring spin pairs that point in opposite directions. The first term (5.184) is a cosmological term (when continued to negative g), while remaining terms are the usual Ising weights. Explicitly, we can write them as

$$\left(\frac{\tilde{c}_2}{c_2}\right)^{\#_\uparrow - \#_\downarrow} \left(\frac{1}{c_2\tilde{c}_2}\right)^{\frac{1}{2}\#_{\uparrow\downarrow}} = e^{-\beta\#_{\uparrow\downarrow} + H(\#_\uparrow - \#_\downarrow)} \tag{5.185}$$

in terms of the inverse temperature

$$\beta := \frac{1}{2}\ln(c_2\tilde{c}_2) \tag{5.186}$$

and the magnetic field

$$H := \ln(\tilde{c}_2/c_2). \tag{5.187}$$

We will now solve the model in the planar limit, using orthogonal polynomials as described above. Since $U(x)$ and $V(y)$ are even, it follows that $Q(t,z)$ and $\tilde{Q}(t,z)$ only contain terms of odd order in z (see discussion around (5.146)), and by (5.141) we then have (similarly for $\tilde{Q}(t,z)$)

$$Q(t,z) = z + Q_{-1}(t)z^{-1} + Q_{-3}(t)z^{-3}. \tag{5.188}$$

Because of the symmetry between $U(x)$ and $V(y)$ equations come in pairs, so we often leave out equations that follow by symmetry. Expanding (5.137) in z

we have

$$
\begin{aligned}
\frac{gt}{z} + O\left(\frac{1}{z^2}\right) &= \left(1 - \frac{\tilde{Q}_{-3}(t)}{v(t)^3}\right) z^3 \\
&\quad + \left(c_2 + 3Q_{-1}(t) - \frac{\tilde{Q}_{-1}(t)}{v(t)}\right) z \\
&\quad + \left(-v(t) + 3Q_{-3}(t) + c_2 Q_{-1}(t) + 3Q_{-1}(t)^2\right)\frac{1}{z} \\
&\quad + O\left(\frac{1}{z^2}\right)
\end{aligned}
\tag{5.189}
$$

We can solve for $Q_{-3}(t)$ and $Q_{-1}(t)$ in terms of $v(t)$,

$$
Q_{-3}(t) = v(t)^3 \qquad \text{and} \qquad Q_{-1}(t) = -\frac{\tilde{c}_2 v(t) + 3c_2 v(t)^2}{9v(t)^2 - 1}. \tag{5.190}
$$

The remaining equation is then

$$
gt = w\big(v(t)\big), \tag{5.191}
$$

where $w\big(v(t)\big)$ is

$$
w(v(t)) := 3v(t)^3 - v(t) + \frac{(c_2 + 3\tilde{c}_2 v(t))(\tilde{c}_2 + 3c_2 v(t))v(t)}{(9v(t)^2 - 1)^2}. \tag{5.192}
$$

Equation (5.191) implicitly defines $v(t)$, and thus the partition function through (5.153).

As discussed in Sec. 5.2.2 the continuum limit is determined once we know $v := v(t = 1)$. As a first step we must find the critical values of the coupling constants $\{g, c_2, \tilde{c}_2\}$. We expect the critical point to respect the spin reversal symmetry, so we set $c_2 = \tilde{c}_2$ for now. The equation for v is then

$$
g = w(v) \tag{5.193}
$$

with

$$
w(v) = 3v^3 - v + \frac{c_2^2 v}{(3v - 1)^2}. \tag{5.194}
$$

At the critical point v should be singular as a function of g. This only happens when

$$
w'(v) = \frac{(3v + 1)([3v - 1]^4 - c_2^2)}{(3v - 1)^3} = 0. \tag{5.195}
$$

Solving this equation, the candidates for critical points are

$$
v^* \in \left\{-\frac{1}{3}, \frac{1}{3}(1 + \sqrt{c_2}), \frac{1}{3}(1 - \sqrt{c_2}), \frac{1}{3}(1 + i\sqrt{c_2}), \frac{1}{3}(1 - i\sqrt{c_2})\right\}. \tag{5.196}
$$

In order to decide which critical point is relevant, we must determine which solution of (5.191) is the correct one. Now, the random matrix integral is convergent for small positive g, and the physical partition function is obtained by an analytical continuation to negative g (to make the first factor of (5.184) positive). For small g the expansion of (5.191) is

$$gt = -g\nu(t) + gc_2^2\nu(t) + O(g^2), \tag{5.197}$$

where we remember that it is $\nu(t) = g^{-1}v(t)$ which is well behaved in the Gaussian $g \to 0$ limit. Since $v(t)$ only depends on t and g through the combination gt, we then have

$$v(t) = \frac{gt}{c_2^2 - 1} + O\big((gt)^2\big). \tag{5.198}$$

The picture is then like this: When

$$g^* < g < 0 \tag{5.199}$$

$v(t)$ is a regular function of $t \in [0,1]$. Since $v(t)$ is the inverse of the function $w(v)$, it follows that $v(t)$ must be monotonous. By (5.198) it starts at $v(t = 0) = 0$ and is decreasing. As $g \to g^*$ the end point $v = v(t = 1)$ moves towards the critical point v^*. We conclude that the physical critical point of v is the first one among (5.196) you reach starting from $v = 0$ and moving towards $v = -\infty$. This narrows the possibilities down to

$$v^* \in \left\{ -\frac{1}{3}, \frac{1}{3}(1 - \sqrt{c_2}) \right\}, \tag{5.200}$$

depending on whether $c_2 < 4$ or $c_2 > 4$. One can check that keeping c_2 fixed and $\neq 4$ leads to either the high temperature or low temperature limit of the Ising model as $g \to g^*$.[61] Here, we are interested in the neighborhood of

$$c_2^* = \tilde{c}_2^* = 4, \tag{5.201}$$

where the two points (5.200) collide, and the Ising model becomes critical.

Inserting the critical values

$$v^* = -\frac{1}{3} \tag{5.202}$$

and (5.201) into (5.191) (with $t = 1$) we get

$$g^* = -\frac{10}{9}. \tag{5.203}$$

To proceed along the lines of Sec. 5.2.2 we need to know the spectrum of operators. We expect that the critical point corresponds to the Ising CFT, which is the minimal model with $(p, q) = (3, 4)$ introduced in Sec. 5.1.2. The non-trivial spinless primary operators are the spin operator $\sigma := \mathcal{O}_{1,2}$ and the energy operator $\varepsilon := \mathcal{O}_{1,3}$. The gravitational dimensions are (see Eq. (5.97))

$$1 - \Delta_{1,2} = \frac{5}{6}, \qquad 1 - \Delta_{1,3} = \frac{1}{3}. \tag{5.204}$$

Let us denote by T and H the coupling constants of ϵ and σ respectively (note that this H is different from the bare quantity in (5.187), also we have dropped the hat on the coupling constants compared to Sec. 5.2.2). We should now take g, c_2 and $\tilde{c}_2$ as the most general[12] regular functions of $a^2\mu$, $a^{2/3}T$ and $a^{5/3}H$,

$$g = g^* + k_1 a^{2/3}T + k_2(a^{2/3}T)^2 + k_3(a^{2/3}T)^3 + k_4 a^{5/3}H + k_5 a^2\mu + O(a^{7/3}), \tag{5.206}$$

$$c_2 = c_2^* + k_6 a^{2/3}T + k_7(a^{2/3}T)^2 + k_8(a^{2/3}T)^3 + k_9 a^{5/3}H + k_{10} a^2\mu + O(a^{7/3}), \tag{5.207}$$

$$\tilde{c}_2 = \tilde{c}_2^* + k_{11} a^{2/3}T + k_{12}(a^{2/3}T)^2 + k_{13}(a^{2/3}T)^3 + k_{14} a^{5/3}H + k_{15} a^2\mu + O(a^{7/3}). \tag{5.208}$$

The string susceptibility is (see Eq. (5.98))

$$\gamma_s = -\frac{1}{3}, \tag{5.209}$$

so we demand that the leading term of v should scale as $a^{2/3}$,

$$v(t=1) = v^* + a^{2/3}u + O(a). \tag{5.210}$$

Plugging these ansätze into (5.191) we obtain

$$12u^3 - \frac{9k_1}{4}Tu^2 + \frac{(k_{14}-k_9)^2 H^2}{108u^2} - \left(\frac{9k_1^3}{128} + \frac{3k_1k_2}{8} + \frac{3k_3 + k_8 + k_{13}}{3}\right)T^3 = \frac{k_{10} + k_{15} + 3k_5}{3}\mu, \tag{5.211}$$

to leading order in a, provided that[13]

$$k_1 = -\frac{k_6 + k_{11}}{3}, \quad k_2 = -\frac{(k_6 + k_{11})^2}{48} - \frac{k_7 + k_{12}}{3}, \quad k_4 = -\frac{k_9 + k_{14}}{3}, \tag{5.213}$$

$$k_{11} = k_6 \qquad \text{and} \qquad k_{12} = k_7. \tag{5.214}$$

We can rescale the coupling constants to obtain[26]

$$u^3 + Tu^2 + \frac{H^2}{u^2} + \kappa T^3 = \mu, \tag{5.215}$$

where κ is some unenlightening function of the k_j. The fifth order equation (5.215) essentially encodes the full non-perturbative spherical partition function of the Ising model coupled to 2d gravity.

[12]One could argue that the ansatz should be invariant under the combined transformation

$$H \to -H, \quad c_2 \to \tilde{c}_2, \quad \tilde{c}_2 \to c_2, \tag{5.205}$$

corresponding to the $\mathbb{Z}_2$ symmetry of the Ising model. However, one can check that with this restriction does not change the end result (5.215).

[13]If these equations are not satisfied, there is no consistent continuum limit. For example, the leading term in the expansion of (5.191) is

$$0 = \frac{(k_{11} - k_6)T^2}{108u^2}, \tag{5.212}$$

so we need to have $k_6 = k_{11}$ if we want to turn on T.

The appearance of the parameter κ in (5.215) is a reflection of the resonance ambiguity mentioned in Sec. 5.2.2. In this particular case it arises because the dimension of μ and T^3 coincide. It is customary to fix κ as follows: We first solve (5.215) for u as an expansion in T and H,

$$u = \mu^{1/3} - \frac{1}{3}T + \frac{1}{9}\frac{T^2}{\mu^{1/3}} - \frac{1}{3}(\frac{2}{27} + \kappa)\frac{T^3}{\mu^{2/3}} + O(T^4) + O(H^2). \tag{5.216}$$

By Eq. (5.179) the coefficient of T^3 should be interpreted as the correlation number

$$\frac{\partial^2}{\partial\mu^2}\langle \textstyle\int \varepsilon \int \varepsilon \int \varepsilon \rangle_\mu. \tag{5.217}$$

For the Ising model the correlation function of an odd number of ε operators is zero. It is natural to demand that the same fusion rule holds for the DT correlation numbers.[71] Looking at (5.216) we see that we must set

$$\kappa = -\frac{2}{27} \tag{5.218}$$

to satisfy the fusion rules. One can show that, at this value of κ, all the fusion rules of the Ising models are satisfied[71]. Note that the T term does not have an unambiguous continuum interpretation, since it is regular in μ. In fact, the only regular term (in μ, T, H) with the same dimension as u is T. We conclude from Eq. (5.179) that the relation between u and the continuum partition function is

$$\frac{\partial^2 \mathcal{Z}(\mu, T, H)}{\partial\mu^2} = u(\mu, T, H) + \frac{1}{3}T, \tag{5.219}$$

where $u(\mu, T, H)$ is the solution to (5.215). Here we have fixed the regular function by enforcing the fusion rule $\langle \int \varepsilon \rangle = 0$.

As a last point, let us note that it is not consistent to let $\{g, c_2, \tilde{c}_2\}$ depend only linearly on T. Indeed, if

$$k_2 = k_3 = k_7 = k_8 = k_{12} = k_{13} = 0 \tag{5.220}$$

then the consistency relations (5.213) and (5.214) imply that

$$k_1 = k_6 = k_{11} = 0, \tag{5.221}$$

but then T drops out completely in the continuum limit.

5.4 The dimer model

As a second example, we turn to the dimer model coupled to a dynamical triangulation[81]. The critical point of the matter theory is known as the Yang-Lee (or Lee-Yang) edge singularity[88, 64]. It was first identified as a critical point of the Ising model in an imaginary magnetic field, but it was later shown that the dimer model has a critical point in the same universality class[52, 63].

As for the Ising model, we consider a surface made of squares. A pair of neighboring squares can be occupied by a dimer, but each square can host at most one dimer. When there is no dimer covering a square, we say that it has a monomer[14]. We define the model by the potentials

$$U(x) = \frac{1}{6}x^6 + \frac{c_4}{4}x^4 + x^2, \qquad V(y) = \frac{1}{2}y^2. \tag{5.222}$$

Since the $V(y)$ potential is quadratic, the integral over the Y matrix is Gaussian, so we effectively have a one-matrix model. We use a two-matrix formulation here, because this allows us to use the formulae of Sec. 5.2.1. The x^4 term represents squares with monomers, while the x^6 term represents two squares joined by a dimer. Note that the dimer exclusion rule is automatically enforced by this formulation.

The weight associated to a triangulation is

$$(-gc_4)^{\#_m}(-g^2)^{\#_d} = (-gc_4)^{\#_4}\left(-\frac{1}{c_4^2}\right)^{\#_d}, \tag{5.223}$$

where $\#_m$ is the number of monomers, $\#_d$ is the number of dimers and $\#_4 = \#_m + 2\#_d$ is the number of squares. We recognize $-1/c_4^2$ as the dimer fugacity, and $-\ln(-gc_4)$ as the bare cosmological constant. We will, however, see that the continuum cosmological constant couples differently to the geometry.

The solution of the model using orthogonal polynomials is similar to that of the Ising model, only simpler. Eqs. (5.137) and (5.138) become

$$\begin{aligned}\frac{gt}{z} + O\left(\frac{1}{z^2}\right) = &\left(1 - \frac{\tilde{Q}_{-5}(t)}{v(t)^5}\right)z^5 \\ &+ \left(c_4 + 5Q_{-1}(t) - \frac{\tilde{Q}_{-3}(t)}{v(t)^3}\right)z^3 \\ &+ \left(2 + 3c_4Q_{-1}(t) + 10Q_{-1}(t)^2 - \frac{\tilde{Q}_{-1}(t)}{v(t)}\right)z \\ &+ (10Q_{-1}(t)^3 + 3c_4Q_{-1}(t)^2 + 2Q_{-1}(t) - v(t))\frac{1}{z} \\ &+ O\left(\frac{1}{z^2}\right),\end{aligned} \tag{5.224}$$

and

$$\begin{aligned}\frac{gt}{z} + O\left(\frac{1}{z^2}\right) = &\left(1 - \frac{Q_{-1}(t)}{v(t)}\right)z \\ &+ (\tilde{Q}_{-1}(t) - v(t))\frac{1}{z} + O\left(\frac{1}{z^2}\right).\end{aligned} \tag{5.225}$$

From the second equation we get

$$Q_{-1}(t) = v(t). \tag{5.226}$$

Plugging this into the first equation, we are left with

$$gt = w(v(t)), \tag{5.227}$$

[14] For this reason the model is also known as the monomer-dimer model.

where

$$w(v) := 10v^3 + 3c_4 v^2 + v. \tag{5.228}$$

The points where $w'(v) = 0$ are

$$v = -\frac{1}{30}\left(c_4 \pm \sqrt{3}\sqrt{3c_4^2 - 10}\right). \tag{5.229}$$

We will be interested in the critical point

$$c_4^* = \sqrt{\frac{10}{3}} \tag{5.230}$$

where the two zeroes (5.229) collide. We then have

$$v^* = -\frac{1}{\sqrt{30}}, \qquad g^* = -\frac{1}{3\sqrt{30}}. \tag{5.231}$$

Note that, at the critical point, the cosmological term $(-gc_4)^{\#_4}$ is positive, while the fugacity $-1/c_4^2$ is negative. This is in accordance with our expectations, since the critical point of the dimer model on a flat lattice is also at negative fugacity. We can also check that the small t behavior of $v(t)$ is

$$v(t) = gt + O\big((gt)^2\big). \tag{5.232}$$

It follows that $v(t)$ will be monotonically decreasing, and thus will hit the singularity at v^*.

The Yang-Lee edge singularity is described by the minimal CFT with $(p, q) = (2, 5)$. There is only one non-trivial spinless primary operator, $\mathcal{O}_{1,2}$, with gravitational scaling dimension (Eq. (5.97))

$$1 - \Delta_{1,2} = \frac{3}{2}. \tag{5.233}$$

We denote by δ the coupling constant of $\mathcal{O}_{1,2}$. We thus set

$$g = g^* + k_1 a^2 \mu + k_2 a^3 \delta + O(a^4), \tag{5.234}$$

$$c_4 = c_4^* + k_3 a^2 \mu + k_4 a^3 \delta + O(a^4). \tag{5.235}$$

The string susceptibility is

$$\gamma_s = -\frac{3}{2}, \tag{5.236}$$

so the leading singularity of v should be of order a,

$$v(t = 1) = v^* + au + O(a^2). \tag{5.237}$$

With these anätze Eq. (5.227) becomes

$$10u^3 + \frac{k_4 - 10k_2}{10}\delta = 2\sqrt{30}k_1 \mu u \tag{5.238}$$

to leading order in a, given that the coefficients satisfy

$$k_3 = 10k_1. \tag{5.239}$$

By a rescaling of the coupling constants, we can completely eliminate the dependence on the k_j, and we simply have

$$u^2 + \frac{\delta}{u} = \mu. \tag{5.240}$$

Since δ is the most relevant coupling constant, the relation between u and the partition function is (Eq. (5.179))

$$\frac{\partial^2 \mathcal{Z}(\mu,\delta)}{\partial \delta^2} = u(\mu,\delta). \tag{5.241}$$

Note that there are no regular terms with the correct dimension.

A peculiar property of this model can be derived from the relation (5.239). If we expand the weight (5.223) (at $\delta = 0$) we find

$$(-gc_4)^{\#_m}(-g^2)^{\#_d} = (-g^* c_4^*)^{\#_m}\left(-(g^*)^2\right)^{\#_d} e^{-\mu 2\sqrt{30}k_1 a^2(\#_m+3\#_d)} + O(a^4). \tag{5.242}$$

We see that μ couples to the combination $\#_m + 3\#_d$, instead of the number of squares $\#_4 = \#_m + 2\#_d$. Taken at face value, this means that the area of the squares depends on the matter configuration, with a square occupied by a dimer having $3/2$ the area of a square with a monomer. This mixing between the geometry and matter is discussed further in Ch. 6.

Chapter 6

Paper: "A note on the Lee-Yang singularity coupled to 2d quantum gravity"

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A note on the Lee-Yang singularity coupled to 2d quantum gravity

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Abstract

We show how to obtain the critical exponent of magnetization in the Lee-Yang edge singularity model coupled to two-dimensional quantum gravity.

1 Introduction

Two-dimensional quantum Liouville gravity and the theory of random triangulations (or matrix models) most likely describe the same theory, two-dimensional quantum gravity coupled to conformal field theories with a central charge $c \leq 1$. The two realizations are sufficiently different that the "proof" that they describe the same theory is basically by comparing the result of calculations of certain "observables". The major problem of such a comparison has been to identify the observables to be compared in the two formulations. This problem has to a large extent been solved in [2] for one and two-point correlation functions and in [3] for three– and four-points correlation functions. Here we will address an observable, the so-called "magnetization" at the Lee-Yang edge singularity. We will show how the general assumptions of operator mixing put forward in [1, 2, 3] allow us to obtain agreement between the critical exponent of the Lee-Yang "magnetization" calculated in quantum Liouville gravity and using matrix models.

The rest of this article is organized as follows: in the next section we recapture how to calculate the magnetization exponent σ in the Ising model and at the Lee-Yang edge singularity using standard conformal field theory. In sec. 3 we then show how to reconcile Liouville and matrix model results.

2 Ising models and dimer models

The Ising model on an arbitrary connected graph G_V with V vertices and L links is defined by

$$Z_{G_V}(\beta, H) = \sum_{\{\sigma_i\}} \exp\Big(\beta \sum_{\langle ij \rangle=1}^{L} \sigma_i\sigma_j + H\sum_{i=1}^{V} \sigma_i\Big), \tag{1}$$

where the Ising spin σ_i (which can take values ± 1) is located at vertex i, $\langle ij \rangle$ symbolizes that vertices i and j are neighbors in G_V, and β and H signify inverse the temperature and a magnetic field, respectively.

If G_V is a regular two-dimensional lattice, e.g. a square lattice, the partition function $Z_{G_V}(\beta, H=0)$ has a second order phase transition for a certain value β_c in the limit $V \to \infty$. Let us calculate

$$\big\langle \mathrm{e}^{H\sum_i \sigma_i} \big\rangle_{\beta=\beta_c, H=0} = \mathrm{e}^{-F_{G_V}(H)}, \tag{2}$$

using the partition function $Z_{G_V}(\beta_c, 0)$. For large V the free energy $F_{G_V}(H)$ becomes extensive and the magnetization m is given by

$$F_{G_V}(H) = f(H)\, V\,(1+o(V)), \quad m = -\frac{df}{dH} \sim |H|^{\sigma}, \quad \sigma = \frac{1}{15}, \tag{3}$$

for small H.

The two-dimensional Ising model at its critical point β_c is a conformal field theory with central charge $c = 1/2$. Let us recall how the above result is derived using conformal field theory. Consider a conformal field theory and let Φ be a primary operator with scaling dimension Δ_0, i.e. $\Phi(\sqrt{\lambda}\, x) = \lambda^{-\Delta_0}\Phi(x)$ (we consider Φ to be the product of its holomorphic and anti-holomorphic parts, i.e. real). Under a scaling $x \to \sqrt{\lambda}\, x$ we thus have

$$A = \int d^2x \to \lambda\, A, \qquad D_0 = \int d^2x\ \Phi(x) \to \lambda^{1-\Delta_0} D_0. \tag{4}$$

We can study a "deformation" away from the conformal point by adding the term

$$\delta\ D_0 = \delta \int \mathrm{d}^2x\ \Phi, \qquad [\delta] = [A]^{\Delta_0 - 1} \tag{5}$$

to the action. The last equation in (5) states the dimension of the coupling constant δ in terms of the dimension of the area A of the 2d universe. As in eq. (2) we can write

$$\left\langle e^{-\delta D_0} \right\rangle_0 = \mathrm{e}^{-F_A(\delta)}, \tag{6}$$

where the average is calculated at the critical point. For large areas A we expect F_A to be extensive. For dimensional reasons we thus have, δ being the only coupling constant,

$$F_A(\delta) = f(\delta) A (1 + o(A)), \qquad f(\delta) = k\, \delta^{\frac{1}{1-\Delta_0}}. \tag{7}$$

The "Φ magnetization" is thus

$$m_\Phi = -\frac{df}{d\delta} \sim \delta^{\Delta_0/(1-\Delta_0)}, \qquad \text{i.e.}\ \ \sigma_\Phi = \frac{\Delta_0}{1-\Delta_0}. \tag{8}$$

Applying this to the spin operator $\Phi_{1,2}$ of the (3,4) minimal conformal field theory which has central charge $c = 1/2$ and corresponds to the Ising model, we have $\Delta_0 = 1/16$ and thus $\sigma_{\Phi_{1,2}} = 1/15$ in agreement with (3). For the (2,5) minimal conformal field theory which has $c = -22/5$ there is only one non-trivial primary operator, again $\Phi_{1,2}$, and $\Delta_0 = -1/5$. The corresponding magnetization exponent is $\sigma_{\Phi_{1,2}} = -1/6$.

Everything said above can be directly transferred to quantum Liouville gravity as long as we consider the partition function for a fixed area which we then take large to avoid finite size effects. More precisely, the partition function for a conformal field theory with central charge c coupled to the Liouville field and with the area of the 2d "universe" fixed to be A is defined as

$$Z_A = \int \mathcal{D}\varphi \mathcal{D}\psi\ \mathrm{e}^{-S_L(\varphi,\hat{g}) - S_c(\psi,\hat{g})}\ \delta\left(\int d^2x \sqrt{\hat{g}}\, \mathrm{e}^{\alpha\varphi} - A \right). \tag{9}$$

In (9) $S_c(\psi)$ is the matter action and $S_L(\varphi)$ the Liouville action. $\hat{g}_{ab}$ is a fiducial metric in the decomposition of the metric $g_{ab} = e^{\varphi}\hat{g}_{ab}$, thereby defining the Liouville field. Changing variables $\varphi \to \varphi + \rho$ in the functional integral allow us to obtain (for surfaces with spherical topology)

$$Z_A \sim A^{\gamma_0 - 3}, \qquad \gamma_0 = \frac{c - 1 - \sqrt{(25-c)(1-c)}}{12}. \tag{10}$$

For a given conformal field theory and a given primary field Φ, the observable D_0 defined above and the area A are changed to

$$D = \int d^2x \sqrt{\hat{g}}\, \mathrm{e}^{\beta\varphi}\Phi, \qquad A = \int d^2x \sqrt{\hat{g}}\, \mathrm{e}^{\alpha\varphi} I \tag{11}$$

In particular the area has become an observable on equal footing with D, associated with the (trivial) primary field I (the identity). The coefficients β, α are determined by the requirement that the observables D and A are invariant under diffeomorphisms and in 2d this implies that they are invariant under conformal transformations [11]. However, D still has a scaling dimension relative to the area A. Let us define the expectation value of an observable $\mathcal{O}$ for fixed area as

$$\langle \mathcal{O} \rangle_A = \frac{1}{Z_A} \int \mathcal{D}\varphi \mathcal{D}\psi\; \mathcal{O}\; \mathrm{e}^{-S_L(\varphi,\hat{g}) - S_c(\psi,\hat{g})}\; \delta\left(\int d^2x \sqrt{\hat{g}}\, \mathrm{e}^{\alpha\varphi} - A \right). \tag{12}$$

One has

$$\langle f(\lambda^{-\beta/\alpha} D) \rangle_{\lambda A} = \langle f(D) \rangle_A \tag{13}$$

for any function f. This follows by the change of integration variable $\varphi \to \varphi + \alpha^{-1} \log \lambda$ in the functional integral (12). In particular we have

$$\langle D \rangle_{\lambda A} = \lambda^{\beta/\alpha} \langle D \rangle_A, \quad \text{i.e.} \quad 1 - \Delta = \frac{\beta}{\alpha}, \tag{14}$$

by analogy with (4). The scaling dimension Δ is thus determined by α and β and is given by the KPZ formula [10]

$$\Delta = \frac{\sqrt{1 - c + 24\Delta_0} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}. \tag{15}$$

As in the ordinary conformal field theory case we can define the "magnetization" related to Φ by considering the perturbation away from the conformal point by the action

$$\delta\, D = \delta \int d^2x\, \sqrt{\hat{g}}\, \mathrm{e}^{\beta\varphi}\, \Phi, \qquad [\delta] = [A]^{\Delta - 1}, \tag{16}$$

in analogy with (5). As in (6) we have

$$\left\langle \mathrm{e}^{-\delta D} \right\rangle_A = \mathrm{e}^{-F_A(\delta)}, \qquad F(\delta) = f(\delta) A (1 + G(A)), \qquad f(\delta) = k\, \delta^{1/(1-\Delta)}. \tag{17}$$

The "magnetization" is thus

$$m = -\frac{df}{d\delta} \sim \delta^{\Delta/(1-\Delta)}, \quad \text{i.e.} \quad \sigma = \frac{\Delta}{1-\Delta}. \tag{18}$$

In the case of the Ising model (i.e. $c = 1/2$) coupled to the Liouville field the exponent Δ_0 changes from 1/16 to $\Delta = 1/6$ according to (16). Thus we find that σ_0 changes from 1/15 to $\sigma = 1/5$. This value was first obtained using the random matrix models in [6] and is a strong test of the equivalence between the continuum limit of the random surface models coupled to matter and quantum Liouville gravity. Applied to the (2,5) minimal conformal field theory coupled to the Liouville field, σ_0 changes from -1/6 to $\sigma = -1/3$.

Finally it can be convenient to consider the grand partition function where the area is not kept fixed

$$Z(\mu, \delta) = \int dA\, Z_A\, \mathrm{e}^{-\mu A} \langle \mathrm{e}^{-\delta D} \rangle_A \sim \left(\mu + k \delta^{1/(1-\Delta)} \right)^{2-\gamma_0}. \tag{19}$$

We obtain

$$Z(\mu, 0) \sim \mu^{2-\gamma_0}, \qquad Z(0, \delta) = \delta^{2-\gamma(\delta)}, \qquad \gamma(\delta) = \frac{\gamma_0 - 2\Delta}{1-\Delta}. \tag{20}$$

We also observe that if the action $\mu A + \delta D$ is viewed as a small perturbation away from the conformal point $\mu = \delta = 0$ and μ and δ are of the same order of magnitude, the singular behavior of $Z(\mu, \delta)$ is dominated by $\mu^{(2-\gamma_0)}$ if the scaling dimension $\Delta > 0$. If $\Delta < 0$, as can be the case for non-unitary conformal field theories, the singular behavior of $Z(\mu, \delta)$ will be dominated by $\delta^{(2-\gamma(\delta))}$. We note for future reference that for the (2,5) minimal conformal field theory $\gamma_0 = -3/2$ and $\gamma(\delta) = -1/3$. In a grand canonical context it is natural to define

$$Z(\mu, \delta) = \mathrm{e}^{-F(\mu, \delta)}, \quad \langle A \rangle_{\mu,\delta} = -\frac{dF}{d\mu}, \qquad M(\delta) = -\frac{dF}{d\delta} = m(\delta) \langle A \rangle_{\mu,\delta}, \tag{21}$$

and we have

$$\langle A \rangle_{\mu,\delta} = \frac{1}{\mu + k\delta^{1/(1-\Delta)}}, \qquad m(\delta) \sim \delta^{\Delta/(1-\Delta)}. \tag{22}$$

For a given value of δ we have

$$\langle A \rangle_{\mu,\delta} \to \infty \quad \text{for} \quad \mu \searrow \bar{\mu}(\delta), \tag{23}$$

where the condition

$$\bar{\mu}(\delta) + k\delta^{1/(1-\Delta)} = 0 \tag{24}$$

determines the "critical" value of the cosmological constant μ for a given value of δ. In particular we have

$$\frac{d\bar{\mu}}{d\delta} \sim m(\delta). \tag{25}$$

2.1 Dimers

Consider the Ising model on the graph G_V defined above. It has a high temperature expansion

$$\begin{aligned} Z_{G_V} \;&=\; (2\cosh H)^V (\cosh\beta)^L \times \\ &\Big[1 + \tanh^2 H[\theta(1)\beta + O(\beta^2)] + \tanh^4 H[\theta(2)\beta^2 + 0(\beta^4)] + \cdots\Big] \end{aligned} \tag{26}$$

where $\theta(n)$ is the number of ways one can put down n dimers on the graph G_V without the dimers touching each other (so-called hard dimers). For imaginary magnetic fields it is thus possible to take the high temperature limit where $\beta \to 0$ and $H = i\tilde{H} \to i\pi/2$ in such a way that $\xi = \beta\,\tanh^2 H$ is kept fixed. In this limit the terms in the bracket $[\cdots]$ in eq. (26) become the partition function

$$Z_{G_V}(\xi) = \sum_n \theta(n)\xi^n, \qquad (\xi = -\beta\,\tan^2\tilde{H}) \tag{27}$$

of a hard dimer model with fugacity ξ (which is negative for $\tilde{H} \in]0, \pi/2[$). For $\beta < \beta_c$ the Ising model is known to have a phase transition at a critical, purely imaginary magnetic field $H_c(\beta) = i\tilde{H}_c(\beta)$, the so-called Lee-Yang edge singularity [7] (assuming as before that we have a regular graph G_V, and that we take $V \to \infty$). It is also known that one can formally associate a "magnetization" to this transition [8]:

$$Z_{G_V}(\beta, \tilde{H}) = \mathrm{e}^{-F_{G_V}(\beta,\tilde{H})}, \qquad F_{G_V}(\beta,\tilde{H}) \sim f(\beta,\tilde{H})\,V, \tag{28}$$

where

$$m(\beta) = -\frac{df}{d(\Delta\tilde{H})} \sim (\Delta\tilde{H})^{\sigma_0}, \qquad \Delta\tilde{H} = \tilde{H} - \tilde{H}_c(\beta). \tag{29}$$

The critical exponent σ_0 is independent of β for $\beta < \beta_c$. $\tilde{H}_c(\beta) \to \pi/2$ for $\beta \to 0$ and at this point we can extract σ from the dimer partition function (27). The dimer model has a critical point ξ_c for a negative value of the fugacity ξ which is precisely the limit of $-\beta\tan^2\tilde{H}(\beta)$ for $\beta \to 0$. Writing

$$Z_{G_V}(\xi) = \mathrm{e}^{-F_{G_V}(\xi)}, \qquad F_{G_V}(\xi) = f(\xi)\,V, \tag{30}$$

we obtain

$$m = -\frac{df}{d\Delta\xi} \sim (\Delta\xi)^{\sigma_0}, \qquad \Delta\xi = \xi - \xi_c. \tag{31}$$

Finally it was shown in [9] that the critical behavior of the Lee-Yang edge singularity or the hard dimer model could be associated with the (2,5) minimal conformal field theory, and from the above arguments, using conformal field theory we know the corresponding $\sigma_0 = -1/6$. This is in agreement with numerical determinations of σ_0 on regular lattices.

Once this is established we can formally couple the Lee-Yang edge singularity to quantum gravity in the sense that the critical behavior is determined by the coupling between between the (2,5) conformal field theory and the Liouville theory. From the above we thus expect the magnetization exponent to change from -1/6 to -1/3, and we would naively expect to obtain that result if we could explicitly solve the Ising model in an imaginary magnetic field or the hard dimer model on the set of random graphs used to represent 2d gravity. In fact one can solve both models on random graphs and one obtains $\sigma = 1/2$ [4].

3 Operator mixing

Let us for simplicity choose to work with the dimer model and discuss how we can re-interpret the result of [4] using the general philosophy outlined in [1, 2, 3]. The coupling of the dimer model to 2d gravity is done by summing over connected random graphs G_V. Here we restrict ourselves to a set of planar graphs, i.e. we define

$$Z_V(\xi) = \sum_{G_V} \frac{1}{C_{G_V}} \; Z_{G_V}(\xi), \tag{32}$$

where C_G denotes the order of the automorphism group of the graph G. We can introduce a grand partition function by also summing over graphs with different number of vertices:

$$Z(g, \xi) = \sum_V g^V Z_V(\xi). \tag{33}$$

Let us choose the simplest set of planar random graphs, namely the set where all vertices have order four. The corresponding $Z(g, \xi)$ can be calculated using matrix model techniques [12, 4]. For details we refer to [4]. Here we are only interested in the result. There exists a critical ξ_c. For each $\xi \geq \xi_c$ there exists a corresponding critical $\bar{g}(\xi)$, the radius of convergence of the power series (33). We write

$$Z_V(\xi) = \mathrm{e}^{-F_V(\xi)}, \quad F_V(\xi) = f(\xi)V(1 + o(V)), \qquad \log \bar{g}(\xi) = f(\xi). \tag{34}$$

On a regular lattice one would clearly identify $f(\xi)$ as the free energy density and expect to calculated the critical exponent σ according to (31). This calculation was performed in [4]:

$$\bar{g}(\xi) = \frac{1}{450\xi^2}\Big[(1+10\xi)^{3/2}-1\Big] - \frac{1}{30\xi} \tag{35}$$

i.e. expanding around $\xi_c = 1/9$ one obtains

$$\Delta\bar{g}(\xi) + \frac{10}{9}\Delta\xi = \frac{20\sqrt{10}}{9}\,\Delta\xi^{3/2} + O(\Delta\xi^2), \tag{36}$$

where

$$\Delta\xi = \xi - \xi_c, \qquad \Delta\bar{g}(\xi) = \bar{g}(\xi) - \bar{g}(\xi_c). \tag{37}$$

Differentiating (36) after $\Delta\xi$ we obtain

$$\frac{df}{d\xi}\Big|_{singular} = \frac{d\log\bar{g}}{d\xi}\Big|_{singular} \sim \Delta\xi^{1/2}. \tag{38}$$

Clearly this is at odds with the KPZ value $\sigma = -1/3$ mentioned above for the Lee-Yang edge singularity. We now explain how this is due to operator mixing of A and D, following the logic outlined in [1, 2, 3].

Denote $\bar{g}(\xi_c)$ by g_c. The first observation is that [12, 4]

$$Z(g,\xi_c)\Big|_{singular} = \Delta g^{-1/3-2}, \quad \Delta g = g_c - g, \tag{39}$$

i.e. one obtains $\gamma(\delta)$ (= -1/3) rather than γ_0 (=-3/2) for the critical susceptibility exponent related Z. Naively one would have made the following identification in (33)

$$\left(\frac{g}{g_c}\right)^V \to \mathrm{e}^{-\mu A} \tag{40}$$

by introducing a scaling parameter a (with the dimension of length relative to A which we define to have the dimension of length squared)

$$\Delta g = \mu\, a^2, \quad A = V\, a^2, \quad a \to 0. \tag{41}$$

But this is clearly too simple as it would imply a critical behavior $\Delta g^{-\gamma_0-2}$ in (39) according to Liouville theory. Δg has to contain some reference to the coupling δ. In some sense this is natural since both A and D appear when we move away from the conformal point $\mu = \delta = 0$. Fixing $\xi = \xi_c$ and changing $g_c \to g_c - \Delta g$ is one way to move away from the point g_c, ξ_c corresponding to $\mu = \delta = 0$. The change (36) is another way, where we move along the critical line with a $\Delta\bar{g}(\xi)$ determined by $\Delta\xi$. It should thus be compared to (24) where $\bar{\mu}(\delta) + k\delta^{1/(1-\Delta)} = 0$, which defines "criticality" in the theory perturbed by the A, D terms in the action. This condition allows us to obtain the relation between $\mu\, a^2, \delta\, a^3$ and $\Delta g, \Delta\xi$ if we, in acordance with [1, 2, 3], assume that we deal with an analytic coupling constant redefinition. To lowest order, which is all we need, we thus have

$$a^2\,\mu = \Delta g(\xi) + c_2\Delta\xi, \qquad a^3\delta = c_3\Delta g(\xi) + \Delta\xi. \tag{42}$$

The condition $\bar{\mu} + k\delta^{2/3} = 0$ implies

$$\Delta\bar{g}(\xi) + c_3^{-1}\Delta\xi = c_3^{-1}\Big(k^{-1}(c_3^{-1} - c_2)\Big)^{3/2}\Delta\xi^{3/2} + O(\Delta\xi^2). \tag{43}$$

Comparing with (36) we obtain

$$a^3\,\delta = \Delta\xi + \frac{9}{10}\,\Delta\bar{g}(\xi), \quad a^2\,\bar{\mu}(\delta) = \Delta\bar{g}(\xi) + d\,\Delta\xi, \tag{44}$$

where $d = 10/9 - k(2\sqrt{10})^{2/3}$. This shows explicitly that Δg couples to δ as anticipated from eq. (39).

By construction we now have $\bar{\mu}(\delta) \sim \delta^{2/3}$ and thus the correct Liouville magnetization. Further, it is amusing to check how the "wrong" result (38) actually becomes correct if one pays attention to the details [1]. (38) is obtained by differentiating (36) after $\Delta\xi$. For the special linear combination (44) eq. (36) can be written as

$$a^3\,\delta(\Delta\xi, \Delta\bar{g}(\xi)) = \frac{20\sqrt{10}}{9}\,\Delta\xi^{3/2} + O(\Delta\xi^2), \tag{45}$$

and differentiating with respect to $\Delta\xi$ leads to

$$a^3\frac{d\delta}{d\Delta\xi} \sim \Delta\xi^{1/2} \quad \text{or} \quad \frac{d\delta}{d\bar{\mu}} \sim \bar{\mu}^{1/2} + O(a), \tag{46}$$

i.e. according to eq. (25) exactly the correct Liouville equation for the magnetization m if $\sigma = -1/3$.

As mentioned one can also solve the Ising model coupled to 2d gravity [5, 6]. The matrix models use the grand canonical ensemble of graphs, i.e. starting with the partition function (1) one performs the same steps as in eqs. (32) and (33) for the dimer model. We thus have a partition function $Z(g, \beta, H)$. Above the critical temperature we find a critical line with a critical imaginary magnetic field

[1] The author of [4] had no motivation to pay attention to details, since his work was done before the understanding of the possibility of operator mixing. In fact his seminal paper was precisely what eventually led to this understanding.

[4] $H_c(\beta) = i\tilde{H}_c(\beta)$, $\beta < \beta_c$, analogous to what we find on a fixed graph. For a fixed value of $\beta < \beta_c$ we have an equation similar to the dimer equation (36) [4]

$$\Delta\bar{g}(\tilde{H}) + d_3\Delta\tilde{H} \sim \Delta\tilde{H}^{3/2}, \quad \Delta\tilde{H} = \tilde{H} - \tilde{H}_c(\beta), \tag{47}$$

from which one would conclude that $\sigma = 1/2$. As for the dimer model, this should be understood as the result of operator mixing, and one should really write

$$a^2\,\bar{\mu} = \Delta\bar{g}(\tilde{H}) + d_2\Delta\tilde{H}, \qquad a^3\delta = d_3^{-1}\Delta\bar{g}(\tilde{H}) + \Delta\tilde{H} \tag{48}$$

in order to recover the KPZ exponent.

Let us briefly mention the ordinary critical point of the Ising model on a dynamical graph. The critical exponents calculated in [5, 6] match the KPZ results, even without accounting for mixing. Regarding σ (and γ_0) one can explicitly check that the naive calculation is unaffected by operator mixing (cf. the discussion after (20)). When the magnetic field is zero the model has a $\mathbb{Z}_2$ symmetry, which guarantees that the spin operator $\Phi_{1,2}$ is not turned on in the continuum language. This, in turn, ensures that also the exponent α associated with the thermal operator $\Phi_{2,1}$ comes out "right" in [6].

4 Discussion

We have shown how the calculation in [4] leads to agreement between the critical exponents of the "magnetization" calculated in the hard dimer model coupled to dynamical triangulations and in quantum Liouville theory coupled to a (2,5) minimal conformal field theory. The price of this agreement is that the naive separation between geometric and matter degrees of freedom which might seem self-evident for models of spins living on dynamical graphs can thus not be taken for granted.

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Chapter 7

Resumé på dansk

Emnet for denne afhandling er kvantegravitation i 1+1 dimensioner. Vi fokuserer på to formalismer: Causal Dynamical Triangulations (CDT) og Dynamical Triangulations (DT). Begge teorier regulariserer gravitations-vej-integralet som en sum over trianguleringer. Forskellen ligger i den klasse af trianguleringer som indgår i summen. Mens CDT-trianguleringerne har en naturlig Lorentz-struktur, er DT-trianguleringerne Euklidiske.

Afhandlingnen er opbygget omkring tre artikler, som vi gengiver som Kapitel 3, 4 og 6. Afhandlingens disposition er som følger: De to første kapitler indeholder baggrundsmateriale om vej-integral-kvantisering og CDT-formalismen. I Kapitel 3 betragter vi en generalisering af CDT (som blev introduceret i [43]) og viser, at kontinuumsgrænsen er den samme som for den almindelige CDT model. Det giver evidens for CDT-universalitetsklassens robusthed. Kapitel 4 giver en analyse af CDT koblet til Yang-Mills. I Kapitel 5 introducerer vi DT-formalismen og grundlæggende aspekter af Liouville-teori. Vi lægger særlig vægt på subtiliteter i forbindelse med kontinuumsgræsen. Vi afslutter, i Kapitel 6, med en diskussion af sammenblandingen mellem geometriske- og stofsfrihedsgrader, når DT er koblet til ikke-unitære CFTer.

Det meste af materialet i Kapitel 1, 2 og 5 er ikke originalt, og vi forsøger at give relevante hendvisninger. Kapitel 3 og 4 er skrevet med J. Ambjørn, mens Kapitel 6 er skrevet med J. Ambjørn, A. Görlich og H.-G. Zhang.

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