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PhD Thesis

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One-point Functions in AdS/dCFT and Integrability

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Abstract

Super Yang-Mills with a co-dimension one defect is studied, in particular, the field theory setup that arises in the D3-probe-D5 brane construction of the Karch-Randal idea. We look at the case where $k \geq 2$ D3-branes are absorbed by the D5, giving rise to a domain wall defect that separates the field theory into an $\mathrm{SU}(N-k)$ theory and a broken $\mathrm{SU}(N)$ theory. The defect allows for interesting one-point functions in the SU(2) sub-sector already at tree-level. One-point functions in this sub-sector are computed, key results include the closed determinant formula at tree-level valid for all k, and subsequently a concise one-loop result for $k = 2$. The one-loop result is conjectured to be exact for the BMN vacuum $\mathrm{tr}\,\Phi_1^L$. A major feat is the diagonalization of the bulk action around the fuzzy-funnel background, as it opens up for many novel tests of the AdS/dCFT correspondence. Results for the BMN one-point functions are compared with string theory in the double-scaling limit. Agreement is found at tree-level and subsequently an all loop conjecture is made based on integrability.

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1 Introduction

When a problem is getting overly complicated, it is usually a good idea to look at the simplest non-trivial example. $\mathcal{N} = 4$ Super Yang-Mills (SYM) is such an example in the space of four-dimensional non-Abelian gauge theories. It is a field theory that enjoys superconformal symmetry in four-dimensions. It possesses conformal symmetry, not only at the level of the action, but also at the level of quantum field theory – the coupling constant does not scale with energy, the beta function is identically zero. This implies that the coupling constant g_{YM} is dimension less, and ultimately implies that the theory lacks any notion of absolute distance or mass scale.

The theory was thought to be overly constrained by its symmetries, and thus not of interest, however recent advances have revealed that in a certain limit this theory is far from boring. The limit is one in which the rank N of the gauge group (for instance $\mathrm{SU}(N)$) is taken large while the coupling constant is sent to zero

$$N \to \infty, \quad g_{\mathrm{YM}} \to 0, \quad \text{while} \quad g_{\mathrm{YM}}^2 N = \text{ fixed.} \tag{1.1}$$

This is the so called planar limit and leads to a new theory called planar SYM whose interactions are governed by the 't Hooft coupling constant

$$\lambda = g_{\mathrm{YM}}^2 N. \tag{1.2}$$

The limit kills all non-planar Feynman graphs as they are suppressed by inverse powers of N as discovered by 't Hooft [1].

This limit of SYM is very special, as it is conjectured that planar SYM is exactly dual to free type-IIB strings floating around in an anti-de Sitter spacetime, namely the background of $\mathrm{AdS}^5 \times \mathrm{S}_5$. This is known as the weak formulation of the archetypal version of the AdS/CFT correspondence conjecture first by Maldacena in 1997 [2].

$$\mathcal{N} = 4 \text{ SYM on } \mathbb{R}^4 \leftrightarrow \text{Type IIB string theory on } \mathrm{AdS}_5 \times \mathrm{S}^5. \tag{1.3}$$

We will refer to the left hand side as the gauge (field) theory side, and the right hand side as the gravity (string) theory side.

There is a lot of excitement surrounding this correspondence due to its potential for applicability [3]. Firstly it relates a gauge theory without gravity to a candidate theory for quantum gravity, and secondly it is a strong-weak duality. The latter entails that when one side of the correspondence becomes difficult to tackle computationally (due to breakdown of perturbation theory), the other side becomes more manageable. This feature of the correspondence has been fruitful, notably in the study of condensed matter physics, but also in aspects of nuclear physics as well as the study of quantum chromodynamics (QCD). In these fields the correspondence has enabled researchers to translated hard to compute problems into more tractable string theory computations.

There is a large amount of literature on the AdS/CFT correspondence, it's triumphs and it's recent history, some pedagogical accounts are [4–6]. Some of the more recent developments involve the discovery of hidden integrable structures within the gauge theory. It is now reckoned that a key feature of the AdS/CFT correspondence in the planar/free limit, is its conjectured integrability. The study of integrability in the context of the AdS/CFT correspondence is vast. A standard starting point, and overview is the very useful review [7].

Integrability has proved to be a very fruitful tool when it comes to computing the spectrum of the theory at various loop orders. Brilliant accounts of the development, in terms of the construction of the dilatation operator of $\mathcal{N} = 4$ SYM, as well as integrability in the context of AdS/CFT correspondence are plentiful, here are two in particular [8, 9].

We will be focusing our attention on the applicability of integrability in a setup that takes a slight departure from the regular AdS/CFT correspondence. The departure is called AdS/dCFT, where the *d* is short for defect. As it turns out there are reasons to believe that certain probe branes on the gravity side, are dual to defects in the field theory. Specifically, we consider the Karch-Randall idea [10, 11], namely that inserting an $\mathrm{AdS}_4 \times \mathrm{S}^2$ probe D5 brane on the gravity side, is equivalent to having a domain wall defect of co-dimension one in the boundary field theory. This brane configuration is special when compared to other setups, in that it keeps in tact half the original supersymmetries.

There are two scenarios to consider in these types of setups. On the one hand, one may consider the case where all the D3 branes are intersecting with the probe brane, which we shall call the standard defect. On the other hand, when a number of D3 branes end on the probe the setup is much richer resulting in a domain wall defect. For the defect case, DeWolfe, Freedman and Ooguria have extensively developed both the gravity and the field theory side of the D3-probe-D5 system [12]. They also provided arguments for conformality of the theory, which have later been reaffirmed in [13]. For the domain wall defect scenario, the field theory has not yet been fully developed, however this is likely to be addressed in the near future [14].

Despite incomplete knowledge of the domain wall scenario, a lot of interesting computations can be carried out. When some of the D3 branes end on the D5, it turns out that the setup allows for non-trivial one-point functions of certain scalar operators at tree-level. Early investigations into this specific scenario were first carried out by Nagasaki et al. In [15] they consider the potential energy between a test particle and the domain wall on the field theory side and find agreement with string theory. Subsequently they look at one-point functions of certain chiral primaries at tree-level [16] and also find a match with string theory. Similar tests have been carried out for chiral primaries in the related D3-probe-D7 setup by Kristjansen et al. [17].

Making tests of AdS/CFT or in this case, the AdS/dCFT correspondence, is notoriously difficult due to the fact that it is a strong-weak duality. To overcome this difficulty it is useful to have other parameters at play. One of the first ways by which such an extra parameter was made available was in the study of long operators, that corresponded with rapidly spinning strings on the string theory side, the study by Berenstein, Maldacena and Nastase (BMN) [18]. When taking this angular momentum quantum number J large, and correspondingly increasing the length of the operators, the two sides of the correspondence start to overlap, and comparison becomes a possibility. We shall see in detail how a similar line of reasoning is what makes it possible to compare with string theory in the present D3-probe-D5 setup.

Such defect theories are interesting in their own right, as the presence of the defect gives rise to interesting new observables, such as one-point functions. The standard AdS/CFT correspondence makes it natural to conjecture that these one-point functions are dual to string states. Moreover, appropriate brane constructions result in defect theories that are relevant as descriptions of condensed matter systems, such as a mono-layer of graphene. In particular the D3-probe-D7 setup has been studied in depth as a means of giving a potentially useful strong coupling description via holography (AdS/dCFT) [19]. Although the defect that we will study is not intended to model graphene like the D3-probe-D7 setup, it will undoubtedly shed light on relevant aspects of such defect theories as a whole, and likely illuminate the general idea of AdS/dCFT. Indeed the D5 brane preserves half of the supersymmetries while the D7 brane breaks all of them. Among other things, this means that the D3-probe-D5 setup potentially retains $\mathcal{N} = 2$ superconformal symmetry in 3 dimensions at the quantum level, which may constrain the theory enough to make it computationally tractable. As we will see in the thesis, there are signs of integrability, even beyond tree-level.

The present thesis is devoted to further elucidating the domain wall defect scenario. Specifically, we will dive deeper into the computation of one-point functions, considering general non-protected operators in the SU(2) sector. Computations will be carried out at both tree-level and loop-level, and comparisons with string theory will be made whenever possible. Major results include the closed determinant formulas at tree-level for one-point functions in the full SU(2) sector, the expansion of the SYM action around the fuzzy-funnel background and an all loop conjecture.

We will start off with some short remarks on integrability in the context of planar SYM, followed by motivating and introducing the details of the D3-probe-D5 brane setup. We then proceed to the one-point functions and compute their expectation values at tree-level. We shall see that these computations boil down to computing overlaps between matrix product states (familiar to condensed matter physics) with Bethe states. The most important parameter to us will be k - the number of D3 branes that end of the D5. This parameter is both a blessing and a curse, in that it both allows for stringy comparison, but also presents the greatest computational difficulties. Despite computational hurdles, a recursion relation will be proved which subsequently gives a closed determinant formula valid for any $k \geq 2$. The final sections of the thesis are devoted to the loop-corrections. These computations firstly require expanding the bulk action around the VEVs ϕ_1, ϕ_2, ϕ_3, and secondly the diagonalization of the resulting mass mixing matrix using fuzzy spherical harmonics. The loop computations are outlined and compared with string theory followed by conclusions and an outlook. Published papers are appended at the very end.

2 Integrability

As a feature of a theory, integrability is very much sought after as it in principle means that the theory is in some sense exactly solvable. To understand what this might look like for a theory, consider the basic example of such a theory, say the Heisenberg $\text{XXX}_{1/2}$ spin-chain.

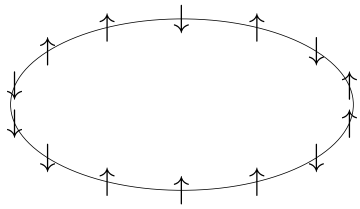


Figure 1. Periodic $\text{XXX}_{1/2}$ spin chain state example, with 14 spin sites.

The number of physical degrees of freedom of the system (L number of spin-cites), is constrained by an equivalent number of symmetries. One of these being the Hamiltonian of the system itself. This feature sets enough constraints on the spectrum of the theory, so that in practice the Hamiltonian is readily diagonalized. In the present case, by a simple plane wave type ansatz as first discovered by Bethe in 1931 [20]. Without the Bethe Ansatz it would be unfeasible to consider the spectrum of spin chains states of chains with very many site.s, i.e $L \gg 1$.

Two dimensional systems have a tendency to be integrable since their kinematics are simple - all scattering processes usually factor into two-body scattering processes (a hallmark of integrability). It is far less trivial to see how planar SYM is integrable. However, the fact that planar SYM is conjecture to be dual to type IIB free strings hints at the origin of it's integrability. The duality makes integrability seem more likely since the dual strings are described by a two-dimensional worldsheet sigma model whose integrability is not as far fetched. Indeed it has been found that classical strings on $\text{AdS}_5 \times \text{S}^5$ are integrable [21]. For a relevant review on the integrability of the $\text{AdS}_5 \times \text{S}^5$ superstring see [22].

The integrability is primarily motivated by the successes of numerous computations that have been carried out under the assumption that planar SYM is integrable. Famous examples include the computation of the cusp anomalous dimension [23] and that of the Konishi operator. Both of which have yielded results that smoothly interpolate between weak and strong coupling. However, as of yet, to the understanding of the author, there is no strict proof. A true proof would involve showing that the theory is invariant with respect to the Yangian, the associated quantum algebra of the underlying symmetries, in the case of planar $\mathcal{N} = 4$ SYM the relevant Yangian is $Y[\text{PSU}(2, 2|4)]$. Recent progress in this direction of a proof has been made [24].

Although the integrability of planar $\mathcal{N} = 4$ SYM may not have a strict proof as of yet. The evidence is mounting, starting with the dilatation operator, that in itself can be viewed as the Hamiltonian of an underlying integrable spin-chain. The integrability of which greatly simplifies the problem of operator mixing. We will see in detail how this works out in the SU(2) sector of single trace operator, which is the simplest case. Amazingly the story that we will present in the SU(2) sector is only a small part of the bigger picture, in which the complete one-loop dilatation operator is realized as the Hamiltonian of an underlying SU(2, 2|4) super spin chain [25]. Beyond one-loop, evidence suggests that the corrections do not spoil integrability as their integrability breaking terms are precisely canceled by terms at the loop order beyond it. Therefore it is believed that the complete all-loop dilatation operator should correspond to the Hamiltonian of a long-range integrable spin chain [26].

3 The D3-probe-D5 Brane-Intersection

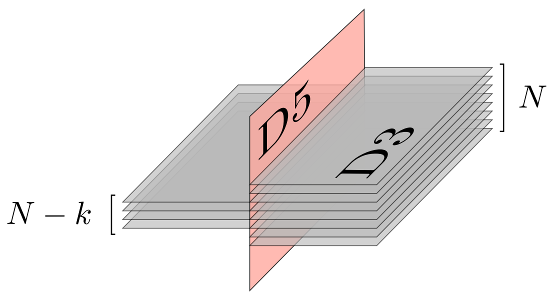


Figure 2. A depiction of how the D5 brane intersects with stacks of D3 branes (gray). The D3 branes are shown separated, in reality they are coincident!

Various brane configurations naturally arise in the setting of 10-dimensional super string theory. A well studied example is a stack of coincident D3-branes, as they make up the ingredients of the fruitful AdS/CFT correspondence. Succinctly, the correspondence is one between two equivalent descriptions of the brane configuration. On the one hand the open strings ending on the D3-branes see it as a super-conformal field theory $\mathcal{N} = 4$ SYM, whilst on the other hand, the closed strings propagating near the D3 branes see them as a black brane, and feel the near horizon geometry which is that of $AdS_5 \times S^5$. The brane configuration that we will consider is the D3-probe-D5

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D5	×	×	×		×	×	×			
D3	×	×	×	×						

Table 1. The dimensions along which the branes are extended. An × indicates that the dimension is occupied by the brane, whilst empty indicates that it is not.

setup, with worldvolume embedding coordinates show in table 1. Stack's of D3-branes are bisected by a probe D5-brane.

As shown in figure 2, we will be considering the case where k of the N D3 branes end on the D5. We will consider a probe brane computation that supports the idea that the D5 brane is wrapping an $AdS_4 \times S^2$ submanifold of $AdS_5 \times S^5$, as depicted in figure 3. This computation will show us how the parameter k appears on the string theory side of the correspondence.

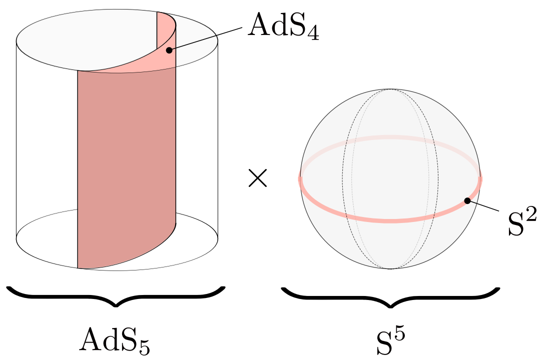


Figure 3. Visualization of the D5 embedding in $AdS_5 \times S^5$.

We follow Karch and Randall who found in investigations of locally localized gravity [10, 11, 27] that in the near horizon limit of this kind of D3-D5 brane system one would have an AdS_4 brane living inside the AdS_5 submanifold. Specifically the D5-brane wraps the S^2 given by $R^2 = x_4^2 + x_5^2 + x_6^2$, and is stretched along the slice $x_3 = 0$. To see this, we first note that the background metric of $AdS_5 \times S^5$ can be written in terms of the embedding coordinates as

$$ds^2 = R^2 \left(-\frac{1}{v^2} dv^2 + v^2 \left(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \right) + d\Omega_5^2 \right) \tag{3.1}$$

with

$$v^2 = \frac{1}{R^4}(x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2), \quad R^2 = \ell_s^2\sqrt{4\pi g_s N}, \tag{3.2}$$

where ℓ_s is the string length scale. The string coupling g_s relates to $g_{\rm YM}$ as

$$4\pi g_s = g_{\rm YM}^2 \tag{3.3}$$

hence

$$R^2 = \ell_s^2\sqrt{\lambda}. \tag{3.4}$$

Following [11], the action relevant (in the large λ strong coupling limit where classical string theory applies) from describing the probe D5 brane consists of the Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) terms

$$S_{\rm D5} = -T_5 \int \sqrt{-\det(G + \mathcal{F})} + T_5 \int \mathcal{F} \wedge C_4. \tag{3.5}$$

Here $\mathcal{F} = 2\pi\ell_s^2 F$ is the D5 world volume gauge flux and G is the induced world-volume metric. We have only background flux through S^2, i.e

$$\int_{\mathrm{S}^2} F = 2\pi k, \qquad F = \tfrac{1}{2}k\,\mathrm{vol}_{\mathrm{S}^2}. \tag{3.6}$$

In addition we also have the N units of five-form flux stemming from the D3 branes that carry charge associated with he Ramond-Ramond four-form potential

$$C_4 = R^4 v^4 dx \wedge dy \wedge dz \wedge dt. \tag{3.7}$$

One may then check an ansatz for the D5 embedding of the form

$$v = v(x_3). \tag{3.8}$$

As is readily apparent, the action $S_{\rm D5}$ factorizes, in particular

$$\det(G + 2\pi\ell_s^2 F) = \det G_{\mathrm{AdS}_4} \times \det(G_{\mathrm{S}^2} + 2\pi\ell_s^2 F). \tag{3.9}$$

The induced world-volume metric G is readily computed from the background metric (3.1), and one finds

$$ds_G^2 = R^2 v^2 (dx_0^2 - dx_1^2 - dx_2^2) - \frac{R^2}{v^2}\left(v^4 + (\partial_3 v)^2\right) dx_3^2. \tag{3.10}$$

The determinants are also straight forward to compute and one finds that the world-volume action (3.5) is equivalent to the four-dimensional action on the profile $v(x_3)$

$$S_{\rm D5}\Big|_{v=v(x_3)} = -4\pi R^4\, T_5 \int d^4x \left(v^2\sqrt{(R^4 + \pi^2 k^2 \ell_s^4)(v^4 + (\partial_3 v)^2)} - \pi k \ell_s^2\, v^4\right). \tag{3.11}$$

The resulting equation of motion for v can then be calculated from the variational principle, and one finds that it reads

$$\frac{v^4 + \frac{1}{2}(\partial_3 v)^2}{\sqrt{v^4 + (\partial_3 v)^2}} - \frac{\kappa\, v^2}{\sqrt{1+\kappa^2}} = \frac{1}{4\, v}\frac{d}{dx_3}\frac{v^2\, \partial_3 v}{\sqrt{v^4 + (\partial_3 v)^2}}. \tag{3.12}$$

As noted by Karch and Randall, this indeed has a particularly simple solution reading

$$v(x_3) = \frac{\kappa}{x_3}, \qquad \kappa = \frac{\pi k}{\sqrt{\lambda}}. \tag{3.13}$$

Note that the relation between R, ℓ_s and λ (and thus also κ and k) is given in (3.4).

This analysis shows that the AdS_4 part of the D5 brane embedding intersects the stack of D3 branes at an angle κ. The interpretation is that the excess k number of D3 branes pull on the D5, bending it into the x_3 direction, the angle being proportional to the number k of extra D3 branes as well as the ratio of brane tensions

$$\frac{T_3}{T_5} = 4\pi^2 \ell_s^2. \tag{3.14}$$

4 The Field Theory

On the one hand, from the open string perspective, the D3-probe-D5 intersection can be viewed as $\mathcal{N} = 4$ SYM with a co-dimension one defect. Whilst on the other hand, the closed strings see a probe brain in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Specifically the field theory setup is that of a domain wall (the defect), that separates the $\mathcal{N} = 4$ SYM theory in two half spaces figure 4. On the left ($x_3 < 0$), we have gauge group $\mathrm{SU}(N-k)$ while on the right ($x_3 > 0$) we have (broken, or Higgsed) $\mathrm{SU}(N)$.

The defect is special in the sense that it preserves half of the supersymmetries of the bulk, which turns out to be the maximum number of supersymmetries, while simultaneously breaking translational invariance in the co-dimension. This is ensured by the vacuum expectation values (VEVs) of the scalars satisfying the relevant Nahm's equations.

Setups like this one, have much in common with the introduction of supersymmetric boundary conditions. In fact the problem of constructing supersymmetry preserving domain walls in $\mathcal{N} = 4$ SYM can be mapped to the construction of supersymmetry preserving boundary conditions. The map involves a simple ”folding” trick, that is, you copy paste the theory on the left over to the right, and discard the left hand side. In the present case, the associated boundary problem is that of describing boundary conditions of $\mathcal{N} = 4$ SYM with gauge group $\mathrm{SU}(N-k) \times \mathrm{SU}(N)$. The trick requires also to change the sign of the three scalars that support the domain wall. This point

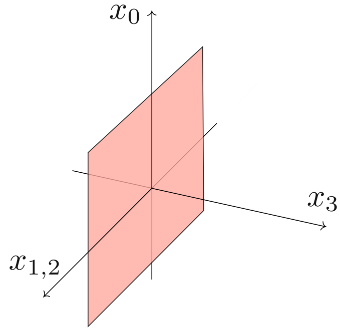


Figure 4. The co-dimension one defect is illustrated, it is located at $x_3 = 0$ and stretches into x_0, x_1, x_2. It separates the field theory into two regions ($x_3 > 0, x_3 < 0$).

is made by Gaiotto and Witten in [28] where they address supersymmetric boundary conditions in $\mathcal{N} = 4$ SYM. It is thus possible to construct a myriad of domain wall setups, similar to the one we are addressing, by simply unfolding boundary conditions on $\mathcal{N} = 4$ SYM with gauge group $G_1 \times G_2$.

The present setup is of particular interest in that it is one of the more minimal ways of departing from the regular AdS/CFT correspondence, while possibly still retaining a duality, an AdS/dCFT correspondence. To be clear, it has been conjectured that this domain wall of type $\mathrm{SU}(N-k)|\mathrm{SU}(N)$ is the field theory dual to the usual $\mathrm{AdS}_5 \times \mathrm{S}^5$ setup but with the introduction of a probe D5-brane that wraps an $\mathrm{AdS}_4 \times \mathrm{S}^2$ submanifold. That is to say that the field theory should describes the decoupling limit of the D3-probe-D5 brane-intersection. [1]

We will be dealing with computations on the field theory side of the AdS/dCFT correspondence. On the field theory side of this setup we have usual $\mathcal{N} = 4$ SYM in the bulk, coupled to a 3D boundary CFT

$$S = S_{\mathcal{N}=4} + S_{D=3}. \tag{4.1}$$

The $S_{D=3}$ part is not well understood for $k > 0$. However, the terms in the action proportional to the boundary theory will not play a role for us. We will at most be computing up to one-loop level for which the $S_{D=3}$ doesn't contribute. We will be focusing entirely on the $S_{\mathcal{N}=4}$ bulk action from now on.

[1] As pointed out in [29] the setup can be viewed as having holography acting twice!

The standard $\mathcal{N} = 4$ SYM action takes the form

$$S_{\mathcal{N}=4} = \frac{2}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4x \, \mathrm{tr} \Big[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} D_\mu\phi_i D^\mu\phi_i + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \frac{1}{2}\bar{\Psi}\tilde{\Gamma}^i[\phi_i, \Psi] + \frac{1}{4}[\phi_i, \phi_j][\phi_i, \phi_j] \Big]. \tag{4.2}$$

Here Greek indices run over the four space-time dimensions $\{0, 1, 2, 3\}$ whilst Latin indices take on the range $\{1, 2, 3, 4, 5, 6\}$ and are understood to implicitly correspond to the space-time dimensions $\{4, 5, 6, 7, 8, 9\}$ from the ten-dimensional perspective.

The spinors Ψ are ten-dimensional Majorana-Weyl spinors, and the gamma matrices $\{\Gamma^\mu, \tilde{\Gamma}^i\}$ satisfy the ten-dimensional Clifford algebra. The components of the field strength are what remain after the reduction of the ten-dimensional ancestors, and the rest have been demoted to scalar fields. The field strength and covariant derivative are given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \\ D_\mu\phi_i &= \partial_\mu\phi_i - i[A_\mu, \phi_i], \\ D_\mu\Psi &= \partial_\mu\Psi - i[A_\mu, \Psi]. \end{aligned} \tag{4.3}$$

The action enjoys the following supersymmetry

$$\begin{aligned} \delta A_\mu &= i\bar{\epsilon}\,\Gamma_\mu\Psi, \\ \delta\phi_i &= i\bar{\epsilon}\,\tilde{\Gamma}_i\Psi, \\ \delta\Psi &= \tfrac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon + D_\mu\phi_i\Gamma^{[\mu}\tilde{\Gamma}^{i]}\epsilon - \tfrac{i}{2}[\phi_i, \phi_j]\tilde{\Gamma}^{ij}\epsilon, \end{aligned} \tag{4.4}$$

which is simply the manifestation of

$$\begin{aligned} \delta A_I &= i\bar{\epsilon}\,\Gamma_I\Psi \\ \delta\Psi &= \frac{1}{2}\Gamma^{IJ}F_{IJ}\epsilon, \end{aligned} \tag{4.5}$$

in four-dimensions. More details on conventions for gamma matrices, and ten-dimensional Majorana-Weyl fermions can be found in appendix B.

The setup involves a co-dimensions one defect located at $x_3 = 0$ whose boundary conditions are such that we retain maximal supersymmetry. Since the commutator of two supersymmetries is a translation generator, it is impossible to preserve all the supersymmetry and at the same time break translation by introducing such a defect. At most one can hope for preserving half the supersymmetries. It will turn out that one of the minimal ways to break the full SUSY is to give three of the scalars $X_i = (\phi_1, \phi_2, \phi_3)$ VEVs that depend on the x_3 coordinate, while the rest of the fields are simply given zero VEVs, in particular $Y_i = (\phi_4, \phi_5, \phi_6) = \vec{0}$. This will break the full $\mathcal{N}=4$ SUSY which is PSU(4|4) to the unbroken subgroup OSp(4|4). The bosonic part of the remaining OSp(4|4) SUSY is $SO(3)_X \times SO(3)_Y \times SO(2,3)$. Where $SO(3)_X$ ($SO(3)_Y$) are the preserved R-symmetry of the symmetry breaking fields X_i (Y_i), and SO(2,3) is the group of conformal transformations that preserves the plane $x_3 = 0$ (the defect) [28]. The equations of motion subject to this type of ansatz reduce to

$$\partial_3^2 X_i = [X_j, [X_j, X_i]]. \tag{4.6}$$

Furthermore (4.4) implies the fermion condition

$$\delta\Psi = 0 \quad \Rightarrow \quad \partial_3 X_i \Gamma^{[3}\tilde{\Gamma}^{i]}\epsilon - \frac{i}{2}[X_i, X_j]\tilde{\Gamma}^{ij}\epsilon = 0. \tag{4.7}$$

It turns out that the only non-trivial way to preserve maximal supersymmetry is to impose Dirichlet boundary conditions on either X_i or Y_i [28]. Clearly we are considering imposing Dirichlet conditions on Y_i, and there is a particular scenario for which the VEVs X_i are compatible with half of the original supersymmetry, namely when they satisfy Nahm's equations:

$$\partial_3 \phi_i = -\frac{i}{2}\epsilon_{ijk}[\phi_j, \phi_k]. \tag{4.8}$$

It is relatively straight forward to see that the fermion condition then reduces to the following constraint [15] on the remaining supersymmetries

$$(1 - \Gamma^{[3}\tilde{\Gamma}^{123]})\epsilon = 0, \tag{4.9}$$

hence showing that indeed satisfying (4.8) is sufficient for maximal SUSY.

We will consider a particular solution to Nahm's equations (4.8) that also satisfies (4.6), namely

$$\phi_i^{\text{cl}} = -\frac{1}{x_3} t_i \oplus 0_{(N-k)\times(N-k)}, \qquad (x_3 > 0). \tag{4.10}$$

Here t_i are SU(2) generators of dimension $k \times k$ satisfying the algebra

$$[t_i, t_j] = i\epsilon_{ijk} t_k, \qquad i,j,k = 1,2,3. \tag{4.11}$$

These particular VEVs correspond to a discontinuity in the number of D3's on either side of the D5, in the brane setup. Specifically k counts the number of D3's ending on the D5, in other words, the world-volume flux is k.

This particular type of co-dimension one defect effectively splits the theory into two domains, hence it is often referred to as a domain wall. On the left ($x_3 < 0$) we have reduced gauge group $SU(N-k)$, whilst on the right ($x_3 > 0$) we have Higgsed $SU(N)$ gauge theory. We will be concerned with the region $x_3 > 0$, and discuss the effects of the Higgsing from the X_i VEVs in greater detail in subsequent sections. It will turn out to yield mass mixing matrices that mix color and flavor, and furthermore give rise to position dependent masses.

5 One-point Functions in the SU(2) Sector

Before computing one-point functions, some general remarks on operators and scaling dimensions are useful to be reminded of.

In a conformal field theory, there is no mass or length scale, instead of a mass spectrum, one speaks of a spectrum of scaling dimensions. The scaling dimension of an operator simply encodes how the operator transforms in response to a scaling of the coordinates:

$$\mathcal{O}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x). \tag{5.1}$$

In a free field theory, Δ is just a number and is equal to the engineering dimension of $\mathcal{O}$. For interacting conformal field theories, it is a function of the coupling constant, that is, the engineering dimension gets quantum corrections

$$\Delta(g) = \Delta_0 + \gamma(g), \qquad \gamma(0) \equiv 0, \tag{5.2}$$

where $\gamma(g)$ is the so called anomalous dimension.

There are several ways to compute scaling dimensions in a conformal field theory. Computing two-point functions of a specific operator will give it's scaling dimension, due to the relation

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{C(g)}{|x-y|^{2\Delta(g)}}, \tag{5.3}$$

where $C(g)$ is an unphysical normalization constant stemming from renormalization of the fields. Similarly one can gain information from computing higher n-point functions.

Another approach, is to consider the generator of dilatations D, which encodes the action of dilatations on the fields themselves, such that operators with well defined scaling dimension are eigenstates

$$D\mathcal{O}_\Delta = \Delta\mathcal{O}_\Delta. \tag{5.4}$$

For $\mathcal{N} = 4$ planar SYM the dilatation operator approach to computing the spectrum of the theory has proven particularly fruitful. Thanks to the rich symmetries of the theory, the full one-loop dilatation operator has been successfully constructed without the need to compute involved higher-loop field theory computations [30]. The dilatation operator approach was in particular sought after in the context of the BMN correspondence, as this correspondence deals with operators consisting of a large number of fields. The construction of a dilatation operator, transforms the problem of solving the spectrum into a purely algebraic one, rendering previously cumbersome field theory computations into comparatively straight forward combinatorial problems

[31]. This approach will be utilized in the following, when we compute one-point functions of operators of well defined anomalous dimension.

Much like two-point functions, one-point functions are constrained as well, thanks to conformal invariance. However, this constraint is usually too strong for one-point functions to be of interest. In our case though, thanks to the co-dimension one defect, the symmetry constraints are reduced and one-point functions may now depend on the distance to the defect

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{C(g)}{x_3^{\Delta(g)}}. \tag{5.5}$$

Although we have introduced a defect, we expect two-point functions far from the defect, that is, $|x-y| \ll x_3$ to coincide with those of regular SYM. It stands to reason that operators of well defined anomalous scaling dimensions in vanilla $\mathcal{N}=4$ are good candidate building blocks for studying the corresponding dCFT. Therefore we will be able to make heavy used of the discoveries already made regarding the dilatation operator.

We will consider single trace operators of the SU(2) sub-sector, that is

$$\mathcal{O}_L = \mathcal{O}^{i_1 i_2 \cdots i_L} \operatorname{tr}[\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_L}], \qquad \mathcal{O}^{i_1 i_2 \cdots i_L} \in \mathbb{C} \tag{5.6}$$

where

$$\Phi_1 = \phi_1 + i\phi_4, \quad \Phi_2 = \phi_2 + i\phi_5. \tag{5.7}$$

This is the simplest non-trivial sector of SYM to consider, as it is closed, meaning that these operators do not mix with other operators, and that this is valid to any order in perturbation theory. This is directly related to the fact that the full dilatation operator should commute with Lorentz generators and the R-symmetry generators implying that the dilatation operator can only mix operators of equal Lorentz or R charges. One can check that it is impossible to reproduce the R charges within the SU(2) sector from other operator constructions outside this sector [32].

We will use a normalization consistent with [16], this normalization is chosen such that two-point functions far from the defect are normalized to unity

$$\langle \mathcal{O}_L(x)\mathcal{O}_L(y) \rangle = \frac{1}{|x-y|^{2L}}. \tag{5.8}$$

Since the two point function of fundamental scalar fields is given by

$$\langle \phi_i(x)\phi_j(y) \rangle = \frac{g_{\mathrm{YM}}^2}{8\pi^2}\frac{\delta_{ij}}{|x-y|^2}, \tag{5.9}$$

it follows via standard computations like the one shown in [32], that we should normalize our operators (5.6) as follows

$$\mathcal{O}_L = \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda}\right)^{L/2} \Psi^{i_1 i_2 \cdots i_L} \, \mathrm{tr}[\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_L}], \qquad \lambda = g_{\mathrm{YM}}^2 N. \tag{5.10}$$

In this way the dependence on g_{YM} and N has been made explicit, that is to say, $\Psi^{i_1 i_2 \cdots i_L}$ are coefficients completely independent of N and g_{YM}.

We will be dealing with the quantum field theory perturbatively, that is, anything depending on the coupling constant will be a polynomial in

$$g^2 = \frac{\lambda}{16\pi^2}. \tag{5.11}$$

In particular we use the following notation for the loop expansion of the dilatation operator and correspondingly the scaling dimension

$$D = \sum_{n=0}^{\infty} D_n g^{2n}, \qquad \Delta = \sum_{n=0}^{\infty} \Delta_n g^{2n}. \tag{5.12}$$

A given loop order corresponds to a given order in g^2, such is the nomenclature, and it's origin lies in the Feynman diagrams.

As first realized by Minahan and Zarembo, in [33], finding operators of well defined anomalous dimension at one-loop ($O(g^2)$), is mapped to the familiar problem of diagonalizing the hamiltonian of a spin-chain. If we think of Φ_1 as representing $\uparrow$ and Φ_2 as representing $\downarrow$, it becomes natural to interpret $\mathcal{O}_L$ as states of a periodic spin chain of length L. Remarkably, it turns out that the one-loop dilatation operator is precisely the Hamiltonian of the simplest spin chain we know of, namely the Heisenberg $XXX_{1/2}$ spin-chain.

$$D_1 = 2\sum_{j=1}^{L} \mathbb{H}_{j,j+1}, \qquad \mathbb{H}_{j,j+1} = \mathbb{I}_{j,j+1} - \mathbb{P}_{j,j+1} \tag{5.13}$$

This spin-chain was first solved by a plane-wave ansatz made by Hans Bethe in 1931 [20]. We will refer to his approach as the coordinate formulation of the Bethe Ansatz, to distinguish it from the more modern algebraic approach pioneered by Faddeev [34]. We shall take a closer look at the algebraic approach later.

The upshot is that operators of well defined anomalous dimension have their coefficients $\Psi^{i_1 i_2 \cdots i_L}$ given by the position basis wave function coefficients of the energy eigenstates of the spin-chain. The operators are thus labeled by their total length L, number of magnons (excitations) M and as will become clear, their momenta (rapidities).

Let us start by introducing basis operators

$$\mathcal{O}_{L|n_1,n_2,\ldots,n_M\rangle} \tag{5.14}$$

that is, e.g

$$\mathcal{O}_{8|1,3,4,8\rangle} = \mathrm{tr}[\Phi_2\Phi_1\Phi_2\Phi_2\Phi_1\Phi_1\Phi_1\Phi_2] \quad \mapsto \quad |{\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow}\rangle. \tag{5.15}$$

Following in Bethe's footsteps, an operator of well defined anomalous dimension would then be a linear combination

$$\mathcal{O}_L(\{p_j\}) = \frac{1}{\sqrt{L}}\left(\frac{8\pi^2}{\lambda}\right)^{L/2} \sum_{1\le n_1<n_2<\cdots<n_M\le L} \Psi(\{p_j\},\{n_j\})\mathcal{O}_{L|n_1,n_2,\ldots,n_M\rangle}, \tag{5.16}$$

the coefficients are those of a plane wave ansatz with scattering phases

$$\Psi(\{p_i\},\{n_i\}) = N_\theta \sum_{\sigma\in S_M} \exp\left[i\sum_j^M p_{\sigma(j)}n_j + \frac{i}{2}\sum_{j<k}^M \theta_{\sigma(j)\sigma(k)}\right]. \tag{5.17}$$

Here S_M denote the set of permutations of M objects, and θ_{jk} are a useful parameterization of the scattering matrix

$$S_{jk} \equiv \exp[\theta_{jk} - \theta_{kj}] = -\frac{1 + e^{ip_j+ip_k} - 2e^{ip_j}}{1 + e^{ip_j+ip_k} - 2e^{ip_k}}. \tag{5.18}$$

However, we shall choose the normalization N_θ such that the $\sigma = \mathbb{1}$ term has a scattering factor equal to one, i.e

$$N_\theta = \exp\left[-\frac{i}{2}\sum_{j<k}^M \theta_{jk}\right]. \tag{5.19}$$

This makes the Bethe wave-function (5.17) explicitly dependent on the scattering matrix elements S_{jk} instead of the phases θ_{jk}, for instance

$$|p_1,p_2\rangle = \sum_{1\le n_1<n_2\le L}\left(e^{ip_1n_1+ip_2n_2} + S_{21}e^{ip_2n_1+ip_1n_2}\right)|n_1,n_2\rangle. \tag{5.20}$$

Subject to periodic boundary conditions, this ansatz only yields proper eigenstates when the momenta satisfy the Bethe Ansatz Equations.

$$e^{ip_jL} = \prod_{k\neq j}^M S_{jk}. \tag{5.21}$$

Which is often viewed as taking a single magnon and transporting it once around the chain, which should be equivalent to scattering it with all of the other magnons.

We will often parametrize the momenta $\{p_j\}$ in terms of so called rapidities $\{u_j\}$

$$p_j = \frac{1}{i} \ln \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}, \tag{5.22}$$

in terms of which (5.21) takes the form

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}. \tag{5.23}$$

This parameterization is the more natural one, and yields more compact expressions. It should be stressed that what has been discussed up to this point is nothing but the usual coordinate Bethe Ansatz in the context of operators in the SU(2) sector of SYM.

One-point functions of general unprotected operators of the SU(2) sector acquire non-trivial expectation values already at the classical level, thanks to the VEVs of ϕ_1, ϕ_2, ϕ_3, in particular

$$\begin{aligned} \langle \Phi_1 \rangle^{(\text{tree})} &= \langle \phi_1 \rangle^{(\text{tree})} = -\frac{1}{x_3} t_1 \oplus 0_{(N-k)\times(N-k)}, \\ \langle \Phi_2 \rangle^{(\text{tree})} &= \langle \phi_2 \rangle^{(\text{tree})} = -\frac{1}{x_3} t_2 \oplus 0_{(N-k)\times(N-k)}. \end{aligned} \tag{5.24}$$

Investigation such one-point functions is a natural starting point for investigating the present dCFT setup. At tree-level, expectation values of SU(2) basis states, like the example

$$\mathcal{O}_{8|1,3,4,8} = \text{tr}[\phi_2\phi_1\phi_2\phi_2\phi_1\phi_1\phi_1\phi_2], \tag{5.25}$$

simply equate to

$$\langle \mathcal{O}_{8|1,3,4,8} \rangle^{(\text{tree})} = \frac{1}{x_3^8} \text{tr}[t_2 t_1 t_2 t_2 t_1 t_1 t_1 t_2]. \tag{5.26}$$

6 Tree-level One-point Functions

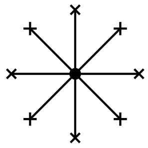


Figure 5. Feynman diagram at tree-level. The operator's location in space-time is represented by a dot whilst the cross' depict the insertions of classical fields. In this case the diagram is equivalent to the tree-level one-point function of an operator consisting of 8 fundamental fields.

From the discussion so far it is apparent that the tree-level computation of one-point functions of SU(2) operators of well defined anomalous dimension, boils down to computing linear combinations of traces whose coefficients are the Bethe coefficients (5.17)

$$\langle \mathcal{O}_L \rangle_{\text{tree}} = \mathcal{O}^{i_1 i_2 \cdots i_L} \, \text{tr}[\phi^{\text{cl}}_{i_1} \phi^{\text{cl}}_{i_2} \cdots \phi^{\text{cl}}_{i_L}]. \tag{6.1}$$

It can be useful to visualize these expectations values at this order in perturbation as done in figure 5. All the fundamental fields are evaluated at the same location in space-time, the location of the dot. The lines that connect a cross to the dot are not propagators. Instead they reflect the fact that each field is inserted into the trace and evaluated at the same location (the dot). This visual representation will be used later on in the section on one-loop corrections.

A concise way to represent this computation is as an overlap between a Bethe state and a matrix product state (MPS). To this end, let us first introduce notation for Bethe states, namely, we denote a position state of given length as

$$|n_1, n_2, \ldots, n_M\rangle_L, \quad \text{e.g} \quad |1, 3, 4, 8\rangle_8 = |{\downarrow}{\uparrow}{\downarrow}{\downarrow}{\uparrow}{\uparrow}{\uparrow}{\downarrow}\rangle. \tag{6.2}$$

A Bethe eigenstate of the $XXX_{1/2}$ spin chain is then denoted

$$|\{p_i\}\rangle_L = \sum_{\{n_i\}} \Psi(\{p_i\}, \{n_i\}) |n_1, n_2, \ldots, n_M\rangle_L, \tag{6.3}$$

which is precisely (5.16), just with a slightly more convenient notation. Now we can express the tree-level computation of the one-point functions as

$$\langle \mathcal{O}_L(\{p_i\}) \rangle^{(\text{tree})} = \frac{1}{x_3^L} \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda} \right)^{L/2} \frac{\langle \text{MPS} | \{p_i\} \rangle_L}{\langle \{p_i\} | \{p_i\} \rangle_L^{\frac{1}{2}}} \tag{6.4}$$

where

$$|\text{MPS}\rangle_L = \text{tr} \prod_{\ell=1}^{L} (t_1 \otimes |{\uparrow_\ell}\rangle + t_2 \otimes |{\downarrow_\ell}\rangle), \tag{6.5}$$

for instance

$$\langle \text{MPS}|0\rangle_4 = \text{tr}[t_1 t_1 t_1 t_1]. \tag{6.6}$$

From now on we will focus our attention to

$$\frac{\langle \text{MPS}_k|\{u_j\}\rangle}{\langle \{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}}, \tag{6.7}$$

where the subscript k indicates the important k dependence of the MPS while the subscript L has been dropped for the sake of brevity. The k dependence of the generators t_1, t_2, t_3 will only be indicated when necessary, by a superscript $t_i^{(k)}$. To the author's knowledge, computing an overlap between this particular matrix product state and Bethe states was first done in [35]. On the other hand, the norm's $\langle \{u_j\}|\{u_j\}\rangle$ are well known from the literature. Our Bethe states are normalized such that the norm takes the form

$$\langle \{u_j\}|\{u_j\}\rangle = \prod_{j=1}^{M} \left(u_j^2 + \tfrac{1}{4}\right) \det_{M\times M} \partial_m f_n, \qquad \partial_m = \frac{\partial}{\partial u_m} \tag{6.8}$$

where

$$f_m = -i \log \left[\left(\frac{u_m - \frac{i}{2}}{u_m + \frac{i}{2}} \right)^L \prod_{m\neq n}^{M} \frac{u_m - u_n + i}{u_m - u_n - i} \right]. \tag{6.9}$$

This was first conjectured by Gaudin [36] and later rigorously proved by Koripin [37]. Notice that f_m is simply the logarithm of the BAE equations (5.23).

Since it is not clear how to construct the matrix product state in the algebraic framework, we will stick to the coordinate framework for now. Our focus will be on computing

$$\langle \text{MPS}|\{p_j\}\rangle = \sum_{1\leq n_1 < n_2 < \cdots < n_M \leq L} \Psi(\{p_j\},\{n_j\})\, \text{tr}[t_3^{n_1-1} t_1 t_3^{n_2-n_1-1} t_1 \cdots t_1 t_3^{L-n_M}], \tag{6.10}$$

where we have taken the liberty to swap around the generators as this is a symmetry of the algebra. Working with t_3 and t_1 is easier.

We will later return to the algebraic framework, in which the Bethe states are constructed from creation operators

$$|\{u_j\}\rangle_{ABA} = \mathcal{B}(u_1)\mathcal{B}(u_2)\cdots\mathcal{B}(u_M)|0\rangle. \tag{6.11}$$

These operators stem from the transfer matrix,

$$\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}. \tag{6.12}$$

For more details see appendix A. For now we turn our attention to the coordinate framework. The connection between the algebraic formalism and the coordinate formalism with our chosen normalization N_θ is [35]

$$|\{u_j\}\rangle_{ABA} = \prod_j^M (u_j - \tfrac{i}{2})^L \left(\frac{i}{u_j + \frac{i}{2}}\right) \prod_{m<n}^M \left(1 + \frac{i}{u_m - u_n}\right) |\{p_j\}\rangle. \tag{6.13}$$

6.1 Properties of the MPS

There are several interesting properties of the MPS (6.5) and it's overlap with a generic Bethe state (6.10). Keep in mind the fact that for a given L the MPS contains terms of the form

$$\mathrm{tr}[t_{j_1} t_{j_2} \cdots t_{j_L}] |s_{j_1} s_{j_2} \cdots s_{j_L}\rangle, \qquad s_j \in \{\uparrow, \downarrow\}, \tag{6.14}$$

where the number of excitations M associated with a given term is allowed to be anything from 0 up to L. It can then be worked out that

1. The overlaps $\langle \mathrm{MPS}|\{p_j\}\rangle = 0$ when $P = \sum_j^M p_j \neq 2\pi\mathbb{Z}$
2. The coefficients $\mathrm{tr}[t_{j_1} t_{j_2} \cdots t_{j_L}] \neq 0$ only when both M and L are even.
3. The third charge kills it : $Q_3|\mathrm{MPS}\rangle = 0$
4. The MPS has positive parity : $\mathcal{P}|\mathrm{MPS}\rangle = |\mathrm{MPS}\rangle$

The first follows straight forwardly from trace-cyclicity and inserting the shift operator into the overlap

$$\langle \mathrm{MPS}|U|\{p_j\}\rangle, \qquad U = e^{iP}. \tag{6.15}$$

The result should be the same regardless of whether U acts to the left or the right, hence

$$P = \sum_{j=1}^M p_j = 2\pi n, \qquad n \in \mathbb{Z}. \tag{6.16}$$

The second property follows from the existence of two similarity transformations that act as trivial automorphisms on the $k \times k$ generators t_i [35]. The third property is a bit involved to arrive at, a proof can be found in [35]. The final property is a statement in terms of a so called parity operation

$$\mathcal{P}|s_1 s_2 \cdots s_L\rangle = |s_L s_{L-1} \cdots s_1\rangle, \qquad s_j \in \{\uparrow, \downarrow\}. \tag{6.17}$$

It implies that we only get contributions to the overlap from Bethe states satisfying

$$\mathcal{P}|\{u_j\}\rangle = |\{u_j\}\rangle. \tag{6.18}$$

The reason this holds has to do with the fact that a similar operation can be defined on the traces in the MPS. Together these properties imply that we only get non-trivial overlaps when the Bethe states are so called un-paired states, that is, when they are labeled by rapidities that come in pairs with opposing sign

$$|\{u_j\}\rangle, \qquad \{u_j\} = \{u_1, \dots, u_{\frac{M}{2}}, -u_1, \dots, -u_{\frac{M}{2}}\}. \tag{6.19}$$

Un-paired Bethe states are annihilated by all of the odd charges Q_{2n+1}.

6.2 Un-paired Bethe Norms

When restricting to un-paired states, the determinant in (6.8) nicely factors into a product of two determinants

$$\langle\{u_j\}|\{u_j\}\rangle = \prod_{j=1}^{M/2} \left(u_j^2 + \tfrac{1}{4}\right)^2 \det G^+ \det G^-. \tag{6.20}$$

This follows from the fact that

$$\partial_m f_n = \begin{pmatrix} G_1 & G_2 \\ G_2 & G_1 \end{pmatrix}, \tag{6.21}$$

where G_1 and G_2 are $\frac{M}{2} \times \frac{M}{2}$ matrices. In this way the determinant is simply

$$\det \partial_m f_n = \det G^+ \det G^-, \quad \text{where} \quad G^\pm = G_1 \pm G_2. \tag{6.22}$$

The G matrices have components given by

$$G^\pm_{jk} = \left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_{n=1}^{M/2} K^+_{jn} \right) \delta_{jk} + K^\pm_{jk} \tag{6.23}$$

where

$$K^\pm_{jk} = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2}. \tag{6.24}$$

6.3 Constructing a $k \times k$ Representation of SU(2)

It turns out that SU(2) generators t_1, t_2, t_3, in particular $k \times k$ matrices satisfying

$$[t_i, t_j] = i\epsilon_{ijk} t_k \tag{6.25}$$

can be written as sums of matrix unities. Matrix unities ${E^i}_j$ of dimension $k \times k$ have only one non-trivial component, namely a value of 1 at location (i, j), i.e, the 2×2 matrix unity

$$({E^1}_1)_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.26}$$

Matrix unities have the useful property that

$${E^i}_j {E^k}_\ell = {\delta^k}_j {E^i}_\ell. \tag{6.27}$$

The construction of the generators is as follows

$$\begin{gathered} t_+ = \sum_{j=1}^{k-1} c_{k,j} {E^j}_{j+1}, \qquad t_- = \sum_{j=1}^{k-1} c_{k,j} {E^{j+1}}_j, \qquad t_3 = \sum_{j=1}^{k} d_{k,j} {E^j}_j, \\ t_1 = \frac{t_+ + t_-}{2}, \qquad t_2 = \frac{t_+ - t_-}{2i} \end{gathered} \tag{6.28}$$

where

$$c_{k,j} = \sqrt{j(k-j)}, \qquad d_{k,j} = \tfrac{1}{2}(k - 2j + 1). \tag{6.29}$$

From (6.27) it is not difficult to show that the above constructed matrices indeed satisfy te SU(2) algebra for any value of $k \geq 2$.

The following matrices provide a trivial automorphism on the algebra

$$U = U^{-1} = \sum_{i=1}^{k} {E^i}_{k-i+1}, \qquad V = V^{-1} = \sum_{i=1}^{k} (-1)^i {E^i}_i, \tag{6.30}$$

$$U t_1 U^{-1} = t_1, \quad U t_{2,3} U^{-1} = -t_{2,3}, \quad V t_3 V^{-1} = t_3, \quad V t_{1,2} V^{-1} = -t_{1,2}. \tag{6.31}$$

6.4 A Determinant Formula for Tree-level

We will start out by considering the simpler cases. In particular it is straight forward to compute the overlap for the vacuum $M = 0$. We will also refer to this as the BMN vacuum and denote it sometimes in the standard notation $\operatorname{tr} \Phi_1^L$, where recall $\Phi_1 = \phi_1 + i\,\phi_4$. At the classical level clearly $\operatorname{tr} \Phi_1^L = \operatorname{tr}[(\phi_1^{\mathrm{cl}})^L]$. The related MPS overlap is readily computed using (6.28) and one finds

$$\begin{aligned} \langle \mathrm{MPS}_k | 0 \rangle &= \operatorname{tr}[t_3^L] \\ &= \sum_{j=1}^{k} d_{k,j}^L \\ &= \zeta_{-L}(\tfrac{1-k}{2}) - \zeta_{-L}(\tfrac{1+k}{2}), \end{aligned} \tag{6.32}$$

where ζ_n are the generalized Riemann-Zeta function. This result is valid for any value of $k \geq 2$, which allowed for an early check with string theory [38]. We will discuss comparisons with string theory further in later sections.

Recall from (6.10), that the general expression we want to compute is

$$\langle \mathrm{MPS}|\{p_j\}\rangle = \sum_{1\leq n_1<n_2<\cdots<n_M\leq L} \Psi(\{p_j\},\{n_j\})\,\mathrm{tr}[t_3^{n_1-1}t_1 t_3^{n_2-n_1-1}t_1\cdots t_1 t_3^{L-n_M}], \tag{6.33}$$

where Ψ was given in (5.17). We start by getting a better handle on the traces, using cyclicity we can write them in the more suggestive form

$$\begin{aligned}\mathrm{tr}[t_3^{n_1-1}t_1 t_3^{n_2-n_1-1}t_1\cdots t_1 t_3^{L-n_M}] &= \mathrm{tr}[t_3^{L-n_{M1}}t_1 t_3^{n_{21}-1}t_1 t_3^{n_{32}-1}\cdots t_3^{n_{M(M-1)}-1}t_1] \\ &= \mathrm{tr}\prod_{j=1}^{M}\left[t_3^{n_{(j+1)j}-1}t_1\right],\end{aligned} \tag{6.34}$$

were the last line is concise at the expense of needing to define

$$n_{ij} = n_i - n_j, \qquad n_{M+1} = n_1 + L. \tag{6.35}$$

What we see is that the separation between the magnons is the important variable, not their absolute positions.

A lot of what follows is all about studying (6.34) for different values of k. At least there is no shortcut found thus far in the computation of these traces for arbitrary k and M. However for $M = 2$ one can make use of the representation (6.28) as done in [35] to find

$$\langle \mathrm{MPS}_k|u_1,-u_1\rangle = L\left(\frac{u_1-\frac{i}{2}}{u_1}\right)\sum_{j=-\frac{k-1}{2}}^{\frac{k-1}{2}} j^L \frac{u_1^2(u_1^2+\frac{k^2}{4})}{\left(u_1^2+(j-\frac{1}{2})^2\right)\left(u_1^2+(j+\frac{1}{2})^2\right)}. \tag{6.36}$$

The computation leading to the above result follows the same style as the ones in the following sections.

6.5 MPS Overlap for $k = 2$

For $k = 2$ we are in luck, since the 2×2 representation is just spin-$\frac{1}{2}$, so they are simply Pauli matrices

$$t_1^{(2)} = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad t_2^{(2)} = \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad t_3^{(2)} = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{6.37}$$

They are nice as they satisfy

$$t_i t_i = \frac{1}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \{t_i,t_j\} = 0, \quad i\neq j. \tag{6.38}$$

In particular this allows us to massage (6.34) to get

$$\begin{aligned}\operatorname{tr}\prod_{j=1}^{M}\left[t_3^{n_{(j+1)j}-1}t_1\right] &= (-1)^{n_1+n_2+\cdots+n_M+M/2}\operatorname{tr}[t_3^{L-M}t_1^{M}]\\ &= 2\times 2^{-L}(-1)^{n_1+n_2+\cdots+n_M+M/2}.\end{aligned} \tag{6.39}$$

This was used in [35] to reduce the inner sum over magnon positions to a set of nested geometric sums that in turn have closed forms

$$\langle \mathrm{MPS}_2|\{p_j\}\rangle = \frac{(-1)^{M/2}}{2^{L-1}}N_\theta\sum_{\sigma\in S_M}e^{\frac{i}{2}\sum_{j<k}^{M}\theta_{\sigma(j)\sigma(k)}}\sum_{1\le n_1<\cdots<n_M\le L}x_{\sigma(1)}^{n_1}x_{\sigma(2)}^{n_2}\cdots x_{\sigma(M)}^{n_M}, \tag{6.40}$$

where $x_j = e^{ip_j}$. The nested geometric sums are given by

$$\sum_{1\le n_1<\cdots<n_M\le L}x_1^{n_1}x_2^{n_2}\cdots x_M^{n_M} = \\ \prod_{n=1}^{M}x_n^{L+1}+\sum_{a=1}^{M}\left[1-\prod_{n=1}^{a}x_n^{L+1}\right]\left[\prod_{m=1}^{a}\frac{x_m^m}{1-\prod_{n=m}^{a}x_n}\right]\left[\prod_{m=a+1}^{M}\frac{x_m^{L+1}}{\prod_{n=a+1}^{m}x_n-1}\right]. \tag{6.41}$$

Using this, the inner sum in (6.40) can be carried out.

It is important to note that if one simply were to impose the Bethe equations at this stage of the computation, one would find that the result vanishes, as it should in general for rapidities that do not satisfy (6.19). Thus to get the full picture for what happens when the rapidities are un-paired, i.e when they do indeed satisfy (6.19), one should take care to impose this constraint from the very start. It should also be noted that the geometric sums are most readily computed before imposing the un-pairing, however, then one should take care to series expand these geometric sums around the un-paired point as they have fictitious poles there. Finally a last caveat is that the normalization chose in (5.19) gives states $|\{p_j\}\rangle$ that are only equivalent up to a phase subject to permutations of the set $\{p_j\}$. To avoid ambiguity we will therefore specifically be un-pairing neighboring momenta, i.e

$$p_{2j}\to -p_{2j-1}, \qquad j=1,\ldots,\tfrac{M}{2}. \tag{6.42}$$

After constraining the rapidities to be un-paired one can proceed to get rid of all factors of the form x_n^L by using the Bethe Equations (5.21). Remarkably when carrying out this computation for $M=4$ for instance, the result collapses to a surprisingly neat result. Overall the structure hints at a determinant formula, and indeed, as was first conjectured and proved in [35], the overlap is proportional to one of the determinants in (6.20)

$$\langle \text{MPS}_2|\{p_j\}\rangle = 2^{1-L}\left[\prod_{j=1}^{M/2}\frac{u_j^2+\frac{1}{4}}{u_j}(u_j-\tfrac{i}{2})\right]\det G^+. \tag{6.43}$$

This implies that

$$\frac{\langle \text{MPS}_2|\{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}} = 2^{1-L}\sqrt{\left[\prod_{j=1}^{M/2}\frac{u_j^2+\frac{1}{4}}{u_j^2}\right]\frac{\det G^+}{\det G^-}}, \tag{6.44}$$

where the residual phase from the numerator was chose to be it's absolute value, i.e

$$u_j+\tfrac{i}{2}\to\sqrt{u_j^2+\tfrac{1}{4}}. \tag{6.45}$$

In general we will of course find results that are ambiguous up to a phase, and will always choose to fix it to its absolute value. Finally, the result (6.44) can be proved to hold for all even M by relating the overlap to the computation of an overlap between a generalized Neél state and a Bethe state [35]. We will discuss this in a bit more detail, but first we shall consider the cases $k=3$ and $k=4$.

6.6 MPS Overlap for $k=3$

For $k=3$ the situation is quite peculiar as we will see. The matrices

$$t_1^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad t_2^{(3)} = \frac{i}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix},\quad t_3^{(3)} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \tag{6.46}$$

have some nice properties, but it is slightly trickier than the $k=2$ case. The properties are

$$t_i^{2n+1}=t_i,\qquad t_i^{2n}=t_i^2,\qquad \text{for}\qquad n\in\mathbb{N}. \tag{6.47}$$

$$t_it_jt_i=0,\qquad \text{for}\quad i\neq j, \tag{6.48}$$

and

$$[t_i^2,t_j^2]=0,\qquad \text{tr}[t_i^2t_j^2]=1. \tag{6.49}$$

From these properties it follows that all traces either vanish or evaluate to unity as follows

$$\text{tr}[t_3^{N_1}t_1^{N_2}t_3^{N_3}t_1^{N_4}\cdots] = \begin{cases}1, & N_i\in 2\mathbb{N}\\ 0 & \text{otherwise.}\end{cases} \tag{6.50}$$

This means that in the computation of the overlap, only spin-chain position states that have local spin states clustered in even bunches of the same spin will contribute. This enables a rather easy counting for $M=4$ magnons, which is the computation that will give the necessary information about the functional dependence on rapidities, while at the same time being computationally feasible.

The position basis states that will contribute in the overlap will be in two categories

$$|n, n+1, n+2\ell+2, n+2\ell+3\rangle \tag{6.51}$$

and

$$|1, 2n, 2n+1, L\rangle, \tag{6.52}$$

where in the first category the position parameters ℓ and n live in

$$\ell \in \{0, 1, \ldots, \tfrac{1}{2}L - 2\} \quad \text{and} \quad n \in \{1, 2, \ldots, L - 2\ell - 3\} \tag{6.53}$$

respectively whilst for the second category $n \in \{1, 2, \ldots, \frac{1}{2}L - 1\}$. In this way the original sum over the four magnon positions $\Sigma(\sigma)$ in the computation of

$$\langle \text{MPS}_3 | p_1, p_2, -p_1, -p_2\rangle = N_\theta \sum_{\sigma \in S_4} \exp\left[\frac{i}{2}\sum_{j<k} \theta_{\sigma(j)\sigma(k)}\right]\Sigma(\sigma) \tag{6.54}$$

becomes a sum over ℓ and n. This sum is readily evaluated and yields the result

$$\begin{aligned}\Sigma(\sigma = \mathbb{1}) = \frac{x_1^1 x_2^2 x_3^3 x_4^4}{1 - x_1 x_2 x_3 x_4}&\left[\frac{1-(x_3x_4)^{L-2}}{1-(x_3x_4)^2} - (x_1x_2x_3x_4)^{L-3}\frac{1-(x_1x_2)^{-(L-2)}}{1-(x_1x_2)^{-2}}\right]\\ &+ x_1^1 x_2^2 x_3^3 x_4^L \frac{1-(x_2x_3)^{L-2}}{1-(x_2x_3)^2}\end{aligned} \tag{6.55}$$

where again $x_j = e^{ip_j}$.

When un-pairing the momenta

$$\begin{aligned} p_2 &\to -p_1, \\ p_4 &\to -p_3 \end{aligned} \tag{6.56}$$

and subsequently imposing the Bethe Equations, the result is summarized nicely in terms of rapidities, and leads one to conjecture the general M result

$$\langle \text{MPS}_3 | \{u_j\}\rangle = 2\left[\prod_{j=1}^{M/2} u_j(u_j - \tfrac{i}{2})\right] \det G^+, \tag{6.57}$$

hence

$$\frac{\langle \text{MPS}_3 | \{u_j\}\rangle}{\langle \{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}} = 2\sqrt{\left[\prod_{j=1}^{M/2} \frac{u_j^2}{u_j^2 + \frac{1}{4}}\right]\frac{\det G^+}{\det G^-}}. \tag{6.58}$$

This closed determinant formula is very reminiscent of the one for $k = 2$, and purely based on the computation for $M = 4$. Nevertheless it can readily be seen to hold for larger values of M using brute force numerical checks. However, a proof similar in nature to the one for $k=2$ has not been found.

This result by itself is very similar to the $k=2$ result, and it is hard to make an all k conjecture purely based on these results. To appreciate the full blown complexity, we have to look at the $k=4$ case, where there are no longer any nice relations among the generators.

6.7 MPS Overlap for $k=4$

For $k=4$ the matrices are not particularly nice

$$t_1^{(4)}=\begin{pmatrix}0&\frac{\sqrt{3}}{2}&0&0\\ \frac{\sqrt{3}}{2}&0&1&0\\ 0&1&0&\frac{\sqrt{3}}{2}\\ 0&0&\frac{\sqrt{3}}{2}&0\end{pmatrix},\quad t_2^{(4)}=i\begin{pmatrix}0&-\frac{\sqrt{3}}{2}&0&0\\ \frac{\sqrt{3}}{2}&0&-1&0\\ 0&1&0&-\frac{\sqrt{3}}{2}\\ 0&0&\frac{\sqrt{3}}{2}&0\end{pmatrix},\quad t_3^{(4)}=\begin{pmatrix}\frac{3}{2}&0&0&0\\ 0&\frac{1}{2}&0&0\\ 0&0&-\frac{1}{2}&0\\ 0&0&0&-\frac{3}{2}\end{pmatrix}. \tag{6.59}$$

However it is still possible to find a closed formula for the traces that appear for the computation of the overlap involving four magnons. For $M=4$ one has

$$\begin{aligned}&\mathrm{tr}[t_3^{n_{54}-1}t_1t_3^{n_{43}-1}t_1t_3^{n_{32}-1}t_1t_3^{n_{21}-1}t_1]\\ &\qquad=\frac{1}{16}\sum_{a,b,c,d}^{k}\sum_{\ell,m,n,o}^{k-1}d_a^{L-n_{41}-1}d_b^{n_{21}-1}d_c^{n_{32}-1}d_d^{n_{43}-1}c_\ell c_m c_n c_o\\ &\qquad\quad\times\mathrm{tr}\Big[E^a{}_a(E^\ell{}_{\ell+1}+E^{\ell+1}{}_\ell)E^b{}_b(E^m{}_{m+1}+E^{m+1}{}_m)\\ &\qquad\qquad\times E^c{}_c(E^n{}_{n+1}+E^{n+1}{}_n)E^d{}_d(E^o{}_{o+1}+E^{o+1}{}_o)\Big].\end{aligned} \tag{6.60}$$

Only 6 of the 16 terms inside the trace over matrix unities survive. The computation is tedious but straight forward, and one finds for $k=4$ a rather involved set of 7 terms that are all simply products of exponentiated integers or rationals, thus the computation boils down to computing geometric sums as previously. It is a tremendous mess, where each of the seven terms in the trace give rise to different nuances to the geometric sums. Remarkable, after taking the limit

$$p_2\to -p_1,\qquad p_4\to -p_3 \tag{6.61}$$

and furthermore imposing the Bethe Ansatz Equations, the entire mess compressed into something just slightly more complicated than what we have found previously. One finds, in particular noting (6.36), that it is natural to conjecture the all k and M

expression

$$\frac{\langle \mathrm{MPS}_k|\{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}} = 2^{L-1}\frac{\langle \mathrm{MPS}_2|\{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}}\sum_{j=-\frac{k-1}{2}}^{\frac{k-1}{2}} j^L \prod_{m=1}^{M/2}\frac{u_m^2(u_m^2+\frac{k^2}{4})}{\left(u_m^2+(j-\frac{1}{2})^2\right)\left(u_m^2+(j+\frac{1}{2})^2\right)}. \tag{6.62}$$

This was first conjectured and proved in [38].

We should also point out that similar analysis has also lead to a conjectured closed determinant formula in the SU(3) subsector [39]. The conjecture therein is based on the analysis of the SU(2) case, however, to the authors knowledge it is purely a conjecture and remains to be proved. The SU(3) subsector is an interesting first step towards the full SO(6) sector. As a sector, the SU(3) sector is only closed to one-loop, thus we should stick to the SU(2) sector when considering higher loop corrections.

As we shall see in the following sections, there is a partial proof of the closed determinant formula for the SU(2) sector.

6.8 Proof of a Recursion Relation

In [38] the determinant formula (6.62) was proved, at least for all even values of k. The caveat being that since the recursion relation jumps by $k+2$, the proof for $k=2$ only carries over to even values of k. Thus far a proof for $k=3$ is still lacking, but there may be ways to arrive at a recursion relation that jumps by $k+1$.

The recursion was originally motivated by the very nature of (6.62). Note how all higher k overlaps are related to the overlap for $k=2$ - the k dependence factors out into the factor

$$\sum_{j=-\frac{k-1}{2}}^{\frac{k-1}{2}} j^L \prod_{m=1}^{M/2}\frac{u_m^2(u_m^2+\frac{k^2}{4})}{\left(u_m^2+(j-\frac{1}{2})^2\right)\left(u_m^2+(j+\frac{1}{2})^2\right)}. \tag{6.63}$$

In particular, it just so happens that for $k=4$ this factor is identical to the transfer matrix eigenvalue for the corresponding Bethe state, evaluated at spectral parameter $\lambda = i$, that is

$$\frac{\langle \mathrm{MPS}_4|\{u_j\}\rangle}{\langle \mathrm{MPS}_2|\{u_j\}\rangle} = \Lambda(i|\{u_j\}) \tag{6.64}$$

where

$$\mathcal{T}(\lambda)|\{u_j\}\rangle = \Lambda(\lambda|\{u_j\})|\{u_j\}\rangle, \tag{6.65}$$

and the eigenvalue is given by

$$\Lambda(\lambda|\{u_j\}) = \left(\frac{\lambda+\frac{i}{2}}{\lambda-\frac{i}{2}}\right)^L \prod_{j=1}^{M}\frac{\lambda-u_j-i}{\lambda-u_j} + \prod_{j=1}^{M}\frac{\lambda-u_j+i}{\lambda-u_j}. \tag{6.66}$$

Playing around a bit one is lead to conjecture the relation

$$\langle \mathrm{MPS}_{k+2}|\{u_j\}\rangle = \Lambda(\tfrac{ik}{2}|\{u_j\})\langle \mathrm{MPS}_k|\{u_j\}\rangle - \left(\frac{k+1}{k-1}\right)^L \langle \mathrm{MPS}_{k-2}|\{u_j\}\rangle, \tag{6.67}$$

with

$$\langle \mathrm{MPS}_0|\{u_j\}\rangle \equiv 0. \tag{6.68}$$

Clearly

$$\Lambda(\tfrac{ik}{2}|\{u_j\})\langle \mathrm{MPS}_k|\{u_j\}\rangle = \langle \mathrm{MPS}_k|\mathcal{T}(\tfrac{ik}{2})|\{u_j\}\rangle, \quad \text{and} \quad S^+|\{u_j\}\rangle = 0 \tag{6.69}$$

and hence the recursion (6.67) implies

$$|\mathrm{MPS}_{k+2}\rangle = \mathcal{T}(\tfrac{ik}{2})|\mathrm{MPS}_k\rangle - \left(\frac{k+1}{k-1}\right)^L |\mathrm{MPS}_{k-2}\rangle + S^-|\cdots\rangle. \tag{6.70}$$

It just so happens that

$$|\mathrm{MPS}_{k+2}\rangle = \mathcal{T}(\tfrac{ik}{2})|\mathrm{MPS}_k\rangle - \left(\frac{k+1}{k-1}\right)^L |\mathrm{MPS}_{k-2}\rangle, \tag{6.71}$$

that is, it is an exact identity on the space of states, not simply a cohomological statement. We will proceed to prove this, after which a proof of (6.67) follows immediately.

We start out by looking at the local construction of the transfer matrix, and correspondingly the MPS, respectively

$$\mathcal{T}(\lambda) = \mathrm{tr}_a\left[\mathcal{L}_{a,1}(\lambda)\cdots\mathcal{L}_{a,L}(\lambda)\right], \qquad \mathcal{L}_{a,n}(\lambda) = \mathbb{I}_{a,n} + \frac{i}{\lambda - \frac{i}{2}}\mathbb{P}_{a,n}, \tag{6.72}$$

$$|\mathrm{MPS}_k\rangle = \mathrm{tr}\prod_{\ell=1}^{L}(t_1^{(k)} \otimes |{\uparrow_\ell}\rangle + t_2^{(k)} \otimes |{\downarrow_\ell}\rangle). \tag{6.73}$$

Thus, when the transfer matrix acts on the matrix product state we get

$$\begin{aligned}\mathcal{T}(\tfrac{ik}{2})|\mathrm{MPS}_k\rangle &= \mathrm{tr}_a\,\mathrm{tr}\prod_{\ell=1}^{L}\mathcal{L}_{a,\ell}(t_1^{(k)} \otimes |{\uparrow_\ell}\rangle + t_2^{(k)} \otimes |{\downarrow_\ell}\rangle)\\ &= \mathrm{tr}_a\,\mathrm{tr}\prod_{\ell=1}^{L}(\tau_1^{(k)} \otimes |{\uparrow_\ell}\rangle + \tau_2^{(k)} \otimes |{\downarrow_\ell}\rangle)\end{aligned} \tag{6.74}$$

where if you work it out you find

$$\tau_1^{(k)} = \begin{pmatrix} \frac{k+1}{k-1}t_1^{(k)} & 0 \\ \frac{2}{k-1}t_2^{(k)} & t_1^{(k)} \end{pmatrix}, \qquad \tau_2^{(k)} = \begin{pmatrix} t_2^{(k)} & \frac{2}{k-1}t_1^{(k)} \\ 0 & \frac{k+1}{k-1}t_2^{(k)} \end{pmatrix}. \tag{6.75}$$

These are matrices in $\mathbb{C}^2$ whose entries are $k \times k$ color space matrices. Note that here tr_a denotes specifically the trace in $\mathbb{C}^2$ whilst tr denotes the trace in color.

Knowing how the transfer matrix acts on the MPS we are now ready to re-examine the proposition (6.72), which is now seen to imply the following identity among traces

$$\operatorname{tr}\operatorname{tr}_a[\tau^{(k)}_{j_1}\cdots\tau^{(k)}_{j_L}] = \operatorname{tr}[t^{(k+2)}_{j_1}\cdots t^{(k+2)}_{j_L}] + \left(\frac{k+1}{k-1}\right)^L \operatorname{tr}[t^{(k-2)}_{j_1}\cdots t^{(k-2)}_{j_L}]. \tag{6.76}$$

This might look strange at first sight, but it works out naturally thanks to the trace in $\mathbb{C}^2$ taken on the left hand side. However, with the caveat that we require the following simultaneous similarity transformation to hold

$$A\tau_j A^{-1} = \begin{pmatrix} t^{(k+2)}_j & 0 \\ \star_j & \frac{k+1}{k-1}t^{(k-2)}_j \end{pmatrix}, \quad j = 1, 3. \tag{6.77}$$

All we need is to prove the existence of such a similarity. A straight forward approach, involving some triagonalizability argument would be nice, but the author does not see one in sight. At present however, a constructive proof exists, as was given in [38].

Such a similarity transformation was found by careful guess work. The trick is to construct similarity transformation for various values of k, by restricting to a sparse checkered type ansatz, with alternating zero entries. A set of 30 or so matrices were found explicitly using Mathematica. Find sequence was then used on patterns in these matrices to find a construction in terms of matrix unities. The construction was then proved to facilitate the similarity for arbitrary values of k using hard coded matrix unity identities in Mathematica. This concludes the proof of the recursion relation (6.72) and hence (6.67). The details of the construction have been deferred to the appendix in [38].

6.9 Proof of Tree-level Determinant Formula

The determinant formula (6.62) conjectured to be valid for all k and M is proved via the recursion relation (6.67) provided that $k = 2$ and $k = 3$ determinant formulas can be proved to hold for all values of M. The all M , $k = 2$ result was proved for the case $M = L/2$ in [35], and later extended to the general M case in [38]. It relied on mapping the computation to a similar one, namely the overlap between a Bethe state and a generalized Neél state. The starting observation is the following,

$$P_M|\mathrm{MPS}_2\rangle = \frac{1}{2^L(\frac{i}{2})^M}|\mathrm{N\acute{e}el}_M\rangle + S^-|\cdots\rangle, \tag{6.78}$$

where P_M is a projector onto the basis states of definite number of M excitations, and

$$|\mathrm{N\acute{e}el}_M\rangle = \sum_{\substack{1\le n_1<\cdots<n_M\le L \\ |n_i-n_j|\,\in\,\mathrm{even}}} |n_1,\cdots,n_M\rangle, \tag{6.79}$$

is the generalized Néel state. This relation is proved in [35] using a certain hybrid state. From this relation it follows that

$$\langle \text{MPS}_2 | \{u_j\} \rangle = \frac{1}{2^L (\frac{i}{2})^M} \langle \text{Néel}_M | \{u_j\} \rangle. \tag{6.80}$$

It was noted first in [38] that this completes the proof for all M since this overlap is known already, hiding in plain sight as the overlap

$$\langle \text{Néel} | (S^-)^{2m} | \{u_j\} \rangle \tag{6.81}$$

which was computed in [40]. The overlap relates directly to the overlap with the generalized Néel state since

$$|\text{Néel}_{\frac{L}{2}-2m}\rangle = \frac{1}{(2m)!} (S^+)^{2m} |\text{Néel}\rangle. \tag{6.82}$$

Less is known in terms of an all M proof for $k = 3$. One idea, which could generalized the recursion (6.67), to one that relates $k+1$ to k, as opposed to $k+2$ to k, is facilitated by Baxters Q-operators. Namely

$$\lim_{\phi \to 0} Q(\tfrac{i}{2})^{-1} Q(0) |\text{MPS}_2\rangle = 2^{-L} |\text{MPS}_3\rangle + S^- |\cdots\rangle. \tag{6.83}$$

However there are issues with performing the limit $\phi \to 0$ in general, as the left-hand-side is divergent, and the divergence is captured by terms proportional to $S^-|\cdots\rangle$ which is hard to track and generalize to a proof valid for all M. The above remains a conjecture, tested for states of length up to $L = 8$. If proved, this relation would immediately imply the relation between the $k = 2$ and $k = 3$ determinant overlap. Further details on this observation can be found in [38].

Another potential route, would be to try to generalize the proof technique used in [41], where Foda and Zarembo present a proof of the $k = 2$ all M determinant formula based on the intimate relationship with partition functions of the six-vertex model. They establish certain reflective boundary conditions satisfying the boundary Yang-Baxter equations, that correspond with the generalized Néel state. And the overlap then follows from the derivation of the partition function subject to the boundary conditions. The unique solution that satisfies 4 constraints dictated by integrability, is then the determinant formula. Perhaps it would be possible to construct boundary states that are cohomologically equivalent to $|\text{MPS}_3\rangle$, and derive the partition function for the six-vertex model in this case.

7 One-loop Corrections to One-point Functions

We now proceed to go beyond tree-level. This is no small task as quantum corrections are generally not very easy to handle. In the present case there are further complications beyond the standard treatments of quantum corrections. We will be presented with position dependent masses, and mass mixing matrices mixing both color and flavor components of the fields. The latter is solved by fuzzy-sphere coordinates, while the former leads to an equivalent description in terms of anti de-Sitter propagators, instead of the usual Minkowski ones. The details of the resolutions to these obstacles are presented in great detail in [42], and in this section we will simply present some of the main results and conclusions. We will begin by expanding the action around the fuzzy-funnel background.

7.1 Expanding the Action around the Fuzzy-funnel

When computing quantum corrections to the the one-point functions, we need to consider quantum fluctuations around the fuzzy-funnel background, i.e

$$\Phi = \Phi^{\text{cl}} + \tilde{\Phi} \tag{7.1}$$

usually $\Phi^{\text{cl}} = 0$ and there's no fuzz to deal with. However, in our case we have non-trivial vacuum expectation values attributed to the co-dimension one defect, that is we have three of the six scalars around which we must expand the action (4.2). In particular

$$\phi_i = \phi_i^{\text{cl}} + \tilde{\phi}_i, \qquad i = 1, 2, 3, \tag{7.2}$$

where ϕ_i^{cl} are those given in (4.10). The expansion gives rise to a rather involved action, not only do mass terms arise that couple different color and flavor components, but also, these masses are position dependent due to the x_3 dependence of the VEVs. We will not look at all the details, as that would be repeating what has already been fleshed out in great detail in the long paper [42]. Let us cite the main results for the expansion and show details behind the bosonic mass terms.

Firstly one notices that the expansion of the scalar kinetic term gives

$$-\frac{1}{2}D_\mu\phi_i D^\mu\phi_i = -\frac{1}{2}\partial_\mu\tilde{\phi}_i\partial^\mu\tilde{\phi}_i + i[A^\mu, \phi_i^{\text{cl}}]\partial_\mu\tilde{\phi}_i + \cdots \tag{7.3}$$

where the dots indicate terms that are either involve a single factor of $\tilde{\phi}_i$ or do not involve derivatives of quantum fields. The terms involving only a single $\tilde{\phi}_i$ do not contribute, as is readily seen that these terms add up to zero when ϕ_i^{cl} is a solution of the equations of motion, as it is.

The second term $i[A^\mu, \phi_i^{\rm cl}]\partial_\mu\tilde{\phi}_i$ is problematic in that it is first order in derivative, however we can cancel it choosing to fix the gauge by adding

$$-\frac{1}{2}\,\mathrm{tr}\,G^2, \qquad G = \partial_\mu A^\mu + i[\tilde{\phi}_i, \phi_i^{\rm cl}], \tag{7.4}$$

to the action. As is readily seen this cancels the unwanted kinetic term, and all together, the kinetic part of the action is given by

$$S_{\rm kin} = \frac{2}{g_{\rm YM}^2}\int d^4x\,\mathrm{tr}\left[\frac{1}{2}A_\mu\partial_\nu\partial^\nu A^\mu + \frac{1}{2}\tilde{\phi}_i\partial_\nu\partial^\nu\tilde{\phi}_i + \cdots\right] \tag{7.5}$$

where again terms have been omitted, this time fermionic and ghost kinetic terms. As explained in [42], the price for introducing the gauge fixing term is that the ghosts acquire now a mass term as well. The ghosts and fermions have been omitted in this recapitulation for the sake of brevity.

The scalar kinetic term and the gauge fixing term also give rise to bosonic mass terms. This along with the expansion of the quartic

$$\frac{1}{4}\,\mathrm{tr}[\phi_i, \phi_j][\phi_i, \phi_j] \tag{7.6}$$

is what gives rise to the terms in $S_{\rm m,b}$. In particular the contributions are

$$\begin{aligned}
\frac{1}{4}\,\mathrm{tr}[\phi_i, \phi_j][\phi_i, \phi_j] &= \frac{1}{2}\,\mathrm{tr}\left[[\phi_i^{\rm cl}, \phi_j^{\rm cl}][\tilde{\phi}_i, \tilde{\phi}_j] + [\phi_i^{\rm cl}, \tilde{\phi}_j][\phi_i^{\rm cl}, \tilde{\phi}_j] + [\phi_i^{\rm cl}, \tilde{\phi}_j][\tilde{\phi}_i, \phi_j^{\rm cl}]\right] + \cdots \\
-\frac{1}{2}\,\mathrm{tr}[G^2] &= \frac{1}{2}\,\mathrm{tr}\left[[\phi_i^{\rm cl}, \tilde{\phi}_i][\phi_j^{\rm cl}, \tilde{\phi}_j] + 2i[A^\mu, \tilde{\phi}_i]\partial_\mu\phi_i^{\rm cl}\right] + \cdots \\
-\frac{1}{2}\,\mathrm{tr}\,D_\mu\phi_i D^\mu\phi_i &= \frac{1}{2}\,\mathrm{tr}\left[[A_\mu, \phi_i^{\rm cl}][A^\mu, \phi_i^{\rm cl}] + 2i[A^\mu, \tilde{\phi}_i]\partial_\mu\phi_i^{\rm cl}\right] + \cdots
\end{aligned} \tag{7.7}$$

where terms that do not contribute to mass terms have been omitted. The sum of these terms is given in equation (2.16) in [42], as well as the various other terms

$$S_{\rm kin} + S_{\rm m,b} + S_{\rm m,f} + S_{\rm cubic} + S_{\rm quartic}, \tag{7.8}$$

that can be found in equations (2.12 - 2.19).

7.2 Diagonalization of the Mass Mixing Matrix

As is apparent already by just looking at the bosonic mass term, there is a lot of mixing of color and flavor going on. However for some of the flavors the mixing is less complex. One finds that the mixing is complicated for the aptly named, complicated scalars

$$C = \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \\ A_3 \end{pmatrix} \tag{7.9}$$

and easy for the rest

$$E = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \tilde{\phi}_4 \\ \tilde{\phi}_5 \\ \tilde{\phi}_6 \end{pmatrix} . \tag{7.10}$$

The mass term for the easy scalars is readily diagonalized, while the one for the complex scalars requires the use of fuzzy-spherical harmonics. The complete mass term can be rewritten in terms of the adjoint action of the generators

$$L_i = \mathrm{Ad}(t_i), \qquad L^2 = L_i L_i \tag{7.11}$$

where L^2 is the Laplacian of the fuzzy sphere. Further changing the basis of the complex bosons and the fermions renders the following expression for the mass term

$$\begin{aligned} S_m = \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \frac{1}{x_3^2} \operatorname{tr} \left[-\frac{1}{2} E^T L^2 E - \bar{c} L^2 c - \frac{1}{2} C_t^\dagger (L^2 - 2\sigma_i L_i) C_t \right] \\ + \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \frac{1}{x_3} \operatorname{tr} \left[\frac{1}{2} \bar{\psi}_t \sigma_i L_i \psi_t \right] + (t \to b) \end{aligned} \tag{7.12}$$

where σ_i are the Pauli matrices acting in flavor space and C_t, C_b, ψ_t, ψ_b are two component vectors in flavor, specifically

$$\begin{pmatrix} \psi_{t,+} \\ \psi_{t,-} \\ \psi_{b,+} \\ \psi_{b,-} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\psi_3 - i\psi_4 \\ +\psi_2 + i\psi_1 \\ -\psi_1 - i\psi_2 \\ -\psi_4 - i\psi_3 \end{pmatrix} , \tag{7.13}$$

$$\begin{pmatrix} C_{t,+} \\ C_{t,-} \\ C_{b,+} \\ C_{b,-} \end{pmatrix} = \begin{pmatrix} +i\tilde{\phi}_1 + \tilde{\phi}_2 \\ -i\tilde{\phi}_3 - A_3 \\ -i\tilde{\phi}_3 + A_3 \\ -i\tilde{\phi}_1 + \tilde{\phi}_2 \end{pmatrix} . \tag{7.14}$$

The Pauli matrices appear via matrices like the G_i matrices derived in appendix B. The rest of the details can be found in [42].

This form of the mass term makes it clear that we need to diagonalize L^2 for the easy scalars and the ghosts, while we need to diagonalize J^2 with

$$J_i = L_i + \tfrac{1}{2}\sigma_i, \tag{7.15}$$

for the complicated fields.

It is already known how to diagonalize L^2 and J^2. The latter is a bit more involved but is equivalent to the addition of spin and orbital angular momentum of the Hydrogen atom, and is diagonalized by fuzzy-sphere harmonics. The details are left for the paper [42] to elaborate on. Here's a table showing the spectrum that arises

Multiplicity	$\nu(\tilde{\phi}_{4,5,6}, A_{0,1,2}, c)$	$m(\psi_{1,2,3,4})$	$\nu(\tilde{\phi}_{1,2,3}, A_3)$
$\ell+1$	$\ell+\frac{1}{2}$	$-\ell$	$\ell-\frac{1}{2}$
ℓ	$\ell+\frac{1}{2}$	$\ell+1$	$\ell+\frac{3}{2}$
$(k+1)(N-k)$	$\frac{k}{2}$	$-\frac{k-1}{2}$	$\frac{k-2}{2}$
$(k-1)(N-k)$	$\frac{k}{2}$	$\frac{k+1}{2}$	$\frac{k+2}{2}$
$(N-k)(N-k)$	$\frac{1}{2}$	0	$\frac{1}{2}$

(7.16)

where $\ell = 1, \ldots, k-1$ and

$$\nu = \sqrt{m^2 + \frac{1}{4}}. \tag{7.17}$$

The relevant Breitenlohner-Freedman bound [43] is

$$m^2 \geq -\frac{1}{4} \tag{7.18}$$

which is seen to hold for all the masses in table 7.16. It is saturated only for $k = 2$. This serves as a sanity check on these preliminary results.

The bosonic mass terms come with factors of $1/x_3^2$ while the fermionic mass terms come with factors of $1/x_3$. This renders the masses position dependent and it is in principle rather tricky to solve the appropriate Klein-Gordon equations. However, as is quite easy to see, the position dependence can be traded for replacing Minkowski propagators with propagators in AdS$_4$, that is

$$\eta_{\mu\nu} \to g_{\mu\nu} = \frac{1}{x_3^2}\eta_{\mu\nu}. \tag{7.19}$$

The x_3 coordinate plays the role of the radial coordinate in this AdS$_4$ space-time where the defect is at it's boundary. The propagators are worked out in full detail in [42].

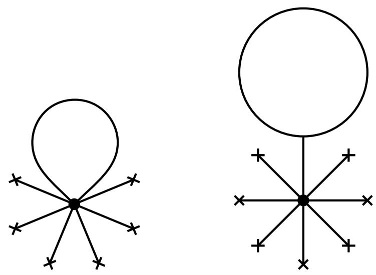
7.3 Contributions at One-loop

We now consider the one-loop corrections to

$$\mathcal{O}_L(x) = \mathcal{O}^{i_1 i_2 \cdots i_L}\, \mathrm{tr}[\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_L}](x), \tag{7.20}$$

where recall that

$$\mathcal{O}^{i_1 i_2 \cdots i_L} = \frac{1}{\sqrt{L}}\left(\frac{8\pi^2}{\lambda}\right)^{L/2} \Psi^{i_1 i_2 \cdots i_L}. \tag{7.21}$$



(a) (b)

Figure 6. Feynman diagrams at one-loop order for an operator of length 8 ((a) tadpole and (b) lollipop). The operators location in space-time is represented by a dot whilst the cross' depict the insertions of classical fields. Lines not touching crosses are propagators.

There are several contributions towards the one-loop expectation value, firstly there are those you get directly from expanding out in powers of the quantum fields, i.e

$$\Phi_i = \phi_i^{\text{cl}} + \tilde{\phi}_i, \tag{7.22}$$

and then there are corrections due to renormalization of $\mathcal{O}_L$ and quantum corrections to $\Psi^{i_1 i_2 \cdots i_L}$. One might worry that there are one-loop corrections that we have omitted here, that involve the 3D theory on the defect. However, the only way to achieve a contribution at one-loop involving an operator in the bulk, and a point on the defect, is to have a loop running in the defect theory. The defect itself has full $\mathcal{N} = 2$ superconformal symmetry, and hence these loops do not contribute.

We will start off considering the expansion in quantum fields (7.22) in the trace. Gathering all terms quadratic in quantum fields one finds many terms, however we're interested in the planar limit, contractions between quantum fields that are not adjacent, will be suppressed. The surviving contributions are thus

$$\langle \mathcal{O}_L \rangle_{\text{tadpole}}(x) = \mathcal{O}^{i_1 i_2 \cdots i_L} \sum_{j=1}^{L} \text{tr}[\phi_{i_1}^{\text{cl}} \cdots \overline{\tilde{\phi}_j \tilde{\phi}_{j+1}} \cdots \phi_{i_L}^{\text{cl}}](x). \tag{7.23}$$

$$\langle \mathcal{O}_L \rangle_{\text{lollipop}}(x) = \mathcal{O}^{i_1 i_2 \cdots i_L} \sum_{j=1}^{L} \text{tr}[\phi_{i_1}^{\text{cl}} \cdots \overline{\tilde{\phi}_j \cdots \phi_{i_L}^{\text{cl}}](x) \int d^4y \sum_{\Phi_1, \Phi_2, \Phi_3} V_3(\Phi_1}, \Phi_2, \Phi_3)(y), \tag{7.24}$$

where

$$\sum_{\Phi_1, \Phi_2, \Phi_3} V_3(\Phi_1, \Phi_2, \Phi_3)(y) \tag{7.25}$$

denotes the sum over all three-point interactions in the theory. These contributions $\langle \mathcal{O}_L \rangle_{\text{tadpole}}$ and $\langle \mathcal{O}_L \rangle_{\text{lollipop}}$ correspond with the Feynman diagrams shown in figure 6.

In addition to these terms we also have a contribution from the one-loop renormalization of the operator itself

$$\mathcal{O}_L \to \mathcal{O}_L + g^2 \mathcal{Z}_1 \mathcal{O}_L + O(g^4) \tag{7.26}$$

as well as finally the one-loop corrections to the Bethe coefficients

$$\Psi^{i_1 \cdots i_L} \to \Psi^{i_1 \cdots i_L} + g^2 \Psi_1^{i_1 \cdots i_L} + O(g^4). \tag{7.27}$$

The one-loop correction to the Bethe coefficients stem from the fact that at one-loop order we need the operators $\mathcal{O}_L$ to have well defined anomalous dimension at two-loop order. This requirement translates to the need for the Bethe states to be eigenstates with respect to the two-loop dilatation operator. These states are referred to as two-loop eigenstates which we will address in a later section.

Let us now consider these terms separately and how they contribute. The contribution $\langle \mathcal{O}_L \rangle_{\text{lollipop}}$ turns out to vanish as it is directly proportional to $\langle \phi_i \rangle_{\text{lollipop}}$ which equates to zero[2], as verified in great detail in [42]. We therefore find that the terms contributing at one-loop amount to

$$\langle \mathcal{O}_L \rangle_{\text{one-loop}} = \langle \mathcal{O}_L \rangle_{\text{tadpole}} + g^2 \mathcal{Z}_1 \langle \mathcal{O}_L \rangle_{\text{tree}} + \mathcal{O}_1^{i_1 i_2 \cdots i_L} \operatorname{tr}[\phi_{i_1}^{\text{cl}} \phi_{i_2}^{\text{cl}} \cdots \phi_{i_L}^{\text{cl}}]. \tag{7.28}$$

Let us start with the tadpole term, and although we will be following [42] we will be skipping some of the details therein concerning propagators and their renormalization. Instead we will focus our attention on aspects pertaining to the color structure, in particular computations of a novel overlap. We have

$$\langle \mathcal{O}_L \rangle_{\text{tadpole}}(x) = \mathcal{O}^{i_1 i_2 \cdots i_L} \sum_{j=1}^{L} \operatorname{tr}[\phi_{i_1}^{\text{cl}} \cdots \overbrace{\tilde{\phi}_j \tilde{\phi}_{j+1}} \cdots \phi_{i_L}^{\text{cl}}](x), \tag{7.29}$$

where the wick contraction implies the insertion of a bosonic propagator. The computation leads to

$$\begin{aligned} \langle \mathcal{O} \rangle_{\text{tadpole}}(x) = \frac{g^2}{(x_3)^2} \sum_j \delta_{s_j = s_{j+1}} \mathcal{O}^{s_1 \ldots s_j\, s_{j+1} \ldots i_L} \operatorname{tr}(\phi_{s_1}^{\text{cl}} \ldots \phi_{s_{j-1}}^{\text{cl}} \phi_{s_{j+2}}^{\text{cl}} \ldots \phi_{s_L}^{\text{cl}})(x) \\ + \tfrac{1}{2} g^2 \Delta_1 \left(2\Psi(\tfrac{k+1}{2}) - 2\log(x_3) - \frac{1}{\epsilon} - \log(4\pi) + \gamma_{\text{E}} \right) \langle \mathcal{O} \rangle_{\text{tree}}(x) \end{aligned} \tag{7.30}$$

where Δ_1 is the eigenvalue of the one-loop dilatation operator, Ψ is the digamma function and γ_{E} the Euler-Mascheroni constant. Furthermore the UV divergence above is precisely canceled by the renormalization of the operator, i.e

$$\mathcal{Z}_1 = \tfrac{1}{2} \Delta_1 (\tfrac{1}{\epsilon} + 1 + \gamma_E + \log \pi). \tag{7.31}$$

[2]Whether or not this holds true beyond one-loop is not immediately clear from that analysis, and remains an open interesting question.

This is the subtraction that gives the one-loop correction to the bulk two-point function standard norm far away from the defect.

The derivation of this result is not straight forward and requires the full machinery of the setup as presented in [42]. We will proceed to take a closer look at the first term

$$\frac{g^2}{(x_3)^2}\sum_j \delta_{i_j=i_{j+1}}\mathcal{O}^{i_1\ldots i_j\, i_{j+1}\ldots i_L}\,\mathrm{tr}[\phi^{\mathrm{cl}}_{i_1}\ldots\phi^{\mathrm{cl}}_{i_{j-1}}\phi^{\mathrm{cl}}_{i_{j+2}}\ldots\phi^{\mathrm{cl}}_{i_L}]. \tag{7.32}$$

This is very similar to the expression for the computation at tree-level (6.1)

$$\langle\mathcal{O}_L\rangle_{\mathrm{tree}} = \mathcal{O}^{i_1 i_2\cdots i_L}\,\mathrm{tr}[\phi^{\mathrm{cl}}_{i_1}\phi^{\mathrm{cl}}_{i_2}\cdots\phi^{\mathrm{cl}}_{i_L}] \tag{7.33}$$

which as we saw was equivalent to computing the overlap between a Bethe state and a matrix product state. Similarly, this one-loop contribution (7.32) can be expressed as the overlap between a Bethe state and an amputated matrix product state (AMPS).

7.4 Amputated Matrix Product States

A concise way to write down the amputated matrix product state is in terms of the matrix product state

$$|\mathrm{AMPS}\rangle = \sum_{\ell=1}^{L}\mathcal{A}_{\ell,\ell+1}|\mathrm{MPS}\rangle, \qquad L+1\sim 1 \tag{7.34}$$

where the operator $\mathcal{A}_{\ell,\ell+1}$ acts on the color space and removes matrices inside the trace at locations ℓ and $\ell+1$ iff they are identical matrices. The first term (7.32) now reads in terms of the AMPS

$$\frac{g^2}{2}\times\frac{1}{x_3^L}\frac{1}{\sqrt{L}}\left(\frac{8\pi^2}{\lambda}\right)^{\frac{L}{2}}\langle\mathrm{AMPS}|\{u_j\}\rangle \tag{7.35}$$

Just as for the overlap between a matrix product state and a Bethe state, we here have the parameters M and k, whose values will dictate whether the computation is tricky or straight forward. For the vacuum, i.e $M=0$ the computation is very straight forward, one has, similar to (6.32)

$$\begin{aligned}\langle\mathrm{AMPS}|0\rangle &= \sum_{\ell=1}^{L}\delta_{i_j,i_{j+1}}\Psi^{i_1\cdots i_j i_{j+1}\cdots i_L}\,\mathrm{tr}[t_{i_1}\cdots t_{i_{j-1}}t_{i_{j+2}}\cdots t_{i_L}]\\ &= L\,\mathrm{tr}[t_3^{L-2}]\\ &= L\left(\zeta_{-(L-2)}(\tfrac{1-k}{2})-\zeta_{-(L-2)}(\tfrac{1+k}{2})\right).\end{aligned} \tag{7.36}$$

For a general number of excitations the first line in the above computation is best expressed as a sum over magnon positions as

$$\langle\mathrm{AMPS}|\{p_j\}\rangle = \sum_{1\le n_1<\cdots<n_M\le L}\Psi(\{p_j\},\{n_j\})\sum_{\ell=1}^{L}\mathcal{A}_{\ell,\ell+1}\,\mathrm{tr}\prod_{j=1}^{M}\left[t_3^{n_{(j+1)j}-1}t_1\right], \tag{7.37}$$

and the first hurdle is to compute the subexpression

$$\sum_{\ell=1}^{L} \mathcal{A}_{\ell,\ell+1} \operatorname{tr} \prod_{j=1}^{M} \left[t_3^{n_{(j+1)j}-1} t_1 \right]. \tag{7.38}$$

We are here using notation introduced in (6.34 - 6.35). This is non-trivial to compute since the matrices are not all of the same kind. In particular it is difficult to compute in general for any number of magnons M and any rank k. It is however doable for special cases, and a conjecture can be made for $k = 2$ that is checked up to $M = 6$ and $L = 16$. It is also possible to find a closed form for $M = 2$ and any k. We will proceed to carry out those two cases separately.

7.5 AMPS Overlap for $k = 2$

For $k = 2$ things are special, as we already noted in section 6.5. These identities allow for easier computation of (7.38) for any M. We will derive a close expression for (7.38) valid for any M, to derive it you need only to consider the case $M = 4$ as it generalizes to all M, it is simply tedious and cluttered to make the argument explicitly valid for all M. For $M = 4$ and any k we have that (7.38) reads

$$\sum_{\ell=1}^{L} \mathcal{A}_{\ell,\ell+1} \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-1} t_2 t_3^{n_{43}-1} t_2 t_3^{n_{54}-1} t_2]. \tag{7.39}$$

Depending on which segments of matrices t_3's or t_2's the amputation operator hits, we get differing results. Working it out one is lead to

$$\begin{aligned}
&\sum_{\ell=1}^{L} \mathcal{A}_{\ell,\ell+1} \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-1} t_2 t_3^{n_{43}-1} t_2 t_3^{n_{54}-1} t_2] \\
&\qquad = \; + \; \delta_{n_{21}>1}(n_{21}-2) \operatorname{tr}[t_3^{n_{21}-3} t_2 t_3^{n_{32}-1} t_2 t_3^{n_{43}-1} t_2 t_3^{n_{54}-1} t_2] \\
&\qquad\quad\; + \; \delta_{n_{32}>1}(n_{32}-2) \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-3} t_2 t_3^{n_{43}-1} t_2 t_3^{n_{54}-1} t_2] \\
&\qquad\quad\; + \; \delta_{n_{43}>1}(n_{43}-2) \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-1} t_2 t_3^{n_{43}-3} t_2 t_3^{n_{54}-1} t_2] \\
&\qquad\quad\; + \; \delta_{n_{54}>1}(n_{54}-2) \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-1} t_2 t_3^{n_{43}-1} t_2 t_3^{n_{54}-3} t_2] \\
&\qquad\quad\; + \; \delta_{n_{21}=1} \operatorname{tr}[t_3^{n_{21}-1} \cancel{t_2} t_3^{n_{32}-1} t_2 t_3^{n_{43}-1} t_2 t_3^{n_{54}-1} \cancel{t_2}] \\
&\qquad\quad\; + \; \delta_{n_{32}=1} \operatorname{tr}[t_3^{n_{21}-1} \cancel{t_2} t_3^{n_{32}-1} \cancel{t_2} t_3^{n_{43}-1} t_2 t_3^{n_{54}-1} t_2] \\
&\qquad\quad\; + \; \delta_{n_{43}=1} \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-1} \cancel{t_2} t_3^{n_{43}-1} \cancel{t_2} t_3^{n_{54}-1} t_2] \\
&\qquad\quad\; + \; \delta_{n_{54}=1} \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-1} t_2 t_3^{n_{43}-1} \cancel{t_2} t_3^{n_{54}-1} \cancel{t_2}],
\end{aligned} \tag{7.40}$$

keep in mind that $n_5 \equiv n_1 + L$. Here the first four terms stem from amputating t_3's, and the last four from amputating t_2's. The delta functions result from the fact that amputation is only possible for certain values of n_i.

Now, so far this is an equality valid for all M and all k. We proceed to make use of the fact that for $k = 2$ the matrices square to $1/4$, in fact this is all that is needed. From the squaring property, it follows that we may insert factors of t_3^2 at will at the expense of multiplying by an overall factor of 4. Simply inserting the missing t_3's in the first four traces makes it clear that the first four traces are identical, and given by

$$4\,\mathrm{tr}[t_3^{n_{21}-1}t_2t_3^{n_{32}-1}t_2t_3^{n_{43}-1}t_2t_3^{n_{54}-1}t_2], \tag{7.41}$$

which we already know how to compute, see (6.39). The last four terms also have identical traces, since they are all of the form

$$\delta_{n_{(m+1)m}=1}\,\mathrm{tr}[\cdots t_3^{n_{m(m-1)}-1}t_3^{n_{(m+1)m}-1}t_3^{n_{(m+2)(m+1)}-1}\cdots], \tag{7.42}$$

hence we can always insert a t_2^2 to the right of the $t_3^{n_{(m+1)m}-1}$ term at the expense of an overall factor of 4. Notice how even though the $t_3^{n_{(m+1)m}-1}$ terms are trivial due to the delta's, they are left in as it makes this argument clearer. We now see that all the traces are identical to (7.41). It is clear that each of the first four terms in (7.40) nicely pair up with each of the last four terms. Indeed, since the traces are all identical, and since

$$\delta_{n>1}(n-2) + \delta_{n=1} = 2(\delta_{n=1} - 1) + n, \tag{7.43}$$

it follows that

$$\begin{aligned}\sum_{\ell=1}^{L} \mathcal{A}_{\ell,\ell+1}\,\mathrm{tr}[t_3^{n_{21}-1}t_2t_3^{n_{32}-1}t_2t_3^{n_{43}-1}t_2t_3^{n_{54}-1}t_2]& \\ = (-1)^{\sum_i n_i}2^{3-L}\Big(L + 2(\delta_{n_{21}=1} + \delta_{n_{32}=1} + \delta_{n_{43}=1} + \delta_{n_{54}=1} - 4)\Big)&,\end{aligned} \tag{7.44}$$

where the expression was simplified further using the definitions given in (6.34 - 6.35), that is

$$n_{ij} = n_i - n_j, \qquad \text{with} \quad n_5 \equiv n_1 + L. \tag{7.45}$$

It is clear that the above arguments generalize in a straight forward manner to larger values of M. The upshot is that for $k = 2$

$$\sum_{\ell=1}^{L} \mathcal{A}_{\ell,\ell+1}\,\mathrm{tr}\prod_{j=1}^{M}\left[t_3^{n_{(j+1)j}-1}t_1\right] = (-1)^{\frac{M}{2}+\sum_i n_i}2^{3-L}\left[L + 2\sum_{m=1}^{M}(\delta_{n_{(m+1)m}=1} - 1)\right] \tag{7.46}$$

gives the amputated trace for any number of magnons M.

Now that we have a closed expression for the amputated trace for $k = 2$ we can plug it into (7.37) and proceed with the rest of the computation. At first sight, one can't help but notice how much of (7.46) is directly proportional to the computation of overlap between the regular matrix product state and a Bethe state. Yet, the part of (7.46) that makes the computation tricky are the delta terms. Unfortunately, to the knowledge of the author there is no obvious way of dealing with these delta terms in full generality. However, it is not too difficult to carry out the computation for $M = 4$ for example, which leads to the result and subsequent conjecture

$$\langle \text{AMPS}_2 | \{u_j\} \rangle = 4 \left(L - \sum_{j=1}^{M/2} \frac{1}{u_j^2 + \frac{1}{4}} \right) \langle \text{MPS}_2 | \{u_j\} \rangle. \tag{7.47}$$

It is a neat little exercise to derive this for $M = 2$ which is surprisingly easy, since in this case the delta terms in (7.46) are very constraining along side the un-pairing. The Bethe wave function for $M = 2$ subject to un-pairing is simply

$$\Psi(n_1, n_2) = e^{-ip\, n_{21}} + e^{ip\,(n_{21}-1)} \tag{7.48}$$

thus

$$\begin{aligned} \langle \text{AMPS}_2 | p \rangle &= 2^{4-L} \sum_{1 \le n_1 < n_2 \le L} (-1)^{n_1 + n_2 + \frac{M}{2}} (\delta_{n_{21}=1} + \delta_{n_{32}=1}) \Psi(n_1, n_2) \\ &= 2^{4-L} L (1 + e^{-ip}) \\ &= 4 \left(L - \frac{1}{u^2 + \frac{1}{4}} \right) \langle \text{MPS}_2 | u \rangle \end{aligned} \tag{7.49}$$

from which (7.47) in the case $M = 2$ follows immediately provided our previous knowledge of the MPS overlap.

Carrying out the computation for $M = 4$ is more involved, as in general the delta terms give rise to modified geometric sums. The conjecture (7.47) has been checked up to $M = 6$ and $L = 16$ numerically. In order to prove this conjecture one would simply need to show that

$$\begin{aligned} 2^{4-L} \sum_{1 \le n_1 < \cdots < n_M \le L} \Psi(\{p_k\}, \{n_j\}) (-1)^{\frac{M}{2} + \sum_i n_i} \sum_{m=1}^{M} (\delta_{n_{(m+1)m}=1} - 1) \\ = -4 \left(\sum_{j=1}^{M/2} \frac{1}{u_j^2 + \frac{1}{4}} \right) \langle \text{MPS}_2 | \{u_j\} \rangle, \end{aligned} \tag{7.50}$$

since the rest of the details follow from the MPS analysis. Finding such a proof remains an interesting open problem.

7.6 AMPS Overlap for $M = 2$ any k

Computing the amputation of the trace for any value of k is tricky. However it is doable for $M = 2$, we find

$$\begin{aligned}\sum_{\ell=1}^{L} \mathcal{A}_{\ell,\ell+1} \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-1} t_2] = {} & \delta_{n_{21}>2}(n_{21}-2) \operatorname{tr}[t_3^{n_{21}-3} t_2 t_3^{n_{32}-1} t_2] \\ & + \delta_{n_{32}>2}(n_{32}-2) \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-3} t_2] \\ & + \delta_{n_{21}=1} \operatorname{tr}[t_3^{n_{21}-1} \cancel{t_2} t_3^{n_{32}-1} \cancel{t_2}] \\ & + \delta_{n_{32}=1} \operatorname{tr}[t_3^{n_{21}-1} \cancel{t_2} t_3^{n_{32}-1} \cancel{t_2}].\end{aligned} \tag{7.51}$$

For general k the traces involved amount to three unique traces (the last two are identical). The three traces are

$$\begin{aligned}\operatorname{tr}[t_3^{n_{21}-3} t_2 t_3^{n_{32}-1} t_2] &= \sum_{i=1}^{k} A_{k,i}(n_{21}), \\ \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-3} t_2] &= \sum_{i=1}^{k} A_{k,i}(n_{21}) \left(\frac{k-2i-1}{k-2i+1}\right)^2, \\ \operatorname{tr}[t_3^{L-2}] &= 2^{2-L} \sum_{i=1}^{k} (k-2i+1)^{L-2},\end{aligned} \tag{7.52}$$

where

$$A_{k,i}(m) = 2^{3-L} \frac{i(k-i)}{(k-2i)^2-1} \left[\frac{k-2i+1}{k-2i-1}\right]^m (k-2i-1)^{L-2}. \tag{7.53}$$

Thus

$$\begin{aligned}&\sum_{\ell=1}^{L} \mathcal{A}_{\ell,\ell+1} \operatorname{tr}[t_3^{n_{21}-1} t_2 t_3^{n_{32}-1} t_2] \\ &\qquad = \sum_{i=1}^{k} \Big[(n_{21} - 2 + \delta_{n_{21}=1}) A_{k,i}(n_{21}) \\ &\qquad\qquad + (L - n_{21} - 2 + \delta_{L-n_{21}=1}) A_{k,i}(n_{21}) \left(\frac{k-2i-1}{k-2i+1}\right)^2 \\ &\qquad\qquad + 2^{2-L}(\delta_{n_{21}=1} + \delta_{L-n_{21}=1})(k-2i+1)^{L-2} \Big].\end{aligned} \tag{7.54}$$

This can then be plugged into (7.37) and one can compute directly symbolically in Mathematica the result

$$\langle \text{AMPS}_k | u_1 \rangle = L\Big(\frac{u_1 + \frac{i}{2}}{u_1}\Big) \sum_{j=-\frac{k-1}{2}}^{\frac{k-1}{2}} j^{L-2} \frac{Q(0)Q(\frac{ik}{2})}{Q(ij_-)Q(ij_+)} \times \Bigg[L - 2 + \frac{Q(j_-)\big(Q(j_+) + Q(\frac{ik}{2})\big)}{Q(\frac{i}{2})Q(\frac{ik}{2})} + \frac{4j^2}{Q(\frac{i}{2})} \Bigg(\frac{j_-^2}{Q(ij_-)} + \frac{j_+^2}{Q(ij_+)} - \frac{\frac{k^2}{4}}{Q(\frac{ik}{2})} + 1 \Bigg) \Bigg], \tag{7.55}$$

where Q is the relevant Baxter polynomial

$$Q(\lambda) = \lambda^2 - u_1^2, \tag{7.56}$$

and

$$j_\pm = j \pm \frac{1}{2}. \tag{7.57}$$

Since the sum is over a symmetric interval, only the symmetric part of the summand contributes. Most of the terms have been symmetrized, those that have not been are left as they were for conciseness. The factor in front of the square brackets is readily recognized as the summand in the regular MPS overlap, and for $k = 2$ the above can be seen to reproduce (7.47) with $M = 2$.

It should be possible to continue in the same vain to get a general k result for $M = 4$. However the computations are rather tedious and involved, and at the end of the day we are not interested in the AMPS overlap by itself, but the total contribution to the one-loop correction. As we will see it turns out that the total contribution takes on a nicer expression than the above result, one that generalizes straight forwardly to higher k and M.

7.7 The Full One-loop One-point Function

The full one-loop one-point function amounts to

$$\langle \mathcal{O}_L \rangle_{\text{tree}} + \langle \mathcal{O}_L \rangle_{\text{one-loop}}, \tag{7.58}$$

where

$$\langle \mathcal{O}_L \rangle_{\text{one-loop}} = \langle \mathcal{O}_L \rangle_{\text{tadpole}} + g^2 \mathcal{Z}_1 \langle \mathcal{O}_L \rangle_{\text{tree}} + \mathcal{O}_1^{i_1 i_2 \cdots i_L} \operatorname{tr}[\phi_{i_1}^{\text{cl}} \phi_{i_2}^{\text{cl}} \cdots \phi_{i_L}^{\text{cl}}]. \tag{7.59}$$

So far we have considered everything but the two-loop Bethe eigenstate term

$$\mathcal{O}_1^{i_1 i_2 \cdots i_L} \operatorname{tr}[\phi_{i_1}^{\text{cl}} \phi_{i_2}^{\text{cl}} \cdots \phi_{i_L}^{\text{cl}}]. \tag{7.60}$$

To our delight the two-loop corrected eigenstates have been carefully studied by Gromov et al. in [44, 45]. They are constructed by means of introducing impurities θ_j on the spin cites. Specifically this involves acting with modified creation operators

$$\hat{\mathcal{B}}(u) = \langle\uparrow| \bigotimes_{j=1}^{L} \left(\mathbb{1}_{j,0} + \frac{i}{u - \theta_j - \frac{i}{2}} \mathbb{P}_{j,0} \right) |\downarrow\rangle \tag{7.61}$$

to construct impurity Bethe states

$$|\theta, \mathbf{u}\rangle = \hat{\mathcal{B}}(u_1) \cdots \hat{\mathcal{B}}(u_M)|0\rangle. \tag{7.62}$$

Note that this is the straightforward generalization of the standard impurity free algebraic creation operator see appendix A.

The two-loop eigenstate may then finally be constructed via taking the theta-morphism of this impurity Bethe state. The action of the theta-morphism is defined via partial derivatives with respect to the impurities θ_j

$$\{f\}_\Theta \equiv f + \frac{g^2}{2} \sum_{j=1}^{L} \left(\frac{\partial}{\partial\theta_j} - \frac{\partial}{\partial\theta_{j+1}} \right)^2 f + \mathcal{O}(g^4) \Big|_{\theta_j \to 0}. \tag{7.63}$$

It turns out that the story is not quite so straight forward, but that the subtle corrections that need to be made to $\{|\theta, u\rangle\}_\Theta$ are simply encoded at the edge, that is, the one-loop corrected eigenstate is given by

$$|\mathbf{u}\rangle \equiv \left(1 - \tfrac{1}{2} g^2 \Delta_1 \, \mathbb{H}_{L,1}\right) \{|\theta, \mathbf{u}\rangle\}_\Theta. \tag{7.64}$$

There is one last caveat though, and that is that the rapidities now have to satisfy the two-loop Bethe equations which belong to the types of Bethe equations applicable for spin chains with longer range interactions [46]. These Bethe equations turn out to be a generalization i.e

$$\left(\frac{x(u_j + \frac{i}{2})}{x(u_j - \frac{i}{2})} \right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \tag{7.65}$$

where

$$x + \frac{1}{x} = \frac{u}{g} \tag{7.66}$$

are the Zhukowski variables [26]. Expanding x to leading order in g gives the first correction to the Bethe equations, the two-loop Bethe equations.

The total one-loop result can then be expressed as

$$\frac{1}{x_3^\Delta} \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda} \right)^{L/2} \left[\frac{\langle \mathrm{MPS}|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle^{\frac{1}{2}}} + g^2 \frac{\langle \mathrm{AMPS}|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle^{\frac{1}{2}}} \right] \left(1 + g^2 \Delta_1 (\Psi(\tfrac{k+1}{2}) + \gamma_E - \log 2 + \tfrac{1}{2})\right) \tag{7.67}$$

where $|\mathbf{u}\rangle$ denotes specifically the one-loop corrected Bethe state, i.e (7.64).

We have already presented the result for the AMPS overlap to leading order, and the MPS overlap with the one-loop corrected Bethe state can be computed with the prescription presented above for various values of k and M as was presented in [47]. The result can be nicely summarized in terms of new $\tilde{G}_\pm$ matrices given by (6.22) with f_m replaced by $\tilde{f}_m$ that is

$$\tilde{G}_\pm = \partial_m \tilde{f}_n \pm \partial_{m+\frac{M}{2}} \tilde{f}_n, \tag{7.68}$$

with

$$\tilde{f}_m = -i \log \left[\left(\frac{x(u_j - \frac{i}{2})}{x(u_j + \frac{i}{2})} \right)^L \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \right]. \tag{7.69}$$

In full the proposed general result is then

$$\langle \mathcal{O}_L(x) \rangle = \frac{1}{x_3^\Delta} \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda} \right)^{L/2} i^L \tilde{\mathcal{T}}_{k-1}(0) \sqrt{\frac{Q(\frac{i}{2})Q(0)}{Q^2(\frac{ik}{2})}} \sqrt{\frac{\det \tilde{G}_+}{\det \tilde{G}_-}} \mathbb{F}_k, \tag{7.70}$$

where

$$\tilde{\mathcal{T}}_n(\lambda) = g^L \sum_{j=-\frac{n}{2}}^{\frac{n}{2}} [x(u+ij)]^L \frac{Q(u + \frac{n+1}{2} j) Q(u + \frac{n+1}{2} j)}{Q(u + (j - \frac{1}{2})i) Q(u + (j + \frac{1}{2})i)} \tag{7.71}$$

is the transfer matrix of the Heisenberg spin chain in the $(n+1)$-dimensional representation. The factor

$$\mathbb{F}_k = 1 + g^2 \left(\Psi(\tfrac{k+1}{2}) + \gamma_E - \log 2 \right) \Delta_1 + O(g^4) \tag{7.72}$$

is a flux factor that captures corrections due to operator renormalization.

This conjecture was originally motivated by its simplicity and the concrete results for $k = 2$, $M = 2$, notably (7.47) and the results for $\langle \text{MPS}_2 | \mathbf{u} \rangle$

$$\frac{\langle \text{MPS}_2 | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle^{\frac{1}{2}}} = \sqrt{\frac{L}{L-1} \frac{u^2 + \frac{1}{4}}{u^2} \frac{1 + g^2 \frac{4}{u^2 + \frac{1}{4}}}{1 + \frac{g^2}{L-1} \frac{6u^2 - \frac{1}{2}}{(u^2 + \frac{1}{4})^2}}} \tag{7.73}$$

presented in [47].

The flux factor is unity for protected operators like the BMN vacuum for which $\Delta_1 = 0$, indeed (7.71) simplifies considerably for this operator

$$\langle \text{tr}[\Phi_1^L](x) \rangle = \frac{1}{x_3^L} \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda} \right)^{L/2} i^L \tilde{\mathcal{T}}_{k-1}(0). \tag{7.74}$$

Moreover the transfer matrix evaluates to

$$\tilde{\mathcal{T}}_{k-1}(0) = \sum_{j=-\frac{k-1}{2}}^{\frac{k-1}{2}} g^L x^L(i\,j) \tag{7.75}$$

and one finds the finite expansion in g

$$\langle \mathrm{tr}[\Phi_1^L](x)\rangle = \frac{1}{x_3^L}\frac{1}{\sqrt{L}}\left(\frac{8\pi^2}{\lambda}\right)^{L/2}\sum_{n=0}^{\frac{L}{2}}\begin{pmatrix} L-n \\ n \end{pmatrix}\frac{L}{L-n}\frac{B_{L-2n+1}(\frac{1-k}{2})}{L-2n+1}g^{2n}. \qquad (7.76)$$

where B_n are Bernoulli polynomials og degree n.

As we will see in the next sections this result seemingly agrees with results from string theory to all loop orders. This seems to be thanks to the fact that the expansion arranges itself in powers of $\frac{\lambda}{k^2}$, however this is unfortunately not true for unprotected operators, the double scaling limit is broken by the flux factor (7.72), in particular by the term

$$\Psi(\tfrac{k+1}{2}) \sim \log k \qquad \text{for} \qquad k \gg 1. \qquad (7.77)$$

8 Comparison to String Theory

We will proceed to make comparisons with string theory for the BMN vacuum as the string counterpart has already been computed for this state [16]. First some short remarks on how to compare results on the two sides of the correspondence are in order.

8.1 Circumventing the Strong-Weak Obstacle

The GKPW (Gubser-Klebanov-Polyakov-Witten) prescription [48, 49] relates computations on the two sides of the correspondence. In particular it tells us that the generating functional on the field theory is related to the path integral in the string theory

$$\left\langle e^{\int d^4x s_0(x)\mathcal{O}(x)} \right\rangle_{\text{CFT}} = e^{-S_{\text{cl}}(s_0)}.$$

On the left hand side is a gauge theory expectation value, and on the right hand side the supergravity counterpart. The source on the field theory side s_0 is identified with the boundary condition on the scalar field s, which is the scalar field on the gravity side corresponding to $\mathcal{O}$. It should be stressed that the right hand side is a simplification of a path integral in the gravity theory under the assumption that it is dominated by the classical contribution S_{cl}. This is a valid approximation when the background curvature R is much larger than the string length scale ℓ_s, i.e when $\lambda \gg 1$.

A priori, the classical computation in string theory therefore relates to a computation in the gauge theory at strong coupling. For this reason, it is usually difficult to verify the correspondence. However, additional tuneable parameters of the theory can circumvent this. This was the idea in the BMN limit, where the additional parameter J made available a double scaling limit. Presently the idea is similar, our additional parameter will be the world-volume flux k, which as we saw appears on the gravity side as an angle

$$\kappa = \frac{\pi k}{\sqrt{\lambda}}. \tag{8.1}$$

On the field theory side, as explained in [15] the gauge theory computation should arrange itself in powers of λ/k^2. Thus we can keep $\lambda \gg 1$ justifying the GKPW, and at the same time consider

$$\frac{\lambda}{k^2} \ll 1, \tag{8.2}$$

by sending

$$k \to \infty, \qquad \text{but} \qquad k \ll N. \tag{8.3}$$

Implicitly we are doing a double scaling limit, as it is understood that we are working in the planar limit, and must therefore send also $N \to \infty$, while $\lambda =$ fixed.

This can lead to an order of limits problem, as turned out to be the case for the BMN limit, however, whether this is the case presently remains to be seen.

Such a double scaling limit was taken in [16] where they compared the one-point functions of the chiral primary operator to string theory. Agreement between the field theory computation and the string theory computation are found at leading order in $\frac{\lambda}{k^2}$.

In [38] we are able to make use of the fact that the vertex operator is known for the chiral primary $\mathrm{tr}[\Phi_1^L]$ to make a direct comparison with string theory at tree-level. That string theory computation however, is only carried out to exponential accuracy and doesn't hold at next to leading order, except for in the large L limit. Therefore we will later make use of another independent result. Namely we will use the string theory computation done in [16], which can be expanded beyond tree-level.

8.2 Comparing Results with String Theory

The GKPW relation (8.1) implies that one-point functions in the field theory are given by

$$\langle \mathcal{O}(x) \rangle = -\frac{\delta S_{\mathrm{cl}}}{\delta s_0(x)}. \tag{8.4}$$

Normally this would simply equate to zero, since the supergravity background is a classical solution to the equations of motion of S_{cl}, and hence all variations on it vanish. However, presently we have the case where there is also a D5 brane in the game, which from the string theory point of view is what gives rise to non-trivial one-point functions.

In [16] the computation of the variations of the D5 brane part of the action, namely

$$S_{\mathrm{D5}} = -T_5 \int \sqrt{-\det(G+\mathcal{F})} + T_5 \int \mathcal{F} \wedge C_4 \tag{8.5}$$

is carried out. It is a rather lengthy computation, but at the end of the day it amounts to

$$-\frac{\delta S_{\mathrm{cl}}}{\delta s_0(x)} = C_L \frac{\sqrt{\lambda}2^{L/2}\Gamma(L+1/2)}{\pi^{3/2}\sqrt{L}\Gamma(L)} \frac{1}{(x_3)^L} \int_0^\infty du \frac{u^{L-2}}{[(1-\kappa u)^2 + u^2]^{L+1/2}}. \tag{8.6}$$

This result is specifically for a variation with respect to a source scalar that corresponds to the SO(3) × SO(3) chiral primary field theory operator

$$\hat{\mathcal{O}}_L = C_L \left\{ \Big(\sum_{i=1}^{3} \phi_i^2 \Big)^{L/2} + \Big(\sum_{i=4}^{6} \phi_i^2 \Big) Q_{L-2} \Big(\sum_{i=1}^{3} \phi_i^2, \sum_{i=4}^{6} \phi_i^2 \Big) \right\} \tag{8.7}$$

where C_L is a normalization constant and $Q_{L-2}(a,b)$ is a homogeneous polynomial of a and b of degree $\frac{1}{2}(L-2)$.

Important to us is that the first term makes it clear that these operators $\hat{\mathcal{O}}_L$ have non-zero overlaps with the chiral primary that we are considering, the BMN vacuum $\operatorname{tr}\Phi_1^L$. We may therefore compare the ratios of the one-loop to the tree-level contributions for these operators, and expect agreement, in particular

$$\left.\frac{\langle \operatorname{tr}\Phi_1^L\rangle_{\text{one-loop}}}{\langle \operatorname{tr}\Phi_1^L\rangle_{\text{tree-level}}}\right|_{\text{gauge}} = \left.\frac{\langle \hat{\mathcal{O}}_L\rangle_{\text{one-loop}}}{\langle \hat{\mathcal{O}}_L\rangle_{\text{tree-level}}}\right|_{\text{string}}. \tag{8.8}$$

This was first noted in [42], where we also found that the results indeed matched to one-loop order. From [16], specifically equation (4.7) therein one finds after identifying 2ℓ with our L

$$\left.\frac{\langle \hat{\mathcal{O}}_L\rangle_{\text{one-loop}}}{\langle \hat{\mathcal{O}}_L\rangle_{\text{tree-level}}}\right|_{\text{string}} = \frac{\lambda}{4\pi^2 k^2}\frac{L(L+1)}{L-1}. \tag{8.9}$$

And on the other hand from (7.76) one finds

$$\left.\frac{\langle \operatorname{tr}\Phi_1^L\rangle_{\text{one-loop}}}{\langle \operatorname{tr}\Phi_1^L\rangle_{\text{tree-level}}}\right|_{\text{gauge}} = \frac{\lambda}{4\pi^2 k^2}\left(\frac{L(L+1)}{L-1} + O(k^{-2})\right). \tag{8.10}$$

To see that simply expand the Bernoulli polynomials in $1/k$ for large k and then finally take the first couple of terms in the sum over n.

Remarkably this game can be continued to higher orders, by expanding the integral in (8.6) and the all-loop conjecture (7.76) and comparing order by order in the λ/k^2 expansion. As can be seen the two continue to agree order by order, in fact they continue to agree to all orders which was first demonstrated in [47]. This can be seen by noting that

$$\int_0^\infty du\,\frac{u^{L-2}}{\Big[(1-\kappa u)^2+u^2\Big]^{L+1/2}} = I(\kappa, L), \tag{8.11}$$

where

$$I(\kappa, L) = \left[\kappa^2+1\right]^{\frac{3}{2}}\int_{-\arctan\kappa}^{\pi/2} d\theta\,\cos^{2L-1}\theta\,(\kappa+\tan\theta)^{L-2}. \tag{8.12}$$

The ratio of higher orders to tree-level on the string theory side is then

$$\frac{I(\kappa, L)}{I(\infty, L)} = \frac{\Gamma(L+\frac{1}{2})}{\kappa^{L+1}\sqrt{\pi}\,\Gamma(L)}\left[\kappa^2+1\right]^{\frac{3}{2}}\int_{-\arctan\kappa}^{\frac{\pi}{2}} d\theta\,\cos^{2L-1}\theta\,(\kappa+\tan\theta)^{L-2}. \tag{8.13}$$

The integral above was carried out in [47] giving the results

$$\left.\frac{\langle \hat{\mathcal{O}}_L\rangle_{\text{one-loop}}}{\langle \hat{\mathcal{O}}_L\rangle_{\text{tree-level}}}\right|_{\text{string}} = \frac{I(\kappa, L)}{I(\infty, L)} = \frac{\left(\kappa+\sqrt{\kappa^2+1}\right)^L\left(L\sqrt{\kappa^2+1}-\kappa\right)}{2^L(L-1)\kappa^{L+1}}. \tag{8.14}$$

Now on the other hand the field theory result (7.76) takes also on a very nice closed form in the double scaling limit, noting that the Bernoulli polynomials simplify greatly

$$B_n\Big(\frac{1-k}{2}\Big) \to -\left(\frac{k}{2}\right)^n \tag{8.15}$$

one finds

$$\left.\frac{\langle \mathrm{tr}\, \Phi_1^L\rangle_{\text{one-loop}}}{\langle \mathrm{tr}\Phi_1^L\rangle_{\text{tree-level}}}\right|_{\text{gauge}} \to \left[1+\sum_{n=1}^{L/2}\binom{L-n}{n-1}\frac{L}{n}\frac{L+1}{L-n}\left(\frac{2g}{k}\right)^{2n}\right]. \tag{8.16}$$

This sum is readily computed and gives

$$\left.\frac{\langle \mathrm{tr}\, \Phi_1^L\rangle_{\text{one-loop}}}{\langle \mathrm{tr}\Phi_1^L\rangle_{\text{tree-level}}}\right|_{\text{gauge}} \to \frac{\left(\sqrt{\frac{\lambda}{\pi^2k^2}+1}+1\right)^L\left(L\sqrt{\frac{\lambda}{\pi^2k^2}+1}-1\right)}{2^L(L-1)}. \tag{8.17}$$

Remarkably this coincides exactly with (8.14) once we remember that κ is the angle of the D5 embedding and given by

$$\kappa = \frac{\pi k}{\sqrt{\lambda}}. \tag{8.18}$$

This makes it an example of a non-trivial computation in support of AdS/dCFT, at least in the context of the present double scaling limit. Outside this special limit, one would possibly need to take into account further corrections from the dressing phase and wrapping interactions. Normally one also needs to take into account Lüscher corrections, however this is probably not necessary in the present case due to the lack of excitations in the vacuum state considered.

9 Conclusion and Outlook

As we have seen, the defect $\mathcal{N} = 4$ SYM theory arising in the D3-probe-D5 brane setup features new and interesting observables, such as one-point functions. In this thesis the focus has been on the SU(2) sector of operators, as they elegantly map to the simplest spin-chain we know. However, there are other observables of interest, both one-point functions of operators that do not belong to the SU(2) sector, as well as Wilson loops and two-point functions between operators of differing conformal dimensions. The results discussed in this thesis have opened up the possibilities for novel exploration into these kinds of observables.

Recently the interface-particle potential has been computed at one-loop level via a Wilson line [50], and furthermore [14, 51] have initiated the study of two-point functions involving operators of differing conformal dimensions, and in particular [14] tested the boundary conformal bootstrap equations. The Wilson loop computation showed perfect agreement with string theory result. Along with the results herein for the BMN vacuum (7.76), the all-loop conjecture, these results make a compelling case for the AdS/dCFT program. In addition it would seem that integrability is still a central piece of the story, without which these results would be hard to obtain. Lastly, the defect in this setup breaks the vanilla $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 2$, and as such, this analysis is at the forefront of holography and integrability in a setting with reduced symmetries, when compared with the standard.

That being said, the tests so far have been for highly special observables, like the chiral primary $\operatorname{tr}\Phi_1^L$, an operator that is protected from quantum corrections in vanilla $\mathcal{N} = 4$ SYM. Although the computations in the defect scenario still allow them to gain quantum corrections, these corrections are less involved than the corrections that the unprotected operators are subject to. In particular, the double scaling limit that enables the success of the comparisons with string theory in the λ/k^2 expansion, break down for the unprotected operators. This is encoded in the flux factor in (7.72) which explicitly breaks the λ/k^2 expansion for operators with non-trivial anomalous dimension.

In general it would be interesting to look at the nature of the proposed AdS/dCFT correspondence for non-protected operators. However, their dual string states have thus far not been obtained. The string state dual to $\operatorname{tr}\Phi_1^L$ is a string stretching between the D5 and the insertion point on the stack of D3's. The equivalent for non-protected SU(2) operators, would be a rapidly spinning string that should somehow also span from the D5 to the stack of D3's, but a solution to this boundary problem is far more involved.

Aside from further advanced tests of the AdS/dCFT correspondence. It would be a good idea to revisit a proof for the tree-level one-point functions. As mentioned there seems a possibility that Baxter Q-operators might facilitate a much more concise proof. Furthermore, the proof rests presently on the existence of a similarity transformation (6.77), which so far only has a rather involved constructive proof. It would be interesting to investigate the nature of (6.77) in further detail, and potentially find a deeper connection.

In general, the results at one-loop order for arbitrary k and M that lead to (7.70) are mostly conjectures. It would be desirable to obtain proofs, in particular of the all-loop conjecture. A first step in that direction would be to address proofs of the results for the amputated matrix product state overlaps (7.37). A natural starting place might be to look for a proof of the special case $k = 2$, where (7.50) is found to hold for arbitrary M, proving this might be easier than the author is able to gauge at present.

Other than the already mentioned desirable continuations of this work, there is the closely related D3-probe-D7 brane setup. A notable key difference in that setup is the fact that there is no longer any remnant supersymmetry, in other words the defect breaks it all. Furthermore the defect is now associated with the probe D7 brane which is still an AdS_4 brane, but that wraps an S^4 or $\text{S}^2 \times \text{S}^2$ inside the S^5. This gives rise to two world volume fluxes, one for each S^2. Thus the k we have seen appearing in the present D3-probe-D5 brane setup is appearing twice, i.e, one gets two ranks k_1, k_2. This case has been studied so far at tree-level in [17]. Going beyond tree-level, following the same ideas as in the one-loop study of the D3-probe-D5 setup, is not out of the question, however it is likely more involved due to lacking symmetries. Aside from that, the setup is very reminiscent, and clearly a potentially interesting candidate for an AdS/dCFT setup that no longer preserves any of the original supersymmetries of the archetypal AdS/CFT correspondence.

In general, it is safe to say that interacting defect conformal field theories in four dimensions have not been studied as much as the very well understood two-dimensional cousins. Indeed there are not many known interacting four-dimensional conformal field theories, in that sense $\mathcal{N} = 4$ SYM is rather unique. It remains to be seen whether or not the present D3-probe-D5 dCFT retains it's classical symmetries at the quantum level, in particular the conformal symmetries that leave the defect invariant. Evidence points towards this being the case when $k = 0$, where [12, 13] make compelling arguments for why this should be the case. However a strict proof still seems desirable, and much less is known in general for $k \geq 2$. Mapping out the complete theory, including the interactions with the defect theory for $k \geq 2$ is necessary to go to higher loop orders, and could present us with a much interesting picture of the present setup. The recent paper [14] is a first step in this direction, but it would seem that a lot of work still lies ahead.

A Algebraic Bethe Ansatz

The works of Faddeev and the Leningrad group have lead to the very useful algebraic approach to the problem of diagonalizing integrable models, such as the $XXX_{1/2}$ spin-chain. We will proceed to give a quick rundown of some constructions relevant to the current context based on the lecture notes [34].

In the algebraic language the spin chain is an object whose state is constructed from the product of local spaces V^n. In the case of spin-$\frac{1}{2}$ these local spaces can be identified with $V^2 = \mathbb{C}^2$. Each of these cites can be in a local state of either spin up $\uparrow$ or spin down $\downarrow$. The Hilbert space is in other words taken to be

$$\mathcal{H}_L = \mathbb{C}^2_1 \otimes \mathbb{C}^2_2 \otimes \cdots \otimes \mathbb{C}^2_L. \tag{A.1}$$

On this space the Hamiltonian takes the neat form

$$H = \sum_{j=1}^{L} \mathbb{H}_{j,j+1}, \qquad \mathbb{H}_{j,j+1} = \mathbb{I}_{j,j+1} - \mathbb{P}_{j,j+1} \tag{A.2}$$

where the indices indicate that the operators work on cites $\mathbb{C}^2_j, \mathbb{C}^2_{j+1}$. The marvelous realization was that all of the conserved charges of this system could be generated by the expansion of a transfer matrix in the spectral parameter. The transfer matrix is constructed via the Monodromy which in turn is a product of Lax matrices. The Lax matrix is admittedly a bit taken out of the blue, and given by

$$\mathbb{L}_{n,a}(\lambda) = (\lambda - \tfrac{i}{2})\mathbb{I}_{n,a} + i\mathbb{P}_{n,a}. \tag{A.3}$$

The Monodromy is then given by

$$M_a(\lambda) = \mathbb{L}_{1,a}(\lambda)\mathbb{L}_{2,a}(\lambda)\cdots\mathbb{L}_{L,a}(\lambda) \tag{A.4}$$

and finally the transfer matrix is simply its trace over the auxiliary space a

$$T(\lambda) = \mathrm{tr}_a\, M_a(\lambda). \tag{A.5}$$

The tower of conserved charges are then found by expanding around $\lambda = \frac{i}{2}$

$$\log T(\lambda) = \sum_{n=1}^{L} Q_n (\lambda - \tfrac{i}{2})^n. \tag{A.6}$$

Notably Q_1 gives the total momentum of the chain, and Q_2 gives the total energy, up to shift, specifically

$$H = \frac{i\, Q_2}{2} - \frac{L}{2}. \tag{A.7}$$

The algebraic framework gives us much more than just the charges. Taking a peek back at the Monodromy, we see that it is a 2×2 matrix in the auxiliary space, that is

$$M_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{A.8}$$

where each entry A, B, C, D can be understood as an operator acting on the entirety of the chain. Indeed,

$$T(\lambda) = A(\lambda) + D(\lambda), \tag{A.9}$$

and since $H \propto T(\lambda)$ we understand that states of definite energy must be eigenstates of the transfer matrix. As explained in more detail in [34], it becomes natural to associate the operators $B(\lambda)$ with creation operators provided that we choose a vacuum state $|0\rangle$ that satisfies

$$C(\lambda)|0\rangle = 0. \tag{A.10}$$

The upshot is that, not only do we get a ladder of commuting charges, but also, a prescription for how to generate eigenstates, namely

$$B(u_1)B(u_2)\cdots B(u_M)|0\rangle, \tag{A.11}$$

which remarkably, like the plane wave ansatz, are only eigenstates provided that the rapidities satisfy

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}. \tag{A.12}$$

To see this requires a bit of work, but it is a very approachable exercise.

Note that the Lax matrix as defined in (A.4) coincides with the one in [34], but it is not the same as the one we use elsewhere in this thesis. Here it is simply used because it is convenient for the story above. Otherwise we use

$$\mathcal{L}_{a,n}(\lambda) = \mathbb{I}_{a,n} + \frac{i}{\lambda - \frac{i}{2}} \mathbb{P}_{a,n}, \tag{A.13}$$

which only means that we get factors of $(u - \frac{i}{2})$. For instance, using (A.13) we would have

$$\mathcal{B}(u) = \langle \uparrow | \bigotimes_{j=1}^{L} \left(\mathbb{I}_{j,0} + \frac{i}{u - \frac{i}{2}} \mathbb{P}_{j,0} \right) |\downarrow\rangle \tag{A.14}$$

whilst using Faddeev prescription we find

$$B(u) = (u - \tfrac{i}{2})^L \mathcal{B}(u). \tag{A.15}$$

In general calligraphic typesetting is used for the objects that are derived from the prescription (A.13).

B SYM from 10 dimensions and getting the fermions right

We proceed to construct $\mathcal{N}=4$ SYM in four spacetime dimensions from the reduction of $\mathcal{N}=1$ SYM in ten spacetime dimensions, following Brink and Schwarz [52].

The 10 dimensional action reads

$$S_{10} = \int d^{10}x \,\mathrm{Tr}\left\{-\tfrac{1}{4}F^{MN}F_{MN} + \tfrac{i}{2}\bar{\Psi}\Gamma^M D_M\Psi\right\}, \tag{B.1}$$

where $N, M = 0, 1, 2, \ldots, 9$ are Lorentz indices. The field-strength tensor is given by

$$F_{MN} = i[D_M, D_N], \qquad D_M = \partial_M - iA_M, \tag{B.2}$$

on adjoint field

$$D_M = \partial_M - i[A_M, \quad], \tag{B.3}$$

and recall that under a gauge transformation U we have

$$A \to A^U = UAU^{-1} + iU\partial U. \tag{B.4}$$

The fermions Ψ satisfy simultaneously the Majorana and the Weyl condition

$$\Psi = \mathcal{C}_{10}\bar{\Psi}^T, \qquad \Gamma_{11}\Psi = -\Psi. \tag{B.5}$$

Lastly Γ^M are 10-dimensional gamma matrices satisfying

$$\{\Gamma^M, \Gamma^N\} = -2\eta^{MN}. \tag{B.6}$$

with mostly positive signature.

We use the following conventions for gamma matrices, and since the aim is to reduce to four-dimensions we begin with those. We denote them γ^μ, $\mu = 0, 1, 2, 3$ and use the representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \tag{B.7}$$

in mostly positive signature. We also have

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{B.8}$$

and the charge conjugation matrix $\mathcal{C}$ is give by

$$\mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \gamma_\mu^T = -\mathcal{C}\gamma_\mu\mathcal{C}^{-1}. \tag{B.9}$$

The following relations hold

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0, \qquad \gamma_\mu^* = -\gamma_0\mathcal{C}\,\gamma_\mu\,\mathcal{C}^{-1}\gamma_0 \tag{B.10}$$

and it follows that the Lorentz invariant reality condition for Dirac spinors in four-dimensions is

$$\Psi^* = \alpha\gamma_0\mathcal{C}\Psi, \qquad |\alpha|^2 = 1. \tag{B.11}$$

This reality condition can be written as

$$\Psi = \Psi^C, \qquad \Psi^C \equiv \mathcal{C}\overline{\Psi}^T, \tag{B.12}$$

where $\overline{\Psi} = \Psi^\dagger\gamma_0$. To see this consider $[\gamma_\mu, \gamma_\nu]\Psi$ and take it's complex conjugate and find a consistent reality condition of the form

$$\Psi^* = \beta\Psi, \qquad \Psi^{**} = \Psi. \tag{B.13}$$

Now on to the ten-dimensional gamma matrices. We choose to express them in terms of the four-dimensional ones as follows

$$\begin{aligned}
\Gamma^\mu &= \gamma^\mu \otimes \mathbb{1}, && \mu = 0,1,2,3,\\
\Gamma^4 &= \gamma_5 \otimes \mathbb{1} \otimes i\gamma^0, &&\\
\Gamma^{a+4} &= \gamma_5 \otimes \mathbb{1} \otimes \gamma^a, && a = 1,2,3\\
\Gamma^8 &= \gamma_5 \otimes \sigma_1 \otimes i\gamma_5, &&\\
\Gamma^9 &= \gamma_5 \otimes \sigma_2 \otimes i\gamma_5. &&
\end{aligned} \tag{B.14}$$

In this basis one finds that

$$\mathcal{C}_{10} = \mathcal{C} \otimes \sigma_2 \otimes \mathcal{C}, \qquad \Gamma_M^T = -\mathcal{C}_{10}\Gamma_M\mathcal{C}_{10}^{-1} \tag{B.15}$$

and

$$\Gamma_{11} = \Gamma^0\Gamma^1\cdots\Gamma^9 = -\gamma_5 \otimes \sigma_3 \otimes \gamma_5. \tag{B.16}$$

This is quite nice, it turns out however that we can do better. Using a unitary transformation

$$U = \mathbb{1}_4 \otimes U_8, \qquad U_8 = \begin{pmatrix}
0&0&0&0&0&-1&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&1&0&0&0&0&0\\
i&0&0&0&0&0&0&0\\
0&i&0&0&0&0&0&0\\
0&0&0&0&0&0&i&0\\
0&0&0&0&0&0&0&i
\end{pmatrix} \tag{B.17}$$

does the job. In the transformed basis $\Gamma \to U\Gamma U^\dagger$ we have

$$\mathcal{C}_{10} = \mathcal{C} \otimes \begin{pmatrix} 0 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{pmatrix}, \qquad \Gamma_{11} = \gamma_5 \otimes \begin{pmatrix} -\mathbb{1}_4 & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}. \tag{B.18}$$

where we also made sure to transform the charge conjugation matrix appropriately, i.e

$$\mathcal{C}_{10} \to U^* \mathcal{C}_{10} U^\dagger, \tag{B.19}$$

such that it still satisfies (B.15) in the new basis. Note here that it would seem that the original Brink and Schwarz paper has a sign missing in Γ_{11}. The tensor product nicely hints at the spinor decomposition

$$\Psi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_8 \end{pmatrix} \tag{B.20}$$

where $\chi_i, i = 1, 2, 3, \ldots 8$ are for the moment unconstrained Dirac spinors in four-dimensions.

We notice that the Weyl condition

$$\Gamma_{11}\Psi = -\Psi \tag{B.21}$$

implies

$$\begin{pmatrix} \chi_1 \\ \vdots \\ \chi_4 \\ \chi_5 \\ \vdots \\ \chi_8 \end{pmatrix} = \begin{pmatrix} +\gamma_5\chi_1 \\ \vdots \\ +\gamma_5\chi_4 \\ -\gamma_5\chi_5 \\ \vdots \\ -\gamma_5\chi_8 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_4 \\ \chi_5 \\ \vdots \\ \chi_8 \end{pmatrix} = \begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_5 \\ \vdots \\ R\psi_8 \end{pmatrix} \tag{B.22}$$

where

$$L = \tfrac{1}{2}(\mathbb{1} + \gamma_5), \qquad R = \tfrac{1}{2}(\mathbb{1} - \gamma_5), \tag{B.23}$$

are respectively left and right projectors that act on the four-dimensional spinors. Note that $\gamma_5 L = L$ while $\gamma_5 R = -R$. Furthermore the Majorana condition reads

$$\Psi = \mathcal{C}_{10}\Gamma_0\Psi^* \quad \Longleftrightarrow \quad \Psi = \Psi^C = \mathcal{C}\overline{\Psi}^T. \tag{B.24}$$

From $\mathcal{C}\gamma_0\gamma_5\psi_i^* = -\mathcal{C}\gamma_0\gamma_5\gamma_0 C\psi_i = -\gamma_5\psi_i$, it follows that

$$\begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_5 \\ \vdots \\ R\psi_8 \end{pmatrix} = \Psi^C = \begin{pmatrix} \mathcal{C}\gamma_0 R\psi_5^* \\ \vdots \\ \mathcal{C}\gamma_0 R\psi_8^* \\ \mathcal{C}\gamma_0 L\psi_1^* \\ \vdots \\ \mathcal{C}\gamma_0 L\psi_4^* \end{pmatrix} = \begin{pmatrix} L\psi_5 \\ \vdots \\ L\psi_8 \\ R\psi_1 \\ \vdots \\ R\psi_4 \end{pmatrix} . \tag{B.25}$$

This implies that

$$\Psi = \begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_1 \\ \vdots \\ R\psi_4 \end{pmatrix} \tag{B.26}$$

is the form of the general ten-dimensional Majorana-Weyl spinor. The spinors sitting inside it are four-dimensional Majorana spinors. We see that we have precisely four of them, $\psi_i, i = 1, 2, 3, 4$ and they are Majorana as they satisfy the four-dimensional Majorana condition $\psi_i = \mathcal{C}\overline{\psi}_i^T$.

Now that we have the Majorana-Weyl spinor decomposed in terms of four-dimensional Majorana spinors, we are almost ready to do the dimensional reduction. We should also decompose the gauge field

$$\begin{aligned} A^\mu &= A^\mu, \quad \mu = 0, 1, 2, 3, \\ A^i &= \phi^i, \qquad i = 4, 5, 6, 7, 8, 9. \end{aligned} \tag{B.27}$$

The reduction ansatz is then to "freeze" all dependence on coordinates x^i, $i = 4, 5, \ldots 9$. We find that the gauge kinetic term reduces to

$$F^{MN}F_{MN} = F^{\mu\nu}F_{\mu\nu} + 2F^{\mu i}F_{\mu i} + F^{ij}F_{ij}, \tag{B.28}$$

with

$$\begin{aligned} F^{\mu i}F_{\mu i} &= \big(\partial^\mu\phi^i - \partial^i A^\mu - i[A^\mu, \phi^i]\big)\big(\partial_\mu\phi_i - \partial_i A_\mu - i[A_\mu, \phi_i]\big) \\ &= D^\mu\phi^i D_\mu\phi_i, \end{aligned} \tag{B.29}$$

and

$$\begin{aligned} F^{ij}F_{ij} &= \big(\partial^i\phi^j - \partial^j\phi^i - i[\phi^i, \phi^j]\big)\big(\partial_i\phi_j - \partial_j\phi_i - i[\phi_i, \phi_j]\big) \\ &= -[\phi^i, \phi^j][\phi_i, \phi_j]. \end{aligned} \tag{B.30}$$

While the fermionic kinetic term reduces to

$$i\bar{\Psi}\Gamma^M D_M \Psi = i\bar{\Psi}\Gamma^\mu D_\mu \Psi + \bar{\Psi}\Gamma^i [\phi_i, \Psi], \tag{B.31}$$

where the term proportional to ∂_i was omitted since it gives no contribution due to reduction ansatz.

So the resulting four-dimensional theory has the action

$$\int d^4x \operatorname{Tr}\left\{-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}D^\mu\phi_i D_\mu\phi_i + i\bar{\Psi}\Gamma^\mu D_\mu\Psi + \bar{\Psi}\Gamma^i[\phi_i,\Psi] + \tfrac{1}{4}[\phi_i,\phi_j][\phi_i,\phi_j]\right\}. \tag{B.32}$$

We can make it more explicit, by using the gamma matrix decomposition (B.14). Specifically $\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}_8$ makes it possible for us to write

$$\begin{aligned} i\bar{\Psi}\Gamma^\mu D_\mu \Psi &= i(L\psi_i)^\dagger \gamma_0 \gamma^\mu D_\mu L\psi_i + i(R\psi_i)^\dagger \gamma_0\gamma^\mu D_\mu R\psi_i \\ &= i\bar{\psi}_i \gamma^\mu D_\mu L\psi_i + i\bar{\psi}_i\gamma^\mu D_\mu R\psi_i \\ &= i\bar{\psi}_i\gamma^\mu D_\mu \psi_i, \end{aligned} \tag{B.33}$$

where we used that $L\gamma_0 = \gamma_0 R$ and $RR = R, LL = L$ and furthermore that $\psi = L\psi + R\psi$. It would seem that in Brink and Schwarz they have forgotten the term involving R and hence have not arrived at as simple of a kinetic term.

For the remaining fermionic term we write

$$\bar{\Psi}\Gamma^i[\phi_i, \Psi] = \bar{\Psi}_j \gamma_5 \tilde{\Gamma}^i_{jk}[\phi_i, \Psi_k] \tag{B.34}$$

where

$$\Psi_j = L\psi_j, \qquad j = 1,2,3,4, \qquad \Psi_j = R\psi_j, \qquad j = 5,6,7,8, \tag{B.35}$$

and it is hopefully understood that γ_5 acts on the four-dimensional spinors sitting inside Ψ_i.

Since $RL = 0$ we conclude that only the blocks of $\tilde{\Gamma}^i_{jk}$ that mix right and left contribute, since $\overline{(R\psi)} = \bar{\psi}L$ and $\overline{(L\psi)} = \bar{\psi}R$. But indeed this is a mute statement since $\tilde{\Gamma}^i$ has the form

$$\tilde{\Gamma}^i = \begin{pmatrix} 0 & (-1)^i G^i \\ G^i & 0 \end{pmatrix}, \qquad i = 4, \dots, 9. \tag{B.36}$$

It will turn out to be convenient for us to shuffle around the definitions of our 10-dimensional gamma matrices as we defined in (B.14). Specifically

$$\begin{aligned}
\Gamma^4 &\to \Gamma^5, \\
\Gamma^5 &\to \Gamma^7, \\
\Gamma^6 &\to \Gamma^9, \\
\Gamma^7 &\to \Gamma^4, \\
\Gamma^8 &\to \Gamma^6, \\
\Gamma^9 &\to \Gamma^8.
\end{aligned} \tag{B.37}$$

Now we have

$$\tilde{\Gamma}^i = \begin{pmatrix} 0 & -G^i \\ G^i & 0 \end{pmatrix}, \quad i = 4, 5, 6, \qquad \tilde{\Gamma}^i = \begin{pmatrix} 0 & G^i \\ G^i & 0 \end{pmatrix}, \quad i = 7, 8, 9. \tag{B.38}$$

Now that means that we can write

$$\begin{aligned}
\bar{\Psi}_j \Gamma^i_{jk}[\phi_i, \Psi_k] &= \sum_{i=4}^{6} \bar{\Psi}_j \gamma_5 \tilde{\Gamma}^i_{jk}[\phi_i, \Psi_k] + \sum_{i=7}^{9} \bar{\Psi}_j \gamma_5 \tilde{\Gamma}^i_{jk}[\phi_i, \Psi_k] \\
&= \sum_{i=4}^{6} \left(\bar{\psi}_{j+4} L \gamma_5 G^i_{jk}[\phi_i, L\psi_k] - \bar{\psi}_j R \gamma_5 G^i_{jk}[\phi_i, R\psi_{k+4}] \right) + \sum_{i=7}^{9} (\{L\} + \{R\})_i \\
&= \sum_{i=4}^{6} \bar{\psi}_j G^i_{jk}[\phi_i, \psi_k] + \sum_{i=7}^{9} \bar{\psi}_j G^i_{jk}[\phi_i, \gamma_5 \psi_k] \\
&\overset{\text{cl}}{=} \sum_{i=4}^{6} \bar{\psi}_j G^i_{jk}[\phi^{\text{cl}}_i, \psi_k] \overset{!}{=} \sum_{i=1}^{3} \bar{\psi}_j G^i_{jk}[\phi^{\text{cl}}_i, \psi_k]
\end{aligned} \tag{B.39}$$

In going from the second to the third line a sign change is induced by the fact that $L\gamma_5 = L$ while $R\gamma_5 = -R$, furthermore note that $\psi_{j+4} = \psi_j$ due to Majorana-Weyl constraint in 10-dimensions. Lastly we see that when we only have $\phi^{\text{cl}}_i \neq 0$ for $i = 4, 5, 6$. From now on we will shift the indies $i, j \cdots$ by -3 so that we sum from 1 to 3, Here are the relevant G^i matrices

$$G^1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad G^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad G^3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{B.40}$$

$$G^4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad G^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad G^6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{B.41}$$

It is straight forward to check that

$$\{G^i, G^j\} = \begin{cases} +2\delta^{i,j}, & i,j = 1,2,3, \\ -2\delta^{i,j}, & i,j = 4,5,6. \end{cases} \tag{B.42}$$

$$[G^i, G^j] = 0, \qquad i = 1,2,3, \quad j = 4,5,6. \tag{B.43}$$

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C Appended Publications

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One-point Functions in AdS/dCFT from Matrix Product States

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Abstract

One-point functions of certain non-protected scalar operators in the defect CFT dual to the D3-D5 probe brane system with k units of world volume flux can be expressed as overlaps between Bethe eigenstates of the Heisenberg spin chain and a matrix product state. We present a closed expression of determinant form for these one-point functions, valid for any value of k. The determinant formula factorizes into the $k = 2$ result times a k-dependent prefactor. Making use of the transfer matrix of the Heisenberg spin chain we recursively relate the matrix product state for higher even and odd k to the matrix product state for $k = 2$ and $k = 3$ respectively. We furthermore find evidence that the matrix product states for $k = 2$ and $k = 3$ are related via a ratio of Baxter's Q-operators. The general k formula has an interesting thermodynamical limit involving a non-trivial scaling of k, which indicates that the match between string and field theory one-point functions found for chiral primaries might be tested for non-protected operators as well. We revisit the string computation for chiral primaries and discuss how it can be extended to non-protected operators.

1 Introduction

Holographic modeling of spontaneously or explicitly broken symmetries typically involves probe branes. An interesting class of quantum field theory set-ups arises when the probe brane breaks translational invariance and introduces a defect in the dual field theory. Internal degrees of freedom on the defect then originate from open strings and belong to the fundamental representation of the gauge group, while the fields in the bulk arise from closed strings and transform in the adjoint. Such defect field theories allow for novel types of correlation functions that are not possible without the defect. Examples are one-point functions of bulk fields and correlation functions involving operators localized on the defect.

In the present paper we concentrate on the defect CFT dual to the D3-D5 probe brane system with k units of background gauge field flux [1]. The brane intersection introduces a domain wall that separates the vacua with respectively unbroken $SU(N)$ and $SU(N-k)$ gauge symmetry in the $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory, with additional degrees of freedom living on the defect [2, 3]. One-point functions in this dCFT were studied in [2, 4, 5, 6] whereas two-point functions of defect operators were considered in [2, 7, 8], where integrability of the underlying $\mathcal{N}=4$ SYM proved particularly useful. The defect operators are mapped to spin chains with open boundary conditions and are dual to open strings attached to the probe D5-brane. We approach the problem from a different angle by picturing the D5-brane as a boundary state that can emit and absorb closed strings. An absorption of a single string state is represented, in the field theory, by a one-point function of a bulk operator.

The non-vanishing flux on the D5-brane represents k D3 banes dissolved in its world-volume, while in the field-theory language the symmetry-breaking is described by a non-zero vacuum expectation value of scalar fields, that form a k-dimensional unitary representation of $\mathfrak{su}(2)$ [9, 10]. In the present paper we continue our study [6] of the one-point functions in the defect CFT resulting from this semiclassical description. We also do some rudimentary analysis on the strong-coupling side of the AdS/CFT duality.

Our work relies in many ways on methods borrowed from solid state physics. It is already well-known that probe brane systems can be used to model various strongly coupled condensed matter systems (see [11] for an overview). Furthermore, the spin-chain picture of the single-trace operators in $\mathcal{N}=4$ SYM uncovers the integrable structure of the theory [12, 13] and paves the way for the use of the Bethe ansatz techniques that greatly facilitate the spectral analysis of theory. Apart from these well-known points of contact we find that so-called matrix product states (MPS), which in the condensed matter context have been used in the evaluation of quantum entanglement in one-dimensional systems, have exactly the right properties to act as a "defect state". The computation of the one-point functions in the dCFT maps to the computation of an overlap between the MPS and the Bethe eigenstates of the spin chain. Finally, the Néel state, i.e. the ground state of the classical Heisenberg anti-ferromagnet, plays a surprisingly central rôle in our investigations.

A simple set of scalar operators in the D3-D5 dCFT with non-trivial one-point functions are the operators of the form tr $Z^{L-M}W^M$, where Z and W are complex scalar fields from the $\mathcal{N} = 4$ supermultiplet. Conformal operators belonging to this $SU(2)$ sub-sector are known to be expressible as Bethe eigenstates of the Heisenberg $\mathrm{XXX}_{1/2}$ spin chain of length L in the sector with $L - M$ spins up and M spins down. Each operator is characterized by a set of M Bethe roots and, as shown in [6], only parity-symmetric operators with paired rapidities $\{u_j, -u_j\}_{j=1}^{M/2}$ and even length, L, can have non-trivial one-point functions at tree level. The one-point functions are constrained by conformal symmetry to take the form

$$\langle \mathcal{O}_L(x) \rangle = \frac{C_k(\{u_j\})}{x^L}, \tag{1.1}$$

where x is the distance to the defect.

In our previous work we found a closed expression for $C_2(\{u_j\})$ valid for any value of L and any value of M [6]:

$$C_2\left(\{u_j\}\right) = 2\left[\left(\frac{2\pi^2}{\lambda}\right)^L \frac{1}{L} \prod_j \frac{u_j^2 + \frac{1}{4}}{u_j^2} \, \frac{\det G^+}{\det G^-}\right]^{\frac{1}{2}}, \tag{1.2}$$

where $G^\pm$ are $\frac{M}{2} \times \frac{M}{2}$ matrices with matrix elements:

$$G^\pm_{jk} = \left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_n K^+_{jn}\right)\delta_{jk} + K^\pm_{jk}, \tag{1.3}$$

and $K^\pm_{jk}$ are defined as

$$K^\pm_{jk} = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2}\,. \tag{1.4}$$

The main result of the present paper is the general formula for the one-point function with arbitrary k:

$$C_k\left(\{u_j\}\right) = 2^{L-1} C_2\left(\{u_j\}\right) \sum_{j=\frac{1-k}{2}}^{\frac{k-1}{2}} j^L \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2\left(u_i^2 + \frac{k^2}{4}\right)}{\left[u_i^2 + (j-\frac{1}{2})^2\right]\left[u_i^2 + (j+\frac{1}{2})^2\right]}\,. \tag{1.5}$$

The multiplicative factor which relates C_{2n} to C_2 is simply the eigenvalue of a product of transfer matrices of the Heisenberg spin chain when acting on the Bethe state in question and C_{2n+1} is related to C_3 in a similar manner. Finally C_3 is related to C_2 via the eigenvalues of a ratio of Q-operators. Apart from being deeply rooted in integrability the formula (1.5) also has the appealing property that it allows us to take a classical, thermodynamical limit which involves scaling u_i in the same way as k. An interesting semi-classical limit with $k \to \infty$, $\lambda \to \infty$ and λ/k^2 finite exists and allows for a comparison of string and gauge theory results. So far, in this limit a match has been found between one-point functions of chiral primaries on the string and the gauge theory side [4, 5]. Formula (1.5) opens the possibility of extending the comparison to massive string states.
The outline of our paper is as follows. In section 2 we describe in slightly more detail the D3-D5 probe brane set-up and, in addition, recapitulate why matrix product states constitute a convenient tool for the calculation of one-point functions. Section 3 contains some additional insights on the $k = 2$ case. Subsequently, in section 4 we proceed to prove the multiplicative relation between C_2 and C_{2n} as well as between C_3 and C_{2n+1} for $n \geqslant 2$. Details are relegated to an appendix. The special case $k = 3$ is treated in section 5. In section 6 we consider the behavior at large-k and in the thermodynamical limit. The latter limit, in principle, allows for a comparison with string theory and in section 7 we revisit calculation of the one-point functions of the chiral primary states, now from the classical string theory perspective. This set-up bears promise of an extension to massive states. Finally, section 8 contains some concluding remarks.

2 One point functions from matrix product states.

As mentioned above, AdS/CFT set-ups relating probe brane systems with fluxes to defect conformal field theories allow for non-trivial one-point functions. In the simplest such set-up, the D3-D5-brane system, the D5-brane has the geometry $AdS_4 \times S^2$ and carries k units of magnetic flux on S^2 [1]. On the field theory side one finds $\mathcal{N} = 4$ SYM with a co-dimension one defect separating a region, $x > 0$, where the gauge group is $SU(N)$ from one where it is $SU(N-k)$ [2, 3]. For $x > 0$ the classical equations of motion then allow for a non-trivial x-dependence for some of the scalar fields, namely [10]

$$\Phi_i^{\text{cl}} = \frac{1}{x}\begin{pmatrix} (t_i)_{k\times k} & 0_{k\times(N-k)} \\ 0_{(N-k)\times k} & 0_{(N-k)\times(N-k)} \end{pmatrix}, \; i = 1,2,3, \qquad \Phi_i^{\text{cl}} = 0, \; i = 4,5,6, \tag{2.1}$$

where the three $k \times k$ matrices t_i constitute a unitary k-dimensional representation of $\mathfrak{su}(2)$, that is, they satisfy

$$[t_i, t_j] = i\varepsilon_{ijk}t_k. \tag{2.2}$$

The remaining bulk fields can consistently be set to zero at the classical level. Hence, at tree level the only operators with non-trivial one-point functions (discarding derivatives) are those which take the form

$$\mathcal{O} = \Psi^{i_1 \dots i_L} \operatorname{tr} \Phi_{i_1} \dots \Phi_{i_L}, \tag{2.3}$$

with $i_1, \dots, i_L \in \{1,2,3\}$ and, obviously, these one-point functions are obtained simply by replacing each field with its classical value, i.e.

$$\Psi^{i_1 \dots i_L} \operatorname{tr} \Phi_{i_1} \dots \Phi_{i_L} \longrightarrow \Psi^{i_1 \dots i_L} \operatorname{tr} t_{i_1} \dots t_{i_L}. \tag{2.4}$$

The natural basis of operators consists of the operators with well-defined conformal dimensions and for simplicity we will restrict our analysis to operators from an $SU(2)$ sub-sector, a sub-sector known to be closed to all loop orders. We therefore define

$$\begin{aligned} Z &= \Phi_1 + i\Phi_4, \\ W &= \Phi_2 + i\Phi_5, \end{aligned} \tag{2.5}$$

and consider only single trace operators built from these two complex scalar fields. Aiming only at tree-level one-point functions it suffices to know the conformal operators of the theory to one-loop order. It is well-known that the conformal operators in the $SU(2)$ sub-sector of $\mathcal{N} = 4$ SYM at one-loop order can be identified with the zero-momentum Bethe eigenstates of the $\text{XXX}_{1/2}$ Heisenberg spin chain upon mapping each Z-field to a spin up and each W-field to a spin down [12]. This result is unchanged by the presence of the defect [7]. Working within the approach of the algebraic Bethe ansatz the eigenstates of the $XXX_{1/2}$ Heisenberg spin chain can be written as a series of creation operators acting on the ferromagnetic vacuum (that we can take to be the state with all spins up), i.e.

$$|\{u_j\}\rangle = B(u_1)\dots B(u_M)\,|0\rangle_L\,, \tag{2.6}$$

where L denotes the length of the chain, the operator $B(u)$ creates an excitation (a flipped spin) of rapidity u and in order for the state to be an eigenstate the rapidities $\{u_j\}$ have to fulfill a set of Bethe equations, see for instance [14]. The state (2.6) has a total of L spins and M of these are down-spins. It maps to an $SU(2)$ operator built of $L-M$ fields of type Z and M fields of type W.
As pointed out in [6] one can implement the transformation (2.4) for a given Bethe eigenstate by taking the inner product of the state with a matrix product state, defined as

$$\langle \mathrm{MPS_k}| = \mathrm{tr}_a \prod_{l=1}^{L} \left(\langle \uparrow_l | \otimes t_1^{(k)} + \langle \downarrow_l | \otimes t_2^{(k)} \right), \tag{2.7}$$

where the index a is an auxiliary space index associated with the t_i's (and thus takes k different values for a representation of dimension k). Choosing the canonical normalization of the field theory two-point functions (from the theory without the defect) one can hence express the desired one-point functions as

$$C_k\left(\{u_j\}\right) = \left(\frac{8\pi^2}{\lambda}\right)^{\frac{L}{2}} L^{-\frac{1}{2}} \frac{\langle \mathrm{MPS_k}|\{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}}. \tag{2.8}$$

Without reference to the dimension of the representation, k, one can show that [6]

- $C_k\left(\{u_j\}\right)$ vanishes unless L and M are both even.
- $C_k\left(\{u_j\}\right)$ vanishes unless $\{u_j\} = \{-u_j\}$.

The states which fulfill the second criterium are the so-called unpaired states which can also be characterized as states being invariant under spin-chain parity, cf. f.inst. [15]. In particular, we note that the one-point function thus effectively depends only on $M/2$ rapidities.

3 The $k = 2$ case

The overlap for $k = 2$ was found in [6] and is given by eq. (1.2) in the introduction. It was observed in [6] that for $M = L/2$ the overlap, up to a simple factor, coincided with the overlap between the Bethe eigenstate and the Néel state, i.e. the state with alternating spins which is the ground state of the anti-ferromagnetic Heisenberg spin chain

$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow \ldots \uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow \ldots \downarrow\uparrow\rangle\,. \tag{3.1}$$

This fact could be exploited to construct a proof of the formula (1.2) for $M = L/2$ as it could be proved that, restricted to the components with half-filling, the matrix product state is cohomologically equivalent to the Néel state

$$|\text{MPS}_2\rangle\Big|_{M=\frac{L}{2}} = \frac{1}{2^L(\frac{i}{2})^M}\,|\text{Néel}\rangle + S^-\,|\ldots\rangle\,. \tag{3.2}$$

Here S_i is the total spin operator, and S^- is its lowering component that flips in turn all the spins in the chain with weight one. Since Bethe eigenstates are highest-weight:

$$S^+\,|\{u_j\}\rangle = 0, \tag{3.3}$$

the second term in (3.2) does not contribute to the inner product between the matrix product state and the Bethe state. In this way the result for the one-point function corresponding to a Bethe eigenstate with half filling followed from the overlap formula for the Néel state derived in [16], see also [17]. Away from half-filling the formula (1.2) continues to hold. This can be understood from an earlier result for the overlap between a Bethe eigenstate and the $(2m)$-fold raised Néel state [18].[1] More precisely, it follows by noting

$$|\text{MPS}_2\rangle\Big|_{M=\frac{L}{2}-2m} = \frac{1}{2^L(\frac{i}{2})^M}\frac{1}{(2m)!}(S^+)^{2m}\,|\text{Néel}\rangle + S^-\,|\ldots\rangle\,. \tag{3.4}$$

This result directly follows from the fact that the $(2m)$-fold raised Néel state is equivalent to the generalized Néel state (compare eq. (5.5) from [6] and eq. (38) from [18]), which was shown to be cohomologically equivalent to the matrix product state in [6]. See [19] for a rederivation of the $k = 2$ overlap formula away from half-filling using reflecting-boundary domain-wall boundary conditions.

[1] We thank Stefano Mori for pointing this out to us.

4 The general k case

As explained in [6] one can explicitly evaluate the overlap (2.8) for lower values of L, M and k by choosing a specific k-dimensional representation of $\mathfrak{su}(2)$ and making use of the well-known coordinate space version of the Bethe eigenstates. It was the results of such evaluations that first lead us to the main result (1.5).
In this section we will prove a recursive relation between matrix product states with different values of k. More precisely, we will shown that all matrix product states with even k are recursively related to the matrix product state with $k = 2$ via the action of a series of transfer matrices of the Heisenberg spin chain. Similarly, all matrix product states with odd k are shown to be recursively related to the matrix product state with $k = 3$, and finally evidence is presented that the matrix product state for $k = 3$ can be generated from the matrix product state for $k = 2$ by the action of a ratio of Baxter's Q-operators. The general result (1.5) then follows from the fact that the Bethe eigenstates are eigenstates of the transfer matrix as well as of Baxter's Q-operator with known eigenvalues.
For illustrative purposes, let us spell out the general formula (1.5) in a few cases

$$C_3(\{u_j\}) = C_2(\{u_j\})\, 2^L \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2}{u_i^2 + \frac{1}{4}}, \tag{4.1}$$

$$C_4(\{u_j\}) = C_2(\{u_j\}) \left[3^L \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2}{u_i^2 + 1} + \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2 + 4}{u_i^2 + 1} \right], \tag{4.2}$$

$$C_5(\{u_j\}) = C_3(\{u_j\}) \left[2^L \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2 + \frac{1}{4}}{u_i^2 + \frac{9}{4}} + \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2 + \frac{25}{4}}{u_i^2 + \frac{9}{4}} \right]. \tag{4.3}$$

The previously announced recursive relation between matrix product states with different values of k takes the following form

$$|\mathrm{MPS_{k+2}}\rangle = T(\tfrac{ik}{2})\,|\mathrm{MPS_k}\rangle - \left(\frac{k+1}{k-1}\right)^L |\mathrm{MPS_{k-2}}\rangle, \tag{4.4}$$

where $k \geqslant 2$ and $|\mathrm{MPS_0}\rangle = 0$. Here $T(v)$ is the transfer matrix of the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain

$$T(v) := \mathrm{tr}_a(\mathcal{L}_{a1} \ldots \mathcal{L}_{aL}), \tag{4.5}$$

with $\mathcal{L}$ the Lax operator

$$\mathcal{L}_{a,i}(v) = 1 + \frac{i}{v - \frac{i}{2}} P, \tag{4.6}$$

which is expressed in terms of the permutation operator P. As usual the label a refers to an auxiliary 2-dimensional space, $\mathbb{C}^2$, which is traced over in the definition of $T(v)$. For details we refer to [14]. The idea behind the proof of formula (4.4) is to consider the local action of the Lax operator. The matrix product state (2.7) is an element of $\mathbb{C}^{2L}$ and it is constructed out of the local building blocks

$$\Big(\langle\uparrow| \otimes t_1^{(k)} + \langle\downarrow| \otimes t_2^{(k)} \Big) \in \mathbb{C}^2 \otimes \mathrm{GL}(\mathbb{C}^k). \tag{4.7}$$

Now, we add an additional auxiliary $\mathbb{C}^2$ space and consider the action of the Lax operator on the physical space and the new auxiliary space which gives

$$\mathcal{L}_{i,a}(\tfrac{ik}{2}) \Big[\langle\uparrow_i| \otimes t_1^{(k)} + \langle\downarrow_i| \otimes t_2^{(k)} \Big] =: \Big(\langle\uparrow_i| \otimes \tau_1^{(k)} + \langle\downarrow_i| \otimes \tau_2^{(k)} \Big) \in \mathbb{C}^2 \otimes \mathrm{GL}(\mathbb{C}^{2k}),$$

where the matrices $\tau_{1,2}^{(k)}$ are given by

$$\tau_1^{(k)} = \begin{pmatrix} \frac{k+1}{k-1} t_1^{(k)} & 0 \\ \frac{2}{k-1} t_2^{(k)} & t_1^{(k)} \end{pmatrix}, \qquad \tau_2^{(k)} = \begin{pmatrix} t_2^{(k)} & \frac{2}{k-1} t_1^{(k)} \\ 0 & \frac{k+1}{k-1} t_2^{(k)} \end{pmatrix}. \tag{4.8}$$

In the appendix we show explicitly for even as well as for odd $k \geqslant 2$, that there exists a similarity transformation A such that

$$A \tau_i^{(k)} A^{-1} = \begin{pmatrix} t_i^{(k+2)} & 0 \\ \star_i & \frac{k+1}{k-1} t_i^{(k-2)} \end{pmatrix}. \tag{4.9}$$

This relation immediately proves the recursion relation (4.4) for $k \geqslant 2$.
The transfer matrix is the key ingredient of the Algebraic Bethe ansatz. In particular, the Bethe states $|\{u_i\}\rangle$ are eigenvectors of the transfer matrix with eigenvalues

$$\Lambda(v|\{u_i\}) = \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right)^L \prod_i \frac{v - u_i - i}{v - u_i} + \prod_i \frac{v - u_i + i}{v - u_i}. \tag{4.10}$$

The recursion relation (4.4) hence allows us to fix all overlap functions C_{2n} with $n \geqslant 2$ in terms of C_2 and $C_0 \equiv 0$, as well as all C_{2n+1} with $n \geqslant 2$ in terms of C_3 and $C_1 \equiv 0$ by means of the following recursion relation

$$C_{k+2} = \Lambda(\tfrac{ik}{2}|\{u_i\})C_k - \left(\frac{k+1}{k-1}\right)^L C_{k-2}. \tag{4.11}$$

It is easily checked that (1.5) (for $k \geqslant 2$) is the unique solution to this equation.

5 The special case $k = 3$.

The analysis of the previous section involving the transfer matrix did not allow us to prove the relation (4.1) for $C_3(\{u_j\})$. However, this relation, which was observed by studying short chains of length $L \leqslant 10$, seems to indicate that a ratio of two so-called Q-operators could relate $|\mathrm{MPS}_3\rangle$ and $|\mathrm{MPS}_2\rangle$. The Q operator was originally introduced by Baxter in connection with his solution of the 8-vertex model [20]. Only recently, an explicit algebraic construction, especially adapted to the $XXX_{1/2}$ Heisenberg chain was carried out [21], see also [22]. The Bethe eigenstates are eigenstates of the Q-operator, i.e. they fulfill a relation like

$$\hat{Q}(u)\,|\{u_j\}\rangle \propto \prod_{j=1}^{M}(u-u_j)\,|\{u_j\}\rangle\,. \tag{5.1}$$

The algebraic construction of the Q-operator from [21] is strictly speaking only well-defined for the Heisenberg spin chain when a certain twist, ϕ, is introduced. The twist can be introduced either at the level of the Hamiltonian or entirely via the boundary conditions. In the latter case the spin chain boundary conditions turn into

$$\mathcal{S}^z_{L+1} = \mathcal{S}^z_1, \qquad \mathcal{S}^{\pm}_{L+1} = e^{\mp i\phi}\mathcal{S}^{\pm}_1. \tag{5.2}$$

In the presence of the twist, the action of the Q operator on a Bethe eigenstate gives rise to a product of not only M, but a larger number of factors of the type $(u-u_j)$, hence involving an extra set of rapidities which, however, all tend to infinity when the twist is sent to zero. The extra rapidities contain information about Bethe eigenstates in the twisted model which become descendent states in the limit $\phi \to 0$. Although the Q-operator itself is thus

ill-defined in the zero twist limit, a ratio of two Q-operators is generically finite and can give rise to exactly the pre-factor in (4.1).
In analogy with the transfer matrix, the Q-operator can be defined as the trace of a certain monodromy matrix [21]. The auxiliary Hilbert space associated with the monodromy is infinite dimensional, namely the Fock space, $\mathcal{F}$, associated with the usual harmonic oscillator algebra

$$\left[\mathbf{a}, \mathbf{a}^\dagger\right] = 1. \tag{5.3}$$

In other words the auxiliary Hilbert space $\mathcal{F}$ is spanned by the vectors $|n\rangle$, $n \in Z_0$ which fullfil

$$\mathbf{a}^\dagger|n\rangle = |n+1\rangle, \qquad \mathbf{a}|n\rangle = n|n-1\rangle. \tag{5.4}$$

The Q operator itself then takes the form

$$Q(u) := \frac{e^{\frac{\phi}{2}u}}{\mathrm{tr}_{\mathcal{F}}\left(e^{-i\phi\mathbf{h}}\right)} \mathrm{tr}_{\mathcal{F}}\left(e^{-i\phi\mathbf{h}}\, L_L(u) \otimes \ldots \otimes L_1(u)\right), \tag{5.5}$$

where ϕ is the twist, $\mathbf{h} = \mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}$, and

$$L_l(u) = \begin{pmatrix} 1 & \mathbf{a}^\dagger \\ -i\,\mathbf{a} & u - i\mathbf{h} \end{pmatrix}_l . \tag{5.6}$$

This explicit form of the Q operator makes it straightforward to implement it in Mathematica and by explicit computations one can demonstrate that for short matrix product states ($L \leqslant 8$) one has

$$\lim_{\phi\to 0} Q(\tfrac{i}{2})^{-1}Q(0)\,|\mathrm{MPS}_2\rangle = 2^{-L}\,|\mathrm{MPS}_3\rangle + S^-|\ldots\rangle. \tag{5.7}$$

Note that $Q(\frac{i}{2})^{-1}Q(0)$ is divergent in the $\phi \to 0$ limit due to the fact that $u = \frac{i}{2}$ corresponds to a singular point for the Bethe equations for any L. However, it turns out that the divergencies in the vector $Q(\frac{i}{2})^{-1}\,Q(0)|\mathrm{MPS}_2\rangle$ appear in prefactors of terms of the type $S^-|\ldots\rangle$ which have zero overlap with a Bethe eigenstate.[2] We would also like to note that the term $S^-|\ldots\rangle$ first appears for $L = 8, M = 4$. In particular, for the other values of L, M that

[2] Thus, strictly speaking in order to have a well-defined version of eqn. (5.7), one would have to redefine the left hand side with a term proportional to S^-. This is a further complication of the $\phi \to 0$ limit.

were checked, the left hand side of (5.7) is finite and the ratio of Q-operators exactly relates the matrix product states.[3]
If formula (5.7) could be proved true for any length it would immediately imply relation (4.1) as the Bethe eigenstates of the untwisted Heisenberg spin chain are highest weight states. The construction of the Q operator as a monodromy matrix makes it tempting to speculate about the possibility of a proof relying only on the local operator $L_l(u)$, similar in idea to the proof of the recursive structure of the overlap formula. However, the need for an inversion of Q, a limiting procedure as well as the appearance of the term involving the lowering operator complicate matters.

6 Large k

The general formula for the one-point function (1.5) is valid for any k under the assumption that $k \ll N$. An interesting limit to consider is to take k very large (but still small compared to N). At strong coupling, k quite naturally scales with λ such that the ratio $k/\sqrt{\lambda}$ remains finite at $\lambda \to \infty$. This ratio controls the field strength of the internal gauge field on the world-volume of the D5-brane, the holographic dual of the domain wall that separates the two vacua. The classical solution for the D5-brane [1] depends only on $k/\sqrt{\lambda}$, but not on λ or k separately.
In this paper we study the weak-coupling regime when scaling k with λ makes little sense, but we can still take $k \gg 1$. The large-k limit of the overlap that involves a small number of excitations (up to $M = 4$) has already been considered in [6]. With the explicit expression at hand, we can now take the large-k limit in full generality, for any M. We can also consider the thermodynamic limit when the length of the spin chain L, the number of excitations M and the rank of the $\mathfrak{su}(2)$ representation k go to infinity simultaneously such that $L \sim M \sim k$.
When k is large, while L and M are of order one, the sum over j in the general

[3] One can also consider the equation

$$Q(0)\,|\mathrm{MPS}_2\rangle = 2^{-L} Q(\tfrac{i}{2})\,|\mathrm{MPS}_3\rangle + S^{-}\,|\ldots\rangle\,, \tag{5.8}$$

which we found to hold for $L \leqslant 10$ and *any* value of ϕ. This equation is also divergent in the limit of vanishing twist and again the divergencies are of the form $S^{-}\,|\ldots\rangle$.

formula (1.5) is saturated on the upper (or lower) limit of summation:

$$C_k(\{u_j\}) \simeq 2^{L-M} C_2(\{u_j\}) \prod_{i=1}^{\frac{M}{2}} u_i^2 \sum_{j=1}^{\frac{k}{2}} j^{L-2M}, \tag{6.1}$$

and yields the following result

$$C_k(\{u_j\}) \simeq \frac{2^{M-1} \prod_{i=1}^{\frac{M}{2}} u_i^2}{L-2M+1} C_2(\{u_j\})\, k^{L-M+1} + \mathcal{O}(k^{L-M}), \tag{6.2}$$

whose dependence on k agrees with the scaling indicated in [6] and reproduces in detail the particular cases $M = 0, 2, 4$ studied there.

Alternatively, we can take k to infinity simultaneously with L and M. The limit when the spin chain becomes infinitely long and is populated by a large number of low-lying excitations is the semiclassical limit of the Heisenberg model. The Bethe roots in this regime scale as $u_j \sim L$, while $M \sim L \to \infty$ [23, 24]. Bethe states of this type describe macroscopic, essentially classical waves of coherent spin precession [25].

While taking the semiclassical limit at weak coupling is not exactly the same as considering classical strings in $AdS_5 \times S^5$, quantities calculated in classical string theory depend on λ through the combination λ/L^2. By re-expanding the string results in this parameter one can often reproduce the weak-coupling perturbation theory up to some fixed order in λ/L^2. The agreement of the BMN spectrum [26] with magnon energies in the spin chain, or comparison of classical spinning strings in S^5 [27] with semiclassical Bethe states [24, 28] are two well-known examples where this approach works. In the context of the defect CFT, the one-point functions of protected operators with small L and $M = 0$ also perfectly agree with the classical supergravity calculation expanded in λ/k^2 [4, 5]. Keeping in mind possible comparison to semiclassical string theory (rather than supergravity), we will compute one-point functions of non-protected operators with $M \sim L$ in the thermodynamic limit, taking in addition $k \sim L$ at $L \to \infty$.

The Bethe roots in the thermodynamic limit condense on a number of cuts in the complex plane and can be characterized by a continuous density

$$\rho(x) = \frac{1}{L} \sum_{j=1}^{\frac{M}{2}} \left(\delta\left(x - \frac{u_j}{L}\right) + \delta\left(x + \frac{u_j}{L}\right)\right). \tag{6.3}$$

The density satisfies a singular integral equation, as a consequence of the Bethe equations for u_j's:

$$2 \fint_{C} \frac{dy\,\rho(y)}{x-y} = \frac{1}{x} + 2\pi n_i, \qquad x \in C_i. \tag{6.4}$$

Each of the cuts C_i is associated with an integer mode number n_i. The normalization of the density is the filling fraction,

$$\int_{C} dx\,\rho(x) = \frac{M}{L} \equiv \alpha, \tag{6.5}$$

and $\alpha \leqslant 1/2$ for physical, highest-weight Bethe states.
The ratio of determinants in (1.2) tends to a constant in the thermodynamic limit [6], while the products over Bethe roots in (1.5) exponentiate and can be replaced by convolution integrals with the density. Approximating summation over j by integration over $\xi = j/L$, we find:

$$C_k \simeq \mathrm{const}\,\sqrt{L}\left(\frac{8\pi^2 L^2}{\lambda}\right)^{\frac{L}{2}} \int_{-\chi}^{\chi} d\xi\ \mathrm{e}^{\,LS_{\mathrm{eff}}(\xi)}, \tag{6.6}$$

where

$$\chi = \frac{k}{2L} \tag{6.7}$$

and

$$S_{\mathrm{eff}}(\xi) = \frac{1}{2}\int dx\,\rho(x)\ln\frac{x^2\left(x^2+\chi^2\right)}{\left(x^2+\xi^2\right)^2} + \ln|\xi|. \tag{6.8}$$

The integral is again saturated at $\xi = \pm\chi$, and we get for the following result for the overlap in the thermodynamic limit:

$$C_k \sim \frac{\mathrm{const}}{\sqrt{L}}\left(\frac{2\pi^2 k^2}{\lambda}\right)^{\frac{L}{2}} \mathrm{e}^{\,-\mathcal{A}L}, \tag{6.9}$$

where

$$\mathcal{A} = \frac{1}{2}\int dx\,\rho(x)\ln\frac{x^2+\chi^2}{x^2}\,. \tag{6.10}$$

The simplest example is the BMN vacuum of the spin chain, the empty state with no Bethe roots that corresponds to the chiral primary operator $\mathrm{tr}\, Z^L$. In this case $\mathcal{A} = 0$, and with exponential accuracy

$$\left\langle \mathrm{tr}\, Z^L \right\rangle_{\mathrm{def}} \simeq \left(\frac{\sqrt{2}\,\pi k}{\sqrt{\lambda}} \right)^L \frac{1}{R^L}\,, \tag{6.11}$$

where R is the distance from the operator insertion to the defect.
The simplest non-trivial solution of the finite-gap (classical Bethe) equations (6.4), which is symmetric under $x \to -x$, has two cuts $(\mathtt{x}_1, \mathtt{x}_2)$ and $(-\mathtt{x}_1, -\mathtt{x}_2)$ symmetrically located in the complex plane, such that $\mathtt{x}_2 = \bar{\mathtt{x}}_1$. The mode numbers of this solution are n and $-n$. The density can be expressed through elliptic integrals [24], but it is more convenient to characterize the solution by the quasi-momentum

$$p(x) = \int \frac{dy\, \rho(y)}{x - y} - \frac{1}{2x}\,, \tag{6.12}$$

whose differential is meromorphic on the two-sheeted cover of the complex plane with cuts. For the two-cut solution [28],

$$dp = \frac{\frac{1}{2} - \alpha - \frac{\mathtt{x}_1 \mathtt{x}_2}{2x^2}}{\sqrt{(x^2 - \mathtt{x}_1^2)\,(x^2 - \mathtt{x}_2^2)}}\, dx. \tag{6.13}$$

The solution is parameterized by a single complex variable

$$r = \frac{\mathtt{x}_1^2}{\mathtt{x}_2^2}\,, \tag{6.14}$$

through which the endpoints are expressed as

$$\mathtt{x}_1 = \frac{1}{4nK}\,, \qquad \mathtt{x}_2 = \frac{1}{4n\sqrt{r}\, K}\,, \tag{6.15}$$

while the filling fraction is given by

$$\alpha = \frac{1}{2} - \frac{E}{2\sqrt{r}\, K}\,, \tag{6.16}$$

where $E \equiv E(1 - r)$ and $K \equiv K(1 - r)$ are the complete elliptic integrals of the second and first kind.

To compute (6.10) we first express its derivative with respect to χ through the quasi-momentum:

$$\frac{\partial \mathcal{A}}{\partial \chi} = i \int_{-i\chi}^{i\chi} dp(x) + \frac{1}{\chi}\,. \tag{6.17}$$

Integrating (6.13) twice we obtain:

$$\begin{aligned}
\mathcal{A} \;=\; & \frac{\chi}{\mathtt{x}_1 K}\left(EF\left(\varphi\right) - KE\left(\varphi\right)\right) - \left(1 - 2\alpha\right)\ln\frac{\sqrt{\chi^2 + \mathtt{x}_1^2} + \sqrt{\chi^2 + \mathtt{x}_2^2}}{\mathtt{x}_1 + \mathtt{x}_2} \\
& + \frac{1}{2}\ln\frac{\chi^2\left(\mathtt{x}_1^2 + \mathtt{x}_2^2\right) + 2\mathtt{x}_1^2\mathtt{x}_2^2 + 2\mathtt{x}_1\mathtt{x}_2\sqrt{\left(\chi^2 + \mathtt{x}_1^2\right)\left(\chi^2 + \mathtt{x}_2^2\right)}}{4\mathtt{x}_1^2\mathtt{x}_2^2} \\
& - \frac{\mathtt{x}_1}{\mathtt{x}_2}\sqrt{\frac{\chi^2 + \mathtt{x}_2^2}{\chi^2 + \mathtt{x}_1^2}} + 1,
\end{aligned} \tag{6.18}$$

where $F(\varphi)$ and $E(\varphi)$ are the incomplete elliptic integrals of the same modulus $1 - r$, and argument

$$\tan\varphi = \frac{\chi}{\mathtt{x}_1}\,. \tag{6.19}$$

The one-point function exponentiates in the thermodynamic limit, which suggests a semiclassical interpretation. Since the exponent is always negative (it is easy to see that $\mathcal{A} > 0$), the overlap is exponentially suppressed and perhaps can be interpreted as a tunneling amplitude of a transition between a Bethe eigenstate and the MPS or generalized Néel state. If this interpretation is correct the transition amplitude could probably be described in the semiclassical regime by an instanton solution of the Landau-Lifshitz equations, the classical equations of motion of the Heisenberg model. We are not in a position to construct such a solution here. Instead we will study the one-point function at strong coupling, where the description from the very beginning is in classical terms.

7 Comparison to string theory

In string theory, the defect that separates the $SU(N)$ and $SU(N-k)$ vacua is described by a D5-brane embedded in $AdS_5 \times S^5$, and carrying k units of magnetic flux on its world-volume. The magnetic flux naturally scales with λ such that

$$\kappa = \frac{\pi k}{\sqrt{\lambda}} \tag{7.1}$$

remains finite in the strong-coupling limit.

The brane embedding is very simple in Poincaré coordinates

$$ds^2 = \frac{dx^2 + dz^2}{z^2}\,. \tag{7.2}$$

The brane intersects AdS_5 along the AdS_4 hyperplane, tilted with respect to the boundary at an angle that depends on the magnetic flux [1, 4]:

$$x = \kappa z, \tag{7.3}$$

where x is the direction perpendicular to the defect (for instance, $x = x^3$ if the defect is the domain wall in the x^1-x^2 plane). The remaining two dimensions of the brane wrap the equatorial two-sphere in S^5. The orientation of the S^2 within S^5 is dictated by the R-symmetry quantum numbers of the defect. The solution (2.1) involves scalar fields Φ_1, Φ_2, Φ_3. When S^5 is represented by a unit sphere in $\mathbb{R}^6$, each Φ_i is dual to the i-th coordinate direction, and consequently the D-brane intersects the S^5 along the 123-plane. The background gauge field is a constant (monopole) magnetic field with k units of field strength on S^2.

The process of emitting or absorbing a string by a D-brane is described by a string world-sheet attached to the D-brane at its constant-τ section, which we take to be $\tau = 0$. The boundary conditions are then of the Dirichlet type for the coordinates transverse to the brane (X^i) and mixed Neumann-Dirichlet for the longitudinal coordinates X^μ:

$$\begin{aligned} &\partial_\sigma X^i = 0, \\ &\partial_\tau X_\mu + \frac{2\pi}{\sqrt{\lambda}}\, F_{\mu\nu}\, \partial_\sigma X^\nu = 0, \end{aligned} \tag{7.4}$$

where $F_{\mu\nu}$ is the internal gauge field on the D-brane world-volume.

The one-point function of a local operator is computed by inserting a vertex operator in the string path integral:

$$\langle \mathcal{O}(x) \rangle_{\text{def}} = \int DX^M \int d^2w\, V_{\mathcal{O}}\left(X(w)|x\right)\, \mathrm{e}^{-\frac{\sqrt{\lambda}}{2\pi} S_{\text{str}}[X]}. \tag{7.5}$$

The vertex operator, schematically, has the following form:

$$V(X|x) = \partial X\, \partial X\, \mathrm{e}^{\Sigma(X)}, \tag{7.6}$$

where, roughly speaking, e^{Σ} is the wave function of the corresponding string mode in $AdS_5 \times S^5$. The exponent is proportional to the quantum number of the string state, and for large quantum numbers, $\Sigma \sim Q \sim \sqrt{\lambda}$ is of the same order of magnitude as the string action.
In the semiclassical approximation, valid at $\lambda \to \infty$, the path integral over X^M, as well as the integration over the position of the vertex operator, are saturated on the saddle point of the integrand:

$$\langle \mathcal{O}(x) \rangle_{\text{def}} \simeq V_{\mathcal{O}}\left(X_{\text{cl}}(w_0)|x\right) e^{-\frac{\sqrt{\lambda}}{2\pi} S_{\text{str}}[X_{\text{cl}}]}, \tag{7.7}$$

where $\simeq$ denotes equality with exponential accuracy, and X^M_{cl} is the solution of the string equations of motion [29]:

$$\frac{\delta S_{\text{str}}}{\delta X^M} = \frac{2\pi}{\sqrt{\lambda}} \frac{\partial \Sigma}{\partial X^M} \delta\left(w - w_0\right). \tag{7.8}$$

The boundary conditions are the Dirichlet-Neumann ones (7.4) on the end of the string which is attached to the D-brane. The delta-function source in the equations of motion can be traded for boundary conditions at the other end (at $w \to w_0$). In the simplest case both the equations of motion and the source can be linearized near $w = w_0$:

$$\begin{aligned} \frac{\delta S_{\text{str}}}{\delta X^M} &= -\partial^2 X_M + \ldots \\ \Sigma &= Q_M X^M + \ldots \end{aligned}$$

The delta-function then produces a logarithmic singularity at w_0 in X^M:

$$X_M = -\frac{Q_M}{\sqrt{\lambda}} \ln|w - w_0| + \ldots \tag{7.9}$$

It is convenient to introduce exponential coordinates near w:

$$w - w_0 = e^{i\sigma - \tau}, \tag{7.10}$$

The boundary conditions then take the familiar form of the string moving in the direction X^M with momentum Q_M:

$$X_M = \frac{Q_M}{\sqrt{\lambda}} \tau + \ldots \tag{7.11}$$

In this paper we only consider the simplest case of the chiral primary operator $\mathcal{O} = \operatorname{tr} Z^L$. The dual vertex operator is known exactly [30], but for our purposes the exponential accuracy would suffice:

$$V_{\rm CPO} \simeq 2^{\frac{L}{2}} z^{-L}\, \mathrm{e}^{iL\varphi}. \tag{7.12}$$

Here φ is the angle in the 14-plane in $\mathbb{R}^6$ (the orientation is again dictated by the R-symmetry quantum numbers of the field Z in (2.5)).
The classical string solution with the boundary conditions described above can be constructed by the method of images, placing a fictitious source at the same distance R on the other side of the defect and considering a two-point function $\left\langle \operatorname{tr} \bar{Z}^L(-R) \operatorname{tr} Z^L(R)\right\rangle$. The classical solution for the latter is the Euclidean continuation of the BMN geodesic [31]:

$$\begin{aligned}
\varphi &= i\omega\tau, \\
x &= R\tanh\omega\left(\tau+\tau_0\right), \\
z &= \frac{R}{\cosh\omega\left(\tau+\tau_0\right)}\,.
\end{aligned} \tag{7.13}$$

We take the $\tau > 0$ portion of the world-sheet as the solution for the string ending on the D-brane. The solution automatically satisfies the right boundary conditions at the operator insertion point ($\tau = \infty$), provided that

$$\omega = \frac{L}{\sqrt{\lambda}}\,, \tag{7.14}$$

which follows from comparing (7.11) with (7.12).
As for the boundary conditions on the D-brane, the solution can be made compatible with them by adjusting the constant of integration τ_0. For a point-like string the boundary conditions are Dirichlet in the transverse directions and purely Neumann in the longitudinal ones (the magnetic field plays no rôle because $\partial_\sigma X^\mu = 0$). Geometrically, these boundary conditions mean that the string world-sheet should meet the D-brane (7.3) at the right angle. And indeed, the string forms a semi-circle centered at zero, for which the D5-brane (projected onto the xz plane) is a radius and so the two are perpendicular at the point of intersection (fig. 1). The same is true on S^5, where the string trajectory approaches the brane, sitting in the 123-plane, perpendicularly, along the 14-plane.

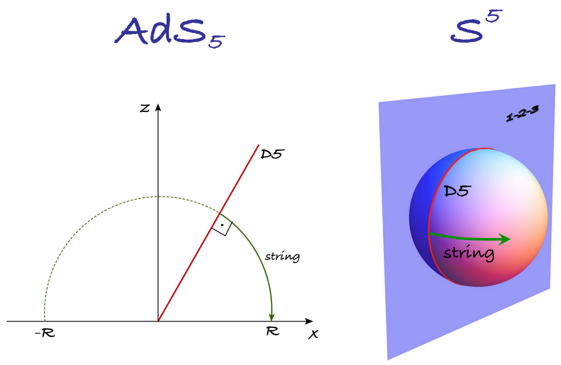

Figure 1: The BMN string ending on the brane can be constructed by the method of images from the solution that describes the two-point function of the operators inserted at points $-R$ and R placed symmetrically on the two sides of the defect. Since the string world-line in AdS_5, geometrically, is a semicircle, it is perpendicular to the D5-brane and hence satisfies the correct Dirichlet-Neumann boundary conditions.

It is only necessary to make sure that string emission is simultaneous in S^5 and AdS_5. This can done by adjusting the parameter τ_0. The condition for $X^M(0)$ to lie on the D-brane worldvolume (7.3) is

$$\kappa = \frac{x(0)}{z(0)} = \sinh \omega\tau_0, \tag{7.15}$$

which determines τ_0 in terms of κ.
The relevant part of the string action (the solution is in the conformal gauge),

$$S = \frac{1}{2}\int d\tau d\sigma \left[\frac{(\partial x)^2 + (\partial z)^2}{z^2} + (\partial\varphi)^2\right], \tag{7.16}$$

evaluates to zero on the classical solution (7.13), so the contribution to the one-point functions comes entirely from the vertex operator. From (7.12) we find:

$$\left\langle \operatorname{tr} Z^L \right\rangle_{\text{def}} \simeq \frac{2^{\frac{L}{2}}}{R^L} \lim_{\tau\to\infty} \cosh^L \omega\left(\tau+\tau_0\right) \mathrm{e}^{-\omega L\tau} = \left(\frac{\kappa+\sqrt{\kappa^2+1}}{\sqrt{2}}\right)^L \frac{1}{R^L}\,, \tag{7.17}$$

where we have used (7.15) in the last equality. This result agrees with the supergravity calculation [4] in their overlapping regime of validity. One can check that at large[4] L , the integral (3.15) in [4] is saturated by the saddle-point which results in (7.17). This is not surprising, since at large L the geodesic approximation should be valid for the supergraviton propagator in AdS_5, making string and supergravity calculations manifestly equivalent.
This result is valid at $\lambda \to \infty$, $k \to \infty$, with $k/\sqrt{\lambda}$ fixed. In this approximation the one-point function does not depend on λ and k separately but only on their ratio, and we have not made any assumptions on whether this ratio is big or small. Assuming that κ is big we can expand the answer in $1/\kappa^2 = \lambda/\pi^2 k^2$ getting a power series that resembles in form the ordinary perturbation theory. To leading order we get:

$$\left\langle \operatorname{tr} Z^L \right\rangle_{\text{def}} \simeq \frac{1}{R^L}\left(\frac{\sqrt{2}\pi k}{\sqrt{\lambda}}\right)^L \left(1+O\left(\frac{\lambda}{k^2}\right)\right), \tag{7.18}$$

in complete agreement with the weak-coupling prediction (6.11).

8 Conclusions

Our general form for the one-point function (1.5) hints that integrability may play a more profound role in the present context than we have so far been able to reveal. First, the recursive structure of the one-point function formula hinges on the properties of the transfer matrix of the Heisenberg spin chain and seems to indicate the possibility of a proof which builds more directly on the algebraic Bethe ansatz approach. Secondly, the relation between the results for the two- and three-dimensional representation points towards a novel application of Baxter's Q-operator.
As pointed out previously, one-point functions of chiral primary operators calculated in dCFT's have been successfully matched to one-point functions calculated in a supergravity approach [4, 5]. The fact that we have derived

[4]The R-charge is denoted by l in [4].

an overlap formula valid for any value of k opens up a vast new arena for the comparison between field theory and string theory, namely the comparison of one-point functions of massive operators. We have taken a first step towards entering this arena by re-formulating the gravity computation of the chiral primary one-point function in a way which in principle allows for a generalization to massive states. Implementing this generalization constitutes an interesting and challenging future line of investigation.

Our work points towards several other possible lines of investigation. One-point functions on the field theory side could be studied at higher loop orders, in bigger sectors or for other types of defect field theories resulting from probe-brane set-ups with fluxes. It would also be interesting to investigate in further detail the exact role of the Q-operator in the present context and to find a proof of (5.7).

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A Similarity transformation

In this section we present a similarity transformation matrix A and the matrix quantities $\star_i$ which fulfill

$$A\tau_i^{(k)}A^{-1} = \hat{t}_i^{(k)}, \qquad i = 1, 2, \tag{A.1}$$

where

$$\hat{t}_i^{(k)} = \begin{pmatrix} t_i^{(k+2)} & 0 \\ \star_i & \frac{k+1}{k-1} t_i^{(k-2)} \end{pmatrix}. \tag{A.2}$$

The quantities A, A^{-1} and $\star_i$ are expressed in terms of the matrix unities E^i_j for which

$$E^i_j\,E^k_l = \delta^k_j\,E^i_l. \tag{A.3}$$

It is then a tedious albeit straightforward computation to show that

$$AA^{-1} = 1, \qquad \text{and} \qquad A\tau^{(k)}_i = \hat{t}_i A. \tag{A.4}$$

Constructing A

We define the following functions

$$K[k,j] = \left(\frac{k+1}{k-1}\right)^{\frac{(j-2)(j+1)}{4}}\left(\frac{k-2}{k}\right)^{\frac{(j-2)(j-1)}{4}}, \tag{A.5}$$

$$F[k] = \frac{k(k-1)}{k+1}\sqrt{\frac{k-1}{k-2}}, \tag{A.6}$$

$$H[k] = \frac{\sqrt{2}}{k+1}\sqrt{\frac{k-1}{k-2}} \tag{A.7}$$

and

$$G[j] = \sqrt{j(j+1)}. \tag{A.8}$$

Furthermore the matrix structure is such that we can write it in terms of the matrices

$$Z^n_m = E^n_m + i\,E^n_{k+m}, \qquad \text{and} \qquad W = Z^T, \tag{A.9}$$

where E^n_m are the $2k \times 2k$ matrix unities – the one appears in the n-th row in column m. We will also use the complex conjugates of these matrices, and we denote them $\bar{Z}, \bar{W}$.

Even k

The similarity transformation for even values of k is given by

$$\begin{aligned} A_{\text{even}} &= Z^{k+3}_1 - \bar{Z}^{2k}_k + H[k]\left(Z^{2k}_{k-2} - \bar{Z}^{k+3}_3\right) \\ &+ \sum_{j=1}^{k}\frac{G[j]}{G[k-1]}\left(Z^{j+2}_j + \bar{Z}^{k-j+1}_{k-j+1}\right) + \sum_{j=2}^{\lfloor k/2-1\rfloor}\frac{F[k]}{G[j]}\left(Z^{2k-j+1}_{k-j-1} - \bar{Z}^{k+j+2}_{j+2}\right) \end{aligned} \tag{A.10}$$

and its inverse by

$$A_{\text{even}}^{-1} = \frac{1}{2}\sum_{j=1}^{3} K[k,j]\left(W_j^j + \bar{W}_{k-j+3}^{k-j+1}\right) + \frac{1}{2}\frac{G[1]}{G[k-1]}\left(W_k^k + \bar{W}_3^1\right) \\ + \frac{1}{2}\sum_{j=1}^{k-2}\frac{G[j]}{F[k]}\left(\bar{W}_{2k-j+1}^{k-j-1} - W_{k+j+2}^{j+2}\right) + \frac{1}{2}\sum_{j=2}^{\lfloor k/2-1\rfloor}\frac{G[k-1]}{G[j]}\left(W_{k-j+1}^{k-j+1} + \bar{W}_{j+2}^{j}\right). \tag{A.11}$$

Odd k

For odd values of k the similarity transformation is given by

$$A_{\text{odd}} = A_{\text{even}} + \frac{F[k]}{2\,G\left[\frac{k-1}{2}\right]}\left(Z_{\frac{k-1}{2}}^{\frac{3(k+1)}{2}} - \bar{Z}_{\frac{k+3}{2}}^{\frac{3(k+1)}{2}}\right) \tag{A.12}$$

and its inverse by

$$A_{\text{odd}}^{-1} = A_{\text{even}}^{-1} + \frac{G[2k]}{G[2k+1]}\left(W_{\frac{k+3}{2}}^{\frac{k+3}{2}} + \bar{W}_{\frac{k+3}{2}}^{\frac{k+3}{2}-2}\right). \tag{A.13}$$

Constructing $\star_i$

We define the following functions

$$F^\star[k,j] = k\left(\frac{k-1}{k+1}\right)\sqrt{\frac{k}{k-2}}\sqrt{\frac{k-2-j}{(j+1)(j+2)(j+3)}}, \tag{A.14}$$

$$G^\star[k] = (k-1)\left(\frac{k-1}{k+1}\right)\sqrt{\frac{k}{k-2}}\sqrt{\frac{k+2}{k-2}}, \tag{A.15}$$

$$H^\star[k] = \frac{k}{2}\left(\frac{k-1}{k+1}\right)\sqrt{\frac{k}{k-2}}\sqrt{\frac{k+3}{k-1}}, \tag{A.16}$$

$$I^\star[k] = \frac{k}{2}\sqrt{\frac{k}{k-2}}\sqrt{\frac{k-1}{k-3}}, \tag{A.17}$$

and

$$J^\star[k] = \frac{k+1}{2}\sqrt{\frac{k}{2}}\sqrt{\frac{k-3}{k-1}}. \tag{A.18}$$

Then $\star_i$ $(i = 1, 2)$ is given by

$$\star_i = (-1)^{\frac{i-1}{2}} \Bigg[\sum_{j=1}^{\lfloor k/2-3 \rfloor} F^\star[k,j](E_{k-j-1}^{2k-j} + (-1)^i E_{j+4}^{k+j+3}) \tag{A.19}$$
$$- J^\star[k](E_3^{k+4} + (-1)^i E_k^{2k-1}) - \frac{k+3}{2k} F^\star[k,0]\left(E_{k-1}^{2k} + (-1)^i E_4^{k+3}\right) + \mathcal{R}_i^\star[k] \Bigg]$$

where

$$\mathcal{R}_i^\star[k] = \begin{cases} G^\star[k](E_{\frac{k+2}{2}}^{\frac{3k+4}{2}} + (-1)^i E_{\frac{k+4}{2}}^{\frac{3k+2}{2}}), & \text{even } k \\ I^\star[k](E_{\frac{k+3}{2}}^{\frac{3k+5}{2}} + (-1)^i E_{\frac{k+3}{2}}^{\frac{3k+1}{2}}) + H^\star[k](E_{\frac{k+1}{2}}^{\frac{3k+3}{2}} + (-1)^i E_{\frac{k+5}{2}}^{\frac{3k+3}{2}}), & \text{odd } k \end{cases} \tag{A.20}$$

Note that we have written $\star_i$ as a $2k \times 2k$ dimensional matrix. To be clear this is just to ease the computation, and strictly speaking $\star_i$ denotes the $(k-2) \times (k+2)$ matrix sitting inside $\hat{t}_i^{(k)}$. It's simply a matter of taking $E_m^n \to \check{E}_m^{n-k-2}$ where the latter is a matrix unity of dimension $(k-2) \times (k+2)$.

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One-loop one-point functions in gauge-gravity dualities with defects

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We initiate the calculation of loop corrections to correlation functions in 4D defect CFTs. More precisely, we consider $\mathcal{N} = 4$ SYM theory with a codimension-one defect separating two regions of space, $x_3 > 0$ and $x_3 < 0$, where the gauge group is $SU(N)$ and $SU(N-k)$, respectively. This set-up is made possible by some of the real scalar fields acquiring a non-vanishing and x_3-dependent vacuum expectation value for $x_3 > 0$. The holographic dual is the D3-D5 probe brane system where the D5 brane geometry is $AdS_4 \times S^2$ and a background gauge field has k units of flux through the S^2. We diagonalise the mass matrix of the defect CFT making use of fuzzy-sphere coordinates and we handle the x_3-dependence of the mass terms in the 4D Minkowski space propagators by reformulating these as standard massive AdS_4 propagators. Furthermore, we show that only two Feynman diagrams contribute to the one-loop correction to the one-point function of any single-trace operator and we explicitly calculate this correction in the planar limit for the simplest chiral primary. The result of this calculation is compared to an earlier string-theory computation in a certain double-scaling limit, finding perfect agreement. Finally, we discuss how to generalise our calculation to any single-trace operator, to finite N and to other types of observables such as Wilson loops.

INTRODUCTION

Introducing boundaries or defects in conformal field theories leads to novel features concerning correlation functions [1]. For instance, one-point functions can be non-vanishing and operators which have different conformal dimensions can have a non-vanishing overlap. Furthermore, such set-ups typically involve additional fields which are confined to the defect and these fields can have overlaps with the bulk fields. Via the Karch-Randall idea [2], several examples of defect conformal field theories (dCFTs) with holographic duals have been identified.

Our focus is on a particular such 4D defect conformal theory, namely $\mathcal{N} = 4$ supersymmetric Yang-Mills ($\mathcal{N} = 4$ SYM) theory with a codimension-one defect separating two regions of space-time where the gauge group is $SU(N)$ and $SU(N-k)$, respectively [3–6]. The holographic dual is the probe D3-D5 brane system involving a single probe D5 brane with geometry $AdS_4 \times S^2$ where a background gauge field has k units of flux on the S^2 [6].

A number of one- and two-point functions involving both bulk and defect fields have been analysed in the zero flux case [7–11], but the study of correlation functions in the presence of flux was only initiated recently. In [12, 13], tree-level one-point functions of chiral primary operators were calculated. For non-protected operators, tree-level one-point functions are only meaningful for operators which are one-loop eigenstates of the dilatation generator. As is well known, such operators can be described as Bethe eigenstates of a certain integrable spin chain [14, 15]. A systematic method for the calculation of tree-level one-point functions of non-protected operators was presented in [16, 17], in which the one-point function was expressed as the overlap between a Bethe eigenstate and a certain matrix product state. Using the tools of integrable spin chains, it was possible to derive a closed expression for the one-point function of any operator in the $SU(2)$ sector valid for any value of the flux, k. The method can be extended to the $SU(3)$ sector, which is a closed sector at the one-loop level [18].

In the present letter, we initiate the calculation of quantum corrections to the observables of the above dCFT. We focus on the one-loop corrections to one-point functions, but our work also paves the way for the analysis of other types of correlators, of Wilson loops and of computations to higher loop orders. The major obstacle in moving on to one-loop level is that the vacuum expectation values (vevs) of the scalar fields, that realise the difference in the gauge group on the two sides of the defect, introduce a highly involved mass matrix, which needs to be diagonalised. We perform this diagonalisation by making use of fuzzy-sphere coordinates. Another complication is that the masses in the spectrum all depend on the distance from the defect, which invalidates many of the traditional field-theoretical methods. We deal with this problem by working with propagators in an auxiliary AdS_4 space instead of usual 4D Minkowski space propagators. For the one-loop corrections to the one-point functions of single-trace operators, we find that only two Feynman diagrams contribute and we regulate these using dimensional reduction. As expected, the dependence of the regulator, ε, drops out and we end up with a finite result. We relegate many details of our analysis to a forthcoming article [19].

THE DEFECT THEORY

Our starting point is the dCFT formulated in [7]. It consists of $\mathcal{N} = 4$ SYM theory coupled to a 3D hypermultiplet of fundamental fields living on a codimension-one defect, a set-up which preserves half of the supersymmetries of $\mathcal{N}=4$ SYM theory as well as the defect-preserving conformal symmetries [7, 20].

The action of the system is the sum of the usual $\mathcal{N}=4$ SYM action and an action describing the self-interactions of the defect fields and their couplings to the fields of $\mathcal{N}=4$ SYM theory. It will turn out that the defect fields play no role at the loop order we consider. We use the $\mathcal{N}=4$ SYM action in the form

$$S_{\mathcal{N}=4}=\frac{2}{g_{\rm YM}^2}\int \mathrm{d}^4x\,\mathrm{tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}\mathrm{D}_\mu\,\phi_i\,\mathrm{D}^\mu\,\phi_i + \frac{i}{2}\bar\psi\varGamma^\mu\,\mathrm{D}_\mu\,\psi+\frac{1}{2}\bar\psi\varGamma^i[\phi_i,\psi]+\frac{1}{4}[\phi_i,\phi_j][\phi_i,\phi_j]\Big]\,, \quad (1)$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu]$, $\mathrm{D}_\mu=\partial_\mu-i[A_\mu,\cdot]$ and $\{\varGamma_\mu,\varGamma_i\}$ are the 10-dimensional gamma matrices in the Majorana-Weyl representation. A situation where the defect separates two regions of space with different ranks of the gauge group is realised by the so-called fuzzy-funnel solution [6], in which three of the scalar fields of $\mathcal{N}=4$ SYM theory acquire a non-vanishing vev on one side of the defect. If the codimension-one defect is placed at $x_3=0$, the vevs of the scalar fields take the form

$$\langle\phi_i\rangle_{\rm tree}=\phi_i^{\rm cl}=-\frac{1}{x_3}t_i\oplus 0_{(N-k)\times(N-k)}\,,\qquad x_3>0\,, \quad (2)$$

where $i=1,2,3$ and where all other classical fields are set to zero. Here, t_1,t_2 and t_3 are generators of the $SU(2)$ Lie algebra in the k-dimensional irreducible representation. With this set-up, the gauge group is (broken) $SU(N)$ for $x_3>0$ and $SU(N-k)$ for $x_3<0$.

To perform perturbative calculations, we expand the scalar fields around their classical values

$$\phi_i=\phi_i^{\rm cl}+\tilde\phi_i\,,\qquad i=1,2,3\,. \quad (3)$$

Furthermore, we fix the gauge by adding the following term to the action (1):

$$S_{\rm gf}=-\frac{1}{2}\frac{2}{g_{\rm YM}^2}\int \mathrm{d}^4x\,\mathrm{tr}(G^2)\,,\quad G=\partial_\mu A^\mu+i[\tilde\phi_i,\phi_i^{\rm cl}]\,. \quad (4)$$

This also cancels an unwanted term linear in the derivative, which arises when expanding (1) around the classical solution.

The resulting gauge-fixed action is

$$S_{\mathcal{N}=4}+S_{\rm gf}+S_{\rm ghost}=S_{\rm kin}+S_{\rm m}+S_{\rm cubic}+S_{\rm quartic}\,, \quad (5)$$

where the Gaussian part consists of the kinetic terms

$$S_{\rm kin}=\frac{2}{g_{\rm YM}^2}\int \mathrm{d}^4x\,\mathrm{tr}\Big[\frac{1}{2}A_\mu\partial_\nu\partial^\nu A^\mu+\frac{1}{2}\tilde\phi_i\partial_\nu\partial^\nu\tilde\phi_i + \frac{i}{2}\bar\psi\varGamma^\mu\partial_\mu\psi+\bar c\,\partial_\mu\partial^\mu c\Big]\,, \quad (6)$$

and the mass terms

$$S_{\rm m}=\frac{2}{g_{\rm YM}^2}\int \mathrm{d}^4x\,\mathrm{tr}\Big[\frac{1}{2}[\phi_i^{\rm cl},\phi_j^{\rm cl}][\tilde\phi_i,\tilde\phi_j]+\frac{1}{2}[\phi_i^{\rm cl},\tilde\phi_j][\phi_i^{\rm cl},\tilde\phi_j] +\frac{1}{2}[\phi_i^{\rm cl},\tilde\phi_j][\tilde\phi_i,\phi_j^{\rm cl}]+\frac{1}{2}[\phi_i^{\rm cl},\tilde\phi_i][\phi_j^{\rm cl},\tilde\phi_j]+\frac{1}{2}[A_\mu,\phi_i^{\rm cl}][A^\mu,\phi_i^{\rm cl}] + 2i[A^\mu,\tilde\phi_i]\partial_\mu\phi_i^{\rm cl}+\frac{1}{2}\bar\psi\varGamma^i[\phi_i^{\rm cl},\psi]-\bar c\,[\phi_i^{\rm cl},[\phi_i^{\rm cl},c]]\Big]\,. \quad (7)$$

The interactions are given by the cubic vertices

$$S_{\rm cubic}=\frac{2}{g_{\rm YM}^2}\int \mathrm{d}^4x\,\mathrm{tr}\Big[i[A^\mu,A^\nu]\partial_\mu A_\nu+i[A^\mu,\tilde\phi_i]\partial_\mu\tilde\phi_i + [\phi_i^{\rm cl},\tilde\phi_j][\tilde\phi_i,\tilde\phi_j]+[A_\mu,\phi_i^{\rm cl}][A^\mu,\tilde\phi_i]+\frac{1}{2}\bar\psi\varGamma^\mu[A_\mu,\psi] + \frac{1}{2}\bar\psi\varGamma^i[\tilde\phi_i,\psi]+i(\partial_\mu\bar c)[A^\mu,c]-\bar c\,[\phi_i^{\rm cl},[\tilde\phi_i,c]]\Big]\,, \quad (8)$$

plus a number of standard quartic vertices which will not play any role. Here, c and its conjugate $\bar c$ are fermionic (but Lorentz scalar) ghost fields.

Note that (7) are not usual mass terms, as they depend on the classical solution $\phi_i^{\rm cl}$ and hence on the distance x_3 to the defect. Moreover, they are non-diagonal in both flavour and colour. Not all flavours mix, though. The colour components of the gauge field A_0 only mix among themselves and not with colour components of any other fields. The same is true for the colour components of A_1 and A_2 as well as for the colour components of the scalars $\tilde\phi_4$, $\tilde\phi_5$ and $\tilde\phi_6$ and the ghosts. For the remaining bosonic fields $\tilde\phi_1$, $\tilde\phi_2$, $\tilde\phi_3$, A_3 and the original fermions, the mixing problem is more complicated and involves both flavour and colour. We find that the mixing problem can be completely solved by making use of fuzzy-sphere coordinates. We present the eigenvalues and corresponding multiplicities in table I, while deferring the detailed derivation to a forthcoming paper [19]. Notice that we have left out the factor $1/x_3$ in table I, which multiplies all masses in the diagonalised action. For the bosonic fields, the mass eigenvalues are expressed in terms of

$$\nu=\sqrt{m^2+\frac{1}{4}}\,. \quad (9)$$

The mass matrix of the fermions ψ has positive as well as negative eigenvalues. In order to obtain the canonical form of the action with positive masses, the sign of the latter can be changed via a chiral rotation of the fermions.

Once we have diagonalised the mass matrix, the propagators are obtained in the usual way. Hence, a scalar propagator $K(x,y)$ is the solution to

$$\left(-\partial_\mu\partial^\mu+\frac{m^2}{(x_3)^2}\right)K(x,y)=\frac{g_{\rm YM}^2}{2}\delta(x-y)\,, \quad (10)$$

where the derivatives are with respect to x. If one compares this to the definition of the propagator $K_{\rm AdS}(x,y)$

Multiplicity	$\nu(\tilde{\phi}_{4,5,6}, A_{0,1,2}, c)$	$m(\psi_{1,2,3,4})$	$\nu(\tilde{\phi}_{1,2,3}, A_3)$
$\ell + 1$	$\ell + \frac{1}{2}$	$-\ell$	$\ell - \frac{1}{2}$
ℓ	$\ell + \frac{1}{2}$	$\ell + 1$	$\ell + \frac{3}{2}$
$(k+1)(N-k)$	$\frac{k}{2}$	$-\frac{k-1}{2}$	$\frac{k-2}{2}$
$(k-1)(N-k)$	$\frac{k}{2}$	$\frac{k+1}{2}$	$\frac{k+2}{2}$
$(N-k)(N-k)$	$\frac{1}{2}$	0	$\frac{1}{2}$

TABLE I. Masses and multiplicities of the different fields with $\ell = 1, \ldots, k-1$, partially given in terms of ν defined in (9).

of a scalar in AdS_4 with mass $\tilde{m}$

$$(-\nabla_\mu \nabla^\mu + \tilde{m}^2) K_{\mathrm{AdS}}(x, y) = \frac{\delta(x-y)}{\sqrt{g}}\,, \qquad (11)$$

with the metric of AdS_4 given as $g_{\mu\nu} = (x_3)^{-2}\,\eta_{\mu\nu}$, one concludes that

$$K(x,y) = \frac{g_{\mathrm{YM}}^2}{2} \frac{K_{\mathrm{AdS}}(x,y)}{x_3 y_3}\,, \qquad (12)$$

with the identification $\tilde{m}^2 = m^2 - 2$. We notice the satisfying fact that none of the scalar masses in table I leads to a violation of the Breitenlohner-Freedman (BF) bound [21], since $\tilde{m}^2 \geq -\frac{9}{4}$, which is exactly the BF bound for AdS_4. The bound is only saturated in the special case $k = 2$. Closed expressions for $K_{\mathrm{AdS}}(x, y)$ in terms of hypergeometric functions can be found in the literature, see e.g. [22, 23]. A representation which is particularly useful for our purpose is [24]

$$\begin{aligned} K(x,y) = {} & \frac{g_{\mathrm{YM}}^2 \sqrt{x_3 y_3}}{2} \\ & \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3}\, \mathrm{e}^{i\vec{k}\cdot(\vec{x}-\vec{y})}\, I_\nu(|\vec{k}|x_3^<) K_\nu(|\vec{k}|x_3^>)\,, \end{aligned} \qquad (13)$$

where I_ν and K_ν are modified Bessel functions with ν given in (9) and with $x_3^<$ ($x_3^>$) the smaller (larger) of x_3 and y_3. Furthermore, $\vec{k} = (k_0, k_1, k_2)$ denotes the directions parallel to the defect. For the propagators of the spinor fields, one finds by similar considerations

$$K^F(x,y) = \frac{g_{\mathrm{YM}}^2}{2} \frac{K_{\mathrm{AdS}}^F(x,y)}{(x_3)^{3/2}(y_3)^{3/2}}\,, \qquad (14)$$

this time with $\tilde{m}_F = m_F$. For more details, we refer to [19]. Our considerations are an elaboration of the statement already made in [25] that the mass terms could be rendered position independent by performing a Weyl transformation to AdS_4 space.

ONE-POINT FUNCTIONS

With the classical fields given by (2), single-trace operators built from the scalar fields ϕ_1, ϕ_2 and ϕ_3 will have non-vanishing one-point functions on one side of the defect, $x_3 > 0$, already at tree level with the expected space-time dependence [1]:

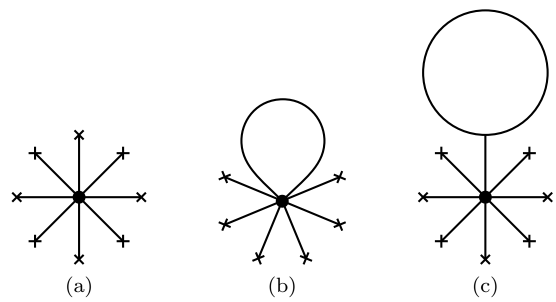

FIG. 1. Tree-level (a) and one-loop ((b) tadpole and (c) lollipop) contributions to one-point functions. A cross stands for the insertion of the classical solution, while the operator is depicted as a dot.

$$\langle \mathcal{O}_\Delta \rangle = \frac{C}{x_3^\Delta}\,, \qquad (15)$$

where C is a constant and Δ denotes the scaling dimension of $\mathcal{O}$. For simplicity, we illustrate our method by considering operators which do not get corrected (in the theory without the defect), i.e. the chiral primaries of $\mathcal{N} = 4$ SYM theory. Furthermore, we will consider the simplest such operator

$$\mathcal{O}(x) = \mathrm{tr}(Z^L)(x)\,, \quad Z(x) = \phi_3(x) + i\phi_6(x)\,. \qquad (16)$$

At tree level, the one-point function of $\mathcal{O}$ is given by inserting (2) into (16), as depicted in figure 1(a). This yields [16]

$$\langle \mathcal{O} \rangle_{\text{tree-level}} = -\frac{2}{x_3^L (L+1)} B_{L+1}\left(\frac{1-k}{2}\right) \qquad (17)$$

for L even and vanishes when L is odd. Here, $B_{L+1}(u)$ are the Bernoulli polynomials. We have not divided by the norm of the two-point function, since this normalisation factor will not play any role in our analysis [26].

At one-loop order, there are two possible Feynman diagrams which we depict in figures 1(b) and 1(c) and denote as the tadpole and the lollipop diagram. Symbolically, the tadpole contribution looks like

$$\langle \mathcal{O} \rangle_{\text{1-loop,tad}} \sim \frac{1}{x_3^{L-2}} \sum_m K(x,x)\,. \qquad (18)$$

The sum is over the spectrum of the relevant (scalar) modes, and we have omitted the similarity transformations that change between the original and mass-diagonal basis. Symbolically, the lollipop diagram contributes as follows:

$$\langle \mathcal{O} \rangle_{\text{1-loop,lol}} \sim \frac{g_{\mathrm{YM}}^{-2}}{x_3^{L-1}} \sum_{m_1, m_2} \int \mathrm{d}^4 y\, K_1(x,y) V K_2(y,y)\,. \qquad (19)$$

Here, m_1 ranges only over bosonic modes, whereas m_2 also includes fermions. The vertex factor V is $\propto 1/y_3$ for scalars, gluons and ghosts in the loop but just a number for fermions. Again, we have neglected many factors. One can convince oneself that the quartic interaction terms do not contribute at one-loop order. Likewise, the defect fields do not play any role at one-loop order; the only way a defect field could contribute at one-loop order would involve a tadpole diagram of the 3D theory living on the defect, which vanishes due to conformal invariance.

Both the scalar and the fermion loop are divergent and require regularisation. We regulate using dimensional reduction [27] in the $d = 3 - 2\varepsilon$ dimensions parallel to the defect and show that all dependence on the regulator, ε, cancels out in the final result. This constitutes a strong consistency check of our calculations. For the scalar loop $K(x, x)$ with $m \neq 0$, dimensional regularisation leads to [19]

$$K(x,x) = \frac{g_{\text{YM}}^2}{2}\frac{1}{16\pi^2 x_3^2}\left(m^2\left[-\frac{1}{\varepsilon} - \log(4\pi) + \gamma_{\text{E}} -2\log(x_3) + 2\Psi(\nu + \tfrac{1}{2}) - 1\right] - 1\right). \tag{20}$$

Here, γ_{E} is the Euler-Mascheroni constant and Ψ is the Euler digamma function. The fermion loop in dimensional regularisation reads

$$\operatorname{tr} K^F(x,x) = \frac{m}{|m|}\frac{g_{\text{YM}}^2}{2}\frac{1}{4\pi^2 x_3^3}\Bigg[|m|^3 + |m|^2 - 3|m| - 1 + |m|(|m|^2 - 1) \times\left(-\frac{1}{\varepsilon} - \log(4\pi) + \gamma_{\text{E}} - 2\log(x_3) + 2\Psi(|m|) - 2\right)\Bigg], \tag{21}$$

where the sign of the mass, $m/|m|$, stems from the aforementioned chiral rotation of the fermions.

In the present letter, we shall restrict ourselves to calculating the large-N contribution to the one-point function. The evaluation of the finite-N contribution poses no conceptual problems but involves colour components of the fields which can be ignored in the large-N limit. We refer to [19] for a more detailed discussion. In the large-N limit, only tadpole diagrams where the tadpole connects neighbouring fields contribute and there are L such terms. The excitations which run in the loop can either be $\tilde{\phi}_3$ or $\tilde{\phi}_6$ and both of the associated contributions can be calculated explicitly. This leads to the following result, valid for even L

$$\langle\mathcal{O}\rangle_{\text{1-loop,tad}} = -\frac{\lambda}{16\pi^2}\frac{2L}{x_3^L(L-1)}B_{L-1}\left(\frac{1-k}{2}\right). \tag{22}$$

The contribution vanishes for odd L.

The evaluation of the contribution from the lollipop diagram is considerably more involved. First, the large-N limit only constrains the type of colour components for the fields which run in the loop and not for the fields which run in the stick. Second, one needs to repeatedly use the similarity transformation which relates the mass eigenstates to the various field components. Finally, the use of a supersymmetry-preserving renormalisation scheme is crucial. Assembling the numerous contributions, we find that the lollipop contribution vanishes:

$$\langle\mathcal{O}\rangle_{\text{1-loop,lol}} = 0\,. \tag{23}$$

For details on the calculation, in particular on the similarity transformation to the mass eigenbasis which features heavily in it, see [19]. Notice that in both (22) and (23) all dependence on the regulator ε has cancelled out and so have the various logarithms and the Euler-Mascheroni constant. Note also that the contribution of the lollipop diagram can be equivalently obtained from the one-loop correction to the vev of the scalars, $\langle\phi_i\rangle_{\text{1-loop}}$, which equally vanishes.

COMPARISON TO STRING THEORY

The present calculations open a new possibility of comparing results between gauge and string theory with less (super) symmetries. In particular, we have at our disposal a novel parameter k. In [12, 25], it was suggested to consider a limit which consists in letting $N \to \infty$ and subsequently $k \to \infty$ (but $k \ll N$) while keeping $\lambda/k^2 \ll 1$.

In the string-theory language, the $N \to \infty$ limit eliminates string interactions and the limit $\lambda \to \infty$ justifies a supergravity treatment. The string configuration dual to a one-point function is that of a string stretching from the boundary of AdS_5 (more precisely from the insertion point of the dual gauge-theory operator) and ending on the $D5$-brane in the interior of $AdS_5 \times S^5$. In the case of a chiral primary, the string can be considered point-like and the one-point function can be computed using a variant of the Witten prescription [12, 17, 28]. In the limit described above, the result organises into a power series expansion in λ/k^2. This led the authors of [12] to suggest that the result might match the result of a perturbative gauge-theory computation, which, however, would require that the gauge-theory perturbative result would likewise organise itself into a power series expansion in λ/k^2. This idea is very reminiscent of the BMN idea [29] fostered in connection with the study of the spectral problem of $\mathcal{N} = 4$ SYM theory. Here, another quantum number, J, which had the interpretation of an S^5 angular momentum of a spinning string, was considered large as well as λ while λ/J^2 was assumed to be finite. In the BMN case, it eventually turned out that starting at four-loop order the perturbative gauge-theory expansion of anomalous dimensions did not organise itself into powers of λ/J^2 [30–32].

The authors of [12] showed that the leading term in the $\frac{\lambda}{k^2}$ expansion matches the tree-level gauge-theory result. Their supergravity result, however, also implies a prediction for the one-loop gauge-theory correction to the one-point function. The chiral primary of length L considered in [12] is not the same as (16) but has a non-vanishing projection onto the latter. Thus, the ratio of the next-to-leading-order term and the leading-order term in λ/k^2 should match the ratio between our one-loop and tree-level result. The prediction for this ratio following from [12] reads

$$\left.\frac{\langle\mathcal{O}\rangle_{\text{1-loop}}}{\langle\mathcal{O}\rangle_{\text{tree-level}}}\right|_{\text{string}} = \frac{\lambda}{4\pi^2k^2}\frac{L(L+1)}{L-1}\,. \tag{24}$$

For the tadpole diagram and the vanishing lollipop diagram, we find

$$\left.\frac{\langle\mathcal{O}\rangle_{\text{1-loop}}}{\langle\mathcal{O}\rangle_{\text{tree-level}}}\right|_{\text{gauge}} = \frac{\lambda}{4\pi^2k^2}\left(\frac{L(L+1)}{L-1}+O(k^{-2})\right)\,, \tag{25}$$

which is identical to the supergravity result in the double-scaling limit. This match provides a highly non-trivial check of the gauge-gravity duality in the case of partially broken supersymmetry as well as conformal symmetry!

CONCLUSIONS & OUTLOOK

With the present work, we have laid the foundation for a detailed analysis of a class of dCFTs based on $\mathcal{N}=4$ SYM theory, which have holographic duals involving background gauge fields with flux. The flux, which is related to the difference in rank of the gauge group on the two sides of a defect, constitutes an interesting extra tunable parameter of the AdS/dCFT set-up. Its presence severely complicates the field-theoretical analysis since some of the scalar fields of $\mathcal{N}=4$ SYM theory acquire non-vanishing and space-time-dependent vevs, which leads to a highly non-trivial mixing both at the flavour and at the colour level. We have solved this mixing problem and diagonalised the mass matrix of the theory. In addition, we have shown how to trade Minkowski space propagators with space-time-dependent mass terms for *AdS* space propagators with standard mass terms. With these two steps accomplished, the perturbative calculation of observables in the dCFT can be carried out by standard methods. We illustrated this by calculating the planar one-loop correction to the one-point function of the chiral primary operator $\text{tr}(Z^L)$. In a certain double-scaling limit, our gauge-theory result perfectly agrees with an earlier prediction for the same quantity from string theory. This provides a strong test of the AdS/dCFT duality at quantum level.

Our analysis can be extended in numerous directions. First, it is straightforward to extend the calculation to finite N. Second, the calculation can be generalised to any operator built of scalars. This might reveal interesting novel structures, as integrability has recently shown its face in the calculation of tree-level one-point functions in the $SU(2)$ sector [16, 17]. It would also be interesting to investigate the types of correlators special to dCFTs such as two-point functions between bulk operators with different conformal dimensions and two-point functions involving both bulk and defect fields. Moreover, one could envision going to higher loop orders where presumably starting from two-loop order the defect fields would come into play and present further challenges. Finally, some simple examples of Wilson loops in the present defect set-up were considered in [25], where a tree-level computation was carried out on the field-theory side and compared to a supergravity computation. As for one-point functions, agreement was observed between the tree-level and the supergravity result in the double-scaling limit described above. It would be interesting to address this at one-loop order.

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A Quantum Check of AdS/dCFT

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Abstract

We build the framework for performing loop computations in the defect version of $\mathcal{N}=4$ super Yang-Mills theory which is dual to the probe D5-D3 brane system with background gauge-field flux. In this dCFT, a codimension-one defect separates two regions of space-time with different ranks of the gauge group and three of the scalar fields acquire non-vanishing and space-time-dependent vacuum expectation values. The latter leads to a highly non-trivial mass mixing problem between different colour and flavour components, which we solve using fuzzy-sphere coordinates. Furthermore, the resulting space-time dependence of the theory's Minkowski space propagators is handled by reformulating these as propagators in an effective AdS_4. Subsequently, we initiate the computation of quantum corrections. The one-loop correction to the one-point function of any local gauge-invariant scalar operator is shown to receive contributions from only two Feynman diagrams. We regulate these diagrams using dimensional reduction, finding that one of the two diagrams vanishes, and discuss the procedure for calculating the one-point function of a generic operator from the SU(2) subsector. Finally, we explicitly evaluate the one-loop correction to the one-point function of the BPS vacuum state, finding perfect agreement with an earlier string-theory prediction. This constitutes a highly non-trivial test of the gauge-gravity duality in a situation where both supersymmetry and conformal symmetry are partially broken.

Keywords: Super-Yang-Mills; Defect CFTs; One-point functions; D5-D3 probe brane

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1 Introduction

Defect conformal field theories (dCFTs) with holographic duals constitute an interesting new arena for precision tests of the AdS/CFT correspondence [1] and for the search for integrable structures [2]. Moreover, for such quantum field theories new types of correlation functions come into play. For instance, fields living on the defect can mix with bulk fields and two-point functions of bulk fields with unequal conformal dimensions need not vanish [3]. Further interesting features emerge if one considers set-ups where some of the bulk fields acquire a vacuum expectation value (vev), in which case the theory can have non-vanishing one-point functions already at tree level [3, 4]. The study of one-point functions is a natural first step when entering the realm of dCFTs. Tree-level studies carried out within the AdS/dCFT framework show that one-point functions, interestingly, have many features in common with three-point functions of the standard AdS/CFT set-up, e.g. determinant-based expressions, integrable structure and an accessible strong-coupling limit [5–7].

In the present paper, we shall develop the necessary tools to go beyond tree-level computations in certain dCFTs with vevs and with holographic duals, an endeavour which will make possible the extraction of large amounts of new data from these theories as well as the initiation of new directions of study. We already briefly presented one example of a one-loop analysis in such a dCFT in the letter [8], where we calculated the one-loop correction to the one-point function of a chiral primary and compared it to the result of a string-theory computation in a certain double-scaling limit, finding exact agreement. Here, we present the derivations which made the field-theoretic part of that computation possible, give the details of the computation and extend these results to finite N as well as to general single-trace operators built out of scalar fields.

The dCFT we are going to consider consists of $\mathcal{N} = 4$ super Yang-Mills ($\mathcal{N} = 4$ SYM) theory with a codimension-one defect inserted at $x_3 = 0$ [4]. Three of the scalar fields of the theory are assigned specific, x_3-dependent vevs on one side of the defect, $x_3 > 0$, while all classical fields vanish for $x_3 < 0$. This Higgsing results in a highly non-trivial mass mixing problem where different colour components for both bosonic and fermionic fields mix with each other and where in addition one space-time component of the gauge field mixes with the scalars. Moreover, all mass terms become x_3-dependent. The motivation for this particular Higgsing comes from the string-theory set-up, where the vevs represent the so-called fuzzy-funnel solution of the probe D5-D3 brane system where the probe-D5 brane is embedded in $AdS_5 \times S^5$ so that it shares three dimensions (the defect) with the N D3 branes. More precisely, the geometry of the D5 brane is $AdS_4 \times S^2$ and a certain background gauge field has a non-vanishing flux, k, on S^2 meaning that k out of the N D3 branes get dissolved in the D5 brane [9–12]. On the gauge theory side, the parameter k appears as the difference in rank of the gauge group on the two sides of the defect, cf. figure 1.

Due to the Higgsing, the theory has non-vanishing one-point functions already at tree level. Tree-level one-point functions of chiral primaries were calculated for this particular theory in [13] as well as in a closely related one in [14], and a match with a string-theory computation was found at the leading order in a certain double-scaling limit. Moreover,

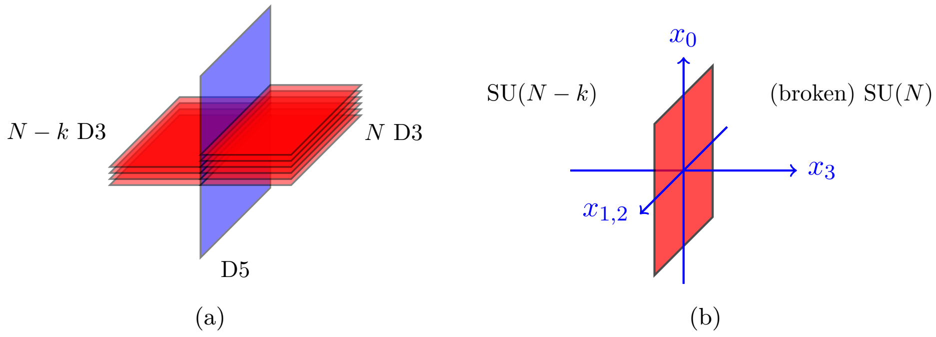


Figure 1. Illustration of the set-up: (a) k of the N D3 branes get dissolved in the D5 probe brane (b) the rank of the gauge group differs on the two sides of the defect.

making use of the tools of integrability, it was possible to derive a closed expression of determinant form for the tree-level one-point functions of non-protected operators belonging to an SU(2) subsector of $\mathcal{N}=4$ SYM theory [5, 6]. An empirically based proposal for how to extend this to an SU(3) sector likewise exists [7].

Due to the mass mixing problem, going beyond tree-level for the Higgsed theory is considerably more complicated than for $\mathcal{N}=4$ SYM theory itself. It turns out, however, that the language of fuzzy-sphere coordinates is tailored for the diagonalisation of the mass matrix. In these coordinates, the mixing problem can literally be viewed as the spin-orbit interaction of the hydrogen atom of the 21st century, $\mathcal{N}=4$ SYM theory. Furthermore, it is possible to avoid the space-time dependence of the masses by formulating the propagators in an effective AdS_4 space. The radial coordinate of this AdS_4 space is x_3, the coordinate perpendicular to the defect, and the defect itself plays the role of the AdS_4 boundary. With these steps accomplished, the theory is in principle amenable to the standard program of perturbation theory. We show that the one-loop correction to any (single-trace) operator built from scalars obtains contributions from only two Feynman diagrams and we calculate these using dimensional regularisation in combination with dimensional reduction carefully adjusted to respect the symmetries of the present set-up. One of the two relevant Feynman diagrams corresponds to the one-loop correction to the vevs of the scalars and cancels exactly.

We discuss in some depth the computation of one-loop corrections to one-point functions in the SU(2) subsector and, in particular, we present the details of the calculation of the planar correction to the one-point function of the BMN vacuum state, the result of which we presented in the letter [8]. Here, we adress the finite-N case as well.

The first step of our perturbative calculation consists in expanding the SYM action around the classical fields and fixing an appropriate gauge. This step is carried out in section 2. Section 3 is devoted to the resolution of the mass mixing problem. First, we rewrite the mass term in terms of irreducible SU(2) representations in flavour space. Then, we explicitly construct the eigenstates via fuzzy-sphere coordinates and a Clebsch-Gordan decomposition. The section closes with a table of the resulting spectrum of the theory, cf. page 13. As all mass terms carry space-time dependence, being proportional to $1/x_3$

for fermions and $1/(x_3)^2$ for bosons, the propagators of the theory are not of standard Minkowskian type. We show in section 4 that the propagators can be viewed as standard propagators of AdS_4 instead. Moreover, we translate the propagators in the mass eigenbasis to the flavour and colour basis. We discuss the dimensional regularisation of the occurring integrals as well as dimensional reduction in section 5. Section 6 deals with the computation of one-loop corrections to one-point functions of scalar operators, first in general, subsequently for operators belonging to the SU(2) subsector and finally for the BMN vacuum state. We are mainly working in the planar limit but include a number of finite N results as well. The computation of the one-loop correction to the vevs of the scalar fields, which is required for the analysis of this section, is relegated to appendix D. Section 7 is devoted to the comparison to string theory and finally section 8 contains a conclusion and outlook, where we discuss a number of other interesting quantum computations for dCFTs which our work makes feasible. Five appendices provide details on various aspects of our work: the irreducible SU(2) representations (A), the fuzzy-sphere coordinates (B), our conventions for the ten-dimensional gamma matrices (C), the aforementioned calculation of the vevs of the scalars (D) and the alternative Hadamard and zeta-function regularisation (E).

2 The action

The action of the dCFT is the sum of the usual $\mathcal{N}=4$ SYM action in the bulk and an action describing the self-interactions of a 3D hypermultiplet of fundamental fields living on the defect and their couplings to the fields of $\mathcal{N}=4$ SYM theory:

$$S = S_{\mathcal{N}=4} + S_{D=3}\,. \tag{2.1}$$

The defect fields will turn out to play no role at the loop order we consider. We will use the action of $\mathcal{N}=4$ SYM theory in the following form

$$\begin{aligned} S_{\mathcal{N}=4} = \frac{2}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\,\mathrm{tr}\biggl[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mathrm{D}_\mu\,\phi_i\,\mathrm{D}^\mu\,\phi_i + \frac{i}{2}\bar{\Psi}\Gamma^\mu\,\mathrm{D}_\mu\,\Psi \\ &+\frac{1}{2}\bar{\Psi}\tilde{\Gamma}^i[\phi_i,\Psi] + \frac{1}{4}[\phi_i,\phi_j][\phi_i,\phi_j]\biggr]\,, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\,, \\ \mathrm{D}_\mu\,\phi_i &= \partial_\mu\phi_i - i[A_\mu,\phi_i]\,, \qquad \mathrm{D}_\mu\,\Psi = \partial_\mu\Psi - i[A_\mu,\Psi]\,. \end{aligned} \tag{2.3}$$

Here, the field Ψ is a ten-dimensional Majorana-Weyl fermion and $\{\Gamma^\mu, \tilde{\Gamma}^i\}$ are the corresponding ten-dimensional gamma matrices, which we explicitly give in appendix C. The ranges of the indices are $\mu,\nu = 0,1,2,3$ and $i,j = 1,2,3,4,5,6$. We are using a mostly-plus convention for the metric.

We wish to expand the fields around the classical solution

$$\langle\phi_i\rangle_{\mathrm{tree}} = \phi_i^{\mathrm{cl}} = -\frac{1}{x_3}t_i \oplus 0_{(N-k)\times(N-k)}\,, \tag{2.4}$$

where $i = 1, 2, 3$ and the t_i constitute a k-dimensional irreducible representation of the Lie algebra SU(2); expressions for the representation matrices in our conventions can be found in appendix A. All other classical fields vanish. This solution is the gauge-theory dual of the fuzzy-funnel solution of the probe D5-D3 brane set-up [12].

We expand the action around the classical solution, writing

$$\phi_i = \phi_i^{\text{cl}} + \tilde{\phi}_i \,, \tag{2.5}$$

where ϕ_i^{cl} denotes the classical part and $\tilde{\phi}_i$ the quantum part. Terms which upon expansion do not depend on any quantum fields can be ignored as can all terms linear in the quantum fields as these should vanish by the equations of motion. This latter fact can also be checked explicitly.

2.1 Gauge fixing

As usual, we have to fix a gauge in order to perform calculations. Moreover, we notice that the expansion of the gauge-kinetic term of the scalar contains

$$i[A_\mu, \phi_i^{\text{cl}}]\partial^\mu \tilde{\phi}_i \,, \tag{2.6}$$

which would lead to complications in computing the propagators. Hence, we want to cancel this term while fixing the gauge. Following [15], this can be achieved by adding the gauge-fixing term

$$-\frac{1}{2}\operatorname{tr}(G^2) \quad \text{with} \quad G = \partial_\mu A^\mu + i[\tilde{\phi}_i, \phi_i^{\text{cl}}] \tag{2.7}$$

to the action. The price for doing this is a massive ghost field that couples to the scalars.

Explicitly, we add to the action (2.2) the BRST exact term

$$S_{\text{gh}} = \frac{2}{g_{\text{YM}}^2}\int \mathrm{d}^4x \operatorname{tr}\left[-s\left(\bar{c}(\partial_\mu A^\mu - i[\phi_i^{\text{cl}}, \tilde{\phi}_i]) + \frac{1}{2}\bar{c}B\right)\right] \,, \tag{2.8}$$

where s is the BRST variation defined by

$$\begin{aligned} sA_\mu &= \mathrm{D}_\mu c = \partial_\mu c - i[A_\mu, c] \,, & s\phi_i &= -i[\phi_i, c] \,, & s\Psi &= i\{\Psi, c\} \,, \\ sc &= ic^2 \,, & s\bar{c} &= -B \,, & sB &= 0 \,. \end{aligned} \tag{2.9}$$

One can check that with this definition $s^2 = 0$. The ghosts $c, \bar{c}$ are fermionic (Lorentz) scalars, while the auxiliary field B is a bosonic scalar. The BRST variation only acts on the quantum part of ϕ_i, i.e.

$$s\phi_i^{\text{cl}} = 0 \,, \qquad s\tilde{\phi}_i = -i[\phi_i^{\text{cl}} + \tilde{\phi}_i, c] \,. \tag{2.10}$$

We now find, noting that moving s past a fermion introduces a sign,

$$S_{\text{gh}} = \frac{2}{g_{\text{YM}}^2}\int \mathrm{d}^4x \operatorname{tr}\left[\bar{c}(\partial_\mu \mathrm{D}^\mu c - [\phi_i^{\text{cl}}, [\phi_i^{\text{cl}} + \tilde{\phi}_i, c]]) + B(\partial_\mu A^\mu - i[\phi_i^{\text{cl}}, \tilde{\phi}_i]) + \frac{1}{2}B^2\right] \,. \tag{2.11}$$

Since B is not dynamical, we can immediately integrate it out; its equation of motion is $B = -\partial_\mu A^\mu + i[\phi_i^{\rm cl}, \tilde{\phi}_i]$. After rearranging the result a bit, this yields

$$S_{\rm gh} = \frac{2}{g_{\rm YM}^2} \int \mathrm{d}^4 x \, \mathrm{tr} \Big[\bar{c}(\partial_\mu \, \mathrm{D}^\mu \, c - [\phi_i^{\rm cl}, [\phi_i^{\rm cl} + \tilde{\phi}_i, c]]) - \frac{1}{2}(\partial_\mu A^\mu)^2 + i[A^\mu, \tilde{\phi}_i]\partial_\mu \phi_i^{\rm cl} + i[A^\mu, \partial_\mu \tilde{\phi}_i]\phi_i^{\rm cl} + \frac{1}{2}[\phi_i^{\rm cl}, \tilde{\phi}_i]^2 \Big] \, . \quad (2.12)$$

We note that this cancels the unwanted mixing between A_μ and $\partial_\mu \tilde{\phi}_i$, as mentioned above. We also see that the kinetic term for the gluons is changed to

$$-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}(\partial_\mu A^\mu)^2 = \frac{1}{2} A_\mu \partial_\nu \partial^\nu A^\mu \, , \quad (2.13)$$

which is invertible and diagonal in the Lorentz index. Notice that for $\phi_i^{\rm cl} = 0$ our gauge choice reduces to Feynman gauge.

2.2 The expanded action

We can write the gauge-fixed action as

$$S_{\mathcal{N}=4} + S_{\rm gh} = S_{\rm kin} + S_{\rm m,b} + S_{\rm m,f} + S_{\rm cubic} + S_{\rm quartic} \, . \quad (2.14)$$

The Gaußian part consists of the kinetic terms

$$S_{\rm kin} = \frac{2}{g_{\rm YM}^2} \int \mathrm{d}^4 x \, \mathrm{tr} \left[\frac{1}{2} A_\mu \partial_\nu \partial^\nu A^\mu + \frac{1}{2} \tilde{\phi}_i \partial_\nu \partial^\nu \tilde{\phi}_i + \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + \bar{c} \partial_\mu \partial^\mu c \right] \, , \quad (2.15)$$

the bosonic mass terms

$$S_{\rm m,b} = \frac{2}{g_{\rm YM}^2} \int \mathrm{d}^4 x \, \mathrm{tr} \Big[\frac{1}{2} [\phi_i^{\rm cl}, \phi_j^{\rm cl}][\tilde{\phi}_i, \tilde{\phi}_j] + \frac{1}{2} [\phi_i^{\rm cl}, \tilde{\phi}_j][\phi_i^{\rm cl}, \tilde{\phi}_j] + \frac{1}{2} [\phi_i^{\rm cl}, \tilde{\phi}_j][\tilde{\phi}_i, \phi_j^{\rm cl}] + \frac{1}{2} [\phi_i^{\rm cl}, \tilde{\phi}_i][\phi_j^{\rm cl}, \tilde{\phi}_j] + \frac{1}{2} [A_\mu, \phi_i^{\rm cl}][A^\mu, \phi_i^{\rm cl}] + 2i[A^\mu, \tilde{\phi}_i]\partial_\mu \phi_i^{\rm cl} \Big] \, , \quad (2.16)$$

and the fermionic mass terms

$$S_{\rm m,f} = \frac{2}{g_{\rm YM}^2} \int \mathrm{d}^4 x \, \mathrm{tr} \left[\frac{1}{2} \bar{\psi} G^i [\phi_i^{\rm cl}, \psi] - \bar{c} [\phi_i^{\rm cl}, [\phi_i^{\rm cl}, c]] \right] \, , \quad (2.17)$$

where we have reduced the ten-dimensional Majorana-Weyl fermion to four four-dimensional Majorana fermions ψ_j, $j = 1, 2, 3, 4$, as explained in appendix C, and the 4×4 matrices G^i that describe their coupling to the scalars are given in (C.10). The interaction is given by the cubic vertices

$$S_{\rm cubic} = \frac{2}{g_{\rm YM}^2} \int \mathrm{d}^4 x \, \mathrm{tr} \Big[i[A^\mu, A^\nu]\partial_\mu A_\nu + [\phi_i^{\rm cl}, \tilde{\phi}_j][\tilde{\phi}_i, \tilde{\phi}_j] + i[A^\mu, \tilde{\phi}_i]\partial_\mu \tilde{\phi}_i + [A_\mu, \phi_i^{\rm cl}][A^\mu, \tilde{\phi}_i] + \frac{1}{2} \bar{\psi} \gamma^\mu [A_\mu, \psi] + \sum_{i=1}^{3} \frac{1}{2} \bar{\psi} G^i [\tilde{\phi}_i, \psi] + \sum_{i=4}^{6} \frac{1}{2} \bar{\psi} G^i [\tilde{\phi}_i, \gamma_5 \psi] + i(\partial_\mu \bar{c})[A^\mu, c] - \bar{c}[\phi_i^{\rm cl}, [\tilde{\phi}_i, c]] \Big] \quad (2.18)$$

and the quartic vertices

$$S_{\text{quartic}} = \frac{2}{g_{\text{YM}}^2} \int \mathrm{d}^4x \, \mathrm{tr} \left[\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}[A_\mu, \tilde{\phi}_i][A^\mu, \tilde{\phi}_i] + \frac{1}{4}[\tilde{\phi}_i, \tilde{\phi}_j][\tilde{\phi}_i, \tilde{\phi}_j] \right] . \quad (2.19)$$

We shall see below that S_{quartic} is not relevant for the one-loop corrections in this article. In the remainder of the paper, we will work in Euclidean signature.

3 The mass matrix

The mass terms of the action (2.16) and (2.17) involve mixing between fields of different flavour as well as mixing between colour components of the same field. To prepare for perturbative calculations of correlation functions, we first have to solve this highly non-trivial mixing problem. Notice that the mass terms are also unconventional in the sense that they depend via the classical fields on the distance x_3 to the defect. This x_3-dependence renders some of the traditional tools of quantum field theory in Minkowski space inapplicable. We will show how to deal with this issue by trading x_3-dependent 4d Minkowski space propagators for x_3-independent propagators in AdS_4 in the next section.

Let us now diagonalise the mass matrix. First, in subsection 3.1 we rewrite the mass terms in close analogy to the spin-orbital interaction of the hydrogen atom, so that they are easy to diagonalise. Subsequently, in subsection 3.2 we explicitly carry out the diagonalisation and read off the spectrum including its degeneracies. We summarise our results on the spectrum in subsection 3.3.

3.1 Rewriting of the mass terms

For a sub-set of the fields, the mass terms are diagonal in the flavor index (but not in the colour index) and we denote the corresponding fields as easy fields. Accordingly, the remaining fields are denoted as complicated fields. The easy fields consist of the three scalars ϕ_4, ϕ_5, ϕ_6, the three gauge fields A_0, A_1, A_2 and the ghost c.

For the easy fields, say A_0 for concreteness, the mass term is proportional to

$$\mathrm{tr}([t_i, A_0][t_i, A_0]) = -\,\mathrm{tr}(A_0[t_i, [t_i, A_0]]) = -\,\mathrm{tr}(A_0 L^2 A_0) \,, \quad (3.1)$$

where

$$L_i = \mathrm{Ad}(t_i) \,, \qquad L^2 = L_i L_i \quad (3.2)$$

are satisfying the well-known commutation relations of angular momenta:

$$[L_i, L_j] = i\epsilon_{ijk} L_k \,. \quad (3.3)$$

The operator L^2 is the Laplacian on the so-called fuzzy sphere. The field A_0 transforms in a – in general reducible – representation of the Lie algebra SU(2). We will decompose this representation into irreducible representations with definite orbital quantum number ℓ and magnetic quantum number m in the next subsection.

The mass term for the complicated bosons, i.e. ϕ_1, ϕ_2, ϕ_3 and A_3, reads

$$S_{\mathrm{m,cb}} = \frac{2}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4x \frac{1}{x_3^2} \operatorname{tr} \left[-\frac{1}{2}\tilde{\phi}_i L^2 \tilde{\phi}_i - \frac{1}{2} A_3 L^2 A_3 + i\epsilon_{ijk}\tilde{\phi}_i L_j \tilde{\phi}_k + i\tilde{\phi}_i L_i A_3 - iA_3 L_i \tilde{\phi}_i \right] , \tag{3.4}$$

where $i = 1, 2, 3$. We can write this in the more suggestive way

$$S_{\mathrm{m,cb}} = \frac{2}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4x \frac{1}{x_3^2} \operatorname{tr} \left[C^T(-\frac{1}{2}L^2 + 2S_i L_i) C \right] , \tag{3.5}$$

where we have introduced the combined field

$$C = \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \\ A_3 \end{pmatrix} , \tag{3.6}$$

and where the matrices S_i acting on the ‘flavour’ index of C are given by

$$S_1 = -\frac{1}{2} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} , \quad S_2 = \frac{i}{2} \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} , \quad S_3 = \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{3.7}$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.8}$$

It is easy to verify that the matrices S_i form a four-dimensional representation of the SU(2) Lie algebra:

$$[S_i, S_j] = i\epsilon_{ijk} S_k . \tag{3.9}$$

This representation is reducible and its explicit decomposition into irreducible representations is

$$U^\dagger S_i U = \begin{pmatrix} \frac{1}{2}\sigma_i & 0 \\ 0 & \frac{1}{2}\sigma_i \end{pmatrix} , \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & 0 & i \\ 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} . \tag{3.10}$$

The eigenvectors of the irreducible representations are

$$\begin{pmatrix} C_{t,+} \\ C_{t,-} \\ C_{b,+} \\ C_{b,-} \end{pmatrix} = U^\dagger C = \frac{1}{\sqrt{2}} \begin{pmatrix} +i\tilde{\phi}_1 + \tilde{\phi}_2 \\ -i\tilde{\phi}_3 - A_3 \\ -i\tilde{\phi}_3 + A_3 \\ -i\tilde{\phi}_1 + \tilde{\phi}_2 \end{pmatrix} , \tag{3.11}$$

which have spin $\frac{1}{2}$ and spin magnetic quantum number $\pm\frac{1}{2}$. It now follows that the complicated boson problem can be solved by the usual procedure of adding angular momentum as it occurs in the well-known spin-orbit interaction of the hydrogen atom. Concretely, we define the total angular momentum operator

$$J_i = L_i + \frac{1}{2}\sigma_i , \tag{3.12}$$

and find that

$$\sigma_i L_i = J^2 - L^2 - \frac{3}{4}\,. \tag{3.13}$$

We will construct the simultaneous eigenstates of L^2, J^2 and J_3 in the next subsection.

The fermionic mass term is proportional to

$$\mathrm{tr}[\bar{\psi} G^i [t_i, \psi]] = \mathrm{tr}[\bar{\psi} G^i L_i \psi]\,, \tag{3.14}$$

where the matrices G^i are given by

$$G^1 = i\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\,, \qquad G^2 = i\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\,, \qquad G^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\,. \tag{3.15}$$

These matrices satisfy the commutation relations

$$[G^i, G^j] = -2i\epsilon^{ijk} G^k \tag{3.16}$$

and thus also form a representation of the Lie algebra SU(2), at least after a rescaling. This representation is equally reducible and explicitly reduced as

$$\tilde{U}^\dagger G^i \tilde{U} = \begin{pmatrix} -\sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}\,, \quad \tilde{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & -1 & 0 \\ 0 & 1 & i & 0 \\ -1 & 0 & 0 & i \\ i & 0 & 0 & -1 \end{pmatrix}\,. \tag{3.17}$$

The eigenvectors of these irreducible representations are

$$\begin{pmatrix} \psi_{t,+} \\ \psi_{t,-} \\ \psi_{b,+} \\ \psi_{b,-} \end{pmatrix} = \tilde{U}^\dagger \psi = \frac{1}{\sqrt{2}}\begin{pmatrix} -\psi_3 - i\psi_4 \\ +\psi_2 + i\psi_1 \\ -\psi_1 - i\psi_2 \\ -\psi_4 - i\psi_3 \end{pmatrix}\,, \tag{3.18}$$

which have spin $\frac{1}{2}$ and spin magnetic quantum number $\pm\frac{1}{2}$. The mixing problem of the fermions can now be solved in complete analogy to the one of the complicated bosons.

To summarise, the complete mass term (2.16), (2.17) can be written as

$$\begin{aligned} S_{\mathrm{m,b}} + S_{\mathrm{m,f}} = &\frac{2}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4x \frac{1}{x_3^2}\,\mathrm{tr}\Big[-\frac{1}{2}E^T L^2 E - \bar{c}L^2 c - \frac{1}{2}C_t^\dagger (L^2 - 2\sigma_i L_i) C_t\Big] \\ &+ \frac{2}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4x \frac{1}{x_3}\,\mathrm{tr}\Big[\frac{1}{2}\bar{\psi}_t \sigma_i L_i \psi_t\Big] + (t \to b)\,, \end{aligned} \tag{3.19}$$

where

$$E = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \tilde{\phi}_4 \\ \tilde{\phi}_5 \\ \tilde{\phi}_6 \end{pmatrix}\,. \tag{3.20}$$

Note that the conjugation here is understood to be outside of the indices, i.e.

$$C_t^\dagger \equiv (C_t)^\dagger\,, \qquad \bar{\psi}_t \equiv (\psi_t)^\dagger\gamma^0\,, \tag{3.21}$$

and similarly for $t \to b$. Correspondingly, $C^\dagger_{t/b,\pm}$ and $\bar{\psi}_{t/b,\pm}$ are related to C and $\bar{\psi}$ via U and $\tilde{U}$, respectively.

3.2 Explicit diagonalisation of the mass matrix

We decompose the different fields with respect to their matrix elements in colour space as

$$\begin{aligned}\Phi = [\Phi]_{n,n'}E^n{}_{n'} + [\Phi]_{n,a}E^n{}_a + [\Phi]_{a,n}E^a{}_n + [\Phi]_{a,a'}E^a{}_{a'}\\ +\Phi_{\rm tr}((N-k)\,\mathbb{1}_{k\times k} + k\,\mathbb{1}_{(N-k)\times(N-k)})\,,\end{aligned} \tag{3.22}$$

where $\Phi \in \{A_0, A_1, A_2, \tilde{\phi}_4, \tilde{\phi}_5, \tilde{\phi}_6, c, C_{t,\pm}, C_{b,\pm}, \psi_{t,\pm}, \psi_{b,\pm}\}$, $n, n' = 1, \ldots, k$ and $a, a' = k+1, \ldots, N$. Moreover, we have split the diagonal components into individually traceless blocks, $\sum_n [\Phi]_{n,n} = 0 = \sum_a [\Phi]_{a,a}$, and a component $\Phi_{\rm tr}$ proportional to the identity in each block. Note that the matrix elements above are not independent degrees of freedom; apart from the aforementioned tracelessness condition, they are also (partially) related to each other via reality conditions.

The matrices $E^a{}_{a'}$ are annihilated by the L_i and the corresponding components $[\Phi]_{a,a'}$ in the $(N-k)\times(N-k)$ block of all fields are hence massless. Moreover, the L_i annihilate $((N-k)\,\mathbb{1}_{k\times k} + k\,\mathbb{1}_{(N-k)\times(N-k)})$ such that $\Phi_{\rm tr}$ is also massless.

The matrices $E^n{}_a$ and $E^a{}_n$ in the off-diagonal $k\times(N-k)$ and $(N-k)\times k$ blocks transform in the irreducible k-dimensional representation of SU(2) with angular momentum $\ell = \frac{k-1}{2}$ and magnetic quantum number $m = \pm\left(\frac{k+1}{2} - n\right)$:

$$L_i E^n{}_a = E^{n'}{}_a [t_i]_{n',n}\,, \qquad L_i E^a{}_n = -[t_i]_{n,n'} E^a{}_{n'}\,. \tag{3.23}$$

The same holds for the corresponding components of the fields.

The standard matrices $E^n{}_{n'}$ in the $k\times k$ block do not transform in an irreducible representation of SU(2) yet. The desired eigenstates yielding the decomposition to irreducible representations are provided by the spherical harmonics $\hat{Y}_\ell^m$ of the fuzzy sphere, where $\ell = 1, \ldots, k-1$ and $m = -\ell, \ldots, \ell$. They are explicitly given in appendix B and satisfy

$$L_3 \hat{Y}_\ell^m = m\hat{Y}_\ell^m\,, \qquad L^2\hat{Y}_\ell^m = \ell(\ell+1)\hat{Y}_\ell^m\,. \tag{3.24}$$

We thus write

$$[\Phi]_{n,n'}E^n{}_{n'} = \Phi_{\ell,m}\hat{Y}_\ell^m\,, \tag{3.25}$$

where the traceless $\hat{Y}_\ell^m$ implement the tracelessness condition $\sum_n [\Phi]_{n,n} = 0$. This concludes the diagonalisation of L^2.

For the easy bosons and ghosts, only L^2 occurs in the mass term, and $\Phi_{\ell,m}$, $[\Phi]_{n,a}$, $[\Phi]_{a,n}$, $[\Phi]_{a,a'}$ and $\Phi_{\rm tr}$ completely diagonalise it. In terms of these components, the mass term reads

$$-\frac{1}{2x_3^2}\,{\rm tr}(A_0L^2A_0) = -\frac{1}{2x_3^2}\left(2\frac{k^2-1}{4}[A_0]^\dagger_{n,a}[A_0]_{n,a} + \ell(\ell+1)(A_0)^\dagger_{\ell,m}(A_0)_{\ell,m}\right)\,, \tag{3.26}$$

where we again have chosen A_0 for concreteness and used (B.11). Here, $[A_0]^\dagger_{n,a} \equiv ([A_0]_{n,a})^\dagger = [A_0]_{a,n}$ and $(A_0)^\dagger_{\ell,m} \equiv ((A_0)_{\ell,m})^\dagger = (-1)^m (A_0)_{\ell,-m}$. Comparing this to the kinetic term

$$-\frac{1}{2}\,\mathrm{tr}(A_0\partial^2 A_0) = -\frac{1}{2}\left(2[A_0]^\dagger_{n,a}\partial^2[A_0]_{n,a} + (A_0)^\dagger_{\ell,m}\partial^2(A_0)_{\ell,m}\right) + \text{massless fields}\,, \qquad (3.27)$$

we immediately see that we have the nonzero mass eigenvalues $\frac{m^2}{x_3^2} = \frac{k^2-1}{4x_3^2}$ with multiplicity $2k(N-k)$ and $\frac{m^2}{x_3^2} = \frac{\ell(\ell+1)}{x_3^2}$ with multiplicity $2\ell+1$ for $\ell = 1,\dots,k-1$. Note that in both equations we have used the first reality condition to remove $[A_0]_{a,n}$, resulting in the relative factor 2 in front of the fields from the $k\times(N-k)$ block compared to those from the $k\times k$ block.

For the complicated bosons and the fermions, we have to diagonalise J^2 with $J_i = L_i + \frac{1}{2}\sigma_i$ in addition to L^2, see the discussion in the previous subsection. Let $\Phi_\pm$ be a field with definite angular momentum ℓ, magnetic quantum number m, spin $\frac{1}{2}$ and spin magnetic quantum number $\pm\frac{1}{2}$, i.e. $[C_{t,\pm}]_{n,a}$, $[C_{t,\pm}]_{a,n}$, $(C_{t,\pm})_{\ell,m}$ as well as the corresponding components of $C_{b,\pm}$, $\psi_{t,\pm}$, $\psi_{b,\pm}$, $\psi_{t,\pm}$ and $\psi_{b,\pm}$. The field can then be written in terms of the desired eigenstates of L^2 and J^2 as

$$\begin{aligned}\Phi_\pm = &+\left\langle j_1=\ell, j_2=\tfrac{1}{2}; m_1=m, m_2=\pm\tfrac{1}{2}\middle| j=j_1-\tfrac{1}{2}, m_j\right\rangle \Phi_{\upharpoonleft\downharpoonleft,m_j}\\ &+\left\langle j_1=\ell, j_2=\tfrac{1}{2}; m_1=m, m_2=\pm\tfrac{1}{2}\middle| j=j_1+\tfrac{1}{2}, m_j\right\rangle \Phi_{\upharpoonleft\upharpoonright,m_j}\,.\end{aligned} \qquad (3.28)$$

Here, $\Phi_{\upharpoonleft\downharpoonleft,m_j}$ denotes the eigenstate with total angular momentum $j=\ell-\frac{1}{2}$ and $\Phi_{\upharpoonleft\upharpoonright,m_j}$ denotes the eigenstate with total angular momentum $j=\ell+\frac{1}{2}$, i.e.

$$\begin{aligned}L^2\Phi_{\upharpoonleft\downharpoonleft,m_j} &= \ell(\ell+1)\Phi_{\upharpoonleft\downharpoonleft,m_j}\,, & L^2\Phi_{\upharpoonleft\upharpoonright,m_j} &= \ell(\ell+1)\Phi_{\upharpoonleft\upharpoonright,m_j}\,,\\ J^2\Phi_{\upharpoonleft\downharpoonleft,m_j} &= (\ell-\tfrac{1}{2})(\ell+\tfrac{1}{2})\Phi_{\upharpoonleft\downharpoonleft,m_j}\,, & J^2\Phi_{\upharpoonleft\upharpoonright,m_j} &= (\ell+\tfrac{1}{2})(\ell+\tfrac{3}{2})\Phi_{\upharpoonleft\upharpoonright,m_j}\,.\end{aligned} \qquad (3.29)$$

The explicit expressions for the occurring Clebsch-Gordan coefficients are

$$\left\langle j_1, j_2=\tfrac{1}{2}; m_1, m_2=+\tfrac{1}{2}\middle| j=j_1+\tfrac{1}{2}, m_j\right\rangle = \delta_{m_j,m_1+m_2}\frac{\sqrt{j_1+m_1+1}}{\sqrt{2j_1+1}}\,, \qquad (3.30)$$

$$\left\langle j_1, j_2=\tfrac{1}{2}; m_1, m_2=-\tfrac{1}{2}\middle| j=j_1+\tfrac{1}{2}, m_j\right\rangle = \delta_{m_j,m_1+m_2}\frac{\sqrt{j_1-m_1+1}}{\sqrt{2j_1+1}}\,, \qquad (3.31)$$

$$\left\langle j_1, j_2=\tfrac{1}{2}; m_1, m_2=+\tfrac{1}{2}\middle| j=j_1-\tfrac{1}{2}, m_j\right\rangle = -\delta_{m_j,m_1+m_2}\frac{\sqrt{j_1-m_1}}{\sqrt{2j_1+1}}\,, \qquad (3.32)$$

and

$$\left\langle j_1, j_2=\tfrac{1}{2}; m_1, m_2=-\tfrac{1}{2}\middle| j=j_1-\tfrac{1}{2}, m_j\right\rangle = \delta_{m_j,m_1+m_2}\frac{\sqrt{j_1+m_1}}{\sqrt{2j_1+1}}\,. \qquad (3.33)$$

Using the above eigenstates, we can write the mass term of the complicated bosons as

$$\begin{aligned}&-\frac{1}{2x_3^2}\,\mathrm{tr}[C^T(L^2-4S_iL_i)C]\\ &\quad= -\frac{1}{2x_3^2}\bigg(2\frac{(k+2)^2-1}{4}C^\dagger_{at\upharpoonleft\downharpoonleft,m_j}C_{at\upharpoonleft\downharpoonleft,m_j} + 2\frac{(k-2)^2-1}{4}C^\dagger_{at\upharpoonleft\upharpoonright,m_j}C_{at\upharpoonleft\upharpoonright,m_j}\\ &\qquad + (\ell^2+3\ell+2)C^\dagger_{\ell t\upharpoonleft\downharpoonleft,m_j}C_{\ell t\upharpoonleft\downharpoonleft,m_j} + (\ell^2-\ell)C^\dagger_{\ell t\upharpoonleft\upharpoonright,m_j}C_{\ell t\upharpoonleft\upharpoonright,m_j} + (t\to b)\bigg)\,,\end{aligned} \qquad (3.34)$$

where $C^\dagger_{at\uparrow\downarrow,m_j} \equiv (C_{at\uparrow\downarrow,m_j})^\dagger$, etc. We have the (mostly) non-zero mass eigenvalues $\frac{m^2}{x_3^2} = \frac{(k+2)^2-1}{4x_3^2}$ with multiplicity $4(k-1)(N-k)$, $\frac{m^2}{x_3^2} = \frac{(k-2)^2-1}{4x_3^2}$ with multiplicity $4(k+1)(N-k)$, $\frac{m^2}{x_3^2} = \frac{\ell^2-\ell}{x_3^2}$ with multiplicity $4(\ell+1)$ and $\frac{m^2}{x_3^2} = \frac{\ell^2+3\ell+2}{x_3^2}$ with multiplicity 4ℓ for $\ell = 1,\ldots,k-1$.

Similarly, we can write the fermion mass term as

$$-\frac{1}{2x_3}\,\mathrm{tr}[\bar{\psi}G^i L_i \psi] = -\frac{1}{2x_3}\Big(2\frac{k+1}{2}\bar{\psi}_{at\uparrow\downarrow,m_j}\psi_{at\uparrow\downarrow,m_j} - 2\frac{k-1}{2}\bar{\psi}_{at\uparrow\uparrow,m_j}\psi_{at\uparrow\uparrow,m_j} \\ + (\ell+1)\bar{\psi}_{\ell t\uparrow\downarrow,m_j}\psi_{\ell t\uparrow\downarrow,m_j} - \ell\bar{\psi}_{\ell t\uparrow\uparrow,m_j}\psi_{\ell t\uparrow\uparrow,m_j} + (t\to b)\Big)\,, \tag{3.35}$$

where $\bar{\psi}_{at\uparrow\downarrow,m_j} \equiv (\psi_{at\uparrow\downarrow,m_j})^\dagger\gamma^0$, etc. In this case, we have the nonzero mass eigenvalues $\frac{m}{x_3} = \frac{k+1}{2x_3}$ with multiplicity $4(k-1)(N-k)$, $\frac{m}{x_3} = -\frac{k-1}{2x_3}$ with multiplicity $4(k+1)(N-k)$, $\frac{m}{x_3} = -\frac{\ell}{x_3}$ with multiplicity $4(\ell+1)$ and $\frac{m}{x_3} = \frac{\ell+1}{x_3}$ with multiplicity 4ℓ for $\ell = 1,\ldots,k-1$.

3.3 Summary of the spectrum

Defining

$$\nu = \sqrt{m^2 + \frac{1}{4}}\,, \tag{3.36}$$

we find the following pattern for the masses and ν's:

Multiplicity	$\nu(\tilde{\phi}_{4,5,6}, A_{0,1,2}, c)$	$m(\psi_{1,2,3,4})$	$\nu(\tilde{\phi}_{1,2,3}, A_3)$
$\ell+1$	$\ell+\frac{1}{2}$	$-\ell$	$\ell-\frac{1}{2}$
ℓ	$\ell+\frac{1}{2}$	$\ell+1$	$\ell+\frac{3}{2}$
$(k+1)(N-k)$	$\frac{k}{2}$	$-\frac{k-1}{2}$	$\frac{k-2}{2}$
$(k-1)(N-k)$	$\frac{k}{2}$	$\frac{k+1}{2}$	$\frac{k+2}{2}$
$(N-k)(N-k)$	$\frac{1}{2}$	0	$\frac{1}{2}$

(3.37)

where $\ell = 1,\ldots,k-1$.

4 Propagators

Having diagonalised the quadratic part of the action, we can derive the propagators of the mass eigenstates. Anticipating the use of dimensional regularisation and taking into account the symmetries of the problem, we will work in $d+1$ dimensions with d referring to the dimension of the codimension-one defect. For notational simplicity, we will keep denoting the coordinate transverse to the defect as x_3. We derive the scalar and fermionic propagators in subsections 4.1 and 4.2, respectively, by expressing them in terms of propagators in AdS_{d+1}. We translate the propagators of the mass eigenstates to those of the flavour and colour eigenstates in subsection 4.3.

4.1 Scalar propagators

The scalar Minkowski space propagator $K(x,y)$ is the solution to

$$\left(-\partial_\mu\partial^\mu + \frac{m^2}{x_3^2}\right)K(x,y) = \frac{g^2_{\mathrm{YM}}}{2}\delta(x-y)\,, \tag{4.1}$$

where the derivatives are all with respect to x, $\mu = 0, 1, \ldots, d$ takes $d+1$ different values and $\frac{m}{x_3}$ is the "mass" coming from the classical expectation value. The factor $g_{\text{YM}}^2/2$ stems from the normalisation of the action in (2.2).

As noted in [13], $K(x,y)$ is basically the usual propagator of a massive scalar in AdS_{d+1}. To see this, we write

$$K(x,y) = \frac{g_{\text{YM}}^2}{2} \frac{\tilde{K}(x,y)}{(x_3 y_3)^{\frac{d-1}{2}}} \,. \tag{4.2}$$

Equation (4.1) then becomes

$$\begin{aligned} \delta(x-y) &= \left(-\partial_\mu \partial^\mu + \frac{m^2}{x_3^2}\right) \frac{\tilde{K}(x,y)}{(x_3 y_3)^{\frac{d-1}{2}}} \\ &= \frac{1}{(x_3 y_3)^{\frac{d-1}{2}}} \left(-\partial_\mu \partial^\mu + (d-1)\frac{1}{x_3}\partial_3 + \frac{m^2 - \frac{d^2-1}{4}}{x_3^2}\right) \tilde{K}(x,y) \,, \end{aligned} \tag{4.3}$$

or

$$\left(-x_3^2 \partial_\mu \partial^\mu + (d-1) x_3 \partial_3 + m^2 - \frac{d^2-1}{4}\right) \tilde{K}(x,y) = (x_3 y_3)^{\frac{d-1}{2}} x_3^2 \, \delta(x-y) = x_3^{d+1} \delta(x-y) \,. \tag{4.4}$$

Let us now compare this to the AdS_{d+1} case. We choose coordinates such that the (Euclidean) metric is

$$g_{\mu\nu} = \frac{1}{x_3^2}\delta_{\mu\nu} \,, \qquad g^{\mu\nu} = x_3^2 \delta^{\mu\nu} \,, \qquad \sqrt{g} = \frac{1}{x_3^{d+1}} \,. \tag{4.5}$$

The AdS propagator with mass $\tilde{m}$ is defined by

$$(-\nabla_\mu \nabla^\mu + \tilde{m}^2) K_{\text{AdS}}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} \,. \tag{4.6}$$

Inserting the explicit expression (4.5) for the metric, we find

$$\begin{aligned} x_3^{d+1} \delta(x-y) &= (-\nabla_\mu \nabla^\mu + \tilde{m}^2) K_{\text{AdS}}(x,y) \\ &= -\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu K_{\text{AdS}}(x,y)) + \tilde{m}^2 K(x,y)_{\text{AdS}} \\ &= (-x_3^2 \, \delta^{\mu\nu} \partial_\mu \partial_\nu + (d-1) x_3 \partial_3 + \tilde{m}^2) K_{\text{AdS}}(x,y) \,. \end{aligned} \tag{4.7}$$

We see that the equations for $\tilde{K}(x,y)$ and $K_{\text{AdS}}(x,y)$ coincide, and hence that

$$K(x,y) = \frac{g_{\text{YM}}^2}{2} \frac{\tilde{K}(x,y)}{(x_3 y_3)^{\frac{d-1}{2}}} = \frac{g_{\text{YM}}^2}{2} \frac{K_{\text{AdS}}(x,y)}{(x_3 y_3)^{\frac{d-1}{2}}} \,, \tag{4.8}$$

with the identification

$$\tilde{m}^2 = m^2 - \frac{d^2-1}{4} \,. \tag{4.9}$$

Notice that the above implies that the coordinate transverse to the defect, x_3, plays the role of the radial coordinate of an AdS_4 space with the defect as its boundary. This interpretation continues to hold when fermions are taken into account, cf. the next subsection. Notice also that none of the scalar masses in (3.37) violate the Breitenlohner-Freedman (BF) bound [16], since $\tilde{m}^2 \geq -9/4$, which is precisely the BF bound for AdS_4. The bound is saturated only for the special case $k = 2$.

Closed expressions for $K_{\mathrm{AdS}}(x, y)$ in terms of hypergeometric functions can be found in the literature, see e.g. [17, 18]. Another representation, which is useful for our purpose, can be found in [19], and reads

$$\begin{aligned} K_{\mathrm{AdS}}(x, y) &= (x_3 y_3)^{d/2} \int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d} \int_0^\infty \mathrm{d}w \frac{w}{w^2 + \vec{k}^2} \, \mathrm{e}^{i\vec{k}\cdot(\vec{x}-\vec{y})} \, J_\nu(w x_3) J_\nu(w y_3), \\ &= (x_3 y_3)^{d/2} \int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d} \, \mathrm{e}^{i\vec{k}\cdot(\vec{x}-\vec{y})} \, I_\nu(|\vec{k}| x_3^<) K_\nu(|\vec{k}| x_3^>)\,, \end{aligned} \tag{4.10}$$

where I and K are modified Bessel functions with $x_3^<$ ($x_3^>$) the smaller (larger) of x_3 and y_3 and ν was defined in (3.36).

4.2 Fermionic propagators

For the fermions, after diagonalisation and when working in Euclidean space where $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$, the propagator $K_F(x, y)$ fulfils

$$\left(-i\gamma^\mu \partial_\mu + \frac{m}{x_3}\right) K_F(x, y) = \frac{g_{\mathrm{YM}}^2}{2} \delta(x - y)\,. \tag{4.11}$$

To relate this propagator to the propagator of fermions on AdS_{d+1}, we introduce

$$K_F(x, y) = \frac{g_{\mathrm{YM}}^2}{2} \frac{\tilde{K}_F(x, y)}{(x_3)^{d/2}(y_3)^{d/2}}\,. \tag{4.12}$$

Then, we find

$$\begin{aligned} \delta(x - y) &= \left(-i\gamma^\mu \partial_\mu + \frac{m}{x_3}\right) \frac{\tilde{K}_F(x, y)}{(x_3)^{d/2}(y_3)^{d/2}} \\ &= \frac{1}{(y_3)^{d/2}} \left(-\frac{i}{(x_3)^{d/2}} \gamma^\mu \partial_\mu + \frac{d}{2} i\gamma^3 \frac{1}{(x_3)^{d/2+1}} + \frac{m}{(x_3)^{d/2+1}}\right) \tilde{K}_F(x, y)\,, \end{aligned} \tag{4.13}$$

or

$$\left(-x_3 i\gamma^\mu \partial_\mu + \frac{d}{2} i\gamma^3 + m\right) \tilde{K}_F(x, y) = (x_3)^{d/2+1} (y_3)^{d/2} \delta(x - y) = (x_3)^{d+1} \delta(x - y)\,. \tag{4.14}$$

Using again the AdS metric given in (4.5), the fermion propagator $K_{F,\mathrm{AdS}}(x, y)$ solves

$$(-i\not{D} + \tilde{m}) K_{F,\mathrm{AdS}}(x, y) = \frac{\delta(x - y)}{\sqrt{g}}\,, \tag{4.15}$$

where

$$\not{D} = x_3 \partial_\mu \gamma^\mu - \frac{d}{2} \gamma^3 \tag{4.16}$$

is the spinor covariant derivative; see [20] and also [21]. Thus, we have

$$K_F(x, y) = \frac{g_{\mathrm{YM}}^2}{2} \frac{\tilde{K}_F(x, y)}{(x_3)^{d/2}(y_3)^{d/2}} = \frac{g_{\mathrm{YM}}^2}{2} \frac{K_{F,\mathrm{AdS}}(x, y)}{(x_3)^{d/2}(y_3)^{d/2}}\,, \tag{4.17}$$

with $m = \tilde{m}$.

In [22], the following useful expression for the fermionic propagator $K_{F,\text{AdS}}$ in AdS_{d+1} in terms of the bosonic one is given:

$$K^m_{F,\text{AdS}}(x,y) = \sqrt{\frac{y_3}{x_3}} \left[i\not{D} + \frac{i}{2}\gamma^3 + m \right] \left[K^{\nu=m-\frac{1}{2}}_{\text{AdS}}(x,y)\mathcal{P}_- + K^{\nu=m+\frac{1}{2}}_{\text{AdS}}(x,y)\mathcal{P}_+ \right] , \quad (4.18)$$

where

$$\mathcal{P}_\pm = \frac{1}{2}(1 \pm i\gamma^3) \,. \quad (4.19)$$

From this, we can express the flat space fermionic propagator in terms of the bosonic one as follows

$$\begin{aligned} K^m_F(x,y) &= x_3^{-\frac{d+1}{2}} \left[x_3 i\gamma^\mu \partial_\mu - \frac{d-1}{2} i\gamma^3 + m \right] x_3^{\frac{d-1}{2}} \left[K^{\nu=m-\frac{1}{2}}(x,y)\mathcal{P}_- + K^{\nu=m+\frac{1}{2}}(x,y)\mathcal{P}_+ \right] \\ &= \left[i\gamma^\mu \partial_\mu + \frac{m}{x_3} \right] \left[K^{\nu=m-\frac{1}{2}}(x,y)\mathcal{P}_- + K^{\nu=m+\frac{1}{2}}(x,y)\mathcal{P}_+ \right] . \end{aligned} \quad (4.20)$$

For future reference, we note that the fermionic propagator enjoys the charge conjugation symmetry

$$\mathcal{C}\big(K_F(x,y)\big)^T \mathcal{C}^{-1} = K_F(y,x) \,, \quad (4.21)$$

where the transpose acts in spinor space, and $\mathcal{C}$ is defined in (C.5).

4.3 Colour and flavour part of propagators

Using the mass eigenstates derived in section 3.2, we can now rewrite the propagators of the fields with definite flavour in terms of the propagators of the mass eigenstates.

We begin with the fields in the $k \times k$ block. For the easy fields, the propagator is already diagonal in the $\hat{Y}^m_\ell$ basis, so we have e.g.

$$\langle (\tilde{\phi}_4)_{\ell,m}(x) (\tilde{\phi}_4)^\dagger_{\ell',m'}(y) \rangle = \delta_{\ell,\ell'} \delta_{m,m'} K^{m^2=\ell(\ell+1)}(x,y) \,. \quad (4.22)$$

Here, $(\tilde{\phi}_4)^\dagger_{\ell,m} \equiv ((\tilde{\phi}_4)_{\ell,m})^\dagger = (-1)^m (\tilde{\phi}_4)_{\ell,-m}$ and K^{m^2} is the propagator for a scalar mode with squared mass m^2, see section 4.1.

Calculating the propagators for the complicated fields takes a little more effort. It is useful to first consider the $C_{t,\pm}$ fields. Using the relation to the diagonal fields (3.28) and suppressing space-time positions for brevity, we find

$$\langle (C_{t,+})_{\ell,m} (C_{t,+})^\dagger_{\ell',m'} \rangle = \delta_{\ell,\ell'} \delta_{m,m'} \left(\frac{\ell+m+1}{2\ell+1} K^{m^2=\ell(\ell-1)} + \frac{\ell-m}{2\ell+1} K^{m^2=(\ell+1)(\ell+2)} \right) , \quad (4.23)$$

$$\langle (C_{t,-})_{\ell,m} (C_{t,-})^\dagger_{\ell',m'} \rangle = \delta_{\ell,\ell'} \delta_{m,m'} \left(\frac{\ell-m+1}{2\ell+1} K^{m^2=\ell(\ell-1)} + \frac{\ell+m}{2\ell+1} K^{m^2=(\ell+1)(\ell+2)} \right) , \quad (4.24)$$

$$\langle (C_{t,+})_{\ell,m} (C_{t,-})^\dagger_{\ell',m'} \rangle = \delta_{\ell,\ell'} \frac{[t_-^{(2\ell+1)}]_{\ell-m+1,\ell-m'+1}}{2\ell+1} \big(K^{m^2=\ell(\ell-1)} - K^{m^2=(\ell+1)(\ell+2)} \big) , \quad (4.25)$$

and

$$\langle (C_{t,-})_{\ell,m} (C_{t,+})^\dagger_{\ell',m'} \rangle = \delta_{\ell,\ell'} \frac{[t_+^{(2\ell+1)}]_{\ell-m+1,\ell-m'+1}}{2\ell+1} (K^{m^2=\ell(\ell-1)} - K^{m^2=(\ell+1)(\ell+2)}) \,. \quad (4.26)$$

Here, $t_i^{(2\ell+1)}$ are the generators of the $(2\ell+1)$-dimensional irreducible representation of the Lie algebra SU(2) defined in appendix A with $k \to 2\ell+1$. The propagators with $t \to b$ are identical, while the mixed ones vanish. Using (3.11), we express the original fields in terms of $C_{t,\pm}$ and $C_{b,\pm}$. We can now compute e.g.

$$\begin{aligned} \langle (\tilde\phi_1)_{\ell,m} (\tilde\phi_2)^\dagger_{\ell',m'} \rangle &= \frac{1}{2} \left(-i \langle (C_{t,+})_{\ell,m} (C_{t,+})^\dagger_{\ell',m'} \rangle + i \langle (C_{b,-})_{\ell,m} (C_{b,-})^\dagger_{\ell',m'} \rangle \right) \\ &= -i \delta_{\ell,\ell'} \frac{[t_3^{(2\ell+1)}]_{\ell-m+1,\ell-m'+1}}{2\ell+1} (K^{m^2=\ell(\ell-1)} - K^{m^2=(\ell+1)(\ell+2)}) \,. \end{aligned} \quad (4.27)$$

Repeating this exercise, we finally find

$$\begin{aligned} \langle (\tilde\phi_i)_{\ell,m} (\tilde\phi_j)^\dagger_{\ell',m'} \rangle &= \delta_{i,j} \delta_{\ell,\ell'} \delta_{m,m'} \left(\frac{\ell+1}{2\ell+1} K^{m^2=\ell(\ell-1)} + \frac{\ell}{2\ell+1} K^{m^2=(\ell+1)(\ell+2)} \right) \quad (4.28) \\ &\quad - i \epsilon_{ijl} [t_l^{(2\ell+1)}]_{\ell-m+1,\ell-m'+1} \delta_{\ell,\ell'} \frac{1}{2\ell+1} (K^{m^2=\ell(\ell-1)} - K^{m^2=(\ell+1)(\ell+2)}) \,, \\ \langle (A_3)_{\ell,m} (A_3)^\dagger_{\ell',m'} \rangle &= \delta_{\ell,\ell'} \delta_{m,m'} \left(\frac{\ell+1}{2\ell+1} K^{m^2=\ell(\ell-1)} + \frac{\ell}{2\ell+1} K^{m^2=(\ell+1)(\ell+2)} \right) \quad (4.29) \end{aligned}$$

and

$$\begin{aligned} \langle (\tilde\phi_i)_{\ell,m} (A_3)^\dagger_{\ell',m'} \rangle &= - \langle (A_3)_{\ell,m} (\tilde\phi_i)^\dagger_{\ell',m'} \rangle \quad (4.30) \\ &= i \delta_{\ell,\ell'} \frac{[t_i^{(2\ell+1)}]_{\ell-m+1,\ell-m'+1}}{2\ell+1} (K^{m^2=\ell(\ell-1)} - K^{m^2=(\ell+1)(\ell+2)}) \,. \end{aligned}$$

Similarly, we obtain the propagators of the fermions as

$$\begin{aligned} \langle (\psi_i)_{\ell,m} \overline{(\psi_j)}_{\ell',m'} \rangle &= \delta_{i,j} \delta_{m,m'} \delta_{\ell,\ell'} \left(\frac{\ell+1}{2\ell+1} K_F^{m=-\ell} + \frac{\ell}{2\ell+1} K_F^{m=\ell+1} \right) \\ &\quad - \delta_{\ell,\ell'} [G^l]_{i,j} \frac{[t_l^{(2\ell+1)}]_{\ell-m+1,\ell-m'+1}}{2\ell+1} \left(K_F^{m=-\ell} - K_F^{m=\ell+1} \right) \,, \end{aligned} \quad (4.31)$$

where $\overline{(\psi_j)}_{\ell',m'} \equiv ((\psi_j)_{\ell',m'})^\dagger \gamma^0 = (-1)^{m'} (\bar\psi_j)_{\ell',-m'}$, G^l are the 4×4 matrices defined in (3.15) and K_F^m denotes the fermionic propagators of definite mass m derived in section 4.2.

To obtain the propagator between the matrix elements, one can write

$$\langle [\Phi_1]_{n_1,n_2} [\Phi_2]_{n_3,n_4} \rangle = [\hat Y_\ell^m]_{n_1,n_2} [(\hat Y_{\ell'}^{m'})^\dagger]_{n_3,n_4} \langle (\Phi_1)_{\ell,m} (\Phi_2)^\dagger_{\ell',m'} \rangle \quad (4.32)$$

and use (B.12) to get an explicit expression. In practice, however, it is often more convenient to work directly in the $\hat Y_\ell^m$ basis.

We have now written all the propagators for the $k \times k$ block. To obtain the corresponding expressions for the $k \times (N-k)$ and $(N-k) \times k$ blocks is mostly a matter of replacing

$(\Phi)_{\ell,m} \to [\Phi]_{n,a}$ and $\ell \to (k-1)/2$ in the above formulae. In particular, we have

$$\langle[\tilde{\phi}_4]_{n,a}[\tilde{\phi}_4]^\dagger_{n',a'}\rangle = \delta_{n,n'}\delta_{a,a'}K^{m^2=\frac{k^2-1}{4}}\,, \tag{4.33}$$

$$\langle[A_3]_{n,a}[A_3]^\dagger_{n',a'}\rangle = \delta_{n,n'}\delta_{a,a'}\left(\frac{k+1}{2k}K^{m^2=\frac{(k-2)^2-1}{4}} + \frac{k-1}{2k}K^{m^2=\frac{(k+2)^2-1}{4}}\right)\,, \tag{4.34}$$

$$\begin{aligned}\langle[\tilde{\phi}_i]_{n,a}[\tilde{\phi}_j]^\dagger_{n',a'}\rangle &= \delta_{i,j}\delta_{n,n'}\delta_{a,a'}\left(\frac{k+1}{2k}K^{m^2=\frac{(k-2)^2-1}{4}} + \frac{k-1}{2k}K^{m^2=\frac{(k+2)^2-1}{4}}\right) \\ &\quad -i\epsilon_{ijl}[t_l]_{n,n'}\delta_{a,a'}\frac{1}{k}\left(K^{m^2=\frac{(k-2)^2-1}{4}} - K^{m^2=\frac{(k+2)^2-1}{4}}\right)\,,\end{aligned} \tag{4.35}$$

$$\langle[\tilde{\phi}_i]_{n,a}[A_3]^\dagger_{n',a'}\rangle = -\langle[A_3]_{n,a}[\tilde{\phi}_i]^\dagger_{n',a'}\rangle = i[t_i]_{n,n'}\delta_{a,a'}\frac{1}{k}\left(K^{m^2=\frac{(k-2)^2-1}{4}} - K^{m^2=\frac{(k+2)^2-1}{4}}\right) \tag{4.36}$$

and

$$\begin{aligned}\langle[\psi_i]_{n,a}\overline{[\psi_j]}_{n',a'}\rangle &= \delta_{a,a'}\delta_{i,j}\delta_{n,n'}\frac{1}{k}\left(\frac{k+1}{2}K_F^{m=-\frac{k-1}{2}} + \frac{k-1}{2}K_F^{m=\frac{k+1}{2}}\right) \\ &\quad -\delta_{a,a'}[G^l]_{i,j}\frac{[t_l]_{n,n'}}{k}\left(K_F^{m=-\frac{k-1}{2}} - K_F^{m=\frac{k+1}{2}}\right)\,,\end{aligned} \tag{4.37}$$

where $[\tilde{\phi}_4]^\dagger_{n',a'} \equiv ([\tilde{\phi}_4]_{n',a'})^\dagger = [\tilde{\phi}_4]_{a',n'}$, $\overline{[\psi_j]}_{n',a'} \equiv ([\psi_j]_{n',a'})^\dagger\gamma^0 = [\bar{\psi}_j]_{a',n'}$, etc.

Fermionic propagators with bars added and/or removed can be obtained from those given above using the Majorana condition $\psi_i = \mathcal{C}\bar{\psi}_i^T$; see appendix C. In particular, we will need the propagator

$$\begin{aligned}\langle[\psi_i]_{a,n}\overline{[\psi_j]}_{a',n'}\rangle &= \delta_{a,a'}\delta_{i,j}\delta_{n,n'}\frac{1}{k}\left(\frac{k+1}{2}K_F^{m=-\frac{k-1}{2}} + \frac{k-1}{2}K_F^{m=\frac{k+1}{2}}\right) \\ &\quad +\delta_{a,a'}[G^l]_{i,j}\frac{[t_l]_{n',n}}{k}\left(K_F^{m=-\frac{k-1}{2}} - K_F^{m=\frac{k+1}{2}}\right)\,.\end{aligned} \tag{4.38}$$

Here, we have used the charge conjugation symmetry (4.21) to simplify the expression.

5 Dimensional regularisation

For our one-loop computation, we need to evaluate $K(x,x)$ as well as $\operatorname{tr} K_F(x,x)$ and we hence need to regulate these quantities. Dimensional regularisation has been used successfully in combination with dimensional reduction in a number of higher loop computations in standard $\mathcal{N}=4$ SYM theory, see e.g. [23, 24] and references therein, but neither have been tested in the defect setup. In this section, we determine $K(x,x)$ as well as $\operatorname{tr} K_F(x,x)$ in dimensional regularisation and discuss the preservation of supersymmetry in analogy to dimensional reduction.

Results for $K(x,x)$ and $\operatorname{tr} K_F(x,x)$ in Hadamard as well as zeta-function regularisation, which are commonly used in AdS, can be found in the literature and for completeness we summarise these in appendix E.

Bosonic fields In order to evaluate $K(x,x)$ using dimensional regularisation, we use as our starting point the expression (4.10), consider the $\vec{k}$ integral in $d = 3-2\varepsilon$ dimensions, set $\vec{x} = \vec{y}$ and go to polar coordinates. The expression (4.8) then turns into

$$K^{m^2=\nu^2-\frac{1}{4}}(x,x) = \frac{g_{\mathrm{YM}}^2}{2}\, x_3\, \frac{2\pi^{3/2-\varepsilon}}{\Gamma(3/2-\varepsilon)} \int_0^\infty \mathrm{d}k\, \frac{k^{2-2\varepsilon}}{(2\pi)^{3-2\varepsilon}} I_\nu(kx_3) K_\nu(kx_3)\,, \tag{5.1}$$

where k denotes the radial component of $\vec{k}$ and $\frac{2\pi^{3/2-\varepsilon}}{\Gamma(3/2-\varepsilon)}$ is the area of the unit sphere in $d = 3-2\varepsilon$ dimensions resulting from the angular integration. Expanding in small ε and dropping terms of $O(\varepsilon)$, we find

$$\int_0^\infty \mathrm{d}k\, k^{2-2\varepsilon} I_\nu(kx_3) K_\nu(kx_3) = \frac{1}{8\, x_3^3}\left(2m^2\left[\frac{1}{2} + \Psi(\nu+\tfrac{1}{2}) - \log 2x_3 - \frac{1}{2\varepsilon}\right] - 1\right). \tag{5.2}$$

This means that the total, regularised propagator is given by

$$K^\nu(x,x) = \frac{g_{\mathrm{YM}}^2}{2}\, \frac{1}{16\pi^2\, x_3^2}\left(m^2\left[-\frac{1}{\varepsilon} - \log(4\pi) + \gamma_{\mathrm{E}} - 2\log(x_3) + 2\Psi(\nu+\tfrac{1}{2}) - 1\right] - 1\right), \tag{5.3}$$

where γ_{E} is the Euler-Mascheroni constant.

The form of the bosonic spectrum found in the previous section means that the digamma function Ψ simplifies. We first observe that the eigenvalues come in two families. The first family is

$$m^2 = \frac{(k+2s)^2-1}{4}\,, \qquad s \in \{-1,0,1\}\,, \tag{5.4}$$

and the second family is

$$m^2 = j(j-1), \qquad j = 1,\ldots,k+1\,. \tag{5.5}$$

The digamma terms then reduce to

$$\Psi\left(\sqrt{\frac{(k+2s)^2-1}{4} + \frac{1}{4}} + \frac{1}{2}\right) = \begin{cases} -\gamma_{\mathrm{E}} - 2\log 2 + \sum_{n=1}^{\frac{k}{2}+s} \frac{2}{2n-1}\,, & k \text{ even}\,, \\ -\gamma_{\mathrm{E}} + \sum_{n=1}^{\frac{k-1}{2}+s} \frac{1}{n}\,, & k \text{ odd}\,, \end{cases} \tag{5.6}$$

and

$$\Psi\left(\sqrt{j(j-1) + \frac{1}{4}} + \frac{1}{2}\right) = -\gamma_{\mathrm{E}} + \sum_{n=1}^{j-1} \frac{1}{n}\,, \tag{5.7}$$

respectively.

Fermionic fields The other quantity that is relevant for our one-loop computation is the trace of the fermionic propagator. In this case, we will use as our starting point the formula (4.20). Since the γ matrices are traceless and furthermore satisfy $\mathrm{tr}(\gamma^i\gamma^3) = 0$, what remains to evaluate is then effectively

$$\mathrm{tr}\, K_F^m(x,y) = 2\left[-\partial_3 + \frac{m}{x_3}\right] K^{\nu=m-\frac{1}{2}}(x,y) + 2\left[\partial_3 + \frac{m}{x_3}\right] K^{\nu=m+\frac{1}{2}}(x,y)\,, \tag{5.8}$$

where we have used that $\operatorname{tr} m = 4\,m$ and $\operatorname{tr}(\gamma^3)^2 = -4$. Now, we have to find the regularised version of this expression at coinciding points, $K_F(x,x)$.

Using the fact that $\operatorname{tr} K_F(x,y)$ and $K(x,y)$ are symmetric under interchanging x and y,[1] we can write

$$\begin{aligned}\operatorname{tr} K_F^m(x,y) = &\left[-\partial_{x_3} - \partial_{y_3} + \frac{m}{x_3} + \frac{m}{y_3}\right] K^{\nu=m-\frac{1}{2}}(x,y) \\ &+ \left[\partial_{x_3} + \partial_{y_3} + \frac{m}{x_3} + \frac{m}{y_3}\right] K^{\nu=m+\frac{1}{2}}(x,y)\,.\end{aligned} \tag{5.9}$$

In the limit $y \to x$, we have $(\partial_{x_3} + \partial_{y_3})K(x,y) \to \partial_{x_3} K(x,x)$, such that

$$\operatorname{tr} K_F^m(x,x) = \left[-\partial_{x_3} + 2\frac{m}{x_3}\right] K^{\nu=m-\frac{1}{2}}(x,x) + \left[\partial_{x_3} + 2\frac{m}{x_3}\right] K^{\nu=m+\frac{1}{2}}(x,x)\,. \tag{5.10}$$

Substituting the regularised expression (5.3) for the boson into this then leads to

$$\begin{aligned}\operatorname{tr} K_F^m(x,x) = \frac{g_{\mathrm{YM}}^2}{2}\frac{1}{4\pi^2 x_3^3}\Bigg[&m^3 + m^2 - 3m - 1 \\ &+ m(m^2-1)\left(-\frac{1}{\varepsilon} - \log(4\pi) + \gamma_{\mathrm{E}} - 2\log(x_3) + 2\Psi(m) - 2\right)\Bigg]\,.\end{aligned} \tag{5.11}$$

The diagonalisation of the fermionic mass terms yields both positive and negative eigenvalues. By chirally rotating the fermion fields, one can argue that the sign of the mass should only affect the overall sign of the fermion loop; cf. also the expression for the propagator in [25]. Hence, the full m dependence of (5.11) is

$$\begin{aligned}\operatorname{tr} K_F^m(x,x) = \operatorname{sgn}(m)\frac{g_{\mathrm{YM}}^2}{2}\frac{1}{4\pi^2 x_3^3}\Bigg[&|m|^3 + |m|^2 - 3|m| - 1 \\ &+ |m|(|m|^2-1)\left(-\frac{1}{\varepsilon} - \log(4\pi) + \gamma_{\mathrm{E}} - 2\log(x_3) + 2\Psi(|m|) - 2\right)\Bigg]\,.\end{aligned} \tag{5.12}$$

Dimensional reduction Dimensional regularisation alone breaks supersymmetry, as the number of components of the gauge field A^μ is changed from $n_A = 4$ to $n_A = D = 4 - 2\varepsilon$ while the numbers of fermions $n_\psi = 4$ and real scalars $n_\phi = 6$ remains unchanged. In usual $\mathcal{N} = 4$ SYM theory, a supersymmetry-preserving alternative to dimensional regularisation is dimensional reduction [26, 27].[2] It uses the fact that $\mathcal{N} = 4$ SYM theory in four dimensions is the dimensional reduction of $\mathcal{N} = 1$ SYM theory in ten dimensions. Dimensionally reducing to $D = 4 - 2\varepsilon$ dimensions instead leads to a supersymmetry-preserving regularisation with $n_\psi = 4$ fermions but $n_\phi = 6 + 2\varepsilon$ real scalars.

Our regularisation will follow the spirit of dimensional reduction adapted to the situation with the defect and the classical vevs. In our dCFT, the gauge fields and scalars are split into easy and complicated fields: $n_A = n_{A,\mathrm{easy}} + n_{A,\mathrm{com.}} = 4 - 2\varepsilon$ and $n_\phi = n_{\phi,\mathrm{easy}} + n_{\phi,\mathrm{com.}} = 6 + 2\varepsilon$.

[1] For $\operatorname{tr} K_F(x,y)$, this follows from (4.21).

[2] Note that dimensional reduction is inconsistent at sufficiently high loop orders though [28–31].

In the calculation above, we have only touched the d dimensions parallel to the defect, such that the codimension of the defect remains one. Thus, we have $n_{A,\text{easy}} = 3 - 2\varepsilon$ and $n_{A,\text{com.}} = 1$. Furthermore, we have left untouched the three scalar fields which acquire vevs as this ensures that the classical equations of motion and the Nahm condition which define the fuzzy-funnel solution continue to be fulfilled away from $d = 3$. Thus, we are led to conclude $n_{\phi,\text{com.}} = 3$ and $n_{\phi,\text{easy}} = 3 + 2\varepsilon$.

Further support for the above conclusion comes from the construction via the D5-D3 probe-brane set-up. The easy gauge fields corresponds to the directions in which both the D5 and the D3 brane extend, while the easy scalars correspond to the directions into which none of the branes extend. The complicated scalars (gauge field) correspond to the directions in which only the D5 (D3) extends. For the D5-D3 probe-brane set-up, supersymmetry requires that the number of Neumann-Dirichlet directions, i.e. the number of dimensions in which only the D5 brane *or* the D3 branes extend, is 0, 4 or 8; see for instance [32, 33]. Thus, supersymmetry requires that we further keep $n_{A,\text{com.}} + n_{\phi,\text{com.}} = 10 - n_{A,\text{easy}} + n_{\phi,\text{easy}} = 4$ fixed, which indeed leads to $n_{\phi,\text{com.}} = 3$ and $n_{\phi,\text{easy}} = 3 + 2\varepsilon$.

6 One-loop corrections to one-point functions

For operators $\mathcal{O}$ with definite scaling dimension Δ, conformal symmetry constrains the one-point function to be of the form [3]

$$\langle \mathcal{O}_\Delta \rangle(x) = \frac{C}{x_3^\Delta}\,, \tag{6.1}$$

where C is a constant and x_3 denotes the distance to the defect.

Let us consider a general single-trace operator built out of L real scalars:

$$\mathcal{O}(x) = \mathcal{O}^{i_1 i_2 \dots i_L} \operatorname{tr}(\phi_{i_1}\phi_{i_2}\dots\phi_{i_L})(x)\,. \tag{6.2}$$

The classical one-point function is simply given by inserting the classical solution (2.4) into (6.2):

$$\langle \mathcal{O} \rangle_{\text{tree}}(x) = \mathcal{O}^{i_1 i_2 \dots i_L} \operatorname{tr}(\phi^{\text{cl}}_{i_1}\phi^{\text{cl}}_{i_2}\dots\phi^{\text{cl}}_{i_L})(x)\,. \tag{6.3}$$

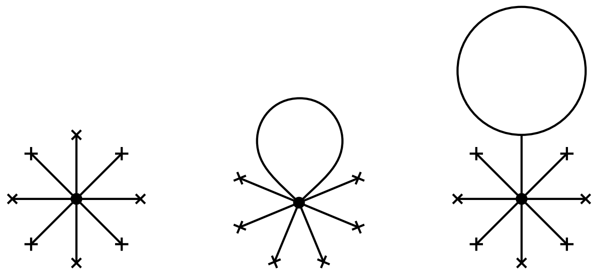
This is depicted in figure 2(a). We now calculate the first quantum correction to this quantity.

6.1 One-loop one-point functions of general operators

At one-loop order, two different diagrams can contribute to the one-point function of any operator. We call them the lollipop diagram and the tadpole diagram and depict them in figure 2(c) and 2(b), respectively.

The lollipop diagram is obtained by expanding the operator to linear order in the quantum fields and connecting this quantum field with a propagator to a quantum field in a cubic vertex whose other two quantum fields are connected with each other by a second propagator:

$$\langle \mathcal{O} \rangle_{\text{1-loop,lol}}(x) = \mathcal{O}^{i_1 i_2 \dots i_L} \sum_{j=1}^{L} \operatorname{tr}(\phi^{\text{cl}}_{i_1}\dots\tilde{\phi}_{i_j}\dots\phi^{\text{cl}}_{i_L})(x) \int \mathrm{d}^4 y \sum_{\Phi_1,\Phi_2,\Phi_3} V_3(\Phi_1,\Phi_2,\Phi_3)(y)\,, \tag{6.4}$$



(a) (b) (c)

Figure 2. The diagrams which contribute to the one-point functions of scalar fields at tree level (a) and one-loop order ((b) tadpole and (c) lollipop). The operator is represented by a dot and a cross symbolises the insertion of the classical solution.

where the second sum is over all cubic vertices V_3 in the theory. Note that this diagram is 1-particle-reducible and effectively is expressed in terms of the contribution of the one-loop correction to the scalar vevs:

$$\langle \mathcal{O} \rangle_{\text{1-loop,lol}}(x) = \mathcal{O}^{i_1 i_2 \dots i_L} \sum_{j=1}^{L} \mathrm{tr}(\phi^{\mathrm{cl}}_{i_1} \dots \langle \phi_{i_j} \rangle_{\text{1-loop}} \dots \phi^{\mathrm{cl}}_{i_L})(x) \,, \tag{6.5}$$

where

$$\langle \phi_i \rangle_{\text{1-loop}}(x) = \tilde{\phi}_i(x) \int \mathrm{d}^4 y \sum_{\Phi_1,\Phi_2,\Phi_3} V_3(\Phi_1, \Phi_2, \Phi_3)(y) \,. \tag{6.6}$$

We calculate $\langle \phi_i \rangle_{\text{1-loop}}$ in appendix D, finding

$$\langle \phi_i \rangle_{\text{1-loop}}(x) = 0 \,. \tag{6.7}$$

Thus,

$$\langle \mathcal{O} \rangle_{\text{1-loop,lol}}(x) = 0 \,, \tag{6.8}$$

independently of which operator we are looking at.

The tadpole diagram is obtained by expanding the operator to quadratic order in the quantum fields and connecting the resulting two quantum fields with a propagator:

$$\langle \mathcal{O} \rangle_{\text{1-loop,tad}}(x) = \sum_{j_1,j_2} \mathcal{O}^{i_1 \dots i_{j_1} \dots i_{j_2} \dots i_L} \, \mathrm{tr}(\phi^{\mathrm{cl}}_{i_1} \dots \tilde{\phi}_{i_{j_1}} \dots \tilde{\phi}_{i_{j_2}} \dots \phi^{\mathrm{cl}}_{i_L})(x) \,. \tag{6.9}$$

In the large-N limit, the tadpole integral only contributes when the two quantum fields are neighbouring, i.e. when $j \equiv j_1 = j_2 - 1$; the components in the off-diagonal $k \times (N-k)$ and $(N-k) \times k$ blocks can contribute only in this case, and only they scale with N.[3] Inserting

[3]Recall that the fields in the $(N-k) \times (N-k)$ block do not directly couple to the classical fields. Moreover, they are massless such that their tadpole integrals vanish.

the decomposition (3.22), we find

$$\langle\mathcal{O}\rangle_{\text{1-loop,tad}}(x) = \sum_j \mathcal{O}^{i_1\ldots i_j\, i_{j+1}\ldots i_L}\,\mathrm{tr}(\phi^{\mathrm{cl}}_{i_1}\ldots E^n{}_a E^a{}_{n'}\ldots\phi^{\mathrm{cl}}_{i_L})(x)\langle[\tilde{\phi}_{i_j}]_{n,a}(x)[\tilde{\phi}_{i_{j+1}}]_{a,n'}(x)\rangle$$
$$+ (k\times k)\text{-contributions}\,. \tag{6.10}$$

The occurring propagator is only non-vanishing for $i_j = i_{j+1} = 4,5,6$ and $i_j, i_{j+1} = 1,2,3$. All required cases are given in subsection 4.3.

At one-loop order, the one-point functions do not receive contributions from the quartic vertices as the occurrence of such a vertex would require an additional propagator in comparison with a cubic vertex. The one-point functions do not receive any contributions from the fields living on the defect either. This is due to the fact that any such one-loop diagram would involve a loop consisting of a single propagator of a defect field, which vanishes due to conformal invariance.

In general, there are two further contributions at one-loop level. The first originates from the need to renormalise the operator via the renormalisation constant $\mathcal{Z} = 1+\mathcal{Z}_{\text{1-loop}}+O(\lambda^2)$:

$$\langle\mathcal{O}\rangle_{\text{1-loop},\mathcal{Z}}(x) = \langle\mathcal{Z}_{\text{1-loop}}\mathcal{O}\rangle_{\text{tree}}(x)\,. \tag{6.11}$$

This contribution cancels the UV divergence in (6.10), see also the discussion underneath (6.17). The second additional contribution originates from the first quantum correction to the one-loop eigenstate, i.e. the two-loop eigenstate, if we are looking at operators of definite scaling dimension Δ:

$$\langle\mathcal{O}\rangle_{\text{1-loop},\mathcal{O}}(x) = \mathcal{O}^{i_1 i_2\ldots i_L}_{\text{2-loop}}\,\mathrm{tr}(\phi^{\mathrm{cl}}_{i_1}\phi^{\mathrm{cl}}_{i_2}\ldots\phi^{\mathrm{cl}}_{i_L})(x)\,. \tag{6.12}$$

Thus, we have for the planar one-loop one-point function of *any* single-trace operator built out of scalar fields:

$$\langle\mathcal{O}\rangle_{\text{1-loop}}(x) = \langle\mathcal{O}\rangle_{\text{1-loop,tad}}(x) + \langle\mathcal{O}\rangle_{\text{1-loop},\mathcal{Z}}(x) + \langle\mathcal{O}\rangle_{\text{1-loop},\mathcal{O}}(x)\,. \tag{6.13}$$

6.2 One-loop one-point functions in the SU(2) sector

Let us now consider operators in the SU(2) sector, which are built from the complex scalars $\Phi_\downarrow \equiv X = \phi_1 + i\phi_4$ and $\Phi_\uparrow \equiv Z = \phi_3 + i\phi_6$. Consider the operator

$$\mathcal{O}(x) = \mathcal{O}^{s_1 s_2\ldots s_L}\,\mathrm{tr}(\Phi_{s_1}\Phi_{s_2}\ldots\Phi_{s_L})(x)\,, \tag{6.14}$$

where $s_i =\uparrow,\downarrow$. The tree-level one-point functions of these operators were computed using integrability in [5, 6].

Of the above diagrams contributing to the one-loop one-point function, only the tadpole diagram simplifies further if we restrict ourselves to the SU(2) sector. Using the explicit expressions for the propagators given in section 4.3, we find

$$\begin{aligned}\langle\mathcal{O}\rangle_{\text{1-loop,tad}}(x) = {}& \frac{\lambda}{16\pi^2}\frac{1}{(x_3)^2}\sum_j \delta_{s_j=s_{j+1}}\mathcal{O}^{s_1\ldots s_j\, s_{j+1}\ldots s_L}\,\mathrm{tr}(\phi^{\mathrm{cl}}_{s_1}\ldots\phi^{\mathrm{cl}}_{s_{j-1}}\phi^{\mathrm{cl}}_{s_{j+2}}\ldots\phi^{\mathrm{cl}}_{s_L})(x)\\ &+\frac{\lambda}{8\pi^2}\left(-\frac{1}{2\varepsilon}-\frac{1}{2}\log(4\pi)+\frac{1}{2}\gamma_{\mathrm{E}}-\log(x_3)+\Psi(\tfrac{k+1}{2})\right)\\ &\times\sum_j \mathcal{O}^{s_1\ldots s_j\, s_{j+1}\ldots s_L}\,\mathrm{tr}(\phi^{\mathrm{cl}}_{s_1}\ldots\phi^{\mathrm{cl}}_{s_{j-1}}[\phi^{\mathrm{cl}}_{s_j},\phi^{\mathrm{cl}}_{s_{j+1}}]\phi^{\mathrm{cl}}_{s_{j+2}}\ldots\phi^{\mathrm{cl}}_{s_L})(x)\,.\end{aligned} \tag{6.15}$$

We observe that the third line is precisely proportional to the one-loop dilatation operator in the SU(2) sector originally obtained in [34]. For one-loop eigenstates, the third line is proportional to the one-loop anomalous dimension multiplied by the tree-level one-point function:

$$\begin{aligned}\langle\mathcal{O}\rangle_{\text{1-loop,tad}}(x) &= \frac{\lambda}{16\pi^2}\frac{1}{(x_3)^2}\sum_j \delta_{s_j=s_{j+1}}\mathcal{O}^{s_1\dots s_j\, s_{j+1}\dots i_L}\,\mathrm{tr}(\phi^{\text{cl}}_{s_1}\dots\phi^{\text{cl}}_{s_{j-1}}\phi^{\text{cl}}_{s_{j+2}}\dots\phi^{\text{cl}}_{s_L})(x)\\ &+\frac{\lambda}{8\pi^2}\left(-\frac{1}{2\varepsilon}-\frac{1}{2}\log(4\pi)+\frac{1}{2}\gamma_{\text{E}}-\log(x_3)+\Psi(\tfrac{k+1}{2})\right)\frac{\Delta_{\text{1-loop}}}{2}\langle\mathcal{O}\rangle_{\text{tree}}(x)\,. \end{aligned}\tag{6.16}$$

As $\mathcal{Z}_{\text{1-loop}}=\frac{\lambda}{16\pi^2}\frac{\Delta_{\text{1-loop}}}{2\varepsilon}$ when using minimal subtraction, we have

$$\langle\mathcal{O}\rangle_{\text{1-loop},\mathcal{Z}}(x)=\frac{\lambda}{16\pi}\frac{\Delta_{\text{1-loop}}}{2\varepsilon}\langle\mathcal{O}\rangle_{\text{tree}}(x)\,.\tag{6.17}$$

Thus, this contribution cancels the divergence above.[4] Moreover, the prefactor of $\log(x_3)\Delta_{\text{1-loop}}$ has the expected form coming from the one-loop correction to the scaling dimension.

The two-loop eigenstates are also known and can be efficiently obtained using one of the two recently developed technologies [35, 36] and [37, 38], both of which build on the manipulation of an inhomogeneous version of the Heisenberg spin chain. Hence, it only remains to calculate two overlaps, one involving a matrix-product state and an amputated one-loop Bethe state, and the other one involving a matrix product state and a two-loop correction to a Bethe state. These calculations should be doable [39] adapting the technique developed in [5, 6].

6.3 One-loop one-point functions of $\mathrm{tr}(Z^L)$

Finally, let us consider the special case of the BPS operator $\mathrm{tr}(Z^L)$, i.e. $\mathcal{O}^{i_1\dots i_L}=\prod_{j=1}^L(\delta_{i_j=3}+i\delta_{i_j=6})$.

At tree level, we have [5]

$$\langle\mathrm{tr}(Z^L)\rangle_{\text{tree}}(x)=\frac{(-1)^L}{x_3^L}\sum_{i=1}^k d_{k,i}^L \;=\begin{cases}0\,, & L\text{ odd}\,,\\ -\frac{2}{x_3^L(L+1)}B_{L+1}\left(\frac{1-k}{2}\right)\,, & L\text{ even}\,,\end{cases}\tag{6.18}$$

where $d_{k,i}$ given in (A.3) denotes the diagonal entries of t_3 and $B_{L+1}(u)$ is the Bernoulli polynomial of degree $L+1$.

The one-loop contributions $\langle\mathcal{O}\rangle_{\text{1-loop},\mathcal{Z}}(x)$ and $\langle\mathcal{O}\rangle_{\text{1-loop},\mathcal{O}}(x)$ vanish for this operator, and (6.10) reduces to

$$\begin{aligned}\langle\mathrm{tr}(Z^L)\rangle_{\text{1-loop,tad}}(x) =&L\,\mathrm{tr}((\phi_3^{\text{cl}})^{L-2}E^n{}_aE^a{}_{n'})(x)\left(\langle[\tilde\phi_3]_{n,a}[\tilde\phi_3]_{a,n'}\rangle-\langle[\tilde\phi_6]_{n,a}[\tilde\phi_6]_{a,n'}\rangle\right)\\ &+(k\times k)\text{-contributions}\,,\end{aligned}\tag{6.19}$$

where we have suppressed the argument x of both propagators and the trivial summation over j has produced a factor L. Inserting (4.35) and (4.33), the summation over a produces

[4] When using modified minimal subtraction, $\mathcal{Z}_{\text{1-loop}}=\frac{\lambda}{16\pi^2}\frac{\Delta_{\text{1-loop}}}{2\varepsilon}\,\mathrm{e}^{-\varepsilon\gamma_{\text{E}}}(4\pi)^\varepsilon$ and also the $-\frac{1}{2}\log(4\pi)+\frac{1}{2}\gamma_{\text{E}}$ is cancelled.

a factor $(N-k)$ and the summation over n, n' reduces the matrix unities to a unit matrix. Thus, we find[5]

$$\langle \mathrm{tr}(Z^L)\rangle_{\text{1-loop}}(x) = \langle \mathrm{tr}(Z^L)\rangle_{\text{1-loop,tad}}(x) = \langle \mathrm{tr}(Z^{L-2})\rangle_{\text{tree}}(x)\frac{1}{x_3^2}\frac{\lambda}{16\pi^2}L + O(\tfrac{1}{N})\,. \tag{6.20}$$

6.4 Finite-N results

In order to check our formalism and results, we have also computed the one-point functions explicitly in colour components for small N, k using `Mathematica`. In this way, we explicitly diagonalised the mass matrix and used the mass eigenstates to find the propagators in colour space. We find that the mass spectrum perfectly matches (3.3). Moreover, from our explicit results for $N, k < 9$, we were able to extract closed formulas for the one-point functions for any N, k. We find that they agree with (6.8) and (6.20) in the large-N limit. The cancellations of divergencies for small mass, the regulator and irrational terms like γ_{E} all provide non-trivial consistency checks of our approach.

One-loop correction to vev From computations for $N, k < 9$, we were able to find a closed expression for the vev of the scalar fields. In particular, our explicit computations show that the planar result

$$\langle \phi_i \rangle_{\text{1-loop}} = 0 \tag{6.21}$$

is actually exact.

Tadpole correction to $\mathrm{tr}(Z^L)$ Similarly, we have explicitly checked the tadpole diagrams for $N, k < 9$. Again, we were able to find an exact expression for any N, k, L. It is given by

$$\begin{aligned}\langle \mathrm{tr}(Z^L)\rangle_{\text{1-loop,tad}}(x) = L\frac{g_{\mathrm{YM}}^2}{8\pi^2}\frac{1}{x_3^L}\Bigg\{ &\frac{B_{L-1}\left(\frac{k+1}{2}\right)}{1-L}\left[N-k+\frac{k-1}{k}\frac{L-1}{2}\right] \\ &+ \sum_{i=0}^{\lfloor \frac{k-2}{2}\rfloor}(H_{k-i-1}-H_i)\left[\frac{k-2i-1}{2}\right]^{L-1}\Bigg\},\end{aligned} \tag{6.22}$$

where $H_n = \sum_{i=1}^{n} i^{-1}$ are the harmonic numbers. Notice that (6.22) reduces to (6.20) in the large-N limit.

7 Comparison to string theory for $\langle \mathrm{tr}(Z^L)\rangle$

When we wish to compare our perturbative, planar gauge-theory results to string theory, we are of course facing the eternal problem (and virtue) of the AdS/CFT correspondence that it is a strong-weak coupling duality. A proposal for how to circumvent this issue in the present set-up was put forward by Nagasaki, Tanida and Yamaguchi [13]. They pointed out that, compared to the usual AdS/CFT scenario, we here have at our disposal one extra

[5] Recall that the lollipop contribution vanishes for all operators, cf. (6.7).

tunable parameter, namely k, which plays the role of the background gauge-field flux in the string-theory picture and corresponds to the dimension of the SU(2) representation associated with the classical fields around which we expand on the gauge-theory side. Hence, one can consider the double-scaling limit

$$\lambda \to \infty, \qquad k \to \infty, \qquad \lambda/k^2 \quad \text{finite}, \tag{7.1}$$

and furthermore consider λ/k^2 to be small. The limit $\lambda \to \infty$ justifies a supergravity approximation on the string-theory side, whereas the assumption of λ/k^2 being small might bring one to the realm of perturbation theory for the field theory. This, however, requires that the gauge-theory perturbation series for the observables of interest organises into an expansion in powers of λ/k^2. This idea is analogous to the BMN construction [40], where another large quantum quantum number, J, with the interpretation of an angular momentum, was considered to be large and was combined with λ to form the double-scaling parameter λ/J^2. In the study of the spectral problem of $\mathcal{N} = 4$ SYM theory, it was found that the perturbative expansion ceased to be an expansion in the parameter λ/J^2 at four loops [41–43].

In [13], the authors calculated in a supergravity approximation the one-point function of a special chiral primary of even length L, namely the unique one which carries $\mathrm{SO}(3)\times\mathrm{SO}(3)$ symmetry:

$$\mathcal{O}(x) = C_L \operatorname{tr}\left(\left(\sum_{i=1}^{3}\phi_i^2\right)^{L/2} + \left(\sum_{i=4}^{6}\phi_i^2\right) Q_{L-2}\left(\sum_{i=1}^{3}\phi_i^2, \sum_{i=4}^{6}\phi_i^2\right)\right)(x)\,, \tag{7.2}$$

where C_L is a normalisation constant and $Q_{L-2}(y,z)$ is a homogeneous polynomial of degree $\frac{L-2}{2}$ in y and z. This was done by considering the bulk-to-boundary propagator carrying the quantum numbers characteristic of the chiral primary, fixing one of its endpoints to the point x in the AdS boundary and integrating the other one over all points belonging to the D5-brane in the interior of $AdS_5 \times S^5$. We note in passing that the computation can be considerably simplified, not necessitating any integration, if one is only interested in the leading large-L behaviour [6]. However, we will include finite-L corrections in the following discussion. The result for the string-theory one-point function found in [44] turned out to be expandable as a series in the double-scaling parameter λ/k^2 and the leading term in this expansion was shown to agree with the result of a tree-level computation in the gauge theory, which simply amounts to inserting the classical value for the fields into (7.2). The string-theory result of [44] also implies a prediction for the gauge-theory result for the one-point function of the operator above at next-to-leading order in the double-scaling parameter. The chiral primary (7.2) differs from the one we focused on in section 6.3, namely $\operatorname{tr}(Z^L)$, but one can easily convince oneself that the latter has a non-vanishing projection on the former. This implies that the ratio between the next-to-leading-order contribution and the leading-order contribution in λ/k^2 should be the same for the two operators. The prediction for this ratio following from the analysis of [44] reads

$$\left.\frac{\langle\mathcal{O}\rangle_{\text{1-loop}}}{\langle\mathcal{O}\rangle_{\text{tree-level}}}\right|_{\text{string}} = \frac{\lambda}{4\pi^2k^2}\frac{L(L+1)}{L-1}\,. \tag{7.3}$$

Combining (6.18) and (6.20), we likewise have a result for this quantity:

$$\left.\frac{\langle\mathcal{O}\rangle_{\text{1-loop}}}{\langle\mathcal{O}\rangle_{\text{tree-level}}}\right|_{\text{gauge}} = \frac{\lambda}{4\pi^2 k^2}\left(\frac{L(L+1)}{L-1} + O(k^{-2})\right), \tag{7.4}$$

which perfectly matches the string-theory prediction. This constitutes a highly nontrivial test of the AdS/dCFT correspondence! Whether the field theory result continues to organise into a power series expansion in the double-scaling parameter λ/k^2 at higher loop order is obviously a question which requires further investigation. As already mentioned, the BMN expansion broke down at four-loop order. Nevertheless, the BMN idea was instrumental in catalysing the integrability approach to AdS/CFT. One could dream that the present double-scaling idea would play a similarly instrumental role for the study of AdS/dCFT.

8 Conclusion and outlook

With the present paper, we have performed a non-trivial, positive test of the gauge-gravity correspondence in a set-up where both the supersymmetry and the conformal symmetry are partially broken. In order to carry out the test, we had to set up the framework for loop computations in a Higgsed defect version of $\mathcal{N}=4$ SYM theory, dual to a D5-D3 probe brane system with flux. This framework now opens the possibility of calculating a large amount of observables of the theory and hence obtaining more insight into the properties of the AdS/dCFT setup in general and the specific dCFT in particular. As an application, we formulated the precise line of action for calculating the one-loop correction to any scalar operator, leaving only a combinatorial problem that should be solvable invoking the tools of integrability. In particular, we have found that only two Feynman diagrams are relevant for the calculation and we have evaluated these using dimensional regularisation finding that one of them vanishes. So far, we have completed the calculation of the one-loop correction to the one-point function of the BMN vacuum which we previously summarised in [8]. For this particular correlator, a comparison with string theory is possible in a certain double-scaling limit and a perfect match is found. A similar situation occurs in a calculation of the expectation value of a straight Wilson line [45].

Apart from the two simple observables just mentioned, there exist at the time of writing no other string-theory results that one could compare to and it would be interesting and important to extend the string-theory computations to other cases. The most immediate one would be one-point functions of spinning strings corresponding to non-protected operators of the SU(2) subsector.

One-point functions only constitute one out of several novel types of correlators specific to dCFTs. Another class of such operators are two-point functions between operators with different conformal dimensions. General arguments constrain the space-time dependence of such two point functions [3] and it would be interesting to demonstrate by explicit computation that the constraints are met both from the particular dCFT considered here and from its string-theory counterpart.

Until now, we have focused on one-loop computations for which the defect fields do not play any role. A natural new direction of investigation would be to consider situations where the defect fields come into play. We expect that this will happen if the present calculation is carried on to higher-loop order. Defect fields can of course also appear in correlation functions either with other defect fields or with bulk fields. Correlation functions between defect and bulk fields again constitute a novel type of observables for which only very few explicit results are known [4].

The D5-D3 probe brane set-up is only one out of a number of probe brane set-ups which have dual dCFTs, see for instance [33]. Another set-up which is very reminiscent of the one considered here is the D7-D3 probe brane system where the geometry of the D7 brane is either $AdS_4 \times S^4$ or $AdS_4 \times S^2 \times S^2$ and where again a certain background gauge field has a non-vanishing flux through either S^4 or $S^2 \times S^2$, making possible the definition of a double-scaling parameter. The dual dCFT is again a defect version of $\mathcal{N} = 4$ SYM theory but the set-up is no longer supersymmetric. So far, for this dCFT only tree-level one-point functions of chiral primaries have been calculated and these were found to match a string-theory prediction to the leading order in the double-scaling parameter [14]. It would be interesting to extend this study to non-protected operators [46] as well as to generalise the approach presented in this paper to proceed to one-loop order. The latter endeavour, however, is likely to involve novel complications and subtleties due to the complete absence of supersymmetry.

The development of the last 15 years has lead to numerous discoveries of novel features of $\mathcal{N} = 4$ SYM theory and the AdS/CFT correspondence as well as novel techniques applicable to this set-up, such as integrability [2], localisation [47], the conformal bootstrap [48] and the duality between Wilson loops and correlators [49]. The tools of integrability have already proven useful in the present set-up, in particular at tree level where they permitted the derivation of a close form for the one-point function valid for any operator in the SU(2) subsector and for any value of the parameter k [5, 6], but also for the present one-loop considerations where they come into play for instance in section 6.2. Whether integrability tools will facilitate going to higher loop orders or to other subsectors remains to be seen. A generalisation of the conformal bootstrap approach to the defect set-up has been studied in [50–53]. It would be interesting to investigate in more detail how far this as well as the other above mentioned techniques can be taken in the context of the present dCFT.

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A Explicit form of the representation matrices

We present here explicit expressions for the representation matrices t_i in the k-dimensional irreducible representation of the Lie algebra SU(2).

Following [5], we define the standard matrices ${E^i}_j$ satisfying

$$ {E^i}_j {E^k}_l = {\delta^k}_j {E^i}_l \,. \tag{A.1} $$

We define

$$ t_+ = \sum_{i=1}^{k-1} c_{k,i} {E^i}_{i+1} \,, \qquad t_- = \sum_{i=1}^{k-1} c_{k,i} {E^{i+1}}_i \,, \qquad t_3 = \sum_{i=1}^{k} d_{k,i} {E^i}_i \,, \tag{A.2} $$

where

$$ c_{k,i} = \sqrt{i(k-i)} \,, \qquad d_{k,i} = \frac{1}{2}(k-2i+1) \,. \tag{A.3} $$

The standard k-dimensional representation of the Lie algebra SU(2) is then given by

$$ t_1 = \frac{t_+ + t_-}{2} \,, \qquad t_2 = \frac{t_+ - t_-}{2i} \qquad \text{and} \qquad t_3 \,. \tag{A.4} $$

B 'Spherical' colour basis and the fuzzy sphere

In this appendix, we summarise some properties of the spherical harmonics of the fuzzy sphere, which are used in the diagonalisation of the mass matrix in section 3.2.

Let Φ be any adjoint field. It transforms naturally under SU(2) as

$$ \Phi \to \mathrm{e}^{-i\lambda_i t_i} \, \Phi \, \mathrm{e}^{i\lambda_i t_i}, \tag{B.1} $$

or infinitesimally

$$ \delta\Phi = -i\lambda_i \, \mathrm{Ad}(t_i)\Phi = -i\lambda_i [t_i, \Phi] \,. \tag{B.2} $$

As usual, we can decompose this representation into a sum of irreducible representations. To do this explicitly for the components in the $k \times k$ block, we use the spherical harmonics Y_ℓ^m; see [54, 55]. We start by remembering that $r^\ell Y_\ell^m$ can be written as a homogeneous polynomial of order ℓ in the Cartesian coordinates. In detail, we have

$$ r^\ell Y_\ell^m = (-1)^m \sqrt{2\ell+1} \bar{\Pi}_\ell^m (x_1 + ix_2)^m \,, \qquad r^\ell Y_\ell^{-m} = \sqrt{2\ell+1} \bar{\Pi}_\ell^m (x_1 - ix_2)^m \,, \tag{B.3} $$

for $m \geq 0$ and with

$$ \bar{\Pi}_\ell^m = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \sum_{s=0}^{\lfloor(\ell-m)/2\rfloor} (-1)^s 2^{-\ell} \binom{\ell}{s} \binom{2\ell-2s}{\ell} \frac{(\ell-2s)!}{(\ell-2s-m)!} r^{2s} x_3^{\ell-2s-m} \,. \tag{B.4} $$

Note that x_1, x_2, x_3 have nothing to do with the physical coordinates. It follows that there is a symmetric set of coefficients $f^{\ell m}_{i_1, i_2, \ldots i_\ell}$ such that

$$ r^\ell Y_\ell^m = \sum_{\{i\}} f^{\ell m}_{i_1, i_2, \ldots i_\ell} x_{i_1} \cdots x_{i_\ell} \,. \tag{B.5} $$

We now want to define a $N \times N$ matrix corresponding to Y_ℓ^m. We rescale the SU(2) generators to

$$\hat{x}_i = \sqrt{\frac{4}{k^2-1}}\, t_i \,. \tag{B.6}$$

These are coordinates on the fuzzy unit sphere. In particular, we have

$$\hat{x}^2 = \hat{x}_i \hat{x}_i = 1 \tag{B.7}$$

as an operator identity. Substituting these operators into (B.5), we obtain the operators[6]

$$\tilde{Y}_\ell^m = \sum_{\{i\}} f^{\ell m}_{i_1,i_2,\dots i_\ell} \hat{x}_{i_1} \cdots \hat{x}_{i_\ell} \,, \qquad \ell = 1, \dots, k-1 \,. \tag{B.8}$$

These operators achieve the decomposition of the SU(2) representation (3.25) in the $k \times k$ block, cf. [54, 55]. In particular, they satisfy (3.24).

The $\tilde{Y}_\ell^m$ form a orthogonal basis for the traceless $k \times k$ matrices, but they are not normalised. If we define[7]

$$\hat{Y}_\ell^m = \sqrt{\frac{(k-\ell-1)!}{(k+\ell)!}}\, 2^\ell \left(\frac{k^2-1}{4}\right)^{\ell/2} \tilde{Y}_\ell^m \,, \tag{B.9}$$

we have

$$\mathrm{tr}[(\hat{Y}_\ell^m)^\dagger \hat{Y}_{\ell'}^{m'}] = \delta_{\ell\ell'}\delta_{mm'} \,, \quad \text{where} \quad (\hat{Y}_l^m)^\dagger = (-1)^m \hat{Y}_l^{-m} \,, \tag{B.10}$$

and thus

$$\mathrm{tr}[\hat{Y}_\ell^m \hat{Y}_{\ell'}^{m'}] = (-1)^m \delta_{\ell\ell'} \delta_{m+m',0} \,. \tag{B.11}$$

The matrix elements of the fuzzy spherical harmonics can be found in [56] up to normalisation; we normalise them to satisfy (B.10). They are given explicitly by

$$[\hat{Y}_\ell^m]_{n,n'} = (-1)^{k-n}\sqrt{2\ell+1} \begin{pmatrix} \frac{k-1}{2} & \ell & \frac{k-1}{2} \\ n - \frac{k+1}{2} & m & -n' + \frac{k+1}{2} \end{pmatrix} \,, \qquad n, n' = 1, \dots, k \,, \tag{B.12}$$

where the large parenthesis denote Wigner's $3j$ symbol. Hence,

$$\hat{Y}_\ell^m = [\hat{Y}_\ell^m]_{n,n'} E^n{}_{n'} \,. \tag{B.13}$$

Inverting this equation using the orthogonality and normalisation of $\hat{Y}_\ell^m$ and $E^n{}_{n'}$, we find

$$E^n{}_{n'} = [\hat{Y}_\ell^m]_{n,n'} \hat{Y}_\ell^m \,. \tag{B.14}$$

Note that $\hat{Y}_\ell^m$ transforms in the spin-ℓ representation under L_i, i.e.

$$L_i \hat{Y}_\ell^m = [t_i^{(k)}, \hat{Y}_\ell^m] = \hat{Y}_\ell^{m'} [t_i^{(2\ell+1)}]_{\ell-m'+1,\ell-m+1} \,, \tag{B.15}$$

[6] Note that for $\ell \geq k$ this construction simply gives zero.

[7] The normalisation constant follows from [54].

where $t_i^{(k)} \equiv t_i$ denotes the generators of the k-dimensional irreducible representation given in appendix A and $t_i^{(2\ell+1)}$ denotes the analogous generators of the $(2\ell+1)$-dimensional irreducible representation.

Finally, for $\ell = 1$ the spherical harmonics can be explicitly related to our t_i matrices:

$$\begin{aligned} t_1 &= \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{6}}(\hat{Y}_1^{-1} - \hat{Y}_1^1)\,, \\ t_2 &= i\frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{6}}(\hat{Y}_1^{-1} + \hat{Y}_1^1)\,, \\ t_3 &= \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{3}}\hat{Y}_1^0\,. \end{aligned} \tag{B.16}$$

C Decomposition of 10-D Majorana-Weyl fermions

In this appendix, we present our conventions for the decomposition of the ten-dimensional fermion into the four-dimensional fermions and the corresponding gamma matrices.

The ten-dimensional Majorana-Weyl fermions satisfy

$$\Psi = \mathcal{C}_{10}\bar{\Psi}^T\,, \qquad \Gamma^{11}\Psi = -\Psi\,, \tag{C.1}$$

where Γ^M are ten-dimensional gamma matrices satisfying[8]

$$\{\Gamma^M, \Gamma^N\} = -2\eta^{MN}\,. \tag{C.2}$$

We proceed to decompose the ten-dimensional gamma matrices in term of four-dimensional ones. The four-dimensional gamma matrices are γ^μ, $\mu = 0, 1, 2, 3$, and we choose the representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\,, \qquad \{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\,, \tag{C.3}$$

where $\sigma^\mu = (\mathbb{1}_2, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}_2, -\sigma^i)$. We also have

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{C.4}$$

and the charge conjugation matrix

$$\mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\,, \qquad \gamma_\mu^T = -\mathcal{C}\gamma_\mu\mathcal{C}^{-1}\,. \tag{C.5}$$

It follows that a Lorentz invariant reality condition is

$$\psi = \psi^C, \qquad \psi^C \equiv \mathcal{C}\bar{\psi}^T\,, \tag{C.6}$$

where $\bar{\psi} = \psi^\dagger\gamma_0$.

[8]Recall that we are using mostly-positive signature.

We adopt the following representation for the ten-dimensional Clifford algebra

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}_8\,, \qquad \mu = 0,1,2,3, \tag{C.7}$$

$$\Gamma^{i+3} = \tilde{\Gamma}^i = \gamma^5 \otimes \begin{pmatrix} 0 & -G^i \\ G^i & 0 \end{pmatrix}, \qquad i = 1,2,3, \tag{C.8}$$

$$\Gamma^{i+3} = \tilde{\Gamma}^i = \gamma^5 \otimes \begin{pmatrix} 0 & G^i \\ G^i & 0 \end{pmatrix}, \qquad i = 4,5,6, \tag{C.9}$$

where G^i are the 4×4 matrices

$$\begin{aligned} G^1 &= i\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad G^2 = i\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad G^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \\ G^4 &= i\begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad G^5 = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad G^6 = i\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \end{aligned} \tag{C.10}$$

The latter satisfy

$$\{G^i, G^j\} = \begin{cases} +2\delta^{i,j}, & i,j = 1,2,3, \\ -2\delta^{i,j}, & i,j = 4,5,6, \end{cases} \tag{C.11}$$

$$[G^i, G^j] = \begin{cases} -2i\,\epsilon^{ijk} G^k, & i,j = 1,2,3, \\ +2\,\epsilon^{ijk} G^k, & i,j = 4,5,6, \\ 0, \qquad i = 1,2,3, & j = 4,5,6. \end{cases} \tag{C.12}$$

Finally, the ten-dimensional charge conjugation matrix and Γ_{11} are given by

$$\mathcal{C}_{10} = \mathcal{C} \otimes \begin{pmatrix} 0 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{pmatrix}, \qquad \Gamma_{11} = \gamma_5 \otimes \begin{pmatrix} -\mathbb{1}_4 & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}. \tag{C.13}$$

Imposing the Majorana-Weyl constraint (C.1) on a ten-dimensional fermion is now seen to imply

$$\Psi = \begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_1 \\ \vdots \\ R\psi_4 \end{pmatrix}, \tag{C.14}$$

where

$$L = \tfrac{1}{2}(\mathbb{1} + \gamma_5), \qquad R = \tfrac{1}{2}(\mathbb{1} - \gamma_5) \tag{C.15}$$

act on four-dimensional Majorana fermions ψ_i satisfying (C.6).

Using the above decomposition of the ten-dimensional fermions and gamma matrices, we find

$$\frac{1}{2}\bar{\Psi}_j \tilde{\Gamma}^i_{jk}[\phi_i, \Psi_k] = \frac{1}{2}\sum_{i=1}^{3} \bar{\psi}_j G^i_{jk}[\phi_i, \psi_k] + \frac{1}{2}\sum_{i=4}^{6} \bar{\psi}_j G^i_{jk}[\phi_i, \gamma_5\psi_k]\,, \tag{C.16}$$

and hence the fermion mass term reads

$$-\frac{1}{2x_3}\sum_{i=1}^{3} \bar{\psi}_j G^i_{jk}[t_i, \psi_k]\,. \tag{C.17}$$

D One-loop correction to the scalar vevs

In this appendix, we compute the one-loop correction to the vevs of the scalar fields. To this loop order, we only need to take cubic vertices into account as only diagrams of lollipop type contribute. The one-loop correction takes the form

$$\langle \phi_i \rangle_{\text{1-loop}}(x) = \tilde{\phi}_i(x) \int \mathrm{d}^4 y \sum_{\Phi_1,\Phi_2,\Phi_3} V_3(\Phi_1(y), \Phi_2(y), \Phi_3(y)) \, . \tag{D.1}$$

There are three parts to the computation of the above vev: the contractions of the fields in the vertex, the integral and the external contraction corresponding to the stick of the lollipop. However, we will see that the sum of all the contractions in the vertex already vanishes after partial integration, and thus

$$\langle \phi_i \rangle_{\text{1-loop}}(x) = 0 \, . \tag{D.2}$$

Moreover, the one-loop corrections to the vevs of all other individual fields also vanish.

D.1 Contractions of the fields in the loop

From the cubic interaction terms in the action (2.18) and the form of the propagators in section 4.3, we find the externally contracted field in the vertex can be either $\Phi_1 = \tilde{\phi}_i$ or $\Phi_1 = A_\mu$.[9] There are then three possible types of loops. We can have easy bosons E and ghosts, complicated bosons C or fermions running in the loop. When we evaluate the loop, all the propagators are taken at the same point y in space-time. Moreover, we will also work in the planar limit.

Contribution of easy scalars, easy gauge fields and ghosts in the loop Let us first consider the contribution of easy scalars, easy gauge fields and ghosts running in the loop of the lollipop diagrams, where we restrict ourselves to the off-diagonal $k \times (N-k)$ and $(N-k) \times k$ blocks that contribute in the large-N limit.

We start with diagrams for which $\Phi_1 = \tilde{\phi}_i$. For the sake of concreteness, we focus on the easy scalar $\tilde{\phi}_4$ running in the loop; the contributions of all other easy fields are essentially the same. The corresponding interaction term is (2.18)

$$+\operatorname{tr}([\phi_i^{\text{cl}}, \tilde{\phi}_4][\tilde{\phi}_i, \tilde{\phi}_4]) = +\operatorname{tr}(\tilde{\phi}_i[\tilde{\phi}_4, [\phi_i^{\text{cl}}, \tilde{\phi}_4]]) = -\frac{1}{y_3}\operatorname{tr}(\tilde{\phi}_i[\tilde{\phi}_4, [t_i, \tilde{\phi}_4]]) \, . \tag{D.3}$$

From the decomposition (3.22) of $\tilde{\phi}_4$, we find

$$\operatorname{tr}(\tilde{\phi}_i[\tilde{\phi}_4, [t_i, \tilde{\phi}_4]]) \simeq -\langle [\tilde{\phi}_4]_{n,a} [\tilde{\phi}_4]_{a,n'} \rangle \left(\operatorname{tr}(\tilde{\phi}_i E^n{}_{n'} t_i) + \operatorname{tr}(\tilde{\phi}_i t_i E^n{}_{n'}) \right) , \tag{D.4}$$

where we have dropped the contributions from the components in the $k \times k$ block, which are irrelevant in the large-N limit. We denote the restriction to terms relevant in the large-N limit by $\simeq$. Using the explicit form of the propagator (4.33), the matrices $E^n{}_{n'}$ become unit

[9] We have no non-vanishing contraction for $\Phi_1 = \psi$, which would lead to a potentially non-vanishing vev of a single fermion.

matrices after the summation over n, n', the a summation yields a factor $N-k$ and we find in the large-N limit

$$+\operatorname{tr}(\tilde\phi_i[\wick{\c{\tilde\phi}_4,[\phi_i^{\rm cl},\c{\tilde\phi}_4}]]) \simeq \frac{2N}{y_3} K^{m^2=\frac{k^2-1}{4}} \operatorname{tr}(\tilde\phi_i t_i)\,. \tag{D.5}$$

In total, this contribution has a prefactor of $n_{\phi,\text{easy}} + n_{A,\text{easy}} - n_c$.

Let us now turn to the effective vertices that involve $\Phi_1 = A_\mu$. We again focus on the easy scalar $\tilde\phi_4$ running in the loop. The corresponding vertex is

$$i\operatorname{tr}([A^\mu, \tilde\phi_4]\partial_\mu\tilde\phi_4) = i\operatorname{tr}(A^\mu[\tilde\phi_4, \partial_\mu\tilde\phi_4])\,. \tag{D.6}$$

We contract the scalar fields and obtain

$$i\operatorname{tr}(A^\mu[\wick{\c{\tilde\phi}_4,\partial_\mu\c{\tilde\phi}_4}]) \simeq i\big[\langle[\tilde\phi_4]_{n,a}\partial_\mu[\tilde\phi_4]_{a,n'}\rangle - i\langle\partial_\mu[\tilde\phi_4]_{n,a}[\tilde\phi_4]_{a,n'}\rangle\big]\operatorname{tr}(A^\mu E^n{}_{n'}) = 0\,, \tag{D.7}$$

where the last step follows from the symmetry of the propagator. Similarly, the contractions of

$$i[A^\mu, A^\nu]\partial_\mu A_\nu\,, \qquad\qquad i(\partial_\mu\bar c)[A^\mu, c] \tag{D.8}$$

with the easy gauge fields and ghosts running in the loop are also vanishing.

Contribution from complicated bosons in the loop For the case of complicated bosons contracted in the loop, there are two vertices with insertions of the classical fields that can contribute:

$$\begin{aligned} +\operatorname{tr}([\phi_i^{\rm cl}, \tilde\phi_j][\tilde\phi_i, \tilde\phi_j]) &= -\frac{1}{y_3}\operatorname{tr}(\tilde\phi_i[\tilde\phi_j,[t_i,\tilde\phi_j]])\,,\\ +\operatorname{tr}([A^\mu, \phi_i^{\rm cl}][A_\mu, \tilde\phi_i]) &= -\frac{1}{y_3}\operatorname{tr}(\tilde\phi_i[A^\mu,[t_i,A_\mu]])\,. \end{aligned} \tag{D.9}$$

The requirement that the boson in the loop is complicated effectively fixes $i, j = 1, 2, 3$ and $\mu = 3$.

The fields at the vertex can be contracted in three different ways. Let us for simplicity restrict to the vertex with $\Phi_1 = \phi_i$. We can connect $\tilde\phi_j$ to $\tilde\phi_j$ and there are two ways we can connect $\tilde\phi_j$ to $\tilde\phi_i$:

$$\operatorname{tr}(\tilde\phi_i[\wick{\c{\tilde\phi}_j,[t_i,\c{\tilde\phi}_j}]]), \qquad \operatorname{tr}(\wick{\c{\tilde\phi}_i[\tilde\phi_j,[t_i,\c{\tilde\phi}_j}]]), \qquad \operatorname{tr}(\wick{\c{\tilde\phi}_i[\c{\tilde\phi}_j},[t_i,\tilde\phi_j]])\,. \tag{D.10}$$

The terms with A_3 can be contracted analogously.

Out of the above three contractions, the easiest one to compute is the first one. Again, we work in the planar limit and the computation is similar to the easy bosons discussed above. From (4.35), we then immediately find

$$\operatorname{tr}(\tilde\phi_i[\wick{\c{\tilde\phi}_1,[t_i,\c{\tilde\phi}_1}]]) \simeq -N\left[\frac{k+1}{k}K^{m^2=\frac{(k-2)^2-1}{4}} + \frac{k-1}{k}K^{m^2=\frac{(k+2)^2-1}{4}}\right]\operatorname{tr}(\tilde\phi_i t_i)\,. \tag{D.11}$$

From (4.35), it is easy to see that all the complicated bosons give the same contribution, which results in an overall factor of $n_{\phi,\text{com.}} + n_{A,\text{com.}}$.

The other two contractions are more involved but share a similar structure. Let us work out the last one first. We obtain

$$\mathrm{tr}(\tilde{\phi}_i[\tilde{\phi}_j,[t_i,\tilde{\phi}_j]]) \simeq (\langle[\tilde{\phi}_i]_{a,n}[\tilde{\phi}_j]_{n',a}\rangle - \langle[\tilde{\phi}_j]_{a,n}[\tilde{\phi}_i]_{n',a}\rangle)\,\mathrm{tr}(E^{n}{}_{n'}[t_i,\tilde{\phi}_j])\,. \tag{D.12}$$

Inserting the explicit form of the propagator (4.35), it is easy to see that the contribution of the term with $\delta_{n,n'}$ cancels and we are left with

$$\begin{aligned}\mathrm{tr}(\tilde{\phi}_i[\tilde{\phi}_j,[t_i,\tilde{\phi}_j]]) &\simeq -2i\frac{N}{k}\left(K^{m^2=\frac{(k-2)^2-1}{4}} - K^{m^2=\frac{(k+2)^2-1}{4}}\right)\epsilon_{ijk}\,\mathrm{tr}(t_k[t_i,\tilde{\phi}_j]) \\ &= 2\frac{N}{k}\left(K^{m^2=\frac{(k-2)^2-1}{4}} - K^{m^2=\frac{(k+2)^2-1}{4}}\right)\epsilon_{ijk}\epsilon_{kil}\,\mathrm{tr}(t_l\tilde{\phi}_j) \\ &= 2(n_{\phi,\mathrm{com.}}-1)\frac{N}{k}\left(K^{m^2=\frac{(k-2)^2-1}{4}} - K^{m^2=\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(t_i\tilde{\phi}_j)\,. \end{aligned} \tag{D.13}$$

The final contraction gives

$$\mathrm{tr}(\tilde{\phi}_i[\tilde{\phi}_j,[t_i,\tilde{\phi}_j]]) \simeq \langle[\tilde{\phi}_i]_{a,n}[\tilde{\phi}_j]_{n',a'}\rangle\,\mathrm{tr}(E^{n}{}_{n'}t_i\tilde{\phi}_j) + \langle[\tilde{\phi}_j]_{n,a}[\tilde{\phi}_i]_{a',n'}\rangle\,\mathrm{tr}(E^{n}{}_{n'}\tilde{\phi}_j t_i)\,. \tag{D.14}$$

The second term in the propagator (4.35) evaluates in the same way as above, but the $\delta_{n,n'}$ term now also contributes and we obtain

$$\begin{aligned}\mathrm{tr}(\tilde{\phi}_i[\tilde{\phi}_j,[t_i,\tilde{\phi}_j]]) &\simeq N\left(\frac{k+1}{k}K^{m^2=\frac{(k-2)^2-1}{4}} + \frac{k-1}{k}K^{m^2=\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(\tilde{\phi}_i t_i) \\ &\quad + (n_{\phi,\mathrm{com.}}-1)\frac{N}{k}(K^{m^2=\frac{(k-2)^2-1}{4}} - K^{m^2=\frac{(k+2)^2-1}{4}})\,\mathrm{tr}(\tilde{\phi}_i t_i)\,. \end{aligned} \tag{D.15}$$

The vertices from (D.9) with $\Phi_1 = A_3$ instead of $\Phi_1 = \tilde{\phi}_i$ contribute with

$$\mathrm{tr}(\tilde{\phi}_i[A_3,[t_i,A_3]]) = \mathrm{tr}(\tilde{\phi}_i[A_3,[t_i,A_3]]) \simeq 0\,, \tag{D.16}$$

as can be seem from a short analogous calculation.

Finally, there is a non-trivial contribution from the vertex

$$\mathrm{tr}(i[A^\mu,\tilde{\phi}_i]\partial_\mu\tilde{\phi}_i)\,, \tag{D.17}$$

which can be contracted non-trivially in two different ways that contribute for $\Phi_1 = \phi_i$:

$$\mathrm{tr}(i[A^3,\tilde{\phi}_i]\partial_3\tilde{\phi}_i)\,, \qquad\qquad \mathrm{tr}(i[A^3,\tilde{\phi}_i]\partial_3\tilde{\phi}_i)\,. \tag{D.18}$$

In the large-N limit, the only terms that survive are

$$\begin{aligned}\mathrm{tr}(i[A^3,\tilde{\phi}_i]\partial_3\tilde{\phi}_i) &\simeq 2i\langle[A^3]_{n,a}[\tilde{\phi}_i]_{a,n'}\rangle\,\mathrm{tr}(E^{n}{}_{n'}\partial_3\tilde{\phi}_i) \\ &\simeq 2\frac{N}{k}\left(K^{m^2=\frac{(k-2)^2-1}{4}} - K^{m^2=\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(t_i\partial_3\tilde{\phi}_i) \end{aligned} \tag{D.19}$$

and

$$\mathrm{tr}(i[\overbracket{A^3, \tilde{\phi}_i]\partial_3\tilde{\phi}_i}) \simeq 2i\langle[\partial_3\tilde{\phi}_i]_{n,a}[A^3]_{a,n'}\rangle\,\mathrm{tr}(E^n{}_{n'}\tilde{\phi}_i)$$
$$\simeq -\frac{N}{k}\partial_3\left(K^{m^2=\frac{(k-2)^2-1}{4}} - K^{m^2=\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(t_i\tilde{\phi}_i)\,. \tag{D.20}$$

In the last line, we expressed the propagator with a derivative on the field as a derivative of the propagator. It follows from the identity

$$\lim_{x\to y}\langle[A^3(x)]_{n,a}[\partial_3\tilde{\phi}_i(y)]_{a,n'}\rangle = \tfrac{1}{2}\partial_{y_3}\lim_{x\to y}\langle[A^3(x)]_{n,a}[\tilde{\phi}_i(y)]_{a,n'}\rangle\,, \tag{D.21}$$

which follows from the explicit form of the propagator (4.10) and the following property of the Bessel functions

$$\tfrac{1}{2}\partial_x\Big[I_{\nu-1}(x)K_{\nu-1}(x) - I_{\nu+1}(x)K_{\nu+1}(x)\Big] = \Big[\partial_x I_{\nu-1}(x)\Big]K_{\nu-1}(x) - \Big[\partial_x I_{\nu+1}(x)\Big]K_{\nu+1}(x). \tag{D.22}$$

The third contraction of (D.17), which corresponds to $\Phi_1 = A_3$, vanishes in complete analogy to (D.7).

Contribution of fermions in the loop The relevant vertices read

$$\frac{1}{2}\sum_{i=1}^{3}\mathrm{tr}(\bar{\psi}_j[G^i]_{jk}[\tilde{\phi}_i,\psi_k]) + \frac{1}{2}\sum_{i=4}^{3+n_{\phi,\text{easy}}}\mathrm{tr}(\bar{\psi}_j[G^i]_{jk}[\tilde{\phi}_i,\gamma_5\psi_k]) + \frac{1}{2}\mathrm{tr}(\bar{\psi}_j\gamma^\mu[A_\mu,\psi_j])\,, \tag{D.23}$$

which contribute for $\Phi_1 = \tilde{\phi}_{i,\text{com.}}$, $\Phi_1 = \tilde{\phi}_{i,\text{easy}}$ and $\Phi_1 = A_\mu$, respectively. The first term gives

$$\frac{1}{2}\,\mathrm{tr}(\overbracket{\bar{\psi}_j[G^i]_{jk}[\tilde{\phi}_i,\psi_k}]) \simeq \frac{1}{2}[G^i]_{jk}\left(\langle[\bar{\psi}_j]_{a,n}[\psi_k]_{n',a}\rangle\,\mathrm{tr}(E^{n'}{}_n\tilde{\phi}_i) - \langle[\bar{\psi}_j]_{n,a}[\psi_k]_{a,n'}\rangle\,\mathrm{tr}(E^n{}_{n'}\tilde{\phi}_i)\right)$$
$$= N[G^i]_{jk}[G^l]_{kj}\frac{[t_l]_{n,n'}}{k}\left(\mathrm{tr}\,K_F^{m=-\frac{k-1}{2}} - \mathrm{tr}\,K_F^{m=\frac{k+1}{2}}\right)\mathrm{tr}(E^n{}_{n'}\tilde{\phi}_i)\,, \tag{D.24}$$

where we used the fermionic propagator (4.37) and the trace of K_F is with respect to its spinor indices. Using the anti-commutator relation (C.11) for the G^i matrices, we then find

$$\frac{1}{2}\,\mathrm{tr}(\overbracket{\bar{\psi}_j[G^i]_{jk}[\tilde{\phi}_i,\psi_k}]) \simeq \frac{N}{2k}\,\mathrm{tr}(\{G^i,G^l\})(\mathrm{tr}\,K_F^{m=-\frac{k-1}{2}} - \mathrm{tr}\,K_F^{m=\frac{k+1}{2}})\,\mathrm{tr}(t_l\tilde{\phi}_i)$$
$$= \frac{N}{k}n_\psi(\mathrm{tr}\,K_F^{m=-\frac{k-1}{2}} - \mathrm{tr}\,K_F^{m=\frac{k+1}{2}})\,\mathrm{tr}(t_i\tilde{\phi}_i)\,. \tag{D.25}$$

The evaluation of the second and third term in (D.23) is similar to the discussion above, but with G^i replaced by G^i with easy index i and γ^μ, respectively. It then follows directly that this contribution vanishes because of the orthogonality of these matrices, cf. appendix C.

D.2 Total effective vertex

All vertices come with an overall factor of $\frac{2}{g^2_{\rm YM}}$. Adding all the contributions derived above, we arrive at the following total contribution

$$
\begin{aligned}
V_{\rm eff}(y) = & \, n_{\rm easy}\frac{2N}{y_3}K^{m^2=\frac{k^2-1}{4}}\,\mathrm{tr}(\tilde{\phi}_i t_i)\frac{2}{g^2_{\rm YM}} \\
& + n_{\phi,{\rm com.}}\frac{N}{y_3}\left(\frac{k+1}{k}K^{m^2=\frac{(k-2)^2-1}{4}}+\frac{k-1}{k}K^{m^2=\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(\tilde{\phi}_i t_i)\frac{2}{g^2_{\rm YM}} \\
& - 3(n_{\phi,{\rm com.}}-1)\frac{N}{y_3}\frac{1}{k}\left(K^{m^2=\frac{(k-2)^2-1}{4}}-K^{m^2=\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(\tilde{\phi}_i t_i)\frac{2}{g^2_{\rm YM}} \\
& + n_{A,{\rm com.}}\frac{2N}{k}\left(K^{m^2=\frac{(k-2)^2-1}{4}}-K^{m^2=\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(t_i\partial_3\tilde{\phi}_i)\frac{2}{g^2_{\rm YM}} \\
& - n_{A,{\rm com.}}\frac{N}{k}\partial_3\left(K^{m^2=\frac{(k-2)^2-1}{4}}-K^{m^2=\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(t_i\tilde{\phi}_i)\frac{2}{g^2_{\rm YM}} \\
& + n_\psi\frac{N}{k}\left(\mathrm{tr}\,K_F^{m=-\frac{k-1}{2}}-\mathrm{tr}\,K_F^{m=\frac{k+1}{2}}\right)\mathrm{tr}(t_i\tilde{\phi}_i)\frac{2}{g^2_{\rm YM}}\,,
\end{aligned}
\tag{D.26}
$$

where all propagators are taken at y and for conciseness we introduced $n_{\rm easy}=n_{\phi,{\rm easy}}+n_{A,{\rm easy}}-n_c$. In particular, the total contribution from all externally contracted fields except for $\Phi_1=\phi_{i,{\rm com.}}$ vanishes.

When contracting the effective vertex (D.26) with a propagator such as in (D.1), the derivative term can be partially integrated. When we then substitute the dimensional regularised expressions for the propagator from section 5, the effective vertex becomes

$$
\begin{aligned}
V_{\rm eff}(y) = \frac{N\,\mathrm{tr}(t_i\tilde{\phi}_i)}{16\pi^2 y_3^3}\Bigg[& \frac{k^2(n_{\rm easy}+n_{\phi,{\rm com.}}-2n_\psi)+n_{\rm easy}-11n_{\phi,{\rm com.}}-2n_\psi+24n_{A,{\rm com.}}+12}{2} \\
& \times\left\{\frac{1}{\varepsilon}-\gamma_{\rm E}+\log(4\pi)+2\log(y_3)-2\Psi(\tfrac{k+1}{2})\right\} \\
& - \frac{k^2(n_{\rm easy}+n_{\phi,{\rm com.}}-2n_\psi)+5n_{\phi,{\rm com.}}-3n_{\rm easy}+6n_\psi-24n_{A,{\rm com.}}}{2}\Bigg]\,.
\end{aligned}
\tag{D.27}
$$

We see that the above vanishes exactly when

$$
n_{A,{\rm com.}}=1, \qquad n_{\phi,{\rm com.}}=3, \qquad n_{\rm easy}=2n_\psi-3\,. \tag{D.28}
$$

In four dimensions, we have $n_{\rm easy}\equiv n_{\phi,{\rm easy}}+n_{A,{\rm easy}}-n_c=3+3-1=5$ and $n_\psi=4$, which satisfies (D.28) such that the effective vertex vanishes. In dimensional regularisation, however, the number of easy gauge fields is $d=3-2\varepsilon$. In dimensional reduction, the number of easy scalars is also changed in order to preserve supersymmetry, cf. the discussion at the end of section 5, and the total number of easy fields stays five. In other words, the one-loop correction to the vacuum expectation value of all fields vanishes. For the scalar fields, this happens exactly because of supersymmetry. It would be interesting to see whether there is a general argument based on supersymmetry that implies that the quantum corrections to (scalar) vevs vanish also at higher loop orders.

E Hadamard and zeta-function regularisation

In this appendix, we summarise the results for $K(x,x)$ and $\operatorname{tr} K_F(x,x)$ obtained in section 5 in the alternative Hadamard as well as zeta-function regularisation, which are commonly used in AdS.

Bosonic fields The expression for the scalar loop $K(x,x)$ in zeta-function renormalisation can be found in [57], and it reads

$$K^m(x,x) = \frac{g_{\mathrm{YM}}^2}{2x_3^2}\left(-\frac{\frac{1}{3}+m^2}{16\pi^2} + \frac{m^2}{8\pi^2}\left[\Psi\left(\nu+\frac{1}{2}\right) - \log\mu\right]\right) . \tag{E.1}$$

Here, μ is the renormalisation (mass) scale, and Ψ is the digamma function. In [58], $K(x,x)$ is found using Hadamard renormalisation:

$$K^m(x,x) = \frac{g_{\mathrm{YM}}^2}{2x_3^2}\left(-\frac{\frac{1}{3}+m^2}{16\pi^2} + \frac{m^2}{8\pi^2}\left[\Psi\left(\nu+\frac{1}{2}\right) - \log\left(\sqrt{2}M\,\mathrm{e}^{-\gamma_{\mathrm{E}}}\right)\right]\right) , \tag{E.2}$$

where M is the Hadamard renormalisation scale. We notice, as also pointed out in [58], that the two expressions agree with the identification

$$\mu = \sqrt{2}M\,\mathrm{e}^{-\gamma_{\mathrm{E}}} \ . \tag{E.3}$$

Fermionic fields The trace of the fermion loop in the Hadamard renormalisation scheme can be extracted from [59]:[10]

$$\operatorname{tr} K_F^m(x,x) = \frac{g_{\mathrm{YM}}^2}{2x_3^3}\left(\frac{1}{4\pi^2}\left[m^3+m^2+\frac{m}{6}-1\right] + \frac{m(m^2-1)}{2\pi^2}\left[\Psi\left(m\right) - \log(\sqrt{2}M\,\mathrm{e}^{-\gamma_{\mathrm{E}}})\right]\right) . \tag{E.4}$$

In [59], it is likewise stated (for the stress-energy tensor) that the Hadamard renormalisation for fermions agrees with the zeta-function one via the identification (E.3). However, note that the fermion loop is also calculated using Schwinger-de Witt renormalisation in [59], and this result does *not* match with the Hadamard expression. Zeta-function renormalisation for fermions was first carried out in [60]. The same remark as made under the discussion of dimensional regularisation concerning the chiral rotation of fermions with negative mass applies here.

Implementation For the tadpole diagram, zeta function regularisation gives the same result as dimensional regularisation, presented in (6.20). However, zeta-function regularisation of the lollipop diagram does not reproduce (6.8) but gives a non-vanishing result. More precisely, inserting (E.1) and (E.4) into the effective vertex (D.26) yields a non-vanishing result, which remains non-vanishing after the contraction with the quantum scalar and the subsequent integration over the vertex position. The reason for this appears to be that zeta function regularisation breaks supersymmetry as observed in other situations [60, 61]; recall that supersymmetry in the form of dimensional reduction was crucial for the vanishing of the lollipop diagram in dimensional regularisation.

[10]There is a misprint in [59] in the overall sign in the equivalent of (E.4). We thank the authors for communications on this point.

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Asymptotic one-point functions in AdS/dCFT

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We take the first step in extending the integrability approach to one-point functions in AdS/dCFT to higher loop orders. More precisely, we argue that the formula encoding all tree-level one-point functions of SU(2) operators in the defect version of $\mathcal{N}=4$ SYM theory, dual to the D5-D3 probe-brane system with flux, has a natural asymptotic generalization to higher loop orders. The asymptotic formula correctly encodes the information about the one-loop correction to the one-point functions of non-protected operators once dressed by a simple flux-dependent factor, as we demonstrate by an explicit computation involving a novel object denoted as an amputated matrix product state. Furthermore, when applied to the BMN vacuum state, the asymptotic formula gives a result for the one-point function which in a certain double-scaling limit agrees with that obtained in the dual string theory to all orders in the double-scaling parameter.

INTRODUCTION

Apart from observables which are protected by supersymmetry, the AdS/CFT correspondence has not provided us with many examples of quantities which can be explicitly calculated to all orders in the coupling constant in both string theory and field theory and successfully matched. The main examples are the cusp anomalous dimension [1] and the expectation value of the circular Maldacena-Wilson loop [2–4]. An instructive attempt to arrange for a situation which could allow an all-order comparison between gauge and string theory was made with the invention of the Berenstein-Maldacena-Nastase (BMN) limit, where a certain double-scaling parameter combining the 't Hooft coupling constant λ with a large angular momentum quantum number was introduced and certain observables being close to protected were considered [5]. However, it turned out that for the observables considered the BMN expansion became inconsistent starting at four-loop order in the field theory [6–8].

In a variant of the AdS/CFT correspondence which involves a D5-D3 probe-brane set-up on the string-theory side and a codimension-one defect in $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory, another double-scaling limit has recently been proposed [9]. It consists of sending the 't Hooft coupling as well as a certain background gauge field flux k to infinity while keeping a certain ratio involving the two parameters fixed. While the study of the BMN expansion acted as a seed for the development of the integrability approach to $\mathcal{N}=4$ SYM theory [10], at the present stage we already have available a vast amount of integrability tools that we can make use of when investigating the defect set-up and the associated novel double-scaling limit. In addition, in the defect case we have an entirely new collection of observables including one-point functions, two-point functions between operators of unequal conformal dimension and correlators between bulk and boundary fields [11]. In particular, we can consider the BMN vacuum states, BPS states of $\mathcal{N}=4$ SYM theory whose two- and three-point functions do not get quantum corrections in pure $\mathcal{N}=4$ SYM theory but whose one-point functions are non-vanishing and receive quantum corrections in the defect theory.

One-point functions of protected operators were calculated at tree level in the above mentioned defect CFT in [12] and in a closely related theory building on a non-supersymmetric D7-D3 probe-brane system in [13]. Furthermore, exploiting the integrability structure of $\mathcal{N}=4$ SYM theory and introducing an appropriate boundary state in the form of a matrix product state, one-point functions of non-protected operators were calculated at tree level for the SU(2) sector in [14, 15]. This approach was generalized to the SU(3) sector [16] as well as to the SO(6) sector of the non-supersymmetric cousin [17].[18] Most recently, the one-loop correction to the one-point function of the BMN vacuum was calculated [19, 20] and shown to match the string-theory prediction of [12]. In addition, a strategy for computing the one-loop correction to the one-point functions of non-protected operators was presented [20]. This involved the introduction of a new object denoted as the amputated matrix product state.

In the present letter, we will argue that the integrability approach to one-point functions suggests a certain generalization of the tree-level formula for the SU(2) sector to higher loop orders. We shall furthermore concretely implement the above mentioned strategy for the calculation of one-loop corrections to one-point functions and show that the results can be accounted for by the suggested asymptotic formula when dressed by a simple flux-dependent factor. This flux factor leads to a breakdown of the above mentioned double-scaling limit for non-protected operators at one-loop order. For protected operators, the flux factor is absent and we will show that the proposed formula implies that the one-point function of the BMN vacuum state has an expansion in the double-scaling limit that to all orders matches the corresponding expansion found on the string-theory side by a supergravity calculation.

OUR PROPOSAL

The defect version of $\mathcal{N}=4$ SYM theory which is dual to the D5-D3 probe-brane system with flux k is characterized by having a codimension-one defect, say at $x_3=0$, separating two regions of space, $x_3>0$ and $x_3<0$, where the gauge group is respectively (broken) U(N) and U($N-k$). The difference in the rank of the gauge group implies assigning the following vacuum expectation values to three out of the six scalar fields of $\mathcal{N}=4$ SYM theory:

$$\langle\phi_i\rangle_{\text{tree}} = -\frac{1}{x_3}t_i \oplus 0_{(N-k)\times(N-k)}, \qquad i=1,2,3, \tag{1}$$

where the t_i are the generators of a k-dimensional irreducible representation of SU(2). For a precise description of the holographic set-up, we refer to [20] as well as the original papers [21, 22].

As usual, we identify two complex scalars of $\mathcal{N}=4$ SYM theory with spins of an integrable SU(2) spin chain as $\uparrow\equiv X=\phi_1+i\phi_4$ and $\downarrow\equiv Y=\phi_2+i\phi_5$. A Bethe (eigen)state of this spin chain is characterized by two Dynkin labels L,M corresponding respectively to the length and the number of excitations, and in addition by M rapidities $\{u_i\}$ that satisfy certain Bethe equations. For a given eigenstate $|\mathbf{u}\rangle$, we define the corresponding single-trace operator from the SU(2) sector as

$$\mathcal{O}\equiv\left(\frac{4\pi^2}{\lambda}\right)^{\frac{L}{2}}\frac{\mathcal{Z}}{\sqrt{L}}\frac{\operatorname{tr}\prod_{l=1}^{L}\Big(\langle\uparrow_l|\otimes X+\langle\downarrow_l|\otimes Y\Big)|\mathbf{u}\rangle}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle}}. \tag{2}$$

Far away from the defect, the tree-level two-point function of $\mathcal{O}$ is normalized to unity, and we will use the freedom in the choice of the finite part of the renormalization constant $\mathcal{Z}$ to enforce this also at loop level. The one-point function then takes the form

$$\langle\mathcal{O}(x)\rangle=\left(\frac{4\pi^2}{\lambda}\right)^{\frac{L}{2}}\frac{C_k}{\sqrt{L}}\frac{1}{x_3^{\Delta}}, \tag{3}$$

where Δ denotes the scaling dimension of the operator. The calculation of C_k will be the subject of this letter.

Tree level At tree level, the one-point function can be written as the overlap of a Bethe eigenstate of the Heisenberg spin chain with a matrix product state [14, 15]. The corresponding Bethe equations read

$$1=\left(\frac{u_k-\frac{i}{2}}{u_k+\frac{i}{2}}\right)^{L}\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{u_k-u_j+i}{u_k-u_j-i}\equiv\exp[i\Phi_k]. \tag{4}$$

Using the algebraic Bethe ansatz approach [23], the Bethe state can be built from the ferromagnetic vacuum $|0\rangle_L$ with all spins up via the creation operators $B(u)$:

$$|\mathbf{u}\rangle=B(u_1)\cdots B(u_M)|0\rangle_L. \tag{5}$$

Defining the matrix product state as

$$\langle\text{MPS}|=\operatorname{tr}\prod_{l=1}^{L}\Big(\langle\uparrow_l|\otimes t_1+\langle\downarrow_l|\otimes t_2\Big), \tag{6}$$

the tree-level one-point function of $\mathcal{O}$ is given as

$$C_k=\frac{\langle\text{MPS}|\mathbf{u}\rangle}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle}}. \tag{7}$$

In [14], it was shown that only operators with L and M even and with paired rapidities $\{u_i\}=\{-u_i\}$ have non-trivial one-point functions [24]. For $k=2$, the tree-level one-point function can be elegantly described in terms of the Bethe function Φ introduced above. Let us order the roots as $\{u_1,\ldots,u_{\frac{M}{2}},-u_1,\ldots,-u_{\frac{M}{2}}\}$ and introduce the following $\frac{M}{2}\times\frac{M}{2}$ dimensional matrices $G_\pm$:

$$G_\pm=\partial_m\Phi_n\pm\partial_{m+\frac{M}{2}}\Phi_n, \tag{8}$$

with $\partial_m\equiv\frac{\partial}{\partial u_m}$. Then, the one-point function for $k=2$ can be written as

$$C_2=2^{1-L}\sqrt{\frac{Q(\frac{i}{2})}{Q(0)}}\sqrt{\frac{\det G_+}{\det G_-}}, \tag{9}$$

where $Q(u)=\prod_{i=1}^{M}(u-u_i)$ is the Baxter polynomial.

According to [15], the one-point function for $k>2$ then takes the form

$$C_k=i^L T_{k-1}(0)\sqrt{\frac{Q(\frac{i}{2})Q(0)}{Q^2(\frac{ik}{2})}}\sqrt{\frac{\det G_+}{\det G_-}}, \tag{10}$$

where

$$T_n(u)=\sum_{a=-\frac{n}{2}}^{\frac{n}{2}}(u+ia)^L\frac{Q(u+\frac{n+1}{2}i)Q(u-\frac{n+1}{2}i)}{Q(u+(a-\frac{1}{2})i)Q(u+(a+\frac{1}{2})i)} \tag{11}$$

can be identified as the transfer matrix of the Heisenberg spin chain in the $(n+1)$-dimensional representation [25].

Quantization Bearing in mind the integrability approach to the spectral problem of $\mathcal{N}=4$ SYM theory, it is natural to introduce the coupling constant dependence via the Zhukovsky variable x [26]:

$$x+\frac{1}{x}=\frac{u}{g}, \qquad x=\frac{u}{g}-\frac{g}{u}+O(g^2), \tag{12}$$

where the effective planar coupling constant g^2 is related to the 't Hooft coupling $\lambda=Ng_{\text{YM}}^2$ as $g^2=\frac{\lambda}{16\pi^2}$. This entails the following all-loop asymptotic Bethe equations [27]:

$$1=\left(\frac{x(u_k-\frac{i}{2})}{x(u_k+\frac{i}{2})}\right)^{L}\prod_{j\neq k}\frac{u_k-u_j+i}{u_k-u_j-i}\equiv\exp[i\tilde{\Phi}_k]. \tag{13}$$

The natural generalization of (9) is then obtained by replacing the classical Bethe function Φ by the quantum Bethe function $\tilde{\Phi}$. Furthermore, the transfer matrix should get modified accordingly:

$$\tilde{T}_n(u) = g^L \sum_{a=-\frac{n}{2}}^{\frac{n}{2}} x(u+ia)^L \times \frac{Q(u+\frac{n+1}{2}i)Q(u-\frac{n+1}{2}i)}{Q(u+(a-\frac{1}{2})i)Q(u+(a+\frac{1}{2})i)} . \quad (14)$$

This gives us a natural expression for (10) at the quantum level. Of course, the roots u_i appearing in the Baxter polynomials satisfy the all-loop Bethe equations (13). Finally, we also need to allow for a flux factor $\mathbb{F}_k$, such that we find

$$C_k = i^L \tilde{T}_{k-1}(0) \sqrt{\frac{Q(\frac{i}{2})Q(0)}{Q^2(\frac{ik}{2})}} \sqrt{\frac{\det \tilde{G}_+}{\det \tilde{G}_-}} \mathbb{F}_k . \quad (15)$$

The flux factor $\mathbb{F}_k$ is 1 for protected operators and its general form at one-loop order turns out to be

$$\mathbb{F}_k = 1 + g^2 \Big[\Psi(\tfrac{k+1}{2}) + \gamma_E - \log 2 \Big] \Delta^{(1)} + O(g^4) , \quad (16)$$

where $\Delta^{(1)} = 2\sum_{i=1}^{M} \frac{1}{u_i^2+\frac{1}{4}}$ is the one-loop correction to the scaling dimension. Note that the Euler digamma function Ψ can be reexpressed in terms of the harmonic number H, which is generalized to non-integer arguments via $H_x = \Psi(x+1) + \gamma_E$.

CHECKS

We now check our proposal – first at one-loop order for non-protected operators in the SU(2) sector and then at all loop orders for the BMN vacuum. Finally, we will discuss the flux factor and the fate of the double-scaling limit.

SU(2) at one-loop

In [20], we have shown that the one-loop one-point function is given by the sum of three contributions: a) the manifestly finite overlap of the Bethe eigenstate with a special spin-chain state, denoted as an amputated matrix product state, b) an ultraviolet (UV)-divergent contribution proportional to the one-loop dilatation operator, which requires operator renormalization, and c) the one-loop correction to the Bethe state. Demanding that the two-point function far away from the defect remains unit-normalized also at one-loop order fixes the renormalization constant to be $\mathcal{Z} = 1 + g^2 \frac{\Delta^{(1)}}{2} \left(\frac{1}{\epsilon} + 1 + \gamma_E + \log \pi \right) + O(g^4)$, see for instance [28]. The one-loop one-point function then reads [20]

$$C_k = \frac{(\langle \mathrm{MPS}| + g^2 \langle \mathrm{AMPS}|)|\mathbf{u}\rangle}{\sqrt{\langle \mathbf{u}|\mathbf{u}\rangle}} \times \Big[1 + g^2 \left(\Psi(\tfrac{k+1}{2}) + \gamma_E - \log 2 + \tfrac{1}{2} \right) \Delta^{(1)} \Big] + O(g^4) , \quad (17)$$

where $|\mathrm{AMPS}\rangle$ denotes the amputated matrix product state, to be explicated below, and $|\mathbf{u}\rangle$ denotes the loop-corrected Bethe state. In order to evaluate (17) explicitly, we need two ingredients. We need to evaluate the overlap of $\langle \mathrm{AMPS}|$ with the Bethe state and we need to compute the first correction to the Bethe state, i.e. the two-loop Bethe eigenstate.

Overlap with $\langle \mathrm{AMPS}|$ The amputated matrix product state $\langle \mathrm{AMPS}|$ is defined as [20]

$$\langle \mathrm{AMPS}| = \sum_{l=1}^{L} \mathcal{A}_{l,l+1} \langle \mathrm{MPS}| , \quad (18)$$

where $\mathcal{A}_{i,i+1}$ removes the matrices at positions i and $i+1$ (with $L+1 \sim 1$) if they are identical and otherwise kills the trace, cf. (6).

Let us consider the overlap between a Bethe state and the amputated matrix product state. The overlap is only non-zero for an even number of magnons M, and in the coordinate formulation it reads

$$\langle \mathrm{AMPS}|\mathbf{u}\rangle = \sum_{n \in \{n\}_M} \Psi_B(n, \mathbf{u}) \sum_{l=1}^{L} \mathcal{A}_{l,l+1} \times \mathrm{tr} \left[\prod_{i=1}^{M} \left(t_3^{n_{(i+1)i}-1} t_2 \right) \right] , \quad (19)$$

where $\{n\}_M$ denotes the usual set of ordered magnon positions ($n_1 < \cdots < n_M$) and $\Psi_B(n, \{u_i\})$ is the Bethe wave-function. Furthermore, the shorthand notation $n_{ij} \equiv n_i - n_j$ and $n_{M+1} \equiv n_1 + L$ is used throughout.

For any even M and $k = 2$, one can compute directly the action of $\sum \mathcal{A}_{l,l+1}$ on the traces in (19):

$$\sum_{l=1}^{L} \mathcal{A}_{l,l+1} \, \mathrm{tr} \prod_{i=1}^{M} \left(t_3^{n_{(i+1)i}-1} t_2 \right) \overset{k=2}{\equiv} (-1)^{\frac{M}{2} + \sum_i n_i} 2^{3-L} \left[L + 2 \sum_{i=1}^{M} \left(\delta_{n_{(i+1)i}=1} - 1 \right) \right] . \quad (20)$$

Using this, the rest of the computation can be carried out symbolically by brute force in Mathematica, at least for smaller values of M. This was done for $M = 2, 4$ and leads to the conjecture

$$\langle \mathrm{AMPS}|\mathbf{u}\rangle \overset{k=2}{\equiv} \left(4L - \Delta^{(1)} \right) \langle \mathrm{MPS}|\mathbf{u}\rangle , \quad (21)$$

which was subsequently tested numerically up to and including $M = 6$ and $L = 16$. A closed formula for $M = 2$ and any k can likewise be obtained.

Two-loop Bethe states The first loop correction to the Bethe state, i.e. the two-loop Bethe state, can be generated via the so-called Θ-morphism [29]. To this end, we consider the Heisenberg spin chain with impurities θ_i. The one-loop Bethe state can again be constructed using the algebraic Bethe ansatz approach:

$$|\theta;\mathbf{u}\rangle = \hat{B}(u_1)\dots\hat{B}(u_M)|0\rangle_L\,, \tag{22}$$

where the $\hat{B}$-operator is

$$\hat{B}(u) = \langle\uparrow|\bigotimes_{j=1}^{L}\left(\mathbb{1}_{j,0} + \frac{i}{u-\theta_j-\frac{i}{2}}\mathbb{P}_{j,0}\right)|\downarrow\rangle\,. \tag{23}$$

The two-loop eigenstate is then

$$|\mathbf{u}\rangle \equiv \left(1 - g^2\frac{\Delta^{(1)}}{2}\mathbb{H}_{L,1}\right)\Big\{|\theta;\mathbf{u}\rangle\Big\}_{\Theta}\,, \tag{24}$$

where $\mathbb{H}_{j,j+1} = \mathbb{1}_{j,j+1} - \mathbb{P}_{j,j+1}$ is the Heisenberg spin chain Hamiltonian density. The Θ-morphism $\{\}_\Theta$ is defined via

$$\Big\{f\Big\}_{\Theta} \equiv f + \frac{g^2}{2}\sum_{i=1}^{L}\left[\frac{\partial}{\partial\theta_i} - \frac{\partial}{\partial\theta_{i+1}}\right]^2 f + O(g^4)\bigg|_{\theta_j\to 0}\,. \tag{25}$$

The rapidities $\{u_i\}$ have to satisfy the two-loop Bethe equations (13). For instance, the easiest case is $M = 2, k = 2$, where we find for the overlap with the matrix product state

$$\frac{\langle\text{MPS}|\mathbf{u}\rangle}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle}} = \sqrt{\frac{L}{L-1}\frac{u^2+\frac{1}{4}}{u^2}\frac{1+g^2\frac{4}{u^2+\frac{1}{4}}}{1+\frac{g^2}{L-1}\frac{6u^2-\frac{1}{2}}{(u^2+\frac{1}{4})^2}}}\,. \tag{26}$$

A closed expression for $M = 2$ and any k can similarly be derived.

General formula Now that we have all the ingredients, we are ready to check if (15) reproduces (17). Indeed, one can analytically show that for $M = 2$ both formulas agree. Moreover, we numerically compared (15) and (17) for $L = 8$ and $M = 4$ excitations for various values of k and again find perfect agreement.

BMN vacuum at all loop orders

A particularly simple situation arises if we consider the spin-chain vacuum, which corresponds to the protected operator $\text{tr}(X^L)$.

Our proposal For the vacuum, there are no Bethe roots and our proposal (10) greatly reduces:

$$C_k = i^L T_{k-1}(0)\,, \tag{27}$$

i.e. the only contribution stems from the transfer matrix for the vacuum

$$T_{k-1}(0) = \sum_{a=\frac{1-k}{2}}^{\frac{k-1}{2}} g^L x(ia)^L. \tag{28}$$

In particular, the contribution from the flux factor trivializes. For even k, the one-point function formula can be readily expanded as a power series in g with the result

$$C_k(g) = -2\sum_{n=0}^{\frac{L}{2}}\binom{L-n}{n}\frac{L}{L-n}\frac{B_{L-2n+1}(\frac{1-k}{2})}{L-2n+1}g^{2n}\,, \tag{29}$$

where B_n is the Bernoulli polynomial with index n. At one-loop level, we find that this exactly agrees with [20]. Moreover, notice that, remarkably, this is a finite sum [30].

String theory We can compare this result to a string-theory prediction in the double-scaling limit proposed in [9]. This limit consists in taking

$$\lambda\to\infty\,,\qquad k\to\infty\,,\qquad \frac{\lambda}{k^2}\ \text{fixed and small}, \tag{30}$$

on top of the planar limit. In [12], the one-point function of a specific SO(3)×SO(3)-invariant chiral primary was calculated by a variant of the Witten prescription, in particular implying a supergravity approximation, which is justified here due to the assumption of $\lambda\to\infty$. As explained in [20], the result of this computation can be turned into a prediction for the one-point function we are considering divided by its tree-level value.

The prediction from string theory reads

$$\frac{C_k(g)}{C_k(0)}\bigg|_{st} = \frac{\Gamma(L+\frac{1}{2})}{\kappa^{L+1}\sqrt{\pi}\Gamma(L+1)}\left[\kappa^2+1\right]^{\frac{3}{2}} \times\int_{-\arctan\kappa}^{\frac{\pi}{2}} \mathrm{d}\theta\cos^{2L-1}\theta\,(\kappa+\tan\theta)^{L-2}\,. \tag{31}$$

The leading two terms of the integral above in the large $\kappa = \frac{\pi k}{\sqrt{\lambda}}$ expansion were already given in [12] and we can even evaluate the integral exactly to get

$$\frac{C_k(g)}{C_k(0)}\bigg|_{st} = \frac{\left(\kappa+\sqrt{\kappa^2+1}\right)^L\left(L\sqrt{\kappa^2+1}-\kappa\right)}{2^L(L-1)\kappa^{L+1}}\,. \tag{32}$$

Comparison In the double-scaling limit, we have $B_n(\frac{1-k}{2}) \to -(\frac{k}{2})^n$ and the gauge-theory one-point function (29) becomes a power series in $\left(\frac{g}{k}\right)^2$, which we can sum directly:

$$\frac{C_k(g)}{C_k(0)}\bigg|_{gt} \to \left[1+\sum_{n=1}^{\frac{L}{2}}\binom{L-n}{n-1}\frac{L}{n}\frac{L+1}{L-n}\left(\frac{2g}{k}\right)^{2n}\right] = \frac{\left(\sqrt{\frac{(4g)^2}{k^2}+1}+1\right)^L\left(L\sqrt{\frac{(4g)^2}{k^2}+1}-1\right)}{2^L(L-1)}\,. \tag{33}$$

Remarkably, we find agreement with the string-theory prediction to all loops after identifying $\kappa = \frac{k}{4g} = \frac{\pi k}{\sqrt{\lambda}}$!

Flux factor

The flux factor in our proposal (15) has no counterpart at tree level and depends on the anomalous scaling dimension $\Delta - L$ such that it vanishes for protected operators.

At one-loop order, the corresponding contribution in (17) has been calculated in [20]. It is the finite part of the UV-divergent integral whose UV divergence is subtracted by the renormalization constant and yields the one-loop scaling dimension $\Delta^{(1)}$. Since UV divergences exponentiate, it seems well motivated to propose

$$\mathbb{F}_k = 2^{L-\Delta} \exp\left[(\Delta - L)(\Psi(\tfrac{k+1}{2}) + \gamma_E)\right]. \tag{34}$$

A direct field-theoretic check of (34) at two-loop order would clearly be desirable, though very demanding.

An independent consequence of the flux factor is that it leads to a breakdown of the double-scaling limit for non-protected operators starting already at one-loop order. As an example, let us consider the Konishi operator, which has $L = 4, M = 2$ and $u_1 = -u_2 = \frac{1}{2\sqrt{3}} + O(g^2)$. Its one-loop one-point function can be explicitly worked out to be

$$C_k = \frac{k(k^2-1)}{12\sqrt{3}}\Big(1 + 12g^2\left[\Psi(\tfrac{k+1}{2}) + \gamma_E - \log 2 + \tfrac{5}{6}\right]\Big), \tag{35}$$

where we used that $\Delta^{(1)} = 12$. Since $\Psi(\frac{k+1}{2}) \sim \log k$ for large k, the perturbative expansion in the double-scaling limit does not arrange itself in powers of $\frac{\lambda}{k^2}$.

CONCLUSIONS & OUTLOOK

We have argued that the recently derived, integrability-based formula for tree-level one-point functions in the SU(2) sector of a specific defect version of $\mathcal{N} = 4$ SYM theory points towards a natural higher-loop generalization. The generalization is based on an idea which worked successfully for the spectral problem of $\mathcal{N} = 4$ SYM theory, and which consists of introducing the coupling constant via a Zhukovski transformation of the Bethe roots characterizing the conformal operators. More precisely, the Zhukovski variables should replace the Bethe roots both in the Bethe equations and the transfer matrix of the system. Furthermore, here an additional flux factor contributing to the higher-loop one-point function formula is needed.

We have performed a number of non-trivial tests of the generalized one-point function formula and these have come out positive. First, we have compared the higher-loop one-point function formula to an honest field-theory calculation of the one-loop one-point function of non-protected operators in the SU(2) sector. This calculation is technically demanding, involving the evaluation of the overlap of an uncorrected Bethe eigenstate and a so-called amputated matrix product state as well as the overlap between a loop-corrected Bethe eigenstate and an uncorrected matrix product state [20]. Results can be obtained analytically for BMN operators, whereas for more complicated operators one has to resort to numerical computations. For all cases tested, the field-theory computation agreed with the proposed higher-loop formula. As a second test, we have carried out a detailed analysis of the higher-loop formula when applied to the BMN vacuum state $\mathrm{tr}(X^L)$. For this state, it is possible to impose a certain double-scaling limit, proposed in [9], and to write the result for the one-point function as a power series in the double-scaling parameter. This power series expansion can be compared to a similar expansion obtained by a string-theory analysis using a supergravity approximation and exact agreement is found to all loop orders in the double-scaling parameter. These two positive tests constitute a strong indication that we are on the right track when trying to move towards higher loop orders.

The flux factor we propose depends on the anomalous dimension of the operator considered and leads to a breakdown of the double-scaling limit in the case of non-protected operators starting at one-loop order. While the proposed exponentiation of the one-loop term is certainly natural, an explicit field-theoretic check at two-loop order is clearly required.

The presented higher-loop one-point function formula is expected to be only an asymptotic formula in the sense that we expect there to be further corrections from the dressing phase of $\mathcal{N} = 4$ SYM theory [31, 32] and from wrapping interactions, likewise known from $\mathcal{N} = 4$ SYM theory [27, 33]. Nevertheless, for the BMN vacuum state we have found an all-loop match between our formula and the string-theory prediction of [9] in the earlier mentioned double-scaling limit. This is presumably due to the fact that neither Lüscher corrections, nor corrections from the dressing phase of $\mathcal{N} = 4$ come into play for states with no excitations. The one-point function of the BMN vacuum hence seems to provide us with a novel example of an observable which can be calculated to all orders in the coupling constant both in string theory and field theory, at least in a certain limit, and successfully matched.

It would be interesting to investigate whether the integrability approach can be used to infer some properties of the higher-loop contributions to other observables in the present defect CFT such as Wilson loops [9, 34, 35] or less studied objects such as two-point functions of operators of unequal conformal dimension [36].

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