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HEAVY PARTICLES AND THE BINARY INSPIRAL PROBLEM

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Heavy Particles and the Binary Inspiral Problem
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Abstract

This thesis comprises five novel publications in the fields of scattering amplitudes and effective field theories (EFTs) which focus on describing binary inspiralling systems in general relativity. These publications fall into two groups. The first group formulates and studies a new EFT: the Heavy Black hole Effective Theory (HBET). Three publications fit into this category, and present novel results pertaining to the post-Minkowskian scattering of spinning black holes, new on-shell variables tailored to computations in the large mass limit, and a prescription for how to obtain HBET amplitudes by double copying those of the Heavy Quark Effective Theory. Lessons learned from these studies are combined with other EFT techniques in the remaining two publications. This second group approaches tidal effects from within a quantum-field-theoretic framework, first without spin, then for spin-1/2 particles (which can be thought of as slowly spinning celestial bodies).

Each publication is introduced by an initiation describing key tools and concepts pertinent to the publication.

SAMMENFATNING

Denne afhandling indeholder resultaterne beskrevet i fem publikationer indenfor emnet spredningsamplituder og effektive feltteorier (EFTs) med specielt fokus på at beskrive par af tunge objekter, der spiralerer mod hinanden som beskrevet af generel relativitetsteori. Disse fem publikationer falder i to grupper. Den første gruppe formulerer og undersøger en ny EFT: Heavy Black Hole Effective Theory (HBET). Tre publikationer falder indenfor denne kategori, og de præsenterer nye resultater der hører til såkaldt post-Minkowskian spredning af sorte huller med spin, nye variabler der er skabt til perfekt at beskrive grænsen hvor masserne af spredningsobjekterne er store, samt reglerne for hvorledes man kan beregne amplituder i HBET ved at lave en dobbelt kopi af amplituder fra Heavy Quark Effective Theory. Ud fra disse tre publikationer, og i kombination med andre EFT-teknikker, har jeg arbejdet mig frem til yderligere to publikationer. Denne anden gruppe af publikationer beskriver gravitationelle 'tidevands'-effekter fra et kvantefeltteoretisk synspunkt, først uden spin, siden for spin-$1/2$ partikler (der kan opfattes som langsomt spinnende astronomiske objekter).

Hver af de fem publikationer bliver introduceret med en indledende tekst, der beskriver de nøglebegreber, der er nødvendige for at forstå hver publikation.

ACKNOWLEDGEMENTS

My experience during my Ph.D studies, as for so many others, has been filled with ups and downs. Throughout this journey, what has made the highs so much happier and the lows much more bearable are the people with whom I could share the joys and lean on in times of need. I feel extremely privileged to have been surrounded by so many incredible people – new friends, old friends, family, and colleagues – who have made my time in Copenhagen fun, happy, and fruitful.

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I feel very lucky to have met Poppy Goddard and Anabel Soriano.[1] Not only was meeting them rather improbable, but it happened at a time when they were exactly the people I needed to meet. They have since become two of my best friends and two of the funnest people I've met during my time here. I can always count on them for both support and celebration.

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[1] ¿Donde esta la fiesta?

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List of Publications

This thesis is based on, and consists of reprints of, the following publications:

1. P.H. Damgaard, **K. Haddad**, A. Helset, *Heavy black hole effective theory*, *JHEP* **11** (2019) 070 [arXiv:1908.10308].

2. R. Aoude, **K. Haddad**, A. Helset, *On-shell heavy particle effective theories*, *JHEP* **05** (2020) 051 [arXiv:2001.09164].

3. **K. Haddad**, A. Helset, *The double copy for heavy particles*, *Phys. Rev. Lett.* **125** (2020) 181603 [arXiv:2005.13897].

4. **K. Haddad**, A. Helset, *Tidal effects in quantum field theory*, *JHEP* **12** (2020) 024 [arXiv:2008.04920].

5. R. Aoude, **K. Haddad**, A. Helset, *Tidal effects for spinning particles*, *JHEP* **03** (2021) 097 [arXiv:2012.05256].

Contents

CHAPTER

1 INTRODUCTION

General relativity (GR) and quantum field theory (QFT) are two of the most extensively experimentally verified physical theories currently known. The success of the latter is encapsulated by the astounding accuracy of the Standard Model of particle physics, which, to the chagrin of many, has not yet produced a result that is inconsistent with experiments. Similarly, the predictions of the former continue to be proven true, with two of the most recent confirmations of the theory being the detection of gravitational waves (GWs) in 2015 [1] and the observation of a black hole in 2019 [2].

Over the past several decades, one of the most active fields of research in high-energy theoretical physics has been concerned with the unification of GR and QFT, aiming to describe gravitational interactions at all energies. An intuitive way to attempt such a unification is to quantize the Einstein-Hilbert (EH) action while decomposing the metric into a small deviation away from flat space. One then treats these small deviations as the quanta of the gravitational field – that is, gravitons – and computes scattering amplitudes perturbatively in the expansion parameter $\kappa = \sqrt{32\pi G} = 1/M_{\text{pl}}$, where G is Newton's constant and M_{pl} is the Planck mass. Following this procedure, one quickly runs into problems: the EH action describing GR is non-renormalizable at two loops, requiring new couplings to absorb these divergences [3–5]. The situation is even worse when one attempts to describe the high-energy behavior of gravity coupled to a scalar particle, in which case the non-renormalizability already appears at the one-loop level [6]. We will refer to this framework of quantized gravity with or without matter as quantized GR (QGR).

While numerous alternative descriptions of gravity have been put forth to remedy this loss of predictive power at large energy scales, the "naïve" approach described above is far from obsolete; QGR is ill-suited to high-energy processes, but it is applicable if the energies in the problem are *small* (compared to the Planck mass). Indeed, a community has formed around taking advantage of the region of validity of QGR to study *macroscopic* gravitational systems.

A central concept facilitating the description of macroscopic physics using scattering amplitudes is the following: the amplitude describing the $2 \to 2$ scattering of massive particles through the exchange of massless bosons contains terms *non-analytic* in the square of the transfer momentum to all loop orders that dominate the amplitude in the *long-range* limit [7, 8]. Such terms are of the form $S \equiv \pi^2/\sqrt{-q^2}$ or $L \equiv \log(-q^2)$, and they tend to (plus or minus) infinity when the transfer momentum squared, q^2, tends to zero. We can see that the limit $q^2 \to 0$ is the correct limit to describe long-range dynamics in two ways. The first is by noting that the transfer momentum is related to the inverse of the impact parameter, $q \sim 1/b$, so the limit $q^2 \to 0$ is conjugate to the limit $b \to \infty$. Another, more intuitive explanation is that only on-shell modes can exist long enough to travel long distances. Since by assumption the processes we're interested in are mediated by massless bosons, the on-shell limit is precisely $q^2 \to 0$.

Through a careful restoration of factors of $\hbar$ to the scattering amplitude [9, 10], these non-analytic structures can be shown to produce an $\hbar$ scaling that leads to *classical* contributions to observables. Concretely, "classical" means that these contributions are finite in the limit $\hbar \to 0$, and classical amplitudes are the portions of scattering amplitudes that yield classical contributions.[1] While there has been plenty of work deriving long-range and classical corrections to the Newtonian potential (see the references in subsequent chapters), the discovery of GWs has turbocharged studies into the link between amplitudes and classical observables in recent years.

A significant portion of this work is explicitly motivated by the binary inspiral problem in GR. The system under consideration comprises two very dense celestial bodies (either black holes, neutron stars, or one of each) locked in orbit around one another. As these two objects revolve around each other, they lose energy by the emission of GWs, consequently causing a shrinking of the orbital radius in the so-called *inspiral* phase. Eventually, the objects will collide and combine into a new, unified body during the merger phase, which continues to emit gravitational waves in the ringdown phase until it reaches equilibrium [11, 12]. Earth-based detectors measure the GWs emitted in these three phases and identify them through comparisons to templates derived from GR. Conventionally, the dynamics used to compute GW predictions have been calculated using general-relativistic techniques, with the inspiral and ringdown handled analytically while the merger is tackled numerically.

Scattering amplitudes present a complimentary approach to the inspiral phase of the problem. In this phase the bodies are well separated, so the gravitational field between them is small enough to decompose the metric as a perturbation around flat space. The actions of QGR can then be used to calculate the amplitude for $2 \to 2$ scattering, expanded in powers of G. After isolating the classical portions of the scattering amplitudes, several observable quantities can be derived using one of a variety of techniques. Since amplitudes are naturally computed with fully-relativistic kinematics, the derived observables are obtained expanded in the so-call *post-Minkowskian* (PM) expansion, which is an expansion only in Newton's contant. This is to be contrasted with the post-Newtonian (PN) expansion more prevalent in general-relativistic approaches, in which calculations are done by expanding in both Newton's constant and the relative velocity of the objects in the binary.

The work presented in this thesis is situated in this context; it comprises five original publications that applied scattering amplitude and effective field theory (EFT) techniques to develop insight into and produce novel results about the binary inspiral problem in GR. Having taken an amplitudes approach, the inspiral phase is the subject of these studies. We omit an overview of the current state of the field here since the introductions of most of the publications already include such a summary.

The first three publications formulated (in Chapter 2) and analyzed (in Chapters 3 and 4) a new EFT: the Heavy Black hole Effective Theory (HBET). This EFT is the gravitational analog to the well-established Heavy Quark Effective Theory (HQET), and achieves an operator expansion where the small expansion parameter is proportional to $\hbar$. The work in Chapter 3 expressed the external states of Heavy Particle Effective Theories (HPETs) in terms of on-shell variables, while Chapter 4 investigated the double-copy relationship between HQET and HBET for matter with various spin quantum numbers.

The next two chapters, Chapters 5 and 6, focus on the description of tidal effects using QFT methods.

[1]Which non-analytic structure produces classical effects depends on the order in perturbation theory one is considering. As a rule of thumb, "non-analytic" can be thought of as qualitatively synonymous with "long-range," while "classical" is a subset of these two.

Both publications presented there employed the Hilbert series to determine all operators describing a certain class of these effects, and presented results at one-loop order. The former considered the spinless case while the latter incorporated spin.

A summary and discussion of potential future work is contained in Chapter 7.

With an eye to bridging the gap between the knowledge of a new researcher and the contents of the publications presented herein, each chapter begins with an initiation that presents some key tools used in, concepts addressed by, and motivations for the work in the publication. These initiations are not to be confused with the subsequent introductions, the latter of which begin the published manuscripts.

Note that in most places in this thesis we work in natural units, setting $\hbar = c = 1$. We will, however, occasionally restore factors of $\hbar$. When we do so, the rules governing these restorations will be explicitly stated.

CHAPTER

2 HEAVY BLACK HOLE EFFECTIVE THEORY

ABSTRACT: We formulate an effective field theory describing large mass scalars and fermions minimally coupled to gravity. The operators of this effective field theory are organized in powers of the transfer momentum divided by the mass of the matter field, an expansion which lends itself to the efficient extraction of classical contributions from loop amplitudes in both the post-Newtonian and post-Minkowskian regimes. We use this effective field theory to calculate the classical and leading quantum gravitational scattering amplitude of two heavy spin-1/2 particles at the second post-Minkowskian order.

2.1 INITIATION

HBET was formulated with the goal of describing a classical system comprising two black holes interacting gravitationally. We will thus begin this initiation by expanding on the encoding of classical physics in scattering amplitudes at all loop orders. In particular, the scattering amplitude of interest describes the $2 \to 2$ scattering of massive and potentially spinning particles. Then we will highlight the concepts behind HQET and HBET, motivating why HBET is appropriate for this system and what benefits it brings. Finally, we summarize the results of the publication.

2.1.1 Classical physics from loop amplitudes

As mentioned in the introduction, the application of scattering amplitudes to classical systems is possible because amplitudes at all loop orders contain long-range and classical effects. This is an unexpected fact; it has long been understood that the loop expansion of scattering amplitudes is an expansion in $\hbar$. From this perspective, higher loop orders scale with more powers of $\hbar$, therefore describing evermore quantum effects. An $\hbar$ counting leading to this conclusion is detailed in ref. [13], which we recapitulate here. First, the number of loops L in a diagram is given by the number of independent internal momenta in the diagram. The total number of internal momenta is equal to the number of internal propagators, I, but several of these are related through momentum conservation at vertices. Specifically, in a diagram with V vertices there are as many constraints that conserve momentum, one of which can be discarded as it imposes conservation of external momenta. All in all, the number of independent internal momenta in a diagram is $L = I - (V - 1)$.

Now, each propagator carries a factor of $\hbar$ from the commutator of a field and its conjugate momentum, contributing I factors of $\hbar$ to the amplitude.[1] Moreover, Green's functions (in position space) are

[1] Ref. [13] counts $\hbar$ for non-truncated Green's functions, so they include factors of $\hbar$ from propagator factors for external lines. We ignore these as we work exclusively with truncated Green's functions in this thesis. In any case, these factors are common to all perturbative orders for fixed external states, so are irrelevant to relating the $\hbar$ scaling to the loop expansion.

described by correlations functions of the form

$$G(x_1,\ldots,x_n)=\frac{\langle 0|T\phi_1(x_1)\ldots\phi_n(x_n)\exp\left[\frac{i}{\hbar}\int d^4y\,\mathcal{L}_{\text{int}}(y)\right]|0\rangle}{\langle 0|\exp\left[\frac{i}{\hbar}\int d^4y\,\mathcal{L}_{\text{int}}(y)\right]|0\rangle}, \tag{2.1}$$

where $|0\rangle$ is the vacuum state at $t=\pm\infty$, the ϕ_i are the external states of the scattering, T represents time ordering, and $\mathcal{L}_{\text{int}}$ is the interaction Lagrangian. The denominator here cancels with vacuum contributions to the numerator, so it will not affect the final $\hbar$ counting. V vertices are generated by expanding the exponential in the numerator to the V^{th} order. Therefore, a diagram with V vertices also acquires a factor $\hbar^{-V}$. In total, the $\hbar$ scaling of a diagram is $\hbar^{I-V}=\hbar^{L-1}$, suggesting higher loops scale with more powers of $\hbar$.

There are, however, counterexamples to this counting. For one, we can consider the $2\to 2$ scattering of spinless electrically charged particles up to one-loop order. Keeping only the non-analytic terms $1/\sqrt{-q^2}$ at one loop, the amplitude for this process is (see Section 2.C)

$$\mathcal{A}=\frac{e^2}{q^2}\omega+\frac{e^4}{16\pi^2 m_1 m_2}\frac{\pi^2}{\sqrt{-q^2}}(m_1+m_2), \tag{2.2}$$

where q^μ is the transfer momentum, $m_{1,2}$ are the masses of the two massive particles, and $\omega\equiv v_{1\mu}v_2^\mu$ is the scalar product of the particles' four-velocities. Also, e is the electric coupling constant. The first term in eq. (2.2) comes from the tree-level diagram in fig. 2.4, while the second term comes from the sum of the one-loop diagrams in figs. 2.5b and 2.5c. Then, restoring $\hbar$ using the counting of ref. [13] gives that the first term should scale with $\hbar^{-1}$ while the second with $\hbar^0$. Noting that e has dimensions of $\hbar^{1/2}$ through its relation to the fine-structure constant, this suggests that the two terms above have different dimensions, which cannot be true. We conclude that the scaling $\hbar^{L-1}$ cannot be correct for this process.

Clearly the $\hbar$ counting employed above has missed some factors. To find them, we look to the action for a scalar field coupled to electromagnetism for $\hbar\neq 1$ (but $c=1$):[2]

$$S=\int d^4x\left[\left(\partial_\mu-\hbar^{c_1}ieA_\mu\right)\phi^*\left(\partial^\mu+\hbar^{c_1}ieA^\mu\right)\phi-\hbar^{c_2}m^2\phi^*\phi-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right]. \tag{2.3}$$

We have introduced the electromagnetic field strength tensor $F^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu$ for the electromagnetic field A^μ. Our task is to find the values $c_{1,2}$ such that this has the correct dimensions of an action, namely $[S]=[\hbar]=[E][L]$. Here $[\cdot]$ denotes the dimension of some quantity, $[E]$ represents dimensions of energy, and $[L]$ dimensions of length. Noting that the dimensions of the fields are $[A^\mu]=[\phi]=[E]^{1/2}[L]^{-1/2}$ and keeping in mind that $[e]=[\hbar]^{1/2}=[E]^{1/2}[L]^{1/2}$, it's not hard to find $c_1=-1$ and $c_2=-2$. These restored factors of $\hbar$ in the action, as well as the fact that derivatives of fields produce momenta divided by $\hbar$, give three additional rules for counting $\hbar$ in an amplitude:

1. factors of the mass come with $\hbar^{-1}$,
2. factors of e come with $\hbar^{-1}$,
3. momenta come with $\hbar^{-1}$.

[2] We have not allowed for factors of $\hbar$ on the kinetic terms as any such factors can always be absorbed by a redefinition of the fields ϕ and A^μ.

Applying these three rules on top of the $\hbar$-scaling from loop counting gives a uniform dimension to both terms:

$$\mathcal{A} = \frac{e^2}{\hbar q^2}\omega + \frac{e^4}{16\pi^2\hbar^2 m_1 m_2}\frac{\pi^2}{\sqrt{-q^2}}(m_1 + m_2). \tag{2.4}$$

Despite all of these rules for restoring $\hbar$, we have still not accounted for all factors as some remain hidden in the momentum q of the massless boson. There is, however, a more efficient method for restoring all factors of $\hbar$ to an amplitude in a way that yields well-defined classical observables. The proceeding arguments follow those made in refs. [9, 14].

Maintaining a dimensionless coupling when restoring factors of $\hbar$ means rescaling $e \to e/\sqrt{\hbar}$. Moreover, to expose all factors of $\hbar$ we must express the momenta of massless modes in terms of their wavenumber. The transfer momentum in this context is carried by a photon, so this means we write $q^\mu = \hbar\bar{q}^\mu$ where $\bar{q}^\mu$ is the wavenumber. With these adjustments, the amplitude is

$$\mathcal{A} = \frac{e^2}{\hbar}\left[\frac{\omega}{\hbar^2\bar{q}^2} + \frac{e^2}{16\pi^2\hbar m_1 m_2}\frac{\pi^2}{\hbar\sqrt{-\bar{q}^2}}(m_1 + m_2)\right]. \tag{2.5}$$

One can check that both terms carry the same dimensions (when $c = 1$), so there are no more relative factors of $\hbar$ to be restored. Critically, we note that the $\hbar$ scaling is uniform amongst both terms! Thus there are indeed contributions at one-loop order that do not carry an additional factor of $\hbar$ relative to tree level. Exposing the factors of $\hbar$ hidden in q in eq. (2.4), we find the same scaling as in eq. (2.5). It is well known that the tree-level $2 \to 2$ amplitude is all that's needed to produce the Coulomb potential. This means that, somewhat counter-intuitively, classical portions of scattering amplitudes in this counting scheme scale as $\hbar^{-3}$. A sanity check for this is given in Section 2.3.

This example makes clear that the non-analytic structure $1/\sqrt{-q^2}$ is paramount to producing effects with a classical $\hbar$ scaling at one-loop order. Indeed, there is no other Lorentz invariant quantity that can provide the requisite factor of $\hbar^{-1}$.[3] It is a general feature that non-analytic structures constitute classical effects at all loop orders. One reason for this is that, as mentioned in Chapter 1, non-analytic pieces are dominant in the long-range limit. Another essential point is that at odd loop orders one always needs an odd power of $\hbar$ from the loop integrals to yield the classical $\hbar$ scaling, which can only result from some odd power of $\sqrt{-q^2}$.

The necessity of non-analytic structures for obtaining long-range portions of amplitudes restricts which types of diagrams must be considered at a given loop order. Non-analytic structures arise from loop integrals containing at least two massless propagators [9]. If we are to focus further on the classical, conservative portion of an L-loop $2 \to 2$ amplitude – that is, the portion that contributes classically to the interaction potential between the two bodies – only diagrams with at least two massless and at least one massive propagator per loop are necessary [15–17]. The fact that such diagrams – and others contributing non-conservatively – contain classical pieces can be confirmed by counting factors of $\hbar$, recalling that the classical $\hbar$ scaling is $\hbar^{-3}$.[4,5] As an easy method for determining whether a diagram can contribute

[3] On-shell conditions relate contractions of the external momenta with q^μ to q^2.

[4] At two loops and above, diagrams with different loop structures – describing radiation reaction effects – give classical corrections necessary for rendering observables finite in the ultra-relativistic limit [18–22]. In this thesis we focus only on the conservative sector, and only up to one loop, so we do not consider such effects.

[5] A diagram scaling with more negative powers of $\hbar$ contains classical effects at subleading orders in $\hbar$. Note that diagrams with internal matter loops are always ignored in a classical context, whether or not they have a classical $\hbar$ scaling.

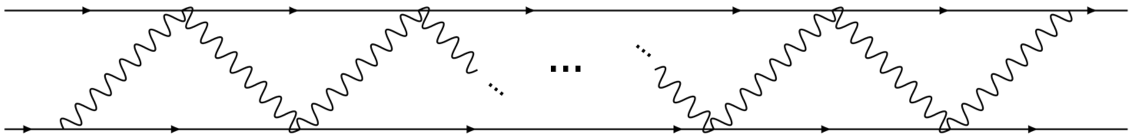


Figure 2.1: The triangle-ladder diagram with an aritrary number of loops. Every boson line that is not shown ends at a two-matter-two-boson vertex. Such a diagram always scales as $\hbar^{-3}$ regardless of the loop order.

classically, it is useful to demonstrate how this counting proceeds for a general diagram with massless modes propagating in each loop.

First, at L loops we pick up $-(L+1)$ factors of $\hbar$ from coupling constants. This is because the $2 \to 2$ amplitude carries two powers of the coupling constant at tree level and acquires two additional powers at each order in perturbation theory, while each power of the coupling constant brings with it a factor of $\hbar^{-1/2}$. From the loop integration measures we acquire $4L$ factors of $\hbar$, since a loop momentum can always be associated to a massless propagator. Each massive propagator provides a factor of $\hbar^{-1}$ (see Section 2.3), while massless propagators contribute $\hbar^{-2}$ – suppose we have n_m of the former and n_0 of the latter. The final consideration we must make is the number of pure-graviton vertices. As can be seen from the pure-gravitational action of ref. [23], which includes the EH action as well as gauge fixing and ghost terms, such a vertex always brings with it two positive powers of $\hbar$. If we have n_g pure-graviton vertices a generic L-loop diagram scales as

$$L\text{-loop diagram } \hbar \text{ scaling:} \quad \hbar^{3L-n_m-2n_0+2n_g-1}. \tag{2.6}$$

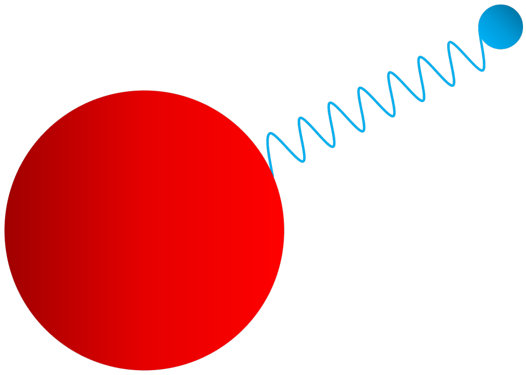
This counting assumes that the graviton-matter vertices scale as $\hbar^0$. While this is true at leading order in $\hbar$, subleading terms in the vertex can scale with more positive powers of $\hbar$ (see for example the Feynman rules in Section 2.D). Certain portions of the vertex Feynman rule can be dropped in a classical context if they carry too many factors of $\hbar$.

Applying eq. (2.6) to the diagrams used to compute classical conservative effects up to the three-loop order in refs. [16, 17, 23, 24], one indeed finds that they all carry $\hbar^{-(n\geq 3)}$. The additional self-energy and vertex-correction diagrams needed to account for classical radiation reaction effects at two loops in ref. [22] scale as $\hbar^{-5}$. There are diagrams with a classical $\hbar$ scaling at every loop order. For example, the triangle-ladder diagram in fig. 2.1 scales as $\hbar^{-3}$ for any number of loops.

To summarize the discussion in this section, we simply state that classical corrections to $2 \to 2$ scattering can be extracted from loop amplitudes at any loop order. In the conservative sector, the contributing diagrams must have at least two massless and one massive propagator per loop. Any diagram yielding classical terms must scale as $\hbar^{-(n\geq 3)}$.

2.1.2 Motivating HBET

HQET [25–28] is an effective theory formulated to describe the strong-force interactions between two quarks, one of which has mass around or below the QCD scale, Λ_{QCD}, and the other being much heavier than this scale. A hierarchy of scales suggests itself in such a system. The majority of the system's momentum is carried by the heavy quark with mass M and four-velocity v^μ. Interactions with the light



$$p^\mu = Mv^\mu + k^\mu$$

Figure 2.2: A depiction of the system described by HQET. The heavy quark (red) carries a large momentum Mv^μ, while the remaining degrees of freedom (blue) carry the residual momentum $|k^\mu| \ll M$. The momentum of the system is the sum of these two.

quark are on the order of the QCD scale, and modify the total momentum by a small residual momentum $|k^\mu| \sim \Lambda_{\text{QCD}} \ll M$, where each component of the residual momentum satisfies this inequality; see fig. 2.2. Expanding the action coupling the heavy quark to gluons in powers of $\Lambda_{\text{QCD}}/M \ll 1$, one obtains HQET.

Conceptually, the process of deriving HQET (see Section 2.4) involves shifting the reference energy of the system from 0 to M. What this achieves is to make the particle modes massless while attributing a mass of $2M$ to the antiparticle. Since the dynamic modes of HQET carry energies much smaller than M, antiparticle modes cannot be created and particle modes cannot be annihilated in a scattering process. This is expressed formally by integrating the antiparticle modes out of the action and keeping only particles and gluons in the theory. The particle can be thought of as a source of gluons with approximately constant momentum that is perturbed only by small residual momenta.

What about gravitational systems? Though we don't have an analog to Λ_{QCD} against which we can compare masses and interaction energies for a general system, in macroscopic systems we can find such a scale. Let us think of the transfer momentum k^μ between two classical, gravitating objects as being carried by gravitons (fig. 2.3). The typical wavelength of such gravitons will be proportional to the separation between the bodies, R. In turn, the momenta of the gravitons satisfy $|k^\mu| \sim \hbar/R$ since the momenta of waves scale as $\hbar$ times the inverse of their wavelengths. For macroscopic separations and masses, it is then clear that $|k^\mu|/M \sim \hbar/RM \ll 1$. The same hierarchy therefore exists for macroscopic gravitational systems as for systems involving heavy quarks: the momentum carried by the messenger boson is much smaller than the mass of the matter it interacts with.

Through this reasoning we realize that there are indeed real gravitational systems that can be subjected to an HQET-like treatment. The next question one may ask is: why should we pursue this treatment? For many, a satisfying answer is simply that we can. There is, however, a better answer. Recall that we are ultimately interested in describing classical physics. As such, since we are beginning with a quantum framework, it is beneficial to be able to identify classical and quantum portions of observables as soon as possible. But we have just argued that in macroscopic gravitational systems there exists a small parameter that is proportional to $\hbar$.[6] Therefore, expanding a gravitational action involving matter

[6] The HQET expansion parameter is also proportional to $\hbar$; we elaborated on this in the publication.

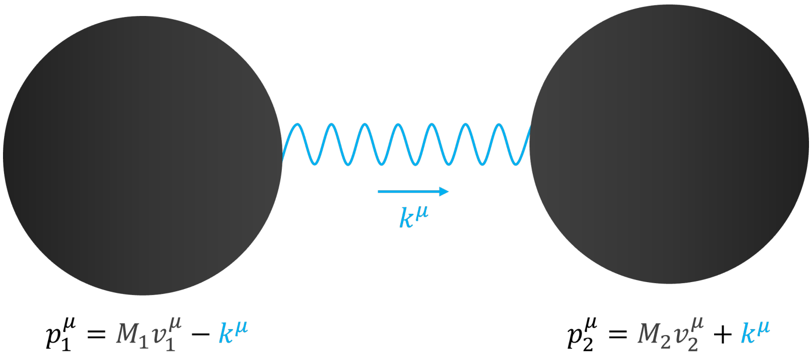


Figure 2.3: A depiction of the system described by HBET. The two black holes interact by exchanging a graviton with momentum $|k^\mu| \ll M_{1,2}$. The HBET Lagrangian describes a single black hole interacting with gravitons; the interaction in this image requires a sum of two HBET Lagrangians, one for each black hole. Note that the wavy line between the black holes recalls the Feynman diagrammatic representation of gravitons, and is not meant to suggest a wavelength for the graviton.

in terms of the small parameter $|k^\mu|/M$ can also be thought of as an expansion in $\hbar$! Being able to assign an $\hbar$ scaling to operators in an action allows us to identify which operators contribute to a given scattering amplitude with which powers of $\hbar$, meaning that classical effects can be targeted before even beginning the calculation. Moreover, it is natural to work in a framework that excludes antiparticles as this is in line with treating a classical system. As a bonus, HBET (and HQET as well) separate spinless and spin effects at the level of the action, allowing one to also target terms in the amplitude at a specific order in the spin-multipole expansion.

2.1.3 Overview of main results

In this publication we derived the action describing HBET for spins 0 and $1/2$, starting from the actions for scalar and spinor fields minimally coupled to gravity. The HBET actions are

$$\sqrt{-g}\mathcal{L}^{s=0}_{\text{HBET}} = \sqrt{-g}\chi^* \left[g^{\mu\nu} i v_\mu \partial_\nu + \frac{1}{2} m (g^{\mu\nu} v_\mu v_\nu - 1) - \frac{1}{2m} g^{\mu\nu}\partial_\mu\partial_\nu \right] \chi + \mathcal{O}(1/m^2), \tag{2.7}$$

$$\sqrt{-g}\mathcal{L}^{s=1/2}_{\text{HBET}} = \sqrt{-g}\bar{Q} \left[(i\not{\nabla} + \mathcal{B}) + (i\not{\nabla} + \mathcal{B})P_- \frac{1}{2m - (i\not{\nabla} + \mathcal{B})P_-}(i\not{\nabla} + \mathcal{B}) \right] Q, \tag{2.8}$$

where $\not{\nabla} \equiv \delta^\mu_a \gamma^a \nabla_\mu$ and $\mathcal{B} \equiv (e^\mu_a - \delta^\mu_a)(i\gamma^a\nabla_\mu + m\gamma^a v_\mu)$.

We explained how to assign factors of $\hbar$ to the operators of the action and to the loop integrals that arise in the scattering of heavy particles. Using these insights, we derived the classical portion of the amplitude for the scattering of two spin-$1/2$ particles up to one-loop (2PM) order. At the time of publication, this was the first computation of the 2PM contribution to this amplitude (later corroborated by ref. [29]). We found this contribution to be (with factors of $\hbar$ restored, hence expressing the result in terms of the wavenumber $\bar{q}^\mu$ instead of the transfer momentum q^μ)[7]

$$\mathcal{M}^{\text{2PM}} = \frac{G^2}{\hbar^3}\frac{\pi^2}{\sqrt{-\bar{q}^2}} \left\{ \frac{3}{2} m_1 m_2 (m_1 + m_2)(5\omega^2 - 1)\mathcal{U}_1\mathcal{U}_2 + \frac{(3m_1 + 4m_2)}{2m_2}\frac{\omega(5\omega^2 - 3)}{\omega^2 - 1} i\mathcal{U}_1\mathcal{E}_2 \right.$$

[7] In addition to the rules for restoring $\hbar$ mentioned above, each factor of the spin vector scales as $\hbar^{-1}$ in the classical limit.

$$+(m_1+m_2)\left[\frac{(20\omega^4-21\omega^2+3)}{2(\omega^2-1)}(\bar{q}\cdot S_1\bar{q}\cdot S_2-\bar{q}^2S_1\cdot S_2)+\frac{2\bar{q}^2\omega^3(5\omega^2-4)}{m_1m_2(\omega^2-1)^2}p_2\cdot S_1p_1\cdot S_2\right]\Bigg\}. \tag{2.9}$$

We went a step further, computing also the leading long-range quantum contributions to the amplitude.

In the appendices, we derived the relation between the heavy spinors and Dirac spinors, which was necessary for the formulation of the variables in Chapter 3. We also performed the identical calculation for electromagnetism using an abelian version of HQET.

2.2 INTRODUCTION

The direct detection of gravitational waves (GWs) from the merging of two black holes by LIGO and Virgo in 2015 [1] has placed a spotlight on GW astronomy as a novel channel through which to test general relativity (GR). As the detection rate of GWs becomes more frequent in the years ahead, it is necessary to improve the analytical predictions on which the GW templates used in the observations are based. To do so requires knowledge of the interaction Hamiltonian of a gravitationally bound binary system to high accuracy. This necessarily entails the calculation of higher orders in the post-Newtonian (PN) and post-Minkowskian (PM) expansions.

Much of the work related to GWs has been done from the relativistic approach to GR; some notable developments are the effective-one-body approach [30–32], numerical relativity [33–35], and effective field theoretic methods [36, 37] (see Refs. [38, 39] for comprehensive reviews summarizing most of the analytical aspects of these methods). Also, there has been substantial work done using traditional and modern scattering amplitude techniques to calculate classical gravitational quantities, including the non-relativistic classical gravitational potential [7, 8, 16, 17, 23, 40–47]. Moreover, techniques were recently presented in Refs. [48, 49] to convert fully relativistic amplitudes for scalar-scalar scattering to the classical potential, and for obtaining the scattering angle directly from the scattering amplitude [50]. The prescription of Ref. [48] was combined there with modern methods in amplitude computations to obtain the 2PM, and elsewhere the state-of-the-art 3PM Hamiltonian for classical scalar-scalar gravitational scattering [16, 17].[8] This large body of work, facilitated by classical effects arising at all loop orders [7, 9] (see Section 2.3), suggests that quantum field theory methods can reliably be used instead of direct computation from GR, particularly when the latter becomes intractable. Following in this vein, we apply here the machinery of effective field theory (EFT) to compute classical gravitational scattering amplitudes.

Computations of classical quantities from quantum scattering amplitudes are inherently inefficient. Entire amplitudes must first be calculated — which are comprised almost entirely of quantum contributions — and then classical terms must be isolated in a classical limit. One of the advantages of EFT methods is that they allow the contributions of certain effects to be targeted in amplitude calculations, thus excluding terms that are not of interest from the outset. From the point of view of classical gravity, it is then natural to ask whether an EFT can be formulated that isolates classical from quantum contributions already at the operator level. Indeed, we find that a reinterpretation of the operator expansion of the well-established Heavy Quark Effective Theory (HQET) [25, 26] (for a review, see, *e.g.*, Ref. [27]) leads us down the right path.

[8] Since the publication of this article, binary dynamics have been computed up to the 4PM level [24, 51].

HQET has been used extensively to describe bound systems of one heavy quark — with mass M large relative to the QCD scale Λ_{QCD} — and one light quark — with mass $m \lessapprox \Lambda_{\text{QCD}}$. Interactions between the light and heavy quarks are on the order of the QCD scale, $q \sim \Lambda_{\text{QCD}}$. Thus the heavy quark can, to leading order, be treated as a point source of gluons, with corrections to the motion of the heavy quark arising from higher-dimensional effective operators organized in powers of $q/M \sim \Lambda_{\text{QCD}}/M$.

A similar hierarchy of scales exists when considering the long-range (classical) gravitational scattering of two heavy bodies; for long-range scattering of macroscopic objects the momentum of an exchanged graviton q is much smaller than the mass of each object. This can be seen by noting that, once powers of $\hbar$ are restored, the transfer momentum is $q = \hbar\bar{q}$, where $\bar{q}$ is the wavenumber of the mediating boson [10]. Consequently, the expansion parameter of HQET — and its gravitational analog, which we refer to as the Heavy Black Hole Effective Theory (HBET) — can be recast as $\hbar\bar{q}/M$. The magnitude of the wavenumber is proportional to the inverse of the separation of the scattering bodies, hence for macroscopic separations and masses, $\hbar\bar{q}/M \ll 1$. The presence of this separation of scales in classical gravitational scattering further motivates the development of HBET. The explicit $\hbar$ power counting of its operators makes HBET a natural framework for the computation of classical gravitational scattering amplitudes.

This work shares conceptual similarity with the Non-Relativistic General Relativity (NRGR) EFT approach to the two-body problem introduced in Ref. [36] (extended to the case of spinning objects in Ref. [37]). As in the case of NRGR, the interacting objects of HQET and HBET are sources for the mediating bosons, and are not themselves dynamical; in HQET and HBET, this can be seen from the fact that derivatives in the Lagrangians produce residual momenta (see Sec. 2.4) in the Feynman rules, not the full momenta of the objects in the scattering. However the EFTs differ in what they describe. NRGR is organized in powers of velocity, facilitating the computation of the Post-Newtonian expansion. In contrast, the operator expansions of HQET and HBET are expansions in $\hbar$, allowing us to target terms in the amplitudes with a desired $\hbar$ scaling. Being derived directly from a relativistic quantum field theory, a Post-Minkowskian expansion is naturally produced by the amplitudes of HBET. Moreover, while NRGR computes the non-relativistic interaction potential directly, HBET is intended for the computation of the classical portions of scattering amplitudes, which must then be converted to classical observables [48–50].

In this paper, we derive HBET in two forms, describing separately the interactions of large mass scalars and fermions minimally coupled to gravity. By restoring $\hbar$ we demonstrate how to determine which operators contribute classically to $2 \to 2$ scattering at n loops. Using the developed EFT we compute the $2 \to 2$ classical scattering amplitude for both scalars and fermions up to 2PM order. We include in our calculations the leading quantum contributions to the amplitudes that originate from the non-analytic structure of the loop integrals.

The structure of this paper is as follows. In Section 2.3 we explain the procedure by which we restore $\hbar$ in the amplitudes. We give a brief review of HQET in Section 2.4, and outline the derivation of the HQET Lagrangian. Our main results are presented in Sections 2.5 and 2.6. In the former we derive the HBET Lagrangians for heavy scalars and heavy fermions, whereas the latter presents the $2 \to 2$ scattering amplitudes for each theory up to 2PM. We conclude in Section 2.7. Technical details of the HQET spinors are discussed in Appendix 2.A. In Appendix 2.B we include the effective theory of a heavy scalar coupled to electromagnetism, and in Appendix 2.C we use HQET to compute the classical and leading quantum contributions to the $2 \to 2$ electromagnetic amplitude up to one-loop. Appendices 2.D and 2.E contain respectively the Feynman rules and a discussion on the one-loop integrals needed to perform the 2PM calculations. We also discuss in Appendix 2.E the circumvention of the so-called pinch singularity, which appears in some HQET loop integrals.

2.3 COUNTING $\hbar$

In quantum field theory we are accustomed to working with units where both the reduced Planck constant $\hbar$ and the speed of light c are set to unity, thus obscuring the classical limit $\hbar \to 0$. We must therefore systematically restore the powers of $\hbar$ in scattering amplitudes so that a classical limit may be taken. We follow Ref. [10] to do so.

The first place we must restore $\hbar$ is in the coupling constants such that their dimensions remain unchanged: in both gravity and QED, the coupling constants are accompanied by a factor of $\hbar^{-1/2}$. Second, as mentioned above, we must distinguish between the momentum of a massless particle p^μ and its wavenumber $\bar{p}^\mu$. They are related through

$$p^\mu = \hbar \bar{p}^\mu. \tag{2.10}$$

In the classical limit, the momenta and masses of the massive particles must be kept constant, whereas for massless particles it is the wavenumber that must be kept constant. While this result is achieved formally through the consideration of wavefunctions in Ref. [10], an intuitive way to see this is that massless particles are classically treated as waves whose propagation can be described by a wavenumber, whereas massive particles are treated as point particles whose motion is described by their momenta.

In this work, we are interested in the scattering of two massive particles, where the momentum q is transferred via massless particles (photons or gravitons). Letting the incoming momenta be p_1 and p_2, the amplitudes will thus take the form

$$i\mathcal{M}(p_1, p_2 \to p_1 - \hbar\bar{q}, p_2 + \hbar\bar{q}). \tag{2.11}$$

As the momentum transfer is carried by massless particles, the wavenumber $\bar{q}$ remains fixed in the classical limit, whereas the momentum q scales with $\hbar$, as indicated in Eq. (2.11). The classical limit of the kinematics is therefore associated with the limit $|q| \to 0$.

2.3.1 Counting at one-loop

With these rules for restoring powers of $\hbar$ in amplitudes, we can preemptively deduce which operators from the EFT expansion can contribute classically at one-loop level. First we must determine the $\hbar$-scaling that produces classical results.

The usual Newtonian potential can be obtained from the Fourier transform of the leading order non-relativistic contribution to the tree-level graviton exchange amplitude (see Fig. 2.4). Using a non-relativistic normalization of the external states,

$$\langle p_1 | p_2 \rangle = (2\pi)^3 \delta^3(\vec{p}_1 - \vec{p}_2), \tag{2.12}$$

this contribution to the amplitude is

$$\mathcal{M}^{(1)} \approx -\frac{\kappa^2 m_1 m_2}{8q^2}, \tag{2.13}$$

where $\kappa = \sqrt{32\pi G/\hbar}$ and G is Newton's constant. Here q is the four-momentum of the mediating graviton. Following the discussion above, we can thus make all factors of $\hbar$ explicit by writing q in terms of the graviton wavenumber. We find

$$\mathcal{M}^{(1)} \approx -\frac{4\pi G m_1 m_2}{\hbar^3 \bar{q}^2}. \tag{2.14}$$

We conclude that classical contributions to scattering amplitudes in momentum space with the current conventions scale as $\hbar^{-3}$. A quantum mechanical term is thus one that scales with a more positive power of $\hbar$ than this, as such a term will be less significant in the $\hbar \to 0$ limit.

Indeed, this must be the $\hbar$-scaling of any term in the amplitude contributing classically to the potential. At tree-level, the relation between the amplitude and the potential is simply

$$V = -\int \frac{d^3 q}{(2\pi)^3} e^{-\frac{i}{\hbar}\vec{q}\cdot\vec{r}} \mathcal{M} = -\hbar^3 \int \frac{d^3 \bar{q}}{(2\pi)^3} e^{-i\vec{\bar{q}}\cdot\vec{r}} \mathcal{M}, \tag{2.15}$$

where we have made factors of $\hbar$ explicit. The scaling of classical contributions from the amplitude must be such that they cancel the overall $\hbar^3$ in the Fourier transform.

Central to the applicability of the Feynman diagram expansion to the computation of classical corrections to the interaction potential is the counterintuitive fact that loop diagrams can contribute classically to scattering amplitudes [7, 9]. Which loop diagrams may give rise to classical terms can be determined by requiring the same $\hbar$-scaling as in Eq. (2.14).

Diagrams at one-loop level have four powers of the coupling constant, which are accompanied by a factor of $\hbar^{-2}$. This implies that classical contributions from one-loop need to carry exactly one more inverse power of $\hbar$, arising from the loop integral. The only kinematic parameter in the scattering that can bring the needed $\hbar$ is the transfer momentum q, and even then only in the non-analytic form $1/\sqrt{-q^2}$. Non-analytic terms at one-loop arise from one-loop integrals with two massless propagators [7, 9]. There are three topologies at one-loop that have two massless propagators per loop, and hence three topologies from which the requisite non-analytic form can arise: the bubble, triangle, and (crossed-)box topologies. We will determine the superficial $\hbar$-scaling of these topologies.

First we note that the loop momentum l can always be assigned to a massless propagator, and hence should scale with $\hbar$. The bubble integral is thus

$$\begin{aligned} i\mathcal{M}^{(2)}_{\text{bubble}} &\sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l} \frac{1}{\hbar^2 \bar{l}^2} \frac{1}{\hbar^2(\bar{l}+\bar{q})^2} + \mathcal{O}(\hbar^{-1}) \\ &= \mathcal{O}(\hbar^{-2}). \end{aligned} \tag{2.16}$$

We conclude that the bubble contains no classical pieces.

Triangle integrals must have an extra HQET/HBET matter propagator, which, as will be seen below, is linear in the residual momentum. Therefore, triangle diagrams scale as

$$i\mathcal{M}^{(2)}_{\text{triangle}} \sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l}\, \frac{1}{\hbar^2\bar{l}^2}\frac{1}{\hbar^2(\bar{l}+\bar{q})^2}\frac{1}{\hbar v\cdot(\bar{l}+\bar{k})} + \mathcal{O}(\hbar^{-2}) \\ = \mathcal{O}(\hbar^{-3}). \tag{2.17}$$

Here, v is the velocity of the heavy quark, and k is the residual HQET/HBET momentum. These quantities and their $\hbar$-scaling are discussed in Section 2.4. The scaling of the triangle integral suggests that triangles must contain classical pieces.

Finally, box and crossed-box integrals scale as

$$i\mathcal{M}^{(2)}_{\text{(crossed}-\text{)box}} \sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l}\, \frac{1}{\hbar^2\bar{l}^2}\frac{1}{\hbar^2(\bar{l}+\bar{q})^2}\frac{1}{\hbar v\cdot(\bar{l}+\bar{k}_1)}\frac{1}{\hbar v\cdot(\bar{l}+\bar{k}_2)} + \mathcal{O}(\hbar^{-3}) \\ = \mathcal{O}(\hbar^{-4}). \tag{2.18}$$

There are potentially classical pieces in the subleading terms of the (crossed-)box – that is, in higher rank (crossed-)box loop integrals. However, the leading terms in the box and crossed-box diagrams look to be too classical, scaling as $1/\hbar^4$. In order for the amplitude to have a sensible classical limit, such contributions must cancel in physical classical quantities. Two types of cancellations occur at one-loop level: cancellations between the box and crossed-box, and cancellations due to the Born iteration of lower order terms when calculating the potential [15, 23, 49, 52].

In this paper we compute long-range effects arising from one-loop integrals, which are proportional to the non-analytic factors $S \equiv \pi^2/\sqrt{-\bar{q}^2}$ and $L \equiv \log(-q^2)$.[9] When considering only spinless terms at one-loop order, those proportional to S are classical, and those proportional to L are quantum. With the established $\hbar$ counting, classical terms at one-loop can arise from operators with at most one positive power of $\hbar$, and quantum terms arise from operators with at most two positive powers of $\hbar$. In the operator expansion of HQET/HBET, powers of $\hbar$ come from partial derivatives.

The inclusion of spin slightly complicates this counting. In order to identify spin multipoles with those of the classical angular momentum, we must allow the spin to be arbitrarily large while simultaneously taking the classical limit. More precisely, for a spin S_i the simultaneous limits $S_i \to \infty$, $\hbar \to 0$ must be taken while keeping $\hbar S_i$ constant [14, 53].[10] When considering spin-inclusive parts of the amplitude we must therefore neglect one positive power of $\hbar$ for each power of spin when identifying the classical and quantum contributions. To make the expansion in classical operators explicit, in this paper we keep track only of the factors of $\hbar$ that count towards the determination of the classicality of terms in the amplitudes. Practically, this amounts to rescaling the Dirac sigma matrices in the operators as $\sigma^{\mu\nu} \to \sigma^{\mu\nu}/\hbar$, or the spins in the amplitudes as $S_i \to S_i/\hbar$. At linear order in spin, this leads again to the interpretation at 2PM order of terms proportional to S as being classical, and those proportional to L as being quantum. At quadratic order in spin, however, terms such as $q^3 S$ and qL begin arising, which respectively have quantum and classical $\hbar$ scaling.

Altogether, operators contributing classically contain either up to one deriative, or up to two derivatives and a Dirac sigma matrix, which will be seen to be related to the spin vector.

[9] In contrast to Refs. [23, 52], we define S in terms of the wavenumber $\bar{q}$ to make powers of $\hbar$ explicit in the amplitude.

[10] The universality of the multipole expansion in gravitational interactions ensures that the expansion remains unchanged in this limit [23, 53].

2.3.2 Counting at n-loops

We can extend this analysis to determine which operators can produce classical terms at arbitrary loop order. First we consider two-loop diagrams, contributing 3PM corrections to the classical potential.

The highest order operator needed is determined by the most classical $\hbar$-scaling attainable at a given loop order, *i.e*, by the $\hbar$-scaling of the diagram that scales with the most inverse powers of $\hbar$. In Appendix 2.D we show that the leading order $\hbar$-scaling of a graviton-matter vertex is always $\hbar^0$. The Einstein-Hilbert action governing pure graviton vertices involves two derivatives of the graviton field, so that pure graviton vertices always scale as $\hbar^2$. It follows that the most classical diagrams at 3PM are the box and crossed-box with four massive and three massless propagators; we refer to these as ladder diagrams. The overall coupling is $G^3/\hbar^3$, and the two integrals over loop momenta contribute eight positive powers of $\hbar$. In total, the amplitude superficially scales as

$$\mathcal{M}^{(3)}_{\text{ladder}} \sim \frac{1}{\hbar^3}\hbar^8\frac{1}{\hbar^{10}} = \frac{1}{\hbar^5}. \tag{2.19}$$

At nPM — corresponding to $n-1$ loops — the dominant diagrams in the $\hbar \to 0$ limit are still the ladder diagrams, with n massless propagators and $2(n-1)$ massive propagators. The scaling is then

$$\mathcal{M}^{(n)}_{\text{ladder}} \sim \frac{1}{\hbar^n}\hbar^{4(n-1)}\frac{1}{\hbar^{2n}}\frac{1}{\hbar^{2(n-1)}} = \frac{1}{\hbar^{2+n}}. \tag{2.20}$$

From the HBET point of view, this means that we need to include operators that scale with one more power of $\hbar$ whenever we go from nPM to $(n+1)$PM order. Starting with the observation from the previous section of classical operators at 2PM, we will need operators with at most $n-1$ derivatives, or n derivatives and one Dirac sigma matrix, to obtain the full classical correction at nPM. Furthermore, to have a sensible classical limit, all superclassical contributions must cancel in physical quantities. We see that the order of cancellation scales with the number of loops.

Note that, according to this counting, starting at 3PM, spinless terms proportional to L can contribute classically. This is consistent with the classical 3PM scalar-scalar amplitude in Refs. [16, 17].

2.4 HEAVY QUARK EFFECTIVE THEORY

As the concepts and methods we will use to derive HBET are based on those of HQET, we give a brief review of the latter here.

HQET is used in calculations involving a bound state of a heavy quark Q with mass $m_Q \gg \Lambda_{\text{QCD}}$, and a light quark with mass smaller than Λ_{QCD}. The energy scale of the interactions between the light and heavy quark is on the order of the QCD scale, and is thus small compared to the mass of the heavy quark. The momentum p^μ of the system is decomposed into a large part, $m_Q v^\mu$, and a small residual momentum, k^μ. Altogether,

$$p^\mu = m_Q v^\mu + k^\mu, \qquad |k^\mu| \sim \mathcal{O}(\Lambda_{\text{QCD}}) \text{ where } \Lambda_{\text{QCD}} \ll m_Q. \tag{2.21}$$

A hierarchy of scales is present, and we can organize an effective theory which expands in this hierarchy.

An interesting feature of HQET, as will be seen below, is that its propagating degrees of freedom are massless. The propagating degrees of freedom carry the residual momentum k^μ. Therefore, since we are interested in classical scattering, we can rewrite the residual momentum according to Eq. (2.10):

$$p^\mu = m_Q v^\mu + \hbar \bar{k}^\mu. \tag{2.22}$$

The procedure we will use to derive the HBET Lagrangian for spinors in the next section is identical to that used to derive the HQET Lagrangian. As such, we outline the derivation of the HQET Lagrangian for one quark coupled to a $U(1)$ gauge field.[11] Our starting point is the QED Lagrangian,

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(i\not{D} - m\right)\psi, \qquad \text{where} \quad D^\mu\psi \equiv (\partial^\mu + ieA^\mu)\psi. \tag{2.23}$$

Next, following the pedagogical derivation in Ref. [54], we introduce the projection operators

$$P_\pm \equiv \frac{1 \pm \not{v}}{2}, \tag{2.24a}$$

and two eigenfunctions of these operators

$$Q \equiv e^{imv\cdot x} P_+ \psi, \tag{2.24b}$$

$$\tilde{Q} \equiv e^{imv\cdot x} P_- \psi. \tag{2.24c}$$

This allows us to decompose the spinor field as

$$\psi = \frac{1+\not{v}}{2}\psi + \frac{1-\not{v}}{2}\psi = e^{-imv\cdot x}\left(Q + \tilde{Q}\right). \tag{2.25}$$

The details pertaining to the external states of the fields Q and $\tilde{Q}$ are discussed in Appendix 2.A.

Substituting Eq. (2.25) into Eq. (2.23), using some simple gamma matrix and projection operator identities, and integrating out $\tilde{Q}$ using its equation of motion, we arrive at the HQET Lagrangian,

$$\mathcal{L}_{\text{HQET}} = \bar{Q}\left(iv\cdot D - \frac{D_\perp^2}{2m} - \frac{e}{4m}\sigma^{\mu\nu}F_{\mu\nu}\right)Q + \frac{1}{2m}\bar{Q}i\not{D}_\perp \sum_{n=1}^{\infty}\left(-\frac{iv\cdot D}{2m}\right)^n i\not{D}_\perp Q. \tag{2.26}$$

Here, $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ is the Dirac sigma matrix and $D_\perp^\mu \equiv D^\mu - v^\mu(v\cdot D)$ is the covariant derivative orthogonal to v^μ.

The redundant operators proportional to the leading order equation of motion can be removed by the field redefinition [28]

$$\begin{aligned} Q \to \Bigg(1 &- \frac{D_\perp^2}{8m^2} - \frac{e}{16m^2}\sigma_{\mu\nu}F^{\mu\nu} + \frac{1}{16m^3}D_\perp^\mu(iv\cdot D)D_{\perp\mu} - \frac{e}{16m^3}v_\mu D_{\perp\nu}F^{\mu\nu} \\ &- \frac{i}{16m^3}\sigma_{\mu\nu}D_\perp^\mu(iv\cdot D)D_\perp^\nu - \frac{ie}{16m^3}v_\rho\sigma_{\mu\nu}D_\perp^\mu F^{\nu\rho}\Bigg)Q \end{aligned} \tag{2.27}$$

to order $\mathcal{O}(1/m^4)$, leading to the Lagrangian

$$\mathcal{L}_{\text{HQET}} = \bar{Q}\Bigg(iv\cdot D - \frac{D_\perp^2}{2m} + \frac{D_\perp^4}{8m^3} - \frac{e}{4m}\sigma_{\mu\nu}F^{\mu\nu} - \frac{e}{8m^2}v^\mu[D_\perp^\nu F_{\mu\nu}] \tag{2.28}$$

[11] The non-abelian case is discussed in *e.g.* Ref. [28].

$$+\frac{ie}{8m^2}v_\rho\sigma_{\mu\nu}\{D_\perp^\mu, F^{\rho\nu}\} + \frac{e}{16m^3}\{D_\perp^2, \sigma_{\mu\nu}F^{\mu\nu}\} + \frac{e^2}{16m^3}F_{\mu\nu}F^{\mu\nu}\Big) Q + \mathcal{O}(m^{-4}).$$

Square brackets enclosing a derivative denote that the derivative acts only within the brackets.

Once Fourier transformed, partial derivatives produce the momentum of the differentiated field. In the specific case of HQET, the partial derivatives produce either a residual momentum (when acting on the spinor field) or a photon momentum (when acting on the vector field) in the Feynman rules. As both types of momenta correspond to massless modes, they both scale with $\hbar$, and hence partial derivatives always result in one positive power of $\hbar$.

2.5 HEAVY BLACK HOLE EFFECTIVE THEORY

We now turn to the case of a heavy particle minimally coupled to gravity. The derivation of the Lagrangian for a heavy scalar coupled to gravity differs from the derivation of the spinor theory, because the scalar field whose heavy-mass limit we are interested in describing is real. The initial Lagrangian is that of a minimally coupled scalar matter field:

$$\sqrt{-g}\mathcal{L}_{\text{sc-grav}} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{2.29}$$

The metric is given by a small perturbation around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where the perturbation $h_{\mu\nu}$ is identified with the graviton.

The heavy-field limit of a real scalar field can be expressed in terms of a complex scalar field χ by employing a suitable field-redefinition. Motivated by earlier analyses in Refs. [55–57], we decompose

$$\phi \to \frac{1}{\sqrt{2m}}\left(e^{-imv\cdot x}\chi + e^{imv\cdot x}\chi^*\right). \tag{2.30}$$

Substituting this into Eq. (2.29) and dropping quickly oscillating terms (those proportional to $e^{\pm 2imv\cdot x}$) gives the HBET Lagrangian for scalars:

$$\sqrt{-g}\mathcal{L}_{\text{HBET}}^{s=0} = \sqrt{-g}\chi^*\left[g^{\mu\nu}iv_\mu\partial_\nu + \frac{1}{2}m(g^{\mu\nu}v_\mu v_\nu - 1) - \frac{1}{2m}g^{\mu\nu}\partial_\mu\partial_\nu\right]\chi + \mathcal{O}(1/m^2). \tag{2.31}$$

Comparing the Feynman rules for this theory in Appendix 2.D with the Feynman rules for the full theory in Ref. [23], we see that they are related by simply decomposing the momenta as in Eq. (2.21) and dividing by $2m$.

Next, we consider the case of a heavy spin-1/2 particle. We begin with the Lagrangian of a minimally coupled Dirac field ψ

$$\sqrt{-g}\mathcal{L}_{\text{grav}} = \sqrt{-g}\bar{\psi}\left(ie^\mu{}_a\gamma^a D_\mu - m\right)\psi, \tag{2.32}$$

where $e^\mu{}_a$ is a vierbein, connecting curved space (with Greek indices) and flat space (with Latin indices) tensors. The expansion of the vierbein in terms of the metric perturbation is given in Ref. [23]. The covariant derivative is [58]

$$D_\mu \psi \equiv \left(\partial_\mu + \frac{i}{2} \omega_\mu{}^{ab} \sigma_{ab} \right) \psi, \tag{2.33}$$

where the spin connection $\omega_\mu{}^{ab}$ is given in terms of vierbeins in Eq. (41) of Ref. [58]. To quadratic order in the graviton field, the spin-connection is [23]

$$\omega_\mu{}^{ab} = -\frac{\kappa}{4} \partial^b h_\mu{}^a - \frac{\kappa^2}{16} h^{\rho b} \partial_\mu h^a{}_\rho + \frac{\kappa^2}{8} h^{\rho b} \partial_\rho h_\mu{}^a - \frac{\kappa^2}{8} h^{\rho b} \partial^a h_\mu{}^\rho - (a \leftrightarrow b). \tag{2.34}$$

Eq. (2.34) differs from that in Ref. [23] by a factor of $-1/2$. The spin connection of Ref. [58] differs from that of Ref. [23] by this same factor, and we use the connection of Ref. [58].

We make the same decomposition of the fermion field ψ as in HQET, Eq. (2.25), and integrate out the anti-field by substituting its equation of motion. As for HQET, the HBET Lagrangian has a non-local form:

$$\sqrt{-g} \mathcal{L}_{\text{HBET}}^{s=1/2} = \sqrt{-g} \bar{Q} \left[(i\not{\nabla} + \mathcal{B}) + (i\not{\nabla} + \mathcal{B}) P_- \frac{1}{2m - (i\not{\nabla} + \mathcal{B}) P_-} (i\not{\nabla} + \mathcal{B}) \right] Q, \tag{2.35}$$

where $\not{\nabla} \equiv \delta_a^\mu \gamma^a \nabla_\mu$ and

$$\mathcal{B} \equiv (e_a^\mu - \delta_a^\mu)(i\gamma^a \nabla_\mu + m\gamma^a v_\mu). \tag{2.36}$$

This is the main result of the paper.

We can recover a local form of this Lagrangian by expanding the denominator in both $1/m$ and κ. We will only need vertices involving two spinors and at most two gravitons, so we expand up to $\mathcal{O}(\kappa^2)$. The resulting Feynman rules are given in Appendix 2.D for reference.

Although we started with massive matter fields, Eqs. (2.31) and (2.35) contain no mass terms for the matter fields. The propagating modes of HBET are therefore massless, so their momenta scale with $\hbar$ in the classical limit. As in the case of HQET, this allows us to interpret the operator expansion of HBET as an expansion in $\hbar$.

The Feynman rules of both theories (Appendix 2.D) are suggestive of the universality of the multipole expansion from Ref. [23]; all terms present in the scalar Feynman rules also appear in the spinor Feynman rules. There are, of course, extra terms in the spinor Feynman rules which encode spin effects. Moreover, we find additional spin-independent terms in the spinor Feynman rules that do not appear in the scalar rules. This is not necessarily inconsistent with Ref. [23]: as will be discussed further below, we expect these additional terms to not contribute to the properly defined potential at one-loop level.

2.6 LONG RANGE $2 \to 2$ GRAVITATIONAL SCATTERING AMPLITUDES

We will demonstrate the utility of the above EFTs for systems of two heavy particles. We do so by calculating the amplitudes for the scattering of scalars and fermions mediated by gravitons up to the leading quantum order at one-loop level. To maximize the efficiency of the computation of the following amplitudes, one could obtain them as double copies of HQET amplitudes. Focusing on the validation of HBET, however, we compute them using standard Feynman diagram techniques applied directly to the

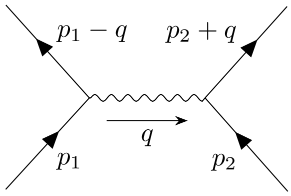


Figure 2.4: Classical scattering of two particles at tree-level.

HBET Lagrangians in Eqs. (2.31) and (2.35), with graviton dynamics described by the usual Einstein-Hilbert action,

$$S_{\text{GR}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R. \tag{2.37}$$

To obtain the classical portions of the amplitudes, we use only the HBET operators described in Section 2.3. The leading quantum terms arise by also including operators that scale with one more factor of $\hbar$.

In what follows we make use of the reparameterization invariance of HBET [59–61] to work in a frame in which the initial momenta are $p_i^\mu = m_i v_i^\mu$, where v_i^μ is the initial four-velocity of particle i. We then define $\omega \equiv v_{1\mu}v_2^\mu$, which, in such a frame, is related to the Mandelstam variable $s = (p_1 + p_2)^2$ via

$$s - s_0 = 2m_1m_2(\omega - 1), \tag{2.38}$$

where $s_0 \equiv (m_1 + m_2)^2$. From Eq. (2.38) it is evident that the non-relativistic limit of the kinematics of both particles, $s - s_0 \to 0$, is equivalent to the limit $\omega \to 1$. As a check on the results, we reproduce the amplitudes in Ref. [23] in the non-relativistic limit.

Amplitudes for scalar-scalar scattering arise as a portion of the fermion-fermion scattering amplitude [23]. For this reason we present here the amplitudes for fermion-fermion scattering.

2.6.1 First Post-Minkowskian Order

At 1PM order, the relevant diagram is the tree-level graviton exchange diagram, shown in Fig. 2.4. Using the $\hbar$-counting in Section 2.3, we see that the coupling constants provide one inverse power of $\hbar$, while the graviton propagator scales as $1/\hbar^2$. The leading tree-level amplitude becomes

$$\begin{aligned}\mathcal{M}_t^{(1)} = -\frac{4\pi m_1 m_2 G}{\hbar^3 q^2}\bigg[&(2\omega^2 - 1)\mathcal{U}_1\mathcal{U}_2 + \frac{2i\omega}{m_1^2 m_2}\mathcal{E}_1\mathcal{U}_2 + \frac{2i\omega}{m_1 m_2^2}\mathcal{E}_2\mathcal{U}_1 \\ &- \frac{1}{m_1^3 m_2^3}\mathcal{E}_1\mathcal{E}_2 + \frac{\omega}{m_1^2 m_2^2}\mathcal{E}_1^\mu\mathcal{E}_{2\mu}\bigg].\end{aligned} \tag{2.39}$$

This is in agreement with Ref. [23] at leading order in $\mathcal{O}(|q|)$. We use the shorthand notation

$$\mathcal{U}_1 \equiv \bar{u}(p_1 - q)u(p_1) \equiv \bar{u}_2 u_1, \tag{2.40a}$$
$$\mathcal{U}_2 \equiv \bar{u}(p_2 + q)u(p_2) \equiv \bar{u}_4 u_3, \tag{2.40b}$$
$$\mathcal{E}_i \equiv \epsilon^{\mu\nu\alpha\beta}p_{1\mu}p_{2\nu}\bar{q}_\alpha S_{i\beta}, \tag{2.40c}$$
$$\mathcal{E}_i^\mu \equiv \epsilon^{\mu\nu\alpha\beta}p_{i\nu}\bar{q}_\alpha S_{i\beta}, \tag{2.40d}$$

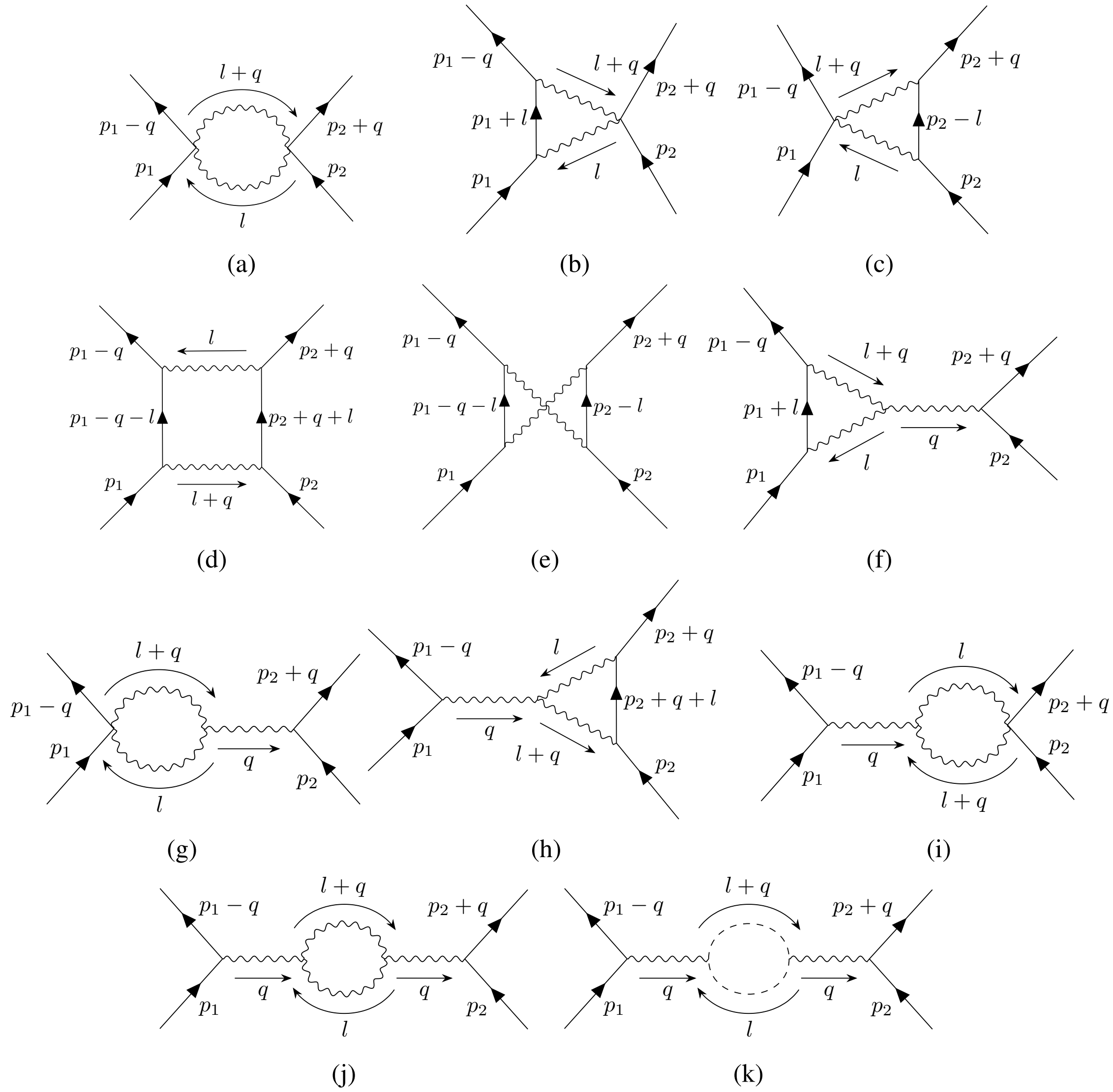


Figure 2.5: The one-loop Feynman diagrams containing non-analytic pieces that contribute to the classical scattering of two particles in GR. Solid lines represent fermions, wavy lines represent gravitons, and dashed lines represent the ghost field arising from working in the harmonic gauge [23].

with the relativistic normalization of the spinors, $\bar{u}(p)u(p) = 2m$. The Levi-Civita tensor is defined by $\epsilon^{0123} = 1$. The spin vector is defined as

$$S_i^\mu \equiv \frac{1}{2}\bar{u}_{2i}\gamma_5\gamma^\mu u_{2i-1}, \tag{2.40e}$$

where $\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$. The definition of the HQET spinor in Eq. (2.24b) automatically imposes the orthogonality of the spin vector and the momentum of the corresponding particle, since it implies the relation $\not{v}u = u$.

2.6.2 Second Post-Minkowskian Order

At 2PM order, eleven one-loop diagrams can contribute, shown in Fig. 2.5. Only triangles or box diagrams contribute to the classical amplitude, but as we also compute the leading quantum contributions, all eleven diagrams are needed.

For clarity, we split the 2PM amplitude into three parts: the spinless, spin-orbit, and spin-spin contributions. These are, respectively,

$$\begin{aligned}
&\mathcal{M}^{(2)}_{\text{spinless}} = \\
&\frac{G^2}{\hbar^3} m_1 m_2 \mathcal{U}_1 \mathcal{U}_2 S \frac{3}{2}(5\omega^2-1)(m_1+m_2) \\
&+ \frac{G^2 \mathcal{U}_1 \mathcal{U}_2 L}{30\hbar^2(\omega^2-1)^2}\Big[2m_1 m_2(18\omega^6 - 67\omega^4 + 50\omega^2 - 1) \\
&-60 m_1 m_2 \omega(12\omega^4 - 20\omega^2 + 7) L_\times(\omega)\sqrt{\omega^2-1} - 15 i\pi(m_1^2+m_2^2)(24\omega^4 - 37\omega^2 + 13)\sqrt{\omega^2-1} \\
&- \frac{120}{\hbar^2 \bar{q}^2} i\pi m_1^2 m_2^2 (4\omega^6 - 8\omega^4 + 5\omega^2 - 1)\sqrt{\omega^2-1}\Big],
\end{aligned} \tag{2.41a}$$

$$\begin{aligned}
&\mathcal{M}^{(2)}_{\text{spin-orbit}} = \\
&\frac{G^2 m_1 m_2 \omega(5\omega^2-3)S}{2\hbar^3(\omega^2-1)}\left[(3m_1+4m_2)\frac{i\mathcal{U}_1\mathcal{E}_2}{m_1 m_2^2}\right] \\
&+ \frac{G^2 L}{10\hbar^2(\omega^2-1)^2}\Big\{2m_1 m_2 \omega(\omega^2-1)(46\omega^2-31) - 20 m_2^2 i\pi\omega(\omega^2-2)\sqrt{\omega^2-1} \\
&- \frac{80}{\hbar^2\bar{q}^2} i\pi m_1^2 m_2^2 \omega(\omega^2-1)(2\omega^2-1)\sqrt{\omega^2-1} - 5 m_1^2 i\pi\omega(12\omega^4 - 10\omega^2 - 5)\sqrt{\omega^2-1} \\
&-5 m_1 m_2[(40\omega^4 - 48\omega^2 + 7)L_\times(\omega) - (8\omega^4-1)(L_\square(\omega)+i\pi)]\sqrt{\omega^2-1}\Big\}\frac{i\mathcal{U}_1\mathcal{E}_2}{m_1 m_2^2} + (1\leftrightarrow 2),
\end{aligned} \tag{2.41b}$$

$$\begin{aligned}
&\mathcal{M}^{(2)}_{\text{spin-spin}} = \\
&G^2(m_1+m_2)\frac{S}{\hbar^3}\left[\frac{(20\omega^4 - 21\omega^2 + 3)}{2(\omega^2-1)}(\bar{q}\cdot S_1 \bar{q}\cdot S_2 - \bar{q}^2 S_1\cdot S_2) + \frac{2\bar{q}^2\omega^3(5\omega^2-4)}{m_1 m_2(\omega^2-1)^2} p_2\cdot S_1 p_1 \cdot S_2\right] \\
&+ \frac{G^2 L}{\hbar^3 m_1 m_2}[m_1 C_1(m_1,m_2) p_2\cdot S_1 \bar{q}\cdot S_2 - m_2 C_1(m_2,m_1)\bar{q}\cdot S_1 p_1\cdot S_2] \\
&+ \frac{G^2 L}{60 m_1 m_2 \hbar^2(\omega^2-1)^2}(2C_2 \bar{q}\cdot S_1 \bar{q}\cdot S_2 + C_3 \bar{q}^2 S_1\cdot S_2) + \frac{G^2\bar{q}^2 L}{20\hbar^4 m_1^2 m_2^2(\omega^2-1)^{5/2}} C_4 p_2\cdot S_1 p_1\cdot S_2,
\end{aligned} \tag{2.41c}$$

where

$$L_\square(\omega) \equiv \log\left|\frac{\omega - 1 - \sqrt{\omega^2-1}}{\omega - 1 + \sqrt{\omega^2-1}}\right|, \tag{2.41d}$$

$$L_\times(\omega) \equiv \log\left|\frac{\omega + 1 + \sqrt{\omega^2-1}}{\omega + 1 - \sqrt{\omega^2-1}}\right|, \tag{2.41e}$$

$$C_1(m_i, m_j) \equiv \frac{(8\omega^4 - 8\omega^2 + 1)}{(\omega^2-1)^{3/2}} i\pi(m_i + \omega m_j), \tag{2.41f}$$

$$\begin{aligned}
C_2 \equiv\ & 60 m_1 m_2 \omega\big((L_\square(\omega)+i\pi)(4\omega^4 - 2\omega^2 - 1) + L_\times(\omega)(-8\omega^4 + 14\omega^2 - 5)\big)\sqrt{\omega^2-1} \\
&- \frac{120}{\hbar^2\bar{q}^2} i\pi m_1^2 m_2^2 (\omega^2-1)(1-2\omega^2)^2\sqrt{\omega^2-1} \\
&- 30 i\pi(m_1^2+m_2^2)(2\omega^6 - 4\omega^4 - \omega^2 + 2)\sqrt{\omega^2-1} \\
&+ 2 m_1 m_2(\omega^2-1)\big(258\omega^4 - 287\omega^2 + 29\big),
\end{aligned} \tag{2.41g}$$

$$\begin{aligned}
C_3 \equiv\ & 60 m_1 m_2 \omega\big((L_\square(\omega)+i\pi)(3-4\omega^2) + 12 L_\times(\omega)(2\omega^4 - 3\omega^2 + 1)\big)\sqrt{\omega^2-1} \\
&+ \frac{120}{\hbar^2\bar{q}^2} i\pi m_1^2 m_2^2(\omega^2-1)(8\omega^4 - 8\omega^2 + 1)\sqrt{\omega^2-1} \\
&+ 15 i\pi(m_1^2+m_2^2)(8\omega^6 - 12\omega^4 - 3\omega^2 + 5)\sqrt{\omega^2-1} \\
&+ 4 m_1 m_2(\omega^2-1)\big(-258\omega^4 + 287\omega^2 - 44\big),
\end{aligned} \tag{2.41h}$$

$$C_4 \equiv -\frac{40}{\vec{q}^2} i\pi m_1^2 m_2^2 \omega \left(\omega^2 - 1\right)\left(8\omega^4 - 8\omega^2 + 1\right). \tag{2.41i}$$

The classical contributions are in the first lines of each of Eqs. (2.41a)-(2.41b), and in the first and second lines of Eq. (2.41c). The classical spinless contribution is in agreement with Ref. [48]. The classical spin-orbit contribution is consistent with the spin holonomy map of Ref. [62]. The classical spin-spin contribution compliments the results in Ref. [46]. In particular, we find that the coefficient of $(-q \cdot S_1 q \cdot S_2)$ in Eq. (2.41c) agrees with $A_{1,1}^{2\,\mathrm{PM}}$ in Eq. (7.18) in Ref. [46], which is the corresponding coefficient in the Leading Singularity approach [42, 43], whereas the remainder of the terms are not presented therein. To the best of our knowledge, this is the first presentation of the leading quantum contributions to the spinless, spin-orbit and spin-spin amplitudes at 2PM order. There are additional spin-spin terms at the quantum level proportional to $p_i \cdot S_j$ for $i \neq j$ that we have not included in our calculation. We note that additional spin quadrupole terms are also present at the second order in spin, which can be calculated from vector-scalar scattering.

To obtain the results in this section we have made use of the identity

$$\bar{u}_{2i}\sigma^{\mu\nu}u_{2i-1} = -2\epsilon^{\mu\nu\alpha\beta}v_{i\alpha}S_{i\beta}, \tag{2.42}$$

which is valid for HQET spinors. This identity merits some discussion. Replacing the HQET spinors by Dirac spinors (denoted with a subscript D), the identity becomes

$$\bar{u}_{2i,\mathrm{D}}\sigma^{\mu\nu}u_{2i-1,\mathrm{D}} = -2ip_{2i-1}^{[\nu}\bar{u}_{2i,\mathrm{D}}\gamma^{\mu]}u_{2i-1,\mathrm{D}} - \frac{2}{m}\epsilon^{\mu\nu\alpha\beta}p_{2i-1,\alpha}S_{i,\mathrm{D}\beta}. \tag{2.43}$$

The second term above is the same as in Eq. (2.42). The first, by contrast, arises only with Dirac spinors, and through the Gordon decomposition contains both a spinless term involving only the spinor product $\mathcal{U}_i$, and a term like that on the left hand side of the equation. Eq. (2.43) thereby mixes spinless and spin-inclusive effects. This is an advantage of this EFT approach, at least at one-loop level. Eq. (2.42) allows one to target spinless or spin-inclusive terms in the amplitude simply by ignoring or including operators involving the Dirac sigma matrix. It is also consistent with the universality of the spin-multipole expansion observed in Refs. [23, 52], where spin effects were found to not mix with, and to be corrections to the universal spin-independent amplitude.

At face value, there is one complication to this interpretation of Eq. (2.42). Due to the heavy propagators, terms such as $\overline{u}\sigma^{\mu\nu}(1+\not{v})\sigma^{\alpha\beta}u$ begin to arise at one-loop level. Through some gamma matrix manipulations, it can be shown that these terms contain spinless (containing no sigma matrices) and spin-inclusive (containing one sigma matrix) components. At one-loop level the spinless components contribute to the classical and leading quantum portions of the spinless part of the amplitude only through the term proportional to $(m_1^2 + m_2^2)L$. As this term is purely imaginary, we expect it to be subtracted by the Born iteration when extracting the potential. Thus, if one is interested only in non-imaginary terms at one-loop, spinless or spin-inclusive terms can be independently targeted by exploiting the separation of spin effects at the level of the Lagrangian.

While spinless and spin-inclusive effects are cleanly separated in spinor HBET, the presence of additional spin-independent operators in spinor HBET compared to scalar HBET makes it ostensibly possible that the spinless parts of its amplitudes differ from the amplitudes of scalar HBET. In fact, calculating scalar-scalar scattering explicitly with scalar HBET, we find that the term proportional to $(m_1^2 + m_2^2)L$ in Eq. (2.41a) does not arise. In addition to receiving contributions from the $\overline{u}\sigma^{\mu\nu}(1 + \not{v})\sigma^{\alpha\beta}u$ tensor structure in the loop amplitudes — a structure that certainly does not arise in scalar HBET — it is also the only term that is affected by the spin-independent operators in spinor HBET that are not present in scalar HBET. We therefore find that we preserve the universality of the multipole expansion from Ref. [23] in the one-loop relativistic regime as well, up to terms which are subtracted by the Born iteration.

As a check on the validity of our results, we compare their non-relativistic limits with what exists in the literature, simply by taking the limit $\omega - 1 \to 0$ in the PM amplitudes. At 1PM order we find that our results agree with those in Ref. [23]. At 2PM the amplitudes above contain those in Ref. [23], but there are two discrepancies:

1. We find an additional spinless term that we expect to be subtracted by Born iteration, arising from the imaginary term proportional to $(m_1^2 + m_2^2)L$.

2. The contraction $p_i \cdot S_j$ for $i \neq j$ vanishes in the non-relativistic limit. However, these terms in Eq. (2.41c) also have denominators that vanish in this limit. Without knowing explicitly how $p_i \cdot S_j \to 0$, we therefore cannot say that these terms will not remain in the limit.

We note that this limit only represents the non-relativistic limit of the kinematics; the non-relativistic limit of the spinors must also be taken in order to obtain the fully non-relativistic amplitude.

2.7 CONCLUSION

While significant progress has been made in understanding the relationship between gravitational scattering amplitudes and classical gravitational quantities, it remains uneconomical to extract the few classically contributing terms from the multitude of other terms that constitute the full amplitude. With an eye to addressing this inefficiency, we have introduced HBET, an EFT which describes the interactions of heavy scalars and heavy fermions with gravity. By restoring $\hbar$ at the level of the Feynman rules, we have been able to infer the $\hbar$-scaling of HBET operators, and exploit it to determine which operators can contribute to the classical amplitude at arbitrary loop order. One may see the present construction as a step towards isolating just those terms of the scattering amplitude that will contribute to the classical scattering of two massive objects, order by order in the loop expansion. Crucially, a method does not yet exist to convert fully relativistic amplitudes including spin to interaction potentials.[12]

We used HBET to directly calculate the 2PM classical gravitational scattering amplitude for the scattering of two fermions, and checked that the spinless part of the amplitude matches the amplitude for scalar-scalar scattering, up to terms that we expect to be subtracted from the potential. To validate the EFT, we compared the fully relativistic amplitudes and their non-relativistic limits with what has been previously calculated, and found agreement. We presented the classical and leading quantum spinless, spin-orbit and spin-spin contributions at 2PM order, up to terms proportional to $p_i \cdot S_j$ where $i \neq j$ for the spin-spin contribution, complementing and extending the results in the literature.

While we derived HBET only for heavy particles of spin $s \leq 1/2$, we believe it is possible to also derive an HBET for heavy higher-spin particles: as long as a Lagrangian can be written for a massive particle of spin s, we can apply similar techniques to those herein to derive the HBET applicable for spin

[12] Since the publication of this article, ref. [29] presented a method for doing exactly this using an EFT matching procedure.

s. This would allow the computation of the classical amplitude for higher order terms in the multipole expansion.

For full efficiency, the HBET formalism should be used in combination with modern scattering amplitude techniques. First, the Feynman rules of scalar and spinor HBET, and the property in Eq. (2.42), are suggestive of the universality of the multipole expansion presented in Ref. [23]. An interesting next step is to express the degrees of freedom of HBET in terms of massive on-shell variables [63]. It would be interesting to study whether this universality can be made manifest in such variables, and how the observed separation of spinless and spin-inclusive effects arises. An on-shell formulation of HBET should also include an explicit $\hbar$ expansion, further elucidating the classical limit for amplitudes computed using on-shell variables. Moreover, the work in Refs. [46, 53, 63] suggests that massive on-shell variables may facilitate the extension of HBET to higher spins. Second, as HQET is derived from QCD, and HBET is derived from GR, we expect the double copy structure of the scattering amplitudes to still hold as a relation between the effective theories. While certainly not the only way to study such a relation, we expect it to be more readily apparent in on-shell versions of HQET and HBET. We leave the on-shell formulation of HBET for future work.

By combining the power counting (which includes the $\hbar$ counting) and multipole expansion of the effective field theories with the on-shell formalism, unitarity methods, and the double copy, we believe that higher order calculations are within reach.

APPENDIX

2.A HQET SPINORS

In this section, we make precise the external states of the HQET spinor field by expressing them in terms of the external states of the original Dirac spinor field ψ. To do so, we begin with the mode expansion of ψ:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s_{\mathrm{D}}(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} w^s_{\mathrm{D}}(p) e^{ip\cdot x} \right), \tag{2.44}$$

where $\mathbf{p}$ represents the three-momentum, $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, s is a spin index, and $a^s_{\mathbf{p}}$ and $b^{s\dagger}_{\mathbf{p}}$ are annihilation and creation operators for the particle and antiparticle respectively. We use the unconventional notation w_D for the antiparticle spinor to differentiate it from the four-velocity. The spinors $u^s_{\mathrm{D}}(p)$ and $w^s_{\mathrm{D}}(p)$ satisfy the Dirac equation,

$$(\not{p} - m) u^s_{\mathrm{D}}(p) = 0, \tag{2.45a}$$

$$(\not{p} + m) w^s_{\mathrm{D}}(p) = 0. \tag{2.45b}$$

Recall the definition of the HQET spinor field Q_v,

$$Q_v = e^{imv\cdot x} \frac{1+\not{v}}{2} \psi, \tag{2.46}$$

where v^μ is defined by the HQET momentum decomposition in Eq. (2.21). The mode expansion for Q_v is then

$$Q_v(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} \frac{1+\not{v}}{2} u^s_{\mathrm{D}}(p) e^{-ik\cdot x} + b^{s\dagger}_{\mathbf{p}} \frac{1+\not{v}}{2} w^s_{\mathrm{D}}(p) e^{i(2mv+k)\cdot x} \right). \tag{2.47}$$

After the decomposition in Eq. (2.21), the Dirac equation can be rewritten as

$$\not{v} u^s_{\mathrm{D}}(p) = \left(1 - \frac{\not{k}}{m}\right) u^s_{\mathrm{D}}(p), \tag{2.48a}$$

$$\not{v} w^s_{\mathrm{D}}(p) = -\left(1 - \frac{\not{k}}{m}\right) w^s_{\mathrm{D}}(p). \tag{2.48b}$$

Using this in the mode expansion for Q_v we find

$$Q_v(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s_v(p) e^{-ik\cdot x} + b^{s\dagger}_{\mathbf{p}} w^s_v(p) e^{i(2mv+k)\cdot x} \right). \tag{2.49a}$$

where

$$u^s_v(p) \equiv \left(1 - \frac{\not{k}}{2m}\right) u^s_{\mathrm{D}}(p), \tag{2.49b}$$

$$w^s_v(p) \equiv \frac{\not{k}}{2m} w^s_{\mathrm{D}}(p). \tag{2.49c}$$

Similarly, the mode expansion of $\tilde{Q}_v$ is

$$\tilde{Q}_v = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} \frac{\not{k}}{2m} u^s_D(p) e^{-ik\cdot x} + b^{s\dagger}_{\mathbf{p}} \left(1 - \frac{\not{k}}{2m}\right) w^s_D(p) e^{i(2mv+k)\cdot x} \right). \tag{2.50}$$

The mode expansion in Eq. (2.49a) makes it apparent that, when considering only particles and not antiparticles, the derivative of Q_v translates to a factor of the residual momentum k^μ in the Feynman rules.

2.B HEAVY SCALAR EFFECTIVE THEORY

For completeness we include the derivation of an effective theory for Scalar Quantum Electrodynamics (SQED). That is, we want the effective theory that arises when ϕ in

$$\mathcal{L}_{\text{SQED}} = (D_\mu \phi)^* D^\mu \phi - m^2 \phi^2, \qquad D^\mu \phi = (\partial^\mu + ieA^\mu)\phi, \tag{2.51}$$

is very massive. To do so, we simply make the field redefinition [54]

$$\phi \to \frac{e^{-imv\cdot x}}{\sqrt{2m}} (\chi + \tilde{\chi}). \tag{2.52}$$

The anti-field $\tilde{\chi}$ is to be integrated out. At leading order, we can drop this term. Inserting Eq. (2.52) into Eq. (2.51), and performing a field redefinition to eliminate redundant operators, we obtain Heavy Scalar Effective Theory (HSET):

$$\mathcal{L}_{\text{HSET}} = \chi^* \left(iv \cdot D - \frac{D^2}{2m} \right) \chi + \mathcal{O}(1/m^3). \tag{2.53}$$

Higher order terms can be restored by keeping contributions coming from integrating out the anti-field.

2.C LONG RANGE $2 \to 2$ ELECTROMAGNETIC SCATTERING AMPLITUDES

In this section we demonstrate that HSET and HQET can be used to calculate the classical and leading quantum contributions to the $2 \to 2$ scattering amplitudes. We present here the results up to one-loop order. As in the gravity case, electromagnetic interactions also possess a universal spin-multipole expansion [52], so we present this calculation using HQET.

At tree level, the diagram in Fig. 2.4 is once again the only one that contributes. The amplitude is, up to leading order in $|q|$,

$$\mathcal{A}^{(0)} = \frac{4\pi\alpha}{\hbar^3 q^2} \left[\omega \mathcal{U}_1 \mathcal{U}_2 + \frac{i\mathcal{U}_1 \mathcal{E}_2}{m_1 m_2^2} + \frac{i\mathcal{E}_1 \mathcal{U}_2}{m_1^2 m_2} + \frac{\omega}{m_1 m_2} \mathcal{E}_1^\mu \mathcal{E}_{2\mu} \right]. \tag{2.54}$$

This amplitude is in agreement with Ref. [52] in the relativistic and non-relativistic regimes.

At one-loop level, the abelian nature of QED reduces the number of relevant diagrams compared to the gravity case. There are only five relevant diagrams in the electromagnetic case: they are diagrams (a) to (e) in Fig. 2.5. Of course, the wavy lines are reinterpreted as photons. We find the amplitude

$$\begin{aligned}
\mathcal{A}^{(1)}_{\text{spinless}} &= \frac{\alpha^2}{\hbar^3 m_1 m_2}\Bigg[S(m_1+m_2) - \frac{\hbar L}{2m_1m_2(\omega^2-1)^2}\Big(2m_1m_2(\omega^4-1)\\
&\quad +4m_1m_2\omega(\omega^2-2)L_\times(\omega)\sqrt{\omega^2-1}\\
&\quad +(m_1^2+m_2^2)i\pi(\omega^2-1)^2\sqrt{\omega^2-1}\\
&\quad +\frac{8i\pi}{\hbar^2\bar{q}^2}m_1^2m_2^2\omega^2(\omega^2-1)\sqrt{\omega^2-1}\Big)\Bigg]\mathcal{U}_1\mathcal{U}_2, && (2.55a)\\
\mathcal{A}^{(1)}_{\text{spin-orbit}} &= \frac{\alpha^2}{\hbar^3 m_1m_2(\omega^2-1)}S\omega(m_1+2m_2)\frac{i\mathcal{U}_1\mathcal{E}_2}{m_1m_2^2}\\
&\quad +\frac{\alpha^2 L}{2\hbar^2 m_1^2m_2^2(\omega^2-1)^2}\Big[-m_2^2\omega(\omega^2-3)\sqrt{\omega^2-1}(i\pi)\\
&\quad +m_1m_2\sqrt{\omega^2-1}((2\omega^2+1)(L_\Box(\omega)+i\pi)+(3-2\omega^2)L_\times(\omega)+\sqrt{\omega^2-1}(\omega^2+2\omega-1))\\
&\quad -m_1^2\omega(2\omega^2-3)\sqrt{\omega^2-1}(2\omega^2-3)(i\pi)\Big]\frac{i\mathcal{U}_1\mathcal{E}_2}{m_1m_2^2}\\
&\quad -\frac{4i\pi\alpha^2L\omega}{\hbar^4\bar{q}^2\sqrt{\omega^2-1}}\frac{i\mathcal{U}_1\mathcal{E}_2}{m_1m_2^2}+(1\leftrightarrow 2), && (2.55b)\\
\mathcal{A}^{(1)}_{\text{spin-spin}} &= \frac{\alpha^2 S(m_1+m_2)}{\hbar^3m_1^2m_2^2(\omega^2-1)}\left[(2\omega^2-1)(q\cdot S_1 q\cdot S_2 - q^2 S_1\cdot S_2)+\frac{2q^2\omega^3}{m_1m_2(\omega^2-1)}p_2\cdot S_1 p_1\cdot S_2\right]\\
&\quad +\frac{\alpha^2}{\hbar^2 m_1m_2}[m_1C_1'(m_1,m_2)p_2\cdot S_1 q\cdot S_2 - m_2C_1'(m_2,m_1)q\cdot S_2 p_1\cdot S_2]\\
&\quad +\frac{\alpha^2 L}{2\hbar^2 m_1^3m_2^3(\omega^2-1)^2}(C_2' q\cdot S_1 q\cdot S_2+2C_3'q^2S_1\cdot S_2)\\
&\quad +\frac{\alpha^2 L}{2m_1^3m_2^3(\omega^2-1)^{5/2}}C_4'p_2\cdot S_1p_1\cdot S_2, && (2.55c)
\end{aligned}$$

where

$$\begin{aligned}
C_1'(m_i,m_j) &\equiv -\frac{q^2 S}{m_i^3m_j^3(\omega^2-1)^2}[m_i^2(3\omega^4+8\omega^2-3)+4m_im_j(\omega+1)^2(2\omega-1)+2m_j^2\omega(5\omega^2-1)]\\
&\quad +\frac{L(2\omega^2-1)}{\hbar m_i^2m_j^2}i\pi(m_i+\omega m_j), && (2.55d)\\
C_2' &\equiv 4m_1m_2\omega\left(L_\times(\omega)+L_\Box(\omega)\omega^2+i\pi\omega^2\right)\sqrt{\omega^2-1}\\
&\quad -i\pi(m_1^2+m_2^2)\left(2\omega^4-5\omega^2+1\right)\sqrt{\omega^2-1}\\
&\quad +6m_1m_2\left(\omega^2-1\right)^2-\frac{8}{\hbar^2\bar{q}^2}i\pi m_1^2m_2^2\omega^2\left(\omega^2-1\right)\sqrt{\omega^2-1}, && (2.55e)\\
C_3' &\equiv 2m_1m_2\omega\left(-L_\Box(\omega)+2L_\times(\omega)\left(\omega^2-1\right)-i\pi\right)\sqrt{\omega^2-1}\\
&\quad +i\pi(m_1^2+m_2^2)\left(2\omega^4-4\omega^2+1\right)\sqrt{\omega^2-1}\\
&\quad +2m_1m_2\left(\omega^2-1\right)\left(2-3\omega^2\right)+\frac{4}{\hbar^2\bar{q}^2}i\pi m_1^2m_2^2\left(\omega^2-1\right)\left(2\omega^2-1\right)\sqrt{\omega^2-1}, && (2.55f)\\
C_4' &\equiv -\frac{4}{\hbar^2\bar{q}^2}i\pi m_1^2m_2^2\omega\left(\omega^2-1\right)\left(2\omega^2-1\right)+6m_1m_2\omega^3\sqrt{\omega^2-1}\\
&\quad +m_1m_2\left(\left(L_\Box(\omega)+i\pi\right)\left(2\omega^4+5\omega^2-1\right)+L_\times(\omega)\left(-2\omega^4+3\omega^2-1\right)\right)\\
&\quad -i\pi\omega\left(2\omega^4-6\omega^2+1\right)\left(m_1^2+m_2^2\right). && (2.55g)
\end{aligned}$$

The non-relativistic limit of this amplitude is in agreement with Ref. [52], with discrepancy number 2 from the gravitational case applying here as well.

Calculating explicitly the amplitude for scalar-scalar scattering using HSET, we find the same amplitude as in Eq. (2.55a), but without the imaginary term proportional to $(m_1^2 + m_2^2)L$. This term vanishes in the non-relativistic limit, thus preserving the non-relativistic universality of the multipole expansion in Ref. [52]. Furthermore, we expect it to be subtracted by the Born iteration when calculating the potential, thus extending the multipole univerality to the relativistic potential.

2.D FEYNMAN RULES

We list here the Feynman rules used to perform the calculations in this paper. Below we denote the matter wave vector entering the vertex by k_1 and the matter wave vector leaving by k_2. q_1 and q_2 are incoming photon (graviton) wave vectors with indices μ, ν $(\mu\nu, \alpha\beta)$, respectively.

We use the photon propagator in the Feynman gauge. The graviton propagator, three graviton vertex, as well as the ghost propagator and two-ghost-one-graviton vertex are given in the harmonic gauge in Ref. [23].

2.D.1 Abelian HSET

Starting with HSET, the one- and two-photon vertex Feynman rules are

$$\tau^{\mu}_{\chi\chi^*\gamma}(m, v, k_1, k_2) = -\frac{ie}{\sqrt{\hbar}}\left[v^\mu + \frac{\hbar}{2m}(k_1^\mu + k_2^\mu) + \mathcal{O}\left(\frac{\hbar^3}{m^3}\right)\right], \tag{2.56a}$$

$$\tau^{\mu\nu}_{\chi\chi^*\gamma\gamma}(m, v, k_1, k_2) = \frac{ie^2}{m\hbar}\left[\eta^{\mu\nu} + \mathcal{O}\left(\frac{\hbar^2}{m^2}\right)\right]. \tag{2.56b}$$

2.D.2 Scalar HBET

For HSBET the one- and two-graviton vertex Feynman rules are

$$\begin{aligned}\tau^{\mu\nu}_{\chi\chi^*h}(m, v, k_1, k_2) = -\frac{i\kappa}{2\sqrt{\hbar}}\Big\{ & m v^\mu v^\nu - \frac{\hbar}{2}[\eta^{\mu\nu} v_\rho (k_1^\rho + k_2^\rho) - v^\mu (k_1^\nu + k_2^\nu) - v^\nu (k_1^\mu + k_2^\mu)] \\ & + \frac{\hbar^2}{2m}\left[(k_1^\mu k_2^\nu + k_2^\mu k_1^\nu) - \eta^{\mu\nu} k_{1\alpha} k_2^\alpha\right] + \mathcal{O}\left(\frac{\hbar^3}{m^2}\right)\Big\},\end{aligned} \tag{2.57a}$$

$$\begin{aligned}\tau^{\mu\nu,\alpha\beta}_{\chi\chi^*hh}(m, v, k_1, k_2) = \frac{i\kappa^2}{\hbar}\Big\{ & m v_\tau v_\lambda \left[I^{\mu\nu,\tau\gamma} I_\gamma{}^{\lambda,\alpha\beta} - \frac{1}{4}(\eta^{\mu\nu} I^{\alpha\beta,\tau\lambda} + \eta^{\alpha\beta} I^{\mu\nu,\tau\lambda})\right] \\ & + \frac{\hbar}{4}\{-P^{\mu\nu,\alpha\beta} v_\rho (k_1^\rho + k_2^\rho) - (\eta^{\mu\nu} I^{\alpha\beta,\tau\lambda} + \eta^{\alpha\beta} I^{\alpha\beta,\tau\lambda}) v_\tau (k_{1\lambda} + k_{2\lambda}) \\ & + 2 I^{\mu\alpha,\tau\gamma} I_\gamma{}^{\lambda,\nu\beta}[v_\tau (k_{1\lambda} + k_{2\lambda}) + v_\lambda (k_{1\tau} + k_{2\tau})]\} \\ & + \frac{\hbar^2}{4m}[-P^{\mu\nu,\alpha\sigma} k_{1\rho} k_2^\rho - (\eta^{\mu\nu} I^{\alpha\beta,\tau\lambda} + \eta^{\alpha\beta} I^{\mu\nu,\tau\lambda}) k_{1\tau} k_{2\lambda} \\ & + 2 I^{\mu\alpha,\tau\gamma} I_\gamma{}^{\lambda,\nu\beta}(k_{1\tau} k_{2\lambda} + k_{2\tau} k_{1\lambda})] + \mathcal{O}\left(\frac{\hbar^3}{m^2}\right)\Big\},\end{aligned} \tag{2.57b}$$

where

$$P_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}). \tag{2.57c}$$

The propagator in both scalar theories is

$$D_v^{s=0}(k) = \frac{i}{\hbar v \cdot k}. \tag{2.58}$$

2.D.3 Abelian HQET

The one- and two-photon Feynman rules in HQET are

$$\begin{aligned}
\tau^{\mu}_{\bar{Q}Q\gamma}(m,v,k_1,k_2) = -\frac{ie}{\sqrt{\hbar}}\Big\{ & v^\mu + \frac{i}{2m}\sigma^{\mu\nu}(k_{2\nu}-k_{1\nu}) + \frac{\hbar}{2m}(k_1^\mu + k_2^\mu)_\perp \\
& + \frac{i\hbar}{8m^2} v_\rho \sigma_{\alpha\beta}(k_1^\alpha + k_2^\alpha)_\perp \left[(k_2^\rho - k_1^\rho)\eta^{\mu\beta} - (k_2^\beta - k_1^\beta)\eta^{\mu\rho}\right] \\
& + \frac{\hbar^2}{8m^2}[v^\mu (k_2-k_1)^2 - v_\rho(k_2^\rho - k_1^\rho)(k_2^\mu - k_1^\mu)] \\
& + \frac{i\hbar^2}{8m^3}(k_{1\perp}^2 + k_{2\perp}^2)\sigma^{\mu\rho}(k_{2\rho}-k_{1\rho}) + \mathcal{O}\left(\frac{\hbar^3}{m^3}\right)\Big\},
\end{aligned} \tag{2.59a}$$

$$\begin{aligned}
\tau^{\mu\nu}_{\bar{Q}Q\gamma\gamma}(m,v,k_1,k_2) = \frac{ie^2}{m\hbar}\Big\{ & \eta_\perp^{\mu\nu} + \frac{i}{4m}[\sigma^{\mu\nu}v_\rho(q_2^\rho - q_1^\rho) + v^\mu v^\nu \sigma^{\lambda\rho} v_\lambda (q_{1\rho} + q_{2\rho}) \\
& - v^\mu v_\lambda \sigma^{\lambda\nu}(v\cdot q_2) - v^\nu \sigma^{\mu\lambda} q_{2\lambda} - v^\nu v_\lambda \sigma^{\lambda\mu}(v\cdot q_1) - v^\mu \sigma^{\nu\lambda} q_{1\lambda}] \\
& - \frac{i\hbar}{4m^2}(k_{2\rho} + k_{1\rho})\sigma_{\alpha\beta}(\eta_\perp^{\rho\mu}\eta^{\beta\nu}q_1^\alpha + \eta_\perp^{\rho\nu}\eta^{\beta\mu}q_2^\alpha) + \mathcal{O}\left(\frac{\hbar^2}{m^2}\right)\Big\},
\end{aligned} \tag{2.59b}$$

where $k_\perp^\mu = k^\mu - v^\mu (v\cdot k)$ and $\eta_\perp^{\mu\nu} = \eta^{\mu\nu} - v^\mu v^\nu$.

2.D.4 Spinor HBET

Finally, the one- and two-graviton Feynman rules in HBET are

$$\begin{aligned}
\tau^{\mu\nu}_{\bar{Q}Qh}(m,v,k_1,k_2) = \frac{i\kappa}{2\sqrt{\hbar}}\Big\{ & -m v^\mu v^\nu + \frac{i}{4}(v^\mu \sigma^{\rho\nu} + v^\nu \sigma^{\rho\mu})(k_{2\rho} - k_{1\rho}) \\
& + \frac{\hbar}{2}[v_\alpha (k_1^\alpha + k_2^\alpha)\eta^{\mu\nu} - v^\mu (k_1^\nu + k_2^\nu) - v^\nu (k_1^\mu + k_2^\mu) \\
& + 3v^\mu (k_2^\nu - k_1^\nu) + 3v^\nu (k_2^\mu - k_1^\mu) - 6\eta^{\mu\nu} v_\rho (k_2^\rho - k_1^\rho)] \\
& + \frac{i\hbar}{4m}[(k_{2\rho} - k_{1\rho})(k_1^\mu \sigma^{\rho\nu} + k_1^\nu \sigma^{\rho\mu}) - v^\mu v^\nu \sigma^{\rho\tau} k_{2\rho} k_{1\tau} \\
& - \frac{1}{2} v_\rho (k_2^\rho k_{2\tau} - k_1^\rho k_{1\tau})(v^\mu \sigma^{\tau\nu} + v^\nu \sigma^{\tau\mu}) \\
& + (k_{2\rho} - k_{1\rho})(k_2^\mu - k_1^\mu)\sigma^{\rho\nu} + (k_{2\rho} - k_{1\rho})(k_2^\nu - k_1^\nu)\sigma^{\rho\mu}] \\
& + \frac{\hbar^2}{2m}\Big[-(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu) + \eta^{\mu\nu} k_1^\rho k_{2\rho} + \frac{1}{2} v^\mu v^\nu k_{1\rho} k_2^\rho \\
& + \frac{1}{2}\eta^{\mu\nu}(k_{2\rho} - k_{1\rho})(k_2^\rho - k_1^\rho) - \frac{1}{2}(k_2^\mu - k_1^\mu)(k_2^\nu - k_1^\nu) \\
& + \frac{1}{4} v_\rho v^\mu (k_2^\nu k_2^\rho + k_1^\nu k_1^\rho) + \frac{1}{4} v_\rho v^\nu (k_2^\mu k_2^\rho + k_1^\mu k_1^\rho)\Big] + \mathcal{O}\left(\frac{\hbar^2}{m^2}\right)\Big\},
\end{aligned} \tag{2.60a}$$

$$\begin{aligned}
\tau^{\mu\nu,\alpha\beta}_{\bar{Q}Qhh}(m,v,k_1,k_2) = \frac{i\kappa^2}{\hbar}\Big\{ & m v_\kappa v_\lambda \left[I^{\mu\nu,\kappa\gamma} I_\gamma{}^{\lambda,\alpha\beta} - \frac{1}{4}(\eta^{\alpha\beta} I^{\mu\nu,\kappa\lambda} + \eta^{\mu\nu} I^{\alpha\beta,\kappa\lambda})\right] \\
& - \frac{i}{16}\epsilon^{\lambda\rho\tau\delta}\gamma_\delta \gamma_5 (I^{\mu\nu,\kappa}{}_\lambda I^{\alpha\beta,}{}_{\tau\kappa} q_{2\rho} + I^{\alpha\beta,\kappa}{}_\lambda I^{\mu\nu,}{}_{\tau\kappa} q_{1\rho})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{i}{16}v_\kappa v_\sigma v_\rho \sigma_{\lambda\tau}[I^{\mu\nu,\kappa\lambda}I^{\alpha\beta,\sigma\tau}(q_{2\rho}+k_{1\rho})+I^{\alpha\beta,\kappa\lambda}I^{\mu\nu,\sigma\tau}(q_{1\rho}+k_{1\rho})]\\
&-\frac{i}{8}v_\kappa\sigma_{\lambda\tau}(k_{1\sigma}-k_{2\sigma})(I^{\mu\nu,\kappa\lambda}I^{\alpha\beta,\sigma\tau}+I^{\alpha\beta,\kappa\lambda}I^{\mu\nu,\sigma\tau})\\
&-\frac{3i}{16}v_\kappa\sigma_{\sigma\rho}(k_1^\rho-k_2^\rho)(I^{\mu\nu,\kappa\tau}I^{\alpha\beta,\sigma\tau}+I^{\alpha\beta,\kappa\tau}I^{\mu\nu,\sigma\tau})\\
&+\frac{i}{8}v_\kappa\sigma_{\lambda\rho}(k_1^\rho-k_2^\rho)(\eta^{\mu\nu}I^{\alpha\beta,\kappa\lambda}+\eta^{\alpha\beta}I^{\mu\nu,\kappa\lambda})\\
&+\frac{i}{16}v_\kappa\sigma_{\lambda\rho}(k_1^\rho-k_2^\rho)(v^\mu v^\nu I^{\alpha\beta,\kappa\lambda}+v^\alpha v^\beta I^{\mu\nu,\kappa\lambda})\\
&+\frac{i}{8}v_\kappa\sigma_{\lambda\rho}(\eta^{\mu\nu}I^{\alpha\beta,\kappa\lambda}q_1^\rho+\eta^{\alpha\beta}I^{\mu\nu,\kappa\lambda}q_2^\rho)\\
&+\frac{i}{8}v_\rho\sigma_{\lambda\tau}(I^{\mu\nu,\kappa\lambda}I^{\alpha\beta,\rho\tau}q_{1\kappa}+I^{\alpha\beta,\kappa\lambda}I^{\mu\nu,\rho\tau}q_{2\kappa})+\mathcal{O}(\hbar)\Bigg\}\,,
\end{aligned}
\tag{2.60b}
$$

where

$$
I^{\mu\nu,\alpha\beta}=\frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}). \tag{2.60c}
$$

Based on the $\hbar$ counting, there are additional terms that could contribute to the amplitude, but we find that they contribute only at subleading quantum levels, and thus don't include them.

The propagator in both spinor theories is

$$
D_v^{s=\frac{1}{2}}(k)=\frac{i}{\hbar v\cdot k}\frac{1+\not{v}}{2}. \tag{2.61}
$$

2.E ONE-LOOP INTEGRAL BASIS

In this section, we point out some subtleties that arise from the linear matter propagators characteristic of HQET/HBET. We first address the appearance of non-analytical contributions to loop integrals when using linear matter propagators instead of quadratic ones. Then we discuss how we circumvent the infamous pinch singularity of HQET.

2.E.1 Non-analytic portions of loop integrals

Consider, for example, the box integral with quadratic massive propagators:

$$
I_{\text{quad}}=\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2(l+q)^2\left[(p_1-l-q)^2-m_1^2+i\epsilon\right]\left[(p_2+l+q)^2-m_2^2+i\epsilon\right]}. \tag{2.62}
$$

Letting the incoming momenta be $p_1^\mu=m_1v_1^\mu$ and $p_2^\mu=m_2v_2^\mu$, and making explicit the factors of $\hbar$ from the massless momenta,

$$
I_{\text{quad}}=\int\frac{d^4\bar{l}}{(2\pi)^4}\frac{1}{\bar{l}^2(\bar{l}+\bar{q})^2\left[-2m_1\hbar v_1\cdot(\bar{l}+\bar{q})+\hbar^2(\bar{l}+\bar{q})^2+i\epsilon\right]\left[2m_2\hbar v_2\cdot(\bar{l}+\bar{q})+\hbar^2(\bar{l}+\bar{q})^2+i\epsilon\right]}. \tag{2.63}
$$

Note that the massive propagators remain quadratic in the loop momentum.

The box integral with the linear massive propagators of HQET/HBET takes the form

$$I_{\text{HQET}} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 (l+q^2) \left[-v_1 \cdot (l+q) + i\epsilon\right] \left[v_2 \cdot (l+q) + i\epsilon\right]}. \tag{2.64}$$

We are concerned with addressing how the non-analytic pieces of the integrals in Eqs. (2.62) and (2.64) are related.

We see from Eq. (2.64) that the HQET integral is, up to a factor of $1/4m_1m_2$, the leading term of the integral in Eq. (2.62) when it has been expanded in $\hbar$ or $1/m$ — the equivalence of the two expansions is once again manifest. However, when including subleading terms in the expansion of Eq. (2.62), additional factors of $(l+q)^2$ appear in the numerator, cancelling one of the massless propagators. We conclude that all non-analytic contributions to Eq. (2.62) must be produced by the leading term of its expansion in $\hbar$ $(1/m)$. The same argument holds in the cases of triangle and crossed-box integrals, so the non-analytic pieces of integrals with quadratic massive propagators are reproduced (up to a factor of $2m$ for each propagator of mass m) by the HQET integrals. Another way of seeing why this should be the case is to invoke generalized unitarity. Upon cutting two massless propagators l^2 and $(l+q)^2$, there is no distinction between I_{quad} and I_{HQET}. Consequently, the one-loop integrals needed to perform the calculations in this paper are those in Ref. [52] with $p^\mu \to mv^\mu + k^\mu$ and multiplied by $2m$ for each massive propagator of mass m.[13]

2.E.2 Pinch singularity

HQET box integrals suffer from the so-called pinch singularity, which causes it to be ill-defined and means that HQET cannot be used to describe a bound state of two heavy particles beyond tree level. The cause of this issue is that, in such a scenario, the two heavy particles would have the same velocity, $v_1^\mu = v_2^\mu = v^\mu$. The HQET box integral in Eq. (2.62) then becomes

$$I_{\text{HQET}} = -\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 (l+q^2) \left[v \cdot (l+q) - i\epsilon\right] \left[v \cdot (l+q) + i\epsilon\right]}. \tag{2.65}$$

Any contour one tries to use to evaluate this integral is then "pinched" in the $\epsilon \to 0$ limit by the singularities above and below the real axis at $v \cdot (l+q) = 0$ [64].

For bound systems, the resolution is to reorganize the power counting expansion in terms of v/c instead of q/m. The resulting effective theory is non-relativistic QCD (NRQCD), which restores the quadratic pieces of the propagators. In the case at hand, however, we are considering the scattering of two unbound heavy particles, the crucial difference being that the velocities of the heavy particles are in general distinct, $(v_1^\mu \neq v_2^\mu)$. Thus, the HQET integral remains well defined.[14] Note that the limit where the HQET box integral becomes ill-defined $(v_1^\mu \to v_2^\mu)$ is precisely the limit in which the box integral with quadratic massive propagators obtains the singularity which is removed by the Born iteration [52].

[13]The integrals in Ref. [52] contain only IR and UV finite terms. It was shown in Ref. [17] that the interference of such terms does not contribute to the classical potential, so we have omitted them from our calculations.

[14]We thank Aneesh Manohar for discussions on this point.

CHAPTER

3 ON-SHELL HEAVY PARTICLE EFFECTIVE THEORIES

ABSTRACT: We introduce on-shell variables for Heavy Particle Effective Theories (HPETs) with the goal of extending Heavy Black Hole Effective Theory to higher spins and of facilitating its application to higher post-Minkowskian orders. These variables inherit the separation of spinless and spin-inclusive effects from the HPET fields, resulting in an explicit spin-multipole expansion of the three-point amplitude for any spin. By matching amplitudes expressed using the on-shell HPET variables to those derived from the one-particle effective action, we find that the spin-multipole expansion of a heavy spin-s particle corresponds exactly to the multipole expansion (up to order $2s$) of a Kerr black hole, that is, without needing to take the infinite spin limit. Finally, we show that tree-level radiative processes with same-helicity bosons emitted from a heavy spin-s particle exhibit a spin-multipole universality.

3.1 INITIATION

Despite the $\hbar$ counting and isolation of spin effects in the HBET action, computing amplitudes at higher loop orders using Feynman diagrammatic methods remains laborious. On top of this, if one is aiming to describe classical, spinning black holes, one must be able to capture all orders in the spin of each black hole. Using a QFT approach, this requires scattering particles with arbitrary spins. Though actions for such particles have been constructed – see refs. [65, 66] – they are themselves quite cumbersome, containing towers of auxiliary fields.

Both of these difficulties are assuaged in an on-shell approach. Unitarity cuts and recursion relations eliminate the need for Feynman diagrams, and knowledge of the Lorentz transformation properties of a spin-s particle facilitate its description using an on-shell language. With an eye to combining the perks of HBET with the computational advantages of on-shell methods, this publication presented on-shell variables specifically describing large-mass particles.

The key technology employed in this work was the massive spinor-helicity formalism of ref. [63] (see also refs. [67, 68]). We review the massless and massive spinor-helicity formalisms in this section and show how little group constraints fix the form of the three-point amplitude in both cases. The three-point amplitude is special because for certain theories – dubbed cut-constructable theories – it is sufficient for building all higher-point amplitudes through recursive techniques. We then summarize the main results of the publication.

3.1.1 The spinor-helicity formalism

The use of spinor-helicity variables greatly simplifies calculations and compactifies the forms of amplitudes. A famous example of this is the remarkable closed form of the maximally helicity violating n-gluon color-ordered amplitude[1] at tree level, given by the Parke-Taylor formula [69, 70]:

$$A(1^+ \dots i^- \dots j^- \dots n^+) = \frac{i\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle (n-1)n\rangle\langle n1\rangle}, \tag{3.1}$$

where the superscripts on the left-hand side represent the gluon helicities and the dots on the left-hand side contain labels for only positive helicity particles. Since gluons are massless, this amplitude is expressed in terms of spinor-helicity variables for massless particles, contractions of which form the $\langle\cdot\rangle$ brackets.

Both massless [71, 72] (see e.g. ref. [73] and the references therein for more details about the massless case) and massive [63, 67, 68] spinor-helicity variables are related to four-momenta by contracting with the Pauli matrix four-vector:

$$q_{\alpha\dot{\alpha}} = q^\mu(\sigma_\mu)_{\alpha\dot{\alpha}}, \quad q^{\dot{\alpha}\alpha} = q^\mu(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}. \tag{3.2}$$

Our conventions are stated in Section 3.A. The indices $\alpha, \dot{\alpha}$ represent the rows and columns of the Pauli matrices. Each index takes the values 1 or 2, and transforms under $SL(2,\mathbb{C})$.[2] Contracting the momentum with the Pauli matrix four-vector therefore re-expresses the $SO(1,3)$ Lorentz index structure of q^μ in terms of the double covering $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. The dot – or lack thereof – on top of the indices indicates that the index belongs to one or the other of the $SL(2,\mathbb{C})$. Belonging to separate groups, the undotted indices do not interact with the dotted ones. Lorentz invariant objects are now formed by contracting (un)dotted indices with one another.

Writing these momentum matrices explicitly,

$$q_{\alpha\dot{\alpha}} = \begin{pmatrix} q^0 - q^3 & -q^1 + iq^2 \\ -q^1 - iq^2 & q^0 + q^3 \end{pmatrix}, \quad q^{\dot{\alpha}\alpha} = \begin{pmatrix} q^0 + q^3 & q^1 - iq^2 \\ q^1 + iq^2 & q^0 - q^3 \end{pmatrix}, \tag{3.3}$$

it's clear that $q_\mu q^\mu = \det q_{\alpha\dot{\alpha}} = \det q^{\dot{\alpha}\alpha}$. At this point, the treatment of massless and massive momenta diverges. We consider each case separately.

Massless momenta

When the momentum q^μ is massless, these matrices will have vanishing determinant, and therefore cannot have full rank. On the other hand, a physical four-momentum can never be identically 0: a massive four-momentum will always have non-zero temporal component, while a massless four-momentum can additionally never have vanishing three-momentum. Combining these two facts means that the matrices $q_{\alpha\dot{\alpha}}$ and $q^{\dot{\alpha}\alpha}$ must have rank 1 if $q_\mu q^\mu = 0$. This allows us to decompose each matrix as an outer product

[1] See Section 4.1.1 for a definition of color-ordered amplitudes.

[2] Transforming the indices under $GL(2,\mathbb{C})$ could change the invariant mass of the momentum q^μ, so such a transformation would not represent a physical Lorentz transformation.

of two two-dimensional vectors:

$$q_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \equiv |q\rangle_\alpha [q|_{\dot{\alpha}}, \quad q^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}} \lambda^\alpha \equiv |q]^{\dot{\alpha}} \langle q|^\alpha, \tag{3.4}$$

with indices raised and lowered using the two-dimensional Levi-Civita symbol.[3] These λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ are the massless spinor-helicity variables, or helicity variables for short.

Helicity variables satisfy the on-shell conditions

$$q_{\alpha\dot{\alpha}} |q]^{\dot{\alpha}} = \langle q|^\alpha q_{\alpha\dot{\alpha}} = [q|_{\dot{\alpha}} q^{\dot{\alpha}\alpha} = q^{\dot{\alpha}\alpha} |q\rangle_\alpha = 0. \tag{3.5}$$

Each condition can be easily verified by writing $q_{\alpha\dot{\alpha}}, q^{\dot{\alpha}\alpha}$ in terms of the helicity variables and noting that contractions are antisymmetric since indices are raised and lowered using the Levi-Civita symbol. As a result of manifesting these conditions, the helicity variables are sometimes referred to as on-shell variables. The on-shell conditions are an expression of the massless Dirac equation, as can be seen by writing the helicity spinors in terms of left- and right-handed projections of Dirac spinors [74]:

$$|q\rangle_\alpha = \frac{1}{2}(\mathbb{I} - \gamma_5) u(q), \quad |q]^{\dot{\alpha}} = \frac{1}{2}(\mathbb{I} + \gamma_5) u(q). \tag{3.6}$$

These relations imply that the λ_α spinors carry helicity $-1/2$ and the $\tilde{\lambda}_{\dot{\alpha}}$ carry $+1/2$.

Equation (3.4) does not uniquely define the helicity variables. The reason for this is the little group of the momentum q_μ, the subgroup of Lorentz transformations under which q_μ is invariant. A massless four-momentum is unchanged by rotations in the plane perpendicular to the direction of its three-momentum, so its little group is $SO(2) = U(1)$. This means that the helicity variables are defined up to some overall phase, transforming as

$$\lambda \to t\lambda, \quad \tilde{\lambda} \to t^{-1}\tilde{\lambda}, \tag{3.7}$$

for $t = e^{-i\theta/2} \in U(1)$. Combining the transformation properties, the rescaling of a spinor with helicity h is given by t^{-2h}.

We embark now on a brief but essential aside. The little group tells us something about the composition of an amplitude containing particles with any mass. Particle states (as opposed quantum fields) transform in irreducible representations of the little groups corresponding to their momenta [75–77]. This means that the external states entering an amplitude are covariant under little group transformations; for example, consider the polarization vector for a massless spin-1 particle [74]:

$$\epsilon^+_{\alpha\dot{\alpha}}(q, \xi) = \sqrt{2} \frac{|\xi\rangle_\alpha [q|_{\dot{\alpha}}}{\langle \xi q \rangle}, \quad \epsilon^-_{\alpha\dot{\alpha}}(q, \xi) = \sqrt{2} \frac{|q\rangle_\alpha [\xi|_{\dot{\alpha}}}{[q\xi]}. \tag{3.8}$$

Here ξ^μ is a reference momentum encoding gauge transformations. These polarization vectors describe particles with helicities ± 1 and rescale as $t^{\mp 2}$ respectively under a transformation in the little group of q^μ. Since vertices and propagators are comprised of four-momenta and their contractions, little group weights can only be carried by external states. Moreover, a particle can only carry its own little group weight. Putting everything together, we conclude that an amplitude must be separately covariant under

[3] A factor of the Levi-Civita symbol exists in one of the two $SL(2, \mathbb{C})$. This means that both indices on a Levi-Civita symbol must either be dotted or undotted.

the little groups of all particles involved in the scattering. Heuristically, little group covariance can be thought of as the amplitude depending on the polarizations of the initial and final state particles.

Specializing again to the massless case, the irreducible representations of the little group are all one-dimensional and are labelled by integers and half-integers. The (half-)integer labelling a representation corresponds to the spin of the massless particle. Then, by the above, a massless particle state with spin s transforms as $\epsilon^{\pm s} \to t^{\mp 2s}\epsilon^{\pm s}$ depending on its helicity $h = \pm s$. Connecting this to amplitudes, if a massless spin-s particle is involved in a scattering process, the amplitude describing the scattering must rescale as t^{-2h} when the helicity variables are rescaled under the particle's little group.

The little-group scaling of the amplitude, combined with some physical considerations, is sufficient to fix the massless three-point amplitude, up to an overall constant. Let us outline the argument for this following refs. [67, 73]. First, momentum conservation at three points is restrictive enough to set either all angle- or all square-bracket contractions to 0. Taking all momenta incoming, momentum conservation is $q_1^\mu + q_2^\mu = -q_3^\mu$. Squaring both sides of this – keeping in mind that the momenta are massless – yields $\langle 12\rangle[21] = 0$. Supposing we choose to keep the angle bracket non-zero, the square bracket $[12]$ must therefore be the vanishing factor. It is easy to show that the remaining two square brackets $[23]$ and $[31]$ also vanish by considering the quantities $\langle 12\rangle[23]$ and $[31]\langle 12\rangle$.

In our chosen setup, where all square brackets vanish, a three-point amplitude is forced to take the form

$$A(1^{h_1}2^{h_2}3^{h_3}) \propto \langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3}. \tag{3.9}$$

Choosing instead to keep square brackets and allow angle brackets to vanish, we would have been lead to conclude that

$$A(1^{h_1}2^{h_2}3^{h_3}) \propto [12]^{-h_3+h_1+h_2}[13]^{-h_2+h_1+h_3}[23]^{-h_1+h_2+h_3}, \tag{3.10}$$

where the exponents switched signs because the square brackets transform in the opposite manner to the angle brackets under the little group.

Now, by looking at the explicit kinematics for the helicity variables (e.g. in ref. [74]), it is clear that for real momenta the angle and square spinors are conjugates to one another. This means that in real three-point kinematics we cannot choose one type of bracket to vanish while keeping the other type non-zero. For real momenta, the order of the pole/root in eq. (3.9) is $-h_1 - h_2 - h_3$, while that of eq. (3.10) is $h_1 + h_2 + h_3$. In the interest of keeping the amplitude finite in all cases, we are motivated to choose eq. (3.9) when $h_1 + h_2 + h_3 < 0$, and eq. (3.10) otherwise. If all helicities have the same sign, the three-point amplitude vanishes whether or not the momenta are real.

Allowing for complex kinematics, the complex-conjugation relation between the square and angle brackets no longer holds. In this case, one can have one of eqs. (3.9) and (3.10) be non-vanishing. Though unphysical on their own, three-point amplitudes with complex kinematics appear ubiquitously in the recursive computation of physical, higher-point amplitudes.

When one or more of the particles in the three-point amplitude are massive the amplitude will no longer be unique, but it can still be written in a general form. The three-point amplitude involving two equal-mass legs and one massless leg played a central role in the results of this publication. Let us then shift our focus to the treatment of massive momenta using on-shell variables.

Massive momenta

If the momentum q^μ is massive the matrices $q_{\alpha\dot{\alpha}}$ and $q^{\dot{\alpha}\alpha}$ have rank 2 and we can no longer perform the decomposition in eq. (3.4). However, using the fact that a rank-n matrix can be written as a sum of n rank-1 matrices, we can still decompose the momentum matrices in a very similar way to eq. (3.4) [63, 67, 68]:

$$q_{\alpha\dot{\alpha}} = \lambda_\alpha^1 \tilde{\lambda}_{1\dot{\alpha}} + \lambda_\alpha^2 \tilde{\lambda}_{2\dot{\alpha}} \equiv \lambda_\alpha^I \tilde{\lambda}_{I\dot{\alpha}}, \quad q^{\dot{\alpha}\alpha} = \tilde{\lambda}_I^{\dot{\alpha}} \lambda^{I\alpha}, \tag{3.11}$$

where $I = 1, 2$. An angle- and square-bracket notation analogous to the massless case is also employed with the massive on-shell variables λ^I and $\tilde{\lambda}_I$.

Thinking of the two-indexed objects λ_α^I and $\tilde{\lambda}_{I\dot{\alpha}}$ as being 2×2 matrices, the mass m of the momentum q^μ can be written in terms of the determinants of these matrices:

$$m^2 = \det \lambda_\alpha^I \det \tilde{\lambda}_{I\dot{\alpha}} = \det \lambda^{I\alpha} \det \tilde{\lambda}_I^{\dot{\alpha}}. \tag{3.12}$$

There is a freedom in the values of the determinants in the second two expressions above. However, we do not need to make a distinction between these two determinants, so we simply fix $\det \lambda = \det \tilde{\lambda} = m$. With this choice, these variables obey the on-shell conditions in eq. (3.133).

The I indices of λ and $\tilde{\lambda}$ transform oppositely and therefore leave the momentum q^μ unchanged. For physical applications, then, the on-shell variables transform under the little group of the massive momentum. By going to the rest frame of the massive particle, it can be seen that the subset of Lorentz transformations leaving the three-momentum invariant are the rotations in three dimensions. This means the little group for a massive particle is $SO(3) \simeq SU(2)/\mathbb{Z}_2$. The I are therefore $SU(2)$ little group indices.[4] They can be raised or lowered with the two-dimensional Levi-Civita symbol of $SU(2)$.

Now, as discussed above, amplitudes transform covariantly under the little groups of the external states. Since a particle can only carry its own little group indices, contractions cannot occur between little group indices of different particles. This means that the little group index structure of an amplitude is determined solely by the identities of the external particles, and is unaffected by the specifics of the scattering. What remains is thus for us to understand the little group indices carried by an arbitrary massive particle.

Recall that a massive particle with spin s transforms in an irreducible representation of its little group, $SU(2)$ [68, 75–77]. The irreducible representations of $SU(2)$ are furnished by tensors with definite symmetry properties. But we can say more about the transformation properties of a spin-s particle. Consider the space of totally symmetric $SU(2)$ tensors with rank $2s$. We argue that this is the correct representation for a spin-s particle. The first way to see this is on the grounds of the dimension of the representation. This dimension is the number of independent components of a totally symmetric, rank-$2s$ tensor, equal to the number of ways of choosing $2s$ items from a set of 2 where order is ignored and repetition is allowed. The sought-after dimension is thus

$$\binom{2 + 2s - 1}{2s} = 2s + 1, \tag{3.13}$$

[4] Ref. [63] points out that the I indices can generally transform under $GL(2, \mathbb{C})$. This transformation group is restricted to $SL(2, \mathbb{C})$ if $\det \lambda$ and $\det \tilde{\lambda}$ are held fixed, and further to $SU(2) \subset SL(2, \mathbb{C})$ to leave the momentum unchanged.

precisely as expected for the representation of a spin-s particle.

Second, we can confirm that such a tensor carries the correct (non-relativistic) spin by acting on the tensor with the square of the generator of rotations, as done in ref. [63].[5] Filling in the details of this computation, the generator of rotations in the $2s+1$ dimensional representation takes the form

$$\vec{J} = \frac{1}{2}\vec{\sigma}\otimes\underbrace{\mathbb{I}\otimes\cdots\otimes\mathbb{I}}_{2s-1}+\underbrace{\ldots}_{2s-2}+\underbrace{\mathbb{I}\otimes\cdots\otimes\mathbb{I}}_{2s-1}\otimes\frac{1}{2}\vec{\sigma}, \tag{3.14}$$

since a rank-$2s$ tensor can be thought of as the direct product of $2s$ rank-1 tensors. Here, $\vec{\sigma}$ is the vector of Pauli matrices. Its action on the tensor $T_{i_1\ldots i_{2s}}$ is

$$\left(\vec{J}T\right)_{i_1\ldots i_{2s}} = \frac{1}{2}\vec{\sigma}^{j_1}_{i_1}T_{j_1 i_2\ldots i_{2s}} + \cdots + \frac{1}{2}\vec{\sigma}^{j_{2s}}_{i_{2s}}T_{i_1\ldots i_{2s-1}j_{2s}}. \tag{3.15}$$

Acting a second time,

$$\begin{aligned}\left(\vec{J}^2T\right)_{i_1\ldots i_{2s}} =& \left(\frac{1}{4}\vec{\sigma}^{k_1}_{i_1}\cdot\vec{\sigma}^{j_1}_{k_1}T_{j_1 i_2\ldots i_{2s}} + \cdots + \frac{1}{4}\vec{\sigma}^{k_{2s}}_{i_{2s}}\cdot\vec{\sigma}^{j_1}_{i_1}T_{j_1\ldots i_{2s-1}k_{2s}}\right) + \ldots \\ &+ \left(\frac{1}{4}\vec{\sigma}^{k_1}_{i_1}\cdot\vec{\sigma}^{j_{2s}}_{i_{2s}}T_{k_1\ldots i_{2s-1}j_{2s}} + \cdots + \frac{1}{4}\vec{\sigma}^{k_{2s}}_{i_{2s}}\cdot\vec{\sigma}^{j_{2s}}_{k_{2s}}T_{i_1\ldots i_{2s-1}j_{2s}}\right).\end{aligned} \tag{3.16}$$

Now, there are $2s$ parentheses here, each of which contains $2s$ terms. Each parenthesis contains exactly one term where the Pauli matrices are contracted with one another, and $2s-1$ where they are not. Using the identity $\vec{\sigma}^i_j\cdot\vec{\sigma}^k_l = 2\delta^i_l\delta^k_j - \delta^i_j\delta^k_l$, the terms with contracted Pauli matrices will sum up to $3\cdot\frac{1}{4}\cdot 2sT_{i_1\ldots i_{2s}}$, while those with no such contraction give $1\cdot\frac{1}{4}\cdot 2s\cdot(2s-1)T_{i_1\ldots i_{2s}}$. Adding the two types of contributions, we find

$$\left(\vec{J}^2T\right)_{i_1\ldots i_{2s}} = s(s+1)T_{i_1\ldots i_{2s}}, \tag{3.17}$$

corroborating that such a tensor transforms in the spin-s representation of $SU(2)$.

Concluding this point, the external state for a massive spin-s particle carries $2s$ totally-symmetrized little group indices. As such, an amplitude inherits these $2s$ indices, and is totally symmetric in the little group indices of each massive particle separately. When writing an amplitude explicitly in terms of these massive on-shell variables, the symmetrization over all the little group indices can be cumbersome. Addressing this inconvenience, a bold notation – i.e. $\langle\mathbf{pq}\rangle$ – was introduced in ref. [63] as a stand-in for explicit symmetrization. See Section 3.A for an example.

We wrap up this section with a discussion of the three-point amplitude with two equal-mass legs and one massless one. Let us begin in this case by again considering the three-particle kinematics. If all momenta are incoming, momentum conservation is $p_1^\mu + q^\mu = -p_2^\mu$, where q^μ is massless and the others have mass m. Squaring both sides gives that p_1^μ and q^μ must be orthogonal. Re-expressing this orthogonality in on-shell variables,

$$\langle q|^\alpha p_{1\alpha\dot{\alpha}}|q]^{\dot{\alpha}} = 0. \tag{3.18}$$

[5]Note that the relativistic spin appearing in computations is defined in terms of the generator of the whole Lorentz group, not the generators of $SU(2)$. Acting with the $\vec{J}$ generators of $SU(2)$ can be thought of as measuring the total spin of the particle in its rest frame. The covariant spin vector is reached by boosting the rest-frame spin vector; see refs. [29, 78].

This means that the massless angle bracket $|q\rangle_\alpha$ is proprotional to $p_{1\alpha\dot{\alpha}}|q]^{\dot{\alpha}}$. Through this proportionality, we can define the constant x [63]:

$$x|q\rangle_\alpha = \frac{1}{m} p_{1\alpha\dot{\alpha}}|q]^{\dot{\alpha}}. \tag{3.19}$$

Inverting this definition by noting that $p_1^{\dot{\beta}\alpha} p_{1\alpha\dot{\alpha}} = m^2 \mathbb{I}^{\dot{\beta}}{}_{\dot{\alpha}}$ yields an analogous expression for x^{-1}:

$$x^{-1}|q]^{\dot{\alpha}} = \frac{1}{m} p_1^{\dot{\alpha}\alpha}|q\rangle_\alpha. \tag{3.20}$$

When the constant x and its inverse appear in computations they stand for non-local forms of eqs. (3.19) and (3.20), obtained by contracting these equations with a reference spinor and isolating the constant.

Suppose now, as is relevant for our purposes, that the two massive legs have spin s, and the massless leg carries helicity h. We outline the derivation of the most general three-point amplitude for such a configuration given in refs. [46, 63] (an alternative derivation of this amplitude can be found in ref. [67]). The on-shell conditions allow us to convert all spinors in the amplitude to one type, say λ_α^I. Each massive leg carries $2s$ such spinors and, as explained above, the spinors appear in the amplitude with their little group indices symmetrized. We can therefore factor out the spinors for each leg to leave behind a spinor-stripped amplitude with $2s$ symmetrized $SL(2,\mathbb{C})$ indices for each leg:

$$M^{\{I_1,\ldots,I_{2s}\},\{J_1,\ldots,J_{2s}\},h} = \lambda_{1\alpha_1}^{I_1} \ldots \lambda_{1\alpha_{2s}}^{I_{2s}} \lambda_{2\beta_1}^{J_1} \ldots \lambda_{2\beta_{2s}}^{J_{2s}} M^{\{\alpha_1,\ldots,\alpha_{2s}\},\{\beta_1,\ldots,\beta_{2s}\},h}. \tag{3.21}$$

Curly brackets represent symmetrization. We have explicitly included a label for the helicity of the massless particle.

Our task is thus to determine the most general form of the spinor-stripped amplitude. This can be done given two independent $SL(2,\mathbb{C})$ objects to span the space of each index. In this setup, the only relevant (i.e. related to the quantities in the problem) objects we have, on account of the on-shell condition, are $\langle q|^\alpha$ and $\epsilon^{\alpha\beta}$. Moreover, since the spinor-stripped amplitude is symmetric in each set of the spinor-indices, any contribution where the Levi-Civita carries two indices from the same set vanishes. Finally, we have a quantity that carries the helicity weight of the massless particle: $x^{\pm 1}$ carries helicity weight ± 1. All things considered, the spinor-stripped amplitude takes the most general form

$$M^{\{\alpha_1,\ldots,\alpha_{2s}\},\{\beta_1,\ldots,\beta_{2s}\},h} = x^h \left[c_0 \epsilon^{2s} + c_1 \epsilon^{2s-1} \frac{x\langle q|\langle q|}{m} + \cdots + c_{2s} \left(\frac{x\langle q|\langle q|}{m} \right)^{2s} \right]^{\{\alpha_1,\ldots,\alpha_{2s}\},\{\beta_1,\ldots,\beta_{2s}\}}. \tag{3.22}$$

Returning the massive spinors, accounting for the little group symmetrization by bolding, and absorbing negative signs into the coefficients,

$$M^{\{I_1,\ldots,I_{2s}\},\{J_1,\ldots,J_{2s}\},h} = x^h \left[c_0 \langle \mathbf{21} \rangle^{2s} + c_1 \langle \mathbf{21} \rangle^{2s-1} \frac{x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle}{m} + \cdots + c_{2s} \left(\frac{x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle}{m} \right)^{2s} \right]. \tag{3.23}$$

This is the most general form of the three-point amplitude involving two equal-mass and equal-spin massive legs and a massless leg with helicity h.

As in the massless case, this three-point amplitude is unphysical for real kinematics. To see this, consider an annihilation of two massive particles in the center-of-momentum frame that results in one massless particle. The total momentum in this frame is zero before the annihilation. However, since the final state is massless it cannot be created at rest, violating momentum conservation. Despite the non-physicality of this amplitude, it can still be used to build physical higher-point amplitudes recursively.

3.1.2 Overview of main results

The primary purpose of this publication was to introduce on-shell variables to describe heavy on-shell states. Such a definition was achieved in terms of the traditional massive on-shell variables presented above:

$$\begin{pmatrix} |\mathbf{p}_v\rangle \\ |\mathbf{p}_v] \end{pmatrix} = \left(\mathbb{I} - \frac{\not{k}}{2m}\right) \begin{pmatrix} |\mathbf{p}\rangle \\ |\mathbf{p}] \end{pmatrix}. \tag{3.24}$$

Here, k^μ is the residual momentum of the heavy particle and m is its mass.

In terms of these variables, the most general three-point amplitude for the emission of a boson of helicity h from a massive particle of spin s cleanly separates spin and spinless effects:

$$\mathcal{M}_3^{+|h|,s} = (-1)^{2s+h} \frac{x^{|h|}}{m^{2s}} \sum_{k=0}^{2s} g^{\rm H}_{s,k} \underbrace{\langle \mathbf{2}_v \mathbf{1}_v\rangle}_{\text{spinless}}{}^{2s-k} \underbrace{\left(\frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q \mathbf{1}_v\rangle\right)^k}_{\text{spin}}, \tag{3.25a}$$

$$\mathcal{M}_3^{-|h|,s} = (-1)^{h} \frac{x^{-|h|}}{m^{2s-1}} \sum_{k=0}^{2s} \tilde{g}^{\rm H}_{s,k} \underbrace{[\mathbf{2}_v \mathbf{1}_v]}_{\text{spinless}}{}^{2s-k} \underbrace{\left(\frac{x^{-1}}{2m}[\mathbf{2}_v q][q \mathbf{1}_v]\right)^k}_{\text{spin}}. \tag{3.25b}$$

This is in contrast to the traditional massive on-shell variables, where spin and spinless effects are mixed by the on-shell conditions. By matching the minimally coupled[6] version of the heavy three-point amplitude to the one derived from the worldline action for a spinning object, we showed that we were able to exactly reproduce the spin-multipole expansion of a Kerr black hole. This is a direct consequence of the separation of spin and spinless effects.

Finally, we computed amplitudes at tree level. In these variables, it was straightforward to show that the emission of n same-helicity photons or gravitons from a massive spin-s particle is described by an amplitude that exhibits a spin universality:

$$\begin{aligned} M^s_{n+2} &= \frac{(-1)^{nh}}{m^{2s}} M^{s=0}_{n+2} \langle \mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^n q_i \cdot S\right] |\mathbf{1}_v\rangle^{2s} \\ &= \frac{(-1)^{nh}}{m^{2s}} M^{s=0}_{n+2} [\mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^n q_i \cdot S\right] |\mathbf{1}_v]^{2s}. \end{aligned} \tag{3.26}$$

Along the same lines, we demonstrated that the leading-in-$\hbar$ portion of the opposite-helicity Compton

[6]Note that, as explained in the publication, minimal coupling in this case has a different meaning than in Chapter 2.

amplitude is

$$\mathcal{M}^{\text{cl.}}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-2}, q_4^{+2}) = \frac{(-1)^{2s}}{m^{2s}}\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{+2})\langle\mathbf{2}_v|^{2s}\exp\left[\frac{(q_4 - q_3 + w)\cdot S}{m}\right]|\mathbf{1}_v\rangle^{2s}, \tag{3.27}$$

for gravity. The definition of the unphysical four-momentum w^μ, eq. (3.106d), shows that the spurious singularity of the opposite-helicity Compton amplitude (see refs. [46, 63]) is not avoided by taking the $\hbar \to 0$ limit.

3.2 INTRODUCTION

The relationship between quantum scattering amplitudes and classical physics has enjoyed a surge of attention in recent years, in large part due to the observation of gravitational waves by the LIGO and Virgo collaborations as of 2015 [1]. Motivating studies in this direction has been the realization that perturbative techniques from quantum field theory are well suited to the computation of the complementary post-Newtonian (PN) and post-Minkowskian (PM) expansions of the binary inspiral problem in General Relativity (GR). Indeed, the effective field theory (EFT) of GR [7, 8] has been used extensively to compute classical corrections to the gravitational potential [8, 15, 23, 40–43, 46, 79]. Furthermore, effective-field-theoretic methods have been used to develop EFTs for gravitationally interacting objects whose operator expansions are tailored to computing terms in the PN approximation [36–39, 80]. In fact, using EFT methods, the entire 4PN spinless conservative dynamics were derived in refs. [81, 82], and the computation of the 5PN spinless conservative dynamics was approached in refs. [83, 84]. Including spin, the current state-of-the-art computations from the PN approach were performed in refs. [85, 86] using the EFT of ref. [80].

On the PM side, it has also recently been shown that quantum scattering amplitudes can be used to extract fully relativistic information about the classical scattering process [10, 14, 16, 17, 45, 47–49, 87–90]. Moreover, a direct relationship between the scattering amplitude and the scattering angle has been uncovered in refs. [31, 32, 50, 91].[7] All of these developments suggest that the $2 \to 2$ gravitational scattering amplitude encodes information that is crucial for the understanding of classical gravitational binary systems, to all loop orders [7, 9].

Various methods exist for identifying the classical component of a scattering amplitude [10, 14, 42, 43]. Towards this same end, Heavy Black Hole Effective Theory (HBET) was recently formulated by Damgaard and two of the present authors in ref. [92] with the aim of streamlining the extraction of classical terms from gravitational scattering amplitudes. It was shown there that the operator expansion of HBET is equivalent to an expansion in $\hbar$. Exploiting this fact, the authors were able to identify which HBET operators can induce classical effects at arbitrary loop order, and the classical portion of the $2 \to 2$ amplitude was computed up to one-loop order for spins $s \leq 1/2$. These results were obtained using Lagrangians and Feynman diagram techniques which, while tractable at the perturbative orders and spins considered, become non-trivial and computationally unwieldy to extend to higher spins or loop orders. Nevertheless, the separation of classical and quantum effects and the observed separation of spinless and spin-inclusive effects are desirable features of the EFT that will prove quite convenient

[7] We thank Andrea Cristofoli for bringing earlier work on this relationship to our attention.

when cast as part of a more user-friendly formalism.

We aim in this paper to present such a formalism that will allow the extension of HBET to higher spins and to facilitate its application to higher loop orders. A means to do so comes in the formalism presented in ref. [63]. Spinor-helicity variables were presented there that describe the scattering of massive matter with arbitrary spin. Based solely on kinematic considerations, these variables were used to construct the most general three-point amplitude for a massive spin-s particle emitting a massless boson with a given helicity.[8] In this most general amplitude, the term that is best behaved in the UV limit is termed the minimal coupling amplitude. When $s \leq 1/2$ it reduces to the three-point amplitude arising from the relevant Lagrangian that is minimally coupled in the sense of covariantized derivatives. This terminology is preserved for higher spins; the minimal coupling amplitude for a general spin-s particle is a tensor product of $2s$ factors of spin-1/2 minimal coupling amplitudes. Note that this definition of minimal coupling generally differs from the typical definition from the Lagrangian perspective. Phenomenologically, these minimal coupling amplitudes are those that produce a gyromagnetic ratio of $g = 2$ for all spins [46, 93, 94].

This minimal coupling amplitude has proven to be quite useful in the study of classical Kerr black holes, which have been shown to couple minimally to gravity [43, 45–47, 53, 95]. Such a description of Kerr black holes is in fact not immediately exact when using the variables of ref. [63] due to the difference between the momenta of the initial and final states, leading to an ill-defined matrix element of the spin-operator. This gap has been overcome using various methods in the above references. However we will show that expressing the degrees of freedom of HBET in on-shell variables reduces the discrepancy to a mere choice of the kinematics. The appropriate kinematics can sometimes be imposed (when a process is described by diagrams with no internal matter lines), but are always recovered in the classical limit; $\hbar \to 0$.

In this paper, we express the asymptotic states of Heavy Particle Effective Theories (HPETs) — the collection of effective field theories treating large mass particles — using the massive on-shell spinor-helicity variables of ref. [63]. An explicit $\hbar$ expansion will arise from these variables, which makes simple the task of taking classical limits of amplitudes. Such an expression of the asymptotic states of HPET will also lead to an explicit separation of spinning and spinless effects in the three-point minimal coupling amplitude. From the lens of the classical gravitational scattering of two spinning black holes, this results in the finding that the asymptotic states of HPET are naturally identified with a Kerr black hole with truncated spin-multipole expansion.

Our construction will also allow us to gain insight into this class of effective field theories. We will derive a conjecture for the three-point amplitude arising from an arbitrary HPET, and posit a form for this same amplitude for heavy matter of any spin. Then, in the appendices, we comment on the link between reparameterization invariance of a momentum and its little group, and finally compute the operator projecting onto a heavy particle of spin $s \leq 2$, the derivation of which can be extended to general spin.

The layout of this paper is as follows. We begin with a very brief review of HPETs in Section 3.3. Also, we introduce on-shell variables that describe the heavy field. The three-point amplitudes of HPETs are analyzed in Section 3.4. In particular, we construct the three-point amplitude of HPET resummed to all orders in the expansion parameter. Furthermore, the construction of ref. [63] provides a method of

[8]For alternative approaches to the application of spinor-helicity variables to massive spinning particles see e.g. refs. [67, 68].

extending HPET amplitudes to arbitrary spin. In Section 3.5, we interpret the on-shell HPET variables as Kerr black holes with truncated spin-multipole expansions, and show that heavy spin-s particles possess the same spin-multipole expansion as a Kerr black hole, up to the $2s^{\text{th}}$ multipole. This is in contrast to previous work [46, 53], which found that minimally coupled particles possess the same spin multipoles as Kerr black holes only in the infinite spin limit. Section 3.6 is dedicated to the computation of on-shell amplitudes, and we show the simplicity of taking the classical limit of an amplitude when it is expressed in on-shell HPET variables. The main body of the paper is concluded in Section 3.7. Our conventions are summarized in Appendix 3.A. The question of the uniqueness of the constructed variables is addressed in Appendix 3.B. We then relate the little group of a momentum p to its invariance under the HPET reparameterization (see Section 3.3) in Section 3.C. In Appendix 3.D we use spin-s polarization tensors for heavy particles to explicitly construct propagators and projection operators for heavy particles with spins $s \leq 2$. We then use these results to conjecture the forms of the projection operators for arbitrary spin. Finally, we describe in Appendix 3.E the forms of the spin-1/2 HPET Lagrangians that must be used to match to the on-shell minimal coupling amplitudes. We also show there that the three-point amplitude derived from a Lagrangian for a heavy spin-1 particle is reproduced by the extension of the variables to arbitrary spin in Section 3.4.

3.3 EFFECTIVE THEORIES WITH HEAVY PARTICLES

When describing a scattering process in which the transfer momentum, q^μ, is small compared to the mass of one of the scattered particles, m, we can exploit the separation of scales by expanding in the small parameter $|q|/m$. Heavy Quark Effective Theory (HQET) [25–27] is the effective field theory that employs this expansion in the context of QCD, with HBET being its gravitational analog. Central to the separation of scales is the decomposition of the momentum of the heavy particle as

$$p^\mu = mv^\mu + k^\mu, \tag{3.28}$$

where v^μ is the (approximately constant) four-velocity ($v^2 = 1$) of the heavy particle, and k^μ is a residual momentum that parameterizes the energy of the interaction; it is therefore comparable in magnitude to the momentum transfer, $|k^\mu| \sim |q^\mu|$. When decomposed in this way, the on-shell condition, $p^2 = m^2$, is equivalent to

$$v \cdot k = -\frac{k^2}{2m}. \tag{3.29}$$

As was argued in ref. [92], using results from ref. [10], the residual momentum scales with $\hbar$ in the limit $\hbar \to 0$. We discuss the counting of $\hbar$ in Section 3.6.1.

With some background about the construction and motivation behind HPETs, we introduce in this section on-shell variables that describe spin-1/2 HPET states. Then, the transformation of these variables under a reparameterization of the momentum eq. (3.28) is given. We end the section by defining the spin operator for heavy particles.

3.3.1 On-shell HPET variables

The spinors $u_v^I(p)$ that describe the particle states of HPET are related to the Dirac spinors $u^I(p)$ via [92]

$$u_v^I(p) = \left(\frac{\mathbb{I} + \not{v}}{2}\right) u^I(p) = \left(\mathbb{I} - \frac{\not{k}}{2m}\right) u^I(p), \tag{3.30}$$

where I is an $SU(2)$ little group index, and v^μ and k^μ are defined in eq. (3.28). The operator $P_+ \equiv \frac{1+\not{v}}{2}$ is the projection operator that projects on to the heavy particle states. Writing the Dirac spinor in terms of massive on-shell spinors $|\mathbf{p}\rangle_\alpha$ and $|\mathbf{p}]^{\dot{\alpha}}$, we define on-shell variables for the HPET spinor field:

$$\begin{pmatrix} |\mathbf{p}_v\rangle \\ |\mathbf{p}_v] \end{pmatrix} = \left(\mathbb{I} - \frac{\not{k}}{2m}\right) \begin{pmatrix} |\mathbf{p}\rangle \\ |\mathbf{p}] \end{pmatrix}. \tag{3.31}$$

The bold notation for the massive on-shell spinors was introduced in ref. [63], and represents symmetrization over the little group indices. We refer to the on-shell variables of ref. [63] as the traditional on-shell variables, and those introduced here as the on-shell HPET variables. The on-shell HPET variables are labelled by their four-velocity v. We emphasize that the relation between the traditional and HPET on-shell variables is exact in k/m. See Section 3.A for conventions.

When working with heavy particles, the Dirac equation is replaced by the relation $\not{v}u_v^I = u_v^I$, which can be seen by multiplying the first equation in eq. (3.30) by $\not{v}$. This relates the on-shell HPET variables in different bases through

$$v_{\alpha\dot{\beta}}|\mathbf{p}_v]^{\dot{\beta}} = |\mathbf{p}_v\rangle_\alpha, \quad v^{\dot{\alpha}\beta}|\mathbf{p}_v\rangle_\beta = |\mathbf{p}_v]^{\dot{\alpha}}, \tag{3.32a}$$

$$[\mathbf{p}_v|_{\dot{\alpha}}v^{\dot{\alpha}\beta} = -\langle\mathbf{p}_v|^\beta, \quad \langle\mathbf{p}_v|^\alpha v_{\alpha\dot{\beta}} = -[\mathbf{p}_v|_{\dot{\beta}}. \tag{3.32b}$$

We associate the momentum p_v^μ with the on-shell HPET spinors, where

$$\not{p}_v = \begin{pmatrix} 0 & |p_v\rangle^I{}_I[p_v| \\ |p_v]_I{}^I\langle p_v| & 0 \end{pmatrix} = m_k\not{v}, \tag{3.33}$$

and

$$m_k \equiv \left(1 - \frac{k^2}{4m^2}\right) m. \tag{3.34}$$

We see that the momentum p_v^μ is proportional to v^μ, regardless of the residual momentum. The momentum p_v^μ is related to the momentum p^μ through

$$P_+\not{p}_v = P_+\not{p}P_+. \tag{3.35}$$

The on-shell HPET variables naturally describe heavy particles in a context with no anti-particles. To see this, note that the relation between the HPET spinor and the Dirac spinor in eq. (3.31) can be inverted [61]

$$u^I(p) = \left(\mathbb{I} - \frac{\not{k}}{2m}\right)^{-1} u_v^I(p)$$

$$= \left[1 + \frac{1}{2m}\left(1 + \frac{k\cdot v}{2m}\right)^{-1}(\not{k} - k\cdot v)\right] u_v^I(p). \tag{3.36}$$

In the free theory, this corresponds to the relation between the fields in the full and effective theories once the heavy anti-field has been integrated out by means of its equation of motion. Thus, eq. (3.30) is equivalent to integrating out heavy anti-particle states.

3.3.2 Reparameterization

There is an ambiguity in the choice of v and k in the decompositon of the momentum in eq. (3.28). The momentum is invariant under reparameterizations of v and k of the forms

$$(v, k) \to (w, k') \equiv \left(v + \frac{\delta k}{m}, k - \delta k\right), \tag{3.37}$$

where $|\delta k|/m \ll 1$ and $(v + \delta k/m)^2 = 1$. Given that observables can only depend on the total momentum, observables computed in heavy particle effective theories must be invariant under this reparameterization [59–61]. In particular, the S-matrix is reparameterization invariant.

The on-shell HPET variables transform under the reparameterization of the momentum in eq. (3.37). The HPET spinors $u_v^I(p)$ and $u_w^I(p)$ are related through

$$\begin{aligned} u_v^I(p) &= \frac{1+\not{v}}{2} u^I(p) \\ &= \frac{1+\not{v}}{2}\left[1 + \frac{1}{2m}\left(1 + \frac{k'\cdot w}{2m}\right)^{-1}(\not{k}' - k'\cdot w)\right] u_w^I(p), \end{aligned} \tag{3.38}$$

where the second line is simply eq. (3.36) with $(v, k) \to (w, k')$. Rewriting this in terms of the on-shell HPET variables, we find

$$|\mathbf{p}_v\rangle = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[\left(1 - \frac{k^2}{4m^2} + \frac{\not{k}\delta\not{k}}{4m^2}\right)|\mathbf{p}_w\rangle - \frac{\delta\not{k}}{2m}|\mathbf{p}_w]\right], \tag{3.39a}$$

$$|\mathbf{p}_v] = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[\left(1 - \frac{k^2}{4m^2} + \frac{\not{k}\delta\not{k}}{4m^2}\right)|\mathbf{p}_w] - \frac{\delta\not{k}}{2m}|\mathbf{p}_w\rangle\right]. \tag{3.39b}$$

Similarly,

$$\langle\mathbf{p}_v| = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[\langle\mathbf{p}_w|\left(1 - \frac{k^2}{4m^2} + \frac{\delta\not{k}\not{k}}{4m^2}\right) + [\mathbf{p}_w|\frac{\delta\not{k}}{2m}\right], \tag{3.39c}$$

$$[\mathbf{p}_v| = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[[\mathbf{p}_w|\left(1 - \frac{k^2}{4m^2} + \frac{\delta\not{k}\not{k}}{4m^2}\right) + \langle\mathbf{p}_w|\frac{\delta\not{k}}{2m}\right]. \tag{3.39d}$$

The transformed spinors $|\mathbf{p}_w\rangle$ and $|\mathbf{p}_w]$ are related to the traditional on-shell variables via eq. (3.31), with the replacement $k \to k'$. This transformation is singular at the point where the new residual momentum has magnitude squared $k'^2 = 4m^2$. This pole is ubiquitous when using these variables, and signals the point where fluctuations of the matter field are energetic enough to allow for pair-creation. As we have integrated out the anti-particle through eq. (3.30), such energies are outside the region of validity of this formalism. In fact, the working assumption of the formalism is that the residual momentum is small compared to the mass, so one would expect the formalism to lose predictive power well before this point.

3.3.3 Spin operator

We identify the spin operator with the Pauli-Lubanski pseudovector,

$$S^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}, \tag{3.40}$$

where $J^{\mu\nu}$ is the generator of rotations, p^μ is the momentum with respect to which the operator is defined, and $m^2 = p^2$. For our purposes, it will be convenient to choose $p^\mu = p_v^\mu$: this ensures that, irrespective of the value of the residual momentum, the momentum $p_v^\mu = m_k v^\mu$ will always be orthogonal to the spin operator. Thus, S^μ is the spin vector of a particle with velocity v^μ and any value of residual momentum. With this choice for the reference momentum, the spin-operator is

$$S^\mu = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}v_\nu J_{\alpha\beta}. \tag{3.41}$$

Its action on irreducible representations of $SL(2,\mathbb{C})$ is

$$(S^\mu)_\alpha{}^\beta = \frac{1}{4}\left[(\sigma^\mu)_{\alpha\dot\alpha}v^{\dot\alpha\beta} - v_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\right], \tag{3.42a}$$

$$(S^\mu)^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{4}\left[(\bar\sigma^\mu)^{\dot\alpha\alpha}v_{\alpha\dot\beta} - v^{\dot\alpha\alpha}(\sigma^\mu)_{\alpha\dot\beta}\right]. \tag{3.42b}$$

These two representations of the spin-vector are related via

$$(S^\mu)_\alpha{}^\beta = v_{\alpha\dot\alpha}(S^\mu)^{\dot\alpha}{}_{\dot\beta}v^{\dot\beta\beta}, \quad (S^\mu)^{\dot\alpha}{}_{\dot\beta} = v^{\dot\alpha\alpha}(S^\mu)_\alpha{}^\beta v_{\beta\dot\beta}. \tag{3.43}$$

On three-particle kinematics, the spin-vector can be written more compactly by introducing the x factor for a massless momentum q [63],

$$mx\langle q| \equiv [q|p_1, \tag{3.44a}$$

$$\Rightarrow mx^{-1}[q| = \langle q|p_1. \tag{3.44b}$$

Using this, when the initial residual momentum is $k = 0$, we can re-express the contraction $q \cdot S$ as

$$(q\cdot S)_\alpha{}^\beta = \frac{x}{2}|q\rangle\langle q|, \tag{3.45a}$$

$$(q\cdot S)^{\dot\alpha}{}_{\dot\beta} = -\frac{x^{-1}}{2}|q][q|. \tag{3.45b}$$

For general initial residual momentum, we find an additional term:

$$(q\cdot S)_\alpha{}^\beta = \frac{1}{4}\left(2x|q\rangle\langle q| + \frac{1}{m}[k,q]_\alpha{}^\beta\right), \tag{3.46a}$$

$$(q\cdot S)^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{4}\left(2x^{-1}|q][q| + \frac{1}{m}[k,q]^{\dot\alpha}{}_{\dot\beta}\right). \tag{3.46b}$$

Note that eq. (3.46) reduces to eq. (3.45) when $k = 0$.

When choosing the reference momentum to be p_v^μ, we can identify the spin-vector with the classical spin-vector of a Kerr black-hole with classical momentum $p^\mu_{\text{Kerr}} = \frac{m}{m_k} p^\mu_v$. This is because the Lorentz generator in eq. (3.40) can be replaced with the black hole spin-tensor $S^{\mu\nu} = J^{\mu\nu}_\perp$ which satisfies the condition [80, 96]

$$p^\mu_{\text{Kerr}} S_{\mu\nu} = 0, \tag{3.47}$$

known as the spin supplementary condition.

In ref. [92], the spin vector was defined as

$$S^\mu_v \equiv \frac{1}{2} \bar{u}_v(p_2) \gamma_5 \gamma^\mu u_v(p_1), \tag{3.48}$$

and it was found that this spin vector satisfied the relation

$$\bar{u}_v(p_2) \sigma^{\mu\nu} u_v(p_1) = -2\epsilon^{\mu\nu\alpha\beta} v_\alpha S_{v\beta}. \tag{3.49}$$

We can therefore relate these two definitions of the spin vector:

$$S^\mu_v = \bar{u}_v(p_2) S^\mu u_v(p_1) = -2\langle \mathbf{2}_v | S^\mu | \mathbf{1}_v \rangle = 2[\mathbf{2}_v | S^\mu | \mathbf{1}_v]. \tag{3.50}$$

Thus the two definitions are consistent, with one being the one-particle matrix element of the other.

3.4 THREE-POINT AMPLITUDE

We study in this section the on-shell three-point amplitudes of HPET. The main goal here will be to express the most general three-point on-shell amplitude for two massive particles (mass m, spin s) and one massless boson (helicity h) in terms of on-shell HPET variables. Focusing on the minimal coupling portion of such an expression, we will be left with a resummed form of the HPET three-point amplitude, valid for any spin. Moreover, we will find that a certain choice of the residual momentum results in the exponentiation of the minimally coupled three-point amplitude.

In the traditional on-shell variables, the most general three-point amplitude for two massive particles of mass m and spin s, and one massless particle with momentum q and helicity h is [63]

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h} \frac{x^{|h|}}{m^{2s}} \left[g_0 \langle \mathbf{21} \rangle^{2s} + g_1 \langle \mathbf{21} \rangle^{2s-1} \frac{x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle}{m} + \cdots + g_{2s} \frac{(x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle)^{2s}}{m^{2s}} \right], \tag{3.51}$$

$$\mathcal{M}^{-|h|,s} = (-1)^{h} \frac{x^{-|h|}}{m^{2s}} \left[\tilde{g}_0 [\mathbf{21}]^{2s} + \tilde{g}_1 [\mathbf{21}]^{2s-1} \frac{x^{-1}[\mathbf{2}q][q\mathbf{1}]}{m} + \cdots + \tilde{g}_{2s} \frac{\left(x^{-1}[\mathbf{2}q][q\mathbf{1}]\right)^{2s}}{m^{2s}} \right]. \tag{3.52}$$

The overall sign differs from the expression in ref. [63], due to our convention that p_1 is incoming. The positive helicity amplitude is expressed in the chiral basis, and the negative helicity amplitude in the anti-chiral basis. The minimal coupling portion of this is the amplitude with all couplings except g_0 and $\tilde{g}_0$ set to zero:

$$\mathcal{M}^{+|h|,s}_{\text{min}} = (-1)^{2s+h} \frac{g_0 x^{+|h|}}{m^{2s}} \langle \mathbf{21} \rangle^{2s}, \tag{3.53}$$

$$\mathcal{M}^{-|h|,s}_{\text{min}} = (-1)^h \frac{\tilde{g}_0 x^{-|h|}}{m^{2s}} [\mathbf{21}]^{2s}. \tag{3.54}$$

Thus we see that expressing this in terms of on-shell HPET variables requires that we convert the spinor products $\langle\mathbf{21}\rangle$, $x\langle\mathbf{2}q\rangle\langle q\mathbf{1}\rangle$ (and their anti-chiral basis counterparts) to the on-shell HQET variables.

In the remainder of this section we take $p_1^\mu = mv^\mu + k_1^\mu$ incoming, and q^μ and $p_2^\mu = mv^\mu + k_2^\mu$ outgoing. With this choice of kinematics, the initial and final residual momenta are related by $k_2^\mu = k_1^\mu - q^\mu$. We can relate a spinor with incoming momentum to the spinor with outgoing momentum using analytical continuation, eq. (3.137). Also, the x factor picks up a negative sign when the directions of p_1 or q are flipped, $x \to -x$.

3.4.1 General residual momentum

We start by converting the $s = 1/2$ amplitude to on-shell HPET variables. Inverting eq. (3.31) and simply taking the appropriate spinor products, we can relate the traditional and HPET spinor products:

$$\langle\mathbf{21}\rangle = \frac{m^2}{m_{k_2} m_{k_1}} \left[\frac{m_{k_1}}{m} \langle\mathbf{2}_v \mathbf{1}_v\rangle + \frac{1}{4m} [\mathbf{2}_v q]\langle q\mathbf{1}_v\rangle + \frac{x^{-1}}{4m} [\mathbf{2}_v q][q\mathbf{1}_v] \right], \tag{3.55a}$$

$$\langle\mathbf{2}q\rangle\langle q\mathbf{1}\rangle = \frac{m^2}{4m_{k_2} m_{k_1}} \left(\langle\mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle + x^{-1}\langle\mathbf{2}_v q\rangle[q\mathbf{1}_v] + x^{-1}[\mathbf{2}_v q]\langle q\mathbf{1}_v\rangle + x^{-2}[\mathbf{2}_v q][q\mathbf{1}_v] \right). \tag{3.55b}$$

Similarly, the spinor products in the anti-chiral basis become

$$[\mathbf{21}] = \frac{m^2}{m_{k_2} m_{k_1}} \left[\frac{m_{k_1}}{m} [\mathbf{2}_v \mathbf{1}_v] + \frac{1}{4m} \langle\mathbf{2}_v q\rangle[q\mathbf{1}_v] + \frac{x}{4m} \langle\mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle \right], \tag{3.56a}$$

$$[\mathbf{2}q][q\mathbf{1}] = \frac{m^2}{4m_{k_2} m_{k_1}} \left([\mathbf{2}_v q][q\mathbf{1}_v] + x[\mathbf{2}_v q]\langle q\mathbf{1}_v\rangle + x\langle\mathbf{2}_v q\rangle[q\mathbf{1}_v] + x^2\langle\mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle \right). \tag{3.56b}$$

By substituting eqs. (3.55) and (3.56) in eqs. (3.53) and (3.54) for $s = 1/2$, the minimally coupled amplitudes for positive and negative helicity become

$$\mathcal{M}^{+|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{1+h} g_0 x^{|h|} \frac{m}{m_{k_2} m_{k_1}} \left[\frac{m_{k_1}}{m} \langle\mathbf{2}_v \mathbf{1}_v\rangle + \frac{1}{4m} [\mathbf{2}_v q]\langle q\mathbf{1}_v\rangle + \frac{x^{-1}}{4m} [\mathbf{2}_v q][q\mathbf{1}_v] \right], \tag{3.57a}$$

$$\mathcal{M}^{-|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{h} \tilde{g}_0 x^{-|h|} \frac{m}{m_{k_2} m_{k_1}} \left[\frac{m_{k_1}}{m} [\mathbf{2}_v \mathbf{1}_v] + \frac{1}{4m} \langle\mathbf{2}_v q\rangle[q\mathbf{1}_v] + \frac{x}{4m} \langle\mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle \right], \tag{3.57b}$$

One can expand the m_{k_i} in powers of $|k|/m$, which is the characteristic expansion of HPETs. These three-point amplitudes therefore provide a conjecture for the resummed spin-1/2 HPET amplitude. Comparing the expansions of eq. (3.57) with that computed directly from the spin-1/2 HPET Lagrangians, we have confirmed that they agree at least up to $\mathcal{O}(m^{-2})$ for HQET, and $\mathcal{O}(m^{-1})$ for HBET.[9] Some subtleties of the matching to the Lagrangian calculation are discussed in Section 3.E.

The spin-dependence of these amplitudes can be made explicit by using the on-shell form of $q \cdot S$ in eq. (3.46):

$$\mathcal{M}^{+|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{1+h} g_0 x^{|h|} \frac{m}{m_{k_2} m_{k_1}} \langle\mathbf{2}_v| \left[1 - \frac{\not{v}\not{k}_1\not{k}_2\not{v}}{4m^2} + \frac{q \cdot S}{m} \right] |\mathbf{1}_v\rangle, \tag{3.58a}$$

[9] Note that the power counting of the HBET operators starts one power of m higher than HQET, at $\mathcal{O}(m)$. Thus both of these checks account for the operators up to and including NNLO.

$$\mathcal{M}^{-|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^h \tilde{g}_0 x^{-|h|} \frac{m}{m_{k_2} m_{k_1}} [\mathbf{2}_v| \left[1 - \frac{\not{v}\not{k}_1\not{k}_2\not{v}}{4m^2} - \frac{q \cdot S}{m} \right] |\mathbf{1}_v]. \tag{3.58b}$$

Written in this way, it is immediately apparent how the $k_1 = 0$ parameterization can be obtained from the general case. We turn now to this scenario.

3.4.2 Zero initial residual momentum

We now consider the parameterization where $k_1^\mu = 0$ and $k_2^\mu = -q^\mu$ With zero initial residual momentum, we can switch between the chiral and anti-chiral bases using eq. (3.32):

$$\langle \mathbf{2}_v \mathbf{1}_v \rangle = -[\mathbf{2}_v \mathbf{1}_v], \tag{3.59a}$$

$$\langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle = x^{-2} [\mathbf{2}_v q][q \mathbf{1}_v]. \tag{3.59b}$$

Recognizing eqs. (3.59a) and (3.59b) as directly relating spinless effects and the spin-vector respectively in different bases, we see that, for this parameterization, spin effects are never obscured by working in any particular basis. This is in contrast to the traditional on-shell variables, where the analog to eq. (3.59a) includes a spin term, thus hiding or exposing spin dependence when working in a certain basis. Thus we have gained a basis-independent interpretation of spinless and spin-inclusive terms.

Either setting $k_1 = 0$ in eq. (3.58), or applying eqs. (3.59a) and (3.59b) to eqs. (3.55) and (3.56), the minimally coupled three-point amplitude with zero residual momentum is obtained:

$$\mathcal{M}^{+|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{1+h} \frac{g_0 x^{|h|}}{m} \left[\langle \mathbf{2}_v \mathbf{1}_v \rangle + \frac{x}{2m} \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \right], \tag{3.60a}$$

$$\mathcal{M}^{-|h|,s=\frac{1}{2}}_{\text{HPET,min}} = (-1)^{h} \frac{\tilde{g}_0 x^{-|h|}}{m} \left[[\mathbf{2}_v \mathbf{1}_v] + \frac{x^{-1}}{2m} [\mathbf{2}_v q][q \mathbf{1}_v] \right]. \tag{3.60b}$$

Note the negative signs which come from treating p_1 as incoming.

Three-point kinematics are restrictive enough when $k_1 = 0$ that we can derive the three-point amplitude in eq. (3.60) in an entirely different fashion. The full three-point amplitude for a heavy spin-$1/2$ particle coupled to a photon can be written as[10]

$$\mathcal{A}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, q^h) = f(m, v, q) e v_\mu \epsilon_q^{h,\mu} \bar{u}_v(p_2) u_v(p_1) + g(m, v, q) e q^\mu \epsilon_q^{h,\nu} \bar{u}_v(p_2) \sigma_{\mu\nu} u_v(p_1). \tag{3.61}$$

The negative in the argument of the amplitude signifies an incoming momentum. The three-point operators in the HQET Lagrangian, as well as any non-minimal couplings, modify the functions f and g, but there are no other spinor structures that can arise. We therefore have two spinor contractions in terms of which we would like to express the spinor brackets of interest. We proceed by writing the two contractions in terms of the traditional on-shell variables, and equating this to the contractions expressed in terms of the on-shell HPET variables. Working with, say, a positive helicity photon, this yields

$$v_\mu \epsilon_q^{+,\mu} \bar{u}_v(p_2) u_v(p_1) = -\sqrt{2} x \langle \mathbf{2}_v \mathbf{1}_v \rangle = -\frac{x}{\sqrt{2}} \left(-\frac{x}{m} \langle \mathbf{2} q \rangle \langle q \mathbf{1} \rangle + 2 \langle \mathbf{21} \rangle \right), \tag{3.62a}$$

$$\bar{u}_v(p_2) \sigma_{\mu\nu} u_v(p_1) q^\mu \epsilon_q^{+,\nu} = \sqrt{2} i x^2 \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle = \sqrt{2} i x^2 \langle \mathbf{2} q \rangle \langle q \mathbf{1} \rangle. \tag{3.62b}$$

[10] We use $\mathcal{A}$ to denote a Yang-Mills amplitude.

Solving for the traditional spinor products, we find

$$\langle \mathbf{2}\mathbf{1}\rangle = \langle \mathbf{2}_v\mathbf{1}_v\rangle + \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle, \tag{3.63a}$$

$$\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle = \langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle. \tag{3.63b}$$

Similarly,

$$[\mathbf{2}\mathbf{1}] = [\mathbf{2}_v\mathbf{1}_v] + \frac{x^{-1}}{2m}[\mathbf{2}_v q][q\mathbf{1}_v], \tag{3.64a}$$

$$[\mathbf{2}q][q\mathbf{1}] = [\mathbf{2}_v q][q\mathbf{1}_v]. \tag{3.64b}$$

Note that eqs. (3.63) and (3.64) decompose the spinor brackets into spinless and spin-inclusive terms. Applying eq. (3.59), it is easy to check that this separation of different spin multipoles is independent of the basis used to express the traditional spinor brackets.

With eqs. (3.62a) and (3.62b), we can rewrite eq. (3.61) as

$$\mathcal{A}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, q^+) = \sqrt{2}xe\left(-f(m,v,q)\langle \mathbf{2}_v\mathbf{1}_v\rangle + g(m,v,q)ix\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right). \tag{3.65}$$

The three-point amplitude in QED — with interaction term $\mathcal{L}_{\text{int}} = e\bar{\psi}\not{A}\psi$ — for a positive helicity photon is

$$\begin{aligned}\mathcal{A}_{\text{QED}}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, q^+) &= e\bar{u}(p_2)\gamma_\mu u(p_1)\epsilon_q^{+,\mu}\\ &= \sqrt{2}ex\langle \mathbf{2}\mathbf{1}\rangle,\end{aligned} \tag{3.66}$$

where in the first line we use Dirac spinors instead of HQET spinors. Substituting eq. (3.63a) into the above equation gives

$$\mathcal{A}_{\text{QED}}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, q^+) = \sqrt{2}ex\left(\langle \mathbf{2}_v\mathbf{1}_v\rangle + \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right). \tag{3.67}$$

As abelian HQET is an effective theory derived from QED, it must reproduce the on-shell QED amplitudes when all operators are accounted for. This means that eqs. (3.65) and (3.67) are equal, so we can solve for the functions f and g:

$$f(m,v,q) = -1, \tag{3.68a}$$

$$g(m,v,q) = \frac{i}{2m}. \tag{3.68b}$$

As a consequence of eqs. (3.68a) and (3.68b), we conclude that only the leading spin and leading spinless three-point operators of HQET are non-vanishing on-shell when $k_1 = 0$. Indeed, in this case the transfer momentum q^μ is the only parameter that can appear in the HQET operator expansion. In the three-point amplitude, it can only appear in the scalar combinations $q^2 = 0$ by on-shellness of the photon, $v \cdot q \sim q^2 = 0$ by on-shellness of the quarks, or $q \cdot \epsilon(q) = 0$ by transversality of the polarization.

To sum up, we list the three-point amplitude for two equal mass spin-1/2 particles and an outgoing photon for both helicities, and in both the chiral and anti-chiral bases:[11]

$$\mathcal{A}^{+1,s=\frac{1}{2}} = \sqrt{2}ex\left(\langle \mathbf{2}_v \mathbf{1}_v\rangle + \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right) = -\sqrt{2}ex\left([\mathbf{2}_v \mathbf{1}_v] - \frac{x^{-1}}{2m}[\mathbf{2}_v q][q\mathbf{1}_v]\right), \quad (3.69a)$$

$$\mathcal{A}^{-1,s=\frac{1}{2}} = \sqrt{2}ex^{-1}\left(\langle \mathbf{2}_v \mathbf{1}_v\rangle - \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right) = -\sqrt{2}ex^{-1}\left([\mathbf{2}_v \mathbf{1}_v] + \frac{x^{-1}}{2m}[\mathbf{2}_v q][q\mathbf{1}_v]\right), \quad (3.69b)$$

so $g_0 = \tilde{g}_0 = \sqrt{2}em$. When a graviton is emitted instead of a photon, we simply make the replacement $e \to -\frac{\kappa m}{2\sqrt{2}}$ and square the overall factors of x.

We can obtain the amplitude with general initial residual momentum by reparameterizing the states by means of eq. (3.39).

3.4.3 Most general three-point amplitude

Recall the most general three-point amplitude for two massive particles of spin s and mass m and a massless boson with helicity h in the chiral basis, eq. (3.51):

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h}\frac{x^{|h|}}{m^{2s}}\left[g_0\langle \mathbf{21}\rangle^{2s} + g_1\langle \mathbf{21}\rangle^{2s-1}\frac{x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle}{m} + \cdots + g_{2s}\frac{(x\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle)^{2s}}{m^{2s}}\right]. \quad (3.70)$$

When expressing eq. (3.51) in terms of the on-shell HPET variables, setting the initial residual momentum to zero, and applying the binomial expansion, we find that

$$\mathcal{M}_3^{+|h|,s} = (-1)^{2s+h}\frac{x^{|h|}}{m^{2s}}\sum_{k=0}^{2s} g^{\mathrm{H}}_{s,k}\langle \mathbf{2}_v\mathbf{1}_v\rangle^{2s-k}\left(\frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right)^k, \quad g^{\mathrm{H}}_{s,k} = \sum_{i=0}^{k} g_i\binom{2s-i}{2s-k}. \quad (3.71a)$$

We can express this in the anti-chiral basis using eq. (3.59):

$$\mathcal{M}_3^{+|h|,s} = \frac{x^{|h|}}{m^{2s}}\sum_{k=0}^{2s} g^{\mathrm{H}}_{s,k}(-1)^{k+h}[\mathbf{2}_v\mathbf{1}_v]^{2s-k}\left(\frac{x^{-1}}{2m}[\mathbf{2}_v q][q\mathbf{1}_v]\right)^k. \quad (3.71b)$$

The k^{th} spin-multipole can be isolated by choosing the k^{th} term in the sum. There are $2s+1$ combinations of the spinor brackets in this sum, consistent with the fact that a spin s particle can only probe up to the $2s^{\text{th}}$ spin order term of the spin-multipole expansion. Note also that the coefficient of the spin monopole term is always equal to its value for minimal coupling, making the monopole term universal in any theory.[12]

The minimal coupling amplitudes are those in eqs. (3.53) and (3.54), which correspond to setting $g_{i>0} = 0$. Translating to the on-shell HPET variables, minimal coupling in eqs. (3.71a) and (3.71b) corresponds to $g^{\mathrm{H}}_{s,k} = g_0\binom{2s}{k}$.

We can write the analogous expressions to eqs. (3.71a) and (3.71b) for a negative helicity massless

[11] We abbreviate the arguments of the amplitude here, but still use p_1 incoming.

[12] This is consistent with the reparameterization invariance of HQET, which fixes the Wilson coefficients of the spinless operators in the HQET Lagrangian up to order $1/m$ [59]. As argued above, when the initial residual momentum is set to 0, these are the only operators contributing to the spin monopole.

particle. Expressing eq. (3.52) using eqs. (3.64a) and (3.64b),

$$\mathcal{M}_3^{-|h|,s} = (-1)^h \frac{x^{-|h|}}{m^{2s-1}} \sum_{k=0}^{2s} \tilde{g}^{\mathrm{H}}_{s,k} [\mathbf{2}_v \mathbf{1}_v]^{2s-k} \left(\frac{x^{-1}}{2m} [\mathbf{2}_v q][q \mathbf{1}_v] \right)^k, \quad \tilde{g}^{\mathrm{H}}_{s,k} = \sum_{i=0}^{k} \tilde{g}_i \binom{2s-i}{2s-k}. \tag{3.72a}$$

Converting to the chiral basis,

$$\mathcal{M}_3^{-|h|,s} = \frac{x^{-|h|}}{m^{2s}} \sum_{k=0}^{2s} \tilde{g}^{\mathrm{H}}_{s,k} (-1)^{2s+h+k} \langle \mathbf{2}_v \mathbf{1}_v \rangle^{2s-k} \left(\frac{x}{2m} \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \right)^k. \tag{3.72b}$$

Minimal coupling in this case corresponds to $\tilde{g}_{i>0} = 0$, and thus $\tilde{g}^{\mathrm{H}}_{s,k} = \tilde{g}_0 \binom{2s}{k}$.

3.4.4 Infinite spin limit

Various methods have been used to show that the minimal coupling three-point amplitude in traditional on-shell variables exponentiates in the infinite spin limit [45, 47, 95]. All of them require a slight manipulation of the minimal coupling to do so, with refs. [45, 47] employing a change of basis between the chiral and anti-chiral bases, ref. [45] applying a generalized expectation value, and refs. [47, 95] using a Lorentz boost – analogous to the gauge-fixing of the spin operator in ref. [80] – to rewrite the minimal coupling amplitude. As the on-shell HPET variables inherently make the spin-dependence of the minimal coupling manifest, the exponentiation of the three-point amplitude is immediate.

Consider the minimal coupling three-point amplitude for two massive spin s particles and one massless particle:

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h} \frac{g_0 x^{|h|}}{m^{2s}} \langle \mathbf{2}_v |^{2s} \sum_{k=0}^{2s} \frac{(2s)!}{(2s-k)!} \frac{\left(\frac{x}{2m} |q\rangle\langle q| \right)^k}{k!} |\mathbf{1}_v\rangle^{2s}. \tag{3.73}$$

The quantity in the sum is the rescaled spin-operator $q \cdot S/m$ for a spin s particle, raised to the power of k and divided by $k!$ [46],

$$\left(\frac{q \cdot S}{m} \right)^n = \frac{(2s)!}{(2s-n)!} \left(\frac{x}{2m} |q\rangle\langle q| \right)^n, \tag{3.74}$$

where we have suppressed the spinor indices. The amplitude is therefore

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h} \frac{g_0 x^{|h|}}{m^{2s}} \langle \mathbf{2}_v |^{2s} \sum_{k=0}^{2s} \frac{\left(\frac{q \cdot S}{m} \right)^k}{k!} |\mathbf{1}_v\rangle^{2s}. \tag{3.75}$$

We identify the sum with an exponential, with the understanding that the series truncates at the $2s^{\text{th}}$ term for a spin $2s$ particle:

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h} \frac{g_0 x^{|h|}}{m^{2s}} \langle \mathbf{2}_v |^{2s} e^{q \cdot S/m} |\mathbf{1}_v\rangle^{2s}. \tag{3.76}$$

Taking the infinite spin limit, the exponential is exact as its Taylor series does not truncate. We treat the exponential as a number in this limit and remove it from between the spinors [47]:

$$\lim_{s\to\infty}\mathcal{M}^{+|h|,s}=\lim_{s\to\infty}(-1)^{2s+h}\frac{g_0x^{|h|}}{m^{2s}}e^{q\cdot S/m}\langle\mathbf{2}_v\mathbf{1}_v\rangle^{2s}. \tag{3.77}$$

Note that since the initial residual momentum is 0, both spinors are associated with the same momentum. Then, using the on-shell conditions for these variables,[13]

$$\lim_{s\to\infty}\mathcal{M}^{+|h|,s}=(-1)^hg_0x^{|h|}e^{q\cdot S/m}. \tag{3.78}$$

This amplitude immediately agrees with the three-point amplitude in refs. [45, 47]: it is the scalar three-point amplitude multiplied by an exponential containing the classical spin-multipole moments. Also notable is that the generalized expectation value (GEV) of ref. [45] or the Lorentz boosts of refs. [47, 95] are not necessary here to interpret the spin dependence classically.

For the emission of a negative helicity boson, the n^{th} power of the spin-operator projected along the direction of the boson's momentum is

$$\left(\frac{q\cdot S}{m}\right)^n=\frac{(2s)!}{(2s-n)!}\left(-\frac{x^{-1}}{2m}|q][q|\right)^n. \tag{3.79}$$

Starting with eq. (3.72a), the three-point amplitude exponentiates as

$$\mathcal{M}^{-|h|,s}=(-1)^h\frac{\tilde{g}_0x^{-|h|}}{m^{2s}}[\mathbf{2}_v|^{2s}e^{-q\cdot S/m}|\mathbf{1}_v]^{2s}, \tag{3.80}$$

with the exponential being truncated at the $2s^{\text{th}}$ term. Taking the infinite spin limit, we find

$$\lim_{s\to\infty}\mathcal{M}^{-|h|,s}=\lim_{s\to\infty}(-1)^h\frac{\tilde{g}_0x^{-|h|}}{m^{2s}}e^{-q\cdot S/m}[\mathbf{2}_v\mathbf{1}_v]^{2s}. \tag{3.81}$$

Applying the on-shell conditions for these variables, we get

$$\lim_{s\to\infty}\mathcal{M}^{-|h|,s}=(-1)^h\tilde{g}_0x^{-|h|}e^{-q\cdot S/m}. \tag{3.82}$$

Once again we find the scalar three-point amplitude multiplied by an exponential containing the classical spin dependence.

That the exponentials in this section are functions of $q\cdot S$ instead of $2q\cdot S$, as is the case when the traditional on-shell variables are naïvely exponentiated — that is, without normalizing by the GEV, or Lorentz boosting one of the spinors — is significant. We discuss the implications of this in the next section.

3.5 KERR BLACK HOLES AS HEAVY PARTICLES

In this section, we apply the on-shell HPET variables to the classical gravitational scattering of two spinning black holes. We show that, with the correct momentum parameterization, a heavy spin-s particle minimally coupled to gravity possesses precisely the same spin-multipole expansion as a Kerr black hole,

[13] The validity of using the on-shell conditions can be checked explicitly by rewriting the bracket in terms of traditional on-shell variables, then boosting one of the momenta into the other as in ref. [95].

up to the order $2s$ multipole. The reason for this is that on-shell HPET variables for a given velocity v^μ, residual momentum k^μ, and mass m always correspond to momenta $m_k v^\mu$, where m_k is defined in eq. (3.34).

We begin with a brief review of the effective field theory for spinning gravitating bodies. The action of a particle interacting with gravitational radiation of wavelength much larger than its spatial extent (approximately a point particle) was formulated in ref. [36]. The generalization to the case of spinning particles was first approached in ref. [37]. The effective action formulated in ref. [80] takes the form

$$S = \int d\sigma \left\{ -m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{\text{SI}}[u^\mu, S_{\mu\nu}, g_{\mu\nu}(x^\mu)] \right\}, \tag{3.83}$$

where σ parameterizes the wordline of the particle, $u^\mu = \frac{dx^\mu}{d\sigma}$ is the coordinate velocity, $S_{\mu\nu}$ is the spin operator, $\Omega^{\mu\nu}$ is the angular velocity, and L_{SI} contains higher spin-multipoles that are dependent on the inner structure of the particle through non-minimal couplings.

The first two terms in eq. (3.83) are the spin monopole and dipole terms, and are universal for spinning bodies with any internal configuration. We assign to them respectively the coefficients $C_{S^0} = C_{S^1} = 1$. From an amplitudes perspective, the universality of the spin-monopole coefficient can be seen from the on-shell HPET variables since the coefficient of the spin-monopole term in eqs. (3.71a) and (3.72a) is always equal to its minimal coupling value. The universality of the spin-dipole coefficient was argued in refs. [46, 53] from general covariance, and by requiring the correct factorization of the Compton scattering amplitude. Explicitly, the higher spin-multipole terms L_{SI} are

$$\begin{aligned} L_{\text{SI}} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{S^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \dots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \dots S^{\mu_{2n}} \\ &\quad + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{S^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \dots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \dots S^{\mu_{2n+1}}. \end{aligned} \tag{3.84}$$

See ref. [80] for the derivation and formulation of this action. The Wilson coefficients C_{S^k} contain the information about the internal structure of the object, with a Kerr black hole being described by $C_{S^k}^{\text{Kerr}} = 1$ for all k.

The three-point amplitude derived from this action was expressed in traditional spinor-helicity variables in refs. [46, 53], where it was shown that the spin-multipole expansion is necessarily truncated at order $2s$ when the polarization tensors of spin s particles are used. By matching this three-point amplitude with the most general form of a three-point amplitude, it was found there that in the case of minimal coupling one obtains the Wilson coefficients of a Kerr black hole in the infinite spin limit. Following their derivation, but using on-shell HPET variables instead, we find (with all momenta incoming)

$$\begin{aligned} \mathcal{M}^{+2,s} &= \sum_{a+b\leq s} \frac{\kappa m x^2}{2m^{2s}} C_{S^{a+b}} n_{a,b}^s \langle \mathbf{2}_{-v} \mathbf{1}_v \rangle^{s-a} \left(-x \frac{\langle \mathbf{2}_{-v} q \rangle \langle q \mathbf{1}_v \rangle}{2m} \right)^a [\mathbf{2}_{-v} \mathbf{1}_v]^{s-b} \left(x^{-1} \frac{[\mathbf{2}_{-v} q][q \mathbf{1}_v]}{2m} \right)^b, \\ n_{a,b}^s &\equiv \binom{s}{a} \binom{s}{b}. \end{aligned} \tag{3.85}$$

As in refs. [46, 53], we refer to this representation of the amplitude in a form symmetric in the chiral and anti-chiral bases as the polarization basis. Flipping the directions of p_2 and q (to allow us to directly compare with eq. (3.71a)), then converting the polarization basis to the chiral basis:

$$\mathcal{M}^{+2,s} = \frac{x^2}{m^{2s}}(-1)^{2s} \sum_{a+b\leq 2s} \frac{\kappa m}{2} C_{S^{a+b}} n^s_{a,b} \langle \mathbf{2}_v \mathbf{1}_v\rangle^{2s-a-b} \left(\frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q \mathbf{1}_v\rangle\right)^{a+b}. \tag{3.86}$$

Comparing with eq. (3.71a), we obtain a one-to-one relation between the coupling constants of both expansions:

$$g^{\mathrm{H}}_{s,k} = \frac{\kappa m}{2} C_{S^k} \sum_{j=0}^{k} n^s_{k-j,j}. \tag{3.87}$$

Such a one-to-one relation is consistent with the interpretation of eq. (3.71a) as being a spin-multipole expansion. Focusing on the minimal coupling case, we set $g_{i>0} = 0$, which means $g^{\mathrm{H}}_{s,k} = g_0\binom{2s}{k}$. Normalizing $g_0 = \kappa m/2$, the coefficients of the one-particle effective action for finite spin take the form

$$C^{\text{min}}_{S^k} = \binom{2s}{k}\left[\sum_{j=0}^{k}\binom{s}{k-j}\binom{s}{j}\right]^{-1} = 1. \tag{3.88}$$

The final equality is the Chu-Vandermonde identity, valid for all k. This suggests that the minimal coupling expressed in the on-shell HPET variables produces precisely the multipole moments of a Kerr black hole, even before taking the infinite spin limit.

Using the same matching technique, refs. [46, 53] showed that, when using traditional on-shell variables, the minimal coupling three-point amplitude for finite spin s corresponded to Wilson coefficients that deviated from those of a Kerr black hole by terms of order $\mathcal{O}(1/s)$. Why is it then that the polarization tensors of finite spin HPET possess the same spin-multipole expansion as a Kerr black hole? Analyzing the matching performed in refs. [46, 53], the s dependence there arises from the conversion of the polarization basis to the chiral basis. The reason for this is that new spin contributions arise from this conversion since the chiral and anti-chiral bases are mixed by two times the spin-operator:

$$\langle \mathbf{12}\rangle = -[\mathbf{12}] + \frac{1}{xm}[\mathbf{1}q][q\mathbf{2}], \tag{3.89a}$$

$$[\mathbf{12}] = -\langle \mathbf{12}\rangle + \frac{x}{m}\langle \mathbf{1}q\rangle\langle q\mathbf{2}\rangle. \tag{3.89b}$$

The second terms on the right hand sides of these equations encode spin effects, while the first terms were interpreted to be purely spinless. However, the left hand sides of these equations contradict the latter interpretation; the spinor brackets $\langle \mathbf{12}\rangle$ and $[\mathbf{12}]$ themselves contain spin effects. This is the origin of the observed deviation from $C^{\text{Kerr}}_{S^k}$: eq. (3.89), while exposing some spin-dependence, does not entirely separate the spinless and spin-inclusive effects encoded in the traditional minimal coupling amplitude. The result is the matching of an exact spin-multipole expansion on the one-particle effective action side, to a rough separation of different spin-multipoles on the amplitude side.

A similar mismatch to Kerr black holes was seen in ref. [47], where the minimal coupling amplitude was shown to produce the spin dependence[14]

$$\langle \mathbf{21} \rangle = -[\mathbf{2}|e^{2q\cdot S/m}|\mathbf{1}], \tag{3.90}$$

where S^μ is the Pauli-Lubanski pseudovector defined with respect to p_1. Expanding the exponential and noting that the series terminates after the spin-dipole term in this case, it's easy to see the equivalence between this and eq. (3.89). The spin-dependence here differs from that of a Kerr black hole by a factor of two in the exponential [45, 97]. Motivated by arguments in ref. [80], an exact match to the Kerr black hole spin multipole expansion was obtained in ref. [47] by noting that additional spin contributions are hidden in the fact that the polarization vectors $[\mathbf{2}|$ and $|\mathbf{1}]$ represent different momenta. Writing $[\mathbf{2}|$ as a Lorentz boost of $[\mathbf{1}|$, the true spin-dependence of the minimal coupling bracket was manifested:

$$\langle \mathbf{21} \rangle \sim -[\mathbf{1}|e^{q\cdot S/m}|\mathbf{1}], \tag{3.91}$$

up to an operator acting on the little group index of $[\mathbf{1}|$. The spin-dependence here matches that of a Kerr black hole, and also matches what has been made explicit in Section 3.4.4. Using a similar Lorentz boost, the authors of ref. [95] also showed that the minimal coupling bracket indeed contains the spin-dependence of a Kerr black hole. We see that in the absence of a momentum mismatch between the polarization states used, the full spin-dependence is manifest, and the multipole expansion of a finite spin s particle corresponds exactly to that of a Kerr black hole up to $2s^{\text{th}}$ order.

This mismatch of momenta is avoided entirely when using on-shell HPET variables. Recall that in general the momentum p_v represented by on-shell HPET variables is

$$p_v^\mu = m_k v^\mu. \tag{3.92}$$

Working in the case where the initial residual momentum is zero, as in the rest of this section, this reduces to simply $mv_{\alpha\dot{\alpha}}$ for the case of $p_{v,1}$. For $p_{v,2}$, where $p_2 = p_1 - q$ and q is the null transfer momentum,

$$p_{v,2} = \left(1 - \frac{q^2}{4m^2}\right) mv^\mu = mv^\mu. \tag{3.93}$$

Consequently, although the initial and final momenta of the massive particle differ by q, the degrees of freedom are arranged in such a way that the external states $|\mathbf{1}_v\rangle$ and $|\mathbf{2}_v\rangle$ are associated with the same momentum. This explains why we have recovered precisely the Wilson coefficients of a Kerr black hole. We identify this common momentum with that of the Kerr black hole $p^\mu_{\text{Kerr}} = mv^\mu$. From the point of view of spinor products, eq. (3.59) shows that on-shell HPET variables provide an unambiguous and basis-independent interpretation of spinless and spin-inclusive spinor brackets. Thus, the entire spin dependence of the minimal coupling amplitude is automatically made explicit, and is isolated from spinless terms.

In the case of $k_1 \neq 0$, the three-term structure of the minimal coupling amplitude spoils its exponentiation. The matching to the Kerr black hole spin-multipole moments is therefore obscured, but is recovered in the reparameterization where k_1 is set to 0. This mismatching of the spin-multipole moments can be attributed to the fact that the polarization tensors for the initial and final states no longer

[14] Ref. [47] worked exclusively with integer spin. However the only adaptation that must be made to the results therein when working with half integer spins is the inclusion of a factor of $(-1)^{2s} = -1$.

correspond to the same momentum, since generally $m_{k_1-q} \neq m_{k_1}$.

A similar matching analysis has recently been performed in ref. [98] for the case of Kerr-Newman black holes. It was also found there that minimal coupling to electromagnetism reproduces the classical spin multipoles of a Kerr-Newman black hole in the infinite spin limit, when the matching is performed using traditional on-shell variables. Repeating their analysis, but using on-shell HPET variables instead, we find again that the classical multipoles are reproduced exaclty, even for finite spin.

3.6 ON-SHELL AMPLITUDES

In this section, we compute electromagnetic and gravitational amplitudes for the scattering of minimally coupled spin-s particles in on-shell HPET variables using eqs. (3.55), (3.56), (3.63) and (3.64). Our goal in this section is two-fold: first, we will show how spin effects remain separated from spinless effects, at the order considered in this work, when using on-shell HPET variables. Second, we will exploit the explicit $\hbar$ dependence of eqs. (3.55) and (3.56) to isolate the classical portions of the computed amplitudes. Given that the momenta of the on-shell HPET variables always reduce to the momentum of a Kerr black hole in the classical limit, we expect to recover the spin-multipoles of a Kerr black hole in this limit. We show that, at tree-level, the spin dependence of the leading $\hbar$ portions factorize into a product of the classical spin-dependence at three-points. This is simply a consequence of factorization for boson exchange amplitudes (a result that has already been noted in ref. [47]). For same-helicity tree-level radiation processes this results from a spin-multipole universality that we will uncover, and for the opposite helicity Compton amplitude there will be an additional factor accounting for its non-uniqueness at higher spins.[15]

3.6.1 Counting $\hbar$

Given that we will be interested in isolating classical effects, we summarize here the rules for restoring the $\hbar$ dependence in the amplitude [10], and adapt these rules to the on-shell variables.

Powers of $\hbar$ are restored in such a way so as to preserve the dimensionality of amplitudes and coupling constants. To do so, the coupling constants of electromagnetism and gravity are rescaled as $e \to e/\sqrt{\hbar}$ and $\kappa \to \kappa/\sqrt{\hbar}$. Furthermore, when taking the classical limit $\hbar \to 0$ of momenta, massive momenta and masses are to be kept constant, whereas massless momenta vanish in this limit — for a massless momentum q, it is the associated wave number $\bar{q} = q/\hbar$ that is kept constant in the classical limit. Thus each massless momentum in amplitudes is associated with one power of $\hbar$. Translating this to on-shell variables, we assign a power of $\hbar^{\alpha}$ to each $|q\rangle$, and a power of $\hbar^{1-\alpha}$ to each $|q]$.[16] Momenta that are treated with the massless $\hbar$ scaling are

- photon and graviton momenta, whether they correspond to external or virtual particles;
- loop momenta, which can always be assigned to an internal massless boson;
- residual momenta [92].

[15] We contrast the factorization for radiation processes here with that in ref. [45] by noting that the entire quantum amplitude was factorized there, whereas we show that the factorization holds also for the leading $\hbar$ contribution.

[16] The value of α can be determined by fixing the $\hbar$ scaling of massless polarization tensors for each helicity. Requiring that the dimensions of polarization vectors remain unchanged when $\hbar$ is restored results in the democratic choice $\alpha = 1/2$.

Finally, we come to the case of spin-inclusive terms. When taking the classical limit $\hbar \to 0$, we simultaneously take the limit $s \to \infty$ where s is the magnitude of the spin. These limits are to be taken in such a way so as to keep the combination $\hbar s$ constant. This means that for every power of spin in a term, there is one factor of $\hbar$ that we can neglect when taking the classical limit. Effectively, we can simply scale all powers of spin with one inverse power of $\hbar$, and understand that $\hbar$ is to be taken to 0 wherever it appears in the amplitude.

As in ref. [92], we identify the components of an amplitude contributing classically to the interaction potential as those with the $\hbar$ scaling

$$\mathcal{M} \sim \hbar^{-3}. \tag{3.94}$$

Terms with more positive powers of $\hbar$ contribute quantum mechanically to the interaction potential. Also, we use $\mathcal{M}^{\text{cl.}}$ to denote the leading $\hbar$ portion of an amplitude.

3.6.2 Boson exchange

We begin with the tree-level amplitudes for photon/graviton[17] exchange between two massive spinning particles. We consider first spin-1/2 – spin-1/2 scattering, to show that the spin-multipole expansion remains explicit in these variables at four points. The classical part of the amplitude can be computed by factorizing it into two three-point amplitudes. To simplify the calculation, we are free to set the initial residual momentum of each massive leg to 0, so we will need only eqs. (3.63) and (3.64). Letting particle a have mass m_a and incoming/outgoing momenta p_1/p_2, and particle b have mass m_b and incoming/outgoing momenta p_3/p_4, we find for an exchanged photon

$$\begin{aligned} i\mathcal{A}_{\text{tree}}(-\mathbf{1}_a^{\frac{1}{2}}, \mathbf{2}_a^{\frac{1}{2}}, -\mathbf{3}_b^{\frac{1}{2}}, \mathbf{4}_b^{\frac{1}{2}}) &= \sum_h \mathcal{A}_{\text{tree}}(-\mathbf{1}^{\frac{1}{2}}, \mathbf{2}^{\frac{1}{2}}, -q^h)\frac{i}{q^2}\mathcal{A}_{\text{tree}}(q^{-h}, -\mathbf{3}^{\frac{1}{2}}, \mathbf{4}^{\frac{1}{2}}) \\ &= -\frac{ie^2}{q^2}\Big[4\omega\langle \mathbf{2}_{v_a}\mathbf{1}_{v_a}\rangle\langle \mathbf{4}_{v_b}\mathbf{3}_{v_b}\rangle \\ &\quad -\frac{2}{m_b}\sqrt{\omega^2-1}\,\langle \mathbf{2}_{v_a}\mathbf{1}_{v_a}\rangle x_b\langle \mathbf{4}_{v_b}q\rangle\langle q\mathbf{3}_{v_b}\rangle \\ &\quad +\frac{2}{m_a}\sqrt{\omega^2-1}\,x_a\langle \mathbf{2}_{v_a}q\rangle\langle q\mathbf{1}_{v_a}\rangle\langle \mathbf{4}_{v_b}\mathbf{3}_{v_b}\rangle \\ &\quad -\frac{\omega}{m_a m_b}x_a\langle \mathbf{2}_{v_a}q\rangle\langle q\mathbf{1}_{v_a}\rangle x_b\langle \mathbf{4}_{v_b}q\rangle\langle q\mathbf{3}_{v_b}\rangle\Big], \end{aligned} \tag{3.95}$$

where $\omega \equiv p_1 \cdot p_3/m_a m_b = (x_a x_b^{-1} + x_a^{-1} x_b)/2$, $v_a = p_1/m_a$, $v_b = p_3/m_b$, and negative momenta are incoming. The x variables are defined as

$$x_a = -\frac{[q|p_1|\xi\rangle}{m_a\langle q\xi\rangle}, \quad x_a^{-1} = -\frac{\langle q|p_1|\xi]}{m_a[q\xi]}, \tag{3.96a}$$

$$x_b = \frac{[q|p_3|\xi\rangle}{m_b\langle q\xi\rangle}, \quad x_b^{-1} = \frac{\langle q|p_3|\xi]}{m_b[q\xi]}. \tag{3.96b}$$

The negative sign in the definitions of x_a and x_a^{-1} account for the fact that the massless boson is incoming to particle a.

[17] We will denote an amplitude involving photons by $\mathcal{A}$, and one involving gravitons by $\mathcal{M}$.

The gravitational amplitude is computed analogously:

$$\begin{aligned} i\mathcal{M}_{\text{tree}}(-\mathbf{1}_a^{\frac{1}{2}}, \mathbf{2}_a^{\frac{1}{2}}, -\mathbf{3}_b^{\frac{1}{2}}, \mathbf{4}_b^{\frac{1}{2}}) = -\frac{i m_a m_b \kappa^2}{8q^2} \Big[& 4\left(2\omega^2 - 1\right) \langle \mathbf{2}_{v_a} \mathbf{1}_{v_a} \rangle \langle \mathbf{4}_{v_b} \mathbf{3}_{v_b} \rangle \\ & -\frac{4\omega}{m_a} \sqrt{\omega^2 - 1}\, x_a \langle \mathbf{2}_{v_a} q \rangle \langle q \mathbf{1}_{v_a} \rangle \langle \mathbf{4}_{v_b} \mathbf{3}_{v_b} \rangle \\ & +\frac{4\omega}{m_b} \sqrt{\omega^2 - 1}\, \langle \mathbf{2}_{v_a} \mathbf{1}_{v_a} \rangle x_b \langle \mathbf{4}_{v_b} q \rangle \langle q \mathbf{3}_{v_b} \rangle \\ & -\frac{(2\omega^2 - 1)}{m_a m_b} x_a \langle \mathbf{2}_{v_a} q \rangle \langle q \mathbf{1}_{v_a} \rangle x_b \langle \mathbf{4}_{v_b} q \rangle \langle q \mathbf{3}_{v_b} \rangle \Big] . \end{aligned} \tag{3.97}$$

Both amplitudes agree with known results [23, 52, 92]. Furthermore, the amplitudes as written are composed of terms which each individually correspond to a single order in the spin-multipole expansion. All terms in these amplitudes scale as $\hbar^{-3}$, so these amplitudes are classical in the sense mentioned in the previous section.

Using the exponential forms of the three-point amplitudes in Section 3.4.4, we can write down the boson-exchange amplitudes in the infinite spin case. We find the same result in the gravitational case as ref. [47]. However we have obtained this result immediately simply by gluing together the three-point amplitudes; we had no need to boost the external states such they represent the same momentum. Omitting the momentum arguments, the amplitudes are

$$\lim_{s_a, s_b \to \infty} \mathcal{A}_{\text{tree}}^{s_a, s_b} = -\frac{2e^2}{q^2} \sum_{\pm} (\omega \pm \sqrt{\omega^2 - 1}) \exp\left[\pm q \cdot \left(\frac{S_a}{m_a} + \frac{S_b}{m_b} \right) \right], \tag{3.98a}$$

$$\lim_{s_a, s_b \to \infty} \mathcal{M}_{\text{tree}}^{s_a, s_b} = -\frac{\kappa^2 m_a m_b}{4q^2} \sum_{\pm} (\omega \pm \sqrt{\omega^2 - 1})^2 \exp\left[\pm q \cdot \left(\frac{S_a}{m_a} + \frac{S_b}{m_b} \right) \right]. \tag{3.98b}$$

The gravitational result corresponds to the first post-Minkowskian (1PM) order amplitude.

3.6.3 Compton scattering

Our focus shifts now to the electromagnetic and gravitational Compton amplitudes. These computations will enable the exploitation of the explicit $\hbar$ and spin-multipole expansions to relate the classical limit $\hbar \to 0$ and the classical spin-multipole expansion. Concretely, we will show that the spin-multipole expansion of the leading-in-$\hbar$ terms factorizes into a product of factors of the classical spin-dependence at three-points.

First, consider the spin-s electromagnetic Compton amplitude with two opposite helicity photons, $\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-1}, q_4^{+1})$. To simplify calculations, we can set the initial residual momentum to 0, so that $p_1^\mu = m v^\mu$. Note that it is impossible to set both initial and final residual momenta to 0 simultaneously, so we will need eqs. (3.55) and (3.56). We perform the computation by means of Britto-Cachazo-Feng-Witten (BCFW) recursion [99, 100], using the $[3, 4\rangle$-shift

$$|\hat{4}\rangle = |4\rangle - z|3\rangle, \quad |\hat{3}] = |3] + z|4]. \tag{3.99a}$$

Under this shift, two factorization channels contribute to this amplitude:

$$\mathcal{A}(-\mathbf{1}^s,\mathbf{2}^s,q_3^{-1},q_4^{+1}) = \left.\frac{\mathcal{A}(-\mathbf{1}^s,\hat{q}_3^{-1},\hat{P}_{13}^s)\mathcal{A}(\mathbf{2}^s,\hat{q}_4^{+1},-\hat{P}_{13}^s)}{\langle 3|p_1|3]}\right|_{\hat{P}_{13}^2=m^2} + \left.\frac{\mathcal{A}(-\mathbf{1}^s,\hat{q}_4^{+1},\hat{P}_{14}^s)\mathcal{A}(\mathbf{2}^s,\hat{q}_3^{-1},-\hat{P}_{14}^s)}{\langle 4|p_1|4]}\right|_{\hat{P}_{14}^2=m^2}. \tag{3.100}$$

This shift avoids boundary terms for $s \le 1$ as $z \to \infty$. When expressing the factorization channels in terms of on-shell HPET variables, there is a question about whether new boundary terms arise relative to the traditional on-shell variables for $z \to \infty$, as would generally be expected because of higher-dimensional operators present in EFTs. This is not the case here, since eq. (3.36) shows that the definition of the on-shell HPET variables accounts for the contributions from all higher order HPET operators. Another way to see this is that, since the relation between the traditional and on-shell HPET variables is exact, an amplitude must always have the same large z scaling for any shift when expressed using the on-shell HPET variables as when expressed with the traditional on-shell variables. Consider for example the spinor contraction part of the P_{13} factorization channel. In the traditional variables, this is

$$\langle \mathbf{2}P_{13}\rangle^I{}_I[\hat{P}_{13}\mathbf{1}], \tag{3.101}$$

which scales as z when $z \to \infty$. In the on-shell HPET variables:

$$\frac{m}{m_{q_3+q_4}}\left(\langle \mathbf{2}_v P_{13v}\rangle_I + \frac{1}{4m}[\mathbf{2}_v 4]\langle \hat{4}P_{13v}\rangle_I + \frac{1}{4m\hat{x}_4}[\mathbf{2}_v 4][4\hat{P}_{13v}]_I\right)\langle P_{13v}|^I\left(\mathbb{I} - \frac{1}{2m\hat{x}_3^{-1}}|3\rangle\langle 3|\right)|\mathbf{1}_v\rangle. \tag{3.102}$$

Choosing appropriate reference vectors for $\hat{x}_3^{-1}$ and $\hat{x}_4$ ($|4]$ and $|3\rangle$ respectively), we recover the unshifted x_3^{-1} and x_4. Thus this also scales as z when $z \to \infty$. All other factors involved in the factorization channel are common to both sets of variables.

Adding the P_{13} and P_{14} factorization channels, we find the spin-s Compton amplitude

$$\begin{aligned}\mathcal{A}(-\mathbf{1}^s,\mathbf{2}^s,q_3^{-1},q_4^{+1}) &= (-1)^{2s}\mathcal{A}(-\mathbf{1}^0,\mathbf{2}^0,q_3^{-1},q_4^{+1})[4|p_1|3\rangle^{-2s}\left(1-\frac{q_3\cdot q_4}{2m^2}\right)^{-2s} \\ &\times\left(\langle 3\mathbf{1}_v\rangle[4\mathbf{2}_v] - \langle 3\mathbf{2}_v\rangle[4\mathbf{1}_v] + \frac{[43]}{2m}\langle \mathbf{2}_v 3\rangle\langle 3\mathbf{1}_v\rangle - \frac{\langle 34\rangle}{2m}[\mathbf{2}_v 4][4\mathbf{1}_v]\right)^{2s}, \\ \mathcal{A}(-\mathbf{1}^0,\mathbf{2}^0,q_3^{-1},q_4^{+1}) &= -\frac{e^2[4|p_1|3\rangle^2}{\langle 4|p_1|4]\langle 3|p_1|3]},\end{aligned} \tag{3.103}$$

which is in agreement with the result in ref. [46] for QED when the massive spinors are replaced with on-shell HPET spinors. In the gravitational case, we find

$$\begin{aligned}\mathcal{M}(-\mathbf{1}^s,\mathbf{2}^s,q_3^{-2},q_4^{+2}) &= (-1)^{2s}\mathcal{M}(-\mathbf{1}^0,\mathbf{2}^0,q_3^{-2},q_4^{+2})[4|p_1|3\rangle^{-2s}\left(1-\frac{q_3\cdot q_4}{2m^2}\right)^{-2s} \\ &\times\left(\langle 3\mathbf{1}_v\rangle[4\mathbf{2}_v] - \langle 3\mathbf{2}_v\rangle[4\mathbf{1}_v] + \frac{[43]}{2m}\langle \mathbf{2}_v 3\rangle\langle 3\mathbf{1}_v\rangle - \frac{\langle 34\rangle}{2m}[\mathbf{2}_v 4][4\mathbf{1}_v]\right)^{2s}, \\ \mathcal{M}(-\mathbf{1}^0,\mathbf{2}^0,q_3^{-2},q_4^{+2}) &= -\frac{\kappa^2[4|p_1|3\rangle^4}{8q_3\cdot q_4\langle 4|p_1|4]\langle 3|p_1|3]}.\end{aligned} \tag{3.104}$$

Note the appearance of spurious poles for $s > 1$ in the electromagnetic case, and for $s > 2$ in the gravitational case, consistent with the necessarily composite nature of higher spin particles [63].

Spin effects are isolated in the last two terms in parentheses. This can be seen in two ways. The first is to rewrite these last two terms in the language of ref. [45]:

$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-2}, q_4^{+2}) = \frac{(-1)^{2s}}{m^{2s}}\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{+2}) \left(1 - \frac{q_3 \cdot q_4}{2m^2}\right)^{-2s} \times \langle \mathbf{2}_v|^{2s} \left(\mathbb{I} + \frac{1}{2} i \frac{q_{3,\mu}\varepsilon^-_{3,\nu} J^{\mu\nu}}{p_1 \cdot \varepsilon^-_3} + \frac{1}{2} i \not{v} \frac{q_{4,\mu}\varepsilon^+_{4,\nu} J^{\mu\nu}}{p_1 \cdot \varepsilon^+_4} \not{v}\right)^{2s} |\mathbf{1}_v\rangle^{2s}. \tag{3.105}$$

Alternatively, as is more convenient for our purposes, the factorization into classical three-point amplitudes can be made more visible by application of the Schouten identity to these terms:

$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-2}, q_4^{+2}) = \frac{(-1)^{2s}}{m^{2s}}\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{+2})(\mathcal{N}_1 + \mathcal{N}_2)^{2s}, \tag{3.106a}$$

where

$$\mathcal{N}_1 \equiv \langle \mathbf{2}_v| \left[\mathbb{I} + \frac{(q_4 - q_3)\cdot S}{m_{q_3+q_4}}\right] |\mathbf{1}_v\rangle, \tag{3.106b}$$

$$\begin{aligned}\mathcal{N}_2 &\equiv \langle \mathbf{2}_v| \left[v|4]\langle 3| \frac{p_1 \cdot q_4}{m_{q_3+q_4}[4|p_1|3\rangle} + |3\rangle[4|v \frac{p_1 \cdot q_3}{m_{q_3+q_4}[4|p_1|3\rangle}\right] |\mathbf{1}_v\rangle \\ &= \langle \mathbf{2}_v| \left[v|4]\langle 3| \frac{q_3 \cdot q_4}{m_{q_3+q_4}[4|p_1|3\rangle} + \frac{w \cdot S}{m_{q_3+q_4}}\right] |\mathbf{1}_v\rangle,\end{aligned} \tag{3.106c}$$

and

$$w_{\alpha\dot\alpha} \equiv 2p_1 \cdot q_3 \frac{|3\rangle_\alpha [4|_{\dot\alpha}}{[4|p_1|3\rangle}, \quad w^{\dot\alpha\alpha} = 2p_1 \cdot q_3 \frac{|4]^{\dot\alpha}\langle 3|^\alpha}{[4|p_1|3\rangle}. \tag{3.106d}$$

$\mathcal{N}_2$ is the term that contributes spurious poles for high enough spins. The contraction $w \cdot S$ has been defined through eq. (3.42a). The momentum w^μ scales linearly with $\hbar$, so the contraction $w \cdot S$ does not scale with $\hbar$. Compared to this term, the first term in $\mathcal{N}_2$ is subleading in $\hbar$. Ignoring it in the classical limit, and noting that binomial combinatoric factors must be absorbed into the spin-vector when it is raised to some power, the remaining terms imply an exponential spin structure:

$$\mathcal{M}^{\text{cl.}}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-2}, q_4^{+2}) = \frac{(-1)^{2s}}{m^{2s}}\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{+2})\langle \mathbf{2}_v|^{2s} \exp\left[\frac{(q_4 - q_3 + w)\cdot S}{m}\right] |\mathbf{1}_v\rangle^{2s}. \tag{3.107}$$

The same exponentiation holds in the electromagnetic case, with the spinless amplitude above replaced by the corresponding spinless amplitude for QED.

The leading $\hbar$ scaling for these amplitudes is $\hbar^{-1}$ whereas naïve counting of the vertices and propagators says that the scaling should be $\hbar^{-2}$. The source of this discrepancy is interference between the two factorization channels, yielding a factor in the numerator of $p_1 \cdot (\hbar\bar{q}_3 + \hbar\bar{q}_4) = \hbar^2 \bar{q}_3 \cdot \bar{q}_4$. It is thus possible for the naïve $\hbar$ counting to over-count inverse powers of $\hbar$, and hence overestimate the classicality of an amplitude. This has consequences for the extension of these results to the emission of n bosons: factorization channels with a cut graviton line are naïvely suppressed by one factor of $\hbar$ relative to those with cut matter lines. The interference described here means that both factorizations may actually have the same leading $\hbar$ behavior.

Consider now the same-helicity amplitudes. The two-negative-helicity amplitude for spin-1 has been computed by one of the present authors in ref. [101] by shifting one massive and one massless leg. Extending the amplitude found there to spin s,

$$\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-1}, q_4^{-1}) = \frac{1}{m^{2s}}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1})[\mathbf{21}]^{2s}, \tag{3.108a}$$

$$\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1}) = \frac{e^2 m^2 \langle 34\rangle^2}{\langle 3|p_1|3]\langle 4|p_1|4]} \tag{3.108b}$$

We have replaced the coupling in ref. [101] with e^2, as is appropriate for QED. Expressing this in terms of on-shell HPET variables, we find

$$\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-1}, q_4^{-1}) = \frac{1}{m^{2s}}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1})[\mathbf{2}_v|^{2s}\left(\mathbb{I} - \frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right)^{2s}|\mathbf{1}_v]^{2s}. \tag{3.109}$$

The spin-dependence immediately becomes explicit after the change of variables. The exponential spin structure is obvious:

$$\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-1}, q_4^{-1}) = \frac{1}{m^{2s}}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1})[\mathbf{2}_v|^{2s}\exp\left[-\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v]^{2s}. \tag{3.110}$$

When the gyromagnetic ratio $g = 2$, the arbitrary spin $s = s_1 + s_2$ gravitational Compton amplitude is proportional to the product between the spin s_1 and s_2 electromagnetic amplitudes [102–104]. As we have constructed the electromagnetic Compton amplitude using the minimal coupling three-point amplitude, this condition is satisfied. The same-helicity gravitational Compton amplitude is thus

$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{-2}, q_4^{-2}) = \frac{1}{m^{2s}}\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{-2})[\mathbf{2}_v|^{2s}\exp\left[-\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v]^{2s}, \tag{3.111a}$$

$$\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-2}, q_4^{-2}) = \frac{\kappa^2}{8e^4}\frac{\langle 3|p_1|3]\langle 4|p_1|4]}{q_3\cdot q_4}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{-1}, q_4^{-1})^2. \tag{3.111b}$$

Analogous results hold for the emission of two positive helicity bosons:

$$\mathcal{A}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{+1}, q_4^{+1}) = \frac{1}{m^{2s}}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+1}, q_4^{+1})\langle\mathbf{2}_v|^{2s}\exp\left[\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v\rangle^{2s}, \tag{3.112a}$$

$$\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+1}, q_4^{+1}) = \frac{e^2[34]^2}{\langle 3|p_1|3]\langle 4|p_1|4]}, \tag{3.112b}$$

$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{+2}, q_4^{+2}) = \frac{1}{m^{2s}}\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+2}, q_4^{+2})\langle\mathbf{2}_v|^{2s}\exp\left[\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right]|\mathbf{1}_v\rangle^{2s}, \tag{3.112c}$$

$$\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+2}, q_4^{+2}) = \frac{\kappa^2}{8e^4}\frac{\langle 3|p_1|3]\langle 4|p_1|4]}{q_3\cdot q_4}\mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+1}, q_4^{+1})^2. \tag{3.112d}$$

Taking the classical limit, we can simply replace $m_{q_3+q_4} \to m$ to obtain the leading $\hbar$ behavior of these amplitudes.

To see that the spin-dependence of the leading $\hbar$ portions of the amplitudes in this section factorize into a product of the three-point amplitudes, note that

$$[q_i \cdot S, q_j \cdot S]_\alpha{}^\beta = -\big(v \cdot q_{[i}q_{j]} \cdot S - i q_i^\mu q_j^\nu J_{\mu\nu}\big)_\alpha{}^\beta = \mathcal{O}(\hbar), \tag{3.113}$$

where square brackets around indices represent normalized anti-symmeterization of the indices. We can thus combine exponentials and split exponentials of sums only at the cost of subleading-in-$\hbar$ corrections.

The on-shell HPET variables have made it immediate that the spin exponentiates in the same-helicity Compton amplitudes, and this exponentiation is preserved in the $\hbar \to 0$ limit. In the opposite helicity case, the composite nature of higher spin particles can be seen to influence dynamics already at the leading $\hbar$ level. It does so through the contraction $w \cdot S$ for the unphysical momentum w^μ, which appears in a spin exponential in the leading $\hbar$ term. The focus in this section has been on the emission of two bosons, but we will now show that the exponentiation in the same-helicity case extends to the n bosons scenario.

3.6.4 Emission of n bosons

We can generalize the exponentiation of the spin observed in the same-helicity Compton amplitudes. In particular, focusing on integer spins for simplicity, we show that for the tree-level emission of n same-helicity bosons with a common helicity h from a heavy spin-s particle, the amplitude satisfies

$$\begin{aligned} M^s_{n+2} &= \frac{(-1)^{nh}}{m^{2s}} M^{s=0}_{n+2} \langle \mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^n q_i \cdot S\right] |\mathbf{1}_v\rangle^{2s} \\ &= \frac{(-1)^{nh}}{m^{2s}} M^{s=0}_{n+2} [\mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^n q_i \cdot S\right] |\mathbf{1}_v]^{2s}. \end{aligned} \tag{3.114}$$

We use $q \equiv \sum_{i=1}^n q_i$ throughout this section. Once we have proven the first line, the second follows from the fact that the velocity commutes with the spin-vector. The easiest way to proceed is inductively, constructing the $n+2$ point amplitude using BCFW recursion. The cases $n = 1, 2$ were the focus of previous sections.

First, note that when expressed in terms of traditional on-shell variables, the spin dependence in eq. (3.114) is simply

$$\langle \mathbf{21}\rangle^{2s} = \langle \mathbf{2}_v|^{2s} \exp\left(\frac{q \cdot S}{m_q}\right) |\mathbf{1}_v\rangle^{2s} = [\mathbf{2}_v|^{2s} \exp\left(\frac{q \cdot S}{m_q}\right) |\mathbf{1}_v]^{2s}, \quad \text{for } h > 0, \tag{3.115a}$$

$$[\mathbf{21}]^{2s} = [\mathbf{2}_v|^{2s} \exp\left(-\frac{q \cdot S}{m_q}\right) |\mathbf{1}_v]^{2s} = \langle \mathbf{2}_v|^{2s} \exp\left(-\frac{q \cdot S}{m_q}\right) |\mathbf{1}_v\rangle^{2s}, \quad \text{for } h < 0. \tag{3.115b}$$

Thus the problem becomes to prove that the spin dependence is isolated in these spinor contractions. Having already proven this for the base cases, let us now assume it holds up to the emission of $n-1$ bosons and show that this implies the relations for the emission of n bosons. Constructing the $n+2$-point amplitude using BCFW, the amplitude takes the general form (see fig. 3.1)

$$M^s_{n+2} = \sum_{k=1}^{n-1}\sum_{\sigma(k)}\left[\hat{M}^{s,I}_{\sigma(k),k+2}\frac{i\epsilon_{IJ}}{P^2_{1,\sigma(k)}}\hat{M}^{s,J}_{\sigma(n-k),n-k+2} + \sum_{h=\pm}\hat{M}^{s,h}_{\sigma(k),k+3}\frac{i}{P^2_{0,\sigma(k)}}\hat{M}^{-h}_{\sigma(n-k),n-k+1}\right], \tag{3.116}$$

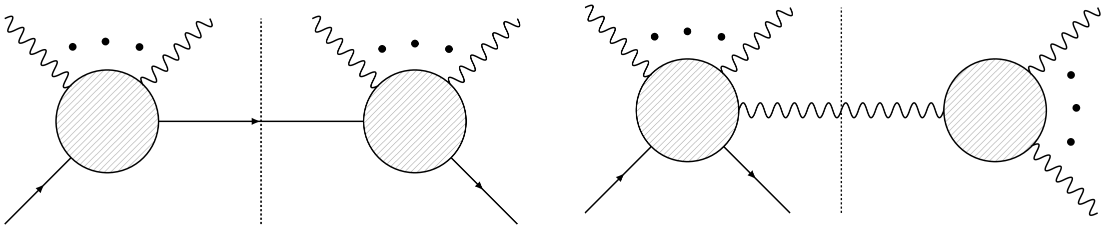


Figure 3.1: The two types of BCFW cuts contributing to the $n+2$-point amplitude. The cut on the left represents the first term in eq. (3.116) while the second term in that equation is represented by the cut on the right.

where $P_{1,\sigma(k)} \equiv p_1 + \sum_{i=1}^{k} q_{\rho(i,\sigma(k))} \equiv p_1 + P_{0,\sigma(k)}$. The permutations $\sigma(k)$ and $\sigma(n-k)$ account for all the ways of organizing the boson legs into $k+2$ and $n-k+2$ point amplitudes, in which shifted legs are never in the same sub-amplitude. $\rho(i,\sigma(k))$ denotes the i^{th} index in the permutation $\sigma(k)$. The notation $\hat{M}$ reminds us that the sub-amplitudes are functions of shifted momenta. The first term in eq. (3.116) represents factorizations where a massive propagator is on-shell, whereas the second accounts for a massless propagator going on-shell — h in this second term is the helicity of the cut boson.

We will treat each term in eq. (3.116) separately. We begin with the first term, which is the only contribution for QED. For the case of n positive-helicity bosons, we shift $|\mathbf{1}]$ and, say, $|q_1\rangle$ as in ref. [101]. Then, applying the induction hypothesis, this term is

$$\begin{aligned}
&\frac{(-1)^{nh}}{m^{4s}} \sum_{k=1}^{n-1} \sum_{\sigma(k)} \hat{M}^{s=0,I}_{\sigma(k),k+2} \frac{i}{P^2_{1,\sigma(k)}} \hat{M}^{s=0,J}_{\sigma(n-k),n-k+2} \langle \mathbf{2}\hat{P}^I_{1,\sigma(k)}\rangle^{2s} \langle \hat{P}_{1,\sigma(k)I}\mathbf{1}\rangle^{2s} \\
&\quad = \frac{(-1)^{nh}}{m^{2s}} \langle \mathbf{21}\rangle^{2s} \sum_{k=1}^{n-1} \sum_{\sigma(k)} \hat{M}^{s=0,I}_{\sigma(k),k+2} \frac{i}{P^2_{1,\sigma(k)}} \hat{M}^{s=0,J}_{\sigma(n-k),n-k+2}
\end{aligned} \tag{3.117}$$

The case of n negative-helicity bosons can be shown similarly by shifting $|\mathbf{1}\rangle$ and, say, $|q_1]$. In particular, choosing an appropriate shift of one massive and one massless leg results in no massive shift appearing in the sub-amplitudes. Applying eq. (3.115) to this, the form of the first term in eq. (3.116) is therefore

$$\frac{(-1)^{nh}}{m^{2s}} \langle \mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^{n} q_i \cdot S\right] |\mathbf{1}_v\rangle^{2s} \sum_{k=1}^{n-1}\sum_{\sigma(k)} \hat{M}^{s=0}_{\sigma(k),k+2} \frac{i}{P^2_{1,\sigma(k)}} \hat{M}^{s=0}_{\sigma(n-k),n-k+2} \tag{3.118}$$

The remaining sum here is the BCFW form of the amplitude for n-photon emission from a massive scalar. Thus we have proven eq. (3.114) for the photon case.

The non-linear nature of gravity allows contributions from the second term in eq. (3.116). The contribution of this term to the amplitude is predictable for unique-helicity configurations. The only non-vanishing factorization channels will involve the product of $(n-1)+2$ point amplitudes with $n-1$ same-helicity gravitons, and a three-graviton amplitude with one distinct helicity graviton, which is the cut graviton. For example, consider the all-plus helicity amplitude. Applying the induction hypothesis,

$$\sum_{k=1}^{n-1}\sum_{\sigma(k)}\sum_{h=\pm} \hat{\mathcal{M}}^{s,h}_{\sigma(k),k+3} \frac{i}{P^2_{0,\sigma(k)}} \hat{\mathcal{M}}^{-h}_{\sigma(n-k),n-k+1} = \sum_{\sigma(n-2)} \hat{\mathcal{M}}^{s,+}_{\sigma(n-2),n+1} \frac{i}{P^2_{0,\sigma(n-2)}} \hat{\mathcal{M}}^{-}_{\sigma(2),3}$$

$$= \frac{1}{m^{2s}}\langle \mathbf{2}_v|^{2s}\text{exp}\left[\frac{1}{m_q}\sum_{i=1}^{n} q_i\cdot S\right]|\mathbf{1}_v\rangle^{2s}\sum_{\sigma(n-2)}\hat{\mathcal{M}}^{s=0,+}_{\sigma(n-2),n+1}\frac{i}{P^2_{0,\sigma(n-2)}}\hat{\mathcal{M}}^{-}_{\sigma(2),3}. \tag{3.119}$$

We have used momentum conservation to write the cut momentum in terms of the sum of the momenta of the gravitons in the all-graviton subamplitude. The argument is identical in the all-negative case. Adding eqs. (3.118) and (3.119) and identifying the remaining sums of sub-amplitudes as the scalar amplitude for the emission of $n+2$ gravitons, we find

$$\begin{aligned}\mathcal{M}^s_{n+2} &= \frac{1}{m^{2s}}\mathcal{M}^{s=0}_{n+2}\langle \mathbf{2}_v|^{2s}\text{exp}\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^{n} q_i\cdot S\right]|\mathbf{1}_v\rangle^{2s} \\ &= \frac{1}{m^{2s}}\mathcal{M}^{s=0}_{n+2}[\mathbf{2}_v|^{2s}\text{exp}\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^{n} q_i\cdot S\right]|\mathbf{1}_v]^{2s}.\end{aligned} \tag{3.120}$$

In amplitudes where this spin universality is manifest, we can eliminate the dependence on the specific states used by taking the infinite spin and classical limits of the result,

$$\lim_{\substack{s\to\infty\\ \hbar\to 0}} M^s_{n+2} = M^{s=0}_{n+2}\,\text{exp}\left[\frac{1}{m}\frac{h}{|h|}\sum_{i=1}^{n} q_i\cdot S\right], \tag{3.121}$$

where we have used that $\lim_{\hbar\to 0} p^{\mu}_{v,2} = \lim_{\hbar\to 0} p^{\mu}_{v,1} = mv^{\mu}$ to apply on-shell conditions. This makes contact between the classical limit of the kinematics, and the classical spin limit: for tree-level same-helicity boson emission processes, the spin dependence of the leading-in-$\hbar$ term factorizes into factors of the classical three-point spin-dependence.

3.7 SUMMARY AND OUTLOOK

We have presented an on-shell formulation of HPETs by expressing their asymptotic states as a linear combination of the chiral and anti-chiral massive on-shell helicity variables of ref. [63]. This expression automatically takes into account the infinite tower of higher-dimensional operators present in HPETs, which result from the integrating out of the anti-field. The variables defined in this manner possess manifest spin multipole and $\hbar$ expansions. Consequently, using the most general three-point amplitude of ref. [63], we have been able to derive a closed form for the amplitude arising from the sum of all three-point operators in an arbitrary spin HPET. This form of the amplitude has been checked explicitly up to NNLO in the operator expansion of spin-1/2 HQET and HBET. We will also show in Section 3.E that the extension to higher spins is suitable for describing the three-point amplitude for zero initial residual momentum for a heavy spin-1 particle coupled to electromagnetism.

We have shown that the spin-multipole expansion of minimally coupled heavy particles corresponds exactly to a truncated Kerr black hole expansion when the initial residual momentum is set to zero. This has been done in two ways. First, we exponentiated the spin dependence of the minimally coupled three-point amplitude in Section 3.4.4. Doing so directly produced the same spin exponential as that in refs. [45, 97] for a Kerr black hole coupled to a graviton. Unlike previous approaches, no further manipulation of the three-point amplitude was needed to match to refs. [45, 97]. An exact match to all spin orders was achieved in the infinite spin limit. An alternative approach to matching the Kerr black hole multipole moments was carried out in refs. [46, 53], by matching to the EFT of ref. [80].

Following this matching procedure but using on-shell HPET variables, an exact match to the Kerr black hole Wilson coefficients was achieved without the need to take an infinite spin limit. The reason that the three-point amplitude in on-shell HPET variables immediately matches the Kerr black hole multipole expansion is that the heavy spinors representing the initial and final states are both associated with the same momentum, which is identified with that of the black hole.

We set out to provide a framework that would enable the extension of HPETs to higher spins, and to enable the application of HPETs to the computation of higher order classical amplitudes. As a step in this direction, we applied recursion relations to the minimal coupling amplitude for heavy particles to build arbitrary-spin higher-point tree amplitudes. Doing so, we showed that the explicit $\hbar$ and spin multipole expansions at three points remained manifest in all amplitudes considered. We also easily constructed the tree-level boson exchange amplitude to all orders in spin for QED and GR, without having to further manipulate the states to produce the correct classical black hole spin multipole expansion.

Moving on to radiative processes, we showed that the same-helicity electromagnetic and gravitational Compton amplitudes exhibit a spin universality: they can be written as

$$M_4^s = M_4^{s=0}\langle \mathbf{2}_v|^{2s}\exp\left[\frac{1}{m_{q_1+q_2}}\frac{h}{|h|}\sum_{i=1}^{2} q_i\cdot S\right]|\mathbf{1}_v\rangle^{2s}. \tag{3.122}$$

This universality extends to the emission of n same-helicity bosons (eq. (3.114)). In the four-point opposite-helicity case, a similar exponential was obtained only in the classical limit. However the sum in the exponential also included an unphysical momentum contracted with the spin, representing the non-uniqueness of the amplitude for large enough spins. It would be interesting to examine whether the opposite-helicity amplitude possesses an n-boson extension analogous to eq. (3.114). Another natural extension is to study how the leading $\hbar$ behaviour changes when a second matter line is included in radiation processes; this is relevant to the understanding of non-conservative effects in spinning binaries. The understanding of radiative processes is paramount to the PM amplitude program, as the construction of higher PM amplitudes using unitarity methods requires knowledge of tree-level radiative amplitudes. Combining radiative amplitudes with the $\hbar$ counting of the on-shell HPET variables in a unitarity-based approach, the classical limits of amplitudes can be easily identified and taken before integration to simplify computations of classical loop amplitudes including spin.

Because of the topicality of the subject, we have focused in the main body of this paper on the application of these variables to their interpretation as spinning black holes and the construction of classical tree-level amplitudes. Nevertheless, they are equally applicable to the QCD systems which HQET was formulated to describe. Moreover an on-shell perspective is useful for the understanding of HPETs as a whole. Indeed, we take an on-shell approach in the appendices to make further statements about HPETs.

APPENDIX

3.A CONVENTIONS

We list here our conventions for reference. In the Weyl basis, the Dirac gamma matrices take the explicit form

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\alpha} \\ (\bar\sigma^\mu)^{\dot\alpha\alpha} & 0 \end{pmatrix}, \tag{3.123}$$

where $\sigma^\mu = (1, \sigma^i)$, $\bar\sigma^\mu = (1, -\sigma^i)$, and σ^i are the Pauli matrices. The gamma matrices obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We use the mostly minus metric convention, $\eta^{\mu\nu} = \text{diag}\{+,-,-,-\}$. The fifth gamma matrix is defined as

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{3.124}$$

The generator of Lorentz transforms is

$$J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{3.125}$$

We express massless momenta in terms of on-shell variables:

$$q_{\alpha\dot\alpha} \equiv q^\mu(\sigma_\mu)_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha} \equiv |q\rangle_\alpha[q|_{\dot\alpha}, \tag{3.126a}$$
$$q^{\dot\alpha\alpha} \equiv q^\mu(\bar\sigma_\mu)^{\dot\alpha\alpha} = \tilde\lambda^{\dot\alpha}\lambda^\alpha \equiv |q]^{\dot\alpha}\langle q|^\alpha. \tag{3.126b}$$

Here α, $\dot\alpha$ are $SL(2,\mathbb{C})$ spinor indices. Spinor brackets are formed by contracting the spinor indices,

$$\langle\lambda_1\lambda_2\rangle \equiv \langle\lambda_1|^\alpha|\lambda_2\rangle_\alpha, \tag{3.127}$$
$$[\lambda_1\lambda_2] \equiv [\lambda_1|_{\dot\alpha}|\lambda_2]^{\dot\alpha}. \tag{3.128}$$

For massive momenta, we have that

$$p_{\alpha\dot\alpha} = \lambda_\alpha{}^I\tilde\lambda_{\dot\alpha I} \equiv |\lambda\rangle_\alpha^I[\lambda|_{\dot\alpha I}, \tag{3.129a}$$
$$p^{\dot\alpha\alpha} = \tilde\lambda_I^{\dot\alpha}\lambda^{\alpha I} \equiv |\lambda]_I^{\dot\alpha}\langle\lambda|^{\alpha I}, \tag{3.129b}$$

where I is an $SU(2)$ little group index. Spinor brackets for massive momenta are also formed by contracting spinor indices, identically to the massless case. We also use the bold notation introduced in ref. [63] to suppress the symmetrization over $SU(2)$ indices in amplitudes:

$$\langle\mathbf{2}q_1\rangle\langle\mathbf{2}q_2\rangle \equiv \begin{cases} \langle 2^I q_1\rangle\langle 2^J q_2\rangle & I = J, \\ \langle 2^I q_1\rangle\langle 2^J q_2\rangle + \langle 2^J q_1\rangle\langle 2^I q_2\rangle & I \neq J. \end{cases} \tag{3.130}$$

The Levi-Civita symbol, used to raise and lower spinor and $SU(2)$ little group indices, is defined by

$$\epsilon^{12} = -\epsilon_{12} = 1. \tag{3.131}$$

Spinor and $SU(2)$ indices are raised and lowered by contracting with the second index on the Levi-Civita symbol. For example,

$$\lambda^I = \epsilon^{IJ}\lambda_J, \quad \lambda_I = \epsilon_{IJ}\lambda^J. \tag{3.132}$$

The on-shell conditions for the massive helicity variables are

$$\lambda^{\alpha I}\lambda_{\alpha J} = m\delta^I{}_J, \quad \lambda^{\alpha I}\lambda_\alpha{}^J = -m\epsilon^{IJ}, \quad \lambda^\alpha{}_I\lambda_{\alpha J} = m\epsilon_{IJ}, \tag{3.133a}$$
$$\tilde{\lambda}^I_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_J = -m\delta^I{}_J, \quad \tilde{\lambda}^I_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}J} = m\epsilon^{IJ}, \quad \tilde{\lambda}_{\dot{\alpha}I}\tilde{\lambda}^{\dot{\alpha}}_J = -m\epsilon_{IJ}. \tag{3.133b}$$

Given eq. (3.31), we can derive the on-shell conditions of the HPET variables, analogous to eq. (3.133). We find

$$\lambda_v^{\alpha I}\lambda_{v\alpha J} = m_k\delta^I{}_J, \quad \lambda_v^{\alpha I}\lambda_{v\alpha}{}^J = -m_k\epsilon^{IJ}, \quad \lambda^\alpha_{vI}\lambda_{v\alpha J} = m_k\epsilon_{IJ}, \tag{3.134a}$$
$$\tilde{\lambda}^I_{v\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{vJ} = -m_k\delta^I{}_J, \quad \tilde{\lambda}^I_{v\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}J}_v = m_k\epsilon^{IJ}, \quad \tilde{\lambda}_{v\dot{\alpha}I}\tilde{\lambda}^{\dot{\alpha}}_{vJ} = -m_k\epsilon_{IJ}, \tag{3.134b}$$

where

$$m_k \equiv \left(1 - \frac{k^2}{4m^2}\right)m. \tag{3.134c}$$

In Section 3.C we will decompose massive momenta into two massless momenta, as in eq. (3.145). When identifying

$$\lambda_\alpha{}^1 = |a\rangle_\alpha, \quad \lambda_\alpha{}^2 = |b\rangle_\alpha, \tag{3.135a}$$
$$\tilde{\lambda}_{\dot{\alpha}1} = [a|_{\dot{\alpha}}, \quad \tilde{\lambda}_{\dot{\alpha}2} = [b|_{\dot{\alpha}}, \tag{3.135b}$$

we use $\langle ba\rangle = [ab] = m$.

On-shell variables can be assigned to the upper and lower Weyl components of a Dirac spinor so that the spinors satisfy the Dirac equation [46],

$$u^I(p) = \begin{pmatrix} \lambda_\alpha{}^I \\ \tilde{\lambda}^{\dot{\alpha}I} \end{pmatrix}, \quad \bar{u}_I(p) = \begin{pmatrix} -\lambda^\alpha{}_I & \tilde{\lambda}_{\dot{\alpha}I} \end{pmatrix}, \tag{3.136}$$

where p is expressed in terms of λ and $\tilde{\lambda}$ as in eq. (3.129).

Using analytic continuation, under a sign flip of the momentum, the on-shell variables transform as

$$|-\mathbf{p}\rangle = -|\mathbf{p}\rangle, \quad |-\mathbf{p}] = |\mathbf{p}], \tag{3.137a}$$

which means

$$|-\mathbf{p}_v\rangle = |\mathbf{p}_{-v}\rangle = -|\mathbf{p}_v\rangle, \quad |-\mathbf{p}_v] = |\mathbf{p}_{-v}] = |\mathbf{p}_v]. \tag{3.137b}$$

3.B UNIQUENESS OF ON-SHELL HPET VARIABLES

In this section, we address the question of uniqueness of the on-shell HPET variables as defined in eq. (3.31). In particular, we relate the on-shell HPET variables $|\mathbf{p}_v\rangle$ and $|\mathbf{p}_v]$ to the traditional on-shell variables under two conditions:

1. The new variables describe a very massive spin-1/2 state that acts as a source for mediating bosons, meaning that the velocity of the state is approximately constant. Since the motion of the particle is always very closely approximated by its velocity, we demand that the new variables satisfy the Dirac equation for a velocity v^μ and mass $v^2 = 1$:

$$\not{v}|\mathbf{p}_v\rangle = |\mathbf{p}_v], \quad \not{v}|\mathbf{p}_v] = |\mathbf{p}_v\rangle. \tag{3.138}$$

   Clearly these relations can be scaled to give the state an arbitrary mass.

2. When describing a heavy particle with mass m and velocity v^μ, the new variables must reduce to the traditional on-shell variables with $p^\mu = mv^\mu$ when $k = 0$.

We express the on-shell HPET variables in the basis of traditional on-shell variables:

$$|\mathbf{p}_v\rangle = a(k)|\mathbf{p}\rangle + \not{\Gamma}_1(k)|\mathbf{p}], \tag{3.139a}$$

$$|\mathbf{p}_v] = b(k)|\mathbf{p}] + \not{\Gamma}_2(k)|\mathbf{p}\rangle. \tag{3.139b}$$

The fact that the functions $a,\ b,\ \Gamma_1,\ \Gamma_2$ can, without loss of generality, be assumed to be functions of only k^μ (and m) follows from on-shellness and the Dirac equation. Any dependence on v^μ must be either in a scalar form, $v \cdot v = 1$ or $v \cdot k = -k^2/2m$, or in matrix form $\not{v}$, which can be eliminated for $\not{k}/m$ using the Dirac equation for $\not{p}$. This also means that we can rewrite $\Gamma^\mu_{1,2} = c_{1,2}(k)k^\mu$, where the $c_i(k)$ are scalars and potentially functions of k^2. Moreover, given that a and b are functions only of k, they must also be scalars; the only possible matrix combinations they can contain to preserve the correct spinor indices are even powers of $\not{k}$, which would reduce to some power of k^2. Condition 2 provides a final constraint on these four functions:

$$a(0) = b(0) = 1, \tag{3.140a}$$

$$\Gamma_1(0) = \Gamma_2(0) = 0. \tag{3.140b}$$

Since $\Gamma^\mu_i = c_i(k)k^\mu$, the second line imposes that the $c_i(k)$ are regular at $k = 0$. From now on we drop the arguments of these functions for brevity.

Applying condition 1 to eqs. (3.139), we derive relations among the four functions $a,\ b,\ c_1,\ c_2$:

$$b = a, \tag{3.141a}$$

$$c_2 = -\frac{a}{m} - c_1. \tag{3.141b}$$

The most general on-shell HPET variables are thus

$$|\mathbf{p}_v\rangle = a|\mathbf{p}\rangle + c_1\not{k}|\mathbf{p}], \tag{3.142a}$$

$$|\mathbf{p}_v] = a|\mathbf{p}] - \left(\frac{a}{m} + c_1\right) \not{k}|\mathbf{p}\rangle. \tag{3.142b}$$

The momentum associated with these states is

$$\not{p}_v = \begin{pmatrix} 0 & |p_v\rangle^I{}_I[p_v| \\ |p_v]_I{}^I\langle p_v| & 0 \end{pmatrix} = m\left[a^2 + c_1\left(\frac{a}{m} + c_1\right)k^2\right]\not{v}. \tag{3.143}$$

The functions a and c_1 cannot be constrained further by conditions 1 and 2. However we can choose $c_1 = -a/2m$ to describe non-chiral interactions. Then, from an off-shell point of view, the function a simply corresponds to the (potentially non-local) field redefinition $Q \to Q/a$ in the spin-1/2 HPET Lagrangian. We are free to redefine our fields such that $a = 1$. The final result is

$$|\mathbf{p}_v\rangle = |\mathbf{p}\rangle - \frac{\not{k}}{2m}|\mathbf{p}], \tag{3.144a}$$

$$|\mathbf{p}_v] = |\mathbf{p}] - \frac{\not{k}}{2m}|\mathbf{p}\rangle. \tag{3.144b}$$

Thus we recover the on-shell HPET variables in eq. (3.31). We conclude that, up to scaling by an overall function of k^2, eq. (3.31) is the unique decomposition in terms of traditional variables of non-chiral heavy particle states. The overall scalings correspond to field redefinitions in the Lagrangian formulation.

3.C REPARAMETERIZATION AND THE LITTLE GROUP

As is apparent from eq. (3.28), reparameterization transformations leave p^μ unchanged. It is therefore reasonable to expect that there exists a relation between reparameterizations and the little group of p^μ. There is indeed a relationship between infinitesimal little group transformations of $\lambda_\alpha{}^I$ and $\tilde{\lambda}^{\dot{\alpha}}_I$ and reparameterizations of the total momentum. The focus of this section is the derivation of such a connection, which is easy to explore by employing the so-called Light Cone Decomposition (LCD) [105, 106] of massive momenta.

The LCD allows any massive momentum to be written as a sum of two massless momenta. That is, for a momentum p^μ of mass m, there exist two massless momenta a^μ and b^μ such that

$$p^\mu = a^\mu + b^\mu. \tag{3.145}$$

When p^μ is real, we can assume wihtout loss of generality that a^μ and b^μ are real as well, since any imaginary components must cancel anyway. The condition $p^2 = m^2$ then implies $a \cdot b = m^2/2$. Expressing this in on-shell variables,

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha{}^I\tilde{\lambda}_{\dot{\alpha}I} = |a\rangle_\alpha[a|_{\dot{\alpha}} + |b\rangle_\alpha[b|_{\dot{\alpha}}, \tag{3.146a}$$

$$p^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}}_I\lambda^{\alpha I} = |a]^{\dot{\alpha}}\langle a|^\alpha + |b]^{\dot{\alpha}}\langle b|^\alpha. \tag{3.146b}$$

This allows us to make the identifications

$$\lambda_\alpha{}^1 = |a\rangle_\alpha, \quad \lambda_\alpha{}^2 = |b\rangle_\alpha, \quad \tilde{\lambda}_{\dot{\alpha}1} = [a|_{\dot{\alpha}}, \quad \tilde{\lambda}_{\dot{\alpha}2} = [b|_{\dot{\alpha}}. \tag{3.146c}$$

In the spirit of the momentum decomposition in eq. (3.28) we can break this up into a large and a small part

$$p^\mu = \alpha a^\mu + \beta b^\mu + (1-\alpha)a^\mu + (1-\beta)b^\mu, \tag{3.147}$$

where $|\alpha|, |\beta| \sim 1$. We identify

$$mv^\mu \equiv \alpha a^\mu + \beta b^\mu, \quad k^\mu \equiv (1-\alpha)a^\mu + (1-\beta)b^\mu. \tag{3.148}$$

Since v^μ is a four-velocity, it must satify $v^2 = 1$, which constrains α and β to obey $\alpha\beta = 1$. Once we require this, the on-shell condition that $2mv \cdot k = -k^2$ is automatically imposed.

Now, consider a reparameterization of the momentum as in eq. (3.37). We can use the LCD to rewrite the shift momentum as

$$\delta k^\mu = c^\mu + d^\mu, \tag{3.149}$$

where $|c+d|/m \ll 1$. For this to be a reparameterization, the new velocity $v^\mu + \delta k^\mu/m$ must have magnitude 1, which means c^μ and d^μ must be such that

$$(\alpha a + \beta b) \cdot (c+d) = -c \cdot d. \tag{3.150}$$

Contracting the shift momentum with the gamma matrices and using the Schouten identity,

$$\begin{aligned}\delta k_{\alpha\dot\alpha} = {}& \frac{2}{m^2} b\cdot(c+d)|a\rangle_\alpha [a|_{\dot\alpha} + \frac{2}{m^2} a\cdot(c+d)|b\rangle_\alpha [b|_{\dot\alpha} \\ & - \frac{[a|(\not{c}+\not{d})|b\rangle}{m^2}|a\rangle_\alpha [b|_{\dot\alpha} - \frac{[b|(\not{c}+\not{d})|a\rangle}{m^2}|b\rangle_\alpha [a|_{\dot\alpha}.\end{aligned} \tag{3.151}$$

Note that setting $k = 0$ is always allowed for an on-shell momentum by reparameterization: indeed, choosing $c^\mu = (1-\alpha)a^\mu$ and $d^\mu = (1-\beta)b^\mu$ trivially satisfies eq. (3.150).

Consider an infinitesimal little group transformation of the on-shell variables $W^I{}_J$ where $W \in SU(2)$. Then we can write

$$W^I{}_J = \mathbb{I}^I{}_J + i\epsilon^j U^{jI}{}_J, \tag{3.152}$$

where ϵ^j are real and infinitesimal parameters, and $U^{jI}{}_J$ is traceless and Hermitian. We suppress the color index j below. Under this transformation, the on-shell variables transform as [63]

$$\lambda_\alpha{}^I \to W^I{}_J \lambda_\alpha{}^J, \tag{3.153a}$$

$$\tilde\lambda_{\dot\alpha I} \to (W^{-1})^J{}_I \tilde\lambda_{\dot\alpha J}. \tag{3.153b}$$

Up to linear order in the infinitesimal parameter, the momentum transforms as

$$\begin{aligned}p_{\alpha\dot\alpha} = \lambda_\alpha{}^I\tilde\lambda_{\dot\alpha I} \to {}& (1+i\epsilon U^1{}_1)\lambda_\alpha{}^1\tilde\lambda_{\dot\alpha 1} + (1+i\epsilon U^2{}_2)\lambda_\alpha{}^2\tilde\lambda_{\dot\alpha 2} + i\epsilon U^2{}_1\lambda_\alpha^1\tilde\lambda_{\dot\alpha 2} + i\epsilon U^1{}_2\lambda_\alpha^2\tilde\lambda_{\dot\alpha 1} \\ & - i\epsilon U^2{}_1\lambda_\alpha^1\tilde\lambda_{\dot\alpha 2} - i\epsilon U^1{}_2\lambda_\alpha^2\tilde\lambda_{\dot\alpha 1} - i\epsilon U^1{}_1\lambda_\alpha{}^1\tilde\lambda_{\dot\alpha 1} - i\epsilon U^2{}_2\lambda_\alpha{}^2\tilde\lambda_{\dot\alpha 2}.\end{aligned} \tag{3.154}$$

Comparing with eq. (3.151), we would like to identitfy the following map to the reparameterization in eq. (3.37):

$$i\epsilon U^I{}_J \to R^I{}_J \equiv \frac{1}{m}\begin{pmatrix} 2b\cdot\frac{\delta k}{m} & -[b|\frac{\delta k}{m}|a\rangle \\ -[a|\frac{\delta k}{m}|b\rangle & 2a\cdot\frac{\delta k}{m}\end{pmatrix}. \tag{3.155}$$

The reparameterization matrix $R^I{}_J$ is infintesimal because of the appearance of $\delta k^\mu/m$ in each entry. Moreover, $R^I{}_J$ is traceless up to corrections of order $\mathcal{O}(\delta k^2/m^2)$ because of eq. (3.150). However, we cannot equate it to $i\epsilon U^I{}_J$ because the latter is always anti-Hermitian, whereas $R^I{}_J$ need not be. Indeed, when δk^μ is real $R^I{}_J$ is Hermitian, and when δk^μ is imaginary it is anti-Hermitian. It can thus be seen that the condition for equality is that δk^μ is imaginary:

$$\delta k^\mu \in i\mathbb{R} \Rightarrow \mathbb{I}^I{}_J + R^I{}_J \in SU(2), \tag{3.156}$$

where $\mathbb{I}^I{}_J + R^I{}_J$ induces the reparameterization in eq. (3.37). It is straightforward to check that this quantity also has determinant 1, up to infinitesimal corrections of order $\mathcal{O}(\delta k^2/m^2)$.

3.D PROPAGATORS

In ref. [107], massive on-shell variables were used to construct propagators for massive spin-1/2 and spin-1 states. In this section, we use the on-shell HPET variables to do the same for a spin $s \leq 2$ state. We find that the propagator for a heavy particle with spin $s \leq 2$ is

$$D_v^s(p_v) = P^s \frac{N^s(p_v)}{p^2 - m^2} P^s, \tag{3.157}$$

where P^s is the spin-s projection operator whose eigenstate is the HPET state, and $N^s(p_v)$ is the numerator of the propagator for a massive particle of that spin. By recognizing the form of the numerator, this will allow us to extract the higher spin projection operators. The methods used in this section can be applied to arbitrary spin, but become quite cumbersome as the number of little group invariant objects that must be computed grows as $s + 1/2$ for half-integer spins, and as s for integer spins. Nevertheless, we are able to use our results to conjecture projection operators for any spin.

Spin-1/2

We begin with the spin-1/2 propagator, which can be constructed as

$$\frac{1}{p^2 - m^2}\left[\begin{pmatrix} |p_v^I\rangle \\ |p_v^I] \end{pmatrix} \epsilon_{IJ} \begin{pmatrix} \langle -p_v^J| & [-p_v^J| \end{pmatrix}\right] = P_+ \frac{2m_k}{p^2 - m^2} P_+ = P_+ \frac{1}{\not{p} - m} P_+. \tag{3.158}$$

We do indeed recover the projection operator for a heavy spin-1/2 field.

Spin-1

We can do the same for a massive spin-1 field. In this case, we posit that the polarization vector is obtained by replacing $p \to p_v$ and $m \to m_k$ in the usual polarization vector:

$$\varepsilon^{IJ}_{v,\mu}(p) = \frac{1}{2\sqrt{2}m_k}(\langle p^I_v|\gamma_\mu|p^J_v] + \langle p^J_v|\gamma_\mu|p^I_v]). \tag{3.159}$$

It is straightforward to see that the polarization vector satisfies the requisite condition on the heavy spin-1 particle, $v \cdot \varepsilon^{IJ}_v = 0$ for $p^\mu = mv^\mu + k^\mu$, as well as the orthonormality condition

$$\varepsilon^{IJ}_v \cdot \varepsilon^{LK}_v = -\frac{1}{2}(\epsilon^{IL}\epsilon^{JK} + \epsilon^{IK}\epsilon^{JL}). \tag{3.160}$$

The heavy spin-1 propagator is

$$\frac{1}{p^2 - m^2}\left[\varepsilon^{IJ}_{v,\mu}(p)\epsilon_{IK}\epsilon_{JL}\varepsilon^{LK}_{v,\nu}(-p)\right] = (g_\mu{}^\lambda - v_\mu v^\lambda)\frac{-g_{\lambda\sigma} + v_\lambda v_\sigma}{p^2 - m^2}\left(g^\sigma{}_\nu - v^\sigma v_\nu\right). \tag{3.161}$$

From this we can read off that the operator projecting onto the heavy spin-1 particle is $P^{\mu\nu}_-$ in Section 3.E.

Spin-3/2

The spin-3/2 polarization tensor is

$$\varepsilon^{IJK}_{v,\mu}(p) = \varepsilon^{(IJ}_{v,\mu}u^{K)}_v = \frac{1}{\sqrt{2}m_k}\langle p^{(I}_v|\gamma_\mu|p^J_v]\begin{pmatrix}|p^{K)}_v\rangle \\ |p^{K)}_v]\end{pmatrix}, \tag{3.162}$$

where the round brackets around sets of indices denote normalized symmetrization over the indices. Using the symmetry of the spin-1 polarization vector in its little group indices, we have that

$$\varepsilon^{IJK}_{v,\mu}(p) = \frac{1}{3}\left(\varepsilon^{IJ}_{v,\mu}u^K_v + \varepsilon^{JK}_{v,\mu}u^I_v + \varepsilon^{IK}_{v,\mu}u^J_v\right). \tag{3.163}$$

The propagator is

$$\begin{aligned}
&\frac{1}{p^2 - m^2}\left[\varepsilon^{IJK}_{v,\mu}(p)\epsilon_{IA}\epsilon_{JB}\epsilon_{KC}\varepsilon^{ABC}_{v,\nu}(-p)\right] \\
&\quad = \frac{1}{p^2 - m^2}\frac{1}{3}\left(\varepsilon^{IJ}_{v,\mu}\varepsilon_{v,\nu IJ}u^K_v\bar{u}_{v,K} + 2\varepsilon^{IJ}_{v,\mu}\varepsilon_{v,\nu IK}u^K_v\bar{u}_{v,J}\right) \\
&\quad = -P_+P_{-,\mu\alpha}\frac{2m_k}{p^2 - m^2}\left[g^{\alpha\beta} - \frac{1}{3}\gamma^\alpha\gamma^\beta - \frac{1}{3}(\not{v}\gamma^\alpha v^\beta + v^\alpha\gamma^\beta\not{v})\right]P_{-,\beta\nu}P_+.
\end{aligned} \tag{3.164}$$

We recognize the quantity between the projection operators as the propagator for a massive spin-3/2 particle with momentum $m_k v^\mu$ [108, 109]. The heavy spin-3/2 projection operator can thus be identified as

$$P^{\mu\nu}_{\frac{1}{2},-} \equiv P_+P^{\mu\nu}_-. \tag{3.165}$$

Spin-2

The spin-2 polarization tensor is

$$\varepsilon^{I_1J_1I_2J_2}_{v,\mu_1\mu_2}(p) = \varepsilon^{(I_1J_1}_{v,\mu_1}\varepsilon^{I_2J_2)}_{v,\mu_2} = \frac{1}{2m_k^2}\langle p_v^{(I_1}|\gamma_{\mu_1}|p_v^{J_1}]\langle p_v^{I_2}|\gamma_{\mu_2}|p_v^{J_2)}]. \tag{3.166}$$

Using the symmetry of each spin-1 polarization vector in its little group indices, we find that

$$\varepsilon^{I_1J_1I_2J_2}_{v,\mu_1\mu_2}(p) = \frac{1}{3}\left(\varepsilon^{I_1J_1}_{v,(\mu_1}\varepsilon^{I_2J_2}_{v,\mu_2)} + \varepsilon^{I_1I_2}_{v,(\mu_1}\varepsilon^{J_1J_2}_{v,\mu_2)} + \varepsilon^{I_1J_2}_{v,(\mu_1}\varepsilon^{I_2J_1}_{v,\mu_2)}\right). \tag{3.167}$$

The propagator is

$$\begin{aligned} &\frac{1}{p^2-m^2}\left[\varepsilon^{I_1J_1I_2J_2}_{v,\mu\nu}(p)\epsilon_{I_1K_1}\epsilon_{J_1L_1}\epsilon_{I_2K_2}\epsilon_{J_2L_2}\varepsilon^{K_1L_1K_2L_2}_{v,\alpha\beta}(-p)\right] \\ &\quad = \frac{1}{p^2-m^2}\frac{1}{3}\left(\varepsilon^{I_1J_1}_{v,(\mu}\varepsilon^{I_2J_2}_{v,\nu)}\varepsilon_{v,\alpha I_1J_1}\varepsilon_{v,\beta I_2J_2} + 2\varepsilon^{I_1J_1}_{v,(\mu}\varepsilon^{I_2J_2}_{v,\nu)}\varepsilon_{v,\alpha I_1J_2}\varepsilon_{v,\beta I_2J_1}\right) \\ &\quad = \frac{1}{p^2-m^2}P_{-,\mu\mu'}P_{-,\nu\nu'}\left[-\frac{1}{2}(P_-^{\mu'\alpha'}P_-^{\nu'\beta'} + P_-^{\mu'\beta'}P_-^{\nu'\alpha'}) + \frac{1}{3}P_-^{\mu'\nu'}P_-^{\alpha'\beta'}\right]P_{-,\alpha'\alpha}P_{-,\beta'\beta}. \end{aligned} \tag{3.168}$$

The quantity in square brackets is the numerator of the massive spin-2 propagator with momentum $m_k v^\mu$ [110]. We therefore identify the heavy spin-2 projection operator:

$$P_-^{\mu\nu,\alpha\beta} \equiv P_-^{\mu\nu}P_-^{\alpha\beta}. \tag{3.169}$$

3.D.1 Spin-s Projection Operator

Based on the above discussion, as well as the properties of a general spin heavy field, we conjecture the projection operator for a spin-s field. An integer spin-s field $Z^{\mu_1\cdots\mu_s}$ must be symmetric and traceless [65]. When the mass of the particle is very large, the particle component $\mathcal{Z}$ must satisfy [111]

$$v_{\mu_1}\mathcal{Z}^{\mu_1\cdots\mu_s} = 0. \tag{3.170}$$

By symmetry, this condition holds regardless of the index with which the velocity is contracted. The general spin-s projection operator for a field satisfying eq. (3.170), and which reduces to the above cases for $s = 1$ and $s = 2$ is

$$P_-^{\mu_1\nu_1,\ldots,\mu_s\nu_s} = \prod_{i=1}^{s} P_-^{\mu_i\nu_i}. \tag{3.171}$$

The integer spin projection operator is simply a product of spin-1 projection operators.

A half-integer spin-$(s+1/2)$ field $\Psi^{\mu_1\cdots\mu_s}$ must be symmetric and γ-traceless [66],

$$\gamma_{\mu_1}\Psi^{\mu_1\cdots\mu_s} = 0. \tag{3.172}$$

Symmetry ensures that the condition holds for any index the γ matrix is contracted with. When the mass of the field becomes very large, its particle component $\mathcal{Q}$ must satisfy [111]

$$\not{v}\mathcal{Q}^{\mu_1\cdots\mu_s} = \mathcal{Q}^{\mu_1\cdots\mu_s}. \tag{3.173}$$

These constraints also imply, among other things, the v-tracelessness of the heavy field. The general spin-$(s+1/2)$ projection operator that results in a field satisfying these conditions, and that reduces to the above cases for spin-1/2 and spin-3/2, is

$$P_{\frac{1}{2},-}^{\mu_1\nu_1,\dots,\mu_s\nu_s} \equiv P_+ P_-^{\mu_1\nu_1,\dots,\mu_s\nu_s}. \tag{3.174}$$

From this we see that knowledge of the spin-1/2 heavy particle states is enough to construct the polarization tensors and projection operators for higher spin states. In this sense, HPETs are unified in terms of the basic building blocks in eq. (3.31).

3.E Matching to HPET Lagrangians

In this section, we address the matching of on-shell amplitudes to those derived from HPET Lagrangians. First, there is a subtlety that must be accounted for when matching the minimal coupling in eqs. (3.55) and (3.56) to an HPET Lagrangian. We focus the discussion of this to the case of spin-1/2 HPET. Next, we confirm explicitly that the general spin three-point amplitude derived from the Zeeman coupling in ref. [46] reproduces the amplitude derived from spin-1 abelian HQET when expressed using on-shell HPET variables.

3.E.1 Matching spin-1/2 minimal coupling

For any quantum field theory, the form of the Lagrangian that produces a given S-matrix is not unique: indeed the S-matrix is invariant under appropriate redefinitions of the fields composing the Lagrangian [112]. Generally, a field redefinition will alter the Green's function for a given process. To relate the Green's functions of two forms of a Lagrangian, the relation between both sets of external states must be specified. The same holds for HQET, which has been presented in various forms in the literature.

Fortunately, the definition of the heavy spinors in eq. (3.30) specifies for us the form of the spin-1/2 HPET Lagrangian whose external spinors are expressible as such. By inverting eq. (3.30), we see that the field redefinition converting the full theory to its HPET form must reduce to

$$\psi(x) = e^{-imv\cdot x}\left[\frac{1+\not{v}}{2} + \frac{1-\not{v}}{2}\frac{1}{iv\cdot\partial + 2m}i\not{\partial}\right]Q_v(x), \tag{3.175}$$

in the free-field limit. For spin-1/2 HQET, this means we must match the minimal coupling to the Lagrangian in the form

$$\mathcal{L}_{\text{HQET}}^{s=\frac{1}{2}} = \bar{Q}iv\cdot DQ + \bar{Q}i\not{D}P_-\frac{1}{2m+iv\cdot D}i\not{D}Q. \tag{3.176}$$

This form of the Lagrangian appears in e.g. ref. [60], and differs from the forms in refs. [92, 113] by the presence of a projection operator in the non-local term. The Lagrangian of HBET presented in ref. [92] must similarly be modified to compare to the minimal coupling amplitude. The suitable form for spin-1/2

HBET is

$$\mathcal{L}_{\text{HBET}}^{s=\frac{1}{2}} = \sqrt{-g}\bar{Q}i\mathcal{D}Q + \frac{\sqrt{-g}}{2m}\bar{Q}i\mathcal{D}P_{-}\sum_{n=0}^{\infty} G_n[h]\frac{F[h]^n}{m^n}i\mathcal{D}Q, \tag{3.177a}$$

where

$$i\mathcal{D} \equiv ie^{\mu}{}_{a}\gamma^{a}\, D_{\mu} + mv_{\mu}\gamma^{a}(e^{\mu}{}_{a} - \delta^{\mu}_{a}), \tag{3.177b}$$

and all other notation is described in ref. [92].

3.E.2 Matching spin-1 Zeeman coupling

We demonstrate explicitly the applicability of the on-shell HPET variables to spin-1 heavy particle systems. To do so, we will show that the same variables are suitable for describing the three-point amplitude arising from the Proca action. First, we note that a massive spin-1 particle described by the Proca action has a gyromagnetic ratio $g = 1$ [93]. As such, it should not be expected that the corresponding three-point amplitude matches with the minimal coupling amplitude for $s = 1$. To understand which three-point amplitude we should match with, we recast the three-point amplitude derived from the Zeeman coupling in ref. [46] into on-shell HPET variables (with $k_1 = 0$):

$$\mathcal{A}^{+,s} = \frac{g_0 x}{m^{2s}}\left[\langle \mathbf{2}_v \mathbf{1}_v\rangle^{2s} + x\frac{sg}{2m}\langle \mathbf{2}_v \mathbf{1}_v\rangle^{2s-1}\langle \mathbf{2}_v 3\rangle\langle 3\mathbf{1}_v\rangle + \dots\right], \tag{3.178}$$

where the dots represent higher spin multipoles. When $g = 2$ we recover the spin-dipole term from $2s$ factors of the spin-1/2 minimal coupling amplitude. Setting $s = g = 1$ for the Proca action,

$$\mathcal{A}^{+,1} = \frac{g_0 x}{m^2}\left[\langle \mathbf{2}_v \mathbf{1}_v\rangle^{2} + \frac{x}{2m}\langle \mathbf{2}_v \mathbf{1}_v\rangle\langle \mathbf{2}_v 3\rangle\langle 3\mathbf{1}_v\rangle + \dots\right]. \tag{3.179}$$

This is the three-point amplitude that we expect from a very heavy spin-1 Proca particle.

Consider now the Proca Lagrangian for a massive vector field B^{μ} coupled to electromagnetism:

$$\mathcal{L} = -\frac{1}{4}F^{*}_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 B^{*}_{\mu}B^{\mu}, \tag{3.180a}$$

where

$$F^{\mu\nu} = D^{\mu}B^{\nu} - D^{\nu}B^{\mu}, \quad D^{\mu}B^{\nu} = (\partial^{\mu} + ieA^{\mu})B^{\nu}, \tag{3.180b}$$

and A^{μ} is the $U(1)$ gauge field. We now need a condition that splits the light component $\mathcal{B}^{\mu}$ from the heavy (anti-field) component $\tilde{\mathcal{B}}^{\mu}$. Furthermore, the light component has to satisfy $v_{\mu}\mathcal{B}^{\mu} = 0$ [111]. The appropriate decomposition of the massive vector field is

$$\mathcal{B}^{\mu} = e^{imv\cdot x}P_{-}^{\mu\nu}B_{\nu}, \tag{3.181a}$$

$$\tilde{\mathcal{B}}^{\mu} = e^{imv\cdot x}P_{+}^{\mu\nu}B_{\nu}, \tag{3.181b}$$

where $P_{-}^{\mu\nu} \equiv g^{\mu\nu} - v^{\mu}v^{\nu}$ — this is the projection operator that has been derived explicitly in Appendix 3.D — and $P_{+}^{\mu\nu} \equiv v^{\mu}v^{\nu}$. Next, we substitute eq. (3.181) into the Proca Lagrangian, and integrate out $\tilde{\mathcal{B}}^\mu$ using its equation of motion to find

$$\mathcal{L}_{\text{HQET}}^{s=1} = -m\mathcal{B}_\mu^*(iv \cdot D)\mathcal{B}^\mu - \frac{1}{4}\mathcal{B}_{\mu\nu}^*\mathcal{B}^{\mu\nu} + \frac{1}{2}\mathcal{B}_\nu^* D^\nu D_\mu \mathcal{B}^\mu + \mathcal{O}(m^{-1}), \tag{3.182}$$

where $\mathcal{B}^{\mu\nu} = D^\mu\mathcal{B}^\nu - D^\nu\mathcal{B}^\mu$. Computing the three-point amplitude with this Lagrangian for $k_1 = 0$ and expressing it using on-shell HPET variables, we find agreement with eq. (3.179) for $g_0 = -em/\sqrt{2}$. This supports the hypothesis that the on-shell information of spin-1/2 HPET is sufficient to extend HPETs to higher spins.

CHAPTER

4 THE DOUBLE COPY FOR HEAVY PARTICLES

ABSTRACT: We show how to double-copy Heavy Quark Effective Theory (HQET) to Heavy Black Hole Effective Theory (HBET) for spin $s \leq 1$. In particular, the double copy of spin-s HQET with scalar QCD produces spin-s HBET, while the double copy of spin-$1/2$ HQET with itself gives spin-1 HBET. Finally, we present novel all-order-in-mass Lagrangians for spin-1 heavy particles.

4.1 INITIATION

HQET is derived from QCD (or QED in the abelian case) and HBET is derived from QGR. Since QCD double copies to QGR, invariance of the S-matrix under field redefinitions (see e.g. ref. [114]) implies that HQET and HBET must also be related to each other through the double copy. As in the case of QCD and QGR, amplitudes are much easier to compute in HQET than in HBET. Knowing how to exploit the double copy to obtain HBET amplitudes is therefore preferable to direct computation using HBET. The difficulty with realizing this double copy is that the external states are not cleanly separated from the scattering dynamics in these EFTs, since the heavy matter states are not canonical. The double-copy procedure must account for this in one form or another, and doing so was the focus of this publication.

The central concept in this publication is the double copy, so we begin this initiation with an introduction to this powerful computational tool. We then discuss the non-canonical nature of the heavy matter states and how it affects the double copy. This is understood through the lens of what is termed the wavefunction normalization factor (WNF) in ref. [114] (though, as we'll explain, our usage of the term WNF differs from the usage in ref. [114]). We illustrate the non-canonicality of heavy particles by deriving the WNF for a heavy scalar particle in two ways: by reference to the WNF for a canonical scalar particle and by considering the propagator for heavy scalars. Next, we elaborate on the subtleties in the double copy introduced by the WNFs and different methods for handling them, before finally summarizing the results of the publication.

4.1.1 The double copy

Very schematically, the double copy states that gravitational amplitudes are squares of the analogous gauge theory amplitudes. Despite its susceptibility to being summarized in one sentence, the double copy relationship between gauge and gravity theories is an intricate one with deep implications and applications to classical as well as quantum systems. We will explore in this section some of its features that are most pertinent to scattering amplitude computations.

It should go without saying that gravitational amplitudes are not literally the squares of gauge theory ones. For one, unlike gauge theories, gravity is a colorless theory. A second issue is that squaring a tree-level gauge theory amplitude produces double poles because of the squares of propagators. Thirdly, there is a discrepancy in the number of degrees of freedom (DoFs) of a graviton's polarization tensor compared to the product of two gluon polarizations. Double-copy prescriptions must account for all of these issues if they are to produce sensible gravitational amplitudes. There are two methods for double-copying gauge theories that are used in amplitudes computations, both of which we have employed in this publication. The first are the KLT relations [115], and the second is the BCJ double copy [116, 117].

The double copy was first realized in the form of the KLT relations – relations between closed string amplitudes and products of open string ones. The KLT relations link pure Yang-Mills (YM) (i.e. containing only massless, adjoint particles) and gravity amplitudes at tree-level in the field-theory limit, when the string length vanishes. Explicitly, the field-theory limit of the relations takes the form [115, 118–122][1]

$$M(1,\dots,n) = -i\sum_{\sigma,\rho} A_n(1,\sigma(2,\dots,n-2),n-1,n)\mathcal{S}[\sigma|\rho]\tilde{A}_n(1,\rho(2,\dots,n-2),n,n-1), \tag{4.1}$$

where $\sigma(2,\dots,n-2)$ and $\rho(2,\dots,n-2)$ are permutations of the particle labels $2,\dots,n-2$. The sum is over S_{n-3}, the group of permutations of the $n-3$ elements. $\mathcal{S}[\sigma|\rho]$ is the KLT momentum kernel [118–122],

$$\mathcal{S}[\sigma|\rho] \equiv \prod_{i=2}^{n-2}\left[2q_1\cdot q_{\sigma_i} + \sum_{j=2}^{i} 2q_{\sigma_i}\cdot q_{\sigma_j}\theta(\sigma_j,\sigma_i)_\rho\right]. \tag{4.2}$$

The function $\theta(\sigma_j,\sigma_i)_\rho$ picks out the terms in the sum where σ_j precedes σ_i in the permutation ρ, in which case $\theta(\sigma_j,\sigma_i)_\rho = 1$. This kernel is in fact the inverse of the matrix of double color-ordered massless biadjoint scalar amplitudes [123].

Let us describe now how the three issues raised above are addressed when using this double-copy prescription. The KLT relations produce a gravitational amplitude without any color generators because the gauge theory amplitudes have been expressed in terms of color-ordered amplitudes. Concretely, the amplitude for the tree-level scattering of n gluons decomposes into color-ordered amplitudes as

$$\mathcal{A}(1,\dots,n) = \sum_{\sigma\in S_{n-1}} \text{Tr}(T^{\sigma(1)}\dots T^{\sigma(n-1)}T^n)A(\sigma(1),\dots,\sigma(n-1),n). \tag{4.3}$$

It is thus clear that the color-ordered amplitudes do not contain any color generators. When representing color-ordered amplitudes diagrammatically, the ordering of the labels in the amplitude dictates the clockwise ordering of the external legs in the diagram.

Next, the presence of the KLT kernel ensures that the correct propagator structure emerges from the double copy. This is perhaps easiest to see with an example: consider the KLT relation for $n=4$ external states. The sum is over the group $S_{n-3}=S_1$ so there is only one term, and the KLT kernel also only contains one factor: $\mathcal{S}[2|2] = 2q_1\cdot q_2 = (q_1+q_2)^2 \equiv s_{12}$, where $s_{ij}\equiv(q_i+q_j)^2$. Therefore,

$$M(1,2,3,4) = -is_{12}A(1,2,3,4)\tilde{A}(1,2,4,3). \tag{4.4}$$

[1] In this section, amplitudes written without a calligraphic font do not include coupling constants, with M representing gravitational amplitudes and A gauge theory ones. Moreover, the amplitudes A are color-ordered amplitudes, which we will define shortly.

The gravitational amplitude contains three propagators, s_{12}, s_{13}, and s_{14}, from the s, t, and u channel Feynman diagrams. On the other hand, the propagators of a color ordered amplitude can only be formed by sums of the momenta of adjacent legs. In this example, A contributes poles s_{12} and s_{14}, whereas $\tilde{A}$ contributes s_{12} and s_{13}. Clearly, then, the KLT kernel has provided the correct Mandelstam invariant to cancel the additional factor of s_{12} and ensure agreement between the poles on both sides of the equation.

Finally, there is the mismatch in the number of DoFs of gravitons compared to proucts of gluons. A massless spinning particle in four spacetime dimensions has two DoFs corresponding to the positive and negative helicity polarizations. This gives gravitons two DoFs and products of two gluon polarization vectors four DoFs. To reconcile this, let us decompose the product of two gluon polarization vectors with polarizations $\lambda_{1,2}$ and momentum q^μ in the following way [124]:

$$\epsilon_\mu^{\lambda_1}\epsilon_\nu^{\lambda_2} = \frac{1}{2}\left[\epsilon_\mu^{\lambda_1}\epsilon_\nu^{\lambda_2} + \epsilon_\mu^{\lambda_2}\epsilon_\nu^{\lambda_1} - \delta^{\lambda_1\lambda_2}\left(\eta_{\mu\nu} - \frac{q_\mu\xi_\nu + q_\nu\xi_\mu}{q\cdot\xi}\right)\right] + \frac{1}{2}\left(\epsilon_\mu^{\lambda_1}\epsilon_\nu^{\lambda_2} - \epsilon_\mu^{\lambda_2}\epsilon_\nu^{\lambda_1}\right) + \delta^{\lambda_1\lambda_2}\frac{1}{2}\left(\eta_{\mu\nu} - \frac{q_\mu\xi_\nu + q_\nu\xi_\mu}{q\cdot\xi}\right), \tag{4.5}$$

where ξ_μ is a gauge-fixing reference vector. The quantity in the first set of brackets is symmetric and traceless in the polarization indices, and therefore has the correct number of DoFs (two) to be identified with the graviton. The second set of brackets is antisymmetric in the polarization labels, carries one DoF, and is dubbed the axion. The last term carries one DoF and is a dilaton. As we might have expected, the graviton states are a subset of all states contained in the product of two gluon polarizations. When only adjoint particles are involved in the double copy, or when adjoint particles are only present in the external states, it is sufficient to project onto the symmetric, traceless part of the products of gluon polarization vectors [122, 124].[2]

The relations eq. (4.1) are valid for double copying amplitudes involving only massless particles transforming in the adjoint representations of their gauge groups. Extensions of eq. (4.1) to amplitudes involving massive, fundamental particles are possible [125, 126]. Ref. [125] showed that QCD amplitudes involving a single massive line for a spin-$1/2$ particle emitting $n-2$ gluons can be double copied through the KLT-like formula

$$M_n^{\frac{1}{2}\otimes\frac{1}{2}} = \frac{1}{2^{\lfloor d/2\rfloor -1}}\sum_{\alpha,\beta}\mathcal{K}[\alpha|\beta]\mathrm{Tr}[A_{n,\alpha}^{\mathrm{QCD}}(\not{p}_1 - m)\not{\epsilon}_1^*\bar{A}_{n,\beta}^{\mathrm{QCD}}(\not{p}_2 - m)\not{\epsilon}_2^*], \tag{4.6}$$

for all momenta outgoing. Here, the KLT kernel $\mathcal{S}$ has been replaced by its massive counterpart $\mathcal{K}$. Much like the massless kernel, $\mathcal{K}$ is related to inverses of amplitude matrices from massive biadjoint scalar theories [125, 126]. The massive kernel was given up to 5-points in ref. [126]. We specialized eq. (4.6) to the large-mass case (and to our choice of kinematics) in eq. (4.59).

The BCJ double copy provides an alternative method for double copying gauge theory amplitudes to gravitational ones. Unlike the KLT relations, the BCJ double copy is valid at all orders in the perturbative expansion [122]. This version of the double copy relies on the color-kinematics duality. An overview of this duality and the consequent double-copy is provided in Section 4.3, so we don't reproduce it here. It is much more transparent how the BCJ double copy avoids the issues highlighted above. Since the color generators are replaced only with the kinematic numerators, the resulting amplitude will have no color

[2]Put plainly, we identify $\epsilon^\mu\epsilon^\nu \to \epsilon^{\mu\nu}$.

dependence, nor any double poles. Moreover, this double copy prescription was originally formulated for adjoint matter, in which case the graviton modes can simply be selected from the product of gluon polarizations.

Since its introduction, the BCJ double copy has been shown to hold also when spinning, massive matter in an arbitrary representation of the gauge group is included in the scattering [127–129]. In these situations, one must be more careful in isolating the portions of the double-copied amplitudes induced by internal gravitons. One method for doing so was applied in ref. [124] to the $2 \to 2(3)$ scattering of massive scalar particles (emitting a graviton). Dilaton contamination was subtracted in these cases by explicitly computing the analogous amplitudes with gravitons replaced by dilatons, then subtracting the result from the double copied amplitude. Yet another method was employed in the computation of the 3PM $2 \to 2$ gravitational amplitude in ref. [16]. There, the two-loop integrands were built using unitarity cuts. The building-block gravitational amplitudes were then written as double copies of gauge theory amplitudes, and contaminating states were eliminated by projecting products of gluons onto graviton states.

The double-copied amplitudes in this chapter's publication did not require the subtraction of contaminating states from the amplitude. At three points, the reason is clear: there are no internal gluons in the relevant diagram. For the Compton amplitude, however, one would expect the double-copied amplitude to contain graviton and dilaton effects, since one of the diagrams for this process possesses a gluon propagator. This is not the case, though, because the spin-s gravitational Compton amplitude can be obtained simply by double copying appropriate-spin Compton amplitudes from QED [93],[3] where there are no diagrams with internal photons.

While the double copy involving massive amplitudes has been demonstrated to produce physical amplitudes, this outcome is not always guaranteed, as was shown through examples in ref. [126]. Having a set of kinematic numerators that are dual to the color factors is sufficient to obtain physical double-copied amplitudes in the massless case, but ref. [126] argued that this condition is insufficient in the massive case. Indeed, they demonstrated that it is always possible to take any set of kinematic numerators to a color-dual form using only generalized gauge transformations, which are shifts of the kinematic numerators that leave the amplitude unchanged. In some cases, the result of the double copy may contain spurious poles, despite the color-kinematics duality being satisfied. In order to have meaningful quantities resulting from the double copy, ref. [126] conjectured the necessary condition that the massive KLT kernel for the process must have minimal rank. This imposes spectral conditions on the masses of the internal particles. For the Compton amplitude, the spectral condition is

$$m_s^2 + m_t^2 + m_u^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2, \tag{4.7}$$

where the masses on the left-hand side are the masses of the propagators in the s-, t-, and u-channels, and on the right-hand side are the external masses. In this case, both sides are equal to $2m^2$. The QCD Compton amplitude therefore satisfies the necessary spectral condition to give a meaningful double copy.

[3] This is visible from eqs. (3.103) and (3.104).

4.1.2 Wavefunction normalization factors

WNFs do not always arise in the computations of scattering amplitudes, but they are crucial for matching amplitudes before and after a field redefinition. We begin our discussion of WNFs by demonstrating their identification through the propagator, which will provide a method for finding the WNF given an action. We will then illustrate how they enter in matching calculations, and finally derive the WNF for a heavy scalar.

Before we begin, let us distinguish our usage of the term WNF from its traditional usage. Generically, a scalar field satisfies [13, 130]

$$\langle\Omega|\phi(x)|\lambda_p\rangle = \sqrt{Z}e^{-ip\cdot x}, \tag{4.8}$$

where $|\Omega\rangle$ is the vacuum state in an interacting theory and λ_p is some state with momentum p^μ. The quantity Z is what is typically referred to as the WNF, and quantifies the mismatch between an interacting field and its free counterpart [13, 114, 130, 131]. This is not the usage we employ, but we found it necessary to apply a similar concept to redefinitions of fields in order to match amplitudes after said redefinitions. A crucial point in this regard is that amplitudes with the same external states are matched to one another. In turn, conceptually, the WNFs we discuss in this chapter define the asymptotic states of the theory — we say two asymptotic states are equal if they have the same WNF — and as such are described in the free-field limit. As is usual in the computation of amplitudes, we do not explicitly handle the portion of the WNF arising from interactions; this portion is dealt with by the LSZ reduction [131].

The time-ordered two-point function for a free (scalar) field ϕ gives the Fourier transform of its momentum-space propagator, $D(p)$ [13, 130],

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\, D(p)\, e^{-ip\cdot(x-y)}. \tag{4.9}$$

Here $|0\rangle$ is the vacuum state of the free theory. A free scalar field is said to be canonically normalized if it satisfies $\langle 0|\phi(x)|p\rangle = e^{-ip\cdot x}$ [13, 130]. In this case, we write $\phi = \phi_c$, and the field's momentum-space propagator is given by

$$D_c(p) \equiv \frac{i}{p^2 - m^2 + i\epsilon}, \tag{4.10}$$

where $+i\epsilon$ represents the Feynman prescription. More practically, the momentum-space propagator can be immediately read off from the free-field Lagrangian as the Fourier transform of the Green's function of the differential operator in the Lagrangian. In equations, the Lagrangian for a real canonical scalar field is

$$\mathcal{L} = \frac{1}{2}\phi_c\left(-\Box - m^2\right)\phi_c, \tag{4.11}$$

where $\Box \equiv \partial_\mu\partial^\mu$ is the d'Alembertian. The Green's function for the differential operator, $\tilde{D}_c(x-y)$, satisfies [130]

$$\left(\Box + m^2\right)\tilde{D}_c(x-y) = -i\delta^{(4)}(x-y). \tag{4.12}$$

Note that the overall $1/2$ in the Lagrangian does not appear here as it is just a symmetry factor. This Green's function is simply the propagator in position space, that is, the time-ordered two-point function. Fourier transforming gives the propagator in momentum space.

Let us consider what happens to the momentum-space propagator under a field redefinition. Suppose we rescale the canonical scalar field as $\phi_c = A^{-1}\phi$ for A some non-zero quantity that is potentially an operator.[4] The Lagrangian describing this field after such a redefinition is

$$\mathcal{L} = \frac{1}{2} A_{\text{free}}^{-1}\phi \left(-\Box - m^2\right) A_{\text{free}}^{-1}\phi + \text{interactions}, \tag{4.13}$$

where A_{free} is the field-independent part of A and the interactions are generated by the parts of A that contain fields. If A_{free}^{-1} is an operator it can only comprise terms with an even number of partial derivatives, so we can commute the leftmost factor with ϕ by integrating by parts. Partial derivatives also commute with the d'Alembertian, so there is no ambiguity in writing the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\phi \frac{\left(-\Box - m^2\right)}{A_{\text{free}}^2}\phi + \text{interactions}. \tag{4.14}$$

Reading off the propagator,

$$D(p) = \frac{i\tilde{A}_{\text{free}}^2}{p^2 - m^2 + i\epsilon}, \tag{4.15}$$

where $\tilde{A}_{\text{free}}$ is the Fourier transform of A_{free} and we've explicitly included the Feynman prescription.

From this example we see that the form of a propagator is not fixed under field redefinitions. In general, the propagator is[5]

$$D(p) = \frac{i\mathcal{R}}{p^2 - m^2 + i\epsilon}, \tag{4.16}$$

where $\mathcal{R}$ is the square of the WNF. For our purposes, we have defined the square of the WNF as the on-shell residue of the propagator in the free theory. The denominator of the propagator need not be of the form $p^2 - m^2$ for us to define the WNF in this way. We will see an example of this soon.

For the sake of clarity, we point out two distinctions between this usage of the term "WNF" and that of refs. [114, 130]. First, we stress again that we define the WNF only with reference to the free theory, whereas refs. [114, 130] use WNF to refer to the on-shell residue of the two-point function in an interacting theory. Second, what we refer to as the WNF is the square-root of the analogous quantity in refs. [114, 130].

The above provides a method for determining the WNF given a general action. Additionally, as is visible from that example, the WNF is related to the field redefinition from a canonical field ϕ_c to ϕ. This gives us a hint of a subtle way that the WNF enters an amplitude calculation. Consider ϕ_c^3 theory with interaction term $\lambda\phi_c^3$. The $2 \to 2$ amplitude for canonical fields at tree-level is

$$i\mathcal{A}_c = -\lambda^2 \frac{i}{q^2 - m^2}, \tag{4.17}$$

[4] If A is an operator its inverse is understood as a Taylor expansion.

[5] In interacting theories the propagator has corrections from terms that do not possess poles [114, 130]. In such cases, the WNF is identified as the residue of the propagator when its momentum goes on shell.

where q is the momentum transferred in the scattering. We now employ the field redefinition $\phi_c = \phi/a$, and consider that a is just a constant for simplicity. The amplitude in these new variables is

$$i\mathcal{A} = -\frac{\lambda^2}{a^6}\frac{ia^2}{q^2 - m^2}\tilde{\phi}(p_1)\tilde{\phi}(p_1 - q)\tilde{\phi}(p_2)\tilde{\phi}(p_2 + q), \tag{4.18}$$

where $\tilde{\phi}$ is the external state of the field ϕ in momentum space. We have kept these factors explicit in the amplitude because, unlike their canonical counterparts, they are not equal to 1. In fact, converting both sides of the relation $\phi_c = A^{-1}\phi$ to their momentum-space external states, we find that $\tilde{\phi}(p) = \tilde{A}_{\text{free}}$ since the external states are determined at times $t = \pm\infty$, when there are no interactions. Apart from affecting the propagator and the vertices, the field redefinition also affected the Feynman rule for the external states! This point is crucial to matching the amplitudes from both actions; plugging in $\tilde{\phi} = \tilde{A}_{\text{free}} = a$ we reach $\mathcal{A}_c = \mathcal{A}$, as we must.

Let us state explicitly the relation between the WNF and a field redefinition from a canonical state. Given a relation between a field and its canonical counterpart such as $\phi_c = A^{-1}\phi$, the WNF for the field ϕ is given by $\mathcal{R}^{1/2} = \tilde{A}_{\text{free}}$.

To close the section, we will illustrate how we derived the WNFs identified in the publication by showing the scalar case explicitly. Beginning with the Lagrangian for a canonical, free, complex scalar,

$$\mathcal{L} = \partial_\mu\phi_c^*\partial^\mu\phi_c - m^2\phi_c^*\phi_c, \tag{4.19}$$

we employ the redefinition [54]

$$\phi_c = \frac{e^{-imv\cdot x}}{\sqrt{2m}}\left(\phi_v + \Phi_v\right), \tag{4.20a}$$

where

$$\phi_v = \frac{e^{imv\cdot x}}{\sqrt{2m}}(iv\cdot\partial + m)\phi, \tag{4.20b}$$

$$\Phi_v = \frac{e^{imv\cdot x}}{\sqrt{2m}}(-iv\cdot\partial + m)\phi. \tag{4.20c}$$

The WNF is not clear from this redefinition because there are two fields on the right-hand side. However, in the heavy limit we must integrate out the Φ_v field. In the process we will be able to express Φ_v in terms of ϕ_v, making the WNF apparent.

Plugging in the field redefinition, the resulting form of the Lagrangian is

$$\mathcal{L} = i\phi_v^*(v\cdot\partial)\phi_v - \Phi_v^*(2m + iv\cdot\partial)\Phi_v - \frac{1}{2m}(\phi_v^* + \Phi_v^*)\partial_\perp^2(\phi_v + \Phi_v). \tag{4.21}$$

The perpendicular derivative is defined below eq. (2.26). Now, the massive mode Φ_v is eliminated by substituting its equation of motion in its place. Its equation of motion is

$$\Phi_v = -\frac{1}{2m + iv\cdot\partial + \frac{\partial_\perp^2}{2m}}\frac{\partial_\perp^2}{2m}\phi_v. \tag{4.22}$$

Performing the substitution gives the Lagrangian for a free heavy scalar field — see eq. (4.66).

However, at this stage, we're more interested in the field redefinition in eq. (4.20). Substituting the equation of motion for Φ_v into eq. (4.20), we find that the entire process of integrating out this field is equivalent to the redefinition

$$\phi_c = \frac{e^{-imv\cdot x}}{\sqrt{2m}} \left(1 - \frac{1}{2m + iv\cdot\partial + \frac{\partial_\perp^2}{2m}} \frac{\partial_\perp^2}{2m}\right) \phi_v. \tag{4.23}$$

We can now read off the WNF for the field ϕ_v:

$$\mathcal{R}^{-1/2} = \frac{1}{\sqrt{2m}} \left(1 + \frac{k_\perp^2}{4m^2 + 2mv\cdot k - k_\perp^2}\right). \tag{4.24}$$

We've dropped the overall phase since any operator will contain one factor of ϕ_v and one of ϕ_v^*.

Alternatively, let us read the WNF off of the Lagrangian after having applied eq. (4.23):

$$\begin{aligned}\mathcal{L} = \phi_v^* \frac{1}{\sqrt{2m}} &\left(1 - \frac{1}{2m + iv\cdot\partial + \frac{\partial_\perp^2}{2m}} \frac{\partial_\perp^2}{2m}\right) (2miv_\mu\partial^\mu - \Box) \\ &\times \frac{1}{\sqrt{2m}} \left(1 - \frac{1}{2m + iv\cdot\partial + \partial_\perp^2} \frac{\partial_\perp^2}{2m}\right) \phi_v.\end{aligned} \tag{4.25}$$

Note that the middle parentheses in momentum space give $2mv\cdot k + k^2$, which is precisely $p^2 - m^2$ after the heavy momentum decomposition $p^\mu = mv^\mu + k^\mu$. We thus identify the middle parentheses as the denominator of the two-point function, and hence

$$\mathcal{R} = \left[\frac{1}{\sqrt{2m}} \left(1 + \frac{k_\perp^2}{4m^2 + 2mv\cdot k - k_\perp^2}\right)\right]^{-2}, \tag{4.26}$$

in agreement with eq. (4.24).

A final remark: we did not compute with the propagator in the form $i\mathcal{R}/(2mv\cdot k + k^2)$. We simply used $i/v\cdot k$ and treated the rest of the contributions as momentum insertions.

4.1.3 WNFs and the double copy

When double copying, we must separate the color from the kinematic information; a straightforward task. When using non-canonically normalized states, though, one must also be aware that the kinematic portion of the amplitude is contaminated by the WNF. The appearance of the WNF is not an obstacle to satisfying the color-kinematics duality. This is because the form taken by the WNF depends solely on the external kinematics (since it must be cancelled by the external states in a matching calculation), and so is the same for all diagrams of a given process. However, it does present two ways of double copying an amplitude, both of which result in different external states in the double-copied amplitude.

Let us consider a simple example of an amplitude where the matter state is not canonically normalized: the three point amplitude for the emission of a gluon from a heavy scalar particle. In the publication, we found this amplitude to be

$$\mathcal{A}_3^{\mathrm{H},s=0} = -\mathbf{T}^a_{ij}\epsilon_q^{*\mu}\phi_v^* \left(1 + \frac{k_1^2 + k_2^2}{4m^2}\right)\phi_v \left[v_\mu + \frac{(k_1+k_2)_\mu}{2m}\right] + \mathcal{O}\left(m^{-4}\right). \tag{4.27}$$

The WNF contribution is the factor in large round brackets. One way to double copy this amplitude is to simply replace the $SU(3)$ generator with the remainder of the amplitude:

$$\mathcal{M}_3^{\mathrm{H},s=0} = \epsilon_q^{*\mu\nu}\phi_v'^{*}\left(1+\frac{k_1^2+k_2^2}{4m^2}\right)^2\phi_v'\left[v_\mu v_\nu + v_\nu\frac{(k_1+k_2)_\mu}{m}+\frac{(k_1+k_2)_\mu(k_1+k_2)_\nu}{4m^2}\right]+\mathcal{O}(m^{-4}). \tag{4.28}$$

We've identified the product of gluon polarizations with a graviton polarization, $\epsilon_q^{*\mu\nu}=\epsilon_q^{*\mu}\epsilon_q^{*\nu}$, and the external scalar states after the double copy are $\phi_v' = \phi_v^2$. The latter identification makes clear what the non-canonical WNF has caused here: this is a gravitational amplitude not for the scalar field ϕ_v, but rather for the field ϕ_v'. Therefore, if we wanted to derive this amplitude directly from a gravitational action, we should compute it from QGR coupled to a canonical scalar that has been redefined twice according to the gravitationally coupled version of eq. (4.23).

While the above comparison is certainly doable, it is rather cumbersome. Moreover, double-copying in this way requires special care to properly combine matter polarizations if the field is not a scalar. Fortunately, there is an alternative to this version of the double copy. It is much easier to derive and compute from HBET if the matter field has only been redefined once, so we would like the matter states before and after the double copy to be identical. This can be achieved if the color generator is replaced by the kinematics from a canonically normalized scalar amplitude. We have chosen this path in most of the publication, and this is why we double copied the spin-s HQET amplitudes with those of scalar QED.

There was one instance in the publication where we used the first of these two approaches: double copying spin-$1/2$ HQET to get spin-1 HBET. In this case, through reference to the on-shell HPET variables, we identified the heavy spin-1 state that corresponded to the "square" of the heavy spinors. This allowed us to compute the WNF of the heavy spin-1 state by combining the spin-$1/2$ WNFs. Covariantizing the spin-1 WNF lead to the identification of the appropriate field redefinitions of the canonical Lagrangians that would describe this state. In this way, we identified the matter state that would result from double copying with two sets of non-canonical states, and used this knowledge to match to amplitudes from an appropriately defined HBET Lagrangian.

4.1.4 Overview of main results

We presented a prescription for double copying HQET amplitudes to HBET ones for matter with spins ≤ 1. At spin 1, this required the derivation of new Lagrangians for these two theories. These Lagrangians are

$$\mathcal{L}^{s=1}_{\text{gluon}} = -B_\mu^*(iv\cdot D)B^\mu - \frac{1}{4m}B^*_{\mu\nu}B^{\mu\nu}+\frac{ig}{2m}F^{\mu\nu}B^*_\mu B_\nu - \left(\mathcal{E}^\lambda_- B^*_\lambda\right)\frac{2}{m+\frac{1}{m}D_\perp^2}\left(\mathcal{E}^\mu_+ B_\mu\right), \tag{4.29a}$$

where

$$\mathcal{E}^\mu_\pm = \left(\pm\frac{i}{2}D^\mu - \frac{1}{2m}D^\mu(v\cdot D)\pm\frac{igv_\nu F^{\nu\mu}}{2m}\right), \tag{4.29b}$$

for HQET, and

$$\begin{aligned}\sqrt{-g}\mathcal{L}^{s=1}_{\text{graviton}} =&\sqrt{-g}\Big[-\frac{m}{2}(v_\mu B^*_\nu)(v_\rho B_\sigma)\left((g^{\mu\rho}-\eta^{\mu\rho})g^{\nu\sigma}-(g^{\mu\sigma}-\eta^{\mu\sigma})(g^{\nu\rho}-\eta^{\nu\rho})\right)\\&+\frac{i}{2}\left[(\nabla_\mu B^*_\nu)(v_\rho B_\sigma)-(v_\mu B^*_\nu)(\nabla_\rho B_\sigma)\right](g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\&-\frac{1}{4m}B^*_{\mu\nu}B_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}-\left(\mathcal{C}^\alpha_- B^*_\alpha\right)\frac{1}{\mathcal{D}}\left(\mathcal{C}^\beta_+ B_\beta\right)\Big],\end{aligned} \tag{4.30a}$$

where

$$\mathcal{C}^\alpha_\pm =-\frac{m}{2}(g^{\alpha\nu}-\eta^{\alpha\nu})v_\nu \pm \frac{i}{2}v_\nu\left[g^{\mu\rho}g^{\alpha\nu}-g^{\alpha\mu}g^{\nu\rho}\right]\nabla_\mu\left(v_\rho \pm \frac{i}{m}\nabla_\rho\right), \tag{4.30b}$$

$$\mathcal{D} =\frac{m}{2}(v_\nu v_\sigma g^{\nu\sigma})+\frac{1}{2m}v_\nu\left[g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\rho\nu}\right]\nabla_\mu\nabla_\rho v_\sigma, \tag{4.30c}$$

for HBET.

Accounting for the fact that the matter states of HQET and HBET are not canonically normalized, we showed by direct computation at three points and for the Compton amplitude that HQET can be double copied to HBET through the schematic formulae

$$(\text{QCD}_{s=0})\times(\text{HQET}_s)=\text{HBET}_s, \tag{4.31a}$$

$$(\text{HQET}_{s=1/2})\times(\text{HQET}_{s=1/2})=\text{HBET}_{s=1}, \tag{4.31b}$$

for matter with spin $s \leq 1$.

4.2 INTRODUCTION

An expanding family of field theories has been observed to obey double-copy[6] relations [115–117, 127, 132–159]. In particular, scattering amplitudes of gravitational theories with massive matter can be calculated from the double copy of gauge theories with massive matter [16, 17, 104, 124, 125, 128, 129, 160–163].

As Heavy Quark Effective Theory (HQET) [25] is derived from QCD and Heavy Black Hole Effective Theory (HBET) [92] is derived from gravity coupled to massive particles, the amplitudes of HBET should be obtainable as double-copies of HQET amplitudes. Indeed, this is the main result of this paper. We show through direct computation that the three-point and Compton amplitudes of HQET and HBET satisfy the schematic relations

$$(\text{QCD}_{s=0})\times(\text{HQET}_s)=\text{HBET}_s, \tag{4.32a}$$

$$(\text{HQET}_{s=1/2})\times(\text{HQET}_{s=1/2})=\text{HBET}_{s=1}, \tag{4.32b}$$

for $s \leq 1$, where the spin-s HQET and HBET matter states are equal in the free-field limit, and the spin-1 heavy polarization vectors are related to the heavy spinors through eq. (4.60). While we only show here the double copy for three-point and Compton amplitudes, invariance of the S-matrix under field redefinitions implies that eq. (4.32) holds more generally whenever QCD double-copies to gravitationally

[6] For a review of the double-copy program, see ref. [122].

interacting matter. Equation (4.32) expands the double copy in powers of $\hbar$ since the operator expansion for heavy particles can be interpreted as an expansion in $\hbar$ [92]. The $\hbar \to 0$ limit of the double copy is currently of particular relevance [16, 17, 124].

We will begin in Section 4.3 with a brief review of the color-kinematics duality, and we will also discuss double-copying with effective matter fields. In Sections 4.4 to 4.6 we demonstrate the double copy at tree level for three-point and Compton amplitudes for spins 0, 1/2, and 1, respectively. We conclude in Section 4.7. The Lagrangians used to produce the amplitudes in this paper are presented in Section 4.A. Among them are novel all-order-in-mass Lagrangians for spin-1 HQET and HBET given in eqs. (4.68) and (4.73).

4.3 COLOR-KINEMATICS DUALITY AND HEAVY FIELDS

An n-point gauge-theory amplitude, potentially with external matter, can be written as[7]

$$\mathcal{A}_n = \sum_{i\in\Gamma} \frac{c_i n_i}{d_i}, \tag{4.33}$$

where Γ is the set of all diagrams with only cubic vertices. Also, c_i are color factors, n_i encode the kinematic information, and d_i are propagator denominators. A subset of the color factors satisfies the identity

$$c_i + c_j + c_k = 0. \tag{4.34}$$

If the corresponding kinematic factors satisfy the analogous identity,

$$n_i + n_j + n_k = 0, \tag{4.35}$$

and have the same anti-symmetry properties as the color factors, then the color and kinematic factors are dual. In this case, the color factors in eq. (4.33) can be replaced by kinematic factors to form the amplitude

$$\mathcal{M}_n = \sum_{i\in\Gamma} \frac{n'_i n_i}{d_i}, \tag{4.36}$$

which is a gravity amplitude with anti-symmetric tensor and dilaton contamination.[8] In general, n'_i and n_i need not come from the same gauge theory, and only one of the sets must satisfy the color-kinematics duality.

In this paper we are interested in applying the double-copy procedure to HQET. A complicating factor to double-copying effective field theories (EFTs) is that Lagrangian descriptions of EFTs are not unique, as the Lagrangian can be altered by redefining one or more of the fields. The LSZ procedure [131] guarantees the invariance of the S-matrix, and in particular eqs. (4.33) and (4.36), under such field

[7] We omit coupling constants for the sake of clarity. Reinstating them is straight-forward: after double-copying the gauge theory coupling undergoes the replacement $g \to \sqrt{\kappa}/2$.

[8] For an amplitude of arbitrary multiplicity containing massive external states with an arbitrary spectrum, eq. (4.36) may not represent a physical amplitude [126]. However, for the cases under consideration in this paper, the application of the double copy will yield a well-defined gravitational amplitude.

redefinitions by accounting for wavefunction normalization factors (WNFs) $\mathcal{R}^{-1/2}$, which contribute to the on-shell residues of two-point functions.[9] Under the double copy the WNFs from each matter copy combine in a spin-dependent manner, which complicates the matching of the double-copied amplitude to one derived from a gravitational Lagrangian.

In order to ease the double-copying of HQET to HBET, we would like to avoid having to compensate for the WNFs. This can be achieved by ensuring that HQET and HBET have the same WNFs – i.e. that the asymptotic states for the spin-s particles in HQET and HBET are equal – and double-copying HQET with QCD, which has a trivial WNF.

The asymptotic states – that is, the states in the free-field limit – of the canonically normalized theories (given by complex Klein-Gordon, Dirac, and symmetry-broken Proca actions) are related to their respective asymptotic heavy states (labelled by a velocity v) in position-space through

$$\varphi(x) = \frac{e^{-imv\cdot x}}{\sqrt{2m}}\left[1 - \frac{1}{2m + iv\cdot\partial + \frac{\partial_\perp^2}{2m}}\frac{\partial_\perp^2}{2m}\right]\phi_v(x), \tag{4.38a}$$

$$\psi(x) = e^{-imv\cdot x}\left[1 + \frac{i}{2m + iv\cdot\partial}(\not{\partial} - v\cdot\partial)\right]Q_v(x), \tag{4.38b}$$

$$A^\mu(x) = \frac{e^{-imv\cdot x}}{\sqrt{2m}}\left[\delta^\mu_\nu - \frac{iv^\mu\partial_\nu - \partial^\mu\partial_\nu/2m}{m + iv\cdot\partial/2}\right]B^\nu_v(x), \tag{4.38c}$$

where $a^\mu_\perp = a^\mu - v^\mu(v\cdot a)$ for a vector a^μ. Here, the momentum is decomposed as $p^\mu = mv^\mu + k^\mu$ in the usual heavy-particle fashion. The Lagrangians for the heavy fields in eq. (4.38) are given in Section 4.A. Converting to momentum space, eq. (4.38) gives the WNFs

$$\mathcal{R}^{-1/2}_{s=0}(p) = \frac{1}{\sqrt{2m}}\left[1 + \frac{k_\perp^2}{4m^2 + 2mv\cdot k - k_\perp^2}\right], \tag{4.39a}$$

$$\mathcal{R}^{-1/2}_{s=1/2}(p) = 1 + \frac{1}{2m + v\cdot k}(\not{k} - v\cdot k), \tag{4.39b}$$

$$\left(\mathcal{R}^{-1/2}_{s=1}(p)\right)_\mu^{\ \nu} = \frac{1}{\sqrt{2m}}\left[\delta^\nu_\mu - \frac{v_\mu k^\nu + k_\mu k^\nu/2m}{m + v\cdot k/2}\right]. \tag{4.39c}$$

We will demonstrate that spin-s HBET amplitudes can directly be obtained by double-copying spin-s HQET amplitudes with scalar QCD for spins $s \leq 1$. At $s = 1$ there is also the possibility to double-copy using two spin-1/2 amplitudes. We will discuss this point further below.

4.4 SPIN-0 GRAVITATIONAL AMPLITUDES

We begin with the simplest case of spinless amplitudes. Consider first the three-point amplitude. For scalar HQET we have that

$$\mathcal{A}_3^{\mathrm{H},s=0} = -\mathbf{T}^a_{ij}\epsilon_q^{*\mu}\phi_v^*\left(1 + \frac{k_1^2 + k_2^2}{4m^2}\right)\phi_v\left[v_\mu + \frac{(k_1 + k_2)_\mu}{2m}\right] + \mathcal{O}\left(m^{-4}\right), \tag{4.40}$$

[9] Note that $\mathcal{R}^{-1/2} = 1$ for canonically normalized fields. The WNF for an effective state $\tilde{\varepsilon}$ can thus be determined by relating it to a canonically normalized state ε through

$$\varepsilon = \mathcal{R}^{-1/2}\cdot\tilde{\varepsilon}. \tag{4.37}$$

where $k_2 = k_1 - q$. For scalar QCD the amplitude is

$$\mathcal{A}_3^{s=0} = -\,\mathbf{T}^a_{ij}\epsilon_q^{*\mu}\left[2mv_\mu + (k_1+k_2)_\mu\right]. \tag{4.41}$$

Note that we have left the external heavy scalar factor ϕ_v explicit in the HQET amplitude. This is because, in contrast to the canonically normalized scalar fields, the heavy scalar factors are not equal to 1 in momentum space. Indeed, for the HQET amplitude to be equal to the QCD amplitude, the heavy scalar factor in momentum space must be equal to the inverse of eq. (4.39a). This will cancel the extra factor in round brackets in eq. (4.40).

The double copy at three-points is simply given by a product of amplitudes:

$$\begin{aligned}\mathcal{A}_3^{s=0}\mathcal{A}_3^{\mathrm{H},s=0} &= \epsilon_q^{*\mu}\epsilon_q^{*\nu}\phi_v^*\left(1+\frac{k_1^2+k_2^2}{4m^2}\right)\phi_v\\ &\times 2m\left[v_\mu v_\nu + v_\mu\frac{k_{1\nu}+k_{2\nu}}{m}+\frac{(k_1+k_2)_\mu(k_1+k_2)_\nu}{4m^2}\right]+\mathcal{O}(m^{-3}).\end{aligned} \tag{4.42}$$

As the only massless particle in this process is external, we can easily eliminate the massless non-graviton degrees of freedom by identifying the outer product of gluon polarization vectors with the graviton polarization tensor. After doing so, eq. (4.42) agrees with the three-point amplitude derived from eq. (4.71).

As another example, consider the Compton amplitude. The color decomposition for Compton scattering[10] is

$$\mathcal{A}_4^s = \frac{c_s n_s}{d_s} + \frac{c_t n_t}{d_t} + \frac{c_u n_u}{d_u}, \tag{4.43a}$$

where

$$c_s = \mathbf{T}^a_{ik}\mathbf{T}^b_{kj}, \quad c_t = if^{abc}\mathbf{T}^c_{ij}, \quad c_u = \mathbf{T}^b_{ik}\mathbf{T}^a_{kj}, \tag{4.43b}$$

and

$$d_s = -2mv\cdot q_1, \quad d_t = (q_1+q_2)^2, \quad d_u = -2mv\cdot q_2. \tag{4.43c}$$

The kinematic numerators for scalar HQET are

$$n_s^{\mathrm{H},s=0} = -2m\phi_v^*\epsilon_{q_1}^{*\mu}\epsilon_{q_2}^{*\nu}v_\mu v_\nu\left(1+\frac{k_1^2+k_2^2}{4m^2}\right)\phi_v, \tag{4.44a}$$

$$n_t^{\mathrm{H},s=0} = 0, \tag{4.44b}$$

$$n_u^{\mathrm{H},s=0} = n_s^{\mathrm{H},s=0}|_{q_1\leftrightarrow q_2}, \tag{4.44c}$$

where $k_2 = k_1 - q_1 - q_2$. Those for scalar QCD are

$$n_s^{s=0} = -4m^2\epsilon_{q_1}^{*\mu}\epsilon_{q_2}^{*\nu}v_\mu v_\nu, \tag{4.45a}$$

[10] We have computed all Compton amplitudes using NRQCD propagators. It is also possible to perform the computations using HQET propagators: in that case, a comparison to the Compton amplitude for the emission of bi-adjoint scalars from heavy particles (described by the Lagrangians in eqs. (4.63) to (4.65)) – analogous to the treatment in ref. [164] – is necessary to identify kinematic numerators. Both methods produce the same results.

$$n_t^{s=0} = 0, \tag{4.45b}$$
$$n_u^{s=0} = n_s^{s=0}|_{q_1 \leftrightarrow q_2}. \tag{4.45c}$$

For brevity we have written the numerators under the conditions $k_1 = q_i \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0$; the initial residual momentum can always be set to 0 by reparameterizing v, and such a gauge exists for opposite helicity gluons. We have checked explicitly up to and including $\mathcal{O}(m^{-2})$ that the following results hold when relaxing all of these conditions.

Both the HQET and QCD numerators satisfy the color-kinematics duality in the form

$$c_s - c_u = c_t \;\Leftrightarrow\; n_s - n_u = n_t. \tag{4.46}$$

We can therefore replace the color factors in the HQET amplitude with the QCD kinematic numerators,

$$\mathcal{M}_4^{\mathrm{H},s=0} = \frac{n_s^{s=0} n_s^{\mathrm{H},s=0}}{d_s} + \frac{n_t^{s=0} n_t^{\mathrm{H},s=0}}{d_t} + \frac{n_u^{s=0} n_u^{\mathrm{H},s=0}}{d_u}. \tag{4.47}$$

Identifying once again the outer products of gluon polarization vectors with graviton polarization tensors, we find that the Compton amplitude derived from eq. (4.71) agrees with eq. (4.47).

To summarize, we have explicitly verified that

$$(\mathrm{QCD}_{s=0}) \times (\mathrm{HQET}_{s=0}) = \mathrm{HBET}_{s=0} \tag{4.48}$$

for three-point and Compton amplitudes.

4.5 SPIN-1/2 GRAVITATIONAL AMPLITUDES

We now move on to the double copy of spin-1/2 HQET with scalar QCD to obtain spin-1/2 HBET. The three-point spin-1/2 HQET amplitude is

$$\begin{aligned}\mathcal{A}_3^{\mathrm{H},s=\frac{1}{2}} = &-\mathbf{T}^a_{ij}\bar{u}_v u_v \epsilon_q^{*\mu}\left(v_\mu + \frac{k_{1\mu}}{m} + \frac{k_1^2 - k_1 \cdot q}{4m^2} v_\mu\right) \\ &- \frac{i\mathbf{T}^a_{ij}}{2m}\bar{u}_v \sigma^{\alpha\beta} u_v \epsilon_q^{*\mu}\left[q_\alpha \eta_{\beta\mu} - \frac{1}{2m} q_\alpha k_{1\beta} v_\mu\right] + \mathcal{O}(m^{-3})\end{aligned} \tag{4.49}$$

Double-copying with scalar QCD, we find

$$\mathcal{M}_3^{\mathrm{H},s=\frac{1}{2}} = \mathcal{A}_3^{s=0}\mathcal{A}_3^{\mathrm{H},s=\frac{1}{2}}, \tag{4.50}$$

where $\mathcal{M}_3^{\mathrm{H},s=\frac{1}{2}}$ is the amplitude derived from eq. (4.72).

We turn now to Compton scattering. For brevity we write here the amplitudes in the case $k_1 = q_i \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0$. We have checked explicitly that the results hold when these conditions are relaxed. Also, we have performed the calculation up to $\mathcal{O}(m^{-2})$ but only present the kinematic numerators up to $\mathcal{O}(m^{-1})$. They are

$$n_s^{\mathrm{H},s=\frac{1}{2}} = -2m\bar{u}_v\left[v \cdot \epsilon_{q_1}^* v \cdot \epsilon_{q_2}^* - \frac{iv_\rho}{2m}\sigma_{\mu\nu}(\epsilon_{q1}^{*\mu} q_1^\nu \epsilon_{q2}^{*\rho} + \epsilon_{q2}^{*\mu} q_2^\nu \epsilon_{q1}^{*\rho} - q_2^\rho \epsilon_{q2}^{*\mu}\epsilon_{q1}^{*\nu})\right] u_v, \tag{4.51a}$$

$$n_t^{\mathrm{H},s=\frac{1}{2}} = 0, \tag{4.51b}$$

$$n_u^{\mathrm{H},s=\frac{1}{2}} = n_s^{\mathrm{H},s=\frac{1}{2}}|_{q_1\leftrightarrow q_2}. \tag{4.51c}$$

In this case, the color-kinematic duality eq. (4.46) is violated at $\mathcal{O}(m^{-2})$. Nevertheless, since the scalar QCD kinematic numerators satisfy the duality we can use them to double copy the spin-1/2 Compton amplitude. Doing so we find

$$\mathcal{M}_4^{\mathrm{H},s=\frac{1}{2}} = \frac{n_s^{s=0} n_s^{\mathrm{H},s=\frac{1}{2}}}{d_s} + \frac{n_t^{s=0} n_t^{\mathrm{H},s=\frac{1}{2}}}{d_t} + \frac{n_u^{s=0} n_u^{\mathrm{H},s=\frac{1}{2}}}{d_u}, \tag{4.52}$$

where $\mathcal{M}_4^{\mathrm{H},s=\frac{1}{2}}$ is the spin-1/2 HBET Compton amplitude derived from eq. (4.72).

We have seen that

$$(\mathrm{QCD}_{s=0}) \times (\mathrm{HQET}_{s=1/2}) = \mathrm{HBET}_{s=1/2} \tag{4.53}$$

for the three-point and Compton amplitudes.

4.6 SPIN-1 GRAVITATIONAL AMPLITUDES

Gravitational amplitudes with spin-1 matter can be obtained by double-copying two gauge theories with matter in two ways: spin-0 $\times$ spin-1 or spin-1/2 $\times$ spin-1/2 [104, 125, 129]. This fact also holds for heavy particles. We now show this in two examples by deriving the spin-1 gravitational three-point and Compton amplitudes using both double-copy procedures.

4.6.1 0×1 Double Copy

The three-point spin-1 HQET amplitude is

$$\begin{aligned}\mathcal{A}_3^{\mathrm{H},s=1} = \mathbf{T}^a_{ij}\varepsilon_v^{*\beta}\varepsilon_v^{\alpha}\epsilon_q^{*\mu}\Big[&\eta_{\alpha\beta}v_\mu + \frac{1}{2m}(\eta_{\alpha\beta}(k_1+k_2)_\mu - 2q_\beta\eta_{\alpha\mu} + 2q_\alpha\eta_{\beta\mu})\\ &+\frac{1}{2m^2}v_\mu(-k_{1\beta}q_\alpha + q_\alpha q_\beta + q_\beta k_{1\alpha})\Big],\end{aligned} \tag{4.54}$$

where $k_2^\mu = k_1^\mu - q^\mu$. Double-copying with scalar QCD we find

$$\mathcal{M}_3^{\mathrm{H},s=1} = \mathcal{A}_3^{s=0}\mathcal{A}_3^{\mathrm{H},s=1}, \tag{4.55}$$

where $\mathcal{M}_3^{\mathrm{H},s=1}$ is the amplitude derived from eq. (4.73) after applying the field redefinition in eq. (4.74).

Compton scattering for spin-1 HQET is given by the kinematic numerators

$$\begin{aligned}n_s^{\mathrm{H},s=1} = 2m\varepsilon_v^{*\beta}\varepsilon_v^{\alpha}\Big[&v\cdot\epsilon_{q_1}^* v\cdot\epsilon_{q_2}^*\eta_{\alpha\beta} + \frac{v_\rho}{m}(\eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\mu}\eta_{\beta\nu})(\epsilon_{q_1}^{*\mu}q_1^\nu\epsilon_{q_2}^{*\rho} + \epsilon_2^{*\mu}q_2^\nu\epsilon_{q_1}^{*\rho})\\ &-\frac{v\cdot q_2}{2m}(\epsilon_{q_1\alpha}^*\epsilon_{q_2\beta}^* - \epsilon_{q_2\alpha}^*\epsilon_{q_1\beta}^*)\Big],\end{aligned} \tag{4.56a}$$

$$n_t^{\mathrm{H},s=1} = 0, \tag{4.56b}$$

$$n_u^{\mathrm{H},s=1} = n_s^{\mathrm{H},s=1}|_{q_1\leftrightarrow q_2}, \tag{4.56c}$$

where, for brevity, we again write the numerators up to $\mathcal{O}(m^{-1})$ and in the case where $k_1 = \epsilon_i \cdot \epsilon_j = q_i \cdot \epsilon_j = 0$. We have performed the calculation up to $\mathcal{O}(m^{-2})$ and checked the general case explicitly. The double copy becomes

$$\mathcal{M}_4^{\mathrm{H},s=1} = \frac{n_s^{s=0} n_s^{\mathrm{H},s=1}}{d_s} + \frac{n_t^{s=0} n_t^{\mathrm{H},s=1}}{d_t} + \frac{n_u^{s=0} n_u^{\mathrm{H},s=1}}{d_u}, \tag{4.57}$$

where $\mathcal{M}_4^{\mathrm{H},s=1}$ is derived from eq. (4.73) after applying the field redefinition in eq. (4.74).

Thus, we find that

$$(\mathrm{QCD}_{s=0}) \times (\mathrm{HQET}_{s=1}) = \mathrm{HBET}_{s=1} \tag{4.58}$$

for three-point and Compton amplitudes.

4.6.2 $\frac{1}{2} \times \frac{1}{2}$ Double Copy

The spin-1 gravitational amplitudes can also be obtained by double-copying the spin-1/2 HQET amplitudes. To do so, we use the on-shell heavy particle effective theory (HPET) variables of ref. [165] to modify eq. (2.11) of ref. [125] for the case of heavy particles. Using the fact that the on-shell HPET variables correspond to momenta $p_v^\mu = m_k v^\mu$ with mass $m_k = m(1 - k^2/4m^2)$, following the derivation of ref. [125] leads to

$$\mathcal{M}_n^{\mathrm{H},\frac{1}{2}\times\frac{1}{2}} = \frac{m_{k_1} m_{k_2}}{m} \sum_{\alpha,\beta} \mathcal{K}[\alpha|\beta] \mathrm{Tr}[\mathcal{A}_{n,\alpha}^{\mathrm{H},\frac{1}{2}} P_+ \not{\epsilon}_v \bar{\mathcal{A}}_{n,\beta}^{\mathrm{H},\frac{1}{2}} P_- \not{\epsilon}_v^*], \tag{4.59}$$

where $P_\pm = (1 \pm \not{v})/2$, $\mathcal{K}[\alpha|\beta]$ is the massive KLT kernel, and α, β represent color orderings. Here $\mathcal{A}^{\mathrm{H}}$ and $\bar{\mathcal{A}}^{\mathrm{H}}$ are amplitudes with the external states stripped, and $\bar{\mathcal{A}}^{\mathrm{H}} = -\gamma_5 (\mathcal{A}^{\mathrm{H}})^\dagger \gamma_5$. We have also adopted the convention that only the initial matter momentum is incoming. Converting to the on-shell HPET variables, it can be easily seen that

$$\varepsilon_{v\mu}^{IJ}(p) = \frac{1}{2\sqrt{2} m_k} \bar{u}_v^I(p) \gamma_5 \gamma_\mu u_v^J(p), \tag{4.60}$$

with I, J being massive little group indices. Given the WNF for the heavy spinors, the WNF for the polarization vector can easily be computed by comparing eq. (4.60) to its canonical polarization vector analog. We find that it is indeed given by eq. (4.39c).

Applying eq. (4.59) to eq. (4.49) with the three-point KLT kernel $\mathcal{K}_3 = 1$, we immediately recover the left-hand side of eq. (4.55). For Compton scattering the KLT kernel is

$$\mathcal{K}[2|2] = \frac{(-2mv \cdot q_1)(-2mv \cdot q_2)}{2q_1 \cdot q_2}. \tag{4.61}$$

Then, applying eq. (4.59) to the spin-1/2 HQET Compton amplitude with $k_1, q_i \cdot \epsilon_j, \epsilon_i \cdot \epsilon_j \neq 0$ up to and including terms of order $\mathcal{O}(m^{-2})$, we find eq. (4.57) up to $\mathcal{O}(m^{-1})$. When imposing $k_1 = q_i \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0$, cancellations make the double copy valid up to $\mathcal{O}(m^{-2})$. The extension to higher inverse powers of the mass amounts to simply including the contributions of higher-order operators in the HQET and HBET amplitudes.

Therefore, by using eq. (4.59) to convert heavy spinors in amplitudes to heavy polarization vectors, we have shown that

$$(\text{HQET}_{s=1/2}) \times (\text{HQET}_{s=1/2}) = \text{HBET}_{s=1} \tag{4.62}$$

for three-point and Compton amplitudes.

4.7 CONCLUSION

We have shown that the three-point and Compton amplitudes derived from HQET can be double-copied to those of HBET for spins $s \leq 1$. As long as the matter states of HQET and HBET are related through the double copy, in the sense described in Section 4.3, and as long as higher-point amplitudes obey the spectral condition of ref. [126], we see no obstacles to extending the double copy to higher-point amplitudes.

As mentioned in the introduction, due to the operator expansion of HPETs, the double-copy relation between HQET and HBET can be studied at each order in the $\hbar$ expansion, with the classical limit being of special interest. Studying the double copy of HPETs through this lens may provide some insight into the connection between the double copy with matter at the quantum and classical levels. We leave this study for future work.

APPENDIX

4.A LAGRANGIANS FOR HEAVY PARTICLES

We present Lagrangians for heavy particles coupled to bi-adjoint scalars, gluons, and gravitons. The heavy-particle Lagrangians were used to derive the scattering amplitudes in the paper. For clarity, we omit the subscript v for the heavy spin-1 fields.

Bi-adjoint scalars and heavy particles

We couple the bi-adjoint scalars $\mathbf{\Phi}$ to heavy particles with spins $s \leq 1$. The spin-0 Lagrangian is

$$\mathcal{L}^{s=0}_{\mathrm{bi-adjoint}} = \phi^*_v \left[iv\cdot\partial - \frac{\partial_\perp^2 - y_s\mathbf{\Phi}}{2m} + \left(\frac{\partial_\perp^2 - y_s\mathbf{\Phi}}{2m}\right) \frac{1}{2m + iv\cdot\partial + \frac{\partial_\perp^2 - y_s\mathbf{\Phi}}{2m}} \left(\frac{\partial_\perp^2 - y_s\mathbf{\Phi}}{2m}\right)\right]\phi_v. \tag{4.63}$$

The spin-1/2 Lagrangian is

$$\mathcal{L}^{s=1/2}_{\mathrm{bi-adjoint}} = \overline{Q}_v \left[iv\cdot\partial + y_f\mathbf{\Phi} + \left(i\not{\partial}_\perp\right) \frac{1}{2m + iv\cdot\partial - y_f\mathbf{\Phi}} \left(i\not{\partial}_\perp\right)\right] Q_v. \tag{4.64}$$

The spin-1 Lagrangian is

$$\mathcal{L}^{s=1}_{\mathrm{bi-adjoint}} = -B^*_\mu(iv\cdot\partial)B^\mu - \frac{1}{4m}B^*_{\mu\nu}B^{\mu\nu} + \frac{y_v}{2m}B^*_\mu\mathbf{\Phi}B^\mu - \left(\mathcal{F}^\lambda_- B^*_\lambda\right)\frac{2}{m + \frac{1}{m}\partial_\perp^2}\left(\mathcal{F}^\lambda_+ B^*_\lambda\right) \tag{4.65a}$$

where

$$\mathcal{F}^\mu_\pm = \left(\pm\frac{i}{2}\partial^\mu - \frac{1}{2m}\partial^\mu(v\cdot\partial) + \frac{y_v\mathbf{\Phi}}{2m}\right). \tag{4.65b}$$

The coupling constants between the bi-adjoint scalars and the heavy scalars, fermions, and vectors are y_s, y_f, and y_v, respectively.

Gluons and heavy particles

We couple gluons to heavy particles. The covariant derivative in this case is given by $D_\mu = \partial_\mu + ig_s\mathbf{T}^a A^a_\mu$. The scalar Lagrangian is

$$\mathcal{L}^{s=0}_{\mathrm{gluon}} = \phi^*_v \left[iv\cdot D - \frac{D_\perp^2}{2m} + \left(\frac{D_\perp^2}{2m}\right) \frac{1}{2m + iv\cdot D + \frac{D_\perp^2}{2m}} \left(\frac{D_\perp^2}{2m}\right)\right]\phi_v. \tag{4.66}$$

The spin-1/2 Lagrangian is

$$\mathcal{L}^{s=1/2}_{\mathrm{gluon}} = \overline{Q}_v \left[iv\cdot D + \left(i\not{D}_\perp\right) \frac{1}{2m + iv\cdot D} \left(i\not{D}_\perp\right)\right] Q_v. \tag{4.67}$$

The spin-1 Lagrangian [166] with gyromagnetic ratio $g = 2$ can be written as

$$\mathcal{L}^{s=1}_{\text{gluon}} = - B^*_\mu (iv\cdot D) B^\mu - \frac{1}{4m} B^*_{\mu\nu} B^{\mu\nu} + \frac{ig}{2m} F^{\mu\nu} B^*_\mu B_\nu - \left(\mathcal{E}^\lambda_- B^*_\lambda\right) \frac{2}{m + \frac{1}{m} D^2_\perp} \left(\mathcal{E}^\mu_+ B_\mu\right) \quad (4.68a)$$

where

$$\mathcal{E}^\mu_\pm = \left(\pm\frac{i}{2} D^\mu - \frac{1}{2m} D^\mu (v\cdot D) \pm \frac{igv_\nu F^{\nu\mu}}{2m}\right). \quad (4.68b)$$

The heavy spin-1 states described by this Lagrangian are related to the canonical massive spin-1 states through

$$A^\mu(x) = \frac{e^{-imv\cdot x}}{\sqrt{2m}} \left[\delta^\mu_\nu - \frac{1}{1 + iv\cdot\partial/m} \frac{iv^\mu \partial_\nu}{m}\right] B^\nu(x). \quad (4.69)$$

To obtain the desired heavy spin-1 states we apply the field redefinition

$$B_\mu \to \left[\delta^\nu_\mu + \frac{1}{2m^2}\left(-v_\mu v\cdot D + D_\mu\right) D^\nu\right] B_\nu + \mathcal{O}(m^{-3}). \quad (4.70)$$

Gravitons and heavy particles

We couple gravitons to heavy particles. The spin-0 Lagrangian is

$$\sqrt{-g}\mathcal{L}^{s=0}_{\text{graviton}} = \sqrt{-g}\phi^*_v \left[\mathcal{A}_1 + (\mathcal{A}_{2-}) \frac{1}{2m + i(v^\mu \nabla_\mu + \nabla_\mu v^\mu) - \mathcal{A}_1} (\mathcal{A}_{2+})\right] \phi_v, \quad (4.71a)$$

where

$$\mathcal{A}_1 = \frac{1}{2} i g^{\mu\nu} (v_\mu \nabla_\nu + \nabla_\mu v_\nu) + \frac{1}{2} m (g^{\mu\nu} - \eta^{\mu\nu}) v_\mu v_\nu - \frac{1}{2m} \nabla_\mu \left((g^{\mu\nu} - \eta^{\mu\nu})\nabla_\nu + \eta^{\mu\nu}\nabla_{\perp\nu}\right), \quad (4.71b)$$

$$\mathcal{A}_{2\pm} = \frac{1}{2m}(imv_\mu - \nabla_\mu)\left((g^{\mu\nu} - \eta^{\mu\nu})(-imv_\nu + \nabla_\nu)\right) - \frac{1}{2m}\nabla_\mu(\eta^{\mu\nu}\nabla_{\perp\mu}) \pm \frac{1}{2} i \left[\nabla_\mu v^\mu\right], \quad (4.71c)$$

with $v^\mu \equiv \eta^{\mu\nu} v_\nu$ and $\nabla_{\perp\mu} \equiv \nabla_\mu - v_\mu (v^\nu \nabla_\nu)$. The spin-1/2 Lagrangian is

$$\sqrt{-g}\mathcal{L}^{s=1/2}_{\text{graviton}} = \sqrt{-g}\, \overline{Q}_v \left[i\nabla\!\!\!/ + \mathcal{B} + \left(i\nabla\!\!\!/ + \mathcal{B}\right) P_- \frac{1}{2m - (i\nabla\!\!\!/ + \mathcal{B})P_-} \left(i\nabla\!\!\!/ + \mathcal{B}\right)\right] Q_v, \quad (4.72a)$$

where $\nabla\!\!\!/ \equiv \delta^\mu_a \gamma^a \nabla_\mu$ and

$$\mathcal{B} = (e^\mu_a - \delta^\mu_a)(i\gamma^a \nabla_\mu + m\gamma^a v_\mu). \quad (4.72b)$$

The spin-1 Lagrangian can be written as

$$\begin{aligned}\sqrt{-g}\mathcal{L}^{s=1}_{\text{graviton}} =& \sqrt{-g}\left[-\frac{m}{2}(v_\mu B^*_\nu)(v_\rho B_\sigma)\left((g^{\mu\rho} - \eta^{\mu\rho})g^{\nu\sigma} - (g^{\mu\sigma} - \eta^{\mu\sigma})(g^{\nu\rho} - \eta^{\nu\rho})\right)\right.\\ &+ \frac{i}{2}\left[(\nabla_\mu B^*_\nu)(v_\rho B_\sigma) - (v_\mu B^*_\nu)(\nabla_\rho B_\sigma)\right](g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})\\ &\left.- \frac{1}{4m} B^*_{\mu\nu} B_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} - \left(\mathcal{C}^\alpha_- B^*_\alpha\right) \frac{1}{\mathcal{D}} \left(\mathcal{C}^\beta_+ B_\beta\right)\right], \end{aligned} \quad (4.73a)$$

where

$$\mathcal{C}_{\pm}^{\alpha} = -\frac{m}{2}(g^{\alpha\nu} - \eta^{\alpha\nu})v_{\nu} \pm \frac{i}{2}v_{\nu}\left[g^{\mu\rho}g^{\alpha\nu} - g^{\alpha\mu}g^{\nu\rho}\right]\nabla_{\mu}\left(v_{\rho} \pm \frac{i}{m}\nabla_{\rho}\right), \tag{4.73b}$$

$$\mathcal{D} = \frac{m}{2}(v_{\nu}v_{\sigma}g^{\nu\sigma}) + \frac{1}{2m}v_{\nu}\left[g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\rho\nu}\right]\nabla_{\mu}\nabla_{\rho}v_{\sigma}. \tag{4.73c}$$

Note that though the velocity four-vector is constant its covariant derivative does not vanish because of the metric connection. The heavy spin-1 states described by this Lagrangian are related to the canonical massive spin-1 states through eq. (4.69). To obtain the desired heavy spin-1 states we apply the field redefinition

$$B_{\mu} \to \left[\delta_{\mu}^{\nu} + \frac{1}{2m^2}\left(-g^{\alpha\beta}v_{\alpha}D_{\beta}v_{\mu} + D_{\mu}\right)g^{\nu\lambda}D_{\lambda}\right]B_{\nu} + \mathcal{O}(m^{-3}). \tag{4.74}$$

Extending this redefinition to higher orders in $1/m$ is straight-forward.

CHAPTER

5

TIDAL EFFECTS IN QUANTUM FIELD THEORY

ABSTRACT: We apply the Hilbert series to extend the gravitational action for a scalar field to a complete, non-redundant basis of higher-dimensional operators that is quadratic in the scalars and the Weyl tensor. Such an extension of the action fully describes tidal effects arising from operators involving two powers of the curvature. As an application of this new action, we compute all spinless tidal effects at the leading post-Minkowskian order. This computation is greatly simplified by appealing to the heavy limit, where only a severely constrained set of operators can contribute classically at the one-loop level. Finally, we use this amplitude to derive the $\mathcal{O}(G^2)$ tidal corrections to the Hamiltonian and the scattering angle.

5.1 INITIATION

So far, all of the amplitudes methods we have employed treat the particle – and hence the macroscopic object we would like to describe – as point-like. While this is a good approximation, it is not an accurate description of classical bodies. In reality, celestial bodies such as neutron stars have a finite size, and variations in gravitational fields along their spatial extent cause them to deform.[1] These deformations and their effects on dynamics are what we refer to as tidal effects. This publication and the next focused on describing tidal effects in a binary system using EFT and amplitudes methods.

Before jumping in to the publication, we elaborate on how tidal effects are described analytically, then introduce the main tool we have used in this publication: the Hilbert series. We close this section with an overview of the main results in the publication.

5.1.1 Modelling tidal effects

In a classical EFT formulation, the dynamics of a non-rotating point particle in a gravitational field are described by the action [36]

$$S_0 = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R - m\int d\tau, \tag{5.1}$$

where g is the determinant of the metric, R is the Ricci scalar, m is the mass of the particle, and τ is the proper time along the particle's trajectory. The finite size of an object is accounted for by adding operators to this action that involve higher curvature terms [36]. In principle, one can add operators with any power of the curvature, but we restrict ourselves to two powers in this thesis. The tidal corrections

[1] In four dimensions, Schwarzchild black holes do not deform [167]. The results in this publication should therefore be thought of as applying to non-spinning neutron stars.

to eq. (5.1) involving two powers of the curvature are expressed in terms of the mass and current tidal multipole moments, $G_{\mu_1\ldots\mu_l}$ and $H_{\mu_1\ldots\mu_l}$ respectively [168–170]:

$$\Delta S = \int d\tau \sum_{l=2}^{\infty} \frac{1}{2l!} \left[\mu^{(l)} G_{\mu_1\ldots\mu_l} G^{\mu_1\ldots\mu_l} + \frac{l}{l+1} \sigma^{(l)} H_{\mu_1\ldots\mu_l} H^{\mu_1\ldots\mu_l} \right], \tag{5.2}$$

where the multipole moments in terms of the curvature are

$$G_{\mu_1\ldots\mu_l} = -\left[\nabla^\perp_{\langle\mu_1} \ldots \nabla^\perp_{\mu_{l-2}} C_{\mu_{l-1}\underline{\rho}\mu_l\rangle\sigma} \right] v^\rho v^\sigma, \tag{5.3}$$

$$H_{\mu_1\ldots\mu_l} = 2\left[\nabla^\perp_{\langle\mu_1} \ldots \nabla^\perp_{\mu_{l-2}} \tilde{C}_{\mu_{l-1}\underline{\rho}\mu_l\rangle\sigma} \right] v^\rho v^\sigma. \tag{5.4}$$

Angle brackets represent traceless symmetrization exempting underlined indices. $C^{\mu\nu\rho\sigma}$ is the Weyl tensor – the traceless component of the Riemann curvature tensor $R^{\mu\nu\rho\tau}$, see eq. (5.90) – and $\tilde{C}^{\mu\nu\rho\sigma}$ is the dual to the Weyl tensor, defined in Section 5.3.1. The four-velocity of the particle is denoted v^μ, and the covariant derivative in the direction orthogonal to the velocity is $\nabla^\perp_\mu = \nabla_\mu - v_\mu v \cdot \nabla$. Details regarding the derivation of eqs. (5.2) and (5.3) can be found in refs. [168, 171, 172]. Though we don't derive these tidal operators here, let us use EFT arguments to motivate their forms.

There are several plausible choices for operators quadratic in the curvature – expressed in terms of the Riemann tensor and its traces – that could describe tidal effects. However, one restriction on the allowed structures is already imposed by the equation of motion (EOM) for the metric. In a vacuum, Einstein's equations dictate the vanishing of the Ricci tensor ($R^{\mu\nu} = R^{\mu\rho\nu}{}_\rho = 0$), and consequently the Ricci scalar ($R = R^\mu{}_\mu = 0$). A field redefinition of the metric can therefore be employed to remove any higher-curvature operators involving either of these two objects. An explicit redefinition was constructed in a purely classical setting involving matter in ref. [168]. An identical argument holds in the quantum context, as well as in the presence of matter, where contact interactions between the matter fields are generated by the redefinition of the metric [90]. These contact terms are themselves related to the EOM for the matter field, and can similarly be redefined away.

In all circumstances – quantum or classical, with or without matter – the only remaining object that can be used to construct these higher-derivative operators is the Riemann tensor – or, equivalently (up to field redefinitions), its traceless portion, the Weyl tensor. It is advantageous to us to work with the Weyl tensor and its dual. As we will see below, the fields described by the Hilbert series must transform in irreducible representations of $SU(2) \times SU(2)$. Being traceless, the Weyl tensor lends itself more easily than the Riemann tensor to being expressed in terms of such irreducible representations.

So far, we have reached that the new operators must contain two powers of the Weyl tensor or its dual. The question remains of how to contract the indices on the two tensors. First, the second Bianchi identity implies

$$\nabla^\mu C_{\mu\nu\rho\tau} = F[R^{\alpha\beta}, R]_{\nu\rho\tau}, \tag{5.5}$$

where the right-hand side is some function of the Ricci tensor and scalar, which can be redefined away. Therefore, none of the indices on the Weyl tensors can be contracted with covariant derivatives that act on them. The two Weyl tensors can have all indices contracted amongst one another, $C^{\mu\nu\rho\tau}C_{\mu\nu\rho\tau}$, but this will be redundant with another operator that more transparently couples to the deformed object's worldline. To couple to the worldline, since we are considering a spinless particle the only available four-vector is the four-velocity of the object. We must contract two indices of each Weyl tensor with four-velocities; otherwise we arrive at a redundant operator through the identity

$$C^{\mu\nu\alpha\beta}C^{\rho}{}_{\nu\alpha\beta} = \frac{1}{4}g^{\mu\rho}C^{\tau\nu\alpha\beta}C_{\tau\nu\alpha\beta}. \tag{5.6}$$

Contracting more than two velocities with a Weyl tensor results in 0 by the symmetry properties of the Weyl tensor.

Therefore, when no derivatives are involved, the only operators that can be written involving two powers of the curvature are $C_{\mu\nu\rho\tau}v^{\nu}v^{\tau}C^{\mu\alpha\rho\beta}v_{\alpha}v_{\beta}$ or the analog involving dual Weyl tensors; the operator $C^{\mu\nu\rho\tau}C_{\mu\nu\rho\tau}$ emerges from $\tilde{C}_{\mu\nu\rho\tau}v^{\nu}v^{\tau}\tilde{C}^{\mu\alpha\rho\beta}v_{\alpha}v_{\beta}$ upon applying Levi-Civita identities.[2] Higher multipole moments can be derived from these by differentiating each Weyl tensor. Derivatives must come symmetrized – antisymmetric combinations are related to the Riemann tensor and we are only interested in quadratic-in-curvature operators – and in traceless combinations because of eqs. (5.5) and (5.92). When derivatives are involved a derivative can be contracted to the Weyl tensor that it is not differentiating.

We have come a long way towards understanding eqs. (5.2) and (5.3) using strictly effective-field-theoretic intuition. Unfortunately, we cannot go all the way using only this sort of thinking; for example, from an EFT perspective, every independent operator should appear in an action with its own coefficient. This doesn't happen in eq. (5.2). Illustrating with the $l = 3$ mass multipole:

$$G_{\mu_1\mu_2\mu_3} = -\left[\nabla^{\perp}_{\langle\mu_1}C_{\mu_2\underline{\rho}\mu_3\rangle\sigma}\right]v^{\rho}v^{\sigma} = -\left[\nabla_{\langle\mu_1}C_{\mu_2\underline{\rho}\mu_3\rangle\sigma}\right]v^{\rho}v^{\sigma} + \left[v_{\langle\mu_1}v\cdot\nabla C_{\mu_2\underline{\rho}\mu_3\rangle\sigma}\right]v^{\rho}v^{\sigma}. \tag{5.7}$$

Both terms on the right-hand side come with the same coefficient in the action, which *a priori* need not be the case. More information derived from GR has been used to fix the relative proportions of these two terms in the classical setting.

So much for the classical action. Let us now motivate the quantum mechanical description of tidal effects. We stress that this is by no means a derivation of the quantum tidal action, but simply a schematic guide for understanding what operators to expect. The first thing to do is promote the tidal moments above to quantum mechanical operators. Moreover, if they are to describe the deformation of an object, they must interact with the field describing that object. We're ignoring spin here, so the appropriate field to couple to will be a scalar field ϕ_v, which we have labelled suggestively with its corresponding four-velocity. Focusing on the $l = 2$ multipoles, the quantum Lagrangian is schematically

$$\Delta\mathcal{L} \sim \phi_v^*\left[\frac{\mu^{(2)}}{4}C_{\mu\nu\rho\sigma}v^{\nu}v^{\sigma}C^{\mu\alpha\rho\beta}v_{\alpha}v_{\beta} + \frac{2}{3}\sigma^{(2)}\tilde{C}_{\mu\nu\rho\sigma}v^{\nu}v^{\sigma}\tilde{C}^{\mu\alpha\rho\beta}v_{\alpha}v_{\beta}\right]\phi_v + \ldots, \tag{5.8}$$

where the dots represent higher multipole moments. This can be rewritten without the dual of the Weyl

[2] One could also consider operators involving one Weyl tensor and one dual Weyl tensor. Such operators encode parity-odd effects, and we ignore them here. See ref. [170] for calculation of parity-odd tidal effects.

tensor using Levi-Civita identities:[3]

$$\Delta\mathcal{L} \sim \frac{\phi_v^*}{\sqrt{m}}\left[\frac{1}{4}\left(\mu^{(2)}+\frac{8}{3}\sigma^{(2)}\right)C_{\mu\nu\rho\sigma}v^\nu v^\sigma C^{\mu\alpha\rho\beta}v_\alpha v_\beta - \frac{1}{12}\sigma^{(2)}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right]\frac{\phi_v}{\sqrt{m}}+\dots. \tag{5.9}$$

Now, the four-velocity is not typically a parameter in a quantum action, unless we are dealing with actions for heavy particles. In this case, velocities arise from the heavy limit of derivatives of the matter field. Going backwards, then, if we identify ϕ_v as the heavy limit of a (canonical) scalar field ϕ,

$$\Delta\mathcal{L} \sim \left[\frac{1}{4m^4}\left(\mu^{(2)}+\frac{8}{3}\sigma^{(2)}\right)C_{\mu\nu\rho\sigma}C^{\mu\alpha\rho\beta}\nabla^\nu\nabla^\sigma\phi\nabla_\alpha\nabla_\beta\phi - \frac{1}{12}\sigma^{(2)}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\phi^2\right]+\dots. \tag{5.10}$$

Up to a redefinition of the Wilson coefficients and summing over particle identities, this is the Lagrangian of ref. [173]. Higher multipole operators are then related to these two structures through arrangements of additional derivatives among the factors of each operator. There are several possible ways to do this, but we give a prescription for the preferred distribution of derivatives in Section 5.3.2.

5.1.2 The Hilbert series

We now understand that if we wish describe tidal effects from a QFT perspective, we must build the most general action comprising operators with two factors of the scalar field and two factors of the Weyl tensor. The Hilbert series is an indispensable tool for this purpose. We move now to introducing the Hilbert series.

Given one or more compact Lie groups, the Hilbert series counts the number of group invariants that can be built out of a given set of objects that transform under potentially different irreducible representations of those groups. To do so, the Hilbert series relies on the orthonormality of the characters of compact groups [174, 175],

$$\int d\mu(\vec{z})\,\chi_i(\vec{z})\chi_j^*(\vec{z}) = \delta_{ij}. \tag{5.11}$$

Here, μ is the Haar measure of the group, i, j label irreducible representations, and $\vec{z}$ is a vector of complex numbers parametrizing the irreducible representations.

The possibility of knowing the number of group invariants is enticing to effective field theorists. In the construction of EFTs – such as the Standard Model EFT (SMEFT) [174, 176–178] or GR + Standard Model EFT (GRSMEFT) [179], for example – the output of the Hilbert series is the number of operators with a given field (and derivative) content that are invariant under the Lorentz group and whatever gauge groups are relevant to the problem. This guides the effective field theorist in the construction of a Lagrangian at a certain mass dimension; once they find a number of independent operators consistent with the output of the Hilbert series, they know they have a basis for all operators with that field content.[4] We were interested in using the Hilbert series in precisely this manner: with the aim of describing tidal effects involving two powers of the Weyl tensor and two of the matter field, we used the Hilbert series to count the number of such operators with an arbitrary number of covariant derivatives. Given

[3] The dual to the Weyl tensor is defined using the Levi-Civita tensor, but we are only interested in two-graviton-two-matter vertices. Dropping higher-point interactions, we can use identities for the Levi-Civita symbol instead.

[4] On a semantic note, by "operator basis" we mean the minimal set of operators needed to generate all possible on-shell structures involving certain fields and their derivatives, potentially with a restricted mass dimension.

this counting as a guide, we could then construct an operator basis at each mass dimension.

For our applications, the relevant part of the Hilbert series takes the form [175]

$$\mathcal{H}_0 = \int d\mu_{(SU(2)\times SU(2))\times G} \frac{1}{P(\mathcal{D};\alpha,\beta)} \prod_i \text{PE}[\psi_i], \tag{5.12}$$

which is obtained after integrating over the dilations of the four-dimensional conformal group $SO(6,\mathbb{C})$.[5] The role of the conformal group will be elucidated shortly. $P(\mathcal{D};\alpha,\beta)$ is the momentum generating function (MGF), $\mathcal{D}$ represents (covariant) derivatives, and α, β parametrize representations of $SU(2)\times SU(2)$. PE$[\psi_i]$ is the plethystic exponential (PE) for the field ψ_i.[6] The PEs construct all combinations of the fields ψ_i, including those that are not invariant under $(SU(2)\times SU(2))\times G$ – these are projected out upon integration. The MGF also appears in the PEs, and has the effect of producing operators containing derivatives with EOM redundancies – which relate operators of different mass dimensions [112] – taken into account. Its explicit appearance in eq. (5.12) is a consequence of a prior integration over the conformal group that removes total-derivative redundancies. All together, when integrated over the Haar measure, the Hilbert series gives the number of $(SU(2)\times SU(2))\times G$ invariant objects with a specific field content, minus EOM and total-derivative relations.

It will be instructive to elaborate slightly on the EOM and total-derivative redundancies that must be accounted for by the Hilbert series. This will lead us to a better understanding of the PE and the MGF, as well as the role of the conformal group in this construction. Once we have done this, we will briefly discuss the final element in eq. (5.12): the Haar measure. A detailed formulation of the Hilbert series and its application to operator bases can be found in ref. [175], and we summarize only some key concepts to their derivation here. For more introductory-level discussions of the Hilbert series, see [174, 176].

EOM redundancies

The EOM for a field relates derivatives of that field to its mass. For example, a scalar field obeys the EOM $\partial^2\phi = -m^2\phi$. A consequence of this is that, through field redefinitions, operators containing factors on, say, the left-hand side of this EOM can be swapped for operators with the right-hand side as a factor instead. When enumerating operators starting with low mass dimensions and systematically increasing the number of derivatives, higher-dimensional operators containing derivative structures related to the EOM are thus redundant to lower-derivative operators. To avoid redundancy, higher-derivative operators must only contain symmetric, traceless combinations of derivatives acting on a given field.[7,8] This leads to the concept of the single-particle module, representing all the ways a field and its derivatives can

[5] Despite the conformal group not being compact, character orthonormality can still be salvaged to a sufficient extent in the present context [175]. There are corrections, $\Delta\mathcal{H}$, to eq. (5.12) due to this non-compactness, but they don't affect our results.

[6] More accurately, the ψ_i and $\mathcal{D}$ in eq. (5.12) are complex numbers standing in for quantum fields and derivatives respectively. In the literature they are referred to as "spurions", but we stick with calling the ψ_i "fields" and $\mathcal{D}$ "derivatives" in this initiation.

[7] In the case of partial derivatives, antisymmetric combinations of derivatives vanish. For covariant derivatives, terms like $D^2\phi$ are still redundant by the EOM, while antisymmetric combinations are related to field strengths and therefore accounted for elsewhere. The same holds for a Dirac spinor ψ: $D^2\psi = \not{D}\not{D}\psi + \frac{i}{2}\sigma^{\mu\nu}[D_\mu, D_\nu]\psi$, where the first term is related to the EOM and the second to an operator with a field strength.

[8] Effectively, this method sets $\partial^2\phi = 0$, even if the field is massive, with operators containing ϕ (without derivatives) instead already accounted for at lower mass dimensions/derivative multiplicities. The Hilbert series therefore does not distinguish between massive and massless fields.

contribute to operators once the EOM redundancies have been removed:

$$R_\phi = \begin{pmatrix} \phi \\ \partial_\mu \phi \\ \partial_{\langle \mu_1} \partial_{\mu_2 \rangle} \phi \\ \vdots \end{pmatrix}, \tag{5.13}$$

where, again, angle brackets represent traceless symmetrization.

As we have worked exclusively in four dimensions, all single-particle modules relevant to the work in this thesis form unitary conformal representations, with the exception of the Weyl tensor [175, 179]. The reason for the exemption of the Weyl tensor is that the scaling dimensions (equal to the mass dimensions) of its left and right handed components – which are the components transforming in irreducible representations of $SU(2) \times SU(2)$ – violate the unitarity bound dictating whether a conformal representation is unitary. This is not an obstacle to applying eq. (5.12) in the gravitational case. It is only consequential if one wishes to compute the correction to the Hilbert series due to non-compactness of the conformal group, in which case the work-around is simply to assign conformal scaling dimension $\Delta_C = 3$ to the Weyl tensor [179]. Doing so has no bearing on the portion of the Hilbert series we are interested in; as we will shortly elaborate, the only appearance of the scaling dimension in our calculation cancels in the PE. In eq. (5.13), ϕ is the primary of the representation and all other elements are its descendants. Having the fields and their derivatives organized into unitary conformal representations suggests to use the conformal group to count derivative interactions through the Hilbert series.

To better understand how eqs. (5.12) and (5.13) and the conformal group account for EOM redundancies, let us write the PE explicitly for a scalar field ϕ:

$$\text{PE}[\phi] = \exp\left[\sum_{r=0}^{\infty} \frac{\phi^r}{r\mathcal{D}^{r\Delta_\phi}} \chi_\phi(\vec{z}^{\,r})\right]. \tag{5.14}$$

If $\vec{z} = (z_1, \ldots, z_n)$, we denote $\vec{z}^{\,r} \equiv (z_1^r, \ldots, z_n^r)$ in this context. $\Delta_\phi = 1$ is the scaling dimension of ϕ, which is equal to its mass dimension. EOM redundancies are accounted for by the group character $\chi_\phi(\vec{z})$. The group character is a product of the characters for the gauge groups G and the conformal group:

$$\chi_\phi(\vec{z}) = \chi_G(\vec{z}_G)\chi_{[\Delta_\phi,(l_\phi,r_\phi)]}(\mathcal{D};\alpha,\beta) = \chi_G(\vec{z}_G)\chi_{[1,(0,0)]}(\mathcal{D};\alpha,\beta), \tag{5.15}$$

where $\vec{z} \equiv (\vec{z}_G, \mathcal{D}, \alpha, \beta)$. The representation of the conformal group is labelled by the scaling dimension Δ of the primary and the representation (l, r) of the Lorentz group $SU(2) \times SU(2)$ under which the field transforms [175].

Using the character of the conformal group, the PE represents the entire single particle module in eq. (5.13) as opposed to just the primary. In this way, derivative interactions are accounted for without including EOM redundancies. Explicitly for a scalar field, the character of the conformal group is

$$\chi_{[1,(0,0)]}(\mathcal{D};\alpha,\beta) = \mathcal{D}P(\mathcal{D};\alpha,\beta)(\chi_{(0,0)} - \mathcal{D}^2\chi_{(0,0)}). \tag{5.16}$$

The overall factor of $\mathcal{D}$ cancels in the argument of the exponential. What remains is the MGF, which generates all symmetric descendants of the primary (not just the traceless ones; this is easier to see from its definition in eq. (3.6) of ref. [175]), and the factor of $\chi_{(0,0)} - \mathcal{D}^2\chi_{(0,0)}$, which transparently removes descendants involving contracted derivatives acting on the scalar field. Character orthonormality therefore selects precisely those operators which involve ϕ or its descendants in eq. (5.13).

The conformal character of every field we consider in this thesis follows the same pattern as the scalar case. Each one has an overall factor of $\mathcal{D}^\Delta$ which cancels in the PE, a factor of the MGF to generate all symmetric derivatives of the field, and a factor which involves subtracting the EOM (and transversality conditions for bosons) from the character for the field's Lorentz representation. In this way, the result of the Hilbert series includes derivative interactions while accounting for EOM redundancies.

Total-derivative redundancies

Operators – or combinations thereof – that can be expressed as total derivatives do not contribute to scattering amplitudes. This has two implications for a complete operator basis. First, it cannot contain any operators that are total derivatives. Consequently, no operator in the basis can be related to linear combinations of other operators in the basis through a total derivative (see Section 5.B for an example of this). Unitary conformal representations also play a role in accounting for these redundancies.

Operators $\mathcal{O}$ are formed through tensor products of fields. In terms of single particle modules,

$$\bigotimes_i \begin{pmatrix} \psi_i \\ \partial_\mu \psi_i \\ \partial_{\langle\mu_1}\partial_{\mu_2\rangle}\psi_i \\ \vdots \end{pmatrix} = \bigoplus_j \begin{pmatrix} \mathcal{O}_j \\ \partial_\mu \mathcal{O}_j \\ \partial_{\mu_1}\partial_{\mu_2}\mathcal{O}_j \\ \vdots \end{pmatrix}, \tag{5.17}$$

where on the right-hand side the tensor product forming the operators has been decomposed into irreducible representations. Thus, operators and their descendants can be arranged into irreducible representations of the conformal group [175]:

$$R_{[\Delta;(l,r)]} = \begin{pmatrix} \mathcal{O}_{(l,r)} \\ \partial_\mu \mathcal{O}_{(l,r)} \\ \partial_{\mu_1}\partial_{\mu_2}\mathcal{O}_{(l,r)} \\ \vdots \end{pmatrix}, \tag{5.18}$$

where the $SU(2) \times SU(2)$ representation of the operator is made explicit. We are seeking to enumerate operators that are Lorentz scalars, so the Hilbert series must select those with $(l, r) = (0, 0)$. Moreover, descendants are total derivatives of operators, which cannot be part of a basis. We conclude that the desired operators are scalar conformal primaries.[9]

Both the restriction to scalar operators and to primaries of the conformal representation can be tackled in one go by noting that, even if $(l, r) = (0, 0)$, descendants of $\mathcal{O}_{(0,0)}$ are not Lorentz scalars. Character orthonormality thus allows us to target the desired operators by inserting the character $\chi_{[\Delta;(0,0)]}$ under

[9] One might worry that the conformal primary is itself a total derivative of some sub-operator. This cannot be the case. A primary operator is the highest weight operator of its conformal representation, with the weight being lowered by differentiation [175]. Thus, if some operator can be written as $\mathcal{O} = \partial\mathcal{O}'$ then $\mathcal{O}'$ has higher weight than $\mathcal{O}$, and hence the latter cannot be a primary.

the integration over the conformal group [175, 179]:[10]

$$\mathcal{H} \sim \int d\mu_{SO(6,\mathbb{C})\times G} \sum_{\Delta} \chi_{[\Delta;(0,0)]} \prod_i \text{PE}[\psi_i]. \tag{5.19}$$

The sum over the scaling dimension results from decomposing the tensor product of fields into irreducible conformal representations. This integration over the conformal group produces the inverse MGF in eq. (5.12); it comes from the Haar measure for the group [175].

An alternative understanding of the treatment of total-derivative redundancies is given in ref. [178] through a consideration of differential forms in d dimensions. There it is argued that, through character orthonormality, the inverse of the MGF in eq. (5.12) projects onto the operators that are not related by total derivatives.

The Haar measure

Equation (5.12) accounts for the transformation of the fields ψ_i under the Lorentz group $SU(2)\times SU(2)$ as well as a gauge group G, which may itself be a product group. The integration over the Haar measure for a product group is simply the product of integrations over the Haar measures of each group,

$$\int d\mu_{G_1\times G_2} = \int d\mu_{G_1} \int d\mu_{G_2}. \tag{5.20}$$

Therefore, eq. (5.12) contains as many integrations as there are groups in the product group $(SU(2)\times SU(2))\times G$.

The group $SU(2)\times SU(2)$ is what remains after integrating over the dilations of the conformal group. Let us comment briefly on the necessity and viability of integrating over $SU(2)\times SU(2)$ instead of the full Lorentz group $SO(1,3)$. As is clear by now, character orthonormality is essential to the functioning of the Hilbert series. Orthonormality requires a compact group, which $SO(1,3)$ is not. Instead, we can take advantage of the local isomorphism between $SO(1,3)$ and $SU(2)\times SU(2)$ to integrate over the latter group instead, which is compact [174, 180]. This local isomorphism ensures that the Hilbert series will count the correct number of Lorentz invariant structures.

All quantities involved in the computations of the Hilbert series in this publication and the next are collected in Sections 5.A and 6.3.1.

5.1.3 Overview of main results

The first step towards detailing the class of tidal operators with two scalars and two Weyl tensors was to compute the Hilbert series at each mass dimension. Due to the mass dimensions and Lorentz structures of scalar fields and Weyl tensors, operators only carried even mass dimensions in this case. Computing

$$\mathcal{H} = \int d\mu_{SU(2)\times SU(2)} \frac{1}{P(\mathcal{D};\alpha,\beta)} \text{PE}[\phi]\text{PE}[C_L]\text{PE}[C_R], \tag{5.21}$$

[10]This integration also removes scalar total derivative operators of the form $\partial_\mu \mathcal{O}^\mu$ because eq. (5.18) does not carry information about how Lorentz indices are contracted. Such an object arises in eq. (5.18) in the form $\partial_\mu \mathcal{O}_\nu$ and is thus not treated as a scalar [175].

up to $\mathcal{O}(\phi^2 C_{L/R} C_{L/R})$, where $C_{L/R}$ are the left- and right-handed components of the Weyl tensor, we found that the Hilbert series at mass dimension $6+2n$ could be written in closed form as

$$\mathcal{H}_{6+2n}^{C^2} = \left\lfloor \frac{n+2}{2} \right\rfloor \left(C_L^2\phi^2 + C_R^2\phi^2\right)\mathcal{D}^{2n} + \left\lfloor \frac{n}{2} \right\rfloor C_L C_R \phi^2 \mathcal{D}^{2n} \tag{5.22}$$

With this as a guide to the number of operators at each mass dimension, we were able to write down all operators explicitly. We determined the tidal correction involving two Weyl tensors and two scalar fields to the action of a minimally coupled scalar to be

$$\begin{aligned} &\Delta\mathcal{L}_{\mathrm{GR}}^{\mathrm{tidal}} = \\ &\sum_{n=0}^{\infty}\sum_{k=0}^{N}\left\{c_k^{(n)}\left[\nabla^{\mu_1\ldots\mu_k}\phi\right]\left[\nabla_{\nu_1\ldots\nu_k}\phi\right]\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{\rho\sigma\alpha\beta}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C^{\rho\sigma\alpha\beta}\right]\right. \\ &\left.+\, d_k^{(n+2)}\left[\nabla^{\rho\sigma\mu_1\ldots\mu_k}\phi\right]\left[\nabla_{\alpha\beta\nu_1\ldots\nu_k}\phi\right]\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{\lambda\rho\tau\sigma}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C^{\lambda\alpha\tau\beta}\right]\right\}. \end{aligned} \tag{5.23}$$

This correction could then be used to derive all tidal corrections to the 2PM scalar-scalar scattering amplitude. Noticing that the general even-rank triangle integral satisfies (see eq. (5.44) for the explicit form of the left-hand side)

$$v_{1\mu_1\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\ldots\mu_{2k}} = \frac{\left(\frac{1}{2}\right)_k}{4^k\,(1)_k}(\omega^2-1)^k q^{2k}\mathcal{I}_{\triangleleft} + \mathcal{O}(\hbar^{2k}) \tag{5.24}$$

when v_1 is different from the velocity in the massive propagator of the integral, we evaluated the full tidal corrections to the 2PM amplitude:

$$\Delta\mathcal{M}_2 = 4G^2 S m_2^3 \sum_{i=0}^{\infty}\left(-\frac{q^2}{2}\right)^{i+2} g_i(\omega), \tag{5.25a}$$

where

$$\begin{aligned} g_i(\omega) \equiv \sum_{k=0}^{i}\frac{(-1)^k(\frac{1}{2})_k}{2^k(1)_k}(\omega^2-1)^k\Bigg[&16 m_1^{2k} c_k^{(i+k)} \\ &+\frac{m_1^{2k+4} d_k^{(i+k+2)}}{4(k+2)(k+1)}\left[(2k+5)(2k+7)\omega^4 - 6(2k+5)\omega^2 + (4k^2+12k+11)\right]\Bigg]. \end{aligned} \tag{5.25b}$$

Finally, we used this amplitude to calculate the tidal corrections to the interaction potential and the scattering angle at 2PM order.

We also computed some analogous results in the context of electromagnetism – where the two Weyl tensors are replaced by two electromagnetic field strengths – up to the one-loop amplitude.

5.2 INTRODUCTION

There is a long history of relating scattering amplitudes to conservative two-body classical observables. Traditionally, such approaches have made extensive use of the quantum action of gravity [181], and have been used most commonly to compute non-relativistic classical and quantum corrections to the interaction Hamiltonian [7, 8, 23, 40, 182]. Other approaches still have utilized on-shell methods to compute the amplitudes, before producing the interaction Hamiltonian [15, 41, 79]. Other than the interaction Hamiltonian, refs. [15, 79, 183] also extracted information about the metric from the two-to-two scattering amplitude.

Even compared to this illustrious record, tremendous progress on this topic in a relatively short time has been inspired by the detection of gravitational waves (GWs) by the LIGO and Virgo collaborations [1]. Developments in this time have by and large focused on the post-Minkowskian (PM) expansion of amplitudes and observables [184, 185]. This has required new tools for the conversion of PM amplitudes to classical quantities such as the interaction Hamiltonian [29, 48, 49], the linear and angular impulse and radiated momentum [10, 14], the scattering angle [50, 91, 186, 187], and the metric [188]. On the front of the amplitudes themselves, the current state-of-the-art is the third post-Minkowskian (3PM) amplitude for scalar-scalar scattering [16, 17, 189] (extended to include tidal effects in ref. [173]). The 3PM amplitude for massless scattering was also computed in ref. [190]. Moreover, amplitudes techniques have been used to compute observables in modified theories of gravity [88, 89].

There has also been significant progress made on the inclusion of spin effects. The spin-1/2 $\times$ spin-1/2 amplitude was computed up to the second post-Minkowskian order using heavy particle effective theory (HPET) techniques in ref. [92], and was converted to a spinning Hamiltonian as part of the spin-inclusive formalism of ref. [29]. An alternative approach to this amplitude involving the leading singularity was presented in ref. [43]. Making use of the massive on-shell variables of ref. [63], several results including all orders in spin were achieved in refs. [45–47, 53, 95, 165, 191]. Some of the notable results from these works include the interpretation of a Kerr black hole as a minimally-coupled infinite spin particle, the scattering angle at the second post-Minkowskian (2PM) order up to fourth order in spin, an amplitudes interpretation of the Newman-Janis complex deformation of Schwarzchild spacetime, and the full 1PM spinning Hamiltonian. Finally, ref. [192] argued that the scattering of minimally coupled spinning particles minimizes the generated entanglement entropy.

Though a plethora of novel results have been achieved using amplitudes-based approaches, the vast majority of results directly applicable to GW templates have been derived using general relativistic methods. Of particular relevance to this paper are the computations of tidal effects on the binary inspiral problem. In this context, several tools have been applied to the computations of these effects. Two such tools are the post-Newtonian (PN) and PM approximations. In the PN context, tidal moments were first introduced in ref. [193]. Ref. [169] incorporated tidal effects into the effective one-body (EOB) formalism [30], and ref. [194] presented tidal contributions to the binding energy within the EOB. Most recently, tidal effects on the PM scattering angle have been computed in refs. [170, 195]

Up to this point, almost all amplitudes approaches to the binary inspiral problem have ignored finite size and tidal effects. In fact, the recent work of ref. [173] is the first application of amplitudes methods to the calculation of these effects. By focusing on operators quadratic in the Weyl tensor, they computed tidal contributions to spinless amplitudes arising from the electric and magnetic quadrupoles, up to the next-to-leading-PM order ($\mathcal{O}(G^3)$).[11] Converting their amplitudes to classical observables, they found agreement with results derived from conventional general relativistic methods [168–170, 194, 195].

In this paper, we expand on the work of ref. [173]. Through application of the Hilbert series (see e.g. [174–179, 196–198]), we obtain a gravitational action comprising all operators quadratic in the

[11] Note that one-loop is the leading order where tidal effects can contribute to conservative dynamics.

Weyl tensor and quadratic in a real scalar field. This action is sufficient to fully describe all spinless tidal contributions to the amplitude at the leading-PM order ($\mathcal{O}(G^2)$). Since we are only interested in the classical portion of the amplitude, we exploit the manifest $\hbar$ scaling of the heavy limit of the action to isolate only classically contributing operators [92]. This simplifies the computation, and we are able to straightforwardly produce the full classical tidal integrand at the leading-PM order. Integrating the integrand in principle requires knowledge of the general even-rank triangle integral. However, we are able circumvent this issue since we are simply interested in the leading-in-$\hbar$ portion of the integral that is proportional to $S \equiv \pi^2/\sqrt{-q^2}$. This allows us to find a form of the general even-rank triangle integral that we have explicitly checked up to rank 10, and that was proven in ref. [199] while this paper was in review. Applying this results in the complete leading-PM tidal amplitude. We indeed find the leading-PM contribution of ref. [173] as the leading contribution to our amplitude. We then use our amplitude to derive all leading-PM tidal corrections to the Hamiltonian and the scattering angle, comparing to existing results along the way.

The layout of this paper is as follows: We begin in Section 5.3 by presenting the full tidal actions for electromagnetism and gravity coupled to real scalars at quadratic order in the field strength or the Weyl tensor respectively. We include a brief primer on the Hilbert series in this section, as it is the main tool in our construction. With the tidal actions in hand, Section 5.4 focuses on the computation of tidal contributions to the one-loop amplitudes. The heavy limits of the tidal actions are also presented here. We conclude in Section 5.5.

5.3 TIDAL ACTIONS

This section is dedicated to the presentation of the tidal actions up to quadratic order in the field strengths or Weyl tensors respectively for QED or gravity coupled to a real scalar. We achieve the complete forms of these actions through application of the Hilbert series. As such, we begin with a brief introduction to the Hilbert series before presenting the results of the series and corresponding tidal actions for QED and then gravity. Technical details about the Hilbert series are postponed to Section 5.A.

5.3.1 Hilbert series for tidal effects

The Hilbert series uses character orthonormality to count group invariants. It is an important tool for constructing a basis of higher-dimensional operators, and has been applied to the effective-field-theory extension of the Standard Model in refs. [174, 176–178], while ref. [179] also included gravity.

The input for the Hilbert series is the field content and the fields' representations under compact symmetries. The output is the number of invariant operators with a given field content and covariant derivatives. Redundancies coming from integration-by-parts relations and field redefinitions are taken into account.

We first want to construct operators with real scalar fields ϕ coupled to photons. The Lorentz group $SO(1,3)$ is not a compact group, but we can use the Euclidean group $SO(4) \simeq SU(2)_L \times SU(2)_R$ to find the group invariants. We then work with fields transforming in irreducible representations of $SU(2)_L$ and $SU(2)_R$ built from linear combinations of the field strength $F_{\mu\nu}$ and the dual field strength

$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$:

$$F^{\mu\nu}_{L/R} \equiv \frac{1}{2}\left(F^{\mu\nu} \pm i\tilde{F}^{\mu\nu}\right). \tag{5.26}$$

The characters for $F^{\mu\nu}_{L/R}$ and ϕ are the input to the Hilbert series.

We restrict our attention to the operators with two real scalar fields, two field strengths, and an arbitrary number of covariant derivatives. The output of the Hilbert series $\mathcal{H}_d^{F^2}$ for mass dimension $d = 6 + 2n$ is

$$\mathcal{H}^{F^2}_{6+2n} = \left\lfloor \frac{n+2}{2} \right\rfloor \left(F_L^2\phi^2 + F_R^2\phi^2\right)\mathcal{D}^{2n} + \left\lfloor \frac{n+1}{2} \right\rfloor F_L F_R \phi^2 \mathcal{D}^{2n}, \tag{5.27}$$

where $n \geq 0$ is an integer and $\lfloor x \rfloor$ is the floor function.

Now consider the Hilbert series for two real scalars coupled to gravity. As explained in Section 5.B, non-redundant operators quadratic in the curvature can be written in terms of the Weyl tensor $C^{\mu\nu\rho\sigma}$. Thus we need only the group characters of

$$C^{\mu\nu\rho\sigma}_{L/R} = \frac{1}{2}\left(C^{\mu\nu\rho\sigma} \pm i\tilde{C}^{\mu\nu\rho\sigma}\right), \tag{5.28}$$

where $\tilde{C}^{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}C_{\alpha\beta}{}^{\rho\sigma}$ is the dual to the Weyl tensor. The Hilbert series $\mathcal{H}_d^{C^2}$ for two real scalar fields, two Weyl tensors, and an arbitrary number of covariant derivatives is

$$\mathcal{H}^{C^2}_{6+2n} = \left\lfloor \frac{n+2}{2} \right\rfloor \left(C_L^2\phi^2 + C_R^2\phi^2\right)\mathcal{D}^{2n} + \left\lfloor \frac{n}{2} \right\rfloor C_L C_R \phi^2 \mathcal{D}^{2n}, \tag{5.29}$$

for integer $n \geq 0$.

We use the output of the Hilbert series as a guide for constructing a basis of higher-dimensional operators which capture all leading-PM tidal effects in electromagnetism and gravity.

5.3.2 QED

The Lagrangian we are after couples a real scalar to photons through operators quadratic in the field strength:

$$\mathcal{L}_{\text{QED}} = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{m^2}{2}\phi^2 + \Delta\mathcal{L}_{\text{QED}}. \tag{5.30}$$

Here $\Delta\mathcal{L}_{\text{QED}}$ describes the tidal interactions between two real scalars and two field strength tensors. We are interested in using the Hilbert series in eq. (5.27) to construct this contribution at general mass dimension.

Ultimately, there is a freedom in the operator basis we use (see Section 5.B). We choose a basis that is optimized for the computation of classical amplitudes. Such a basis does not include any structures of the form $D^\mu\phi D_\mu\phi\mathcal{O}_{F^2}$. These can be seen to mix with $\phi^2\mathcal{O}_{F^2}$ in the heavy limit, hence one could receive contributions to classically-contributing heavy operators from an infinite number of operators in the full action. Furthermore, we will avoid derivative placements that produce any structure that can be removed by a field redefinition; see Section 5.B for a list of such structures. Accounting for these criteria, we will build our basis out of operators of the following form:

$$\mathcal{O}^{(n)}_{LL,k} = \left[D^{\mu_1\ldots\mu_k}\phi\right]\left[D_{\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{L,\rho\sigma}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F_L^{\rho\sigma}\right], \tag{5.31a}$$

$$\mathcal{O}^{(n)}_{RR,k} = \left[D^{\mu_1\ldots\mu_k}\phi\right]\left[D_{\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{R,\rho\sigma}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F_R^{\rho\sigma}\right], \tag{5.31b}$$

$$\mathcal{O}^{(n+1)}_{LR,k} = \left[D^{\rho\mu_1\ldots\mu_k}\phi\right]\left[D_{\sigma\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{L,\rho\tau}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F_R^{\sigma\tau}\right], \tag{5.31c}$$

where $0 \leq k \leq \lfloor n/2 \rfloor$. This range of k produces the number of operators dictated by the Hilbert series. We have defined $D_{\mu_1\ldots\mu_n} \equiv D_{\mu_1}\ldots D_{\mu_n}$.

We would like to construct our action out of the fields $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$. To do so we simply replace $F^{\mu\nu}_{L,R}$ in terms of the field strength and its dual. After this replacement the operators above become

$$\begin{aligned}\mathcal{O}^{(n)}_{LL,k} &= 2\left[D^{\mu_1\ldots\mu_k}\phi\right]\left[D_{\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{\rho\sigma}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F^{\rho\sigma}\right]\\ &\quad + 2i\left[D^{\mu_1\ldots\mu_k}\phi\right]\left[D_{\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{\rho\sigma}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}\tilde{F}^{\rho\sigma}\right], \end{aligned}\tag{5.32a}$$

$$\begin{aligned}\mathcal{O}^{(n)}_{RR,k} &= 2\left[D^{\mu_1\ldots\mu_k}\phi\right]\left[D_{\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{\rho\sigma}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F^{\rho\sigma}\right]\\ &\quad - 2i\left[D^{\mu_1\ldots\mu_k}\phi\right]\left[D_{\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{\rho\sigma}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}\tilde{F}^{\rho\sigma}\right], \end{aligned}\tag{5.32b}$$

$$\begin{aligned}\mathcal{O}^{(n+1)}_{LR,k} &= 2\left[D^{\rho\mu_1\ldots\mu_k}\phi\right]\left[D_{\sigma\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{\rho\tau}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F^{\sigma\tau}\right]\\ &\quad - \frac{1}{2}\eta^{\rho\sigma}\left[D^{\rho\mu_1\ldots\mu_k}\phi\right]\left[D_{\sigma\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{\rho\tau}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F^{\sigma\tau}\right]. \end{aligned}\tag{5.32c}$$

Both operators in eqs. (5.32a) and (5.32b) contain P-odd terms. We are not interested in such effects, so we ignore these operators. Also note that, by integrating by parts twice, the second term in eq. (5.32c) can be reexpressed in terms of other operators already present and terms that can be removed by field redefinitions, up to contributions cubic in the field strength.

All-in-all, there are two generic structures out of which we build the tidal action. The tidal contribution to the action to all mass dimensions is thus

$$\begin{aligned}\Delta\mathcal{L}_{\text{QED}} = \sum_{n=0}^{\infty}\sum_{k=0}^{N}\Big\{ & a_k^{(n)}\left[D^{\mu_1\ldots\mu_k}\phi\right]\left[D_{\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{\rho\sigma}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F^{\rho\sigma}\right]\\ &+ b_k^{(n+1)}\left[D^{\rho\mu_1\ldots\mu_k}\phi\right]\left[D_{\sigma\nu_1\ldots\nu_k}\phi\right]\left[D_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}F_{\rho\tau}\right]\left[D^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}F^{\sigma\tau}\right]\Big\}, \end{aligned}\tag{5.33}$$

where $N \equiv \lfloor n/2 \rfloor$ and we have introduced the Wilson coefficients $a_k^{(n)}$ and $b_k^{(n+1)}$. Note that the covariant derivatives acting on the real scalars or field strenghts reduce to partial derivatives. One can easily incorporate P-odd operators into this tidal action by including the same operators as in the first line in eq. (5.33) where one of the field strenghts is replaced by a dual field strength.

5.3.3 Gravity

We repeat the procedure from the previous section, only this time for a real scalar coupled to gravity. The relevant action is

$$\sqrt{-g}\mathcal{L}_{\text{GR}} = \sqrt{-g}\left[\frac{g^{\mu\nu}}{2}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - \frac{m^2}{2}\phi^2 + \Delta\mathcal{L}_{\text{GR}}\right]. \tag{5.34}$$

We will find the form of the tidal contribution at general mass dimension using the Hilbert series in eq. (5.29).

The optimal basis for our purposes satisfies the same criteria as in the previous section. As such, our basis comprises operators of the form

$$\mathcal{O}^{(n)}_{LL,k} = [\nabla^{\mu_1\ldots\mu_k}\phi]\,[\nabla_{\nu_1\ldots\nu_k}\phi]\,\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{L,\rho\sigma\alpha\beta}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C_L^{\rho\sigma\alpha\beta}\right], \tag{5.35}$$

$$\mathcal{O}^{(n)}_{RR,k} = [\nabla^{\mu_1\ldots\mu_k}\phi]\,[\nabla_{\nu_1\ldots\nu_k}\phi]\,\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{R,\rho\sigma\alpha\beta}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C_R^{\rho\sigma\alpha\beta}\right], \tag{5.36}$$

$$\mathcal{O}^{(n+2)}_{LR,k} = [\nabla^{\rho\sigma\mu_1\ldots\mu_k}\phi]\,[\nabla_{\alpha\beta\nu_1\ldots\nu_k}\phi]\,\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{L,\lambda\rho\tau\sigma}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C_R^{\lambda\alpha\tau\beta}\right]. \tag{5.37}$$

We introduced the shorthand notation $\nabla_{\mu_1\ldots\mu_n} = \nabla_{\mu_1}\ldots\nabla_{\mu_n}$. In this case as well k is in the range $0 \le k \le N$.

These operators can be expressed in terms of the Weyl tensor and its dual. The exact same procedure as in the QED case, along with covariant conservation of the Levi-Civita tensor [200], results in only two forms of operators comprising the basis, modulo P-odd operators. The tidal contribution to the action is thus

$$\begin{aligned}\Delta\mathcal{L}_{\text{GR}} = \; & \\ \sum_{n=0}^{\infty}\sum_{k=0}^{N}\Big\{ & c_k^{(n)}\,[\nabla^{\mu_1\ldots\mu_k}\phi]\,[\nabla_{\nu_1\ldots\nu_k}\phi]\,\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{\rho\sigma\alpha\beta}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C^{\rho\sigma\alpha\beta}\right] \\ & + d_k^{(n+2)}\,[\nabla^{\rho\sigma\mu_1\ldots\mu_k}\phi]\,[\nabla_{\alpha\beta\nu_1\ldots\nu_k}\phi]\,\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{\lambda\rho\tau\sigma}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C^{\lambda\alpha\tau\beta}\right]\Big\},\end{aligned} \tag{5.38}$$

to all mass dimensions. The coefficients $c_k^{(n)}$ and $d_k^{(n+2)}$ are the Wilson coefficients for the action. Again, the P-odd operators which could be added to the basis take the same form as the first line in eq. (5.38) with one of the Weyl tensors replaced by a dual Weyl tensor.

5.4 TIDAL EFFECTS AT THE LEADING-PM ORDER

The actions in eqs. (5.33) and (5.38) describe all tidal effects that can arise from terms quadratic in the electromagnetic field strength and the curvature, respectively. In fact, these actions are sufficient for describing all tidal effects at the one-loop order, which corresponds to the leading-PM order in the case of gravity. We present in this section the full classical one-loop tidal contributions to both electromagnetic and gravitational amplitudes. Since we are exclusively interested in classical contributions, we can take advantage of the heavy limits of these actions to identify the only operators which contribute classically at this loop order [92]. This results in significant simplifications to the Feynman rules and the loop integrals involved.

We follow the method in refs. [55, 56, 92, 201] to take the heavy limit of real scalars. Namely, we apply the field redefinition

$$\phi \to \frac{1}{\sqrt{2m}}\left(e^{-imv\cdot x}\chi + e^{imv\cdot x}\chi^*\right), \tag{5.39}$$

and drop quickly oscillating terms. Furthermore, by counting the powers of $\hbar$ associated with each operator, and given that the triangle diagram in fig. 5.1 is the only topology of interest at the one-loop

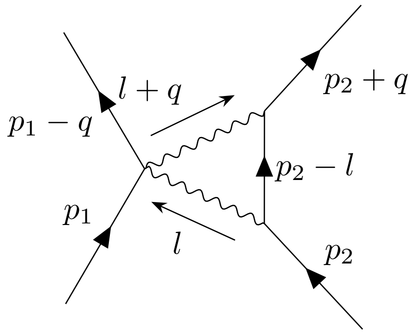


Figure 5.1: The only topology contributing classical tidal effects at one loop. The tidal effects of particle 1 are probed. The wavy lines represent either photons or gravitons.

level, we only need the operators at leading order in the $1/m$ expansion.

We present first the heavy limit of the electromagnetic tidal action, as well as the full classical one-loop tidal contribution to the electromagnetic amplitude, before moving on to the case of gravity. We have normalized all amplitudes by multiplying by $4m_1m_2$. This compensates for the normalization in eq. (5.39).[12]

5.4.1 QED

Beginning with the Lagrangian in eq. (5.30), we apply the field redefinition in eq. (5.39) to obtain

$$\mathcal{L}_{\text{HQET}} = \chi^* iv\cdot\partial\chi + \Delta\mathcal{L}_{\text{HQET}} + \dots, \tag{5.40a}$$

where

$$\begin{aligned}&\Delta\mathcal{L}_{\text{HQET}} = \\ &\sum_{n=0}^{\infty}\sum_{k=0}^{N}\Big\{a_k^{(n)}m^{2k-1}[v^{\mu_1\dots\mu_k}\chi^*][v_{\nu_1\dots\nu_k}\chi][D_{\mu_1\dots\mu_k\alpha_1\dots\alpha_{n-2k}}F_{\rho\sigma}][D^{\nu_1\dots\nu_k\alpha_1\dots\alpha_{n-2k}}F^{\rho\sigma}] \\ &+b_k^{(n+1)}m^{2k+1}[v^{\rho\mu_1\dots\mu_k}\chi^*][v_{\sigma\nu_1\dots\nu_k}\chi][D_{\mu_1\dots\mu_k\alpha_1\dots\alpha_{n-2k}}F_{\rho\tau}][D^{\nu_1\dots\nu_k\alpha_1\dots\alpha_{n-2k}}F^{\sigma\tau}]\Big\} + \dots.\end{aligned} \tag{5.40b}$$

We have defined $v_{\mu_1\dots\mu_n} = v_{\mu_1}\dots v_{\mu_n}$. In these equations, dots represent operators scaling with higher powers of $\hbar$. We can ignore these operators as the computation of classical effects at the one-loop level only requires contributions from the leading-in-$\hbar$ operators.

At this point there is an apparent contradiction in the tidal operators we have claimed to be leading in $\hbar$. Increasing n or decreasing k at fixed n increases the number of derivatives acting on the photon field, thereby increasing powers of $\hbar$ in the resulting contributions to amplitudes [10]. However, the derivative structure of the subleading tidal terms in the worldline action [168, 170, 194] suggests that we are right to keep these terms. Therefore, much like in the case of spin effects where the spin vector absorbs a power of $\hbar$ [14], we propose that the tidal coefficients must scale with $\hbar$ to absorb the factors from the operators. The scalings that cancel those of the operators in eq. (5.40) are

$$a_k^{(n)} \sim \hbar^{-2n+2k-2}, \tag{5.41a}$$

[12] More precisely, this factor is the leading-in-$\hbar$ portion of the heavy scalar external states in momentum space [202].

$$b_k^{(n+1)} \sim \hbar^{-2n+2k-2}. \tag{5.41b}$$

As in the case of spin, this scaling is necessary to make contact between portions of the amplitude and classical quantities.

We proceed now to use the heavy action to compute the classical one-loop tidal amplitude. We let particle i have momentum $p_i = m_i v_i + k_i$, where we have applied the usual heavy-particle decomposition of the momentum. Note that we cannot generate three-point vertices with a photon and two real scalars, so we take particle 2 to be complex in this context, i.e. particle 2 obeys the action given by eq. (B.3) in ref. [92]. The portion of the leading-PM amplitude involving only the $k = 0$ terms in eq. (5.40) is

$$\Delta\mathcal{A}_2^{k=0} = -\frac{e^2}{\pi^2} S m_2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^2}{2}\right)^{n+1} \left[a_0^{(n)} + \frac{m_1^2}{8} b_0^{(n+1)} (3\omega^2 + 1)\right]. \tag{5.42}$$

We have defined $S \equiv \pi^2/\sqrt{-q^2}$ and $\omega \equiv v_1 \cdot v_2$. The notation $\Delta\mathcal{A}$ denotes an electromagnetic amplitude linear in the tidal coefficients in eq. (5.40b).

Let's now extend this result to general k. The unintegrated form of the amplitude is

$$\begin{aligned}\Delta\mathcal{A}_2 = -8ie^2 m_2 \sum_{n=0}^{\infty}\sum_{k=0}^{N} (-1)^n m_1^{2k} \left(\frac{q^2}{2}\right)^{n-2k} &\left[2\left(\frac{q^2}{2}\right) a_k^{(n)} v_{1\mu_1\ldots\mu_{2k}} \mathcal{I}_{\triangleleft}^{\mu_1\ldots\mu_{2k}}\right. \\ &\left.+ m_1^2 b_k^{(n+1)} \left(\omega^2 \frac{q^2}{2} v_{\mu_1\ldots\mu_{2k}} \mathcal{I}_{\triangleleft}^{\mu_1\ldots\mu_{2k}} - v_{\mu_1\ldots\mu_{2k+2}} \mathcal{I}_{\triangleleft}^{\mu_1\ldots\mu_{2k+2}}\right)\right].\end{aligned} \tag{5.43}$$

To integrate the general k amplitude, we need knowledge of integrals of the form

$$v_{1\mu_1\ldots\mu_{2k}} \mathcal{I}_{\triangleleft}^{\mu_1\ldots\mu_{2k}} = \int \frac{d^4 l}{(2\pi)^4} \frac{(v_1 \cdot l)^{2k}}{l^2 (l+q)^2 (-v_2 \cdot l)}. \tag{5.44}$$

This task is simplified since we in fact only need the portion of this integral proportional to the non-analytic structure S, and even then only the leading-in-$\hbar$ contribution to this portion. We observe the following pattern for the portion of the integral we are interested in:

$$v_{1\mu_1\ldots\mu_{2k}} \mathcal{I}_{\triangleleft}^{\mu_1\ldots\mu_{2k}} = \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} (\omega^2 - 1)^k q^{2k} \mathcal{I}_{\triangleleft} + \mathcal{O}(\hbar^{2k}), \tag{5.45}$$

where $(a)_b$ is the Pochhammer symbol and

$$\mathcal{I}_{\triangleleft} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 (l+q)^2 (-v_2 \cdot l)}. \tag{5.46}$$

Note that, since the scalar triangle integral scales as $\hbar^{-1}$, the leading term in eq. (5.45) scales as $\hbar^{2k-1}$. We have explicitly checked eq. (5.45) up to $2k = 10$ using the Passarino-Veltman reduction [203]. Equation (5.45) was proven in ref. [199] while this paper was in review.

Armed with eq. (5.45), we compute a result for general n, k;

$$\Delta\mathcal{A}_2 = -\frac{e^2 S m_2}{\pi^2} \sum_{n=0}^{\infty}\sum_{k=0}^{N} (-1)^n \left(\frac{q^2}{2}\right)^{n-k+1} m_1^{2k} (\omega^2 - 1)^k \tag{5.47}$$

$$\times \left\{ a_k^{(n)} \frac{\left(\frac{1}{2}\right)_k}{2^k\,(1)_k} + \frac{1}{2^{k+1}} m_1^2 b_k^{(n+1)} \left[\frac{\left(\frac{1}{2}\right)_k}{(1)_k}\omega^2 - \frac{\left(\frac{1}{2}\right)_{k+1}}{2\,(1)_{k+1}}(\omega^2-1) \right] \right\}.$$

We can reorganize the sums to make the dependence on q^2 more transparent:

$$\Delta \mathcal{A}_2 = \sum_{i=0}^{\infty} \frac{e^2}{\pi^2} \left(-\frac{q^2}{2}\right)^{i+1} S m_2 f_i(\omega), \tag{5.48}$$

where

$$f_i(\omega) \equiv \sum_{k=0}^{i} \frac{\left(\frac{1}{2}\right)_k}{2^k (1)_k} (1-\omega^2)^k \left[m_1^{2k} a_k^{(i+k)} + \frac{1}{8(k+1)} m_1^{2k+2} b_k^{(i+k+1)} \left[(2k+3)\omega^2 + (2k+1)\right] \right], \tag{5.49}$$

after some algebraic simplification of the Pochhammer symbols.

5.4.2 Gravity

We turn now to the leading-PM gravitational tidal amplitude. Once again the first step is to find the action describing a heavy scalar. Beginning with the Lagrangian in eq. (5.34), we apply the field redefinition in eq. (5.39) to obtain

$$\sqrt{-g}\mathcal{L}_{\text{HBET}} = \sqrt{-g}\left[\frac{1}{2} m (g^{\mu\nu} v_\mu v_\nu - 1)\chi^*\chi + \chi^* i v\cdot\partial\chi + \Delta\mathcal{L}_{\text{HBET}} + \ldots\right], \tag{5.50a}$$

where

$$\Delta\mathcal{L}_{\text{HBET}} = \tag{5.50b}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{N} \Big\{ c_k^{(n)} m^{2k-1} [v^{\mu_1\ldots\mu_k}\chi^*][v_{\nu_1\ldots\nu_k}\chi][\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}} C_{\rho\sigma\alpha\beta}][\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}} C^{\rho\sigma\alpha\beta}]$$

$$+ d_k^{(n+2)} m^{2k+3} [v^{\rho\sigma\mu_1\ldots\mu_k}\chi^*][v_{\alpha\beta\nu_1\ldots\nu_k}\chi][\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}} C_{\lambda\rho\tau\sigma}][\nabla^{\nu_1\ldots\nu_2\alpha_1\ldots\alpha_{n-2k}} C^{\lambda\alpha\tau\beta}] \Big\}$$

$$+ \ldots .$$

In these equations, dots represent operators scaling with higher powers of $\hbar$. Note that we must keep the term $\chi^* i v\cdot\partial\chi$ in the action even though it is subleading in m since it is the kinetic term for the heavy scalar. Following the arguments in section 5.4.1, we propose the following $\hbar$-scaling of the gravitational tidal coefficients:

$$c_k^{(n)} \sim \hbar^{-2n+2k-4}, \tag{5.51a}$$

$$d_k^{(n+2)} \sim \hbar^{-2n+2k-4}. \tag{5.51b}$$

First we reproduce the leading-PM amplitude from ref. [173]. We need only the operators with $n=k=0$ for this task. Thus the amplitude for the leading tidal effect is

$$\Delta\mathcal{M}_2^{n=k=0} = G^2 q^4 S m_2^3 \left[16 c_0^{(0)} + \frac{m_1^4}{8} d_0^{(2)} (35\omega^4 - 30\omega^2 + 11) \right]. \tag{5.52}$$

This agrees with ref. [173] with the identification $c_0^{(0)} \to \lambda/4$ and $d_0^{(2)} \to \eta/(4m_1^4)$. Here $\Delta\mathcal{M}$ is a gravitational amplitude linear in the tidal coefficients in eq. (5.50b).

We easily extend this result by including terms at all orders in n and with $k = 0$:

$$\Delta\mathcal{M}_2^{k=0} = 4G^2 S m_2^3 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^2}{2}\right)^{n+2} \left[16c_0^{(n)} + \frac{m_1^4}{8} d_0^{(n+2)} (35\omega^4 - 30\omega^2 + 11)\right]. \tag{5.53}$$

The result for general k depends on integrals of the form in eq. (5.44). Specifically, the amplitude in terms of these integrals is

$$\begin{aligned}\Delta\mathcal{M}_2 = 512 i\pi^2 G^2 m_2^3 \sum_{n=0}^{\infty}\sum_{k=0}^{N} (-1)^n m_1^{2k} \left(\frac{q^2}{2}\right)^{n-2k+2} &\Big\{2c_k^{(n)} v_{\mu_1\ldots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k}} \\ +m_1^4 d_k^{(n)} \big[q^4(1-2\omega^2)^2 v_{\mu_1\ldots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k}} &+ 4q^2(1-4\omega^2) v_{\mu_1\ldots\mu_{2k+2}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k+2}} \\ +8 v_{\mu_1\ldots\mu_{2k+4}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k+4}}\big]\Big\}.\end{aligned} \tag{5.54}$$

We can integrate this using eq. (5.45) to obtain the result for all n, k:

$$\begin{aligned}\Delta\mathcal{M}_2 = 4G^2 m_2^3 S \sum_{n=0}^{\infty}\sum_{k=0}^{N} (-1)^n m_1^{2k} (\omega^2-1)^k \left(\frac{q^2}{2}\right)^{n-k+2} \Bigg\{16 c_k^{(n)} \frac{\left(\frac{1}{2}\right)_k}{2^k (1)_k} \\ + m_1^4 d_k^{(n+2)} \left[(1-2\omega^2)^2 \frac{\left(\frac{1}{2}\right)_k}{2^{k-1}(1)_k} + (1-4\omega^2)(\omega^2-1) \frac{\left(\frac{1}{2}\right)_{k+1}}{2^{k-1}(1)_{k+1}} + (\omega^2-1)^2 \frac{\left(\frac{1}{2}\right)_{k+2}}{2^k (1)_{k+2}}\right]\Bigg\}.\end{aligned} \tag{5.55}$$

A suggestive structure arises when $k \neq 0$: each contribution is proportional to the factor $(\omega^2-1)^k$. This factor is small in the PN limit, thus we can see already from the PM amplitude level that the corresponding operators must be subleading in the PN limit, in agreement with the constructions in refs. [168, 170, 194]. In fact, this squares perfectly with principles from classical gravitational effective field theories (EFTs). Terms with $k \neq 0$ involve derivatives of the Weyl tensor of the form $v \cdot \nabla$. These reduce to time derivatives in the PN limit, which are subleading compared to spatial derivatives [36].

Once again, we reorganize the sums in powers of the transfer momentum. The advantage of doing so is that contributions are grouped by their significance to observables. We find

$$\Delta\mathcal{M}_2 = 4G^2 S m_2^3 \sum_{i=0}^{\infty} \left(-\frac{q^2}{2}\right)^{i+2} g_i(\omega), \tag{5.56a}$$

where after some simplification

$$\begin{aligned}g_i(\omega) \equiv \sum_{k=0}^{i} \frac{(-1)^k \left(\frac{1}{2}\right)_k}{2^k (1)_k} (\omega^2-1)^k \Bigg[16 m_1^{2k} c_k^{(i+k)} \\ + \frac{m_1^{2k+4} d_k^{(i+k+2)}}{4(k+2)(k+1)} \left[(2k+5)(2k+7)\omega^4 - 6(2k+5)\omega^2 + (4k^2+12k+11)\right]\Bigg].\end{aligned} \tag{5.56b}$$

The amplitude is now presented in an optimal form for conversion to the Hamiltonian or scattering angle. We present these quantities in the next section, and defer comparison of this result with the literature until then.

In this section we have only computed the tidal contribution of particle 1 to the amplitude. If one is interested in the tidal effects from both particles at this order, one must simply symmetrize the results here in the particle labels.

5.4.3 Gravitational Hamiltonian and scattering angle

We use now our leading-PM amplitude in eq. (5.56) to compute the full leading-PM tidal corrections to the Hamiltonian and the scattering angle. Beginning with the Hamiltonian, there are two ways we may proceed. The first is to match to the EFT of ref. [48], and the second is through the Lippmann-Schwinger equation [89]. As we are working to linear order in the tidal coefficients, there will be no contributions from the Born iteration, so we work here with the latter formulation. The Hamiltonian as a function of the center-of-mass momentum and the separation between the bodies is given by

$$H(\mathbf{p},\mathbf{r}) = \sum_{n=1,2} \sqrt{\mathbf{p}^2 + m_i^2} + V(\mathbf{p},\mathbf{r}) + \Delta V(\mathbf{p},\mathbf{r}). \tag{5.57}$$

Here $V(\mathbf{p},\mathbf{r})$ is the point particle potential and can be found up to 3PM order in refs. [16, 17]. $\Delta V(\mathbf{p},\mathbf{r})$ incorporates tidal corrections. At the order to which we have worked, these tidal corrections are simply the Fourier transform of the leading-PM amplitude in the center-of-mass frame:

$$\Delta V(\mathbf{p},\mathbf{r}) = -\int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \Delta\mathcal{M}_2(p,q). \tag{5.58}$$

In this frame the transfer momentum becomes $q^\mu = (0,\mathbf{q})$, so $q^2 = -\mathbf{q}^2$. Substituting now eq. (5.56) into this after incorporating the non-relativistic normalization $1/4E_1E_2$,

$$\Delta V(\mathbf{p},\mathbf{r}) = -\frac{G^2 m_2^3}{E^2\xi} \sum_{i=0}^{\infty} \frac{(-1)^i(2i+4)!}{2^{i+3} r^{2i+6}} g_i(\omega), \tag{5.59}$$

where $E \equiv E_1 + E_2$ is the total energy in the center-of-mass frame and $\xi \equiv E_1E_2/E^2$. The $i = 0$ term is in exact agreement with eq. (10) of ref. [173].

With this in hand we can compute the scattering angle using the method of ref. [91]. Note that V_{eff} in ref. [91] is related to the potential in position space by $V_{\text{eff}} = 2E\xi\Delta V$.[13] Accounting for this, the scattering angle is

$$\Delta\chi = \frac{G^2 m_2^3}{E} \sum_{i=0}^{\infty} \frac{(-1)^i(2i+4)!(i+3)}{2^{i+2} p_\infty^2 b^{2(i+3)}} \frac{\sqrt{\pi}\,\Gamma\left(i+\frac{7}{2}\right)}{\Gamma(i+4)} g_i(\omega), \tag{5.60}$$

where b is the impact parameter and $p_\infty = |\mathbf{p}|$, the magnitude of the center-of-mass three-momentum. Evaluating this at $i = 0$ and noting that $p_\infty b = J$, the angular momentum, we find exact agreement with the $\mathcal{O}(J^{-6})$ portion of eq. (13) in ref. [173]. This also agrees with ref. [170] upon converting to their notation and matching Wilson coefficients:

$$p_\infty \to \frac{m_1 m_2}{E} p_\infty, \quad J \to G m_1 m_2 j, \tag{5.61}$$

[13] We thank Andrea Cristofoli for pointing this out.

$$c_0^{(0)} \to -\frac{1}{12} m \sigma^{(2)}, \quad d_0^{(2)} \to \frac{1}{4m^3}\left(\mu^{(2)} + \frac{8}{3}\sigma^{(2)}\right). \tag{5.62}$$

A similar notation conversion along with the Wilson coefficient map in eq. (5.62) also produces agreement with the sum of eqs. (5.5) and (5.6) of ref. [195]. We remark that the Wilson coefficient matching in eq. (5.62) is equivalent to the matching of ref. [173]. Moreover, this matching can be seen directly from the level of the heavy action: it is the condition that equates the $n = 0, k = 0$ portion of the heavy tidal action eq. (5.50b) with the $l = 2$ term of the classical worldline action in ref. [168], up to factors of $\chi^*\chi$.

To check the $i = 1$ term we have repeated the calculation starting from the worldline action of ref. [170], promoting each term to a quantum-field-theory operator, and multiplying by $\chi^*\chi$. Doing so we find the following matching conditions on the Wilson coefficients of the two operator bases:

$$c_0^{(1)} \to -\frac{1}{32} m_1 \sigma^{(3)}, \tag{5.63a}$$

$$c_1^{(2)} \to \frac{1}{144 m_1}\left(-\mu^{(3)} - 12\sigma'^{(2)} + \frac{9}{2}\sigma^{(3)}\right), \tag{5.63b}$$

$$d_0^{(3)} \to \frac{1}{12 m_1^3}\left(\mu^{(3)} + 3\sigma^{(3)}\right), \tag{5.63c}$$

$$d_1^{(4)} \to \frac{1}{36 m_1^5}\left(9\mu'^{(2)} - \mu^{(3)} + 24\sigma'^{(2)} - 3\sigma^{(3)}\right). \tag{5.63d}$$

This mapping is also consistent with the form factors in eq. (4.39) of ref. [170], reproducing the same ω structure in $g_i(\omega)$ as in the form factors.[14] Note, however, that this mapping is only appropriate up to overall constants when comparing to the form factors, as we are comparing different quantities.

As a final check on the Wilson coefficient matching conditions, we computed the factorizable portion of the tree-level $3 \to 3$ amplitude at linear order in the tidal coefficients. Matching the amplitudes computed from both bases, we indeed find again the matching conditions in eqs. (5.62) and (5.63).

5.5 Summary and outlook

While the application of scattering amplitudes to the binary point-particle inspiral problem has seen much progress in recent years, the description of finite size and tidal effects is a novel and exciting development. We have demonstrated the applicability of powerful EFT tools to this problem. Namely, through the Hilbert series we have been able to write down an action which includes all possible operators involving two real scalars and two Weyl tensors. These operators represent the leading-PM tidal effects, and the action they compose is sufficient to describe all tidal contributions to the 2PM amplitude for scalar-scalar scattering.

The computation of this amplitude was easily performed by taking the heavy limit of the tidal action and isolating only those operators with the correct $\hbar$ scaling to contribute classically. A subtlety arose in this identification of classically contributing operators: operators with an increasing number of derivatives acting on the Weyl tensors would have to be considered classical. This runs counter to the wisdom that more derivatives produce more powers of $\hbar$. To resolve this tension, we proposed that the Wilson coefficients of the action must themselves scale with compensating powers of $\hbar$, analogously to

[14] Note that ω in our notation is equivalent to γ in the notation of ref. [170].

the absorption of $\hbar$ by the spin vector. We presented the unintegrated form of the leading-PM amplitude, and integrated it using the form of the rank-$2k$ triangle integral in eq. (5.45). We found agreement where our amplitude has overlap with existing results. The amplitudes were then converted into a Hamiltonian and scattering angle, and once again we found agreement with known results.

There are two obvious extensions to this work. The first is the inclusion of tidal effects from operators with higher powers of the Weyl tensor. An operator involving n powers of the Weyl tensor contributes to vertices with n or more gravitons and two matter lines, and thus contributes to conservative dynamics starting at the nPM order. Second is the inclusion of spin effects. This point is perhaps the more pressing of the two, since tidal effects for objects of large enough spin may also have implications for the Compton amplitude. The gravitational Compton amplitude acquires a spurious pole for matter with spin $s \geq 2$ [63], an occurrence which is believed to derive from the necessarily composite nature of particles with large spin. If this is true then it is natural to expect that the inclusion of tidal effects may aid in remedying this non-locality. Both of these avenues can be pursued using the same Hilbert series methods we have employed here. We leave these ideas for future research.

APPENDIX

5.A HILBERT SERIES

Below we list the mathematical details we used in the construction of the Hilbert series for tidal effects. For a detailed account of the Hilbert series, see e.g. refs. [174–179, 198].

The Hilbert series $\mathcal{H}$ for a given field content ϕ is the contour integral of the plethystic exponential:[15]

$$\mathcal{H} = \int d\mu \frac{1}{P} \mathrm{PE}[\chi_\phi], \tag{5.64}$$

where the plethystic exponential (PE) generates all symmetric (antisymmetric) tensor products of the representations of the bosonic (fermionic) field content. The factor $1/P$ removes a total derivative, where the momentum generating function P is defined below in eq. (5.78). The plethystic exponential takes the form

$$\mathrm{PE}_\phi = \exp\left[\sum_{r=0}^{\infty} z^{r+1} \frac{\phi^r}{r\mathcal{D}^{r\Delta_\phi}} \chi_\phi(x_1^r, \ldots, x_k^r)\right], \tag{5.65}$$

where Δ_ϕ is the mass dimension of ϕ and $z = \pm 1$ when ϕ is a boson/fermion, respectively. Here χ_ϕ is the character of the representation of ϕ. When we consider several fields, we simply multiply their plethystic exponentials.

We are using the Hilbert series to generate operators with neutral scalars, photons, and gravitons. Thus we need their respective conformal representations:

$$\chi_\phi = \chi_{[1,(0,0)]}(\mathcal{D}; \alpha, \beta), \tag{5.66}$$
$$\chi_{F_L} = \chi_{[2,(1,0)]}(\mathcal{D}; \alpha, \beta), \tag{5.67}$$
$$\chi_{F_R} = \chi_{[2,(0,1)]}(\mathcal{D}; \alpha, \beta), \tag{5.68}$$
$$\chi_{C_L} = \chi_{[3,(2,0)]}(\mathcal{D}; \alpha, \beta), \tag{5.69}$$
$$\chi_{C_R} = \chi_{[3,(0,2)]}(\mathcal{D}; \alpha, \beta). \tag{5.70}$$

We could have included the characters for the $U(1)$ gauge group in electromagnetism, but, since both the scalars and the photons are neutral, their characters would be trivial.

The characters for the unitary conformal representations of interest are [175, 178, 179, 204]

$$\chi_{[1,(0,0)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D} P(\mathcal{D}; \alpha, \beta)(1 - \mathcal{D}^2), \tag{5.71}$$
$$\chi_{[3/2,(1/2,0)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D}^{3/2} P(\mathcal{D}; \alpha, \beta)\left[\chi_{(1/2,0)}(\alpha, \beta) - \mathcal{D}\chi_{(0,1/2)}(\alpha, \beta)\right], \tag{5.72}$$
$$\chi_{[3/2,(0,1/2)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D}^{3/2} P(\mathcal{D}; \alpha, \beta)\left[\chi_{(0,1/2)}(\alpha, \beta) - \mathcal{D}\chi_{(1/2,0)}(\alpha, \beta)\right], \tag{5.73}$$
$$\chi_{[2,(1,0)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D}^2 P(\mathcal{D}; \alpha, \beta)\left[\chi_{(1,0)}(\alpha, \beta) - \mathcal{D}\chi_{(1/2,1/2)}(\alpha, \beta) + \mathcal{D}^2\right], \tag{5.74}$$
$$\chi_{[2,(0,1)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D}^2 P(\mathcal{D}; \alpha, \beta)\left[\chi_{(0,1)}(\alpha, \beta) - \mathcal{D}\chi_{(1/2,1/2)}(\alpha, \beta) + \mathcal{D}^2\right], \tag{5.75}$$
$$\chi_{[3,(2,0)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D}^3 P(\mathcal{D}; \alpha, \beta)\left[\chi_{(2,0)}(\alpha, \beta) - \mathcal{D}\chi_{(3/2,1/2)}(\alpha, \beta) + \mathcal{D}^2\chi_{(1,0)}(\alpha, \beta)\right], \tag{5.76}$$
$$\chi_{[3,(0,2)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D}^3 P(\mathcal{D}; \alpha, \beta)\left[\chi_{(0,2)}(\alpha, \beta) - \mathcal{D}\chi_{(1/2,3/2)}(\alpha, \beta) + \mathcal{D}^2\chi_{(0,1)}(\alpha, \beta)\right], \tag{5.77}$$

[15] The modification term $\Delta\mathcal{H}$ will not be relevant for us as we consider operators with mass dimension greater than 4.

where

$$P(\mathcal{D};\alpha,\beta)=\frac{1}{(1-\mathcal{D}\alpha\beta)(1-\mathcal{D}/(\alpha\beta))(1-\mathcal{D}\alpha/\beta)(1-\mathcal{D}\beta/\alpha)} \tag{5.78}$$

is the momentum generating function [178]. The characters of the Euclidean Lorentz group are simply products of $SU(2)$ characters;

$$\chi_{(l_1,l_2)}(\alpha,\beta)=\chi^{SU(2)}_{l_1}(\alpha)\times\chi^{SU(2)}_{l_2}(\beta). \tag{5.79}$$

The $SU(2)$ characters we need are

$$\chi^{SU(2)}_{0}(\alpha)=1, \tag{5.80}$$

$$\chi^{SU(2)}_{1/2}(\alpha)=\alpha+\frac{1}{\alpha}, \tag{5.81}$$

$$\chi^{SU(2)}_{1}(\alpha)=\alpha^2+1+\frac{1}{\alpha^2}, \tag{5.82}$$

$$\chi^{SU(2)}_{3/2}(\alpha)=\alpha^3+\alpha+\frac{1}{\alpha}+\frac{1}{\alpha^3}, \tag{5.83}$$

$$\chi^{SU(2)}_{2}(\alpha)=\alpha^4+\alpha^2+1+\frac{1}{\alpha^2}+\frac{1}{\alpha^4}. \tag{5.84}$$

Finally, the Haar measure for the Euclidean Lorentz group $SO(4)\simeq SU(2)_L\times SU(2)_R$ is

$$\int d\mu_{\text{Lorentz}}=\left(\frac{1}{2\pi i}\right)^2\oint_{|\alpha|=1}\frac{d\alpha}{2\alpha}(1-\alpha^2)\left(1-\frac{1}{\alpha^2}\right)\oint_{|\beta|=1}\frac{d\beta}{2\beta}(1-\beta^2)\left(1-\frac{1}{\beta^2}\right). \tag{5.85}$$

5.B REDUNDANT OPERATORS

The operator basis for leading-PM tidal effects in eq. (5.38) is a complete, non-redundant basis for all operators involving two scalars and two Weyl tensors. However, the explicit form of the operator basis involves some choices, originating from two types of redundancies: field redefinitions and integration-by-parts relations.

First we consider redundancies from field redefinitions. The free equation of motion (EOM) for the scalar field is

$$\partial^2\phi+m^2\phi=0. \tag{5.86}$$

A composite operator which contains the factor $\partial^2\phi$ can be removed from the operator basis by an appropriate choice of field redefinition which exchanges it for the operator $m^2\phi$. When constructing operators where partial derivatives are acting on the scalar fields, we need only consider symmetric, traceless combinations of the derivatives.[16]

Similarly, the free EOM for the gauge field is

$$\partial_\mu F^{\mu\nu}=0. \tag{5.87}$$

[16] We could also replace the partial derivatives with covariant derivatives. Note that commutators of covariant derivatives are related to field strengths or curvature; $[D_\mu,D_\nu]\sim F_{\mu\nu}$ or $[\nabla_\mu,\nabla_\nu]\sim R$.

Also, we find that

$$\partial^2 F_{\mu\nu} = 0 \tag{5.88}$$

by using the Bianchi identity $\partial_{[\alpha} F_{\mu\nu]} = 0$. Again, we only need symmetric, traceless combinations of derivatives acting on the field strengths, where none of the derivatives are contracted with that field strength.

For gravity, we have Einstein's equation in vacuum:

$$R_{\mu\nu} = 0, \tag{5.89}$$

where $R_{\mu\nu}$ is the Ricci tensor. This means that we don't include the Ricci tensor nor the Ricci scalar in the operator basis as they can be removed by an appropriate redefinition of the metric tensor.[17]

Since the Weyl tensor is the traceless part of the Riemann tensor,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{3} g_{\mu[\rho}g_{\sigma]\nu}R, \tag{5.90}$$

where $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$ for any tensor A, we can freely work with either the Riemann tensor or the Weyl tensor. For our purposes, it is most convenient to work with the Weyl tensor because it transforms in an irreducible representation of the Euclidean Lorentz group; see Section 5.A.

In vacuum, we find that

$$\nabla^\mu C_{\mu\nu\rho\sigma} = 0, \tag{5.91}$$

$$\nabla^2 C_{\mu\nu\rho\sigma} = \mathcal{O}(C^2), \tag{5.92}$$

up to terms with Ricci tensors or Ricci scalars. Thus we need only keep symmetric, traceless combinations of covariant derivatives acting on the Weyl tensors.

Next we will illustrate the redundancies coming from integration-by-parts relations by looking at some possible dimension-8 operators:

$$\mathcal{O}_1 = \phi\phi \left[\nabla_\mu C_{\rho\sigma\alpha\beta}\right] \left[\nabla^\mu C^{\rho\sigma\alpha\beta}\right], \tag{5.93}$$

$$\mathcal{O}_2 = \left[\nabla_\mu \phi\right] \left[\nabla^\mu \phi\right] C_{\rho\sigma\alpha\beta} C^{\rho\sigma\alpha\beta}, \tag{5.94}$$

$$\mathcal{O}_3 = \phi \left[\nabla^\mu \phi\right] \left[\nabla_\mu C_{\rho\sigma\alpha\beta}\right] C^{\rho\sigma\alpha\beta}. \tag{5.95}$$

Here, $\mathcal{O}_1$ corresponds to the operator with coefficient $c_0^{(1)}$ in eq. (5.38), while $\mathcal{O}_2$ and $\mathcal{O}_3$ are absent from eq. (5.38). The Hilbert series in eq. (5.29) informs us that there should be only one P-even operator at this mass dimension, but it does not tell us which one we should choose.

In fact, these operators are related through integration-by-parts relations,

$$\mathcal{O}_1 = \mathcal{O}_2 + \text{EOM} + \mathcal{O}(C^3), \tag{5.96}$$

$$\mathcal{O}_1 = -\mathcal{O}_3, \tag{5.97}$$

[17] From an amplitude perspective, ref. [90] showed that the modification of the Einstein-Hilbert action by the addition of R^2 and $R^{\mu\nu}R_{\mu\nu}$ terms does not affect the amplitude. Ref. [168] found an explicit field redefinition of the graviton field that removes traces of the Riemann curvature from the tidal worldline action, including in the presence of matter.

up to a total derivative, operators proportional to the leading-order EOM, and operators with more than two Weyl tensors. We discard the total derivative due to momentum conservation, and the EOM operators can be removed through a field redefinition. In fact, the Hilbert series have implicitly removed, whenever possible, operators with more derivatives in place of operators with fewer derivatives, i.e. using the EOM.

When we have more than one operator at a given mass dimension, we must carefully include independent operators which cannot be related through integration-by-parts relations or field redefinitions. A systematic way of taking into account integration-by-parts relations is detailed in refs. [176, 205]. We enumerate all the the ways of partitioning the derivatives (ignoring integration-by-parts relations), which we call $\{x_i\}$. Then we enumerate all gauge-invariant operators with one fewer covariant derivative which transform as a Lorentz four-vector, $\{y_i\}$. We can then apply a total derivative to the y_i's, which will generate a relation among the x_i's. The number of independent constraints coming from this procedure is given by the rank of the matrix of constraint equations.

Let's illustrate the procedure for the dimension-8 operators. We assign the $x_i = \mathcal{O}_i$ for $i = 1, 2, 3$. For the operators with one covariant derivative, we can have the covariant derivative act on a scalar or on a Weyl tensor;

$$y_{1,\mu} = \phi\,[\nabla_\mu \phi]\, C_{\rho\sigma\alpha\beta} C^{\rho\sigma\alpha\beta}, \tag{5.98}$$

$$y_{2,\mu} = \phi\phi\,[\nabla_\mu C_{\rho\sigma\alpha\beta}]\, C^{\rho\sigma\alpha\beta}. \tag{5.99}$$

Now we apply the total derivative on the y_i's:

$$\nabla^\mu y_{1,\mu} = x_1 + 2x_2 = 0, \tag{5.100}$$

$$\nabla^\mu y_{2,\mu} = 2x_2 + x_3 = 0. \tag{5.101}$$

Note that we have dropped operators with $D^2\phi$ or $D^2 C$ because they can either be removed by field redefinitions or produce operators with more than two Weyl tensors. We can write the equations in matrix form,

$$M \cdot x \equiv \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{5.102}$$

The number of independent operators is $3 - \mathrm{rank}(M) = 3 - 2 = 1$.

We have applied this method to ensure the operators in our basis are independent up to mass dimension 14. For the higher mass dimensions, we used the on-shell methods discussed in Section 5.C.

We have illustrated the freedom in choosing an operator basis coming from integration-by-parts relations. However, certain operator bases are better suited for calculations in the heavy limit. For example, the heavy limit of $\mathcal{O}_2$ feeds down to the dimension-6 operator $\phi\phi C_{\rho\sigma\alpha\beta}C^{\rho\sigma\alpha\beta}$, so this operator doesn't contribute new information with regards to the classical portion of the amplitude. In fact, we would also need to include subleading-in-$\hbar$ corrections from $\mathcal{O}_2$ to reproduce the correct subleading tidal effects.

The operator basis in eq. (5.38) is chosen to optimally produce all leading-PM tidal effects in the classical limit.

5.C Operator basis from an on-shell perspective

A different approach to constructing the operator basis is to first look at the corresponding on-shell amplitudes. Following the discussion in ref. [206], we consider the non-factorizable part of the two-scalar-two-photon amplitude $\mathcal{A}(\phi\phi;\gamma\gamma)$. We label the momenta for the photons by p_1 and p_2, and the momenta of the massive scalars by p_3 and p_4. For the helicity assignments $\gamma^+(p_1)\gamma^+(p_2)$ and $\gamma^-(p_1)\gamma^+(p_2)$, the structures carrying the correct little group weights are

$$[12]^2 \qquad \text{and} \qquad \langle 1|(p_3-p_4)|2]^2, \tag{5.103}$$

respectively. The amplitudes for the other helicity assignments can be constructed from the same building blocks after exchanging angle and square brackets. The non-factorizable part of the two amplitudes with positive helicity for p_2 are

$$\mathcal{A}(\phi\phi;\gamma^+(p_1),\gamma^+(p_2)) = [12]^2 a(s_{12},s_{13},s_{14}), \tag{5.104}$$
$$\mathcal{A}(\phi\phi;\gamma^-(p_1),\gamma^+(p_2)) = \langle 1|(p_3-p_4)|2]^2 b(s_{12},s_{13},s_{14}), \tag{5.105}$$

where $a(s_{12},s_{13},s_{14})$ and $b(s_{12},s_{13},s_{14})$ are polynomials of the Mandelstam variables $s_{ij} = (p_i + p_j)^2$. Taking into account the relation $s_{12}+s_{13}+s_{14} = 2m^2$, and keeping the symmetry $3 \leftrightarrow 4$, the polynomials take the form

$$a(s_{12},s_{13},s_{14}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{a_{i,j}}{\Lambda^{2i+4j+2}} s_{12}^i (s_{13}s_{14})^j, \tag{5.106}$$
$$b(s_{12},s_{13},s_{14}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{b_{i,j}}{\Lambda^{2i+4j+4}} s_{12}^i (s_{13}s_{14})^j, \tag{5.107}$$

where Λ is some unfixed dimensionful scale and $a_{i,j}$ and $b_{i,j}$ are dimensionless coefficients.

By comparing the non-factorizable part of the on-shell amplitudes with the output of the Hilbert series in eq. (5.27), one can find a correspondence between the Wilson coefficients of the action and the coefficients $a_{i,j}$, $b_{i,j}$. This helps us in inferring the higher-dimensional operators, since we can now construct operators which have the field content given by the Hilbert series and which reduce to the amplitudes when imposing on-shell conditions.

Similarly, we can compare the on-shell amplitudes for two scalars and two gravitons,

$$\mathcal{M}(\phi\phi;g^{2+}(p_1),g^{2+}(p_2)) = [12]^4 c(s_{12},s_{13},s_{14}), \tag{5.108}$$
$$\mathcal{M}(\phi\phi;g^{2-}(p_1),g^{2+}(p_2)) = \langle 1|(p_3-p_4)|2]^4 d(s_{12},s_{13},s_{14}), \tag{5.109}$$

with the output of the Hilbert series in eq. (5.29). The same arguments apply to the polynomials c and d as a and b, so they become

$$c(s_{12},s_{13},s_{14}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{c_{i,j}}{\Lambda^{2i+4j+4}} s_{12}^i (s_{13}s_{14})^j, \tag{5.110}$$
$$d(s_{12},s_{13},s_{14}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{d_{i,j}}{\Lambda^{2i+4j+8}} s_{12}^i (s_{13}s_{14})^j, \tag{5.111}$$

with dimensionless coefficients $c_{i,j}$ and $d_{i,j}$. We see a similar correspondence between the on-shell amplitudes and the effective operators as in the QED case.

CHAPTER

6

TIDAL EFFECTS FOR SPINNING PARTICLES

ABSTRACT: Expanding on the recent derivation of tidal actions for scalar particles, we present here the action for a tidally deformed spin-$1/2$ particle. Focusing on operators containing two powers of the Weyl tensor, we combine the Hilbert series with an on-shell amplitude basis to construct the tidal action. With the tidal action in hand, we compute the leading-post-Minkowskian tidal contributions to the spin-$1/2$ – spin-$1/2$ amplitude, arising at $\mathcal{O}(G^2)$. Our amplitudes provide evidence that the observed long range spin-universality for the scattering of two point particles extends to the scattering of tidally deformed objects. From the scattering amplitude we find the conservative two-body Hamiltonian, linear and angular impulses, eikonal phase, spin kick, and aligned-spin scattering angle. We present analogous results in the electromagnetic case along the way.

6.1 INITIATION

This publication extended the previous one by applying similar methods to derive tidal effects associated with spinning objects. When introducing spin there are many more operator structures that can be written, and relations between these structures are not always obvious off shell. To get a handle on such relations, we combined the Hilbert series with a basis of on-shell amplitudes for two spinors and two gravitons. This on-shell basis is a new tool in our toolbox relative to the previous publication. Therefore, in this initiation we address the construction and application of this on-shell basis to the enumeration of operators at all mass dimensions. As always, we end this section with an overview of the main results of the publication.

6.1.1 On-shell amplitude basis

As mentioned in Section 5.1.2, the Hilbert series counts the minimum number of operators needed to reproduce all on-shell structures at a given mass dimension. In the previous publication, we followed this correspondence unidirectionally: we determined the size of the operator basis through the Hilbert series, constructed the basis with no further input, then computed amplitudes. This is more difficult to do in the spin-$1/2$ case because of the presence of gamma matrices, the multiplicity of which is not elucidated by the Hilbert series. Consequently, there are more plausible ways of creating Lorentz scalars, and relations between off-shell operators can be obscure. As a specific example of the latter point, the on-shell relation in eq. (6.14) suggests the proportionality

$$2D_\lambda\bar{\psi}_R D^\rho\psi_L D_\rho F_{L\mu\nu} D^\lambda F_L^{\mu\nu} - \bar{\psi}_R\psi_L D_\lambda D_\rho F_{L\mu\nu} D^\lambda D^\rho F_L^{\mu\nu} \overset{\text{on-shell}}{\propto} D^\rho\bar{\psi}_R\sigma^{\lambda\tau}D^\sigma\psi_L D_\alpha F_{L\lambda\rho}D^\alpha F_{L\tau\sigma}, \tag{6.1}$$

lowest operator dimension	helicity configuration	helicity structure
7	$v^+v^+f^+f^+$	$[12]^2[34],\ [12]([14][23]+[13][24])$
	$v^+v^+f^-f^-$	$[12]^2\langle 34\rangle$
8	$v^+v^+f^+f^-$	$[12]^2[3\vert(1-2)\vert 4\rangle$
	$v^+v^-f^+f^-$	$[13]\langle 24\rangle[1\vert(3-4)\vert 2\rangle$
9	$v^+v^-f^+f^+$	$[34][1\vert(3-4)\vert 2\rangle^2$

Table 6.1: The helicity basis found in ref. [207] for four-point contact interactions between two vector bosons and two spinors. All other helicity configurations can be obtained from these ones by conjugation. We have taken the notation from ref. [207] where f represents a spinor and v a vector boson.

and its analogs at higher mass dimensions. The L and R subscripts denote respectively left- and right-handed fields. This is why the operators labelled by the Wilson coefficient $b^{(n)}$ in eq. (6.16) and their counterparts in eq. (6.19) only appear every other odd mass dimension. A bidirectional approach to constructing the operator basis – where the Hilbert series and an on-shell amplitude basis are both used to constrain the operator basis – was therefore well-motivated. This was especially true since the relevant massless-amplitude basis had already been constructed in ref. [207].

Recall from Chapter 3 that amplitudes are independently covariant under the little group of each scattering particle. Ref. [207] exploited this extensively to construct local, non-factorizable helicity structures[1] – that is, helicity structures that can emerge from local contact operators. However, little group constraints alone are not sufficient to reduce the set of possible helicity structures to an independent basis. Relations between the spinors arising from momentum conservation and Schouten identities must be accounted for to remove redundancies. Doing so, ref. [207] produced the basis of all helicity structures that can arise from a two-spinor-two-vector contact operator. This basis is reproduced in table 6.1, along with the helicity configuration of each helicity structure and the lowest possible mass dimension of an operator producing each one.[2]

It was verified by the authors of ref. [207] that no new helicity structures arise at higher mass dimensions. All higher-dimensional operators therefore produce the same helicity structures as in table 6.1 multiplied by combinations of Mandelstam variables. These combinations are such that the appropriate spin statistics are obeyed by the helicity structures. In our case, since the vector bosons are both photons, the appropriate on-shell amplitudes must be symmetric under the interchange of the photon momentum labels when both photons carry the same helicity. Moreover, as can be seen by the Hilbert series in Section 6.3.1, the left- and right- handed spinors are described by different fields. As such, no symmetry conditions need to be imposed under an interchange of the fermions. These considerations led to the combinations of Mandelstam variables in table 6.2.

For contact terms involving gravitons instead of vector bosons, ref. [207] listed helicity structures arising from operators with dimension up to and including 8. They found that these structures are the first three in table 6.1 multiplied by $[12]^2$. We obtained the remaining structures by multiplying the last two structures in the table by $[1|(3-4)|2\rangle^2$. Indeed, these two structures arise from operators with mass

[1]Since massless particles transform under irreducible representations of their little groups labelled by their helicities, we use the term "helicity structures" to refer to on-shell strucutres describing the interactions of only massless particles.

[2]Note that the mass dimensions of the operators are not equal to those of the helicity structures they produce. This is because of the discrepancy between the mass dimensions of the fields and the number of on-shell spinors they yield. A two-spinor-two-field strength operator with mass dimension d yields an on-shell structure with mass dimension $d-4$. Replacing the field strengths with Weyl tensors gives an on-shell structure with mass dimension $d-2$. This is consistent with the counting in eq. (9) of ref. [207].

dimensions of at least 10 and 12 respectively; see footnote 2. Spin-statistic considerations are identical to the photon case, leading to the combinations of Mandelstam variables in table 6.3.

The helicity basis in table 6.1 and its gravitational analog describe all possible helicity structures that can comprise a contact interaction between the particles in which we are interested. Crucially, however, the particles involved in this basis are massless, while our interests lie in describing massive matter. Recalling the significance of bolded particle labels for spinor-helicity variables – i.e. that they represent covariantization under the massive little group – refs. [63, 208] showed that amplitudes and helicity structures can be extended to the massive case by appropriately bolding particle labels.

There are caveats to this. First, when extending a massless, factorizable amplitude to a massive one, poles must also be deformed [63]. This issue does not apply to us since we are only interested in local, non-factorizable spinor structures (i.e. contact interactions). Second, the construction of massive contact terms from massless ones also requires a treatment of mass corrections to the bolded spinor structures [207]. Since we are only after an enumeration of spinor structures – as opposed to a construction of amplitudes – any structures appearing in such mass corrections will have already been accounted for at lower mass dimensions. For our purposes, bolding the helicity amplitudes of ref. [207] is thus sufficient.

One way to confirm that we have obtained all possible massive spinor structures by simply bolding is that the number of spinor structures at each mass dimension is precisely equal to the counting of the Hilbert series at the corresponding mass dimension. The sufficiency of bolding massless spinor structures to obtain massive ones is also reflected in the Hilbert series' agnosticism towards particle masses; see Section 5.1.2.

6.1.2 Overview of main results

The layout of this publication is similar to the last one. We began by determining the Hilbert series for operators with arbitrary mass dimensions involving two spinors and two Weyl tensors. In this case, operators existed at both even and odd mass dimensions. The Hilbert series we computed was

$$\mathcal{H} = \int d\mu_{SU(2)\times SU(2)\times U(1)} \frac{1}{P(\mathcal{D};\alpha,\beta)} \mathrm{PE}[\psi_L]\mathrm{PE}[\psi_R]\mathrm{PE}[\psi_L^\dagger]\mathrm{PE}[\psi_R^\dagger]\mathrm{PE}[C_L]\mathrm{PE}[C_R], \tag{6.2}$$

where we assigned $U(1)$ charges to the Weyl spinors to eliminate chiral effects. Once again we found a closed form for the Hilbert series at all mass dimensions:

$$\begin{aligned}\mathcal{H}^{C^2}_{7+2n} &= \lfloor n/2+1 \rfloor (C_L^2 + C_R^2)(\psi_L\psi_R^\dagger + \psi_L^\dagger\psi_R)\mathcal{D}^{2n} + \Theta(n-1)\,(n-1)C_LC_R(\psi_L\psi_R^\dagger + \psi_L^\dagger\psi_R)\mathcal{D}^{2n} \\ &\quad + \frac{1}{2}(1-(-1)^n)(C_L^2\psi_L\psi_R^\dagger + C_R^2\psi_L^\dagger\psi_R)\mathcal{D}^{2n}, \end{aligned}\tag{6.3}$$

$$\mathcal{H}^{C^2}_{6+2n} = \lfloor n/2 \rfloor (C_L^2 + C_R^2)(\psi_L\psi_L^\dagger + \psi_R\psi_R^\dagger)\mathcal{D}^{2n-1} + (n-1)C_LC_R(\psi_L\psi_L^\dagger + \psi_R\psi_R^\dagger)\mathcal{D}^{2n-1}. \tag{6.4}$$

In this case, we found it helpful to enumerate the possible on-shell structures that can arise from a two-spinor-two-graviton interaction. Knowledge of these on-shell structures allowed us to identify relations between operators that were not evident off shell.

The remainder of the main results are lengthy, so we don't reproduce them here.

Combining the Hilbert series with this basis of on-shell structures, we constructed the tidal corrections to the action of a minimally coupled spinor field consisting of two Weyl tensors. These contributions are displayed respectively for odd and even mass dimensions in eqs. (6.19) and (6.20). These tidal operators, along with results for contractions of the arbitrary-rank triangle integral, yielded the complete tidal corrections to the classical portion of the one-loop amplitude – see eq. (6.36).

Various techniques were then applied to this amplitude to extract a number of observables: the interaction potential, the linear and angular impulses, and the aligned-spin scattering angle.

This publication represents the first time tidal distortions of a spinning body were described using QFT methods.

6.2 INTRODUCTION

The surge of attention payed to the binary inspiral problem in general relativity (GR) by the scattering amplitudes community – caused by the observation of gravitational waves by the LIGO/Virgo collaborations [1] – has produced a greater understanding of the description of classical properties of binary systems using quantum-field-theoretic and amplitudes techniques. To date, most of the work has focused on point particles. Some notable results in this direction include elucidating methods for converting scattering amplitudes for relativistic point particles to classical observables [10, 14, 29, 43, 48–50, 91, 186, 209], the description of classical angular momentum from quantum mechanical spin [14, 29, 43, 45–47, 53, 92, 95, 165, 191], and the state-of-the-art computation of the third post-Minkowskian (3PM) dynamics of a spinless binary system [16, 17, 189, 210]. Despite tremendous progress, there remains much to be understood about the connection between scattering amplitudes and classical binary systems.

The quantum description of tidal effects is a topic of particular interest of late. While Schwarzschild black holes do not tidally deform in four spacetime dimensions [167, 211–213], there is still debate whether the same is true for Kerr black holes in a general gravitational environment [214–217]. Nevertheless, such effects impact the gravitational wave signal of a neutron star merger [218, 219], so understanding them is necessary for the full description of these systems. The study of such effects from a scattering amplitudes perspective was only recently initiated in ref. [173]. There, the authors included higher-dimensional operators quadratic in the Weyl tensor in the action of a gravitating scalar particle, and computed the corrections to the Hamiltonian and scattering angle up to next-to-leading post-Minkowskian (PM) order ($\mathcal{O}(G^3)$ for tidal effects). Not long after, two of the present authors applied the Hilbert series to extend the action of ref. [173] to include a complete, non-redundant set of operators quadratic in the Weyl tensor [220]. They subsequently computed all $\mathcal{O}(G^2)$ finite-size[3] contributions to scalar-scalar scattering, including all contributions from higher-derivative operators. Since then, refs. [199, 221] extended the study of finite-size effects by calculating the leading-PM contributions from an infinite set of operators with higher powers of the Weyl tensor, using the geodesic equation and unitarity cuts, respectively.

There have also been efforts to derive fully relativistic information about tidally deformed systems from purely classical frameworks. Refs. [170, 195] incorporated tidal effects into PM worldline actions, and subsequently derived the leading-PM contributions to observables from a subset of tidal effects arising from couplings quadratic in the Weyl tensor. Ref. [222] extended these results to a larger subset of these tidal operators, as well as to the next-to-leading PM order. Incorporating spin, refs. [223, 224] extended the operator basis of ref. [80] to include operators quadratic in the curvature containing up to

[3]We will use the terms "tidal" and "finite-size" interchangably for both gravity and electromagnetism, even though "electric/magnetic susceptibility" is normally used in the electromagnetic context.

four powers of the spin vector. Most recently, ref. [225] presented a relativistic action describing tidally deformed bodies up to linear order in spin.

To date, there has been no amplitudes approach adding spin to the tidally deformed object. In this paper we fill this gap by expanding on the work of ref. [220] to include spin effects. Combining the Hilbert series with on-shell methods, we construct the full action for spinors and two powers of the Weyl tensor. This allows us to compute all classical tidal effects at $\mathcal{O}(G^2)$ for spinor-spinor scattering. Adapting the spinning effective field theory matching of ref. [29], we present the interaction Hamiltonian including spin and tidal effects at $\mathcal{O}(G^2)$ and to linear order in the angular momentum of each body. Then, applying the methods of refs. [10, 14], we use the amplitudes to find the linear and angular impulses. We present the analogous results for quantum electrodynamics (QED). We then compute the eikonal phase to extract additional observables in the gravitational case. First, the eikonal phase allows us to verify the linear impulse through a separate method. Then, it provides a means for computing the spin kick and the tidal corrections to the aligned-spin scattering angle.

This paper is organized as follows. We start by finding the tidal actions in both the electromagnetic and gravitational cases for a massive spin-1/2 particle in section 6.3. This is accomplished by first using the Hilbert series in section 6.3.1, which provides a guide for finding the amplitude bases in sections 6.3.2 and 6.3.3 and the operator bases in sections 6.3.4 and 6.3.5. We then calculate the leading-PM scattering amplitudes in section 6.4. The scattering amplitudes are then used to calculate various classical quantities: the conservative Hamiltonian in section 6.5, linear and angular impulses in section 6.6, and the eikonal phase, spin kick, and aligned-spin scattering angle in section 6.7. We conclude in section 6.8. Section 6.A contains a discussion on the relevant loop integrals, while in section 6.B we show details for the calculation of classical observables.

6.3 TIDAL ACTIONS

Combining the Hilbert series with on-shell amplitudes methods, we construct in this section the full action coupling two photon field strengths or two Weyl tensors to spinor fields.

6.3.1 Hilbert series

We begin with the Hilbert series. The Hilbert series produces the number of group invariants for a given field content, and it is useful when constructing an operator basis in an effective field theory. Notable achievements are the applications of the Hilbert series to the Standard Model effective field theory [174–178], and the extension to include gravity [179]. Non-relativistic effective field theories [226, 227] and effective field theories with non-linearly realized symmetries [228] can also be constructed using Hilbert series techniques.

The Hilbert series was applied to characterize tidal effects for post-Minkowskian scattering in ref. [220]. In addition to the structures described in appendix A of ref. [220], we need the group characters for left- and right-handed Weyl spinors (respectively ψ_L and ψ_R) [179],

$$\chi_{[3/2,(1/2,0)]}(\mathcal{D};x,y) = \mathcal{D}^{3/2}P(\mathcal{D};x,y)\left[\chi_{(1/2,0)}(x,y) - \mathcal{D}\chi_{(0,1/2)}(x,y)\right], \tag{6.5}$$

$$\chi_{[3/2,(0,1/2)]}(\mathcal{D};x,y) = \mathcal{D}^{3/2}P(\mathcal{D};x,y)\left[\chi_{(0,1/2)}(x,y) - \mathcal{D}\chi_{(1/2,0)}(x,y)\right]. \tag{6.6}$$

Moreover, in both the electromagnetic and gravitational cases, we assume the spinor fields are charged under a $U(1)$ gauge group. Thus we also need the gauge group characters $\chi_{U(1)}(\alpha) = \alpha^Q$ for a particle with charge Q and the corresponding Haar measure:

$$\int d\mu_{U(1)} = \frac{1}{2\pi i}\oint_{|\alpha|=1}\frac{d\alpha}{\alpha}. \tag{6.7}$$

We have assigned $Q_{\psi_L} = Q_{\psi_R} = 1$. All other information relevant to our application of the Hilbert series is given in ref. [220] (Section 5.A).

We can now compute the Hilbert series in which we are interested. The Hilbert series for two field strengths coupled to spinors for mass dimension d, $\mathcal{H}_d^{F^2}$, is

$$\begin{aligned}\mathcal{H}^{F^2}_{7+2n} &= \lfloor n/2+1\rfloor(F_L^2+F_R^2)(\psi_L\psi_R^\dagger+\psi_L^\dagger\psi_R)\mathcal{D}^{2n}+nF_LF_R(\psi_L\psi_R^\dagger+\psi_L^\dagger\psi_R)\mathcal{D}^{2n}\\ &\quad+\frac{1}{2}(1-(-1)^n)(F_L^2\psi_L\psi_R^\dagger+F_R^2\psi_L^\dagger\psi_R)\mathcal{D}^{2n}, \qquad (6.8)\\ \mathcal{H}^{F^2}_{6+2n} &= \lfloor n/2\rfloor(F_L^2+F_R^2)(\psi_L\psi_L^\dagger+\psi_R\psi_R^\dagger)\mathcal{D}^{2n-1}+nF_LF_R(\psi_L\psi_L^\dagger+\psi_R\psi_R^\dagger)\mathcal{D}^{2n-1}. \qquad (6.9)\end{aligned}$$

In eq. (6.8) we have $n \geq 0$, whereas $n \geq 1$ in eq. (6.9). Coupling two Weyl tensors to spinors, the Hilbert series for mass dimension d, $\mathcal{H}_d^{C^2}$, is

$$\begin{aligned}\mathcal{H}^{C^2}_{7+2n} &= \lfloor n/2+1\rfloor(C_L^2+C_R^2)(\psi_L\psi_R^\dagger+\psi_L^\dagger\psi_R)\mathcal{D}^{2n}+\Theta(n-1)\,(n-1)C_LC_R(\psi_L\psi_R^\dagger+\psi_L^\dagger\psi_R)\mathcal{D}^{2n}\\ &\quad+\frac{1}{2}(1-(-1)^n)(C_L^2\psi_L\psi_R^\dagger+C_R^2\psi_L^\dagger\psi_R)\mathcal{D}^{2n}, \qquad (6.10)\\ \mathcal{H}^{C^2}_{6+2n} &= \lfloor n/2\rfloor(C_L^2+C_R^2)(\psi_L\psi_L^\dagger+\psi_R\psi_R^\dagger)\mathcal{D}^{2n-1}+(n-1)C_LC_R(\psi_L\psi_L^\dagger+\psi_R\psi_R^\dagger)\mathcal{D}^{2n-1}. \qquad (6.11)\end{aligned}$$

Here, $\Theta(n-1)$ is the Heaviside function and, once again, $n \geq 0$ in eq. (6.10) and $n \geq 1$ in eq. (6.11).

Equations (6.9) and (6.11) are the Hilbert series for even mass dimensions. These operators do not have any analogs in the complex scalar case (which is a slight generalization of the real scalar case discussed in ref. [220]), and we will see that they all contribute spin effects in the PM amplitudes.

The Hilbert series for odd mass dimensions in eqs. (6.8) and (6.10) are very similar to the corresponding Hilbert series for complex scalars coupled to photons or gravitons, respectively (up to a doubling of the number of terms coming from chiral fermions). The main difference is the appearance of the additional term

$$\frac{1}{2}(1-(-1)^n)(F_L^2\psi_L\psi_R^\dagger+F_R^2\psi_L^\dagger\psi_R)\mathcal{D}^{2n}, \tag{6.12}$$

or its analog for gravitons. These terms are present for $d = 9, 13, 17, \ldots$ – where they contribute spin effects – but are absent for $d = 11, 15, 19, \ldots$ In the next section, we will see that this curious behavior can be understood using on-shell spinor-helicity variables.

6.3.2 Amplitude basis for QED

A complementary approach to the characterization of tidal effects is the construction of on-shell amplitudes. It will be useful to us as it will elucidate relations among operators that are not obvious in an off-shell language. The massive spinor-helicity formalism of ref. [63] is ideal for our purposes, and we will make use of it to construct the on-shell amplitude basis. The massive spinors are indicated by a bolding of the momentum labels, which also represents a symmetrization over the massive particle's little group indices. See refs. [207, 208, 229, 230] for recent work constructing on-shell amplitudes.

Our approach makes use of the spinor structures presented in ref. [207]. To extend these results to higher mass dimensions, the various spinor structures are multiplied by combinations of Mandelstam variables, $s_{ij} \equiv (p_i + p_j)^2$, in a way that respects the Bose/Fermi statistics of the system. At four-points there are two independent Mandelstam variables. Labeling the two bosons as 1 and 2 and the two fermions as 3 and 4, we work with the two combinations of Mandelstam variables $x = s_{12}$ and $y = s_{13} - s_{23} + s_{24} - s_{14}$.[4] These combinations manifest symmetry/antisymmetry under the separate exchanges $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$.

We generate all higher-dimensional helicity amplitudes by multiplying the various spinor structures by products of Mandelstam variables, e.g. $x^a y^b$ for $a,\ b$ non-negative integers. As we are interested in amplitudes for two bosons and two spinors, all amplitude structures must be symmetric (antisymmetric) under the exchange $1 \leftrightarrow 2$ ($3 \leftrightarrow 4$) for amplitudes with indistinguishable particles. While Bose symmetry allows the power of x in a helicity amplitude to be arbitrary, it restricts the power of y in certain amplitudes to be either even or odd, i.e. some spinor structures will be multiplied by $x^a y^{2b(+1)}$. Finally, ref. [208] argued that spinor structures for massless particles can be generalized to the massive case by simply bolding the momentum labels of the massive spin-1/2 particles. All things considered, the amplitude basis for two massive spinors coupled to two photons is given in table 6.2.

helicity	spinor structure
$(++++)$	$[12]^2[\mathbf{34}]x^a y^{2b},\ \ [12]([1\mathbf{4}][2\mathbf{3}] + [1\mathbf{3}][2\mathbf{4}])y^{2b+1}$
$(++--)$	$[12]^2\langle\mathbf{34}\rangle x^a y^{2b}$
$(+-++)$	$[1\vert(\mathbf{3}-\mathbf{4})\vert 2\rangle^2[\mathbf{34}]x^a y^b$
$(+++-)$	$[12]^2[\mathbf{3}\vert(1-2)\vert\mathbf{4}\rangle x^a y^{2b+1}$
$(++-+)$	$[12]^2\langle\mathbf{3}\vert(1-2)\vert\mathbf{4}]x^a y^{2b+1}$
$(+-+-)$	$[1\mathbf{3}]\langle 2\mathbf{4}\rangle[1\vert(\mathbf{3}-\mathbf{4})\vert 2\rangle x^a y^b$

Table 6.2: The spinor structure basis for electromagnetic finite-size effects. The helicity labels are ordered as $(\gamma_1\gamma_2\psi_3\psi_4)$. The three first rows are the structures for odd mass dimensions, while the three last rows are for even mass dimensions. Here a and b take integer values from 0 to ∞. The structures for opposite helicities can be obtained by exchanging angle and square brackets.

We are now in a position to discuss the curious operators counted in eq. (6.12). They correspond to the second helicity amplitude in the first row of table 6.2. The reason why they are only present for $d = 9, 13, 17, \ldots$ is a special relation between the helicity amplitudes. The helicity amplitude structures (considering massless fermions for simplicity)

$$[12]^2[34]x^a y^{2b} \qquad \text{and} \qquad [12]([14][23] + [13][24])y^{2c+1} \tag{6.13}$$

are independent for any a, b, c. However, if we multiply the second helicity amplitude in eq. (6.13) by x, then we obtain the relation

$$2[12]([14][23] + [13][24])xy^{2c+1} = -[12]^2[34]y^{2c+2}. \tag{6.14}$$

[4]In massive four-point amplitudes, we must include the mass of the fermions as a further independent structure. However, the mass can always be absorbed into a Wilson coefficient, changing the dimensionality of the amplitude under consideration. Therefore, at a fixed mass dimension, it is sufficient to construct the helicity amplitudes using only x and y.

In the massive case this equivalence is modified by a lower-dimensional spinor structure. Thus, we can choose an amplitude basis where the second term in eq. (6.13) is never multiplied by x. Note that the analogous relation holds for gravitons, where each helicity amplitude is multiplied by $[12]^2$. The relation remains true when exchanging the square brackets for angle brackets.

6.3.3 Amplitude basis for gravity

The amplitude basis for gravity is almost identical to the photon case, with some additional powers of $[12]$, $\langle 1|(\mathbf{3}-\mathbf{4})|2]$, or their conjugates, accounting for the additional little group weights of gravitons relative to photons. The full amplitude basis for gravity is listed in table 6.3.

helicity	spinor structure
$(++++)$	$[12]^4[\mathbf{34}]x^a y^{2b}$, $[12]^3([1\mathbf{4}][2\mathbf{3}]+[1\mathbf{3}][2\mathbf{4}])y^{2b+1}$
$(++--)$	$[12]^4\langle\mathbf{34}\rangle x^a y^{2b}$
$(+-++)$	$[1\|(\mathbf{3}-\mathbf{4})\|2\rangle^4[\mathbf{34}]x^a y^b$
$(+++-)$	$[12]^4[\mathbf{3}\|(1-2)\|\mathbf{4}\rangle x^a y^{2b+1}$
$(++-+)$	$[12]^4\langle\mathbf{3}\|(1-2)\|\mathbf{4}]x^a y^{2b+1}$
$(+-+-)$	$[1\mathbf{3}]\langle 2\mathbf{4}\rangle[1\|(\mathbf{3}-\mathbf{4})\|2\rangle^3 x^a y^b$

Table 6.3: The spinor structure basis for gravitational tidal effects. The helicity labels are ordered as $(g_1 g_2 \psi_3 \psi_4)$. The three first rows are the structures for odd mass dimensions, while the three last rows are for even mass dimensions. Here a and b take integer values from 0 to ∞. The structures for opposite helicities can be obtained by exchanging angle and square brackets.

6.3.4 Operator basis for QED

With the explicit amplitude basis at hand, we can turn to finding the corresponding operator basis. It can be beneficial to have both an amplitude and an operator basis, since then both on-shell and off-shell calculations can be performed directly starting from the appropriate basis. Either approach can be used to calculate leading or subleading PM amplitudes.

In our case, the amplitude basis serves as a guide and as a cross-check. The products of Mandelstam variables correspond to the distribution of covariant derivatives in the operators, and the spinor structure can be simply found by putting various operators on-shell. Moreover, the relation in eq. (6.14) indicates a relation between off-shell operators that we must take into account. As a cross-check, we have verified that the operator basis below matches the amplitude basis in table 6.2 when put on-shell.

The full Lagrangian for fermions coupled to two field strengths is

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi + \Delta\mathcal{L}^{\text{odd}}_{\text{QED}} + \Delta\mathcal{L}^{\text{even}}_{\text{QED}}, \tag{6.15}$$

where $\Delta\mathcal{L}^{\text{odd/even}}_{\text{QED}}$ are the contributions from higher-dimensional operators at odd or even mass dimensions, respectively. Throughout this paper, we will use the prefix Δ to denote tidal contributions, unless otherwise stated. The contribution to the Lagrangian at odd mass dimensions is

$$\Delta\mathcal{L}^{\text{odd}}_{\text{QED}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} a_1^{(n,k)}\left(\bar{\psi}\overleftrightarrow{D}^{\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}F^{\mu\nu}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}F_{\mu\nu}\right)$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} a_2^{(n,k)}\left(\bar\psi \overleftrightarrow{D}^{\mu\nu\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}F_\mu{}^\rho\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}F_{\nu\rho}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} ia_3^{(n,k)}\left(\bar\psi\gamma_5 \overleftrightarrow{D}^{\mu\nu\alpha_1\ldots\alpha_{2k+1}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}F_\mu{}^\rho\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k+1}}D^{\beta_1\ldots\beta_{n-2k}}\tilde F_{\nu\rho}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} ia_4^{(n,k)}\left(\bar\psi\gamma_5 \overleftrightarrow{D}^{\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}F^{\mu\nu}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde F_{\mu\nu}\right)\\
&+\sum_{n=0}^{\infty} ib^{(n)}\left(\bar\psi\sigma^{\mu\nu} \overleftrightarrow{D}^{\rho\alpha_1\ldots\alpha_{2n}}\psi\right)\left(F_{\mu\rho}\overleftrightarrow{D}_{\sigma\alpha_1\ldots\alpha_{2n}}F_\nu{}^\sigma\right). \qquad (6.16)
\end{aligned}
$$

We have only included parity-even operators. We have used the short-hand notation $D^{\mu_1\cdots\mu_k} = D^{\mu_1}\ldots D^{\mu_k}$ and $A\overleftrightarrow{D}^\mu B = A(D^\mu B) - (D^\mu A)B$. In particular, $A\overleftrightarrow{D}^{\mu_1\cdots\mu_k}B = A\overleftrightarrow{D}^{\mu_1\cdots\mu_{k-1}}(D^{\mu_k}B) - (D^{\mu_k}A)\overleftrightarrow{D}^{\mu_1\cdots\mu_{k-1}}B + \mathcal{O}(F^3)$. For our purposes, we only need the part quadratic in the field strengths or Weyl tensors.

The operators labelled by the Wilson coefficient $b^{(n)}$ produce the on-shell structure on the right of eq. (6.13). As a consequence of eq. (6.14), these operators only arise at every second odd mass dimension.

The contribution at even mass dimensions is

$$
\begin{aligned}
\Delta\mathcal{L}_{\rm QED}^{\rm even} =& \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} ic_1^{(n,k)}\left(\bar\psi\gamma_\mu \overleftrightarrow{D}^{\nu\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}F^{\mu\rho}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}F_{\nu\rho}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} ic_2^{(n,k)}\left(\bar\psi\gamma_\mu \overleftrightarrow{D}^{\nu\lambda\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}F^{\mu\rho}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}F_{\nu\rho}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} c_3^{(n,k)}\left(\bar\psi\gamma_5\gamma_\mu \overleftrightarrow{D}^{\nu\lambda\alpha_1\ldots\alpha_{2k}}\psi\right)\Big(D_{\beta_1\ldots\beta_{n-2k}}F^{\mu\rho}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde F_{\nu\rho}\\
&\qquad\qquad -D_{\beta_1\ldots\beta_{n-2k}}\tilde F^{\mu\rho}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}F_{\nu\rho}\Big)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} c_4^{(n,k)}\left(\bar\psi\gamma_5\gamma_\mu \overleftrightarrow{D}^{\nu\lambda\alpha_1\ldots\alpha_{2k}}\psi\right)\Big(D_{\beta_1\ldots\beta_{n-2k}}F^{\mu\rho}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde F_{\nu\rho}\\
&\qquad\qquad +D_{\beta_1\ldots\beta_{n-2k}}\tilde F^{\mu\rho}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}F_{\nu\rho}\Big), \qquad (6.17)
\end{aligned}
$$

where again we only list the parity-even operators.

6.3.5 Operator basis for gravity

We turn now to the tidal action for gravity. The operator basis for fermions coupled to gravitons is very similar to the electromagnetic case. We have verified that our operator basis produces the helicity amplitudes in table 6.3 when placed on-shell.

The full gravitational action includes the minimal coupling for fermions as well as the tidal perturbations to be described:

$$
\sqrt{-g}\mathcal{L}_{\rm GR} = \sqrt{-g}\left[\bar\psi(ie^\mu{}_a\gamma^a D_\mu - m)\psi + \Delta\mathcal{L}_{\rm GR}^{\rm odd} + \Delta\mathcal{L}_{\rm GR}^{\rm even}\right], \qquad (6.18)
$$

where the first part of the action is described in detail in e.g. ref. [92]. The tidal contribution at odd mass dimensions is

$$
\begin{aligned}
\Delta\mathcal{L}^{\rm odd}_{\rm GR} = &\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} d_1^{(n,k)} \left(\bar{\psi}\overleftrightarrow{D}^{\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\nu\rho\sigma}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\mu\nu\rho\sigma}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} d_2^{(n,k)} \left(\bar{\psi}\overleftrightarrow{D}^{\mu\nu\lambda\tau\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C_{\mu\rho\lambda\sigma}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\tau\sigma}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} id_3^{(n,k)} \left(\bar{\psi}\gamma_5\overleftrightarrow{D}^{\mu\nu\lambda\tau\alpha_1\ldots\alpha_{2k+1}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C_{\mu\rho\lambda\sigma}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k+1}}D^{\beta_1\ldots\beta_{n-2k}}\tilde{C}_{\nu\rho\tau\sigma}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} id_4^{(n,k)} \left(\bar{\psi}\gamma_5\overleftrightarrow{D}^{\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\nu\rho\sigma}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde{C}_{\mu\nu\rho\sigma}\right)\\
&+\sum_{n=0}^{\infty} ie^{(n)} \left(\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{D}^{\rho\alpha_1\ldots\alpha_{2n}}\psi\right)\left(C_{\mu\lambda\rho\tau}\overleftrightarrow{D}_{\sigma\alpha_1\ldots\alpha_{2n}}C_{\nu}{}^{\lambda\sigma\tau}\right).
\end{aligned}
\tag{6.19}
$$

The operators labelled by the Wilson coefficient $e^{(n)}$ produce the gravitational analog of the on-shell structure on the right of eq. (6.13). Again, eq. (6.14) means that these operators only arise at every second odd mass dimension.

The tidal contribution at even mass dimensions is

$$
\begin{aligned}
\Delta\mathcal{L}^{\rm even}_{\rm GR} = &\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} if_1^{(n,k)} \left(\bar{\psi}\gamma_\mu\overleftrightarrow{D}^{\nu\gamma\delta\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\rho\gamma\tau}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\delta\tau}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} if_2^{(n,k)} \left(\bar{\psi}\gamma_\mu\overleftrightarrow{D}^{\nu\gamma\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\rho\gamma\tau}\overleftrightarrow{D}_{\delta\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\delta\tau}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} f_3^{(n,k)} \left(\bar{\psi}\gamma_5\gamma_\mu\overleftrightarrow{D}^{\nu\gamma\delta\lambda\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\rho\gamma\tau}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde{C}_{\nu\rho\delta\tau}\right.\\
&\qquad\left.-D_{\beta_1\ldots\beta_{n-2k}}\tilde{C}^{\mu\rho\gamma\tau}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\delta\tau}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} f_4^{(n,k)} \left(\bar{\psi}\gamma_5\gamma_\mu\overleftrightarrow{D}^{\nu\lambda\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\rho\gamma\delta}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde{C}_{\nu\rho\gamma\delta}\right.\\
&\qquad\left.+D_{\beta_1\ldots\beta_{n-2k}}\tilde{C}^{\mu\rho\gamma\delta}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\gamma\delta}\right).
\end{aligned}
\tag{6.20}
$$

The form of the gravitational action is almost identical to the electromagnetic action. However, note that various operators appear at different mass dimensions compared to the electromagnetic case, due to the additional Lorentz index structure of the Weyl tensors.

6.4 LEADING-PM TIDAL EFFECTS

The tidal operators listed in eqs. (6.16), (6.17), (6.19) and (6.20) are all we need to compute the leading-PM tidal contributions to spin-1/2 – spin-1/2 scattering. There are no contributions at tree-level to the

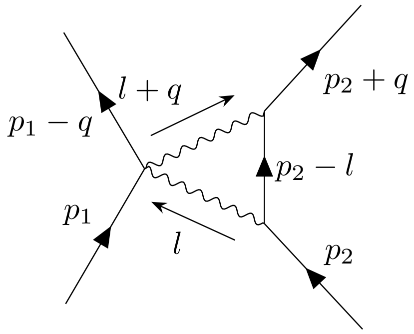


Figure 6.1: The only topology contributing classical tidal effects at one loop. Here, the tidal effects of particle 1 are probed. The wavy lines represent either photons or gravitons.

conservative $2 \to 2$ scattering amplitude, so we must consider the scattering at one loop. The only diagram contributing classically is the triangle diagram, shown in fig. 6.1, where particle 1 is being tidally deformed. Of course, the final result can be symmetrized in the particle labels to obtain the tidal deformation on particle 2. We let the incoming momenta be $p_i^\mu = m_i v_i^\mu$, where v_i^μ are the particles' four-velocities, which satisfy $v_i^2 = 1$. Also, we define $\omega \equiv v_1 \cdot v_2$. As we are interested in the classical portion of the amplitude, we compute the leading-in-$\hbar$ contribution only.

We write the spin effects in terms of the covariant spin vector for heavy particles, defined as

$$S_i^\mu = \frac{1}{2}\bar{u}_{v_i}\gamma_5\gamma^\mu u_{v_i}, \tag{6.21}$$

where $\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$ and u_{v_i} are spinors for a heavy particle with velocity v_i^μ — see refs. [165, 202] for their relation to Dirac spinors. To do so, we convert all Dirac spinors to heavy spinors at the level of the on-shell amplitudes, and keep only the terms at leading order in $\hbar$. As argued in ref. [165], this spin vector is identifiable with the one-particle matrix element of the spin vector of a classical spinning object: it automatically satisfies the covariant spin supplementary condition (SSC) $p_{i\mu}S^{\mu\nu} = 0$ for a spinning object with momentum $p_{i\mu} = mv_{i\mu}$, where $S^{\mu\nu}$ is the classical spin tensor [96].[5]

The computation of all tidal effects requires knowledge of certain projections of the general-rank triangle integral. Details about these integrals are given in section 6.A.

6.4.1 QED

Since we are including spin effects for the tidally deformed particle, as well as for particle 2, the scattering amplitude will be decomposed in terms of spinless, spin-orbit, and spin-spin contributions. In total, the finite-size contributions at one loop to the QED amplitude for spin-1/2 – spin-1/2 scattering are

$$\begin{aligned}\Delta\mathcal{A}_2^{s=1/2} = \frac{e^2 S}{\pi^2}\sum_{j=0}^{\infty}\left(-\frac{q^2}{2}\right)^{j+1}&\Big[\mathcal{U}_1\mathcal{U}_2F_j^{(0)} - i\omega\mathcal{E}_1\mathcal{U}_2F_j^{(1,1)} + i\omega\mathcal{U}_1\mathcal{E}_2F_j^{(1,2)} \\ &+ (q\cdot S_1)(q\cdot S_2)F_j^{(2,1)} - q^2(S_1\cdot S_2)F_j^{(2,2)} + \omega q^2(v_2\cdot S_1)(v_1\cdot S_2)F_j^{(2,3)}\Big],\end{aligned} \tag{6.22}$$

[5]The momentum of a spinning object actually deviates from mv^μ by corrections of $\mathcal{O}(RS^2)$, where R is a stand-in for the Riemann tensor [86]. This modifies the SSC at orders cubic in the object's angular momentum. We can safely ignore such effects, as we are focused on contributions at most linear in the spin of either object.

where e is the electromagnetic coupling, $S \equiv \pi^2/\sqrt{-q^2}$, and S_i^μ is the spin vector of particle i defined in eq. (6.21). The subscript 2 indicates that this is the amplitude at quadratic order in the coupling. There are further spin structures that can appear at quadratic order in spin, but they are subleading in the $\hbar$ expansion. The form factors are functions of ω, the dependence on which we leave implicit. We find the form factors to be

$$\begin{aligned}
F_j^{(0)} &= \sum_{k=0}^{j} a_1^{(j+k,k)}(4m_1)^{2k}(1-\omega^2)^k\frac{\alpha_k}{4} \\
&\quad+\sum_{k=0}^{j}\left(a_2^{(j+k,k)}-\frac{c_1^{(j+k,k)}}{2m_1}\right)(4m_1)^{2k+2}(1-\omega^2)^k\frac{1}{32}\left[(\omega^2-1)\alpha_{k+1}-\omega^2\alpha_k\right], \qquad (6.23)\\
F_j^{(1,1)} &= -\frac{b^{(j)}}{4m_2}(4m_1)^{2j}(1-\omega^2)^j\alpha_j \\
&\quad+\sum_{k=0}^{j}\frac{c_1^{(j+k,k)}}{4m_1m_2}(4m_1)^{2k}(1-\omega^2)^k\left[\alpha_{k+1}-\alpha_k\right] \\
&\quad+\sum_{k=0}^{j}\frac{c_4^{(j+k,k)}-c_3^{(j+k,k)}}{8m_1m_2}(4m_1)^{2k+2}(1-\omega^2)^k\alpha_{k+1}, \qquad (6.24)\\
F_j^{(1,2)} &= \sum_{k=0}^{j}\left(a_2^{(j+k,k)}-\frac{c_1^{(j+k,k)}}{2m_1}\right)\frac{(4m_1)^{2k+2}}{32m_1m_2^2}(1-\omega^2)^k\left[2\alpha_{k+1}-\alpha_k\right], \qquad (6.25)\\
F_j^{(2,1)} &= \sum_{k=0}^{j}\frac{a_4^{(j+k,k)}}{4m_1m_2}(4m_1)^{2k}\left(1-\omega^2\right)^k\alpha_k \\
&\quad+\frac{b^{(j)}}{16m_2}(4m_1)^{2j+1}(1-\omega^2)^j\left[(3\omega^2-2)(\alpha_j+\alpha_{j+1})+(2\omega^2-1)\frac{2}{2j+1}\alpha_{j+1}\right] \\
&\quad+\sum_{k=0}^{j}\frac{c_1^{(j+k,k)}}{8m_2}(4m_1)^{2k}(1-\omega^2)^k\left[(2\omega^2-1)\left(\alpha_k-2\alpha_{k+1}\right)\right] \\
&\quad-\sum_{k=0}^{j}\frac{c_3^{(j+k,k)}}{16m_2}(4m_1)^{2k+2}(1-\omega^2)^{k+1}\frac{1}{k+2}\alpha_{k+1} \\
&\quad+\sum_{k=0}^{j}\frac{c_4^{(j+k,k)}}{8m_2}(4m_1)^{2k+2}(1-\omega^2)^{k+1}\alpha_{k+1}, \qquad (6.26)\\
F_j^{(2,2)} &= \frac{b^{(j)}}{16m_2}(4m_1)^{2j+1}(1-\omega^2)^j\left[(2\omega^2-1)\frac{1}{2j+1}\alpha_{j+1}\right] \\
&\quad+\sum_{k=0}^{j}\frac{c_1^{(j+k,k)}}{8m_2}(4m_1)^{2k}(1-\omega^2)^k\left[(2\omega^2-1)\left(\alpha_k-2\alpha_{k+1}\right)\right] \\
&\quad-\sum_{k=0}^{j}\frac{c_3^{(j+k,k)}}{16m_2}(4m_1)^{2k+2}(1-\omega^2)^{k+1}\frac{1}{k+2}\alpha_{k+1}, \qquad (6.27)\\
F_j^{(2,3)} &= \frac{b^{(j)}}{32m_2}(4m_1)^{2j+1}(1-\omega^2)^j\left(4\alpha_{j+1}-(2\omega^2-1)\alpha_{j+1}\frac{4j}{2j+1}\frac{1}{\omega^2-1}\right) \\
&\quad+\sum_{k=0}^{j}\frac{c_1^{(j+k,k)}}{8m_2}(4m_1)^{2k}(1-\omega^2)^k\left[\alpha_k-\frac{4k}{2k+1}\frac{\omega^2}{(\omega^2-1)}\alpha_{k+1}\right]
\end{aligned}$$

$$+\sum_{k=0}^{j}\frac{c_3^{(j+k,k)}}{4m_2}(4m_1)^{2k+2}(1-\omega^2)^k\left(\alpha_{k+1}-\frac{2(k+1)}{2k+3}\alpha_{k+2}\right). \tag{6.28}$$

We have introduced the notation

$$\alpha_k \equiv \frac{\left(\frac{1}{2}\right)_k}{2^k(1)_k}, \tag{6.29}$$

where $(a)_k$ is the Pochhammer symbol, and

$$u_{v_1}\equiv u(m_1v_1),\quad u_{v_2}\equiv u(m_2v_2),\quad \bar{u}_{v_1}\equiv\bar{u}(m_1v_1-q),\quad \bar{u}_{v_2}\equiv\bar{u}(m_2v_2+q), \tag{6.30}$$

$$\bar{u}_{v_1}u_{v_1}\equiv\mathcal{U}_1,\quad \bar{u}_{v_2}u_{v_2}\equiv\mathcal{U}_2, \tag{6.31}$$

$$v_{2\mu}q_\nu\bar{u}_{v_1}\sigma^{\mu\nu}u_{v_1} = -2v_{2\mu}q_\nu\epsilon^{\mu\nu\alpha\beta}v_{1\alpha}S_{1\beta}\equiv -2\mathcal{E}_1/(m_1m_2), \tag{6.32}$$

$$v_{1\mu}q_\nu\bar{u}_{v_2}\sigma^{\mu\nu}u_{v_2} = -2v_{1\mu}q_\nu\epsilon^{\mu\nu\alpha\beta}v_{2\alpha}S_{2\beta}\equiv 2\mathcal{E}_2/(m_1m_2), \tag{6.33}$$

to simplify the expressions, where the Levi-Civita tensor is defined by $\epsilon^{0123}=1$. We remark on the absence of contributions from the operators with Wilson coefficients $a_3^{(j+k,k)}$ and $c_2^{(j+k,k)}$: the contributions from these operators scale as $(q^2)^{j+2}$, so we treat them as subleading in the $\hbar$ expansion.

Up to a redefinition of Wilson coefficients, the form-factor for the spin-monopole $F_j^{(0)}$ agrees with the amplitude in ref. [220]. The matching conditions, accounting for the normalization of the spinors, are

$$4^{2k}a_1^{(j+k,k)}\to\frac{a_k^{(j+k)}}{m_1}, \tag{6.34}$$

$$4^{2k+2}\left(a_2^{(j+k,k)}-\frac{c_1^{(j+k,k)}}{2m_1}\right)\to -\frac{4}{m_1}b_k^{(j+k+1)}, \tag{6.35}$$

where the Wilson coefficients on the right hand side are those in ref. [220]. This suggests that the spin-multipole universality for long range electromagnetic scattering [52] extends to the case where tidal deformations are accounted for. This means that, at least at the classical level, the tidally-modified spin-monopole is the same regardless of the total spin of the object. The agreement with the scalar amplitude also shows that the additional even-dimensional operators do not contribute unique spinless structures.

6.4.2 Gravity

Again, we split the gravitational scattering amplitude in terms of spinless, spin-orbit, and spin-spin contributions. The leading-in-$\hbar$ tidal contributions at 2PM order are

$$\Delta\mathcal{M}_2^{s=1/2} = G^2m_2^2S\sum_{j=0}^{\infty}\left(-\frac{q^2}{2}\right)^{j+2}\Big[\mathcal{U}_1\mathcal{U}_2G_j^{(0)}-i\omega\mathcal{E}_1\mathcal{U}_2G_j^{(1,1)}+i\omega\mathcal{U}_1\mathcal{E}_2G_j^{(1,2)} \tag{6.36}$$
$$+(q\cdot S_1)(q\cdot S_2)G_j^{(2,1)}-q^2(S_1\cdot S_2)G_j^{(2,2)}+\omega q^2(v_2\cdot S_1)(v_1\cdot S_2)G_j^{(2,3)}\Big],$$

where the form factors are the following functions of ω:

$$\begin{aligned}
G_j^{(0)} &= \sum_{k=0}^{j} d_1^{(j+k,k)} (4m_1)^{2k} (1-\omega^2)^k 16\alpha_k \\
&+ \sum_{k=0}^{j} \left(d_2^{(j+k,k)} - \frac{f_1^{(j+k,k)}}{2m_1} \right) (4m_1)^{2k+4} (1-\omega^2)^k \\
&\times \frac{1}{8} \left[(1-2\omega^2)^2 \alpha_k + 2(1-4\omega^2)(\omega^2-1)\alpha_{k+1} + 2(\omega^2-1)^2 \alpha_{k+2} \right], \qquad (6.37)
\end{aligned}$$

$$\begin{aligned}
G_j^{(1,1)} &= -\frac{e^{(j)}}{m_2} (4m_1)^{2j} (1-\omega^2)^j 16\alpha_{j+1} \\
&+ \sum_{k=0}^{j} \frac{f_1^{(j+k,k)}}{m_1 m_2} (4m_1)^{2k+2} (1-\omega^2)^k \\
&\times \left[2(\omega^2-1)(\alpha_{k+2} - 2\alpha_{k+1}) + (2\omega^2-1)2\alpha_k + (1-4\omega^2)\alpha_{k+1} \right] \\
&+ \sum_{k=0}^{j} \frac{f_3^{(j+k,k)}}{2m_1 m_2} (4m_1)^{2k+4} (1-\omega^2)^k \left[(4\omega^2-3)\alpha_{k+1} - 6(\omega^2-1)\alpha_{k+2} \right] \\
&+ \sum_{k=0}^{j} \frac{f_4^{(j+k,k)}}{m_1 m_2} (4m_1)^{2k+2} (1-\omega^2)^k 8\alpha_{k+1}, \qquad (6.38)
\end{aligned}$$

$$\begin{aligned}
G_j^{(1,2)} &= \sum_{k=0}^{j} \left(d_2^{(j+k,k)} - \frac{f_1^{(j+k,k)}}{2m_2} \right) \frac{(4m_1)^{2k+4}}{2m_1 m_2^2} (1-\omega^2)^k \\
&\times \left[(\omega^2-1)(2\alpha_{k+2} - \alpha_{k+1}) - \frac{1}{2}(2\omega^2-1)(2\alpha_{k+1} - \alpha_k) \right], \qquad (6.39)
\end{aligned}$$

$$\begin{aligned}
G_j^{(2,1)} &= \sum_{k=0}^{j} \frac{d_4^{(j+k,k)}}{m_1 m_2} (4m_1)^{2k} \left(1-\omega^2\right)^k 16\alpha_k \\
&- \frac{e^{(j)}}{m_2} (4m_1)^{2j+1} (1-\omega^2)^{j+1} 4\alpha_{j+1} \\
&- \sum_{k=0}^{j} \frac{f_1^{(j+k,k)}}{m_2} (4m_1)^{2k+2} \left(1-\omega^2\right)^k \frac{4(2k+7)\omega^4 - 3(2k+9)\omega^2 + 3}{8(k+1)(k+2)} \alpha_k \\
&+ \sum_{k=0}^{j} \frac{f_3^{(j+k,k)}}{m_2} (4m_1)^{2k+4} \left(1-\omega^2\right)^{k+1} \frac{3(2k+7)\omega^2 - 3}{8(k+2)(k+3)} \alpha_{k+1} \\
&+ \sum_{k=0}^{j} \frac{f_4^{(j+k,k)}}{m_2} (4m_1)^{2k+2} (1-\omega^2)^{k+1} 8\alpha_{k+1}, \qquad (6.40)
\end{aligned}$$

$$\begin{aligned}
G_j^{(2,2)} &= -\sum_{k=0}^{j} \frac{f_1^{(j+k,k)}}{m_2} (4m_1)^{2k+2} \left(1-\omega^2\right)^k \frac{4(2k+7)\omega^4 - 3(2k+9)\omega^2 + 3}{8(k+1)(k+2)} \alpha_k \\
&+ \sum_{k=0}^{j} \frac{f_3^{(j+k,k)}}{m_2} (4m_1)^{2k+4} \left(1-\omega^2\right)^{k+1} \frac{3(2k+7)\omega^2 - 3}{8(k+2)(k+3)} \alpha_{k+1}, \qquad (6.41)
\end{aligned}$$

$$\begin{aligned}
G_j^{(2,3)} &= \sum_{k=0}^{j} \frac{f_1^{(j+k,k)}}{m_2} (4m_1)^{2k+2} (1-\omega^2)^k \\
&\times \left[\frac{\alpha_k}{4(k+1)(k+2)(\omega^2-1)} \left(-2(2k+7)\omega^4 + (k(2k+11)+17)\omega^2 - 3(k+1)\right) \right]
\end{aligned}$$

$$+\sum_{k=0}^{j}\frac{f_3^{(j+k,k)}}{m_2}(4m_1)^{2k+4}\left(1-\omega^2\right)^k\frac{3(k+4)-(k+6)(2k+7)\omega^2}{4(k+2)(k+3)}\alpha_{k+1}. \tag{6.42}$$

The contributions from the $d_3^{(j+k,k)}$ and $f_2^{(j+k,k)}$ operators scale as $(q^2)^{j+3}$, so we treat them as subleading in the $\hbar$ expansion.

Results for the amplitude describing the scattering of a tidally defomed, spinless object with a spinning point particle were recently presented in ref. [199]. These are to be compared with the function $G_j^{(1,2)}$. After some algebraic manipulations of eq. (6.39) we find agreement with eq. (3.68) in ref. [199] when $k=j$.

As in the electromagnetic case, the form-factor for the spin-monopole $G_j^{(0)}$ agrees with the amplitude in ref. [220]; the two sets of Wilson coefficients can be matched through

$$4^{2k}d_1^{(j+k,k)}\to\frac{c_k^{(j+k)}}{m_1}, \tag{6.43}$$

$$4^{2k+4}\left(d_2^{(j+k,k)}-\frac{f_1^{(j+k,k)}}{2m_1}\right)\to\frac{16}{m_1}d_k^{(j+k+2)}, \tag{6.44}$$

where the coefficients on the right hand side are those in ref. [220]. Under these replacements the form factor here is related to that in ref. [220] through $G_j^{(0)}\to g_j/m_1$. The differing mass dimensions of the Wilson coefficients in each action is because of the different mass dimensions of scalar versus spinor fields. This matching extends the spin-multipole universality for long range gravitational scattering observed in ref. [23] to the tidally deformed setting. More precisely, this provides evidence that the classical tidally-modified spin-monopole is the same regardless of the total spin of the deformed object.

In both the electromagnetic and gravitational point-particle cases, it is well known that classical spin-spin effects of the form $q^2\,S_1\cdot S_2$ and $q\cdot S_1\,q\cdot S_2$ arise in the proportion $q\cdot S_1\,q\cdot S_2-q^2\,S_1\cdot S_2$ through one-loop order [23, 29, 52, 92]. We have shown here that this correlation between spin structures is broken for general values of the Wilson coefficients when finite-size effects are included at the one-loop level.

In the interest of deriving a conservative Hamiltonian in Section 6.5, we rewrite the amplitude in the center-of-mass kinematics of refs. [29, 78]. These are, for all momenta outgoing

$$p_1^\mu=-(E_1,\boldsymbol{p}),\quad p_2^\mu=-(E_2,-\boldsymbol{p}),\quad q^\mu=(0,\boldsymbol{q}),\quad \boldsymbol{p}\cdot\boldsymbol{q}=\frac{\boldsymbol{q}^2}{2},$$
$$S_1^\mu=\left(\frac{\boldsymbol{p}\cdot\boldsymbol{S}_1}{m_1},\boldsymbol{S}_1+\frac{\boldsymbol{p}\cdot\boldsymbol{S}_1}{(E_1+m_1)m_1}\boldsymbol{p}\right),\quad S_2^\mu=\left(-\frac{\boldsymbol{p}\cdot\boldsymbol{S}_2}{m_2},\boldsymbol{S}_2+\frac{\boldsymbol{p}\cdot\boldsymbol{S}_2}{(E_2+m_2)m_2}\boldsymbol{p}\right). \tag{6.45}$$

Here, $\boldsymbol{S}_i$ is the spin vector in the rest frame of particle i. It is also the spatial component of the canonical spin vector, which is the spin vector appearing in the canonical Hamiltonian [97]. We express the spin structures that arise using these kinematics:[6]

$$q\cdot S_i=-\boldsymbol{q}\cdot\boldsymbol{S}_i-\frac{\boldsymbol{q}^2\boldsymbol{p}\cdot\boldsymbol{S}_i}{2m_i(E_i+m_i)},\quad \mathcal{E}_i=E\,(\boldsymbol{p}\times\boldsymbol{q})\cdot\boldsymbol{S}_i,$$

[6] We use the mostly negative metric signature. Note that we find the opposite sign on the first term of $q\cdot S_i$ relative to ref. [29]. This difference is immaterial, however, since these structures only arise in the combination $q\cdot S_1\,q\cdot S_2$, causing the signs to cancel, and since the second term in this inner product is subleading in $\hbar$.

$$p_2 \cdot S_1 = -\frac{E}{m_1}\boldsymbol{p}\cdot\boldsymbol{S}_1, \quad p_1 \cdot S_2 = \frac{E}{m_2}\boldsymbol{p}\cdot\boldsymbol{S}_2,$$
$$S_1 \cdot S_2 = -\boldsymbol{S}_1\cdot\boldsymbol{S}_2 - M_{12}\boldsymbol{p}\cdot\boldsymbol{S}_1\,\boldsymbol{p}\cdot\boldsymbol{S}_2, \tag{6.46}$$

where

$$M_{12} \equiv \frac{(E+M)^2}{2(E_1+m_1)(E_2+m_2)m_1m_2}, \tag{6.47}$$

$E = E_1+E_2$, and $M = m_1+m_2$. Substituting these into eq. (6.36) after non-relativistically normalizing the amplitude, and keeping only the leading-in-$\hbar$ terms, we find

$$\frac{\Delta\mathcal{M}_2^{s=1/2}}{4E_1E_2} = 2G^2S\Big[\mathcal{U}_1\mathcal{U}_2\Delta a^{(0)}_{\text{cov},2} + \Delta a^{(1,1)}_{\text{cov},2}\, i(\boldsymbol{p}\times\boldsymbol{q})\cdot\boldsymbol{S}_1\,\mathcal{U}_2 + \Delta a^{(1,2)}_{\text{cov},2}\,\mathcal{U}_1\, i(\boldsymbol{p}\times\boldsymbol{q})\cdot\boldsymbol{S}_2$$
$$+\Delta a^{(2,1)}_{\text{cov},2}\,(\boldsymbol{q}\cdot\boldsymbol{S}_1\,\boldsymbol{q}\cdot\boldsymbol{S}_2) + \Delta a^{(2,2)}_{\text{cov},2}\,(\boldsymbol{q}^2\boldsymbol{S}_1\cdot\boldsymbol{S}_2) + \Delta a^{(2,3)}_{\text{cov},2}\,\boldsymbol{q}^2\boldsymbol{p}\cdot\boldsymbol{S}_1\,\boldsymbol{p}\cdot\boldsymbol{S}_2\Big], \tag{6.48}$$

where

$$\Delta a^{(0)}_{\text{cov},2} = m_2^2\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\left[\frac{G_j^{(0)}}{8E_1E_2}\right], \tag{6.49}$$

$$\Delta a^{(1,1)}_{\text{cov},2} = m_2^2\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\left[-\frac{\omega E}{8E_1E_2}G_j^{(1,1)}\right], \tag{6.50}$$

$$\Delta a^{(1,2)}_{\text{cov},2} = m_2^2\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\left[\frac{\omega E}{8E_1E_2}G_j^{(1,2)}\right], \tag{6.51}$$

$$\Delta a^{(2,1)}_{\text{cov},2} = m_2^2\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\left[\frac{1}{8E_1E_2}G_j^{(2,1)}\right], \tag{6.52}$$

$$\Delta a^{(2,2)}_{\text{cov},2} = m_2^2\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\left[-\frac{1}{8E_1E_2}G_j^{(2,2)}\right], \tag{6.53}$$

$$\Delta a^{(2,3)}_{\text{cov},2} = m_2^2\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\left[\frac{\omega E^2}{8E_1E_2m_1^2m_2^2}G_j^{(2,3)} - \frac{M_{12}}{8E_1E_2}G_j^{(2,2)}\right]. \tag{6.54}$$

We have borrowed the notation from ref. [29], where the subscript cov denotes that these are the coefficients to the spin structures when the amplitude is written in terms of the covariant spin vectors. We define the notation

$$\Delta a^{A}_{\text{cov},2} \equiv m_2^2\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\Delta a^{A}_{\text{cov},2,j}(\omega), \tag{6.55}$$

for easy reference later on.

6.5 Conservative two-body Hamiltonian

In this section, we use the effective field theory (EFT) matching approach of ref. [29] to derive the two-body spin-dependent conservative Hamiltonian. Working with the spin-coherent states $|\boldsymbol{n}\rangle$ defined therein, the two-body Hamiltonian is given by

$$H(\boldsymbol{q},\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2+m_1^2} + \sqrt{\boldsymbol{p}^2+m_2^2} + \langle \boldsymbol{n}_1\boldsymbol{n}_2| \left[\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_a) + \Delta\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_a)\right] |\boldsymbol{n}_1\boldsymbol{n}_2\rangle, \tag{6.56}$$

where $\Delta\hat{V}$ encodes the tidal contributions to the Hamiltonian for spinning particles. Here, $\boldsymbol{k}$ is the incoming three-momentum, $\boldsymbol{k}'$ is the outgoing three-momentum, $\boldsymbol{q} \equiv \boldsymbol{k} - \boldsymbol{k}'$ is the transferred three-momentum, and $\boldsymbol{p} \equiv (\boldsymbol{k}'+\boldsymbol{k})/2$. Finally, $\hat{\boldsymbol{S}}_a$ is the rest-frame spin operator of particle a, whose expectation value in the spin-coherent state of particle a gives its rest-frame spin vector, $\boldsymbol{S}_a$. The tidal potential can be expanded in the basis of spin operators analogously to the expansion of $\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_a)$ in ref. [29]:

$$\Delta\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_a) = \sum_A \Delta\hat{V}^A(\boldsymbol{k}',\boldsymbol{k})\,\hat{\mathbb{O}}^A. \tag{6.57}$$

A labels the following classical spin structures:

$$\begin{aligned} \hat{\mathbb{O}}^{(0)} &= \mathbb{I}, & \hat{\mathbb{O}}^{(1,1)} &= \boldsymbol{L}_q\cdot\hat{\boldsymbol{S}}_1, & \hat{\mathbb{O}}^{(1,2)} &= \boldsymbol{L}_q\cdot\hat{\boldsymbol{S}}_2, \\ \hat{\mathbb{O}}^{(2,1)} &= \boldsymbol{q}\cdot\hat{\boldsymbol{S}}_1\boldsymbol{q}\cdot\hat{\boldsymbol{S}}_2, & \hat{\mathbb{O}}^{(2,2)} &= \boldsymbol{q}^2\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2, & \hat{\mathbb{O}}^{(2,3)} &= \boldsymbol{q}^2\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_1\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_2, \end{aligned} \tag{6.58}$$

where $\boldsymbol{L}_q \equiv i(\boldsymbol{p}\times\boldsymbol{q})$. The first index in the superscripts labels the number of spin vectors in the operator, whereas the second labels the different structures with that many spin vectors. There are two spin structures that are not included in this basis. They are

$$\boldsymbol{q}\cdot\boldsymbol{p}\,\boldsymbol{q}\cdot\hat{\boldsymbol{S}}_1\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_2, \quad \boldsymbol{q}\cdot\boldsymbol{p}\,\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_1\boldsymbol{q}\cdot\hat{\boldsymbol{S}}_2. \tag{6.59}$$

As discussed in ref. [29], these are omitted from the basis since the on-shell condition $\boldsymbol{q}\cdot\boldsymbol{p}\sim\boldsymbol{q}^2$ means they are subleading in the $\hbar$ expansion.

To match the tidal amplitude to the tidal potential, the coefficients of these operators are also expanded in powers of G:

$$\Delta\hat{V}^A(\boldsymbol{k}',\boldsymbol{k}) = \frac{4\pi G}{\boldsymbol{q}^2}\Delta d_1^A(\boldsymbol{k}',\boldsymbol{k}) + \frac{2\pi^2G^2}{|\boldsymbol{q}|}\Delta d_2^A(\boldsymbol{k}',\boldsymbol{k}) + \mathcal{O}(G^3). \tag{6.60}$$

Tidal effects arise first at $\mathcal{O}(G^2)$, which imposes $\Delta d_1^A(\boldsymbol{k}',\boldsymbol{k}) = 0$. Note also that the tidal effects allow for higher powers of $\boldsymbol{q}^2$ to contribute classically to the potential at a given order in G. This potential is computed from the EFT of ref. [29], whose action for spin-1/2 fermions is given by

$$\begin{aligned} S = &\int_{\boldsymbol{k}} \sum_{a=1,2} \psi_a^\dagger(-\boldsymbol{k})\left(i\partial_t - \sqrt{\boldsymbol{k}^2+m_a^2}\right)\psi_a(\boldsymbol{k}) \\ &- \int_{\boldsymbol{k},\boldsymbol{k}'} \psi_1^\dagger(\boldsymbol{k}')\psi_2^\dagger(-\boldsymbol{k}')\left(\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_a) + \Delta\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_a)\right)\psi_1(\boldsymbol{k})\psi_2(-\boldsymbol{k}). \end{aligned} \tag{6.61}$$

The coefficients $\Delta d_i^A(\boldsymbol{k}',\boldsymbol{k})$ are then found by matching the amplitudes from the above EFT with the amplitudes derived from the tidal actions given in sections 6.3.4 and 6.3.5.

We move now to the computation of the tidal amplitude from this EFT. As noted above, $\Delta d_1^A(\boldsymbol{k}',\boldsymbol{k}) = 0$, which means the leading tidal potential is an $\mathcal{O}(G^2)$ quantity. As we wish to match to the $\mathcal{O}(G^2)$ am-

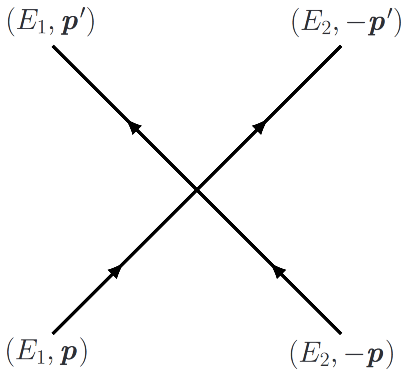


Figure 6.2: The diagram encoding the leading tidal effect in the EFT in eq. (6.61).

plitude from the full theory, this means that we will only need to compute the tidal contribution to the EFT amplitude from the tree-level diagram in fig. 6.2. This amplitude is simply

$$\Delta\mathcal{M}_2^{\text{EFT}} = -\langle \boldsymbol{n}_1\boldsymbol{n}_2|\Delta\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_i)|\boldsymbol{n}_1\boldsymbol{n}_2\rangle \equiv -\Delta V(\boldsymbol{k}',\boldsymbol{k},\boldsymbol{S}_i). \tag{6.62}$$

The subscript 2 on the left-hand side reminds us that this is an amplitude at $\mathcal{O}(G^2)$. We can also expand the EFT amplitude in the basis of eq. (6.58):

$$\begin{aligned}\Delta\mathcal{M}_2^{\text{EFT}} = \frac{2\pi^2G^2}{|\boldsymbol{q}|}\Big[&\Delta a_2^{(0)} + \Delta a_2^{(1,1)}\boldsymbol{L}_q\cdot\boldsymbol{S}_1 + \Delta a_2^{(1,2)}\boldsymbol{L}_q\cdot\boldsymbol{S}_2 \\ &+\Delta a_2^{(2,1)}\boldsymbol{q}\cdot\boldsymbol{S}_1\boldsymbol{q}\cdot\boldsymbol{S}_2 + \Delta a_2^{(2,2)}\boldsymbol{q}^2\boldsymbol{S}_1\cdot\boldsymbol{S}_2 + \Delta a_2^{(2,3)}\boldsymbol{q}^2\boldsymbol{p}\cdot\boldsymbol{S}_1\boldsymbol{p}\cdot\boldsymbol{S}_2\Big].\end{aligned} \tag{6.63}$$

Combining this with eqs. (6.57), (6.60) and (6.62) and matching the coefficients on the different spin operators, we arrive at the simple relation to leading-PM order in tidal effects:

$$\Delta a_2^A = -\Delta d_2^A. \tag{6.64}$$

Now, the EFT in eq. (6.61) is the effective theory where the graviton modes have been integrated out of the full theory. As an effective version of the full theory, it must produce the same low-energy amplitudes as the full theory:

$$\Delta\mathcal{M}_2^{\text{EFT}} = \frac{\Delta\mathcal{M}_2^{s=1/2}}{4E_1E_2}, \tag{6.65}$$

where we have non-relativistically normalized the full theory amplitude. Note that since both of these amplitudes are the leading order where tidal effects arise, there are no iteration terms to be matched. Before we can match these two amplitudes, we must first express the spinor products in the amplitude in eq. (6.48) in terms of the canonical spin vector. The result of this is the mixing of the $\Delta a^A_{\text{cov},2}$ coefficients, which has been worked out in ref. [29].[7] The mixing derived there is slightly modified in our case, as our spinors are dimensionful. Accounting for this mixing, eq. (6.65) yields the matching conditions

$$\Delta a_2^{(0)} = 4m_1m_2\Delta a^{(0)}_{\text{cov},2}, \tag{6.66}$$

[7] We thank Andrés Luna for discussion on this point.

$$\Delta a_2^{(1,1)} = 2m_2\Delta a_{\text{cov},2}^{(1,1)} - \frac{4m_2\Delta a_{\text{cov},2}^{(0)}}{E(p_1)+m_1}, \tag{6.67}$$

$$\Delta a_2^{(1,2)} = 2m_1\Delta a_{\text{cov},2}^{(1,2)} - \frac{4m_1\Delta a_{\text{cov},2}^{(0)}}{E(p_2)+m_2}, \tag{6.68}$$

$$\Delta a_2^{(2,1)} = \Delta a_{\text{cov},2}^{(2,1)} - \frac{2\boldsymbol{p}^2\Delta a_{\text{cov},2}^{(1,1)}}{E(p_2)+m_2} - \frac{2\boldsymbol{p}^2\Delta a_{\text{cov},2}^{(1,2)}}{E(p_1)+m_1} + \frac{4\boldsymbol{p}^2\Delta a_{\text{cov},2}^{(0)}}{[E(p_1)+m_1][E(p_2)+m_2]}, \tag{6.69}$$

$$\Delta a_2^{(2,2)} = \Delta a_{\text{cov},2}^{(2,2)} + \frac{2\boldsymbol{p}^2\Delta a_{\text{cov},2}^{(1,1)}}{E(p_2)+m_2} + \frac{2\boldsymbol{p}^2\Delta a_{\text{cov},2}^{(1,2)}}{E(p_1)+m_1} - \frac{4\boldsymbol{p}^2\Delta a_{\text{cov},2}^{(0)}}{[E(p_1)+m_1][E(p_2)+m_2]}, \tag{6.70}$$

$$\Delta a_2^{(2,3)} = \Delta a_{\text{cov},2}^{(2,3)} - \frac{2\Delta a_{\text{cov},2}^{(1,1)}}{E(p_2)+m_2} - \frac{2\Delta a_{\text{cov},2}^{(1,2)}}{E(p_1)+m_1} + \frac{4\Delta a_{\text{cov},2}^{(0)}}{[E(p_1)+m_1][E(p_2)+m_2]}. \tag{6.71}$$

Combining this with eqs. (6.62) to (6.64), the momentum-space potential at $\mathcal{O}(G^2)$ can be written as

$$\Delta V(\boldsymbol{k}', \boldsymbol{k}, \boldsymbol{S}_a) = -\frac{2\pi^2 m_2^2 G^2}{|\boldsymbol{q}|}\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\sum_A \Delta a_{2,j}^A(\omega)\mathbb{O}^A, \tag{6.72}$$

where the $\Delta a_{2,j}^A(\omega)$ are defined through

$$\Delta a_2^A(\omega) \equiv m_2^2\sum_{j=0}^{\infty}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\Delta a_{2,j}^A(\omega). \tag{6.73}$$

The potential can be expressed in position space by simply Fourier transforming, giving (see Section 6.B for the relevant integrals)

$$\begin{aligned}
\Delta V(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{S}_a) = -\frac{\sqrt{\pi}m_2^2G^2}{2}\sum_{j=0}^{\infty}\left(\frac{2}{\boldsymbol{r}^2}\right)^{j+3}\Gamma(j+3)\Bigg[&\frac{1}{\Gamma(-j-3/2)}\Delta a_{2,j}^{(0)}(\omega)\,\mathbb{I} \\
&-\left(\frac{2}{\boldsymbol{r}^2}\right)\frac{j+3}{\Gamma(-j-3/2)}\left(\Delta a_{2,j}^{(1,1)}(\omega)\boldsymbol{L}\cdot\boldsymbol{S}_1 + \Delta a_{2,j}^{(1,2)}(\omega)\boldsymbol{L}\cdot\boldsymbol{S}_2\right) \\
&-2(j+4)\left(\frac{2}{\boldsymbol{r}^2}\right)\frac{1}{\boldsymbol{r}^2}\frac{j+3}{\Gamma(-j-3/2)}\Delta a_{2,j}^{(2,1)}(\omega)\boldsymbol{r}\cdot\boldsymbol{S}_1\boldsymbol{r}\cdot\boldsymbol{S}_2 \\
&+\left(\frac{2}{\boldsymbol{r}^2}\right)(j+3)\left(\frac{1}{\Gamma(-j-3/2)}\Delta a_{2,j}^{(2,1)}(\omega) + \frac{2}{\Gamma(-j-5/2)}\Delta a_{2,j}^{(2,2)}(\omega)\right)\boldsymbol{S}_1\cdot\boldsymbol{S}_2 \\
&+2\left(\frac{2}{\boldsymbol{r}^2}\right)\frac{j+3}{\Gamma(-j-5/2)}\Delta a_{2,j}^{(2,3)}(\omega)\boldsymbol{p}\cdot\boldsymbol{S}_1\boldsymbol{p}\cdot\boldsymbol{S}_2\Bigg],
\end{aligned} \tag{6.74}$$

where $\boldsymbol{L} \equiv \boldsymbol{r}\times\boldsymbol{p}$ is the angular momentum. The spin-monopole portion of this potential agrees with that found in ref. [220] after performing the matching in eqs. (6.43) and (6.44)

6.6 CLASSICAL OBSERVABLES

The scattering amplitude can be related to various classical observables. For example, refs. [10, 14] derived direct relations between the scattering amplitude and the linear and angular impulses. In this section, we will use the tidally deformed scattering amplitudes for spin-1/2 particles, eqs. (6.22) and (6.36), to find the tidal contributions to these quantities. We remark that the Δ prefixes in the remainder of this section denote changes in the linear and angular momenta.

6.6.1 Linear impulse

Following ref. [10], the leading term for the classical linear impulse is[8]

$$\Delta p_1^\mu = \left\langle\!\!\left\langle i \int \hat{d}^4 q \hat{\delta}(2p_1 \cdot q)\hat{\delta}(2p_2 \cdot q) e^{-ib\cdot q} q^\mu \mathcal{A}(p_1, p_2 \to p_1 + q, p_2 - q) \right\rangle\!\!\right\rangle, \tag{6.75}$$

where b^μ is the impact parameter. Since we are dealing with the leading-PM tidal amplitudes, we do not need to include the second contribution to the linear impulse, i.e. eq. (3.30) in ref. [10]. The effect of the double angle brackets is to take the expectation values for the spins, as well as to impose the replacement $p_i \to m_i v_i$ on the initial momenta. As we have already written the amplitudes by expressing the momenta in this way, the task is reduced to the computation of integrals of the form in eq. (6.126). We present all computational details in section 6.B and write here just the results.

Before computing the impulse, we highlight that we will work with the covariant spin vector in this section. This changes the way the spinor products $\mathcal{U}_i$ are treated compared to Section 6.5. First, note that the spinors in the amplitudes of eqs. (6.22) and (6.36) are normalized such that $\bar{u}(p)u(p) = 2m$. Moreover, as the momenta of the final states differ from those of the initial states by momenta of order $\hbar$, we can expand the final state spinors in powers of $\hbar$:

$$\bar{u}(p \pm \hbar \bar{q}) = \bar{u}(p) + \mathcal{O}(q). \tag{6.76}$$

In fact, when working with the covariant spin vector, this $\mathcal{O}(q)$ correction is an infinitesimal Lorentz boost of the spinor [14]. This applies just as well to the heavy spinors, so, to leading order in q, the spinor products are $\mathcal{U}_i = 2m_i + \mathcal{O}(q^2)$.

We begin with the linear impulse in the gravitational case. Plugging the tidal contribution to the amplitude, eq. (6.36), into the impulse formula we have that[9]

$$\begin{aligned}\Delta p^\mu_{1,\mathrm{GR}} = iG^2 m_2^2 \sum_{j=0}^{\infty} \Big[& 4m_1 m_2 G_j^{(0)} I^\mu_{j+2}(-b) \\ & - 2i\omega m_1 m_2 \epsilon_{\rho\nu\alpha\beta} v_1^\nu v_2^\alpha \left(m_2 \langle S_1^\beta \rangle G_j^{(1,1)} - m_1 \langle S_2^\beta \rangle G_j^{(1,2)} \right) I^{\mu\rho}_{j+2}(-b) \\ & + \left(\langle S_{1\nu} \rangle \langle S_{2\rho} \rangle G_j^{(2,1)} - \eta_{\nu\rho} \langle S_1 \rangle \cdot \langle S_2 \rangle G_j^{(2,2)} + \omega \eta_{\nu\rho} \left(v_2 \cdot \langle S_1 \rangle \right) \left(v_1 \cdot \langle S_2 \rangle \right) G_j^{(2,3)} \right) I^{\mu\nu\rho}_{j+2}(-b) \Big] .\end{aligned} \tag{6.77}$$

By using the results for the integrals in eqs. (6.129), (6.134) and (6.138), we end up with

$$\begin{aligned}\Delta p^\mu_{1,\mathrm{GR}} = \frac{-\pi G^2 m_2}{8m_1\sqrt{\omega^2-1}} \sum_{j=0}^{\infty} \left(-\frac{2}{b^2}\right)^{j+3} \frac{\Gamma[7/2+j]}{\Gamma[-3/2-j]} \Bigg[& -4m_1 m_2 G_j^{(0)} \frac{b^\mu}{|\boldsymbol{b}|} \\ & + 2\omega m_1 m_2^2 \epsilon_{\rho\nu\alpha\beta} v_2^\nu v_1^\alpha \langle S_1^\beta \rangle G_j^{(1,1)} \frac{1}{|\boldsymbol{b}|} \left((7+2j)\frac{b^\mu b^\rho}{b^2} - \Pi^{\mu\rho} \right)\end{aligned}$$

[8] We leave the factors of $\hbar$ implicit.

[9] Note that eq. (6.75) gives the impulse for the particle that absorbs the transfer momentum. In our case, particle 1 is the emitting particle, so we must evaluate the integrals at $-b^\mu$.

$$
\begin{aligned}
&+ 2\omega m_1^2 m_2 \epsilon_{\rho\nu\alpha\beta} v_1^\nu v_2^\alpha \langle S_2^\beta \rangle G_j^{(1,2)} \frac{1}{|\boldsymbol{b}|} \left((7+2j) \frac{b^\mu b^\rho}{b^2} - \Pi^{\mu\rho} \right) \\
&- \left[\langle S_{1\nu} \rangle \langle S_{2\rho} \rangle G_j^{(2,1)} - \eta_{\nu\rho} \left(\langle S_1 \rangle \cdot \langle S_2 \rangle \right) G_j^{(2,2)} + \omega \eta_{\nu\rho} \left(v_2 \cdot \langle S_1 \rangle \right) \left(v_1 \cdot \langle S_2 \rangle \right) G_j^{(2,3)} \right] \\
&\times \left((9/2+j) \frac{b^\mu b^\nu b^\rho}{b^2} - \frac{3}{2} b^{(\mu} \Pi^{\nu\rho)} \right) \frac{2}{|\boldsymbol{b}|} \left(-\frac{2}{b^2} \right) (7/2+j) \Bigg], \qquad (6.78)
\end{aligned}
$$

where $\Pi^{\mu\nu}$ is defined in eq. (6.131). In the electromagnetic case, the impulse is

$$
\begin{aligned}
\Delta p^\mu_{1,\text{EM}} = \frac{-e^2}{8\pi m_1 m_2 \sqrt{\omega^2-1}} \sum_{j=0}^\infty \left(-\frac{2}{b^2} \right)^{j+2} \frac{\Gamma[5/2+j]}{\Gamma[-1/2-j]} \Bigg[-4 m_1 m_2 F_j^{(0)} \frac{b^\mu}{|\boldsymbol{b}|} \\
&+ 2\omega m_1 m_2^2 \epsilon_{\rho\nu\alpha\beta} v_2^\nu v_1^\alpha \langle S_1^\beta \rangle F_j^{(1,1)} \frac{1}{|\boldsymbol{b}|} \left((5+2j) \frac{b^\mu b^\rho}{b^2} - \Pi^{\mu\rho} \right) \\
&+ 2\omega m_1^2 m_2 \epsilon_{\rho\nu\alpha\beta} v_1^\nu v_2^\alpha \langle S_2^\beta \rangle F_j^{(1,2)} \frac{1}{|\boldsymbol{b}|} \left((5+2j) \frac{b^\mu b^\rho}{b^2} - \Pi^{\mu\rho} \right) \\
&- \left[\langle S_{1\nu} \rangle \langle S_{2\rho} \rangle F_j^{(2,1)} - \eta_{\nu\rho} \left(\langle S_1 \rangle \cdot \langle S_2 \rangle \right) F_j^{(2,2)} + \omega \eta_{\nu\rho} \left(v_2 \cdot \langle S_1 \rangle \right) \left(v_1 \cdot \langle S_2 \rangle \right) F_j^{(2,3)} \right] \\
&\times \left((7/2+j) \frac{b^\mu b^\nu b^\rho}{b^2} - \frac{3}{2} b^{(\mu} \Pi^{\nu\rho)} \right) \frac{2}{|\boldsymbol{b}|} \left(-\frac{2}{b^2} \right) (5/2+j) \Bigg]. \qquad (6.79)
\end{aligned}
$$

Note that the electromagnetic result can be obtained from the gravitational result using the following replacements:

$$G^2 m_2^2 \to e^2/\pi^2, \qquad (6.80)$$

$$G_j^A \to F_j^A, \qquad (6.81)$$

$$j \to j-1. \qquad (6.82)$$

The last of these replacements is not applied to the indices of the form factors.

6.6.2 Angular impulse

We turn now to the determination of the angular impulse. The angular impulse for the absorbing particle is related to the amplitude through [14][10]

$$\Delta S_1^\mu = \left\langle\!\!\left\langle i \int \hat{d}^4 q \hat{\delta}(2p_1 \cdot q) \hat{\delta}(2p_2 \cdot q) e^{-ib\cdot q} \left(-\frac{p_1^\mu}{m_1^2} q \cdot S_1(p_1) \mathcal{A}(q) + [S_1^\mu(p_1), \mathcal{A}(q)] \right) \right\rangle\!\!\right\rangle, \qquad (6.83)$$

where p_1^μ is the initial momentum of the absorbing particle. We don't need the second contribution to the impulse, eq. (3.22) in ref. [14], because we are calculating the leading-PM tidal contribution. For the spin-1/2 amplitudes we are considering, this formula will produce terms of $\mathcal{O}(S_i^2)$. We ignore such contributions since one must consider spin-1 scattering to obtain all information at this spin order.

We compute each term individually for particle 1, which is the emitting particle in our setup. The first term is

$$\left\langle\!\!\left\langle i \int \hat{d}^4 q \hat{\delta}(2p_1 \cdot q) \hat{\delta}(2p_2 \cdot q) e^{ib\cdot q} \frac{p_1^\mu}{m_1^2} q \cdot S_1(p_1) \mathcal{A}_2(q) \right\rangle\!\!\right\rangle = -\frac{v_1^\mu}{m_1} \Delta p_1 \cdot \langle S_1 \rangle, \qquad (6.84)$$

[10] Again, we leave the factors of $\hbar$ implicit.

which follows since, to the order we are working, there is only one contribution to the linear impulse. This is true for both the electromagnetic and gravitational cases. Writing this explicitly for the case of gravity,

$$-\frac{v_1^\mu}{m_1}\Delta p_1 \cdot \langle S_1 \rangle = \frac{v_1^\mu \pi G^2 m_2^2}{4m_1|\mathbf{b}|\sqrt{\omega^2-1}} \sum_{j=0}^{\infty} \left(-\frac{2}{b^2}\right)^{j+3} \frac{\Gamma[7/2+j]}{\Gamma[-3/2-j]} \times \left[-2G_j^{(0)} b^\lambda + \omega m_1 \epsilon_{\rho\nu\alpha\beta} v_1^\nu v_2^\alpha \langle S_2^\beta \rangle G_j^{(1,2)} \left((7+2j)\frac{b^\lambda b^\rho}{b^2} - \Pi^{\lambda\rho}\right)\right] \langle S_{1\lambda}\rangle + \mathcal{O}(\langle S_1 \rangle^2). \tag{6.85}$$

To compute the commutator term, we need the following commutator [14]:

$$[S_i^\mu, S_j^\nu] = -\delta_{ij}\frac{i}{m_i}\epsilon^{\mu\nu\rho\sigma} S_{i\rho} p_{i\sigma}. \tag{6.86}$$

With this in hand, the commutator term for gravity is

$$\left\langle\!\!\left\langle i \int \hat{d}^4 q \hat{\delta}(2p_1 \cdot q)\hat{\delta}(2p_2 \cdot q) e^{ib\cdot q} \left[S_1^\mu(p_1), \mathcal{M}_2(q)\right] \right\rangle\!\!\right\rangle$$
$$= iG^2 m_2^2 \sum_{j=0}^{\infty} \left[-2\omega m_1 m_2^2 G_j^{(1,1)} \left[(v_2^\mu - v_1^\mu \omega) I_{j+2}^\alpha - I_{j+2}^\mu v_2^\alpha\right] \langle S_{1\alpha}\rangle \right.$$
$$\left. - i\epsilon^\mu{}_{\nu\lambda\tau} v_1^\tau \left[\delta_\alpha^\nu \delta_\beta^\eta G_j^{(2,1)} - \eta_{\alpha\beta}\eta^{\nu\eta} G_j^{(2,2)} + \eta_{\alpha\beta}\omega v_2^\nu v_1^\eta G_j^{(2,3)}\right] I_{j+2}^{\alpha\beta} \langle S_1^\lambda \rangle \langle S_{2\eta}\rangle \right] + \mathcal{O}(\langle S_1\rangle^2). \tag{6.87}$$

We simply add eq. (6.84) and eq. (6.87) to get the full angular impulse of particle 1 in the gravitational case. Again, the map in eq. (6.80) can be applied to the gravitational impulse to obtain the electromagnetic result. The same calculation can be performed for particle 2, but we don't give the result here as it is almost identical to the calculation for particle 1.

6.7 EIKONAL PHASE

The eikonal phase provides an alternative means for extracting physical observables from the classical portion of scattering amplitudes, and has been successfully applied to systems involving low spins up to $\mathcal{O}(G^2)$ [29, 45]. Relations between classical observables and the eikonal phase were proposed to all perturbative orders in ref. [29]; however, as we are working with tidal effects at the leading-PM order, we only need these relations to leading order. For the linear impulse and the spin kick, they are

$$\Delta \boldsymbol{p}_\perp = \nabla_{\boldsymbol{b}} \chi \quad \text{and} \quad \Delta \boldsymbol{S}_a^i = -\epsilon^{ijk} \frac{\partial \chi}{\partial \boldsymbol{S}_a^j} \boldsymbol{S}_a^k, \tag{6.88}$$

where χ is the eikonal phase, Δ indicates changes in momentum or spin, and $a = 1, 2$ labels the particles and is not summed over. We use Latin letters from the middle of the alphabet to denote spatial indices, which are raised and lowered with the Euclidean metric. The former relation yields the linear impulse in the plane perpendicular to the momentum at negative infinity. Choosing kinematics such that this momentum is oriented along the z-axis, the impulse in the parallel direction is obtained through energy conservation: $\Delta p_z = -(\Delta \boldsymbol{p})^2/2|\boldsymbol{p}|$. Further to the results in Section 6.6, the eikonal phase will allow us to compute the spin kick and the scattering angle for aligned spins. First, we must compute the eikonal phase.

The eikonal phase $\chi = \chi_1 + \chi_2 + \ldots$ is defined as the Fourier transform of the (relativistically normalized) amplitude in the perpendicular plane described above, with the subscripts denoting contributions from the corresponding order in the coupling constant:

$$\chi_i = \frac{1}{4m_1 m_2\sqrt{\omega^2-1}} \int \frac{d^{2-2\epsilon}\boldsymbol{q}}{(2\pi)^{2-2\epsilon}} e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} \mathcal{M}'_i(\boldsymbol{q}), \tag{6.89}$$

where the prime on the amplitude indicates that we ignore iteration pieces. We will use $\Delta\chi$ to denote tidal contributions to the eikonal phase. The leading-PM contributions from the tidal deformations originate from a one-loop amplitude, so $\Delta\chi_1 = 0$ and we will calculate the $\Delta\chi_2$ part of the eikonal phase. This involves integrals of the type

$$\int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} \frac{\pi^2}{|\boldsymbol{q}|} \left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2} \hat{\mathbb{O}}^A(\boldsymbol{p},\boldsymbol{q},\boldsymbol{S}_a) = \hat{\mathcal{O}}^A(\boldsymbol{p},\nabla_{\boldsymbol{b}},\boldsymbol{S}_a) I_{j+2}(\boldsymbol{b}) \frac{\mathcal{N}}{4}, \tag{6.90}$$

where we have shifted $\boldsymbol{q} \to -\boldsymbol{q}$ on the right-hand side to eliminate relative signs with the integrals in section 6.B; $\mathcal{N}$ and I_{j+2} are given there. The classical spin structures $\hat{\mathbb{O}}^A$ are listed in eq. (6.58). They have been removed from the integral by re-expressing them as differential operators in impact-parameter space acting on the integral, yielding

$$\begin{gathered}\hat{\mathcal{O}}^{(0)} = \mathbb{I}, \quad \hat{\mathcal{O}}^{(1,1)} = -(\boldsymbol{S}_1\times\boldsymbol{p})\cdot\nabla_{\boldsymbol{b}}, \quad \hat{\mathcal{O}}^{(1,2)} = -(\boldsymbol{S}_2\times\boldsymbol{p})\cdot\nabla_{\boldsymbol{b}}, \\ \hat{\mathcal{O}}^{(2,1)} = -(\boldsymbol{S}_1\cdot\nabla_{\boldsymbol{b}})(\boldsymbol{S}_2\cdot\nabla_{\boldsymbol{b}}), \quad \hat{\mathcal{O}}^{(2,2)} = -(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\,\nabla_{\boldsymbol{b}}^2, \quad \hat{\mathcal{O}}^{(2,3)} = -(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)\nabla_{\boldsymbol{b}}^2.\end{gathered} \tag{6.91}$$

This allows us to write the eikonal phase $\Delta\chi_2$ compactly as,

$$\Delta\chi_2 = 2G^2 m_2^2 \sum_{j=0}^{\infty} \hat{\mathcal{K}}_j(\omega,\boldsymbol{p},\nabla_{\boldsymbol{b}},\boldsymbol{S}_a)\, I_{j+2}(\boldsymbol{b}), \quad \hat{\mathcal{K}}_j(\omega,\boldsymbol{p},\nabla_{\boldsymbol{b}},\boldsymbol{S}_a) \equiv 4E_1E_2 \sum_A \Delta b^A_{2,j}(\omega)\hat{\mathcal{O}}^A, \tag{6.92}$$

where the dependence on each quantity has been written explicitly. The function $\Delta b^A_{2,j}(\omega)$ is either equal to $m^A \Delta a^A_{\text{cov},2,j}(\omega)$ (no sum over A) or $\Delta a^A_{2,j}(\omega)$, depending on whether we work with the covariant or the canonical spin. The constant m^A accounts for the normalization of the Dirac spinors, and is given by

$$m^A = \begin{cases} 4m_1m_2, & A = (0), \\ 2m_2, & A = (1,1), \\ 2m_1, & A = (1,2), \\ 1, & \text{otherwise}. \end{cases} \tag{6.93}$$

We refer to the operator $\hat{\mathcal{K}}_j$ as the eikonal operator.

We can now substitute this form of the eikonal phase into eq. (6.88) to obtain the impulse and spin kick. For the impulse, the derivative with respect to the impact parameter commutes with the eikonal operator and only acts on the integral. By contrast, for the spin kick, the derivative with respect to the spin vector only acts on the eikonal operator:[11]

$$\Delta \boldsymbol{p}_\perp = -2G^2 m_2^2 \sum_{j=0}^{\infty} \hat{\mathcal{K}}_j(\omega, \boldsymbol{p}, \nabla_{\boldsymbol{b}}, \boldsymbol{S}_a)\left(\nabla_{\boldsymbol{b}} I_{j+2}(\boldsymbol{b})\right), \tag{6.94}$$

$$\Delta \boldsymbol{S}_a = -2G^2 m_2^2 \sum_{j=0}^{\infty} \left(\frac{\partial \hat{\mathcal{K}}_j(\omega, \boldsymbol{p}, \nabla_{\boldsymbol{b}}, \boldsymbol{S}_a)}{\partial \boldsymbol{S}_a} \times \boldsymbol{S}_a \right) I_{j+2}(\boldsymbol{b}). \tag{6.95}$$

Setting $\Delta b^A_{2,j}(\omega) = m^A \Delta a^A_{\text{cov},2,j}(\omega)$, the linear impulse calculated from the eikonal phase agrees with eq. (6.78). We find the spin kick for particle 1 to be

$$\begin{aligned}\Delta \boldsymbol{S}_1^k = -8E_1E_2G^2m_2^2 \sum_{j=0}^{\infty} \Big\{ &-i\Delta b^{(1,1)}_{2,j}[-\boldsymbol{p}^k \boldsymbol{S}_1^m + (\boldsymbol{p}\cdot\boldsymbol{S}_1)\,\delta^{km}] I^m_{j+2} \\ &+ \Big[-\Delta b^{(2,1)}_{2,j}(\varepsilon^{k\ell m}\boldsymbol{S}_1^\ell)\boldsymbol{S}_2^n - \Delta b^{(2,2)}_{2,j}(\boldsymbol{S}_1\times\boldsymbol{S}_2)^k\delta^{mn} + \Delta b^{(2,3)}_{2,j}(\boldsymbol{p}\times\boldsymbol{S}_1)^k(\boldsymbol{p}\cdot\boldsymbol{S}_2)\delta^{mn}\Big] I^{mn}_{j+2}\Big\}.\end{aligned} \tag{6.96}$$

This expression for the spin kick is valid in the center-of-mass frame. Computing the same quantity in the rest frame of particle 1, we find agreement with the commutator portion of the angular impulse, eq. (6.87).

From the eikonal phase it is also possible to obtain the scattering angle. In the case of two non-rotating bodies, the dynamics are constrained to a plane, so a unique scattering angle can be defined. Spinning particles, however, introduce precession effects, which are described by an additional angle. This additional complication can be ignored in the special case of aligned spins, which again restricts motion to a plane. We will calculate the unique scattering angle in this special case, where the spin structures are

$$\boldsymbol{S}_1\cdot\boldsymbol{S}_2 = |\boldsymbol{S}_1||\boldsymbol{S}_2|, \quad \boldsymbol{b}\cdot\boldsymbol{S}_i = 0, \quad \boldsymbol{p}\cdot\boldsymbol{S}_i = 0. \tag{6.97}$$

This scattering angle can be related to the eikonal phase using the stationary phase approximation [231]:

$$2\sin\frac{\theta}{2} \approx \theta = -\frac{1}{|\boldsymbol{p}|}\frac{\partial}{\partial|\boldsymbol{b}|}\chi(\omega,\boldsymbol{b}). \tag{6.98}$$

For the calculation of the scattering angle, we will use the eikonal phase in terms of the canonical spin, as it is to be compared to the scattering angle derived from canonical equations of motion. Thus, $\Delta b^A_{2,j}(\omega) = \Delta a^A_{2,j}(\omega)$.

As in the case of the linear impulse, the derivative with respect to $|\boldsymbol{b}|$ acts only on the integral. Using the relation

$$\frac{\partial}{\partial|\boldsymbol{b}|} I_{j+2}(\boldsymbol{b}) = \frac{2|\boldsymbol{b}|}{(5+2j)} I_{j+3}(\boldsymbol{b}), \tag{6.99}$$

we can write the scattering angle generally as

$$\Delta\theta_2 = -\frac{2G^2m_2^2}{|\boldsymbol{p}|}\sum_{j=0}^{\infty}\frac{2|\boldsymbol{b}|}{(5+2j)}\hat{\mathcal{K}}_j(\omega,\boldsymbol{p},\nabla_{\boldsymbol{b}},\boldsymbol{S}_a)\, I_{j+3}(\boldsymbol{b}). \tag{6.100}$$

[11]The sign for the linear impulse here is because we compute the impulse for the emitting particle.

Using the action of each differential operator on the integral (eq. (6.144)), the tidal corrections to the 2PM aligned-spin scattering angle are

$$\begin{aligned}\Delta\theta_2 = &\frac{\pi G^2 m_2 E_1 E_2}{|\boldsymbol{p}| m_1 \sqrt{\omega^2-1}} \sum_{j=0}^{\infty} \frac{\Gamma[9/2+j]}{\Gamma[-3/2-j]} \left(\frac{2}{\boldsymbol{b}^2}\right)^{j+4} \\ &\times \left[\Delta a_{2,j}^{(0)} \frac{\boldsymbol{b}^2}{(7+2j)} + \Delta a_{2,j}^{(1,1)} (\boldsymbol{S}_1 \times \boldsymbol{p}) \cdot \boldsymbol{b} + \Delta a_{2,j}^{(1,2)} (\boldsymbol{S}_2 \times \boldsymbol{p}) \cdot \boldsymbol{b} \right. \\ &\left. + \left(\Delta a_{2,j}^{(2,1)} - (5+2j)\, \Delta a_{2,j}^{(2,2)}\right) |\boldsymbol{S}_1||\boldsymbol{S}_2|\right]. \end{aligned} \tag{6.101}$$

The spin-monopole portion of this is in agreement with that in ref. [220], upon applying the matching conditions in eqs. (6.43) and (6.44).

6.8 CONCLUSION

As the recent burst in activity suggests, quantum-field-theoretic techniques are well suited for studying tidal deformations, where the tidal effects are characterized by higher-dimensional operators. A full classification of tidal operators relevant for tidally-deformed spinless objects at the one-loop level was presented in ref. [220], and in this paper we have extended this analysis to include effects at linear order in the spin of the deformed object. As in ref. [220], the starting point was the Hilbert series, which counts the number of independent operators – equivalently, the number of independent amplitudes – for a given field content and number of covariant derivatives. Using this as a guide, we wrote down both the amplitude basis and the operator basis for a spin-1/2 particle coupled to photons or gravitons through at most two photon field strengths/Weyl tensors. These operator bases represent the full set of operators coupling two spinors to two photon field strengths or Weyl tensors, describing the complete set of finite-size contributions at one loop.

Employing traditional Feynman diagrammatic methods, we used these actions to calculate the one-loop amplitudes – corresponding to the leading-PM order in the case of gravity – for these finite-size effects. We find that the spin-multipole universality for long-range classical effects observed in refs. [23, 52] extends to tidally deformed systems; the spin-monopole portions of the amplitudes calculated here are in agreement with those found in ref. [220]. For general Wilson coefficients, the finite-size contributions to the amplitudes break the observed correspondence between the $q \cdot S_1\, q \cdot S_2$ and $q^2 S_1 \cdot S_2$ terms in the point-particle case.

We then extracted various classical quantities from these amplitudes. First, we extended the EFT matching formalism of ref. [29] to include tidal effects, and subsequently used this formalism to derive the tidal corrections to the conservative gravitational Hamiltonian at leading-PM order. We then derived the finite-size contributions to the electromagnetic and gravitational linear and angular impulse. The linear impulse was computed in two ways for the gravitational case, producing the same result: it was first calculated using the formalism of ref. [10], then through application of the eikonal phase [29]. The angular impulse was computed using the method of ref. [14]. A portion of this result was corroborated by the extraction of the spin kick from the eikonal phase; when computing this quantity in the rest frame of particle 1, it agrees with the commutator contribution to the angular impulse. Finally, the eikonal phase allowed us to derive the scattering angle in the case of aligned spins.

In the interest of describing real macroscopic systems, one must account for finite-size effects at arbitrary orders in the spin vector. We have demonstrated that EFT techniques such as the Hilbert series and the construction of on-shell helicity amplitudes are suitable for combining spin and finite-size effects. These techniques can yet be extended to higher spins, but this is outside the scope of this paper.

The description of finte-size effects for spinning particles also opens the door for the study of the entanglement entropy generated in the scattering of tidally deformed objects. A similar analysis to that performed in ref. [192] can be applied to the systems described here, potentially shedding some light on the values of tidal Love numbers for Kerr black holes in a general (but weak) gravitational environment.

Finally, it would be helpful to understand links between the tidal action here and worldline actions describing tidal effects at linear order in the angular momentum, perhaps by taking the heavy or non-relativistic limits of the quantum action in eq. (6.18). The action in eq. (6.18) is the most general, non-redundant action describing parity-even four-point contact terms, but not all operators necessarily need to contribute classical effects; for example, we found that certain operators enter only at subleading orders at the one-loop level. A matching to classical quantities is then essential for determining the subset of operators that do indeed contribute classically.

APPENDIX

6.A LOOP INTEGRALS

For the calculation of tidal effects at one loop, we only need to evaluate the triangle integral, but we need to do so for arbitrary even rank,

$$\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}} \equiv \int \frac{d^4l}{(2\pi)^4}\frac{l^{\mu_1\cdots\mu_{2k}}}{l^2(l+q)^2[-v_2\cdot l+i\epsilon]}. \tag{6.102}$$

The Passarino-Veltman reduction [203] allows one to solve this for any rank in terms of the scalar triangle integral[12]

$$\mathcal{I}_{\triangleleft} \equiv \int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2(l+q)^2[-v_2\cdot l+i\epsilon]} = -\frac{iS}{16\pi^2}, \tag{6.103}$$

where $S \equiv \pi^2/\sqrt{-q^2}$, but this becomes cumbersome for high ranks.

Fortunately, the portions of the integrals needed for our purposes are only the leading-in-$\hbar$ pieces. When the tensor integral is dotted into $v_{1\mu_1\ldots\mu_{2k}}$, ref. [220] found by explicit calculation up to rank $2k=10$ that this leading term is

$$v_{1\mu_1\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}} \equiv \int \frac{d^4l}{(2\pi)^4}\frac{(v_1\cdot l)^{2k}}{l^2(l+q)^2[-v_2\cdot l+i\epsilon]} = \frac{\left(\frac{1}{2}\right)_k}{4^k(1)_k}(\omega^2-1)^k q^{2k}\mathcal{I}_{\triangleleft}+\mathcal{O}(q^{2k}). \tag{6.104}$$

This formula was proven for general k in ref. [199] by finding the residue of the matter pole and then calculating the remaining three-dimensional integral using a known expression [232].

When calculating the leading-PM amplitudes for spinning tidal effects, we also need five additional integrals:

$$(\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1}u_1)v_{1\mu_2\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}}, \tag{6.105}$$

$$(\bar{u}_4 q^\alpha \sigma_{\alpha\mu_1}u_3)v_{1\mu_2\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}}, \tag{6.106}$$

$$(\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1}u_1)(\bar{u}_4 q^\beta \sigma_{\beta\mu_2}u_3)v_{1\mu_3\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}}, \tag{6.107}$$

$$S_{1\mu_1}S_{2\mu_2}v_{1\mu_3\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}}, \tag{6.108}$$

$$q\cdot S_1\, S_{2\mu_1}v_{1\mu_2\ldots\mu_{2k+1}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k+1}} = S_{1\mu_1}\, q\cdot S_2 v_{1\mu_2\ldots\mu_{2k+1}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k+1}}. \tag{6.109}$$

The leading terms for these integrals are

$$(\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1}u_1)v_{1\mu_2\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}} = (\bar{u}_2 q^\alpha v_2^\beta \sigma_{\alpha\beta}u_1)\frac{\left(\frac{1}{2}\right)_k}{4^k(1)_k}\omega(\omega^2-1)^{k-1}q^{2k}\mathcal{I}_{\triangleleft}, \tag{6.110}$$

$$(\bar{u}_4 q^\alpha \sigma_{\alpha\mu_1}u_3)v_{1\mu_2\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}} = -(\bar{u}_4 q^\alpha v_1^\beta \sigma_{\alpha\beta}u_3)\frac{\left(\frac{1}{2}\right)_k}{4^k(1)_k}(\omega^2-1)^{k-1}q^{2k}\mathcal{I}_{\triangleleft}, \tag{6.111}$$

$$(\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1}u_1)(\bar{u}_4 q^\beta \sigma_{\beta\mu_2}u_3)v_{1\mu_3\ldots\mu_{2k}}\mathcal{I}_{\triangleleft}^{\mu_1\cdots\mu_{2k}} = \tag{6.112}$$

[12] More precisely, the Passarino-Veltman reduction expresses the rank-$2k$ integral in terms of scalar triangle, bubble, and tadpole integrals, but neither of the latter two contributes classical information.

$$(\bar{u}_2 q^\alpha \sigma_{\alpha\mu} u_1)(\bar{u}_4 q^\beta \sigma_{\beta\nu} u_3) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \times \left[-\frac{1}{2k-1} \eta^{\mu\nu} (\omega^2 - 1)^{k-1} - \frac{2(k-1)}{(2k-1)} v_1^\nu v_2^\mu \omega (\omega^2-1)^{k-2} \right],$$

$$S_{1\mu_1} S_{2\mu_2} v_{1\mu_3\ldots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k}} = S_{1\mu_1} S_{2\nu} \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \times \left[\frac{2k+1}{2k-1} q^\mu q^\nu (\omega^2-1)^{k-1} - \frac{1}{2k-1} q^2 \eta^{\mu\nu} (\omega^2-1)^{k-1} - \frac{2(k-1)}{(2k-1)} q^2 v_1^\nu v_2^\mu \omega (\omega^2-1)^{k-2} \right], \tag{6.113}$$

$$q \cdot S_1 \, S_{2\mu_1} v_{1\mu_2\ldots\mu_{2k-1}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k-1}} = -q \cdot S_1 \, q \cdot S_2 \frac{\left(\frac{1}{2}\right)_{k-1}}{2^{2k-1}(1)_{k-1}} (\omega^2-1)^{k-1} q^{2k-2} \mathcal{I}_\triangleleft. \tag{6.114}$$

where $k \geq 1$. The first three of these contractions rely on the same portion of the rank-$2k$ triangle integral as that of eq. (6.104); namely, only the part of the integral whose tensor structure contains no transfer momenta. As such, these integrals can be derived by a combinatoric analysis of eq. (6.104). We show now this combinatoric analysis.

The leading-in-$\hbar$ portion of the triangle integral proportional to tensor structures containing only factors of the metric and the velocity can be inferred from eq. (6.104). To do this, we expand the binomial $(\omega^2-1)^k$ and "uncontract" the integral by noting that $\omega = v_{1\mu} v_2^\mu$ and $v_{1\mu\nu}\eta^{\mu\nu} = 1$:

$$\begin{aligned} v_{1\mu_1\ldots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k}} &= \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} v_{1\mu_1\ldots\mu_{2k-2n}} \eta^{\mu_1\mu_2,\ldots,\mu_{2k-2n-1}\mu_{2k-2n}} v_{1\alpha_1\ldots\alpha_{2n}} v_2^{\alpha_1\ldots\alpha_{2n}} \\ &= v_{1\mu_1\ldots\mu_{2k}} \left[\frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \sum_{n=0}^{k} \frac{1}{(2k)!} \binom{k}{n} (-1)^{k-n} \eta^{\{\mu_1\mu_2,\ldots,\mu_{2k-2n-1}\mu_{2k-2n}} v_2^{\mu_{2k-2n+1}\ldots\mu_{2k}\}} \right], \end{aligned} \tag{6.115}$$

where $\eta^{\mu_1\mu_2,\ldots,\mu_{2k-1}\mu_{2k}} \equiv \eta^{\mu_1\mu_2}\ldots\eta^{\mu_{2k-1}\mu_{2k}}$. We now identify the quantity in the square brackets with the uncontracted triangle integral. Curly brackets denote symmetrization without normalization. Note that this is not the actual value of the uncontracted triangle integral: even at the order of $\hbar$ at which we are working, tensor structures with an even number of transfer-momentum four-vectors are present. For our purposes, however, these contributions will always vanish or become subleading through the on-shell condition $v_1 \cdot q \sim q^2/m$ when contracted with the spin structures in eqs. (6.105) to (6.107).

We begin with the contraction in eq. (6.105). Equation (6.32) implies that the $n = 0$ term vanishes, and that the only non-vanishing terms have a v_2^μ contracted with the sigma matrix. The remaining indices will all be contracted symmetrically with factors of $v_{1\mu}$, so for a term with $2n$ factors of v_2^μ and $2k$ total symmetrized Lorentz indices there will be $2n(2k-1)!$ non-vanishing and identical distributions of the Lorentz indices. Then,

$$\begin{aligned} &(\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1} u_1) v_{1\mu_2\ldots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k}} \\ &= (\bar{u}_2 q^\alpha v_2^{\mu_{2k}} \sigma_{\alpha\mu_{2k}} u_{2k}) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \sum_{n=1}^{k} \frac{2n(2k-1)!}{(2k)!} \binom{k}{n} (-1)^{k-n} \omega^{2n-1} \end{aligned}$$

$$= (\bar{u}_2 q^\alpha v_2^{\mu_{2k}} \sigma_{\alpha\mu_{2k}} u_{2k}) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \omega \sum_{n=0}^{k-1} \binom{k-1}{n} (-1)^{k-1-n} \omega^{2n-2}$$
$$= (\bar{u}_2 q^\alpha v_2^{\mu_{2k}} \sigma_{\alpha\mu_{2k}} u_{2k}) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} \omega(\omega^2-1)^{k-1} q^{2k} \mathcal{I}_\triangleleft. \tag{6.116}$$

This is what we wanted to prove.

The argument for the contraction in eq. (6.106) is similar. However, because of eq. (6.33), now only terms with at least one metric survive. Thus the $n = k$ term vanishes, and a term with $k - n$ metrics and $2k$ total symmetrized Lorentz indices contributes $2(k-n)(2k-1)!$ identical Lorentz index distributions. Therefore,

$$(\bar{u}_4 q^\alpha v_1^{\mu_1} \sigma_{\alpha\mu_1} u_3) v_{1\mu_2\dots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\cdots\mu_{2k}}$$
$$= (\bar{u}_4 q^\alpha v_1^{\mu_{2k}} \sigma_{\alpha\mu_{2k}} u_3) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \sum_{n=0}^{k-1} \frac{2(k-n)(2k-1)!}{(2k)!} \binom{k}{k-n} (-1)^{k-n} \omega^{2n}. \tag{6.117}$$

We have used the identity $\binom{k}{n} = \binom{k}{k-n}$. Inverting the order of the sum,

$$(\bar{u}_4 q^\alpha v_1^{\mu_1} \sigma_{\alpha\mu_1} u_3) v_{1\mu_2\dots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\cdots\mu_{2k}}$$
$$= (\bar{u}_4 q^\alpha v_1^{\mu_{2k}} \sigma_{\alpha\mu_{2k}} u_3) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \sum_{n=1}^{k} \frac{2n(2k-1)!}{(2k)!} \binom{k}{n} (-1)^n \left(\omega^2\right)^{k-n}$$
$$= (\bar{u}_4 q^\alpha v_1^{\mu_{2k}} \sigma_{\alpha\mu_{2k}} u_3) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \sum_{n=0}^{k-1} \binom{k-1}{n} (-1)^{n+1} \left(\omega^2\right)^{k-1-n}$$
$$= -(\bar{u}_4 q^\alpha v_1^{\mu_{2k}} \sigma_{\alpha\mu_{2k}} u_3) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} \left(\omega^2-1\right)^{k-1} q^{2k} \mathcal{I}_\triangleleft, \tag{6.118}$$

as claimed.

Finally, we move to the contraction of eq. (6.107). Equation (6.33) implies that the $n = k$ term in the sum vanishes, and that we must always have one metric contracted with the spin of particle 2. There are then clearly two types of contributions: those where the second index of that metric contracts with the spin of particle 1, and those where it contracts with a factor of v_1^μ:

$$(\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1} u_1)(\bar{u}_4 q^\beta \sigma_{\beta\mu_2} u_3) v_{1\mu_3\dots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\cdots\mu_{2k}}$$
$$= (\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1} u_1)(\bar{u}_4 q^\beta \sigma_{\beta\mu_2} u_3) v_{1\mu_3\dots\mu_{2k}} \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft$$
$$\times \left[\eta^{\mu_1\mu_2} \sum_{n=0}^{k-1} \frac{N_n}{(2k)!} \binom{k}{n} (-1)^{k-n} \eta^{\{\mu_3\mu_4,\dots,\mu_{2k-2n-1}\mu_{2k-2n}} v_2^{\mu_{2k-2n+1}\cdots\mu_{2k}\}} \right.$$
$$\left. + \eta^{\mu_2\mu_3} \sum_{n=0}^{k-1} \frac{P_n}{(2k)!} \binom{k}{n} (-1)^{k-n} \eta^{\{\mu_1\mu_4,\dots,\mu_{2k-2n-1}\mu_{2k-2n}} v_2^{\mu_{2k-2n+1}\cdots\mu_{2k}\}} \right]. \tag{6.119}$$

Our task is now to determine the integers N_n and P_n – which count the number of identical Lorentz index distributions that do not vanish when contracted with the spin structure – and to resum the sums. In the case of N_n, in a term with $k - n$ metrics, there are $k - n$ metrics that can be contracted with the spin of particle 2, two identical ways to distribute the indices on this metric, and $(2k - 2)!$ identical ways to distribute the remaining Lorentz indices for each of these metric permutations. Therefore, $N_n = 2(k-n)(2k-2)!$.

Now, in the term corresponding to P_n, eq. (6.32) implies that the $n=0$ term also vanishes, and that one factor of v_2^μ must be contracted with the spin of particle 1. For a term with $2n$ factors of v_2^μ, there are then $2n$ identical ways to contract a velocity with the spin. Moreover, such a term will have $k-n$ metrics that can be contracted with the spin of particle 2, each of which has two indices that can be contracted in this way. Finally, as eqs. (6.32) and (6.33) fix two indices, there are $(2k-2)!$ identical permutations of the remaining Lorentz indices. Therefore, $P_n = (2n)[2(k-n)](2k-2)!$.

Putting these together,

$$\begin{aligned}
&(\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1} u_1)(\bar{u}_4 q^\beta \sigma_{\beta\mu_2} u_3) v_{1\mu_3\ldots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k}} \\
&= (\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1} u_1)(\bar{u}_4 q^\beta \sigma_{\beta\mu_2} u_3) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \\
&\quad \times \left[\eta^{\mu_1\mu_2} \sum_{n=0}^{k-1} \frac{(k-n)}{k(2k-1)} \binom{k}{n} (-1)^{k-n} (\omega^2)^n \right. \\
&\qquad \left. + \eta^{\mu_2\mu_3} v_{1\mu_3} v_2^{\mu_1} \omega \sum_{n=0}^{k-2} \frac{2(n+1)(k-1-n)}{k(2k-1)} \binom{k}{n+1} (-1)^{k-1-n} (\omega^2)^n \right].
\end{aligned} \tag{6.120}$$

We recognize the first sum as the one in eq. (6.117) divided by $2k-1$. Focusing on the second sum, we apply the recursive identity for the binomial coefficients, invert the sum, multiply by $(k-1)/(k-1)$, and apply the recursive identity again to find

$$\begin{aligned}
&(\bar{u}_2 q^\alpha \sigma_{\alpha\mu_1} u_1)(\bar{u}_4 q^\beta \sigma_{\beta\mu_2} u_3) v_{1\mu_3\ldots\mu_{2k}} \mathcal{I}_\triangleleft^{\mu_1\ldots\mu_{2k}} \\
&= (\bar{u}_2 q^\alpha \sigma_{\alpha\mu} u_1)(\bar{u}_4 q^\beta \sigma_{\beta\nu} u_3) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \\
&\quad \times \left[-\frac{\eta^{\mu\nu}}{(2k-1)} (\omega^2-1)^{k-1} + v_1^\nu v_2^\mu \omega \sum_{n=0}^{k-2} \frac{2(k-1)}{(2k-1)} \binom{k-2}{n} (-1)^{n+1} (\omega^2)^{k-n-2} \right] \\
&= (\bar{u}_2 q^\alpha \sigma_{\alpha\mu} u_1)(\bar{u}_4 q^\beta \sigma_{\beta\nu} u_3) \frac{\left(\frac{1}{2}\right)_k}{4^k (1)_k} q^{2k} \mathcal{I}_\triangleleft \\
&\qquad \times \left[-\frac{\eta^{\mu\nu}}{(2k-1)} (\omega^2-1)^{k-1} - \frac{2(k-1)}{(2k-1)} v_1^\nu v_2^\mu \omega (\omega^2-1)^{k-2} \right].
\end{aligned} \tag{6.121}$$

This completes the proof.

All integrals in this section were checked explicitly up to rank $2k = 10$.

6.B INTEGRALS FOR CLASSICAL IMPULSES AND THE POTENTIAL

We give here details about the integrals needed to Fourier transform the potential in section 6.5 to position space, as well as the integrals used to derive the linear and angular impulses in sections 6.6 and 6.7.

6.B.1 Fourier transforms

To convert the momentum space tidal potential in section 6.5 to position space, we need knowledge of three-dimensional Fourier integrals up to rank 2:

$$J_j^{i_1\ldots i_k} = -\frac{1}{\sqrt{2}}\int \hat{d}^3\boldsymbol{q}\, e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+3/2}\boldsymbol{q}^{i_1\ldots i_k}, \quad k\leq 2. \tag{6.122}$$

First, for $k=0$,

$$J_j = 2\left(\frac{2}{r^2}\right)^{j+3}\frac{\Gamma(j+3)}{(4\pi)^{3/2}\Gamma(-j-3/2)}, \tag{6.123}$$

where $r\equiv|\boldsymbol{r}|$. In the rank 1 case,

$$J_j^i = i\frac{\partial}{\partial\boldsymbol{r}^i}J_j = -2i\boldsymbol{r}^i\left(\frac{2}{r^2}\right)^{j+4}\frac{\Gamma(j+4)}{(4\pi)^{3/2}\Gamma(-j-3/2)}. \tag{6.124}$$

Similarly, for rank 2,

$$J_j^{ik} = i\frac{\partial}{\partial\boldsymbol{r}^k}J_j^i = 2\left(\frac{2}{r^2}\right)^{j+4}\left[\delta^{ik}-2(j+4)\frac{\boldsymbol{r}^i\boldsymbol{r}^k}{r^2}\right]\frac{\Gamma(j+4)}{(4\pi)^{3/2}\Gamma(-j-3/2)}. \tag{6.125}$$

6.B.2 Linear and angular impulse

The calculation of the classical impulses in sections 6.6 and 6.7 needed the evaluation of integrals of the form

$$I_{j+2}^{\mu_1\ldots\mu_k}(b) = \int \hat{d}^4q\hat{\delta}(2p_1\cdot q)\hat{\delta}(2p_2\cdot q)e^{-ib\cdot q}q^{\mu_1\ldots\mu_k}S\left(-\frac{q^2}{2}\right)^{j+2}, \tag{6.126}$$

where $\hat{d}^nq = d^nq/(2\pi)^n$ and $\hat{\delta}(x) = 2\pi\delta(x)$. These integrals are similar to integrals evaluated in refs. [10, 14], and we will use the same steps here. We go to the rest frame of particle 1, where $v_1 = (1,0,0,0)$ and $v_2 = (\omega,0,0,\omega\beta)$, with β satisfying $\omega^2(1-\beta^2)=1$. After evaluating the delta functions, we find that

$$I_{j+2}^{\mu_1\ldots\mu_k}(\boldsymbol{b}) = \frac{1}{\mathcal{N}}\int d^2q e^{i\boldsymbol{b}\cdot\boldsymbol{q}}q^{\mu_1\ldots\mu_k}\frac{1}{|\boldsymbol{q}|}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}, \quad \text{where} \quad \mathcal{N}\equiv 16m_1m_2\sqrt{\omega^2-1}. \tag{6.127}$$

We will omit the argument of the integral from now on. We parametrize $\boldsymbol{q}$ to be in the 2-dimensional plane perpendicular to v_1 and v_2, $q^\mu = (0,\chi\cos\theta,\chi\sin\theta,0)$, such that the rank-0 integral becomes

$$\begin{aligned}I_{j+2} &= \frac{1}{\mathcal{N}}\int d^2q e^{i\boldsymbol{b}\cdot\boldsymbol{q}}\frac{1}{|\boldsymbol{q}|}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\\ &= \frac{1}{\mathcal{N}}\int_0^\infty d\chi\chi\int_{-\pi}^{\pi}d\theta e^{i|\boldsymbol{b}|\chi\cos\theta}\frac{1}{\chi}\left(\frac{\chi^2}{2}\right)^{j+2}\\ &= \frac{2\pi}{\mathcal{N}}\int_0^\infty d\chi J_0(|\boldsymbol{b}|\chi)\left(\frac{\chi^2}{2}\right)^{j+2}\end{aligned}$$

$$= \frac{\pi}{8m_1m_2\sqrt{\omega^2-1}}\frac{1}{|\boldsymbol{b}|}\left(-\frac{2}{b^2}\right)^{j+2}\frac{\Gamma[5/2+j]}{\Gamma[-3/2-j]}, \tag{6.128}$$

where in the last step we restored Lorentz invariance. The rank-1 integral is also straight-forward to evaluate;

$$\begin{aligned} I^{\mu}_{j+2} &= \frac{1}{\mathcal{N}}\int_0^\infty d\chi\chi\int_{-\pi}^{\pi} d\theta e^{i|\boldsymbol{b}|\chi\cos\theta} q^\mu \frac{1}{\chi}\left(\frac{\chi^2}{2}\right)^{j+2} \\ &= \frac{2\pi i}{\mathcal{N}}\int_0^\infty d\chi\chi J_1(|\boldsymbol{b}|\chi)\hat{\boldsymbol{b}}^\mu\left(\frac{\chi^2}{2}\right)^{j+2} \\ &= \frac{\pi i}{8m_1m_2\sqrt{\omega^2-1}}\frac{b^\mu}{|\boldsymbol{b}|}\left(-\frac{2}{b^2}\right)^{j+3}\frac{\Gamma[7/2+j]}{\Gamma[-3/2-j]}. \end{aligned} \tag{6.129}$$

Next, we have the rank-2 integral, which must take the form

$$I^{\mu\nu}_{j+2} = \alpha_2 b^\mu b^\mu + \beta_2\Pi^{\mu\nu}, \tag{6.130}$$

where

$$\Pi^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \frac{1}{\omega^2-1}\left(v_1^\mu(v_{1\nu}-\omega v_{2\nu}) + v_2^\mu(v_{2\nu}-\omega v_{1\nu})\right) \tag{6.131}$$

is the projector onto the 2-dimensional plane perpendicular to v_1 and v_2 [14]. By taking the trace of eq. (6.130), we have that

$$\begin{aligned} \alpha_2 b^2 + 2\beta_2 &= -\frac{1}{\mathcal{N}}\int_0^\infty d\chi\chi\int_{-\pi}^{\pi} d\theta e^{i|\boldsymbol{b}|\boldsymbol{q}\cos\theta}\chi\left(\frac{\chi^2}{2}\right)^{j+2} \\ &= -\frac{4\pi}{\mathcal{N}}\int_0^\infty d\chi J_0(|\boldsymbol{b}|\chi)\left(\frac{\chi^2}{2}\right)^{j+3} \\ &= -\frac{4\pi}{\mathcal{N}}\frac{1}{|\boldsymbol{b}|}\left(-\frac{2}{b^2}\right)^{j+3}\frac{\Gamma[7/2+j]}{\Gamma[-5/2-j]}. \end{aligned} \tag{6.132}$$

If we contract with $b_\mu b_\nu$ instead, we find that

$$\begin{aligned} \alpha_2 b^4 + \beta_2 b^2 &= -\frac{b^2}{\mathcal{N}}\int_0^\infty d\chi\chi^2\int_{-\pi}^{\pi} d\theta e^{i|\boldsymbol{b}|\chi\cos\theta}(\cos\theta)^2\left(\frac{\chi^2}{2}\right)^{j+2} \\ &= -\frac{4\pi b^2}{\mathcal{N}}\int_0^\infty d\chi\left(J_1(|\boldsymbol{b}|\chi)\frac{1}{|\boldsymbol{b}|\chi} - J_2(|\boldsymbol{b}|\chi)\right)\left(\frac{\chi^2}{2}\right)^{j+3} \\ &= \frac{4\pi b^2}{\mathcal{N}}\frac{1}{|\boldsymbol{b}|}\left(-\frac{2}{b^2}\right)^{j+3}\frac{(j+3)\Gamma[7/2+j]}{\Gamma[-3/2-j]}. \end{aligned} \tag{6.133}$$

Putting this together with eq. (6.132), we end up with

$$I^{\mu\nu}_{j+2} = \left((7+2j)\frac{b^\mu b^\nu}{b^2} - \Pi^{\mu\nu}\right)\frac{\pi}{8m_1m_2\sqrt{\omega^2-1}}\frac{1}{|\boldsymbol{b}|}\left(-\frac{2}{b^2}\right)^{j+3}\frac{\Gamma[7/2+j]}{\Gamma[-3/2-j]}. \tag{6.134}$$

Lastly, the rank-3 integral is

$$I^{\mu\nu\rho}_{j+2} = \alpha_3 b^\mu b^\nu b^\rho + \beta_3 b^{(\mu}\Pi^{\nu\rho)} \tag{6.135}$$

(with normalization $1/3!$ for the second term). Contracting with $b_\mu \eta_{\nu\rho}$, we have that

$$\begin{aligned}\alpha_3 b^4 + \beta_3 \frac{4}{3} b^2 &= \frac{1}{\mathcal{N}} \int_0^\infty d\chi \chi \int_{-\pi}^{\pi} d\theta e^{i|\boldsymbol{b}|\chi\cos\theta} |\boldsymbol{b}|\chi^2 \cos\theta \left(\frac{\chi^2}{2}\right)^{j+2} \\ &= \frac{4\pi i|\boldsymbol{b}|}{\mathcal{N}} \int_0^\infty d\chi J_1(|\boldsymbol{b}|\chi)\chi \left(\frac{\chi^2}{2}\right)^{j+3} \\ &= \frac{8\pi i}{\mathcal{N}} \frac{1}{|\boldsymbol{b}|} \left(\frac{2}{\boldsymbol{b}^2}\right)^{j+3} \frac{\Gamma[9/2+j]}{\Gamma[-5/2-j]}. \end{aligned} \tag{6.136}$$

When we contract with $b_\mu b_\nu b_\rho$, we find that

$$\begin{aligned}\alpha_3 b^6 + \beta b^4 &= -\frac{1}{\mathcal{N}} \int_0^\infty d\chi \chi \int_{-\pi}^{\pi} d\theta e^{i|\boldsymbol{b}|\chi\cos\theta} (|\boldsymbol{b}|\chi\cos\theta)^3 \frac{1}{\chi} \left(\frac{\chi^2}{2}\right)^{j+2} \\ &= -\frac{2\pi i}{\mathcal{N}} |\boldsymbol{b}|^3 \int_0^\infty d\chi \left(J_2(|\boldsymbol{b}|\chi)\frac{3}{|\boldsymbol{b}|\chi} - J_3(|\boldsymbol{b}\chi)\right) \chi^3 \left(\frac{\chi^2}{2}\right)^{j+2} \\ &= -\frac{8\pi i b^2}{\mathcal{N}} \frac{1}{|\boldsymbol{b}|} \left(-\frac{2}{b^2}\right)^{j+3} \frac{(j+3)\Gamma[9/2+j]}{\Gamma[-3/2-j]}. \end{aligned} \tag{6.137}$$

The integral becomes

$$I^{\mu\nu\rho}_{j+2} = \left((9/2+j)\frac{b^\mu b^\nu b^\rho}{b^2} - \frac{3}{2} b^{(\mu}\Pi^{\nu\rho)}\right) \frac{i\pi}{4m_1 m_2\sqrt{\omega^2-1}} \frac{1}{|\boldsymbol{b}|} \left(-\frac{2}{b^2}\right)^{j+4} \frac{\Gamma[9/2+j]}{\Gamma[-3/2-j]}. \tag{6.138}$$

We also need these integrals restricted to spatial indices. We can write the spatial portions of the integrals in as

$$I^{i_1\ldots i_n}_{j+2} = (-i)^n \frac{\partial^n}{\partial b^{i_1}\ldots\partial b^{i_n}} I_{j+2}. \tag{6.139}$$

Computing derivatives of the rank 0 integral gives

$$\begin{aligned} I^m_{j+2} &= \frac{\pi i}{8m_1 m_2\sqrt{\omega^2-1}} \frac{\boldsymbol{b}^m}{|\boldsymbol{b}|} \left(\frac{2}{\boldsymbol{b}^2}\right)^{j+3} \frac{\Gamma[7/2+j]}{\Gamma[-3/2-j]}, && (6.140) \\ I^{mn}_{j+2} &= \left(\Pi^{mn} - (7+2j)\frac{\boldsymbol{b}^m\boldsymbol{b}^n}{\boldsymbol{b}^2}\right) \frac{\pi}{8m_1 m_2\sqrt{\omega^2-1}} \frac{1}{|\boldsymbol{b}|} \left(\frac{2}{\boldsymbol{b}^2}\right)^{j+3} \frac{\Gamma[7/2+j]}{\Gamma[-3/2-j]}, && (6.141) \\ I^{mnl}_{j+2} &= \left(\frac{3}{2}\boldsymbol{b}^{(m}\Pi^{nl)} - (9/2+j)\frac{\boldsymbol{b}^m\boldsymbol{b}^n\boldsymbol{b}^l}{\boldsymbol{b}^2}\right) \frac{i\pi}{4m_1 m_2\sqrt{\omega^2-1}} \frac{1}{|\boldsymbol{b}|} \left(\frac{2}{\boldsymbol{b}^2}\right)^{j+4} \frac{\Gamma[9/2+j]}{\Gamma[-3/2-j]}. && (6.142) \end{aligned}$$

Moreover, note that the projector to the plane perpendicular to the velocities must be modified when restricted to purely spatial components:

$$\Pi^{mn} = \delta^{mn} - \frac{1}{\omega^2-1} \left[\boldsymbol{v}_1^m(\boldsymbol{v}_1^n - \omega\boldsymbol{v}_2^n) + \boldsymbol{v}_2^m(\boldsymbol{v}_2^n - \omega\boldsymbol{v}_1^n)\right]. \tag{6.143}$$

6.B.3 Eikonal operator

The eikonal phase was written in section 6.7 as the action of the operator $\hat{\mathcal{K}}_j$ on the integral I_{j+2}. We list here for reference the action of each term in this operator on this integral:

$$\hat{\mathcal{O}}^{(0)} I_{j+2} = I_{j+2}, \tag{6.144}$$

$$\hat{\mathcal{O}}^{(1,a)} I_{j+2} = -i\,(\boldsymbol{S}_a \times \boldsymbol{p})_m I^m_{j+2}, \tag{6.145}$$

$$\hat{\mathcal{O}}^{(2,1)} I_{j+2} = \boldsymbol{S}_{1m} \boldsymbol{S}_{2n} I^{mn}_{j+2}, \tag{6.146}$$

$$\hat{\mathcal{O}}^{(2,2)} I_{j+2} = (\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)\delta_{mn} I^{mn}_{j+2} = 2(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2) I_{j+3}, \tag{6.147}$$

$$\hat{\mathcal{O}}^{(2,3)} I_{j+2} = (\boldsymbol{p} \cdot \boldsymbol{S}_1)(\boldsymbol{p} \cdot \boldsymbol{S}_2)\delta_{mn} I^{mn}_{j+2} = 2(\boldsymbol{p} \cdot \boldsymbol{S}_1)(\boldsymbol{p} \cdot \boldsymbol{S}_2) I_{j+3}, \tag{6.148}$$

where m, n are spatial indices.

CHAPTER

7

SUMMARY AND OUTLOOK

The work in this thesis has been primarily motivated by an interest in better understanding the binary inspiral problem in GR. The approaches taken were grounded in scattering amplitude and effective field theory methods, and have benefited from and contributed to the rapidly-expanding field of study relating scattering amplitudes to classical physics.

The first three publications presented focused on HBET and some of its properties that make it eminently applicable to the binary inspiral problem. The first two of these properties have to do with the arrangement of operators in the HBET Lagrangian. The operators are organized in an expansion in $\hbar$, while also separating spin and spinless effects. Both of these points were exploited in the publication presented in Chapter 2 to compute – for the first time – the 2PM amplitude describing the scattering of two spin-1/2 objects. The observables derived from such an amplitude describe a black hole binary to linear order in the angular momentum of each black hole.

Next, in Chapter 3, we formulated new on-shell variables that encode the properties of heavy spinors. These variables allowed us to express HPETs in an on-shell language, bringing the above two desireable features of HBET into a more wieldy framework. An immediate consequence of this re-expression of HPETs was that this formulation could be easily extended to arbitrary spin. From the perspective of classical black holes, these variables trivialized the matching to the Kerr black hole stress-energy tensor, as well as to the three-point amplitude derived from the worldline action for a spinning point particle. Moreover, we were easily able to show that an entire class of tree-level amplitudes exhibits a spin-multipole universality, and that the spurious pole of the high-spin Compton amplitude is present even at leading order in $\hbar$.

The third study of HBET, in Chapter 4, involved understanding how the double copy relates HQET to HBET. Such a relation is obfuscated due to the fact that HQET and HBET do not describe canonically normalized matter states: their propagators are not of the form $i/(p^2 - m^2)$. Consequently, if one is interested in comparing the double copy of HQET to HBET amplitudes, they must ensure that the matter states after the double copy are equivalent to those from HBET, from the point of view of their WNFs. Controlling for this, we presented a prescription for double copying between the two theories for matter with spin ≤ 1. This also entailed presenting two novel Lagrangians for HQET and HBET with spin 1 matter.

The next two publications shifted focus away from HBET. While making use of some of the insights gained from HBET, these latter works employed other powerful EFT techniques to address the description of tidal effects through a quantum-field-theoretic lens. The first of these two tackled tidal effects for spinless particles, employing the Hilbert series to aid in writing a Lagrangian containing all possible operators with two real scalar fields and two Weyl tensors. Recognizing a pattern in the arbitrary-even-rank triangle integrals at one loop, we were able to express the full tidal dynamics at the leading PM order.

Having understood the scalar case, we turned our attention to the tidal deformation of a spin-1/2 object. Combining the Hilbert series with an on-shell amplitude basis, we constructed the action involving all operators with two spinors and two Weyl tensors. Similar patterns in the contractions of arbitrary-rank triangle integrals permitted the computation of the corresponding amplitude at one-loop order. Several classical quantities were derived from this amplitude.

These five works have yielded various novel results and opened new avenues from which to approach the binary inspiral problem, the relation between QFT and classical physics, and amplitudes in their own right. As such, there are several possible directions of exploration that build upon these works.

First, the on-shell HPET variables were formulated to facilitate the application of HBET to higher-spin scattering at higher orders in the loop expansion. While this can in principle already be done, referencing the relation between the on-shell HPET variables and the traditional on-shell variables to apply unitarity techniques, a more efficient and satisfying utilization of these variables would handle them as distinct objects unto themselves. With this in mind, an interesting first step in the continued study of the on-shell HPET variables is to better understand their behavior under application of recursion relations and unitarity cuts.

Next, a striking facet of HPETs is the immediacy with which the relation to classical physics is suggested. The loop integrals of HPET are precisely those that contain the non-analytic structures describing long-range physics. The external states arrange the external momenta in such a way as to produce instant agreement with a Kerr black hole's spin multipole expansion at three points. The leading order operator in the HBET action for a scalar particle is nearly identical to the PM worldline action of ref. [195] (not to mention the similarities between the analogous term in scalar HQET and the classical action in ref. [233]). This third point also extends to tidal effects, where the heavy limit of the tidal operators in ref. [173] takes the same form (up to a redefinition of Wilson coefficients) as the leading operators in the worldline action of ref. [169]. All these coincidences suggest that the heavy actions might be more intimately related to classical physics than expected. It would therefore be interesting to look more closely at such a connection, perhaps by applying similar methods to those in ref. [209] and investigating whether a wider class of classical actions can be obtained in a formulaic manner from HPET actions. Building on such a derivation of classical actions, an understanding of the double copy between HQET and HBET may then aid in connecting the classical and quantum double copies.

Extending the studies into tidal effects, the methodologies employed can be used to describe these effects for even higher spin particles. Additionally, the connection between the derived spin-1/2 action and the classical dynamics of tidally deformed objects deserves more attention. This action required the introduction of nine (infinite) families of Wilson coefficients. It is not necessarily the case, however, that all of these coefficients are needed to describe classical physics. Indeed, we already saw that two of these families always yield contributions at the leading-PM level that carry additional factors of $\hbar$. In the spinless case, this idea was addressed by matching the Wilson coefficients to a classical worldline action. This sort of matching of the spin-1/2 action would validate which coefficients are needed if one is only interested in classical physics. In a similar direction, the values of the Wilson coefficients for the spin-1/2 tidal action could be constrained using consistency conditions for scattering amplitudes.

To sum up, there remain myriad open questions on the topic of the binary inspiral and its connection to scattering amplitudes. The work presented here has taken and highlighted novel approaches to tackling some of these problems by introducing a new EFT and by applying powerful formalisms to these problems for the first time. It is my hope that the mentioned topics extending the work herein will soon be addressed, and that the perspectives in this thesis will help motivate further investigations both related to and separate from classical binary systems.

Bibliography

[1] B. P. Abbott et al., LIGO Scientific Collaboration and Virgo Collaboration, *Observation of gravitational waves from a binary black hole merger*, Phys. Rev. Lett. **116** (2016) 061102.

[2] Event Horizon Telescope collaboration, *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, Astrophys. J. Lett. **875** (2019) L1 [1906.11238].

[3] M. H. Goroff and A. Sagnotti, *Quantum gravity at two loops*, Physics Letters B **160** (1985) 81.

[4] M. H. Goroff and A. Sagnotti, *The ultraviolet behavior of einstein gravity*, Nuclear Physics B **266** (1986) 709.

[5] A. E. van de Ven, *Two-loop quantum gravity*, Nuclear Physics B **378** (1992) 309.

[6] G. 't Hooft and M. J. G. Veltman, *One loop divergencies in the theory of gravitation*, *Ann. Inst. H. Poincare Phys. Theor. A* **20** (1974) 69.

[7] J. F. Donoghue, *Leading quantum correction to the Newtonian potential*, Phys. Rev. Lett. **72** (1994) 2996 [gr-qc/9310024].

[8] J. F. Donoghue, *General relativity as an effective field theory: The leading quantum corrections*, Phys. Rev. **D50** (1994) 3874 [gr-qc/9405057].

[9] B. R. Holstein and J. F. Donoghue, *Classical physics and quantum loops*, Phys. Rev. Lett. **93** (2004) 201602 [hep-th/0405239].

[10] D. A. Kosower, B. Maybee and D. O'Connell, *Amplitudes, Observables, and Classical Scattering*, JHEP **02** (2019) 137 [1811.10950].

[11] J. M. Antelis and C. Moreno, *Obtaining gravitational waves from inspiral binary systems using LIGO data*, Eur. Phys. J. Plus **132** (2017) 10 [1610.03567].

[12] P. Schmidt, *Gravitational waves from binary black hole mergers: Modeling and observations*, Frontiers in Astronomy and Space Sciences **7** (2020) 28.

[13] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, International Series In Pure and Applied Physics. McGraw-Hill, New York, 1980.

[14] B. Maybee, D. O'Connell and J. Vines, *Observables and amplitudes for spinning particles and black holes*, JHEP **12** (2019) 156 [1906.09260].

[15] D. Neill and I. Z. Rothstein, *Classical Space-Times from the S Matrix*, *Nucl. Phys.* **B877** (2013) 177 [1304.7263].

[16] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, *Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order*, *Phys. Rev. Lett.* **122** (2019) 201603 [1901.04424].

[17] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, *Black Hole Binary Dynamics from the Double Copy and Effective Theory*, *JHEP* **10** (2019) 206 [1908.01493].

[18] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, *Universality of ultra-relativistic gravitational scattering*, *Phys. Lett. B* **811** (2020) 135924 [2008.12743].

[19] T. Damour, *Radiative contribution to classical gravitational scattering at the third order in G*, *Phys. Rev. D* **102** (2020) 124008 [2010.01641].

[20] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, *The Eikonal Approach to Gravitational Scattering and Radiation at $\mathcal{O}(G^3)$*, 2104.03256.

[21] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, *Radiation Reaction from Soft Theorems*, *Phys. Lett. B* **818** (2021) 136379 [2101.05772].

[22] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté and P. Vanhove, *The Amplitude for Classical Gravitational Scattering at Third Post-Minkowskian Order*, 2105.05218.

[23] B. R. Holstein and A. Ross, *Spin Effects in Long Range Gravitational Scattering*, 0802.0716.

[24] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon et al., *Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$*, *Phys. Rev. Lett.* **126** (2021) 171601 [2101.07254].

[25] H. Georgi, *An Effective Field Theory for Heavy Quarks at Low-energies*, *Phys. Lett. B* **240** (1990) 447.

[26] G. T. Bodwin, E. Braaten and G. P. Lepage, *Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium*, *Phys. Rev.* **D51** (1995) 1125 [hep-ph/9407339].

[27] M. Neubert, *Heavy quark effective theory*, *Subnucl. Ser.* **34** (1997) 98 [hep-ph/9610266].

[28] A. V. Manohar, *The HQET / NRQCD Lagrangian to order alpha / m^{-3}*, *Phys. Rev.* **D56** (1997) 230 [hep-ph/9701294].

[29] Z. Bern, A. Luna, R. Roiban, C.-H. Shen and M. Zeng, *Spinning Black Hole Binary Dynamics, Scattering Amplitudes and Effective Field Theory*, 2005.03071.

[30] A. Buonanno and T. Damour, *Effective one-body approach to general relativistic two-body dynamics*, *Phys. Rev.* **D59** (1999) 084006 [gr-qc/9811091].

[31] T. Damour, *Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory*, *Phys. Rev.* **D94** (2016) 104015 [1609.00354].

[32] T. Damour, *High-energy gravitational scattering and the general relativistic two-body problem*, *Phys. Rev.* **D97** (2018) 044038 [1710.10599].

[33] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz and J. van Meter, *Gravitational wave extraction from an inspiraling configuration of merging black holes*, *Phys. Rev. Lett.* **96** (2006) 111102 [gr-qc/0511103].

[34] M. Campanelli, C. O. Lousto, P. Marronetti and Y. Zlochower, *Accurate evolutions of orbiting black-hole binaries without excision*, *Phys. Rev. Lett.* **96** (2006) 111101 [gr-qc/0511048].

[35] F. Pretorius, *Evolution of binary black hole spacetimes*, *Phys. Rev. Lett.* **95** (2005) 121101 [gr-qc/0507014].

[36] W. D. Goldberger and I. Z. Rothstein, *An Effective field theory of gravity for extended objects*, *Phys. Rev.* **D73** (2006) 104029 [hep-th/0409156].

[37] R. A. Porto, *Post-Newtonian corrections to the motion of spinning bodies in NRGR*, *Phys. Rev.* **D73** (2006) 104031 [gr-qc/0511061].

[38] R. A. Porto, *The effective field theorist's approach to gravitational dynamics*, *Phys. Rept.* **633** (2016) 1 [1601.04914].

[39] M. Levi, *Effective Field Theories of Post-Newtonian Gravity: A comprehensive review*, 1807.01699.

[40] N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, *Quantum gravitational corrections to the nonrelativistic scattering potential of two masses*, *Phys. Rev.* **D67** (2003) 084033 [hep-th/0211072].

[41] N. E. J. Bjerrum-Bohr, J. F. Donoghue and P. Vanhove, *On-shell Techniques and Universal Results in Quantum Gravity*, *JHEP* **02** (2014) 111 [1309.0804].

[42] F. Cachazo and A. Guevara, *Leading Singularities and Classical Gravitational Scattering*, *JHEP* **02** (2020) 181 [1705.10262].

[43] A. Guevara, *Holomorphic Classical Limit for Spin Effects in Gravitational and Electromagnetic Scattering*, *JHEP* **04** (2019) 033 [1706.02314].

[44] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté and P. Vanhove, *General Relativity from Scattering Amplitudes*, *Phys. Rev. Lett.* **121** (2018) 171601 [1806.04920].

[45] A. Guevara, A. Ochirov and J. Vines, *Scattering of Spinning Black Holes from Exponentiated Soft Factors*, *JHEP* **09** (2019) 056 [1812.06895].

[46] M.-Z. Chung, Y.-T. Huang, J.-W. Kim and S. Lee, *The simplest massive S-matrix: from minimal coupling to Black Holes*, *JHEP* **04** (2019) 156 [1812.08752].

[47] A. Guevara, A. Ochirov and J. Vines, *Black-hole scattering with general spin directions from minimal-coupling amplitudes*, *Phys. Rev. D* **100** (2019) 104024 [1906.10071].

[48] C. Cheung, I. Z. Rothstein and M. P. Solon, *From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion*, *Phys. Rev. Lett.* **121** (2018) 251101 [1808.02489].

[49] A. Cristofoli, N. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, *Post-Minkowskian Hamiltonians in general relativity*, *Phys. Rev. D* **100** (2019) 084040 [1906.01579].

[50] G. Kälin and R. A. Porto, *From Boundary Data to Bound States*, *JHEP* **01** (2020) 072 [1910.03008].

[51] C. Dlapa, G. Kälin, Z. Liu and R. A. Porto, *Dynamics of Binary Systems to Fourth Post-Minkowskian Order from the Effective Field Theory Approach*, 2106.08276.

[52] B. R. Holstein and A. Ross, *Spin Effects in Long Range Electromagnetic Scattering*, 0802.0715.

[53] M.-Z. Chung, Y.-T. Huang and J.-W. Kim, *Classical potential for general spinning bodies*, *JHEP* **09** (2020) 074 [1908.08463].

[54] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014.

[55] A. H. Guth, M. P. Hertzberg and C. Prescod-Weinstein, *Do Dark Matter Axions Form a Condensate with Long-Range Correlation?*, *Phys. Rev.* **D92** (2015) 103513 [1412.5930].

[56] E. Braaten, A. Mohapatra and H. Zhang, *Classical Nonrelativistic Effective Field Theories for a Real Scalar Field*, *Phys. Rev.* **D98** (2018) 096012 [1806.01898].

[57] M. H. Namjoo, A. H. Guth and D. I. Kaiser, *Relativistic Corrections to Nonrelativistic Effective Field Theories*, 1712.00445.

[58] J. Yepez, *Einstein's vierbein field theory of curved space*, 1106.2037.

[59] M. E. Luke and A. V. Manohar, *Reparametrization invariance constraints on heavy particle effective field theories*, *Phys. Lett.* **B286** (1992) 348 [hep-ph/9205228].

[60] Y.-Q. Chen, *On the reparameterization invariance in heavy quark effective theory*, *Physics Letters B* **317** (1993) 421 .

[61] M. Finkemeier, H. Georgi and M. McIrvin, *Reparametrization invariance revisited*, *Phys. Rev.* **D55** (1997) 6933 [hep-ph/9701243].

[62] D. Bini and T. Damour, *Gravitational spin-orbit coupling in binary systems at the second post-Minkowskian approximation*, *Phys. Rev.* **D98** (2018) 044036 [1805.10809].

[63] N. Arkani-Hamed, T.-C. Huang and Y.-t. Huang, *Scattering Amplitudes For All Masses and Spins*, 1709.04891.

[64] A. V. Manohar and M. B. Wise, *Heavy quark physics*, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **10** (2000) 1.

[65] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 1. The boson case*, *Phys. Rev.* **D9** (1974) 898.

[66] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 2. The fermion case*, *Phys. Rev.* **D9** (1974) 910.

[67] E. Conde and A. Marzolla, *Lorentz Constraints on Massive Three-Point Amplitudes*, *JHEP* **09** (2016) 041 [1601.08113].

[68] E. Conde, E. Joung and K. Mkrtchyan, *Spinor-Helicity Three-Point Amplitudes from Local Cubic Interactions*, *JHEP* **08** (2016) 040 [1605.07402].

[69] S. J. Parke and T. R. Taylor, *Amplitude for n-gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[70] F. Berends and W. Giele, *Recursive calculations for processes with n gluons*, *Nuclear Physics B* **306** (1988) 759.

[71] P. De Causmaecker, R. Gastmans, W. Troost and Tai Tsun Wu, *Helicity amplitudes for massless qed*, *Physics Letters B* **105** (1981) 215.

[72] Z. Xu, D.-H. Zhang and L. Chang, *Helicity Amplitudes for Multiple Bremsstrahlung in Massless Nonabelian Gauge Theories*, *Nucl. Phys. B* **291** (1987) 392.

[73] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, 1308.1697.

[74] L. J. Dixon, *A brief introduction to modern amplitude methods*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Particle Physics: The Higgs Boson and Beyond*, 10, 2013, DOI [1310.5353].

[75] E. P. Wigner, *On Unitary Representations of the Inhomogeneous Lorentz Group*, *Annals Math.* **40** (1939) 149.

[76] V. Bargmann and E. P. Wigner, *Group Theoretical Discussion of Relativistic Wave Equations*, *Proc. Nat. Acad. Sci.* **34** (1948) 211.

[77] S. Weinberg, *RELATIVISTIC QUANTUM MECHANICS*, vol. 1, p. 49–106. Cambridge University Press, 1995. 10.1017/CBO9781139644167.004.

[78] C. Lorcé, *New explicit expressions for Dirac bilinears*, *Phys. Rev. D* **97** (2018) 016005 [1705.08370].

[79] V. Vaidya, *Gravitational spin Hamiltonians from the S matrix*, *Phys. Rev.* **D91** (2015) 024017 [1410.5348].

[80] M. Levi and J. Steinhoff, *Spinning gravitating objects in the effective field theory in the post-Newtonian scheme*, *JHEP* **09** (2015) 219 [1501.04956].

[81] S. Foffa and R. Sturani, *Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach I: Regularized Lagrangian*, *Phys. Rev.* **D100** (2019) 024047 [1903.05113].

[82] S. Foffa, R. A. Porto, I. Rothstein and R. Sturani, *Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach II: Renormalized Lagrangian*, Phys. Rev. **D100** (2019) 024048 [1903.05118].

[83] J. Blümlein, A. Maier and P. Marquard, *Five-Loop Static Contribution to the Gravitational Interaction Potential of Two Point Masses*, Phys. Lett. **B800** (2020) 135100 [1902.11180].

[84] S. Foffa, P. Mastrolia, R. Sturani, C. Sturm and W. J. Torres Bobadilla, *Static two-body potential at fifth post-Newtonian order*, Phys. Rev. Lett. **122** (2019) 241605 [1902.10571].

[85] M. Levi and J. Steinhoff, *Next-to-next-to-leading order gravitational spin-squared potential via the effective field theory for spinning objects in the post-Newtonian scheme*, JCAP **1601** (2016) 008 [1506.05794].

[86] M. Levi, S. Mougiakakos and M. Vieira, *Gravitational cubic-in-spin interaction at the next-to-leading post-Newtonian order*, JHEP **01** (2021) 036 [1912.06276].

[87] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté and P. Vanhove, *General Relativity from Scattering Amplitudes*, Phys. Rev. Lett. **121** (2018) 171601 [1806.04920].

[88] A. Brandhuber and G. Travaglini, *On higher-derivative effects on the gravitational potential and particle bending*, JHEP **01** (2020) 010 [1905.05657].

[89] A. Cristofoli, *Post-Minkowskian Hamiltonians in Modified Theories of Gravity*, Phys. Lett. B **800** (2020) 135095 [1906.05209].

[90] M. Accettulli Huber, A. Brandhuber, S. De Angelis and G. Travaglini, *Note on the absence of R^2 corrections to Newton's potential*, Phys. Rev. D **101** (2020) 046011 [1911.10108].

[91] N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, *Post-Minkowskian Scattering Angle in Einstein Gravity*, JHEP **08** (2020) 038 [1910.09366].

[92] P. H. Damgaard, K. Haddad and A. Helset, *Heavy Black Hole Effective Theory*, JHEP **11** (2019) 070 [1908.10308].

[93] B. R. Holstein, *How Large is the 'natural' magnetic moment?*, Am. J. Phys. **74** (2006) 1104 [hep-ph/0607187].

[94] S. Ferrara, M. Porrati and V. L. Telegdi, *$g = 2$ as the natural value of the tree-level gyromagnetic ratio of elementary particles*, Phys. Rev. D **46** (1992) 3529.

[95] N. Arkani-Hamed, Y.-t. Huang and D. O'Connell, *Kerr black holes as elementary particles*, JHEP **01** (2020) 046 [1906.10100].

[96] W. Tulczyjew, *Motion of multipole particles in general relativity theory*, Acta Phys. Pol **18** (1959) 393.

[97] J. Vines, *Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings*, Class. Quant. Grav. **35** (2018) 084002 [1709.06016].

[98] M.-Z. Chung, Y.-T. Huang and J.-W. Kim, *Kerr-Newman stress-tensor from minimal coupling*, *JHEP* **12** (2020) 103 [1911.12775].

[99] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B715** (2005) 499 [hep-th/0412308].

[100] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052].

[101] R. Aoude and C. S. Machado, *The Rise of SMEFT On-shell Amplitudes*, *JHEP* **12** (2019) 058 [1905.11433].

[102] S. Y. Choi, J. S. Shim and H. S. Song, *Factorization and polarization in linearized gravity*, *Phys. Rev.* **D51** (1995) 2751 [hep-th/9411092].

[103] Z. Bern, *Perturbative quantum gravity and its relation to gauge theory*, *Living Rev. Rel.* **5** (2002) 5 [gr-qc/0206071].

[104] Y. F. Bautista and A. Guevara, *From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems*, 1903.12419.

[105] R. Kleiss and W. Stirling, *Spinor techniques for calculating* $pp \to w^{\pm}/z^{0}+$ *jets*, *Nuclear Physics B* **262** (1985) 235 .

[106] D. A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, *Phys. Rev.* **D71** (2005) 045007 [hep-th/0406175].

[107] G. Durieux, T. Kitahara, Y. Shadmi and Y. Weiss, *The electroweak effective field theory from on-shell amplitudes*, *JHEP* **01** (2020) 119 [1909.10551].

[108] R. E. Behrends and C. Fronsdal, *Fermi decay of higher spin particles*, *Phys. Rev.* **106** (1957) 345.

[109] H. T. Williams, *Misconceptions regarding spin 3/2*, *Phys. Rev. C* **31** (1985) 2297.

[110] K. Hinterbichler, *Theoretical Aspects of Massive Gravity*, *Rev. Mod. Phys.* **84** (2012) 671 [1105.3735].

[111] J. Heinonen, R. J. Hill and M. P. Solon, *Lorentz invariance in heavy particle effective theories*, *Phys. Rev.* **D86** (2012) 094020 [1208.0601].

[112] A. V. Manohar, *Introduction to Effective Field Theories*, in *Les Houches summer school: EFT in Particle Physics and Cosmology Les Houches, Chamonix Valley, France, July 3-28, 2017*, 2018, 1804.05863.

[113] A. V. Manohar, *The HQET / NRQCD Lagrangian to order alpha / m-3*, *Phys. Rev.* **D56** (1997) 230 [hep-ph/9701294].

[114] A. V. Manohar, *Introduction to Effective Field Theories*, *ArXiv e-prints* (2018) [1804.05863].

[115] H. Kawai, D. Lewellen and S. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1.

[116] Z. Bern, J. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [0805.3993].

[117] Z. Bern, J. J. M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [1004.0476].

[118] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, *Gravity and Yang-Mills Amplitude Relations*, *Phys. Rev. D* **82** (2010) 107702 [1005.4367].

[119] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, *New Identities among Gauge Theory Amplitudes*, *Phys. Lett. B* **691** (2010) 268 [1006.3214].

[120] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, *Proof of Gravity and Yang-Mills Amplitude Relations*, *JHEP* **09** (2010) 067 [1007.3111].

[121] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [1010.3933].

[122] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, 1909.01358.

[123] F. Cachazo, S. He and E. Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, *JHEP* **07** (2014) 033 [1309.0885].

[124] A. Luna, I. Nicholson, D. O'Connell and C. D. White, *Inelastic Black Hole Scattering from Charged Scalar Amplitudes*, *JHEP* **03** (2018) 044 [1711.03901].

[125] Y. F. Bautista and A. Guevara, *On the Double Copy for Spinning Matter*, 1908.11349.

[126] L. A. Johnson, C. R. T. Jones and S. Paranjape, *Constraints on a Massive Double-Copy and Applications to Massive Gravity*, *JHEP* **02** (2021) 148 [2004.12948].

[127] H. Johansson and A. Ochirov, *Pure Gravities via Color-Kinematics Duality for Fundamental Matter*, *JHEP* **11** (2015) 046 [1407.4772].

[128] H. Johansson and A. Ochirov, *Color-Kinematics Duality for QCD Amplitudes*, *JHEP* **01** (2016) 170 [1507.00332].

[129] H. Johansson and A. Ochirov, *Double copy for massive quantum particles with spin*, *JHEP* **09** (2019) 040 [1906.12292].

[130] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995.

[131] H. Lehmann, K. Symanzik and W. Zimmermann, *On the formulation of quantized field theories*, *Nuovo Cim.* **1** (1955) 205.

[132] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev. D* **82** (2010) 065003 [1004.0693].

[133] Z. Bern, A. De Freitas and H. Wong, *On the coupling of gravitons to matter*, Phys. Rev. Lett. **84** (2000) 3531 [hep-th/9912033].

[134] J. Broedel and L. J. Dixon, *Color-kinematics duality and double-copy construction for amplitudes from higher-dimension operators*, JHEP **10** (2012) 091 [1208.0876].

[135] G. Chen and Y.-J. Du, *Amplitude Relations in Non-linear Sigma Model*, JHEP **01** (2014) 061 [1311.1133].

[136] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity*, JHEP **01** (2015) 081 [1408.0764].

[137] F. Cachazo, S. He and E. Y. Yuan, *Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM*, JHEP **07** (2015) 149 [1412.3479].

[138] G. Chen, Y.-J. Du, S. Li and H. Liu, *Note on off-shell relations in nonlinear sigma model*, JHEP **03** (2015) 156 [1412.3722].

[139] Y.-J. Du and C.-H. Fu, *Explicit BCJ numerators of nonlinear simga model*, JHEP **09** (2016) 174 [1606.05846].

[140] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *Spontaneously Broken Yang-Mills-Einstein Supergravities as Double Copies*, JHEP **06** (2017) 064 [1511.01740].

[141] E. Casali, Y. Geyer, L. Mason, R. Monteiro and K. A. Roehrig, *New Ambitwistor String Theories*, JHEP **11** (2015) 038 [1506.08771].

[142] S. Stieberger and T. R. Taylor, *New relations for Einstein–Yang–Mills amplitudes*, Nucl. Phys. B **913** (2016) 151 [1606.09616].

[143] O. Schlotterer, *Amplitude relations in heterotic string theory and Einstein-Yang-Mills*, JHEP **11** (2016) 074 [1608.00130].

[144] D. Nandan, J. Plefka, O. Schlotterer and C. Wen, *Einstein-Yang-Mills from pure Yang-Mills amplitudes*, JHEP **10** (2016) 070 [1607.05701].

[145] C. Cheung, G. N. Remmen, C.-H. Shen and C. Wen, *Pions as Gluons in Higher Dimensions*, JHEP **04** (2018) 129 [1709.04932].

[146] E. Bergshoeff and P. Townsend, *Super D-branes*, Nucl. Phys. B **490** (1997) 145 [hep-th/9611173].

[147] E. Bergshoeff, R. Kallosh, T. Ortin and G. Papadopoulos, *Kappa symmetry, supersymmetry and intersecting branes*, Nucl. Phys. B **502** (1997) 149 [hep-th/9705040].

[148] R. Kallosh, *Volkov-Akulov theory and D-branes*, Lect. Notes Phys. **509** (1998) 49 [hep-th/9705118].

[149] E. Bergshoeff, F. Coomans, R. Kallosh, C. Shahbazi and A. Van Proeyen, *Dirac-Born-Infeld-Volkov-Akulov and Deformation of Supersymmetry*, JHEP **08** (2013) 100 [1303.5662].

[150] F. Cachazo, P. Cha and S. Mizera, *Extensions of Theories from Soft Limits*, *JHEP* **06** (2016) 170 [1604.03893].

[151] S. He and O. Schlotterer, *New Relations for Gauge-Theory and Gravity Amplitudes at Loop Level*, *Phys. Rev. Lett.* **118** (2017) 161601 [1612.00417].

[152] H. Elvang, M. Hadjiantonis, C. R. Jones and S. Paranjape, *Soft Bootstrap and Supersymmetry*, *JHEP* **01** (2019) 195 [1806.06079].

[153] C. Cheung and C.-H. Shen, *Symmetry for Flavor-Kinematics Duality from an Action*, *Phys. Rev. Lett.* **118** (2017) 121601 [1612.00868].

[154] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *Explicit Formulae for Yang-Mills-Einstein Amplitudes from the Double Copy*, *JHEP* **07** (2017) 002 [1703.00421].

[155] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *Gauged Supergravities and Spontaneous Supersymmetry Breaking from the Double Copy Construction*, *Phys. Rev. Lett.* **120** (2018) 171601 [1710.08796].

[156] F. Teng and B. Feng, *Expanding Einstein-Yang-Mills by Yang-Mills in CHY frame*, *JHEP* **05** (2017) 075 [1703.01269].

[157] Y.-J. Du, B. Feng and F. Teng, *Expansion of All Multitrace Tree Level EYM Amplitudes*, *JHEP* **12** (2017) 038 [1708.04514].

[158] C. Cheung, C.-H. Shen and C. Wen, *Unifying Relations for Scattering Amplitudes*, *JHEP* **02** (2018) 095 [1705.03025].

[159] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *Non-Abelian gauged supergravities as double copies*, *JHEP* **06** (2019) 099 [1812.10434].

[160] L. de la Cruz, A. Kniss and S. Weinzierl, *Proof of the fundamental BCJ relations for QCD amplitudes*, *JHEP* **09** (2015) 197 [1508.01432].

[161] R. W. Brown and S. G. Naculich, *KLT-type relations for QCD and bicolor amplitudes from color-factor symmetry*, *JHEP* **03** (2018) 057 [1802.01620].

[162] J. Plefka and W. Wormsbecher, *New relations for graviton-matter amplitudes*, *Phys. Rev. D* **98** (2018) 026011 [1804.09651].

[163] J. Plefka, C. Shi and T. Wang, *Double copy of massive scalar QCD*, *Phys. Rev. D* **101** (2020) 066004 [1911.06785].

[164] C.-H. Shen, *Gravitational Radiation from Color-Kinematics Duality*, *JHEP* **11** (2018) 162 [1806.07388].

[165] R. Aoude, K. Haddad and A. Helset, *On-shell heavy particle effective theories*, *JHEP* **05** (2020) 051 [2001.09164].

[166] C. D. Carone, *Superflavor symmetry for heavy vector mesons and heavy quarks*, *Phys. Lett. B* **253** (1991) 408.

[167] B. Kol and M. Smolkin, *Black hole stereotyping: Induced gravito-static polarization*, *JHEP* **02** (2012) 010 [1110.3764].

[168] Q. Henry, G. Faye and L. Blanchet, *Tidal effects in the equations of motion of compact binary systems to next-to-next-to-leading post-Newtonian order*, *Phys. Rev. D* **101** (2020) 064047 [1912.01920].

[169] T. Damour and A. Nagar, *Effective One Body description of tidal effects in inspiralling compact binaries*, *Phys. Rev. D* **81** (2010) 084016 [0911.5041].

[170] D. Bini, T. Damour and A. Geralico, *Scattering of tidally interacting bodies in post-Minkowskian gravity*, *Phys. Rev. D* **101** (2020) 044039 [2001.00352].

[171] T. Damour, M. Soffel and C.-m. Xu, *General relativistic celestial mechanics. 1. Method and definition of reference systems*, *Phys. Rev. D* **43** (1991) 3273.

[172] T. Damour and G. Esposito-Farese, *Testing gravity to second postNewtonian order: A Field theory approach*, *Phys. Rev. D* **53** (1996) 5541 [gr-qc/9506063].

[173] C. Cheung and M. P. Solon, *Tidal Effects in the Post-Minkowskian Expansion*, *Phys. Rev. Lett.* **125** (2020) 191601 [2006.06665].

[174] L. Lehman and A. Martin, *Hilbert Series for Constructing Lagrangians: expanding the phenomenologist's toolbox*, *Phys. Rev. D* **91** (2015) 105014 [1503.07537].

[175] B. Henning, X. Lu, T. Melia and H. Murayama, *Operator bases, S-matrices, and their partition functions*, *JHEP* **10** (2017) 199 [1706.08520].

[176] L. Lehman and A. Martin, *Low-derivative operators of the Standard Model effective field theory via Hilbert series methods*, *JHEP* **02** (2016) 081 [1510.00372].

[177] B. Henning, X. Lu, T. Melia and H. Murayama, *Hilbert series and operator bases with derivatives in effective field theories*, *Commun. Math. Phys.* **347** (2016) 363 [1507.07240].

[178] B. Henning, X. Lu, T. Melia and H. Murayama, *2, 84, 30, 993, 560, 15456, 11962, 261485, ...: Higher dimension operators in the SM EFT*, *JHEP* **08** (2017) 016 [1512.03433].

[179] M. Ruhdorfer, J. Serra and A. Weiler, *Effective Field Theory of Gravity to All Orders*, *JHEP* **05** (2020) 083 [1908.08050].

[180] S. Willenbrock, *Symmetries of the standard model*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics in $D \geqq 4$*, 10, 2004, hep-ph/0410370.

[181] B. S. DeWitt, *Quantum theory of gravity. ii. the manifestly covariant theory*, *Phys. Rev.* **162** (1967) 1195.

[182] Y. Iwasaki, *Quantum Theory of Gravitation vs. Classical Theory*): Fourth-Order Potential*, *Progress of Theoretical Physics* **46** (1971) 1587 [https://academic.oup.com/ptp/article-pdf/46/5/1587/5271183/46-5-1587.pdf].

[183] N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, *Quantum corrections to the Schwarzschild and Kerr metrics*, *Phys. Rev. D* **68** (2003) 084005 [hep-th/0211071].

[184] B. Bertotti, *On gravitational motion*, *Il Nuovo Cimento* (1956) .

[185] B. Bertotti and J. Plebanski, *Theory of gravitational perturbations in the fast motion approximation*, *Annals of Physics* **11** (1960) 169 .

[186] G. Kälin and R. A. Porto, *From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist)*, *JHEP* **02** (2020) 120 [1911.09130].

[187] A. Cristofoli, P. H. Damgaard, P. Di Vecchia and C. Heissenberg, *Second-order Post-Minkowskian scattering in arbitrary dimensions*, *JHEP* **07** (2020) 122 [2003.10274].

[188] A. Cristofoli, *Gravitational shock waves and scattering amplitudes*, *JHEP* **11** (2020) 160 [2006.08283].

[189] C. Cheung and M. P. Solon, *Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams*, *JHEP* **06** (2020) 144 [2003.08351].

[190] Z. Bern, H. Ita, J. Parra-Martinez and M. S. Ruf, *Universality in the classical limit of massless gravitational scattering*, *Phys. Rev. Lett.* **125** (2020) 031601 [2002.02459].

[191] M.-Z. Chung, Y.-t. Huang, J.-W. Kim and S. Lee, *Complete Hamiltonian for spinning binary systems at first post-Minkowskian order*, *JHEP* **05** (2020) 105 [2003.06600].

[192] R. Aoude, M.-Z. Chung, Y.-t. Huang, C. S. Machado and M.-K. Tam, *Silence of Binary Kerr Black Holes*, *Phys. Rev. Lett.* **125** (2020) 181602 [2007.09486].

[193] T. Damour, M. Soffel and C.-m. Xu, *General relativistic celestial mechanics. 3. Rotational equations of motion*, *Phys. Rev. D* **47** (1993) 3124.

[194] D. Bini, T. Damour and G. Faye, *Effective action approach to higher-order relativistic tidal interactions in binary systems and their effective one body description*, *Phys. Rev. D* **85** (2012) 124034 [1202.3565].

[195] G. Kälin and R. A. Porto, *Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics*, *JHEP* **11** (2020) 106 [2006.01184].

[196] S. Benvenuti, B. Feng, A. Hanany and Y.-H. He, *Counting BPS Operators in Gauge Theories: Quivers, Syzygies and Plethystics*, *JHEP* **11** (2007) 050 [hep-th/0608050].

[197] B. Feng, A. Hanany and Y.-H. He, *Counting gauge invariants: The Plethystic program*, *JHEP* **03** (2007) 090 [hep-th/0701063].

[198] E. E. Jenkins and A. V. Manohar, *Algebraic Structure of Lepton and Quark Flavor Invariants and CP Violation*, *JHEP* **10** (2009) 094 [0907.4763].

[199] Z. Bern, J. Parra-Martinez, R. Roiban, E. Sawyer and C.-H. Shen, *Leading Nonlinear Tidal Effects and Scattering Amplitudes*, *JHEP* **05** (2021) 188 [2010.08559].

[200] N. J. Poplawski, *Classical Physics: Spacetime and Fields*, 0911.0334.

[201] M. H. Namjoo, A. H. Guth and D. I. Kaiser, *Relativistic Corrections to Nonrelativistic Effective Field Theories*, Phys. Rev. D **98** (2018) 016011 [1712.00445].

[202] K. Haddad and A. Helset, *The double copy for heavy particles*, Phys. Rev. Lett. **125** (2020) 181603 [2005.13897].

[203] G. Passarino and M. Veltman, *One-loop corrections for $e+e-$ annihilation into $\mu+\mu-$ in the weinberg model*, Nuclear Physics B **160** (1979) 151 .

[204] A. Barabanschikov, L. Grant, L. L. Huang and S. Raju, *The Spectrum of Yang Mills on a sphere*, JHEP **01** (2006) 160 [hep-th/0501063].

[205] C. Hays, A. Martin, V. Sanz and J. Setford, *On the impact of dimension-eight SMEFT operators on Higgs measurements*, JHEP **02** (2019) 123 [1808.00442].

[206] Y. Shadmi and Y. Weiss, *Effective Field Theory Amplitudes the On-Shell Way: Scalar and Vector Couplings to Gluons*, JHEP **02** (2019) 165 [1809.09644].

[207] G. Durieux and C. S. Machado, *Enumerating higher-dimensional operators with on-shell amplitudes*, Phys. Rev. D **101** (2020) 095021 [1912.08827].

[208] G. Durieux, T. Kitahara, C. S. Machado, Y. Shadmi and Y. Weiss, *Constructing massive on-shell contact terms*, JHEP **12** (2020) 175 [2008.09652].

[209] G. Mogull, J. Plefka and J. Steinhoff, *Classical black hole scattering from a worldline quantum field theory*, JHEP **02** (2021) 048 [2010.02865].

[210] G. Kälin, Z. Liu and R. A. Porto, *Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach*, Phys. Rev. Lett. **125** (2020) 261103 [2007.04977].

[211] T. Binnington and E. Poisson, *Relativistic theory of tidal Love numbers*, Phys. Rev. D **80** (2009) 084018 [0906.1366].

[212] T. Damour and A. Nagar, *Relativistic tidal properties of neutron stars*, Phys. Rev. D **80** (2009) 084035 [0906.0096].

[213] N. Gürlebeck, *No-hair theorem for Black Holes in Astrophysical Environments*, Phys. Rev. Lett. **114** (2015) 151102 [1503.03240].

[214] P. Pani, L. Gualtieri, A. Maselli and V. Ferrari, *Tidal deformations of a spinning compact object*, Phys. Rev. D **92** (2015) 024010 [1503.07365].

[215] P. Landry and E. Poisson, *Tidal deformation of a slowly rotating material body. External metric*, Phys. Rev. D **91** (2015) 104018 [1503.07366].

[216] A. Le Tiec and M. Casals, *Spinning Black Holes Fall in Love*, Phys. Rev. Lett. **126** (2021) 131102 [2007.00214].

[217] A. Le Tiec, M. Casals and E. Franzin, *Tidal Love Numbers of Kerr Black Holes*, *Phys. Rev. D* **103** (2021) 084021 [2010.15795].

[218] LIGO Scientific Collaboration and Virgo Collaboration collaboration, *Gw170817: Observation of gravitational waves from a binary neutron star inspiral*, *Phys. Rev. Lett.* **119** (2017) 161101.

[219] The LIGO Scientific Collaboration and the Virgo Collaboration collaboration, *Gw170817: Measurements of neutron star radii and equation of state*, *Phys. Rev. Lett.* **121** (2018) 161101.

[220] K. Haddad and A. Helset, *Tidal effects in quantum field theory*, *JHEP* **12** (2020) 024 [2008.04920].

[221] C. Cheung, N. Shah and M. P. Solon, *Mining the Geodesic Equation for Scattering Data*, *Phys. Rev. D* **103** (2021) 024030 [2010.08568].

[222] G. Kälin, Z. Liu and R. A. Porto, *Conservative Tidal Effects in Compact Binary Systems to Next-to-Leading Post-Minkowskian Order*, *Phys. Rev. D* **102** (2020) 124025 [2008.06047].

[223] M. Levi, A. J. Mcleod and M. Von Hippel, *NNNLO gravitational quadratic-in-spin interactions at the quartic order in G*, 2003.07890.

[224] M. Levi and F. Teng, *NLO gravitational quartic-in-spin interaction*, *JHEP* **01** (2021) 066 [2008.12280].

[225] P. K. Gupta, J. Steinhoff and T. Hinderer, *Relativistic effective action of dynamical gravitomagnetic tides for slowly rotating neutron stars*, *Phys. Rev. Res.* **3** (2021) 013147 [2011.03508].

[226] A. Kobach and S. Pal, *Hilbert Series and Operator Basis for NRQED and NRQCD/HQET*, *Phys. Lett. B* **772** (2017) 225 [1704.00008].

[227] A. Kobach and S. Pal, *Conformal Structure of the Heavy Particle EFT Operator Basis*, *Phys. Lett. B* **783** (2018) 311 [1804.01534].

[228] L. Graf, B. Henning, X. Lu, T. Melia and H. Murayama, *2, 12, 117, 1959, 45171, 1170086, . . . : a Hilbert series for the QCD chiral Lagrangian*, *JHEP* **01** (2021) 142 [2009.01239].

[229] H.-L. Li, Z. Ren, J. Shu, M.-L. Xiao, J.-H. Yu and Y.-H. Zheng, *Complete Set of Dimension-8 Operators in the Standard Model Effective Field Theory*, 2005.00008.

[230] A. Falkowski, *Bases of massless EFTs via momentum twistors*, 1912.07865.

[231] D. Amati, M. Ciafaloni and G. Veneziano, *Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions*, *Int. J. Mod. Phys. A* **3** (1988) 1615.

[232] V. A. Smirnov, *Analytic tools for Feynman integrals*, vol. 250. Springer, Berlin, Heidelberg, 2012, 10.1007/978-3-642-34886-0.

[233] W. D. Goldberger and A. K. Ridgway, *Radiation and the classical double copy for color charges*, *Phys. Rev. D* **95** (2017) 125010 [1611.03493].