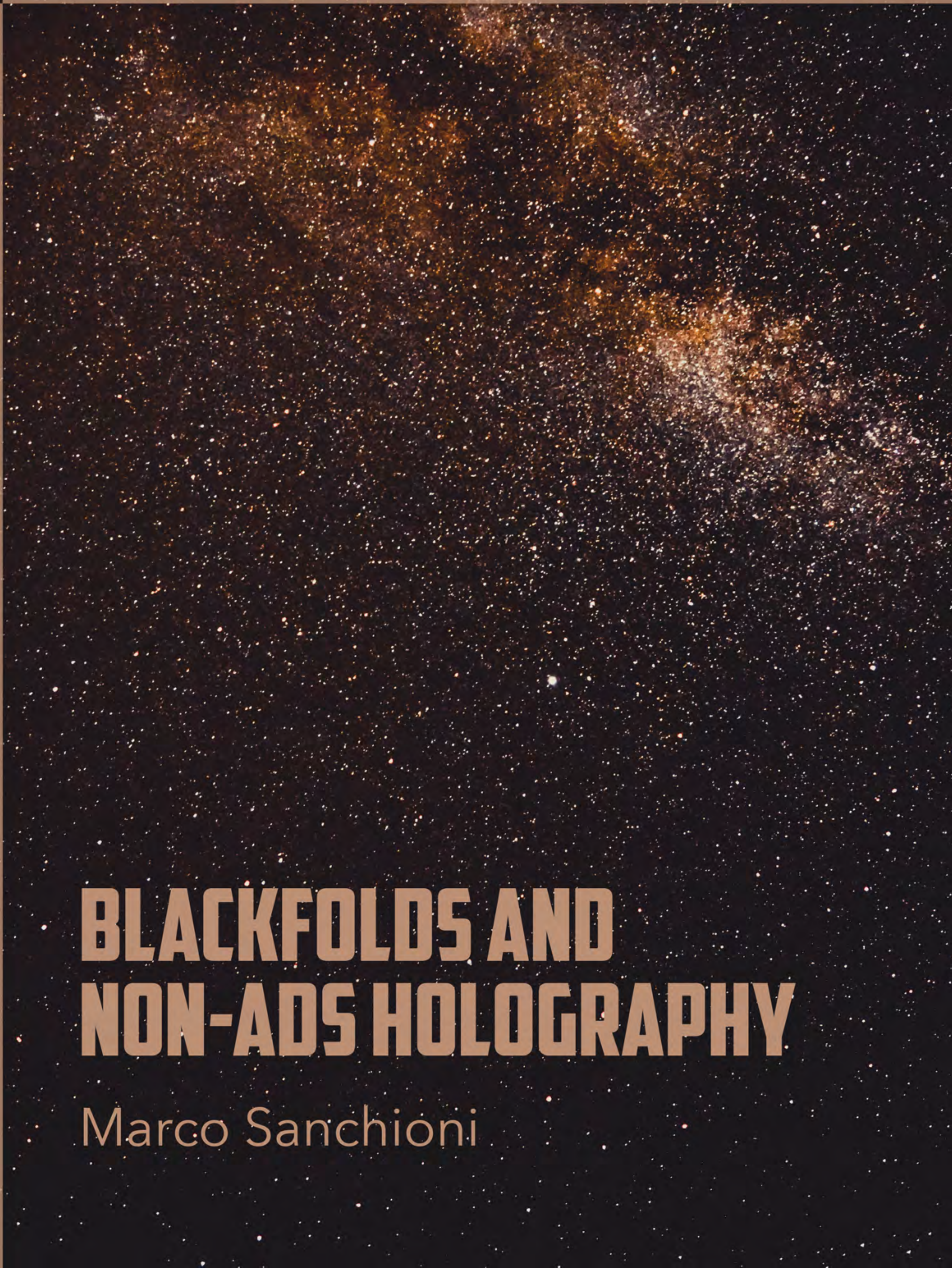
PhD Thesis



BLACKFOLDS AND NON-ADS HOLOGRAPHY

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Preface

Over the past 20 years we learnt that General Relativity is not "just" a theory that describes the 4 dimensional universe we live in, through the study of cosmology and black holes. It is much more. In fact we discovered that General Relativity can be used as a tool to describe also other physical settings, not just astrophysics and cosmology. As it happened with Quantum Field Theory, which was discovered in the realm of quantum electrodynamics but was later applied in many other branches of physics, so it happened with General Relativity. After the discovery of the AdS/CFT correspondence we learnt that General Relativity is the tool for describing strongly coupled system. This is one of the motivations for studying General Relativity in settings that are different with respect to astrophysics and cosmology.

The first part of this work, in particular chapters 1-2-3, is devoted to studying higher dimensional advanced black holes configurations. They can be useful for understanding more about general relativity, and they can be used, through AdS/CFT correspondence, for studying some strongly coupled Quantum Field Theory. Over the last years, effective theories for black holes, together with numerical methods, have been extremely important for studying gravity in higher dimensions, which is particularly difficult to solve. Among these effective theories, there are the blackfold approach [87] and the large D expansion [81]. In this work we analyze the first of these two approaches. This part of the thesis is organized as follows

- The first chapter is a review of the main developments regarding the blackfold approach in the last years. It starts by giving the definition of a blackfold and providing the solutions of the blackfold equations in the regime in which the blackfold is stationary. Then it is explained how build charged blackfolds, in particular particle and string charged blackfolds. It is reviewed how the blackfold effective theory can be derived from the Einstein Equations. At the end of the chapter some examples of charged and uncharged blackfolds are provided.

- The second chapter is based on the paper [17]. The work is about finding novel charged blackfold solutions placed in different background spaces, such as Lifshitz, (A)dS, Plane Waves, but also black hole backgrounds, such as Schwarzshild backgrounds. Beside the standard thermodynamics, also pressure and volume had been computed, following the work developed for black holes in AdS backgrounds of [137, 65, 59, 63, 141, 67, 145].

Studying the blackfold in other background spaces than AdS, it was evident that those definitions of volume and pressure ad they were given in the previous works were not physically interpretable, as it happens in AdS backgrounds. We therefore give a different point of view of such thermodynamics properties, by studying the gravitational tension. We also understood that, in order to compute the proper mass of a blackfold in a curved background, one need to subtract to the blackfold mass a proper factor of the gravitational tension. All of this is explained in chapter 2.

- The third chapter is based on an unpublished work, which is indeed to be published in the near future [18]. Following the work done in [20, 12], we would like to find novel blackfold solution with charge and intrinsic spin, in different backgrounds. For such configurations, the idea is to calculate their gyromagnetic ratio, which was already defined in the literature. Some preliminary results are provided in chapter 3.

The aim of the second part of the work is to study holographic applications which are not relativistic. Not all the systems in nature are in fact relativistic. There are many system, especially in the condensed matter sector, which exhibits non-relativistic symmetries. In the past years many work has been done in order to extend the holographic principle (on which the AdS/CFT correspondence was based) also for non-relativistic physics. These works have been very useful in studying strongly coupled physical systems with non-relativistic symmetries, such as Galielan or Schröedinger symmetry. This second part of the thesis is organized as follows

- The fourth chapter is a review of many developments in the study of a novel geometry, which is called Torsional Newton-Cartan geometry (TNC). This is important for the holographic description because it was found by [56] that this geometry lives in the boundary of a Lifshitz bulk. In chapter 4 there will be an introduction on the basis of the TNC geometry, analyzing some of the methods that were developed in the literature on how to derive such geometry. Then there is also an overview on one "dual" geometry to TNC, which is the Carroll geometry. In the second section of the chapter there is a review on how to couple field theories to TNC background, and the physical interpretation of the non-relativistic energy momentum tensor. The third section is a summary of the work done in [114] where they realize that a version of dynamical (T)NC geometry can be seen as a covariant version of Horava-Lifshitz gravity. At the end of the chapter there is the holographic realization of the geometry explaining an upliftable model which is going to be used in chapter 5.

- The fifth chapter is based on this paper [115]. In this work we developed a gravitational dual realization of the hydrodynamics, or rather the perfect fluid limit, of a class of non-relativistic theories by constructing appropriate Lifshitz black branes.

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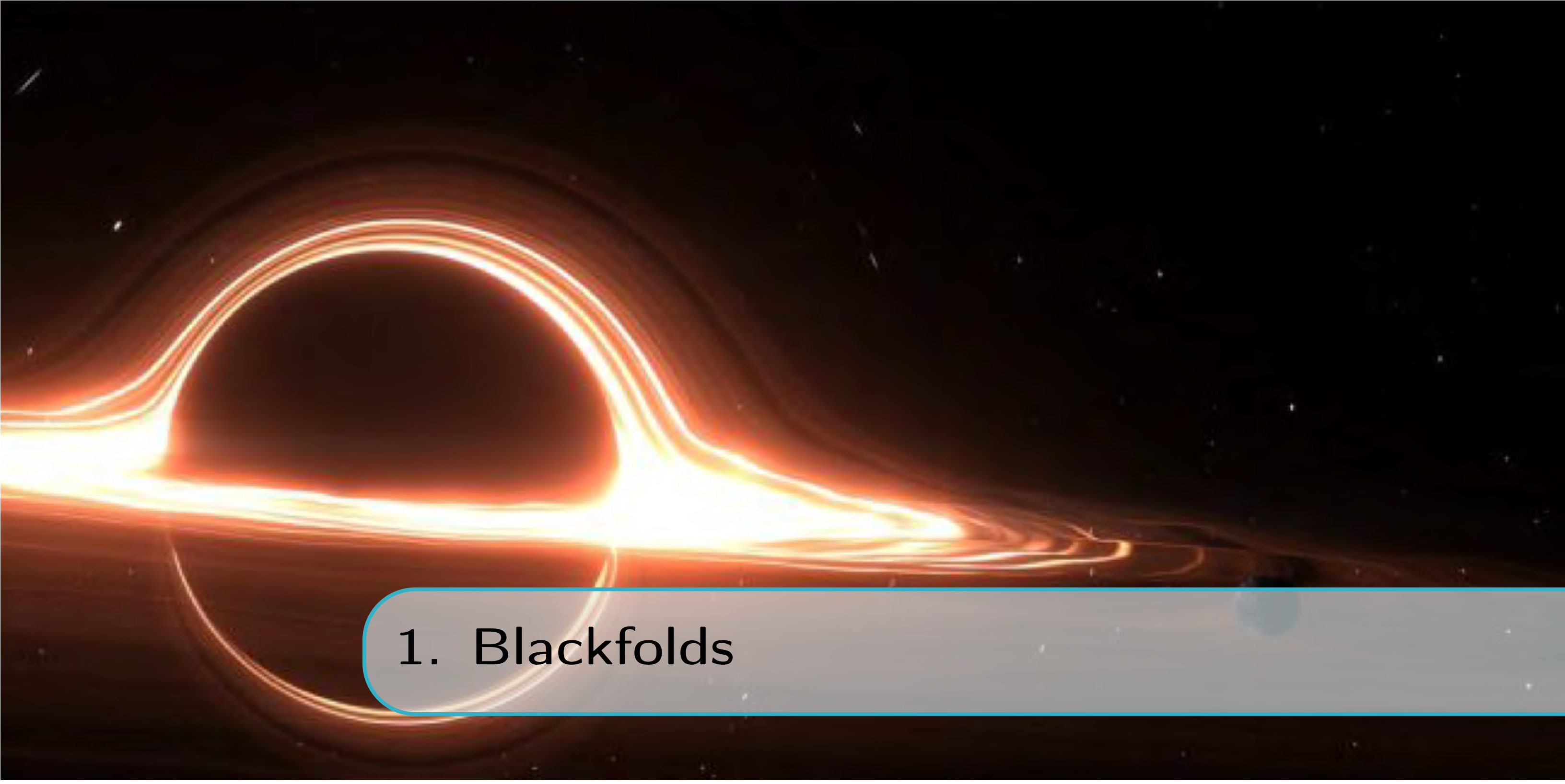


1. Blackfolds

General relativity in higher dimensions has been an intense subject of research in the last decades. There are meanly two reasons why it is important to study gravity in higher dimensions. The first reason is that gravity in higher dimensions is reacher then gravity in four dimensions, and therefore we can learn more about it. The second reason is that since the discovery of the holographic correspondence the study of gravity in higher dimensions became very important for studying field theories. In particular, in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, the development on gravity in 5 dimensions are important to gain more knowledge on conformal field theories in 4 dimensions.

Since the appearance of black rings in 5 dimensions [78, 80] it was clear that General Relativity in higher dimensions permits the possibility of a large class of black holes, with respect to the four dimensional ones. After the black ring in 5 dimensions, other exotic solutions have been discovered, such as black saturn and multi black ring solutions [74, 73, 128, 88]. Unfortunately the techniques we used to construct exact black holes solutions in four and five dimensions [155] cannot be naively extended to higher dimensions. Still, this does not mean that exotic solutions, such as black rings in higher dimensions, do not exist. In fact, following the intuition that such solutions do exist, it could be possible to take smoothly bent black branes, and try to solve the Einstein equation in a perturbative regime. In other words, one can try to work out an effective theory for black branes.

This approach started with the construction of a thin black ring in $D = 5$ in [86] and then a general effective theory for bent black branes was developed in [87, 82] and it took the name of *blackfold effective theory*.

1.1 Blackfold Effective Theory

The blackfold approach is an effective theory that describes the physics of black objects. It can be applied whenever there are two widely separated length scales along the horizon of a black hole: in those situations the natural approach is to integrate out the short distance physics, and deal with an effective theory with long distance degrees of freedom which considers the effects of the short distance degrees of freedom that are encoded into an effective action. The leftover theory for blackfold then will describe a thin black p-brane curved on a submanifold $\mathcal{W}_{p+1}$ embedded in a background spacetime. This effort has been done in [87]. In the following sections are a review of the main developments of this theory. We will work within a *blackfold regime*, considering the horizon size of the black p-brane much smaller than any other scale in the physical system:

$$r_0(\sigma^a) \ll min(R, L) \tag{1.1}$$

where r_0 is the horizon size of the black p-brane, σ^a are the coordinates on the worldvolume $(a = 0, 1, \dots, p)$, R is the curvature radius of the worldvolume, and L is any length scale of the background where the black p-brane can eventually be placed. The blackfold is specified by a set of collective coordinates $X^\mu(\sigma^a)$ that describes the embedding of the $(p+1)$-dimensional worldvolume of a black p-brane into a specific D-dimensional background spacetime. The requirement (1.1) implies that near any small enough region around the blackfold, its geometry looks like that of a boosted flat black brane. The near horizon metric of a boosted black p-brane can be described at leading order by

$$ds^2 = \left(\gamma_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2 \tag{1.2}$$

where $n = D - p - 3$ (D being the number of dimensions), u^a are p independent spatial components of the velocity field, and γ_{ab} is the induced metric on the worldvolume, which depends on σ via the embedding coordinates of the brane

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \tag{1.3}$$

where $g_{\mu\nu}$ is the metric of the background spacetime. Since this is obtained by just boosting the metric of a flat p-brane along the worldvolume coordinates, this is still a solution of Einstein equations.

In order to encode some short distance physics in this metric, the next step is to let X^μ, u^a and r_0 to slowly vary along the worldvolume under perturbations over a length scale $R \gg r_0$. With this assumption, the near-zone geometry will be described by

$$ds^2 = \left(\gamma_{ab} + \frac{r_0^n(\sigma^a)}{r^n} u_a(\sigma^a) u_b(\sigma^a)\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2 \tag{1.4}$$

At this point the blackfold is described by thickness of the black brane $r_0(\sigma)$, the velocity fluid $u^a(\sigma)$ and the set of the embedding coordinates $X^\mu(\sigma)$

$$\phi(\sigma) = \{X^\mu(\sigma), r_0(\sigma), u^a(\sigma)\} \tag{1.5}$$

where ϕ are all the collective variables describing the blackfold. One should also keep in mind also that, because of diffeomorphism invariance, not all the embedding coordinates are independent.

Being more precise, the configuration we are describing is as follows: zooming in to a local neighborhood of the blackfold, the metric is the one of a boosted black p-brane (1.2); at large scales, instead, the space-time is described by the metric $g_{\mu\nu}$ with an infinitely thin p-brane, with embeddings $X^\mu(\sigma)$ and with $r_0(\sigma)$ and $u^a(\sigma)$.

The dynamics of this blackfold system is encoded into the stress-energy tensor $T_{\mu\nu}$ of a black brane, which can be obtained by the Bown-York procedure

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta I}{\delta g^{\mu\nu}} \tag{1.6}$$

The stress-energy tensor of a boosted black p-brane can be computed and the result is that, to leading order, it takes the perfect fluid form

$$T_{ab} = \epsilon u_a u_b + P(\gamma_{ab} + u_a u_b) \tag{1.7}$$

where ϵ is the energy density and P is the pressure of the brane

$$\epsilon = \frac{(n+1)\Omega_{(n+1)}}{16\pi G} r_0^n \qquad P = -\frac{1}{n+1}\epsilon, \tag{1.8}$$

From the stress energy tensor of a black p-brane, it is possible to derive the stress-energy tensor of the blackfold by letting the collective parameters of the blackfold (1.5) to slowly vary along the worldvolume

$$T_{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n(\sigma)\left(n u_a(\sigma)\, u_b(\sigma) - \gamma_{ab}(\sigma)\right) \tag{1.9}$$

and this will be, to leading order, the stress-energy tensor of the blackfold. To complete the local thermodynamics of the brane, the entropy and the temperature can be computed via the Bekenstein-Hawking identification, and the results are the following

$$s = \frac{(n+1)\Omega_{(n+1)}}{4G} r_0^{n+1} \quad , \quad \mathcal{T} = \frac{n}{4\pi r_0} \tag{1.10}$$

These local physical quantities obey the first law of thermodynamics and the Gibbs relation

$$d\epsilon = \mathcal{T} ds \quad , \quad \epsilon + P = \mathcal{T} s \tag{1.11}$$

1.1.1 Worldvolume geometry

In order to derive blackfold equations, it is important to give some basics of the geometry governing the worldvolume of the blackfold itself. The geometry of the submanifold $\mathcal{W}_{p+1}$ is totally encoded in the induced metric $\gamma_{ab}(\sigma)$ (that fixes the intrinsic geometry) and by the extrinsic curvature $K_{\mu\nu}{}^{\rho}(\sigma)$ (that encodes the shape of the embedding). The projector to the tangent space of $\mathcal{W}_{p+1}$ (which is also called first fundamental form) is given by

$$h^{\mu\nu} = \partial_a X^{\mu} \partial_b X^{\nu} \gamma^{ab} \tag{1.12}$$

while the projector of the orthogonal directions is

$$\perp_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu} \tag{1.13}$$

Background tensors $A^{\mu\dots}{}_{\nu\dots}$ can be converted to worldvolume tensors $A^{a\dots}{}_{b\dots}$ (pullback) and viceversa (pushforward) by using $\partial_a X^{\mu}(\sigma)$, as it has been done in (1.3). It is important also to define covariant differentiation of tensor living on the worldvolume. This is done by the following

$$\bar{\nabla}_\mu = h_\mu{}^\nu \nabla_\nu \tag{1.14}$$

The bar will denote the fact that the differentiation will be performed along the worldvolume directions. Using this definitions one can define also the extrinsic curvature (or second fundamental form)

$$K_{\mu\nu}{}^\rho(\sigma) = h_\mu{}^\sigma \bar{\nabla}_\nu h_\sigma{}^\rho \tag{1.15}$$

The trace of the extrinsic curvature is called mean curvature vector

$$K^\rho = h^{\mu\nu} K_{\mu\nu}{}^\rho \tag{1.16}$$

1.1.2 Blackfold equations

Since we said that all the dynamics of the system should be encoded into the energy-momentum tensor, we are going to derive the blackfold equation from it. We know that the stress-energy should be conserved along the worldvolume of the brane, which means that

$$\bar{\nabla}_\mu T^{\mu\rho} = 0 \tag{1.17}$$

Decomposing the equation of the conservation of the stress-energy tensor of the black p-brane into components parallel or orthogonal to $\mathcal{W}_{p+1}$ we obtain the blackfold equations

$$T^{\mu\nu} K_{\mu\nu}{}^\rho = 0 \qquad \text{(extrinsic equation)} \tag{1.18}$$

$$D_a T^{ab} = 0 \qquad \text{(intrinsic equation)} \tag{1.19}$$

where D_a is the covariant derivative on the worldvolume and T^{ab} is the pullback of the stress-tensor on the worldvolume. The intrinsic equation describes the fluid living on the worldvolume, while the extrinsic equation describes the bending of the brane.

It is enlightening to write the extrinsic equation in terms of the embedding coordinates X^μ:

$$T^{ab}(\nabla_a \partial_b X^\rho + \Gamma_\mu{}^\rho{}_\nu \partial_a X^\mu \partial_b X^\nu) = 0 \tag{1.20}$$

These are the generalization of the Geodesics equations for extended relativistic objects, or in other words the generalization of Newton second law of dynamics where T_{ab} represent the mass and K^ρ is the acceleration.

If we insert the explicit form of the stress-energy tensor of the black neutral p-brane (1.7) into the blackfold equations (1.18) and (1.19) we get

$$\dot{u}_a + \frac{1}{n+1} u_a D_b u^b = \partial_a \log r_0 \tag{1.21}$$

$$K^\rho = n \perp^\rho{}_\mu \dot{u}^\mu \tag{1.22}$$

where we defined $\dot{u}^\mu = u^\nu \nabla_\nu u^\mu$. These are the equations we would like to solve in order to find specific blackfold configurations.

1.2 Stationary solutions

In general blackfolds do not have to describe just black branes in equilibrium, in fact we can describe blackfolds not in thermodynamic equilibrium (over distances $\geq$ R). In this section however (and for the rest of this thesis) we restrict ourselves to equilibrium configurations, which are described by the so called stationary blackfolds. In general all stationary (including static) black holes have a vector field k^μ such that it is a Killing vector field and it is null at the horizon

$$\mathcal{L}_k g_{\mu\nu} = 0 \tag{1.23}$$

$$g_{\mu\nu} k^\mu k^\nu \mid_{EH} = 0 \tag{1.24}$$

Also for blackfolds we should require the same in order to reach stationarity. In particular we say that a blackfold is stationary when there exist a Killing vector field k^μ of the background metric such that

1. the pullback of k^μ to the worldvolume, i.e. $k^a = k_\mu \partial_a X^\mu$, is a Killing vector field on the worldvolume.
2. the velocity fluid is set to be proportional to this killing vector field

$$u^a = \frac{k^a}{|k|} \tag{1.25}$$

These two requirements precisely characterize stationary fluid configurations: in particular they are necessary for the absence of dissipative effects, as it was proven in [44].

1.2.1 Solution of the intrinsic equation

Now, since k^μ satisfies the killing equations, and since u^a is proportional to k^a, we can realize that

$$\dot{u}^a = \partial^a \log |k| \qquad D_a u^a = 0 \tag{1.26}$$

Plugging this informations into the intrinsic equation (1.18) we get

$$\frac{r_0}{|k|} = \text{constant} \tag{1.27}$$

which therefore is a solution of the intrinsic blackfold equation for stationary blackfolds. The only thing that it is left is to fix the constat. It has been argued that the killing vector k it is null when approaching the horizon and therefore it is a null generator of the horizon. This means that we can use it to get the surface gravity

$$\kappa = \frac{nk}{2r_0} \tag{1.28}$$

Thus we have fixed the above constant. Furthermore we can compute the temperature from the surface gravity, by $\kappa = 2\pi T$. T is really the global temperature of the blackfold. In conclusion we have

$$r_0(\sigma) = \frac{n}{4\pi T}|k| \tag{1.29}$$

The killing vector k^a, without loss of generality, can be written as a linear combination of the worldvolume coordinates, as

$$k^a \partial_a = \partial_\tau + \sum_{\hat{a}} \Omega_{\hat{a}} \partial_{\phi_{\hat{a}}} \tag{1.30}$$

where τ is the timelike worldvolume directions and $\Omega_{\hat{a}}$ are the angular velocities associated with the Cartan angles $\phi_{\hat{a}}$. We assume now that the background spacetime has Killing vectors ξ and χ_i, that they are canonically normalized at infinity, and that their norms at the horizon are called respectively R_0^2 and R_i^2. This means that

$$\partial_t = \frac{1}{R_0}\xi \quad , \quad \partial_{\phi_i} = \frac{1}{R_i}\chi_i \tag{1.31}$$

We assume also that ξ is hypersurface orthogonal, such that it can foliate the blackfold into spacelike slices $\mathcal{B}_p$, and we define the normal vector to $\mathcal{B}_p$ as

$$n^a = \frac{1}{R_0}\xi^a \tag{1.32}$$

Defining also the "worldvolume velocity field" to be $V_i = \Omega_i R_i$, one can write the modulus of k as

$$|k| = \frac{R_0}{\sqrt{1 - V^2(\sigma)}} \tag{1.33}$$

which is suggestive because it means that $|k|$ can be interpreted as a relativistic Lorentz factor, with a local redshift.

Solution to the extrinsic equation and effective free energy functional

Concerning the extrinsic equation (1.19) for stationary configuration, using the solution of the intrinsic equations, we can rewrite it as

$$K^\mu = \perp^{\mu\nu} \partial_\nu \ln |k|^n \tag{1.34}$$

Since $r_0/|k|$ is required to be constant, using the explicit form of the pressure of the p-brane, we can also write it as

$$K^\mu = \perp^{\mu\nu} \partial_\nu \ln(-P) \tag{1.35}$$

One can show that in general this equation can be integrated to an effective energy functional

$$\mathcal{F}[X^\mu(\sigma)] = -\int_{\mathcal{W}_{(p+1)}} d^{p+1}\sigma \sqrt{-\gamma} P \tag{1.36}$$

where γ is the determinant of the induced metric γ_{ab}. Thus this means that we can find stationary blackfolds configurations by extremizing this action. Then, since ξ is hypersurface orthogonal, in the action (1.36) one can split the integral into one integral over time and an other integral over $\mathcal{B}_p$. Wick rotating to Euclidian time, the integration can be trivially performed along the timelike killing vector generated by ξ (since we are dealing with stationary configurations that are not changing in time, i.e. time is a killing vecotr) and this gives an overall factor β of the time interval, so that the Euclidian action becomes

$$\mathcal{F}[X^\mu(\sigma)] = -\int_{\mathcal{B}_p} dV_{(p)} R_0 P. \tag{1.37}$$

This will be the effective free energy functional that we have to minimize in order to find stationary blackfold configurations (in the next sections there will be many examples).

1.2.2 Thermodynamics of stationary blackfolds

Given the effective free energy (1.37), one can easily extract the conjugate global thermodynamics potentials to T and Ω_a, namely the entropy S the angular momenta $J_{\hat{a}}$, via the corresponding relation

$$S = -\frac{\partial \mathcal{F}}{\partial T} \quad , \quad J_{\hat{a}} = -\frac{\partial \mathcal{F}}{\partial \Omega_{\hat{a}}} \tag{1.38}$$

These thermodynamic relations can be also obtained integrating their conserved currents. For instance, defining $n^a = \xi^a / R_0$, which is the unit normal vector on the worldvolume to $\mathcal{B}_p$, one can define in general

$$M = \int_{\mathcal{B}_p} dV_{(p)} T_{ab} n^a \xi^b \tag{1.39}$$

$$J_{\hat{a}} = \int_{\mathcal{B}_p} dV_{(p)} T_{ab} n^a \chi^b_{\hat{a}} \tag{1.40}$$

$$S = -\int_{\mathcal{B}_p} dV_{(p)} s u_a n^a \tag{1.41}$$

and the two descriptions are equivalent. Having introduced this thermodynamic properties, one realizes that the free energy functional satisfies

$$\mathcal{F} = M - TS - \sum_{\hat{a}} \Omega_{\hat{a}} J_{\hat{a}} \tag{1.42}$$

Furthermore, one can also define a total tension

$$\hat{\mathcal{T}} = -\int_{\mathcal{B}_p} dV_{(p)} (\gamma^{ab} + n^a n^b) T_{ab} \tag{1.43}$$

and the thermodynamics we just wrote down obey a Smarr-like relation and the first law of thermodynamics

$$(D-3)M = (D-2)\left(\sum_{\hat{a}} \Omega_{\hat{a}} J_{\hat{a}} + TS\right) + \hat{\mathcal{T}} \tag{1.44}$$

$$dM = TdS + \sum_{\hat{a}} \Omega_{\hat{a}} dJ_{\hat{a}} \tag{1.45}$$

1.3 Charged Blackfolds

Charge can be naturally incorporated in this framework as explained in [43], we just need to change the seed solution. For instance one can consider charged solution of the Einstein-Maxwell theory

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left(R - 2(\nabla\phi)^2 - \frac{1}{2(q+2)!} e^{-2a\phi} H^2_{[q+2]} \right) \tag{1.46}$$

where ϕ is the dilaton field, a is the dilaton coupling, H is the two form field strenght of A, which is the Maxwell field. It is also convenient to introduce the parameter N defined as

$$N = \frac{4}{N} - \frac{2(D-3)}{D-2} \tag{1.47}$$

When we are dealing with a specific type of branes, called Kaluza-Klein branes, the parameter N is equal to 1. We can think of a p-brane with charge density extend along q spatial directions. When $q = 0$ the p-brane has pointlike electric charge on its worldvolume, instead when $q = 1$ the p-brane has an electric string dipole. In this section we analyze $q = 0$ and $q = 1$ configurations. They can be studied together, in a unified framework. In general, for a p-brane carrying a q-brane charge, we can write the charge current as

$$J = \mathcal{Q}\,\hat{V}_{q+1} \tag{1.48}$$

where $\hat{V}_{q+1}$ is the volume form on $\mathcal{W}_{q+1}$. For charged configuration, along with the stress-energy tensor T_{ab}, the dynamics is encoded in the particle current J.

1.3.1 $0-$charged blackfold

For 0-charged object, the particle current must be proportional to the fluid velocity u

$$J^a = \mathcal{Q}\,u^a\,(\sigma) \tag{1.49}$$

and the form of the stress-energy tensor will be the same of the neutral configuration.

1.3.2 $1-$charged blackfold

When $q = 1$ instead we should introduce an other vector v orthogonal to u which will describe the direction along which the string charge will lie (and should be of norm 1, since it should be spacelike)

$$u \cdot v = 0 \quad , \quad -u^2 = v^2 = 1 \tag{1.50}$$

Then we can write the current as

$$J_{ab} = \mathcal{Q}\,(u_a v_b - u_b v_a) \tag{1.51}$$

Moreover, for 1-charged brane, the vector v brakes spatial isotropy because it characterizes a preferred direction along which the (dissolved) string lays. Therefore the stress-energy tensor will be characterized by a pressure which will be parallel to this direction and a pressure orthogonal to it.

$$T_{ab} = \epsilon u_a u_b + P_{\parallel} v_a v_b + P_{\perp}(\gamma_{ab} + u_a u_b - v_a v_b) \tag{1.52}$$

1.3.3 A unified description

In order to have a unified description of the 0-charge and the 1-charge case, we need to introduce the projector onto the space parallel to the string/particole worldline/worldsheet

$$\hat{h}^{(q)}_{ab} = -u_a u_b + q v_a v_b \tag{1.53}$$

and onto directions orthogonal to it

$$\hat{\perp}_{ab} = \gamma_{ab} - \hat{h}^{(q)}_{ab} \tag{1.54}$$

Moreover we can define the volume form $\hat{V}_{q+1}$ of the worldline/sheet as

$$\hat{V}_{q+1} = \begin{cases} u & \text{for } q = 0 \\ u \wedge v & \text{for } q = 1 \end{cases} \tag{1.55}$$

With this definitions it is possible to write down the stress-energy tensor and the charge current in the following way

$$T_{ab} = \left(\epsilon + P_{\|}\right) u_a\left(\sigma\right) u_b\left(\sigma\right) + \left(P_{\|} - P_{\perp}\right) h^{(q)}_{ab}\left(\sigma\right) + P_{\perp}\gamma_{ab}\left(\sigma\right) \tag{1.56}$$

$$J_{q+1} = \mathcal{Q}_q \hat{V}_{q+1}\left(\sigma\right) \tag{1.57}$$

1.3.4 Local thermodynamics

Now that we have a unified description of the string and particle charge configurations, it is important to look at the local thermodynamics of these black p-brane carrying q-brane charge (with $q = 0,\, 1$). The difference between the pressure parallel to the string and the one perpendicular, which represents the tension that the string produces, can be computed

$$P_{\perp} - P_{\|} = \Phi\mathcal{Q} \tag{1.58}$$

and Φ is called string chemical potential. Since we will be considering configuration in equilibrium, locally on the brane the first law of thermodynamics holds

$$d\epsilon = \mathcal{T} ds + \Phi d\mathcal{Q} \tag{1.59}$$

where $\mathcal{T}$ is the local temperature, as in the neutral case. Also the thermodynamic Gibbs-Duhem relations hold

$$\epsilon + P_{\perp} = \mathcal{T}s + \Phi\mathcal{Q} \quad , \quad dP_{\perp} = sd\mathcal{T} + \mathcal{Q}d\Phi \quad , \quad , dP_{\|} = sd\mathcal{T} - \Phi d\mathcal{Q} \tag{1.60}$$

For these configurations, along with the blackfold equations, we should also require the conservation of the particle current, i.e. the continuity equation

$$d \star J = 0 \tag{1.61}$$

The stress energy tensor for a p-brane carrying a q-brane charge for the action (1.46) can be computed through Brown-York procedure, as it is also for the neutral case. This has been done for any $q < p$ and the result for the local thermodynamic parameters is the following

$$\epsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + 1 + nN \sinh^2\alpha\right) \tag{1.62}$$

$$P_{\perp} = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \tag{1.63}$$

$$P_{\|} = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(1 + nN \sinh^2\alpha\right) \tag{1.64}$$

$$\mathcal{Q} = \frac{\Omega_{(n+1)}}{16\pi G} n\sqrt{N} r_0^n \cosh\alpha \sinh\alpha \tag{1.65}$$

$$\Phi = \sqrt{N}\tanh\alpha \tag{1.66}$$

where α is the charge parameter of the configuration. These local thermodynamics depend on n and N (defined in (1.47)) but not from p and q since these densities are conserved charges and they can depend only on the number of dimensions they live in, and on the dilaton coupling. Therefore, after plugging the local thermodynamics in, the stress energy tensor of the blackfold becomes

$$T_{ab} = \mathcal{T}s\left(\sigma\right)\left(u_a\left(\sigma\right) u_b\left(\sigma\right) - \frac{1}{n}\gamma_{ab}\left(\sigma\right)\right) - \Phi\left(\sigma\right)\mathcal{Q}\, h^{(q)}_{ab}\left(\sigma\right) \tag{1.67}$$

1.3.5 Blackfold equations solution for stationary configurations

The intrinsic blackfold equation together with the continuity equations, and keeping in mind the Frobenius theorem (see reference [79] for details), can be rewritten as

$$\hat{\perp}^{ab}\mathcal{T}\left(\dot{u}_b+\partial_b\log\mathcal{T}\right)-\mathcal{Q}\Phi\left(\hat{K}^a-\perp^{ab}\partial_b\log\Phi\right) = 0 \tag{1.68}$$

$$(\hat{h}^{ab}+u^a u^b)(\dot{u}_b+\partial_b\log\mathcal{T}) = 0 \tag{1.69}$$

where we introduced the mean curvature of the worldline/sheet $\hat{K}=\hat{h}^{bc}D_b\hat{h}_c{}^a$. The general solution of the intrinsic equation (1.69) for 0-brane charge in a stationary configuration can be easily obtained as we explained before for the neutral case, and the solution to the intrinsic blackfold equation is

$$\mathcal{T}(\sigma)=\frac{T}{|k|}\quad,\quad\phi(\sigma)=\frac{\Phi_H}{|k|} \tag{1.70}$$

For the string dipole charge instead, we need to specify the geometry along the currents. We introduce a vector which is spacelike and commutes with k, i.e. $[k,\psi]=0$. Then we need to construct its component orthogonal to k

$$\zeta=\psi-\frac{\psi^a k_a}{k^2}k \tag{1.71}$$

assuming that ζ is spacelike over the blackfold worldvolume. Then we choose v to be proportional to ζ

$$v^a=\frac{\zeta^a}{|\zeta|} \tag{1.72}$$

This is not the most general choice we could make. With this construction we can easily solve the intrinsic equations also for $q=1$, getting

$$\mathcal{T}(\sigma)=\frac{T}{|k|}\quad,\quad\phi(\sigma)=\frac{1}{(2\pi)^q}\frac{\Phi_H}{|\hat{h}|^{(1/2)}} \tag{1.73}$$

where the area element on the string worldline/sheet is

$$|\hat{h}|^{1/2}=\begin{cases}|k| & \text{for } q=0\\ |\zeta||k| & \text{for } q=1\end{cases} \tag{1.74}$$

Inverting this equation and using the potentials for the brane, we get

$$r_0(\sigma)=\frac{n}{4\pi T|k|}\left(1-\frac{1}{(2\pi)^q}\frac{\Phi_H^2}{N|\hat{h}|}\right)\quad,\quad\tanh\alpha(\sigma)=\frac{1}{(2\pi)^q}\frac{\Phi_H}{\sqrt{N}|\hat{h}|^{1/2}} \tag{1.75}$$

This is the solution to the instrinsic blackfold equation (1.69) for stationary configurations in the unifed description. Concerning the extrinsic blackfold equation (1.68), also for stationary charged configurations it can be integrated to an effective free energy, where instead of having just P, we will have $P_\perp$. Also in this case, after a Wick rotation, we can perform the trivial integral along the Killing time generated by ξ so that

$$\mathcal{F}=\frac{\Omega_{(n+1)}}{16\pi G}\beta\int_{\mathcal{B}_p}dV_{(p)}R_0r_0^n \tag{1.76}$$

All the thermodynamics can be computed from the free energy, charge density included.

$$\mathcal{Q} = -\frac{\partial F}{\partial \Phi_H} \tag{1.77}$$

Also for the obtained thermodynamics holds the first law, and there is an extended Smarr relation

$$(D-3)M - (D-2)\left(TS + \sum_{\hat{a}} \Omega_{\hat{a}} J_{\hat{a}}\right) - (D-3-q)\Phi_H \mathcal{Q} = 0 \tag{1.78}$$

1.4 Derivation of the blackfold effective theory

The blackfold equations have been shown to describe correctly many black branes that were already known. The blackfold theory also yielded to new solutions. In order to solidly ground the theory it is important to show that the extrinsic blackfold equations can be derived directly from Einstein equations that describe a black p-brane with a linear perturbation of scale $R \gg r$. Moreover one needs also to show that the black p-brane remains regular under these perturbations. This has been done in [45] and it works as follows. The starting point is the metric of a black p-brane in which the parameters vary slowly along the worldvolume

$$ds^2 = \left(\gamma_{ab}(\sigma) + \frac{r_0^n(\sigma^a)}{r^n} u_a(\sigma^a) u_b(\sigma^a)\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2 + h_{\mu\nu}(x) dx^\mu dx^\nu \tag{1.79}$$

As we said, we would like the perturbations to be slightly away from flatness by terms that are linear in the fluctuations of transverse coordinates. This can be done by using a set of adapted coordinates which are called Fermi normal coordinates, which employs the idea of removing first derivatives of a metric around a given point. We call y^i the orthogonal directions to $\mathcal{W}_{p+1}$. Including terms to first order in y/R, the metric of the perturbed black p-brane written in Fermi normal coordinates is

$$ds^2 = \left(\eta_{ab} - 2K_{ab}{}^i y_i + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2 + h_{\mu\nu}(x) dx^\mu dx^\nu + O\left(r^2/R^2\right) \tag{1.80}$$

where $r = \sqrt{y^i y_i}$. Now we restrict ourselves, without loss of generality, to perturbations that are nonzero only along one specific direction $i = \hat{i}$ and that are locally linearized, meaning that

$$y^{\hat{i}} = r \cos\theta \tag{1.81}$$

Now we look at large $r \gg r_0$, where the gravitational field that the black brane creates is weak. In this region the metric of the brane is then

$$\begin{aligned} ds^2 \;&=\; \left(\eta_{ab} - 2K_{ab}{}^{\hat{i}} r \cos\theta + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \left(1 + \frac{r_0^n}{r^n}\right) dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\Omega_{(n)}^2\right) \\ &+ \;\cos\theta \hat{h}_{\mu\nu}(x) dx^\mu dx^\nu + O\left(r_0^{2n}/r^{2n}\right) \end{aligned} \tag{1.82}$$

By using the form of the stress-energy tensor of a black p-brane

$$T_{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n u^a u^b - \eta^{ab}\right) \tag{1.83}$$

plugging in and keeping the terms in the perturbation at large $r \gg r_0$, the metric can be rewritten as

$$\begin{aligned} ds^2 &= \left(\eta_{ab} - 2K_{ab}{}^{\hat{i}} r\cos\theta \frac{16\pi G}{n\Omega_{(n+1)}}\left(T_{ab} - \frac{1}{D-2}T\eta_{ab}\right)\frac{1}{r^n}\right) d\sigma^a d\sigma^b \\ &+ r^2\left(d\theta^2 + \sin^2\theta d\Omega^2_{(n)}\right) + \cos\theta \hat{h}_{\mu\nu}(x)dx^\mu dx^\nu + O\left(T^2/r^{2n}\right) \\ &+ \left(1 - \frac{16\pi G}{n\Omega_{(n+1)}}\frac{1}{D-2}\frac{T}{r^n}\right) dr^2 \end{aligned} \tag{1.84}$$

Now by a direct computation of the Einstein tensor $G_{\mu\nu}$ one finds a combination that do not involve $\hat{h}_{\mu\nu}$ which is the following

$$G_{r\theta} - \frac{r\tan\theta}{n+1}G_{rr} = \frac{n+2}{n+1}\frac{\sin\theta}{r^2}\frac{8\pi G}{\Omega_{(n+1)}}T^{ab}K_{ab}{}^{\hat{i}} \tag{1.85}$$

Thus the corresponding equation is a constraint which take the form

$$T^{ab}K_{ab}{}^{\hat{i}} = 0 \tag{1.86}$$

where $\hat{i}$ denotes an arbitrary direction transverse to the brane. These are the extrinsic blackfold equations.

1.5 Some examples of blackfold stationary configurations

In this section we review some basics examples of blackfolds, with our without charge, and in AdS background.

1.5.1 Electrically charged odd-spheres in flat background

We would like to consider a flat background in D dimensions, which is described by the following Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{(D-2)} \tag{1.87}$$

We would like to embed p-dimensional sphere with radius R in this flat background. Therefore we place a p dimensional sphere of radius R in this metric by choosing the following embedding

$$t = \tau \ , \quad r = R \ , \quad \theta = 0 \ , \quad \phi_a = \phi_{\hat{a}} \tag{1.88}$$

We set the configuration to rotate with equal angular velocity Ω in any $[(p+1)/2]$ angles of the sphere, which we will label by $\phi_{\hat{a}}$. The geometry of the p-dimensional sphere and the corresponding killing vector field are

$$ds^2 = -d\tau^2 + R^2 d\Omega^2_{(p)} \quad , \quad \mathbf{k}^a\partial_a = \partial_\tau + \sum_{\hat{a}=1}^{[(p+1)/2]} \partial_{\phi_{\hat{a}}} \tag{1.89}$$

With this embedding and killing vector field the free energy of this configuration, defined in (1.76), is

$$\mathcal{F} = \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \tag{1.90}$$

where

$$r_0 = \frac{n}{4\pi T}\left(1 - r^2\Omega^2\right)^{\frac{1-N}{2}}\left(1 - r^2\Omega^2 - \Phi_H^2/N\right)^{N/2} \tag{1.91}$$

and where $V_{(p)} = \Omega_{(p)} R^p$. By minimizing this effective action it is possible to compute the equilibrium condition for a 0-charge odd-spere in flat background, which is the following

$$\Omega^2 R^2 = \frac{p}{nN\sinh^2\alpha + n + p} \quad . \tag{1.92}$$

For completeness, in the appendix A, one can find all the thermodynamics of this configuration.

1.5.2 Uncharged odd-spheres in Ads background

As mentioned in the previous pages it is possible to place blackfolds in many different backgrounds. This possibility is already encoded into our effective blackfold theory by means of R_0 and R_i. One of the characteristic that the background should have is that it should be asimptotically flat and the (A)dS background has this feature. The metric of (A)dS space can be written in the following way

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{(D-2)} \quad , \quad f(r) = 1 + \frac{r^2}{L^2} \quad , \tag{1.93}$$

where L is the AdS radius. In order to obtain the deSitter metric we simply perform the Wick rotation $L \to iL$, therefore what follows encodes informations also about blackfolds in deSitter space (just need to replace L with iL at any time in the description). We would like to place a p dimensional sphere of radius R in the background described by (2.40). We set the configuration to rotate with equal angular velocity Ω in each of the $[(p+1)/2]$ Cartan angles of the p-dimensional sphere, labelled by $\phi_{\hat{a}}$. The embedded geometry and the corresponding Killing vector field are given by

$$\mathbf{ds}^2 = -f(\mathbf{R})d\tau^2 + R^2 d\Omega^2_{(p)} \quad , \quad \mathbf{k}^a\partial_a = \partial_\tau + \Omega\sum_{\hat{a}=1}^{[(p+1)/2]}\partial_{\phi_{\hat{a}}} \quad , \quad f(\mathbf{R}) = 1 + \mathbf{R}^2 \quad , \tag{1.94}$$

where we have defined the dimensionless radius $\mathbf{R} = R/L$. The free energy takes the simple form

$$\mathcal{F}[R] = \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} f(\mathbf{R}) r_0^n \quad , \tag{1.95}$$

Upon variation of this with respect to R, restricting to the cases where p is odd, and solving the resulting equation leads to the equilibrium condition

$$\Omega^2 R^2 = (1 + \mathbf{R}^2)\frac{\mathbf{R}^2(n+p+1) + p}{\mathbf{R}^2(n+p+1) + n + p} \quad . \tag{1.96}$$

For completeness we will write all the thermodynamics in the appendix A.

1.5.3 Black odd-spheres with string dipole in flat background

In this section we consider a related configuration to the previous example where instead of an electric charge, the black hole has a string dipole charge. The background geometry is still flat space, characterized be the metric (1.87) and the free energy (1.98), but it must be supplemented with the polarisation vector

$$v^a\partial_a = \frac{\gamma}{R}\left(\sum_{\hat{a}=1}^{[(p+1)/2]}\partial_{\phi_{\hat{a}}} + \Omega R^2 \partial_\tau\right) \quad , \quad \gamma = \frac{1}{\sqrt{1-\Omega^2 R^2}} \quad . \tag{1.97}$$

The free energy is the same as in the 0-charged configuration

$$\mathcal{F} = \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \tag{1.98}$$

Minimizing this free energy (i.e. solving the extrinsic equation) we get the following equilibrium condition

$$\Omega^2 R^2 = \frac{p + nN\sinh^2\alpha}{n+p+nN\sinh^2\alpha} \quad . \tag{1.99}$$

The thermodynamic properties are given in App. A.

1.5.4 Black charged discs in flat background

In this section we make a perturbative construction of the analogue of the Kerr-Newman black hole in higher-dimensional flat with one single angular momentum. This corresponds, in the blackfold approximation, to an electrically charged rotating disc with induced metric and Killing vector field

$$\mathbf{ds}^2 = -d\tau^2 + d\rho^2 + \rho^2 d\phi^2 \ , \ \mathbf{k}^a\partial_a = \partial_\tau + \Omega\partial_\phi \ . \tag{1.100}$$

This geometry develops a boundary when $\mathbf{k} = 0$, corresponding to the maximum of ρ given by

$$\rho_{\max} = \frac{\sqrt{1-\Phi_{\mathrm{H}}^2/N}}{\Omega} \quad , \tag{1.101}$$

This disc configuration trivially solves the blackfold equations since it is a minimal surface [10]. The thickness of the disc and the charge parameter are given by

$$\begin{aligned} r_0(\rho) &= \frac{n}{4\pi T}\left(1-r^2\Omega^2\right)^{\frac{1-N}{2}}\left(1-r^2\Omega^2-\frac{\Phi_{\mathrm{H}}^2}{N}\right)^{N/2} \quad , \\ \tanh\alpha(\rho) &= \frac{\Phi_{\mathrm{H}}/\sqrt{N}}{\sqrt{1-r^2\Omega^2}} \quad . \end{aligned} \tag{1.102}$$

Therefore, at the boundary $\rho_{\max}$ the thickness of the disc vanishes and hence the resulting black holes have topology $\mathbb{R}\times\mathbb{S}^{(D-2)}$. We now proceed and evaluate the free energy for these configurations from which all thermodynamic properties can be obtained. This is given by

$$\mathcal{F} = \frac{\Omega_{(n+1)}}{8G}\tilde{r}_0^n \frac{{}_2F_1\left(1, \frac{1}{2}(N-1)n; \frac{Nn}{2}+2; 1-\frac{\Phi_{\mathrm{H}}^2}{N}\right)}{\Omega^2(2+Nn)} \quad , \tag{1.103}$$

where we have defined

$$\tilde{r}_0^n = \left(\frac{n}{4\pi T}\right)^n \left(1 - \frac{\Phi_{\mathrm{H}}^2}{N}\right)^{\frac{2+Nn}{2}} \quad . \tag{1.104}$$

The thermodynamic properties are given in App. A.

2. Gravitational Tension

2.1 Pressure, Volume and Tension

The discovery that black holes carry an entropy [29, 118] proportional to the area of the event horizon has lead to view black holes as thermodynamic objects. This point of view provides a simple set of quantities, such as the mass M, the entropy S and the angular momenta J_a, which can be used to characterise many of the properties of black holes. For an uncharged black hole in asymptotically flat spacetime, these quantities satisfy the first law of thermodynamics

$$dM = TdS + \sum_a \Omega_a dJ_a \quad , \tag{2.1}$$

where T is the Hawking temperature and Ω_a is the set of horizon angular velocities. The study of (2.1) leads to a deeper understanding of the dynamics, stability and uniqueness of these objects. Therefore, from a purely gravitational point of view it is important to understand what types and kinds of physical modifications can occur in (2.1).

Many of such modifications are known and they arise due to other *intrinsic* properties that black holes can have, such as an electric/magnetic charge or scalar hair [95]. Other intrinsic properties such as horizon topology can allow for other types of charges, as in the case of the five-dimensional charged black ring [76, 58], which can cary dipole charge.[1] However, there are also *extrinsic* properties that can affect (2.1) such as the length scales characterizing the asymptotic region of a given black hole, or alternatively, the curvature scales characterising the resulting spacetime once the black hole is removed (i.e. when its horizon radius is set to zero).[2]

A concrete physical set up which we have in mind, with potential astrophysical implications, is to understand what the modification of (2.1) is when a black hole is immersed in the gravitational field of another black hole, as in a black hole binary system. In this

[1] For other non-trivial topologies, generalisations of dipole charge were found in [83].

[2] Another external factor that can affect (2.1) is non-trivial spacetime topology when, for example, there are fluxes present in the spacetime [94].

case, each of the black hole horizons will satisfy a first law of thermodynamics of the form (2.1) in which the mass (or horizon radius) of the other black hole appears as an external parameter. A natural question to ask is then: how is the first law (2.1) modified when there are variations in the gravitational field, in which the black hole is immersed, due to variations in the mass (or size) of the other black hole?

This question would be easily addressed if there existed exact analytic solutions of black hole binary systems. In four dimensional asymptotically flat spacetime no such solutions are known, however, perturbative solutions where a small black hole is orbiting a large black hole do exist up to several high orders [153]. Unfortunately, these *distorted* black hole solutions have only been constructed near the horizon and hence are not suitable for studying thermodynamic properties. Luckily, there are several exact or approximate analytic solutions that can be used as toy models for this kind of physics. This includes the first-order corrected solution [101] for localized Kaluza-Klein black holes [104], which due to the periodicity of one of the coordinates and by the method of images, can be viewed as being immersed in their own gravitational field. Moreover, in five spacetime dimensions there are several examples of exact and analytic black hole binary systems in asymptotically flat spacetime. The simplest of these being the black saturn solution [74], in which a black ring horizon orbits the centre Myers-Perry black hole.[3] Furthermore, the blackfold construction [87, 84] provides a general tool to analytically construct perturbative solutions of large classes of black holes in non-trivial backgrounds, such as those considered in [42, 47] along with the novel solutions with non-trivial spacetime asymptotics constructed in this paper. Indeed, we will use these examples to study the modifications to (2.1).

When studying the modifications to (2.1) due to the presence of external gravitational fields, one wishes to introduce/observe new quantities which: **(1)** have a geometric/physical meaning, **(2)** have a thermodynamic interpretation, **(3)** can be defined in the presence of any gravitational field and **(4)** reduce to the same universal result once the gravitational field is removed. We wish to qualify these statements.

By property **(1)** we mean that the quantities appearing in (2.1) can be obtained, for a given black hole spacetime, e.g. by some integration over the horizon involving Killing vector fields (e.g. the Komar mass) or by looking at the asymptotic fall-off of the metric fields (such as the ADM mass). By property **(2)** we mean that all such quantities are clear analogues of classical thermodynamic quantities as the entropy S or temperature T and can be obtained by taking appropriate derivatives of the free energy. By property **(3)** we mean that such quantities can be defined for all black holes immersed in any gravitational field, regardless of what the source of that field might be. In fact, we seek to introduce a set of thermodynamic quantities which can be universally defined, regardless of the field being created by another black hole, by a star or by some cosmological fluid. Finally, by property **(4)** we mean that such quantities must have a universal limit when the gravitational field that surrounds the black hole is removed. Intuitively, one might think that any such extra quantity appearing in (2.1) must vanish when the gravitational field is removed since in that case there are no other quantities characterising the (uncharged) black hole. However, for reasons that will become apparent later, we will not require from the start that such quantities must vanish in that limit but we will ultimately argue that the correct physical picture is one where such quantities do vanish in that limit.

[3] Other examples are bi-rings, di-rings and several multiple combinations of these [75, 129, 88, 128].

2.1.1 Pressure and volume for Anti-de Sitter black holes

As we have mentioned, we look for modifications of (2.1) regardless of what is the source of the gravitational field. One can think of Anti-de Sitter (AdS) black holes in D spacetime dimensions as being immersed in the gravitational field created by a cosmological fluid with pressure

$$P_e = -\frac{\Lambda}{8\pi G} \quad , \quad \Lambda = \frac{(D-1)(D-2)}{2L^2} \quad , \tag{2.2}$$

which sources Einstein equations. In this case, variations in the gravitational field are controlled by variations of the cosmological constant Λ, or alternatively, by variations of the AdS radius L. In order to analyse the modifications of (2.1) we consider the simplest case of the Schwarzschild-AdS black hole in $D=4$ with mass, entropy and temperature given by

$$M = \frac{r_+}{2G}\left(1+\frac{r_+^2}{L^2}\right) \quad , \quad S = \frac{\pi}{G}r_+^2 \quad , \quad T = \frac{1}{4\pi r_+}\left(1+3\frac{r_+^2}{L^2}\right) \quad , \tag{2.3}$$

where r_+ is the horizon radius. Allowing for variations of the length scale L, one can easily verify that these black holes satisfy the first law of thermodynamics

$$dM = TdS + B_k d\mathbb{L}_k \quad , \tag{2.4}$$

where $\mathbb{L}_k$ is proportional to some power of L such that $\mathbb{L}_k = \lambda L^k$, where λ is an arbitrary constant which can depend on Newton's constant G but otherwise cannot depend on any of the thermodynamic variables of the solution such as temperature T. In turn, the response B_k is given by

$$B_k = \left(\frac{\partial \mathcal{F}}{\partial \mathbb{L}_k}\right)_T \quad , \tag{2.5}$$

where $\mathcal{F}$ is the Gibbs free energy $\mathcal{F} = M - TS$. Moreover, this leads to a corresponding Smarr relation which follows from the Euler scaling argument

$$(D-3)M - (D-2)TS = kB_k\mathbb{L}_k \quad . \tag{2.6}$$

From this point of view, any particular choice of k, from the infinite set of quantities B_k and their infinite set of conjugate variables $\mathbb{L}_k$, is as good as any other in describing variations of the external gravitational field - a consideration which has not been previously stated in the literature.

A very popular choice in describing these variations in AdS has been the choice $k=-2$ and the identification $\mathbb{L}_{-2}$ of the extrinsic spacetime pressure, i.e. $\mathbb{L}_{-2} = P_e$ [137, 65, 59, 63, 141, 67, 145].[4] In this case, $\mathbb{L}_{-2}$ has dimensions of pressure and the quantity $B_{-2} = -V_i$ has dimensions of volume. For the particular case of the Schwarzschild-AdS black hole in $D=4$, this intrinsic black hole volume takes the form

$$V_i = \frac{4}{3}\pi r_+^3 \quad . \tag{2.7}$$

There are several interesting aspects of this particular choice. First of all, the quantities P_e and V_i have direct analogues with classical thermodynamic systems, they have dimensions

[4]There is a large literature on considering the cosmological constant as a thermodynamic variable, starting with the early papers [120, 171, 119] and also for example the later work [41, 161, 176, 175, 144, 174]

of pressure and volume, respectively [64, 66, 62]. In particular the phase diagram $P_e(V_i)$ for Schwarzschild-AdS black holes can be recast as a Van der Waals equation [141][5]

$$P_e = \frac{T}{v} - \frac{1}{2\pi v^2} \quad , \quad v = 2\left(\frac{3V_i}{4\pi}\right)^{\frac{1}{3}} \quad , \tag{2.8}$$

strengthening further the analogy with classical thermodynamic systems. Furthermore, the quantity P_e is physically meaningful, since it is the pressure of the cosmological fluid, while V_i has a geometric interpretation, as it can be obtained for a given black hole in AdS by means of evaluating the Killing potential [137]. Moreover, there is something peculiar to the volume V_i. As it may be seen from (2.7), the volume remains constant as the cosmological constant is sent to zero. The fact that this is the case and that the volume (2.7) coincides with the naive volume[6] in the flat spacetime limit has been seen as further strengthening the case for introducing the pair of thermodynamic variables (P_e, V_i), despite the fact that no such quantity enters the first law (2.1) for asymptotically flat black holes.

2.2 Pressure, volume and gravitational tension: a different point of view

We would like to understand whether or not the introduction of the set of variables (P_e, V_i) is unique to AdS or can actually satisfy the desired properties **(1)**-**(4)** which we have described above. Regarding property **(2)** it is clear that it is satisfied by this set of variables, which also has property **(3)** since the exercise that we have performed for the Schwarzschild-AdS black hole can be carried out for any black hole with non-trivial asymptotics and because extra terms in the first law of the form (2.4) can be obtained by performing the Legendre transform

$$M \to M + B_k \mathbb{L}_k \quad , \tag{2.9}$$

therefore moving onto extended phase space in which the length scale $\mathbb{L}_k$ is allowed to vary.

However, property **(1)** is not satisfied in a straightforward way. By this we mean that one cannot use Killing potentials to obtain the volume V_i for any black hole in an arbitrary gravitational field, since the volume obtained by a suitable integral of the Killing potential is non-zero if there is a non-zero cosmological constant [137]. As we will show in the course of this work, defining V_i does not require a cosmological constant nor introducing matter in Einstein equations. In fact, it can be defined in backgrounds with non-trivial length scales which are solutions of the vacuum Einstein equations, such as plane wave solutions or black hole solutions. Moreover, the pair of variables (P_e, V_i) does not satisfy property **(4)**, since as we will show in this paper, the volume V_i in flat spacetime, obtained via the limit in which the background length scales are removed, is meaningless. We show this by taking two different charged black hole solutions both of which, when the length scales are removed, reduce to the same charged rotating asymptotically flat black hole but lead to two different volumes. In addition, one may study asymptotic plane wave black holes with an arbitrary number of length scales L_a which, when $L_a = 0$, lead to an arbitrary number of volumes

[5]For earlier work in the same spirit see [49, 41].

[6]By the *naive volume* we mean the volume that can be obtained by taking the metric and performing a volume integration up to the horizon radius. For other black holes like rotating black holes there have been proposals for how to define the volume [152, 25, 26] and we refer the reader to the review [61] for a more detailed explanation of these cases.

describing an asymptotically flat black hole. In this sense, one would have to argue that an asymptotic flat black hole is characterized by an infinite set of volumes.

These considerations demand another point of view and the introduction of new quantities which can be generally applied to any context where a gravitational field surrounding the black hole is present. We consider borrowing a concept which has its roots in the study of black branes and Kaluza-Klein (KK) black holes, namely, gravitational tension (or gravitational binding energy) [173, 172, 105] and applying it in a broader context.

Gravitational tension can be thought of as the contribution to the black hole energy due to the energy stored in the surrounding gravitational field. In the case of the black hole being a black brane, it is the same as the brane tension associated with a given non-compact direction. This notion of energy is described by the simplest choice of quantities in (2.4), namely, $k=1$ and $\lambda=1$ for which $\mathbb{L}_1 = L$ and $B_1 = \boldsymbol{\mathcal{T}}$. The first law of thermodynamics then takes the following form

$$dM = TdS + \sum_a \Omega_a dJ_a + \Phi_H dQ_{(p)} + \sum_a \boldsymbol{\mathcal{T}}_a dL_a \quad , \tag{2.10}$$

where we have allowed for the presence of a p-form charge $Q_{(p)}$ and corresponding chemical potential Φ_H as well as the existence of several length scales L_a and their corresponding tensions per unit length $\boldsymbol{\mathcal{T}}_a$. The corresponding Smarr relation (2.6) reads

$$(D-3)M - (D-2)\left(TS + \sum_a \Omega_a J_a\right) - (D-3)\Phi_{\mathrm{H}} Q = \hat{\boldsymbol{\mathcal{T}}} \quad , \tag{2.11}$$

where $\hat{\boldsymbol{\mathcal{T}}}$ is the total tension (or gravitational binding energy) given by

$$\hat{\boldsymbol{\mathcal{T}}} = \sum_a \boldsymbol{\mathcal{T}}_a L_a = \sum_a L_a \left(\frac{\partial \mathcal{F}}{\partial L_a}\right)_{T,\Omega_a,\Phi_H} \quad . \tag{2.12}$$

From here we note that the total tension is obtained by summing the result of acting with the scaling operators $d/d\log L_a$ on the free energy. If we apply this to the case of the Schwarzschild-AdS black hole in $D=4$ for which $L_1 = L$ and $\boldsymbol{\mathcal{T}}_1 = \boldsymbol{\mathcal{T}}$ we obtain

$$\boldsymbol{\mathcal{T}} = -\frac{r_+^3}{L^3} \quad . \tag{2.13}$$

This quantity vanishes, as well as $\hat{\boldsymbol{\mathcal{T}}}$, in the limit $L \to \infty$, in which the surrounding gravitational field is removed.

The introduction of these new pairs of variables $(L_a, \boldsymbol{\mathcal{T}}_a)$ has several advantages. First of all, they have a well defined physical and thermodynamic meaning. If we take AdS spacetime as an example then $L_1 = L$ is a measure of the *spacetime volume* associated with each spacetime direction[7], while $\boldsymbol{\mathcal{T}}_a$ is a measure of the energy stored per unit spacetime volume. In fact, when considering the case of black branes, $\boldsymbol{\mathcal{T}}_a$ is the brane tension per unit length, which is equal to minus the brane pressure. The variables $(L_a, \boldsymbol{\mathcal{T}}_a)$ can be thought as the reverse of the variables (P_e, V_i), in which brane tension has replaced spacetime pressure and

[7]Note that since we are using the word *volume* associated to a given spacetime direction then this is equivalent to using the word *length*.

spacetime volume has replaced black hole volume. Indeed, such point of view had already been taken in [104, 106, 139, 103] for the particular case of KK black holes.

The variables $(L_a, \boldsymbol{\mathcal{T}}_a)$ besides having a physical meaning, also have a well defined geometrical meaning. The length scales L_a are simply the length scales associated with the curvature of spacetime along given spacetime directions. The gravitational tensions $\boldsymbol{\mathcal{T}}_a$, in turn, can be obtained in several different ways. If the spacetime has a compact direction, such as KK spacetimes, or if the the horizon is non-compact then one can simply apply the prescription of [105].[8] If the black hole admits a blackfold limit, as large classes of higher-dimensional black holes do [85, 16, 98, 43, 83, 13, 10, 11], then the prescription of [105] also applies, once we zoom locally into the horizon rendering it brane-like. The total tension is subsequently obtained by integrating the local tension over the blackfold worldvolume. If the black hole does not admit such limit, as is the case of the Schwarzschild-AdS black hole, then the prescription of [105] needs to be generalised and we leave this generalization for future work. For the moment, when dealing with these cases, we simply apply (2.12).

Finally, the set of variables $(L_a, \boldsymbol{\mathcal{T}}_a)$ can be introduced in the presence of an arbitrary gravitational field and in this respect it is not different than the set of variables (P_e, V_i). However, the variables $(L_a, \boldsymbol{\mathcal{T}}_a)$ satisfy property **(4)** which the set (P_e, V_i) does not. More precisely, when removing the gravitational field we find the universal result $\boldsymbol{\mathcal{T}}_a \to 0$, which naturally does not lead to extra quantities describing an asymptotically flat black hole.

2.2.1 Brief summary

In order to illustrate the ideas expressed above we will first consider the case of distorted black holes in Sec. 2.3, i.e., localized KK black holes which can be seen as black objects surrounded by the presence of their own gravitational field. In this context, we will show that the concept of tension is much more natural to introduce than the notion of black hole volume and we will already give evidence for the non-universality of black hole volume in the flat spacetime limit.

In Sec. 2.4 we construct a series of new non-trivial and perturbative charged black hole solutions in Anti-de Sitter, plane wave and Lifshitz spacetimes using the blackfold approach. In here we study examples of spacetimes with multiple length scales, such as plane waves, and construct analogues of the higher-dimensional Kerr-Newman solution of [43] in Anti-de Sitter and plane wave spacetimes. These new black hole solutions are interesting in their own right, in particular, the Kerr-Newman solution in AdS for which there is no corresponding exact solution. Furthermore, we provide the first example of a black hole with non-trivial horizon topology in Lifshitz spacetimes. The reader may skip this section entirely if he/she is only concerned with the implications of these results to the modifications of (2.1).

In Sec. 2.5 we construct perturbative solutions in backgrounds with a black hole. One of these solutions corresponds to a specific limit of the black saturn solution. We then study in detail the example of this black hole binary system in five spacetime dimensions using the exact and analytic solution of [74]. Here we focus on the case in which the Myers-Perry black hole in the centre is not rotating and show that the black ring horizon satisfies a first law of the form (2.10).

Finally, in Sec. 2.6, we briefly use the new solutions of Sec. 2.4 to show that the notion of volume in flat spacetime is non-universal, while in Sec. 6.1 we discuss some of the

[8] We note that this prescription does not require introducing Killing potentials.

limitations of this work and future extensions of these research directions. We also provide some interesting results in the appendices, namely, in App. A we have collected all the thermodynamic properties of the new perturbative solutions constructed in this paper, while in App. A.10 we have the explicit construction of a family of black holes carrying string charge. In App. A.12 we give the thermodynamic quantities of the black saturn solution in the blackfold regime and compare it with our blackfold constructions.

2.3 Black holes on cylinders: the Kaluza-Klein case

In this section we analyse the thermodynamic properties of localized KK black holes which were constructed in a perturbative expansion in dimensions greater than four [101]. These objects provide examples of black holes immersed in their own non-trivial gravitational field. We review that the thermodynamics of these objects follow (2.10), as noted in [106, 139, 101], and that the tension can be extracted from the free energy using (2.12). We then show that the concept of black hole volume is not the desirable one when analysing variations in the surrounding gravitational field, which are controlled by variations in L - the KK compactification parameter. In the end, we also take a look at the case of KK black strings.

2.3.1 The localised black hole

The localised black hole in KK spacetime is a static and perturbative solution found in [101] in $D \geq 5$, obtained by perturbing the Schwarzschild black hole to leading order in the parameter r_0/L where r_0 is the horizon radius and L the size of the KK circle. Its thermodynamic properties can be found in [101] and read

$$M = \frac{\Omega_{(D-2)}}{16\pi G}(D-2)r_0^{D-3}\left(1+\frac{1}{2}\beta\left(\frac{r_0}{L}\right)^{D-3}\right) \quad , \tag{2.14}$$

$$T = \frac{(D-3)}{r_0}\left(1-\frac{(D-2)}{(D-3)}\beta\left(\frac{r_0}{L}\right)^{D-3}\right) \quad , \tag{2.15}$$

$$S = \frac{\Omega_{(D-2)}}{16\pi G}r_0^{D-2}\left(1+\frac{(D-2)}{(D-3)}\beta\left(\frac{r_0}{L}\right)^{D-3}\right) \quad , \tag{2.16}$$

where we have defined[9]

$$\beta = \frac{\zeta(D-3)}{(2\pi)^{D-3}} \quad . \tag{2.17}$$

As noted already in [104, 101], even though it is a localised black hole, it has a tension which can be obtained via the Smarr relation (2.11) and the above thermodynamic quantities. It reads

$$\hat{\boldsymbol{\mathcal{T}}} = \frac{\Omega_{(D-2)}}{32\pi G}\beta(D-2)(D-3)r_0^{D-3}\left(\frac{r_0}{L}\right)^{D-3} \quad . \tag{2.18}$$

We note that this quantity vanishes once we take the decompactification limit $L \to \infty$. This result can also be obtained using (2.12) by evaluating the free energy

$$\mathcal{F} = \frac{\Omega_{(D-2)}}{16\pi G}r_0^{D-3}\left(1+\frac{1}{2}(D-2)\beta\left(\frac{r_0}{L}\right)^{D-3}\right) \quad , \tag{2.19}$$

[9] Here $\zeta(s)$ is the Riemann Zeta function defined as $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$.

and provides a non-trivial check of formula (2.12). The existence of this tension justifies the thermodynamic interpretation in terms of brane tension per unit length $\boldsymbol{\mathcal{T}} = \hat{\boldsymbol{\mathcal{T}}}/L$ and spacetime volume L.

Defining spacetime pressure and black hole volume

We now consider the possibility of defining a spacetime pressure and black hole volume. As we have noted in the previous section, black holes in spacetimes with non-trivial length scales satisfy (2.1) for an infinite set of quantities. Following the same footsteps as in the introduction, we can attempt to define the black hole volume by choosing $k = -2$. This leads to the pressure $P_e = \lambda L^{-2} G^{-1}$ and the volume

$$V_i = -\frac{\Omega_{(D-2)}}{64\pi}\frac{L^2}{\lambda}\beta(D-2)(D-3)r_0^{D-3}\left(\frac{r_0}{L}\right)^{D-3} \quad . \tag{2.20}$$

Looking at the above expression, we notice that in dimensions $D > 5$, the volume V_i goes to zero in the decompactification limit $L \to \infty$, when one would expect that it would reduce to the non-zero volume of the Schwarzschild black hole in $D > 5$ (see e.g. [6]) which was obtained by taking the flat spacetime limit of the volume of the Schwarzschild-AdS black hole,

$$V_i^{\text{sch}} = \frac{\Omega_{(D-2)}}{16\pi(D-1)} r_0^{D-1} \quad . \tag{2.21}$$

In $D = 5$ we find that $V_i \propto r_0^4$ and hence we could choose λ appropriately so that V_i for KK black holes in $D = 5$ would be equal to (2.21) for the Schwarzschild black hole in the same number of spacetime dimensions.

This definition of black hole volume, besides only making some sense in $D = 5$, would also loose its geometric interpretation since in $D = 5$ one would expect the black hole volume to be $V_i \propto r_0^3 L$. It is possible to attempt defining the black hole volume by introducing a new length scale and defining a shifted mass. This new length scale $\tilde{L}$ leads to the right scaling of the volume in the decompactification limit $\tilde{L} \to \infty$ (or $L \to \infty$), namely,

$$\left(\frac{\tilde{L}}{r_0}\right)^2 = \left(\frac{L}{r_0}\right)^{D-3} \quad . \tag{2.22}$$

In this way, we introduce the spacetime pressure $P_e = \lambda G^{-1}\tilde{L}^{-2}$ and define a new mass $\tilde{M}$ by shifting the mass M by a fraction b of the tension such that

$$\tilde{M} = M + b\hat{\boldsymbol{\mathcal{T}}} \quad . \tag{2.23}$$

By requiring the correct thermodynamic behaviour, namely,

$$\frac{\partial \tilde{M}}{\partial S}|_{P_e} = T \quad , \tag{2.24}$$

this implies that we must choose

$$b = \frac{1}{D-3} - \frac{2}{D-1} \quad . \tag{2.25}$$

Using (2.5) with $k = -2$ and $\tilde{L}$ as the new length scale we obtain the black hole volume

$$V_i = -\frac{\Omega_{(D-2)}}{16\pi}\frac{\beta}{\lambda}\frac{(D-2)(D-3)}{(D-1)}r_0^{D-1} \quad . \tag{2.26}$$

Comparing this volume with the volume of the Schwarzschild black hole in D dimensions (2.21) we fix the factor λ such that

$$\lambda = -\beta(D-2)(D-3) \quad . \tag{2.27}$$

While we see that introducing a new mass $\tilde{M}$, with a priori no inherent physical meaning, allows us to recover the Schwarzschild black hole volume, we believe that such possibility is not so natural. The concept of black hole volume and spacetime pressure is useful if the ADM mass M would satisfy the first law (2.10). However, this is not the case for this particular example. Consequently, we propose that the concept of tension $\hat{\boldsymbol{\mathcal{T}}}$ is a more useful one for studying variations in the external gravitational field for KK black holes.

2.3.2 The black string

Here we briefly consider the case of the KK black string. This solution is simply obtained by taking an asymptotically flat Schwarzschild black string and compactifying the infinitely extended direction on a circle of radius L. The resulting free energy for $D \geq 5$ is given by

$$\mathcal{F} = \frac{\Omega_{(n+1)}}{16\pi G}r_0^{D-4}L \quad , \tag{2.28}$$

and once applying formula (2.12) leads to the tension $\hat{\boldsymbol{\mathcal{T}}} = \mathcal{F}$. This tension naturally behaves like $\hat{\boldsymbol{\mathcal{T}}} \to \infty$ when we take the decompactification limit $L \to \infty$. This is expected since the total tension of an asymptotically flat black string diverges while the tension density $\boldsymbol{\mathcal{T}} = \hat{\boldsymbol{\mathcal{T}}}/L$ remains finite. This exercise had the purpose of showing that the tension per unit length of asymptotically flat black branes can be obtained by compactifying the infinitely extended directions on a circle and applying formula (2.12).

As in the previous case, we could introduce a naive definition of volume $V_i \propto r_0^{D-4}L^3$ or, by introducing a new length scale $\tilde{L} = r_0^2 L^{-1}$ and a new mass $\tilde{M}$ as in (2.23), define a new volume of the form $V_i \propto r_0^{D-4}\tilde{L}$, which would give rise to (2.21) in $D-1$ dimensions once we take the limit $\tilde{L} \to 0$ and evaluate the black hole volume per unit length $V_i/\tilde{L}$. The latter would require $b = -2/(D-2)$ and $\lambda = (D-4)/2$. However, for the same reasons as for the localized black hole, we find both of these possibilities unnatural.

2.4 Blackfolds in background spacetimes with intrinsic length scales

In this section we construct new perturbative (charged) black hole solutions in (Anti)-de Sitter, plane wave and Lifshitz spacetimes using the blackfold approach, which we first describe in Sec. 2.4.1. Of special importance is the perturbative construction of the Kerr-Newman solution in higher-dimensions both in (Anti)-de Sitter and plane wave spacetimes. Furthermore, we introduce the pair of variables $(\boldsymbol{\mathcal{T}}, L_a)$ for all these solutions in order to describe variations in the surrounding gravitational field. This pertubative black hole solutions are interesting in their own right and provide evidence for the existence of a rich phase space of black hole solutions in these non-trivial spacetimes. If the reader is interested in the implications of these solutions to the notion of spacetime pressure and black hole volume, he/she can skip this section entirely and move on to Sec. 2.5.

2.4.1 Blackfold essentials

In the blackfold approach [87, 84], one constructs stationary black objects with compact horizons by starting with black brane solutions, with horizon scale r_0, of some (super)gravity theory and wrapping these on compact submanifolds with a characteristic scale R. In the regime $r_0 \ll R$ the near-horizon region is well approximated by a perturbed black brane geometry and can be corrected order-by-order in a derivative expansion [86, 20, 45, 8, 9, 14]. The prototypical case to keep in mind is the construction of an asymptotically flat black ring in any dimension $D \geq 5$ by wrapping a thin black string with horizon radius r_0 on a large circle with radius $R \gg r_0$ [86].

This construction, however, is completely general and can be applied to the wrapping of black branes in any asymptotic background. Since it is convenient to work with known exact black brane solutions, the explicit examples studied so far in the literature involve the bending/wrapping of asymptotically flat black branes. In this work, we are interested in bending asymptotically flat black branes carrying electric charge, which were found in [43].[10] These are black branes which are exact solutions of the following Einstein-Maxwell-dilaton action

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\left(R - 2(\nabla\phi)^2 - \frac{1}{4}e^{-2a\phi}F^2\right) \quad , \tag{2.29}$$

where ϕ is the dilaton field, with a being the dilaton coupling and F is the two-form field strength dA, with A being the 1-form gauge field. It is convenient to introduce the parameter N defined as

$$a^2 = \frac{4}{N} - \frac{2(D-3)}{(D-2)} \quad . \tag{2.30}$$

The perturbative black hole solutions constructed from these brane geometries are not exclusively solutions of the action (2.29). Instead, other terms, such as the cosmological constant and other field content, can be added to the above action without affecting this construction as long as the curvature scales associated with each new field are much larger than the horizon size r_0. More precisely, if the blackfold is being constructed in backgrounds with a set of intrinsic length scales L_a then we must require that $r_0 \ll \min(R, L_a)$.[11] This implies that to leading order neither the curvature of the worldvolume nor the curvature associated with background scales are felt near the horizon and hence that locally the blackfold is still described by an asymptotically flat black brane solution of the action (2.29).

In order to locally wrap black branes, and in the absence of couplings to gauge/dilaton external fields, one must satisfy the local constraint equation, which can be derived from Einstein equations [86, 45], namely,

$$T^{ab}{K_{ab}}^{i} = 0 \quad , \tag{2.31}$$

where ${K_{ab}}^{i}$ is the extrinsic curvature tensor of the embedding geometry and T^{ab} is the stress-energy tensor corresponding to the charged black brane, which in this case takes the perfect fluid form [43]

$$\begin{aligned} &T^{ab} = (\epsilon + P)u^a u^b + P\gamma^{ab} \quad , \quad (\epsilon + P) = -nP + \Phi\mathcal{Q} \quad , \\ &P = -\frac{\Omega_{(n+1)}}{16\pi G}r_0^n \quad , \quad \Phi = \tanh\alpha \quad , \quad \mathcal{Q} = \frac{\Omega_{(n+1)}}{16\pi G}r_0^n n\sqrt{N}\sinh\alpha\cosh\alpha \quad . \end{aligned} \tag{2.32}$$

[10] We consider one example of branes carrying string charge in App. A.10.

[11] A more rigorous determination of the regime of validity of blackfold configurations consists in evaluating second order world volume scalar invariants [10].

Here we have introduced the fluid pressure P, energy density ϵ, chemical potential Φ, electric charge density $\mathcal{Q}$ and the induced metric on the $(p+1)$-dimensional geometry γ_{ab} in dimensions $D=n+p+3$. The fluid variables $P,\epsilon,\mathcal{Q}$ depend only on the local temperature $\mathcal{T}$ and the local chemical potential Φ, which are in turn functions of the coordinates σ^a along the world volume. We have also introduced α, which is the charge parameter of the brane and sometimes more convenient to parameterise blackfold solutions than the chemical potential Φ.

In stationary equilibrium, which is the case we are interested here, as they give rise to stationary black holes, the fluid velocities u^a must be aligned with a world volume Killing vector field $\mathbf{k}^a$ with modulus $\mathbf{k}$, which we can write, without loss of generality, as

$$\mathbf{k}^a\partial_a=\partial_\tau+\sum_{\hat{a}}\Omega_{\hat{a}}\partial_{\phi_{\hat{a}}} \quad , \tag{2.33}$$

where τ is the time-like world volume direction and $\Omega_{\hat{a}}$ is the angular velocity associated with each of the Cartan angles $\phi_{\hat{a}}$.[12] Furthermore, in equilibrium we also have that the global temperature T and global chemical potential Φ_{H} are determined via a redshift of the local thermodynamic potentials such that $T=\mathbf{k}\mathcal{T}$ and $\Phi_{\mathrm{H}}=\mathbf{k}\Phi$. This leads to the relation between the horizon size r_0 and the global thermodynamic potentials [43],

$$r_0=\frac{n}{4\pi T}\mathbf{k}\left(1-\frac{\Phi_{\mathrm{H}}^2}{N\mathbf{k}^2}\right)^{\frac{N}{2}} \quad . \tag{2.34}$$

A given stationary blackfold configuration is thus described by an induced world volume line element $\mathbf{ds}^2=\gamma_{ab}d\sigma^a d\sigma^b$, a world volume Killing vector field $\mathbf{k}^a$ and the global potentials T and Φ_{H}.

Stationary configurations may also have boundaries. In this case, the constraint equation (2.31) must also be supplemented by the boundary condition

$$\mathbf{k}|_{\partial\mathcal{W}_{p+1}}=0 \quad , \tag{2.35}$$

which, in the uncharged case ($\Phi_{\mathrm{H}}=0$) translates into the condition that the fluid must be moving at the speed of light on the boundary, while for non-zero charge, it has the physical interpretation that the brane must be extremal at the boundary. Stationary configurations have topologies of the form $\mathbb{R}\rtimes\mathbb{B}^{(p)}\rtimes\mathbb{S}^{(n+1)}$, where $\mathbb{B}^{(p)}$ is the topology of the spatial part $\mathcal{B}_p$ of the world volume geometry $\mathcal{W}_{p+1}$ and $\mathbb{S}^{(n+1)}$ denotes the topology of the transverse spherical space associated with the black brane geometry with properties (2.32).[13] In the case of the existence of boundaries, $\mathbb{B}^{(p)}$ is not the topology of $\mathcal{B}_p$ but instead the result of a non-trivial fibration over $\mathcal{B}_p$.

In global thermodynamic equilibrium, the constraint equation (2.31) can be equally derived from an effective free energy functional $\mathcal{F}$ given by [83, 43, 98]

$$\mathcal{F}[X^i]=-\int_{\mathcal{B}_p}R_0P\left(\mathcal{T},\Phi\right) \quad , \tag{2.36}$$

where X^i denotes the set of transverse scalars describing the position of the surface in the ambient space and R_0 is the modulus of the time-like worldvolume Killing vector field ∂_τ.

[12]The Killing vector field (2.33) is required to map to a background Killing vector field [84].

[13]Here we have assumed that the world volume geometry is of the form $\mathbb{R}\times\mathcal{B}_p$.

Given the effective free energy, one can easily extract the conjugate global thermodynamic potentials to $T, \Omega_{\hat{a}}, \Phi_{\rm H}$ of the blackfold configuration, namely, the entropy S, the angular momenta $J_{\hat{a}}$ and the electric charge Q via the corresponding relations

$$S = -\frac{\partial \mathcal{F}}{\partial T} \quad , \quad J_{\hat{a}} = -\frac{\partial \mathcal{F}}{\partial \Omega_{\hat{a}}} \quad , \quad Q = -\frac{\partial \mathcal{F}}{\partial \Phi_{\rm H}} \quad . \tag{2.37}$$

These thermodynamic properties can also be obtained via the integration of appropriate conserved currents [84, 43]. With these quantities introduced, we note that the free energy (2.36) satisfies

$$\mathcal{F} = M - TS - \sum_a \Omega_a J_a - \Phi_{\rm H} Q_{(p)} \quad . \tag{2.38}$$

The total tension can also be obtained using (2.12), and using appropriate conserved currents, it is possible to derive its general form

$$\hat{\boldsymbol{\mathcal{T}}} = -\int_{\mathcal{W}_{p+1}} dV_{(p)} R_0 \left(\gamma^{ab} + n^a n^b\right) T_{ab} \quad , \quad n^a \partial_a = R_0^{-1} \partial_\tau \quad . \tag{2.39}$$

By the same token, it is possible to derive, from general principles, the Smarr relation (2.11) and the first law (2.10). The Smarr relation (2.11) with the tension (2.39) was first derived in [85] and using the Euler argument, it follows that the first law (2.10) is satisfied. We thus see that blackfolds naturally exhibit this universal thermodynamic behavior in spacetimes with non-trivial asymptotics.

2.4.2 (Anti)-de Sitter background

Here we construct novel black holes with electric charge in global (A)dS and in App. A.10 we consider the case of black holes with dipole charge. We write the global (A)dS metric in the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{(D-2)} \quad , \quad f(r) = 1 + \frac{r^2}{L^2} \quad , \tag{2.40}$$

where L is the AdS radius. In order to obtain the de Sitter metric we simply perform the Wick rotation $L \to iL$.

Charged black odd-spheres

This type of configurations are obtained by embedding a p-dimensional sphere with radius R in the background (2.40). We set the configuration to rotate with equal angular velocity Ω in each of the $[(p+1)/2]$ Cartan angles of the p-dimensional sphere, labelled by $\phi_{\hat{a}}$. The embedded geometry and the corresponding Killing vector field are given by

$$\mathbf{ds}^2 = -f(\mathbf{R})d\tau^2 + R^2 d\Omega^2_{(p)} \quad , \quad \mathbf{k}^a \partial_a = \partial_\tau + \Omega \sum_{\hat{a}=1}^{[(p+1)/2]} \partial_{\phi_{\hat{a}}} \ , \ f(\mathbf{R}) = 1 + \mathbf{R}^2 \ , \tag{2.41}$$

where we have defined the dimensionless radius $\mathbf{R} = R/L$. We choose to parametrise the resulting configuration in terms of the variables $r_0, \mathbf{R}, \alpha$. In terms of these, the free energy takes the simple form

$$\mathcal{F}[R] = \frac{\Omega_{(n+1)} V_{(p)}}{16\pi G} f(\mathbf{R}) r_0^n \quad , \tag{2.42}$$

where $V_{(p)} = \Omega_{(p)} R^p$. Upon variation of this with respect to R, restricting to the cases where p is odd, and solving the resulting equation leads to the equilibrium condition

$$\Omega^2 R^2 = (1+\mathbf{R}^2)\frac{\mathbf{R}^2\left(nN\sinh^2\alpha + n + p + 1\right) + p}{\mathbf{R}^2\left(nN\sinh^2\alpha + n + p + 1\right) + nN\sinh^2\alpha + n + p} \quad . \tag{2.43}$$

This expression connects to several others in the literature. In the uncharged limit $\alpha = 0$, this yields the result obtained in [16], while in the flat space limit $L \to \infty$, this yields the result of [43]. When taking both limits, $\alpha \to 0$ and $L \to \infty$ we obtain the result of [85].

Properties of the solution

There are several interesting cases that should be noted. First of all, these black holes with horizon topology $\mathbb{R}\times\mathbb{S}^{(p)}\times\mathbb{S}^{(n+1)}$ admit an extremal limit. Taking $\alpha \to \infty$ we obtain the equilibrium condition

$$\Omega^2 R^2 = \frac{1+\mathbf{R}^2}{2} \quad , \tag{2.44}$$

for extremal black odd-spheres. In the flat space case for which $\mathbf{R} = 0$ this reduces to the analysis of [43].

Secondly, in the deSitter case for which $L \to iL$ and hence $\mathbf{R} \to i\mathbf{R}$, not all values of $\mathbf{R}$ are allowed. In fact we find the two possible regimes

$$\mathbf{R} \leq 1 \quad \vee \quad \frac{p}{nN\sinh^2\alpha + n + p + 1} \leq \mathbf{R} \leq \frac{nN\sinh^2\alpha + n + p}{nN\sinh^2\alpha + n + p + 1} \quad . \tag{2.45}$$

The extremal branch of solutions lies within the regime $\mathbf{R} \leq 1$. Furthermore, as noted in [16], in deSitter spacetime there can exist static solutions ($\Omega = 0$) and, as noted in [10], they are valid for all $p \geq 1$ and not only for odd p. These are solutions for which the radius takes the specific value of

$$\mathbf{R} = \frac{p}{nN\sinh^2\alpha + n + p + 1} \quad . \tag{2.46}$$

We note here that this branch of static solutions does not admit an extremal limit.

Gravitational tension

Using (2.42) we can obtain all thermodynamic properties which we collect in App. A while here we present expressions for the tension. Using formula (2.12) together with (2.42), we obtain the total tension given by

$$\hat{\boldsymbol{\mathcal{T}}} = -\frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \mathbf{R}^2\sqrt{1+\mathbf{R}^2}\left(nN\sinh^2\alpha + n + p + 1\right) \quad , \tag{2.47}$$

which is a function of T, Ω and L, where Ω was given in (2.43). From here we can introduce the tension per unit spacetime volume such that

$$\boldsymbol{\mathcal{T}} = \frac{\hat{\boldsymbol{\mathcal{T}}}}{L} = -\frac{\Omega_{(n+1)}V_{(p)}}{16\pi G}\frac{r_0^n}{L}\mathbf{R}^2\sqrt{1+\mathbf{R}^2}\left(nN\sinh^2\alpha + n + p + 1\right) \quad . \tag{2.48}$$

These quantities satisfy the first law (2.10) and the Smarr relation (2.11). Furthermore, both these quantities vanish once we take the limit $L \to \infty$, leaving their corresponding asymptotically flat counterparts with no extra quantities characterizing them.

Charged black discs: analogue of the Kerr-Newman black hole

In this section we make a perturbative construction of the analogue of the Kerr-Newman black hole in higher-dimensional (A)dS with one single angular momentum. This corresponds, in the blackfold approximation, to an electrically charged rotating disc with induced metric and Killing vector field

$$\mathbf{ds}^2 = -f(\rho)d\tau^2 + f(\rho)^{-1}d\rho^2 + \rho^2 d\phi^2 \ , \ \mathbf{k}^a\partial_a = \partial_\tau + \Omega\partial_\phi \ , \ f(\rho) = 1 + \frac{\rho^2}{L^2} \ . \tag{2.49}$$

This geometry develops a boundary when $\mathbf{k} = 0$, for which the brane becomes locally extremal, corresponding to the maximum of ρ given by

$$\rho_{\max} = \frac{\sqrt{1-\Phi_{\mathrm{H}}^2/N}}{\Omega\sqrt{\xi}} \ \ , \ \ \xi = 1 - \frac{1}{L^2\Omega^2} \ \ , \tag{2.50}$$

which implies in the AdS case that $\Omega L \geq 1$. This disc configuration trivially solves the blackfold equations (2.31) since it is a minimal surface [10]. The thickness of the disc and the charge parameter are given by

$$\begin{aligned} r_0(\rho) &= \frac{n}{4\pi T}\left(1-\xi\rho^2\Omega^2\right)^{\frac{1-N}{2}}\left(1-\xi\rho^2\Omega^2-\frac{\Phi_{\mathrm{H}}^2}{N}\right)^{N/2} \ \ , \\ \tanh\alpha(\rho) &= \frac{\Phi_{\mathrm{H}}/\sqrt{N}}{\sqrt{1-\xi\rho^2\Omega^2}} \ \ . \end{aligned} \tag{2.51}$$

Therefore, at the boundary $\rho_{\max}$ the thickness r_0 of the disc vanishes and hence the resulting black holes have topology $\mathbb{R}\times\mathbb{S}^{(D-2)}$. The thickness remains finite for all values of ξ and hence this configuration lies within the regime of validity $r_0 \ll L$.[14] In the uncharged case $\Phi_{\mathrm{H}} = 0$, this reduces to the construction of [16] and when $L \to \infty$, hence when $\xi \to 1$, it reduces to the higher-dimensional Kerr-Newman solution perturbatively constructed in [43].

This configuration has several interesting properties. In particular, it admits an extremal limit, for which $\Phi_{\mathrm{H}} \to \sqrt{N}$. In this case it seems that the size of the disc (2.50) would shrink to zero. As noted in [43] for the asymptotically flat case, one must also have that $\Omega \to 0$ such that the ratio $\sqrt{1-\Phi_{\mathrm{H}}^2/N}/\Omega$ remains finite. However, in the presence of the cosmological constant this is not possible. Instead, attaining extremal regimes is only possible in AdS for which one must send $\Omega L \to 1$, hence $\xi \to 0$, such that the ratio $\sqrt{1-\Phi_{\mathrm{H}}^2/N}/\sqrt{\Omega^2L^2-1}$ remains constant. From (2.51) it implies that the temperature must also approach zero such that the ratio $(1-\Phi_{\mathrm{H}}^2/N)^{N/2}/T$ remains finite. In this situation, the disc is not rotating at very slow speeds as in the flat space case. This is not possible in deSitter spacetime as there one has that $\xi \geq 1$. On the other hand we note that in dS, static solutions where the disc has finite size exist for which its size is given by

$$\rho_{\max}|_{\Omega\to 0} = L\sqrt{1-\Phi_{\mathrm{H}}^2/N} \ \ . \tag{2.52}$$

When the disc is uncharged and static, it ends on the cosmological horizon as noted in [16]. However, we observe here that if the disc is charged it does not reach the cosmological

[14] The boundary of the blackfold deserves special attention, see [10] for a discussion of its validity.

horizon. One may also conclude from here that static and extremal discs do not lie within the regime of validity of our method.

We now proceed and evaluate the free energy for these configurations from which all thermodynamic properties can be obtained. This is given by

$$\mathcal{F} = \frac{\Omega_{(n+1)}}{8G}\tilde{r}_0^n \frac{{}_2F_1\left(1, \frac{1}{2}(N-1)n; \frac{Nn}{2}+2; 1-\frac{\Phi_{\rm H}^2}{N}\right)}{\xi\Omega^2(2+Nn)} \ , \tag{2.53}$$

where we have defined

$$\tilde{r}_0^n = \left(\frac{n}{4\pi T}\right)^n \left(1-\frac{\Phi_{\rm H}^2}{N}\right)^{\frac{2+Nn}{2}} \ . \tag{2.54}$$

The thermodynamic properties are given in App. A. From here we also extract the tension

$$\hat{\boldsymbol{\mathcal{T}}} = -\frac{\Omega_{(n+1)}}{4G}\tilde{r}_0^n \frac{{}_2F_1\left(1, \frac{1}{2}(N-1)n; \frac{Nn}{2}+2; 1-\frac{\Phi_{\rm H}^2}{N}\right)}{(2+Nn)\xi^2 L^2\Omega^4} \ , \tag{2.55}$$

which, as expected, vanishes in the limit $L \to \infty$. Its thermodynamic properties satisfy (2.10) and (2.11).

2.4.3 Plane wave background

In this section we consider the perturbative construction of new black holes in a plane wave background with metric

$$ds^2 = -(1+\sum_{q=1}^{D-2} A_q x_q^2)dt^2 + (1-\sum_{q=1}^{D-2} A_q x_q^2)dy^2 - 2\sum_{q=1}^{D-2} A_q x_q^2 dt dy + \sum_{q=1}^{D-2} dx_q^2 \ . \tag{2.56}$$

Here the constants A_q define the curvature length scales $1/\sqrt{A_q}$ of the spacetime. We furthermore assume that this is a solution of the vacuum Einstein equations, which requires that $\mathrm{Tr}A_q = 0$. This background is of special importance, since contrary to (A)dS, it is in general anisotropic as there is a set of $(D-3)$ independent length scales, each of them associated with a particular spacetime direction.

The configuration studied in (A)dS in Sec. 2.4.2 is also a solution in the background (2.56) as long as it is embedded such that $y = 0$ [10]. To leading order in this perturbative construction all the solution properties are the same provided we define $\mathbf{R} = R\sqrt{A_1}$ where A_1 is the value of A_q for all the directions involved in the planes of rotation of the odd-sphere. In order to exhibit the differences between this spacetime and (A)dS, we consider the case of product of odd-spheres, to highlight its multi-scale properties, and the analogue of the disc solution of Sec. 2.4.2.

Products of black odd-spheres: a multi scale example

In this section we consider the simple case of the product of m uncharged odd-spheres in the background (2.56). The corresponding black holes solutions have horizon topology $\mathbb{R}\times\prod_{\hat{a}=1}^{n}\mathbb{S}^{(p_{\hat{a}})}\times\mathbb{S}^{(n+1)}$ and were in fact constructed in [10]. These are described by the induced metric and Killing vector field

$$\mathbf{ds}^2 = -R_0^2 d\tau^2 + \sum_{\hat{a}=1}^{m} R_{\hat{a}}^2 d\Omega_{(p_{\hat{a}})}^2 \ , \ R_0^2 = 1+\sum_{\hat{a}=1}^{m} A_{\hat{a}} R_{\hat{a}}^2 \ , \ \mathbf{k}^a\partial_a = \partial_\tau + \sum_{\hat{a}=1}^{[(p+1)/2]} \Omega^{\hat{a}}\partial_{\phi_{\hat{a}}} \ , \tag{2.57}$$

where $\Omega^{\hat{a}}$ is the angular velocity associated with all the $[(p_{\hat{a}}+1)/2]$ Cartan angles of each $p_{\hat{a}}$-dimensional sphere. It's free energy is given by

$$\mathcal{F}[R_{\hat{a}}] = -V_{(p)} R_0 P \quad , \quad V_{(p)} = \prod_{\hat{a}=1}^{m} \Omega_{(p_{\hat{a}})} R_{\hat{a}}^{p_{\hat{a}}} \quad , \tag{2.58}$$

while the equilibrium condition reads [10]

$$(\Omega^{\hat{a}})^2 R_{\hat{a}}^2 = R_0^2 \frac{\hat{p}_a + \mathbf{R}_{\hat{a}}^2 (n+p+1)}{(n+p) + (n+p+1)\mathbf{R}^2} \quad , \quad \mathbf{R}_{\hat{a}}^2 = A_{\hat{a}} R_{\hat{a}}^2 \quad , \quad \mathbf{R} = \sum_{\hat{a}=1}^{m} \mathbf{R}_{\hat{a}}^2 \quad . \tag{2.59}$$

The thermodynamic properties, namely, the mass, angular momenta and entropy can actually be found in [16] since these configurations to leading order exhibit the same properties as in (A)dS if one identifies $A_{\hat{a}} = L^{-2}$ for all $\hat{a} = 1, .., m$. Note that here we assume that there is another non-zero scale A_{m+1} such that $\mathrm{Tr} A_q = 0$ but which does not enter in the thermodynamics of the configuration due to the choice of embedding. Hence all the scales $A_{\hat{a}}$ with $\hat{a} = 1, .., m$ are independent scales.

We now proceed and analyse the total tension $\hat{\boldsymbol{\mathcal{T}}}$ from (2.58), which is given as a sum of tensions, one for each $p_{\hat{a}}$-sphere, which we label by $\hat{\boldsymbol{\mathcal{T}}}_{\hat{a}}$. We find the simple result[15]

$$\hat{\boldsymbol{\mathcal{T}}}_{\hat{a}} = -\frac{\Omega_{(n+1)} V_{(p)}}{16\pi G} r_0^n \mathbf{R}_{\hat{a}}^2 \sqrt{1+\mathbf{R}^2}\,(n+p+1) \quad , \tag{2.60}$$

which vanishes when $A_{\hat{a}} \to 0$. This example shows that these black holes are characterised by a set of tensions, satisfying the first law (2.10).

Charged black discs

In this section we construct the analogue of the disc solution of Sec. 2.4.2, which will reveal the non-universal character of the spacetime pressure for a given spacetime. These configurations are charged versions of those found in [10], obtained via an embedding such that $y = 0$ with induced metric and Killing vector field

$$\mathbf{ds}^2 = -R_0^2 d\tau^2 + d\rho^2 + \rho^2 d\phi^2 \quad , \quad R_0^2 = 1 + A_1 \rho^2 \quad , \quad \mathbf{k}^a \partial_a = \partial_\tau + \Omega \partial_\phi \quad , \tag{2.61}$$

where we have chosen $A_2 = A_1$ such that the Killing vector field presented above is a Killing vector field of the background (2.56). This is trivially a solution of the blackfold equations as it also represents a minimal surface in these spacetimes [10]. The thickness and charge parameter of the disc are given by (2.51) but with the replacement $A_1 = L^{-2}$. Hence, the disc has a maximum size given by (2.50) and the discussion regarding extremal limits in AdS also holds in this case provided $A_1 > 0$. The case $A_1 < 0$ is similar to deSitter spacetime.

The thermodynamic properties, due to the different induced metric, are however altogether distinct from its (A)dS counterpart. It is possible to derive them for any N, however, the resulting expressions are slightly cumbersome. Therefore we focus on the simplest case of $N = 1$ which captures all the essential physics.[16] In this case, the free energy takes the

[15] Note that here we had to use instead the formula $\hat{\boldsymbol{\mathcal{T}}}_{\hat{a}} = -2A_{\hat{a}} \frac{\partial \mathcal{F}}{\partial A_{\hat{a}}}$ due to the the fact that $A_{\hat{a}}$ has dimensions of inverse length square.

[16] The case $N = 1$ corresponds to a Kaluza-Klein reduction.

following form

$$\mathcal{F} = \frac{\Omega_{(n+1)}}{8G}\tilde{r}_0^n \frac{{}_2F_1\left(-\frac{1}{2},1;\frac{n+4}{2};\frac{A_1(\Phi_{\rm H}^2-1)}{\Omega^2-A_1}\right)}{(n+2)\left(\Omega^2-A_1\right)} \quad , \quad \tilde{r}_0^n = \left(\frac{n}{4\pi T}\right)^n \left(1-\Phi^2\right)^{\frac{n}{2}+1} \quad , \tag{2.62}$$

while the remaining thermodynamic properties are given in App. A. From here we extract, as previously, the total tension and find

$$\begin{aligned}\hat{\boldsymbol{\mathcal{T}}} =& -\frac{\Omega_{(n+1)}}{2G}\tilde{r}_0^n \frac{\Gamma\left(\frac{n}{2}+3\right)}{(n+2)(n+4)\left(A_1-\Omega^2\right)^2}\left(\Omega^2\, {}_2\tilde{F}_1\left(-\frac{1}{2},2;\frac{n+4}{2};\frac{A_1(\Phi_{\rm H}^2-1)}{\Omega^2-A_1}\right)\right.\\ &\left.+\left(A_1-\Omega^2\right)\, {}_2\tilde{F}_1\left(-\frac{1}{2},1;\frac{n+4}{2};\frac{A_1(\Phi_{\rm H}^2-1)}{\Omega^2-A_1}\right)\right) \quad ,\end{aligned} \tag{2.63}$$

which vanishes, as expected, when $A_1 \to 0$.

2.4.4 Lifshitz background

We now consider one of the Lifshitz spacetimes found in [168] as the background for perturbatively constructing new solutions. This spacetime has a spherically symmetric metric of the form

$$ds^2 = -\frac{r^{2z}}{L^{2z}}f(r)dt^2 + \frac{L^2}{r^2}f(r)^{-1}dr^2 + r^2 d\Omega^2_{(D-2)} \quad , \quad f(r) = 1+\beta\frac{L^2}{r^2} \quad , \tag{2.64}$$

where we have defined the constant β as

$$\beta = \left(\frac{(D-3)}{(D+z-4)}\right)^2 \quad . \tag{2.65}$$

Here, z is the Lifshitz exponent which can lie in the interval $z \geq 1$. In the limit $z \to 1$ we obtain the AdS metric (2.40). This spacetime is in fact supported by several gauge fields and dilaton in order to be a solution of the model studied in [168]. However, since we will be constructing an uncharged solution, we do not need to take into account possible couplings to these background fields.

We focus on the analogue configurations of the odd-spheres of Sec. 2.4.2, hence of a class of black holes with horizon topology $\mathbb{R}\times\mathbb{S}^{(p)}\times\mathbb{S}^{(n+1)}$ which includes the case of the black ring ($p = 1$). This configuration has an induced metric and Killing vector field given by

$$\mathbf{ds}^2 = -\frac{R^{2z}}{L^{2z}}f(R)d\tau^2 + R^2 d\Omega^2_{(p)} \ , \ f(R) = 1+\beta\frac{L^2}{R^2} \ , \ \mathbf{k}^a\partial_a = \partial_\tau + \Omega\sum_{\hat{a}=1}^{[(p+1)/2]}\partial_{\phi_{\hat{a}}} \ . \tag{2.66}$$

The blackfold equations (2.31) are easily solved. By defining $\mathbf{R} = R/L$ we obtain the equilibrium condition

$$\Omega R = \frac{\sqrt{\beta+\mathbf{R}^2}\mathbf{R}^{z-1}\sqrt{(n+2)\left(\mathbf{R}^2 z+\beta(z-1)\right)+p\left(\beta+\mathbf{R}^2\right)}}{\sqrt{\left(\beta+\mathbf{R}^2\right)\quad .(n+p+2z-1)-\beta}} \quad . \tag{2.67}$$

With this we evaluate the horizon size of the odd-sphere and find

$$r_0 = \frac{n}{4\pi T}\left(-\frac{n\left(\beta+\mathbf{R}^2\right)\mathbf{R}^{2z-2}\left(\mathbf{R}^2(z-1)+\beta(z-2)\right)}{\left(\beta+\mathbf{R}^2\right)(n+p+2z-1)-\beta}\right)^{n/2} \quad . \tag{2.68}$$

We see that for this to take real values we must require that

$$\mathbf{R}^2(z-1)+\beta(z-2)<0 \quad , \tag{2.69}$$

which implies that we must restrict to values of z within the interval $1 \leq z < 2$. This we assume from now on and in the remaining of this section.

With this we obtain the total tension which takes the form

$$\hat{\mathcal{T}} = -\frac{\Omega_{(n+1)}}{16\pi G}V_{(p)}r_0^n\frac{(n+p+z)\sqrt{\beta+\mathbf{R}^2}\left(z\left(\beta+\mathbf{R}^2\right)-\beta\right)\mathbf{R}^{z-1}}{|\mathbf{R}^2(z-1)+\beta(z-2)|} \quad , \tag{2.70}$$

which vanishes when $L \to \infty$. The thermodynamic quantities non-trivially satisfy the first law (2.10). This is the first example of a black hole solution with non-trivial topology in these spacetimes.

2.5 Blackfolds in background spacetimes with a black hole

In this section we present the perturbative construction of toy models of black hole binary systems, namely, the generalisation of the black saturn solution to higher-dimensions and with electric charge. Focusing on the case in which the centre black hole is not rotating and is surrounded by a black ring, leaving the rotating case for future work, we consider introducing a definition of black hole volume. We show that such definition does not give rise to the expected scaling once the centre black hole is removed. Therefore, using gravitational tension as the natural modification of (2.1), we take the exact black saturn solution in five dimensions and extract the tension to one higher-order in the ultraspinning regime. This requires enforcing the first law (2.10) to hold on the black ring horizon.

Finally, we also consider the generalisation of these solutions to AdS backgrounds, thereby illustrating the case in which there are two different tensions, where one is associated with the centre black hole mass and the other with the cosmological constant.

2.5.1 Schwarzschild black hole background

We begin with the Schwarzschild black hole background which has a spherically symmetric metric as in (2.40) but with a function $f(r)$ given by

$$f(r) = 1-\frac{\mu^{D-3}}{r^{D-3}} \quad , \tag{2.71}$$

where μ is the horizon location of the Schwarzschild black hole. In this background we place a charged odd-sphere geometry at a fixed $r=R$ and hence the induced metric and Killing vector field are the same as in Sec. 2.4.2 as well as the free energy (2.42) but with the function (2.71). It is straightforward to obtain the equilibrium condition, which reads

$$\Omega^2R^2 = f(R)\frac{2pf(R)+(1+n+nN\sinh^2\alpha)Rf'(R)}{2(nN\sinh^2\alpha+n+p)f(R)+Rf'(R)} \quad . \tag{2.72}$$

This equilibrium condition is in fact valid for any spherically symmetric spacetime of the form (2.40) for some function $f(r)$. In the limit $\alpha \to 0$ it reduces to that obtained in [16].

Defining the dimensionless ratio $\mathbf{R} = \mu/R$, which vanishes when the black hole is removed $\mu \to 0$, and using (2.71) we can rewrite the above condition as

$$\Omega^2 R^2 = \frac{(\mathbf{R}^{n+p} - 1)\left(\mathbf{R}^{n+p}\left(nN\sinh^2\alpha(n+p) + n(n+p+1) - p\right) + 2p\right)}{\left(2nN\sinh^2\alpha\,(\mathbf{R}^{n+p}-1) + (n+p)\,(\mathbf{R}^{n+p}-2)\right)} \quad . \tag{2.73}$$

From here we see that extremal configurations exist when $\alpha \to \infty$. Furthermore, using the free energy (2.42) in order to compute the conserved charges of this configuration, which are presented in App. A, and the Smarr relation (2.11), we obtain the total tension

$$\hat{\boldsymbol{\mathcal{T}}} = -\frac{\Omega_{(n+1)} V_{(p)} r_0^n}{16\pi G}\mathbf{R}^{n+p}\sqrt{1-\mathbf{R}^{n+p}}\frac{(n+p)\left(nN\sinh^2\alpha + n + p + 1\right)}{2 - \mathbf{R}^{n+p}(n+p+2)} \quad , \tag{2.74}$$

where we have defined $V_{(p)} = \Omega_{(p)} R^p$. We note that in $D = 5$ and for a ring geometry ($p = 1$), that is, $n = 1$, and in the uncharged limit ($\alpha = 0$), this describes the black ring surrounding the spherical black hole in the black saturn solution of [74]. In App. A.12 we show explicitly, by taking the blackfold limit of [74], that this is indeed the case.[17] We note that the thermodynamic properties of this solution and considering the tension per unit length $\boldsymbol{\mathcal{T}} = \hat{\boldsymbol{\mathcal{T}}}/\mu$, it is straightforward to check that the first law (2.10) is satisfied and, furthermore, both $\boldsymbol{\mathcal{T}}$ and $\hat{\boldsymbol{\mathcal{T}}}$ vanish when $\mu \to 0$.

Spacetime pressure and black hole volume in black hole backgrounds

As we have shown in the introduction, these solutions satisfy the first law (2.1) for an infinite set of quantities. We may introduce a notion of black hole volume by choosing $k = -2$, which leads to the black hole volume

$$V_i = \frac{\Omega_{(n+1)} V_{(p)}}{32\pi G}\frac{r_0^n}{\lambda}\mu^2\mathbf{R}^{n+p}\sqrt{1-\mathbf{R}^{n+p}}\frac{(n+p)\left(nN\sinh^2\alpha + n + p + 1\right)}{2 - \mathbf{R}^{n+p}(n+p+2)} \quad . \tag{2.75}$$

From here we see that once we remove the black hole $\mu \to 0$, the volume (2.75) vanishes. This is not the expected result for the volume in this limit. More precisely, the volume for the black ring ($p = 1$) has been computed in [6] by taking the flat spacetime limit of the perturbative construction of an uncharged AdS black ring in the ultraspinning regime [42]. The volume for this case scales like $V_i \propto r_0^n R^2$ in the flat spacetime limit. Since this solution corresponds to the case $\mu \to 0$ and $\alpha = 0, p = 1$ in (2.75), we see that the volume introduced in (2.75) does not scale in the expected way. We could introduce by hand a new length scale $\tilde{L}$ such that $(\tilde{L}/R)^2 = (\mu/R)^{D-3}$ which would lead to the right scaling in the flat spacetime limit. However, this would not satisfy the first law (2.1) with a pressure of the form $P_e = \lambda\tilde{L}^{-2}$, even if we would try to define a new mass $\tilde{M}$ as in Sec. 2.3. Again, therefore, we conclude that the notion of gravitational tension is more natural than the notion of black hole volume.

Blackfold mass in asymptotically flat black hole backgrounds

We note that the definition of gravitational tension is related to the generalised first law of thermodynamics (2.10), and hence to a specific definition of mass/energy, namely, the definition of mass that enforces (2.10) for a given black hole horizon in the presence of

[17] In the charged case for Kaluza-Klein coupling ($N = 1$) this potentially describes a particular limit of the charged black saturn solution obtained in [99]. We leave this check for future work.

surrounding gravitational fields. When using the free energy (2.42) and the thermodynamic relations (2.37),(2.38) we obtain the mass of this blackfold construction, which is given by

$$M = \frac{\Omega_{(n+1)} V_{(p)}}{8\pi G} r_0^n (1 - \mathbf{R}^{n+p})^{\frac{3}{2}} \frac{\left(nN \sinh^2 \alpha + n + p + 1\right)}{2 - \mathbf{R}^{n+p}(n+p+2)} \quad . \tag{2.76}$$

However, as we show in App. A.12 this mass, with the appropriate values of the several constants involved, does not correspond to the Komar mass M^{BR} measured near the black ring horizon of the black saturn solution. In fact we find that

$$M^{\mathrm{BR}} = M - \frac{\hat{\boldsymbol{\mathcal{T}}}}{(D-3)} \quad . \tag{2.77}$$

This is expected since the authors of [74] have shown that the first law (2.1) holds for the black saturn solution where M is the total ADM mass, which is the sum of the Komar masses of the centre black hole and of the black ring. However, if we require the existence of a first law that holds for each individual horizon then we must introduce a new mass measured in connection with each separate horizon. For the black ring horizon, this is precisely (2.76), that is, the mass that is directly obtained from the blackfold approach. Note that this is rather different than the KK case of Sec. 2.3, since there both the original mass M and the shifted mass $\tilde{M}$ satisfied the first law of thermodynamics. In this case, only the blackfold mass (2.76), obtained from general principles, satisfies the first law.

Free energy for the black ring in the black saturn solution

From the above discussion about the mass we are lead to an intriguing consequence for the thermodynamics of disconnected horizons, namely, the free energy for a given horizon, obtained via local computations of the conserved quantities, does not behave in a thermodynamically correct way.

In the case of the black saturn solution in $D = 5$, if we denote the black ring mass and angular momentum, which can be obtained via Komar integrations near the horizon [74], by M^{BR} and J^{BR} , and furthermore, denoting the black ring horizon temperature, angular velocity and entropy by T^{BR}, Ω^{BR} and S^{BR}, then the free energy $\mathcal{F}^{BR} = M^{BR} - T^{BR}S^{BR} - \Omega^{BR}J^{BR}$ can be seen as the free energy of a black ring in a non-trivial background for which the background length scale is that associated with the mass M^{BH} of the black hole in the center. However, we find that the following expected thermodynamic relations do not hold, i.e.,

$$S^{BR}|_{M^{BH},\Omega^{BR}} \neq -\frac{\partial \mathcal{F}^{BR}}{\partial T^{BR}} \quad , \quad J^{BR}|_{M^{BH},T^{BR}} \neq -\frac{\partial \mathcal{F}^{BR}}{\partial \Omega^{BR}} \quad . \tag{2.78}$$

Indeed this could have been anticipated, due to the fact that in the ultraspinning limit, the black ring mass, obtained from the blackfold approach, does not coincide with the Komar integration on the black ring horizon. Therefore, in order to define a proper free energy for a given horizon in a black hole solution with multiple disconnected horizons, we must introduce another notion of mass for that specific horizon.

We now give an explicit construction of this mass for the black ring in the black saturn solution for which the angular momentum of the center black hole vanishes. All thermodynamic quantities can be parametrised by 3 parameters L, β, k_2 (see App. A.12), where β controls the mass of the center black hole. If $\beta = 0$ we obtain the pure black ring solution. The relation between these parameters, in the ultraspinning limit, and those used to parametrize the solution (2.73), is given in (A.41)-(A.42). We now proceed by introducing a new mass M by adding to the Komar mass M^{BR} a term proportional to a function $f(\beta, k_2)$ such that

$$M = \frac{3\pi}{4G} L^2 \left(k_2 + f(\beta, k_2)\right) \quad , \tag{2.79}$$

and we want to demand that the resulting free energy $\mathcal{F} = M - T^{BR}S^{BR} - \Omega^{BR}J^{BR}$ satisfies the thermodynamic relations (2.78). Due to the cumbersome expressions inherent to the black saturn solution, we will show how this is done to first order in the ultraspinning approximation, i.e., in an expansion around $k_2 = 0$. We first decompose $f(\beta, k_2)$ as

$$f(\beta, k_2) = k_2 f(\beta)_{(0)} + k_2^2 f(\beta)_{(1)} + \mathcal{O}(k_2^3) \quad . \tag{2.80}$$

To leading order in the ultraspinning limit, there is already a correction, as we have seen above, which one can easily corroborate from the analytic solution and hence obtaining $f(\beta)_{(0)} = \beta/(\beta - 2)$. Proceeding to next order we find,

$$f(\beta)_{(1)} = \frac{\beta\left(\beta(17 + \beta(2\beta - 5)) - 12\right)}{6(\beta - 2)^3(\beta - 1)} \quad . \tag{2.81}$$

Now, using the Smarr relation (2.11) we find the total tension

$$\hat{\boldsymbol{\mathcal{T}}} = -\frac{3\pi L^2}{2G} k_2 \beta \left(\frac{1}{(2-\beta)} + \frac{(\beta(17 + \beta(2\beta - 5)) - 12)}{6(\beta - 2)^3(1 - \beta)} k_2 + \mathcal{O}(k_2^2) \right) \quad , \tag{2.82}$$

which when using (A.41)-(A.42) coincides with (2.74). This procedure can be iteratively continued to arbitrary orders in k_2. It would be interesting to obtain an exact expression for the shifted mass to all orders, and consequently obtain the exact gravitational tension for the black ring in the black saturn solution. Furthermore, a similar analysis can be carried out for the free energy of the centre black hole horizon, which can be thought of as a black hole placed in the gravitational field of a black ring. We leave this interesting analysis for future work.

2.5.2 Schwarzschild-(A)dS black hole background

We now briefly consider a similar construction in (A)dS spacetimes by placing a ring surrounding a Schwarzschild-(A)dS black hole. The metric of the Schwarzschild-(A)dS black hole takes the same form as in (2.40) but with the blackening factor

$$f(r) = 1 + \frac{r^2}{L^2} - \left(\frac{\mu}{r}\right)^{n+p} , \tag{2.83}$$

where μ is the horizon location of the Schwarzschild black hole and L is the size of (A)dS. In this background we place a neutral odd-sphere geometry at a fixed $r = R$ and hence the induced metric and Killing vector field are the same as in Sec. 2.4.2 as well as the free energy (2.42), but with the function (2.83). The equilibrium angular velocity is given by (2.72) and using (2.83) we can rewrite the equilibrium condition as

$$\Omega^2 R^2 = \frac{\left(1 + \mathbf{R}^2 - \tilde{\mathbf{R}}^{n+p}\right)\left((n+1)\left(2\mathbf{R}^2 + (n+p)\tilde{\mathbf{R}}^{n+p}\right) + 2p\left(1 + \mathbf{R}^2 - \tilde{\mathbf{R}}^{n+p}\right)\right)}{\left(\left(2\mathbf{R}^2 + (n+p)\tilde{\mathbf{R}}^{n+p}\right) + 2(n+p)\left(1 + \mathbf{R}^2 - \tilde{\mathbf{R}}^{n+p}\right) + 2p\right)} \quad , \tag{2.84}$$

where we have defined the dimensionless quantities $\mathbf{R} = R/L$ and $\tilde{\mathbf{R}} = (\mu/R)$. We now proceed and analyse the total integrated tension $\hat{\mathcal{T}}$ from (2.58), which is given as sum of tensions, one for the black hole and one for the the (A)dS radius. The results are the following

$$\hat{\mathcal{T}}_\mu = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \tilde{\mathbf{R}}^{n+p} \frac{(n+p+1)(n+p)\sqrt{1+\mathbf{R}^2-\tilde{\mathbf{R}}^{n+p}}}{(n+p+2)\tilde{\mathbf{R}}^{n+p}-2} \ , \tag{2.85}$$

$$\hat{\mathcal{T}}_L = \frac{V_{(p)}\Omega_{(n+1)}}{8\pi G} \mathbf{R}^2 \frac{(n+p+1)\sqrt{1+\mathbf{R}^2-\tilde{\mathbf{R}}^{n+p}}}{(n+p+2)\tilde{\mathbf{R}}^{n+p}-2} r_0^n \ . \tag{2.86}$$

Since each scale is independent, we can view this spacetime as having two distinct tensions: the one associated with AdS spacetime and the one associated with Schwarzschild spacetime. By defining the corresponding tensions per unit length, it is straightforward to see that the first law (2.10) holds. The remaining thermodynamic properties can also be easily obtained as in the previous examples.

2.6 Non-universality of black hole volume

In this section we analyse the consequences of the existence of the perturbative solutions of Sec. 2.4 to the notion of black hole volume. We have already argued that this notion was unnatural and did not lead to he expected scaling for both the KK black holes of Sec. 2.3 and the higher-dimensional black saturn solutions of Sec. 2.5.1. However, since a sceptic reader might consider those examples too exotic or far removed from the case of black holes in AdS, we focus in this section on two simpler examples. In particular, we look at the family of (charged) black odd-spheres of Secs. 2.4.2, 2.4.3 and 2.4.4 in AdS, plane wave and Lifshitz spacetimes respectively, as well as the family of charged rotating black holes in Secs. 2.4.2 and 2.4.3 in AdS and plane wave spacetimes respectively.

Our methodology consists in analysing the flat spacetime limit of each of these families of solutions. Since in this limit all the different cases within each familiy reduce to the same flat spacetime black hole solution, then, if the notion of black hole volume is to be meaningful in this limit, it must be universal. If we denote the black hole volume for a given family of solutions in the flat spacetime limit by $\tilde{V}_i$, then we must require that

$$V_i|_{L_a\to\infty} \to \tilde{V}_i \ . \tag{2.87}$$

However, as we will see, if we require this to be the case for the family of black odd-spheres (which contains black rings as a particular case) then we are forced to accept that black odd-spheres in flat spacetime are characterized by an infinite set of volumes. Furthermore, we will see that it is not possible to demand (2.87) for the family of charged black discs (charged rotating black holes).

2.6.1 Black odd-spheres

We first consider the case of the black odd-spheres in AdS, which we constructed in Sec. 2.4.2. Since we are in AdS, we take (2.8) as a working definition of pressure in AdS. Therefore we find the black hole volume

$$V_i = \frac{\Omega_{(n+1)}V_{(p)}}{2(n+p+1)(n+p+2)} r_0^n R^2 \sqrt{1+\mathbf{R}^2}\left(nN\sinh^2\alpha + n + p + 1\right) \ , \tag{2.88}$$

for this particular class of AdS solutions. In the uncharged case ($\alpha = 0$) and for the ring ($p = 1$) this volume had been obtained in [6]. From here we obtain the non-zero flat spacetime limit of the volume (2.88) by taking $\mathbf{R} \to 0$. For simplicity, focusing on the uncharged case $\alpha = 0$, we obtain

$$\tilde{V}_i = \frac{\Omega_{(n+1)} V_{(p)}}{2(n+p+2)} r_0^n R^2 \quad . \tag{2.89}$$

We now consider, by the same token, obtaining the black hole volume for the class of Lifshitz solutions of Sec. 2.4.4. We note that in Lifshitz spacetimes, the authors of Ref. [39] have recently proposed the following definition of the spacetime pressure

$$P_e = \frac{(D+z-2)(D+z-3)}{16\pi L^2} \quad , \tag{2.90}$$

which reduces to (2.8) when $z = 1$. Using (2.90), the black hole volume for this class of solutions is given by

$$V_i = \frac{\Omega_{(n+1)}}{2} V_{(p)} r_0^n \frac{\sqrt{\alpha + \mathbf{R}^2}\left(z\left(\beta + \mathbf{R}^2\right) - \beta\right)\mathbf{R}^{z-1}}{(n+p+z+1)\left|\mathbf{R}^2(z-1) + \beta(z-2)\right|} \quad . \tag{2.91}$$

This indeed reduces to the volume (2.88) when $z = 1$ and hence has the same flat spacetime limit (2.89). This corroborates the choice of spacetime pressure (2.90) by the authors [39].

However, let us consider the case of the odd-spheres in plane wave spacetimes constructed in Sec. 2.4.3. These solutions reduce to the same flat spacetime odd-spheres as the previous two cases when $A_{\hat{a}} \to 0$. Since in plane wave spacetimes we have m length scales and each length scale is independent, then, associated with each tension computed in (2.60), we have a specific black hole volume $V_i^{\hat{a}}$. Taking the pressure on each direction to be given by

$$P_e^{\hat{a}} = \frac{(D-1)(D-2)}{16\pi} A_{\hat{a}} \quad , \tag{2.92}$$

we obtain a set of black hole volumes

$$V_i^{\hat{a}} = \frac{\Omega_{(n+1)} V_{(p)}}{2(n+p+2)} r_0^n R_{\hat{a}}^2 \sqrt{1 + \mathbf{R}^2} \quad , \tag{2.93}$$

each associated with one of the $p_{\hat{a}}$-spatial parts of the worldvolume. The existence of this set of volumes is a direct consequence of the anisotropy of the spacetime. The definition of pressures (2.92) is indeed the correct one for these configurations since the flat spacetime limit of the volume ($A_{\hat{a}} \to 0$) coincides with (2.89). However, since we have an arbitrary number of volumes $V_i^{\hat{a}}$ associated with each non-trivial plane wave direction, then we must conclude that the corresponding flat spacetime black holes would be characterised by an arbitrary number of black hole volumes. For this reason, we argue that it is more natural to work with the gravitational tension associated with each spacetime direction, all of which vanish when taking the flat spacetime limit.

2.6.2 Charged rotating black holes

We first consider the analogue of the Kerr-Newman solution in higher-dimensions in AdS constructed in 2.4.2. Using the results of that section we obtain black hole volume

$$(V_i)_{\mathrm{AdS}} = \frac{2\Omega_{(n+1)}}{(D-1)(D-2)} \tilde{r}_0^n \frac{{}_2F_1\left(1, \frac{1}{2}(N-1)n; \frac{Nn}{2}+2; 1 - \frac{\Phi_{\mathrm{H}}^2}{N}\right)}{(2+Nn)\xi^2\Omega^4} \quad , \tag{2.94}$$

where we have used the definition of pressure (2.8). This volume in the limit $L \to \infty$ and for $\Phi_{\mathrm{H}} = 0$ yields the black hole volume corresponding to the Myers-Perry black hole in the ultraspinning regime in $D \geq 6$, which has been analysed in [59]. In the uncharged case and in (A)dS, this volume has also been computed in [59, 67, 6] for the entire family of higher-dimensional Kerr-(A)dS black holes. The exact form obtained here for the ultraspinning regime was only analysed in [6] and for the special case where $\xi \to 0$ for which the black hole saturates the BPS bound [57] in AdS.

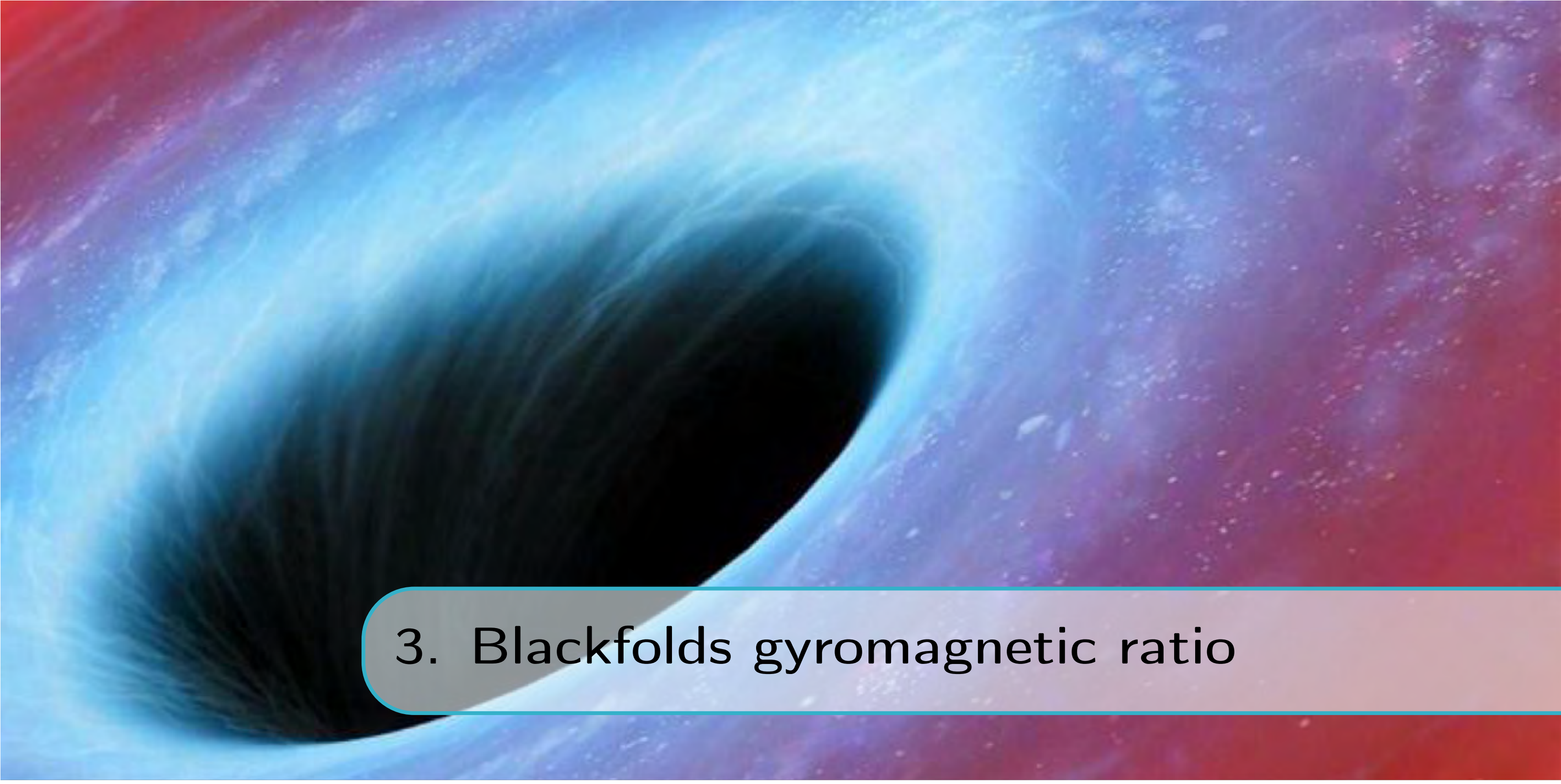
We now consider the case of the charged black holes in plane wave spacetimes constructed in Sec. 2.4.3 which have the same flat spacetime limit as the case above. In this case we take the pressure to be of the form $P_e = \lambda A_1 G^{-1}$. Using (2.63), we express the black hole volume in terms of λ and the tension $\hat{\boldsymbol{\mathcal{T}}}$ such that

$$(V_i)_{\mathrm{pp}} = -\frac{\hat{\boldsymbol{\mathcal{T}}}}{2\lambda A_1} \quad . \tag{2.95}$$

In order to obtain the correct factor λ we compare the volume (2.94) in the flat spacetime limit $L \to \infty$ and the volume (2.95) in the same flat spacetime limit $A_1 \to 0$. We find the ratio

$$\frac{(\tilde{V}_i)_{\mathrm{AdS}}}{(\tilde{V}_i)_{\mathrm{pp}}} = \frac{16\pi\lambda}{(n+3)(n+5-\Phi_{\mathrm{H}}^2)} \quad . \tag{2.96}$$

If we demand the black hole volume to be universal in the flat spacetime limit we must require the above ratio to be equal to unity. However, we would have to require λ to have a dependence on the chemical potential Φ_{H} which is not possibe since λ is a constant by definition. If we were to allow such dependence then the first law (2.1) would bot be satisfied. Therefore, we conclude, black hole volume in the flat spacetime limit is not universal.



3. Blackfolds gyromagnetic ratio

The gyromagnetic ratio of a particle (or g-factor) is a dimensionless quantity that relates the magnetic moment to the angular momentum. It is basically defined by the following formula

$$\mu = g\frac{q}{2m}j \tag{3.1}$$

where μ is the magnetic moment, q is the charge, m is the mass and j is the angular momentum. The gyromagnetic factor has been computed for various particles in the standard model, and actually these computations is its major success, since it reproduce with very high precision the experimental results. QED predicts, up to radiative corrections, that $g = 2$ for charged fermions [90]. The g-factor plays an important role also for physics beyond the standard model. In fact the exact result $g = 2$ is related to unbroken supersymmetry and therefore the factor $\lambda = g - 2$ is a measure of SUSY-breaking effects [165].

The gyromagnetic ratios could also play some role in the AdS/CFT correspondence. It is well known that there is a duality between the thermodynamic features of a black hole living on the bulk and the properties of the boundary conformal field theory. For instance [117] shows that the thermodynamic of a Kerr-AdS black hole coincides with the ones of a conformal field theory at the boundary of a Einstein universe at the critical limit (meaning the limit in which the Einstein universe is rotating at the speed of light). Since the gyromagnetic ratio is a thermodynamic property of the black hole living in the bulk, it could play a role in this description. For instance it could be interesting to see if there is a connection between the gyromagnetic ratio of a black hole in the bulk and some properties of the boundary conformal field theory, it could be somehow connected to the gyro of the particle living in the boundary CFT.

Charged black holes with intrinsic spin and charge can have a gyromagnetic ratio and this has been computed. A remarkable fact is that the gyromagnetic ratio of a rotating asymptotically flat black hole is equal to 2 [48, 159, 156]. It takes the same value of the fermions in the standard model. The g-factor for black holes is defined in the same way of equation (3.1), meaning

$$\mu = g\frac{Q}{2M}J \tag{3.2}$$

where μ is the magnetic dipole moment of a black hole, Q is the total electric charge, M is the total mass and J is the total angular momentum. The value of the gyromagnetic ratio for black holes in 4 dimensions is universal, meaning that it is 2 for any black hole configuration. The analysis changes when the g-factor is computed for higher dimensions [5, 4, 148, 43]: in fact, already for 5 dimensional Myers-Perry black holes, the gyromagnetic ratio is find to be 3. A more recent paper [3] computed the g-factor for the Kerr-Newman-AdS black hole solution in higher dimensions, and it has been found the following remarkable formula

$$g = 2 + (d-4)\xi \tag{3.3}$$

where d is the number of spacetime dimensions (so when $d = 4$ the formula gives back the universal result of four dimensions) and ξ is defined to be

$$\xi = 1 - \frac{a^2}{l^2} \tag{3.4}$$

where a is the spin parameter of the black hole, and l is the AdS length scale. There is an other important limit: when $l \to \infty$, i.e. when the cosmological constant is zero, the formula recovers the value of g previously computed for charged Myers-Perry black holes in $d = 4$.

In the rest of this chapter we would like to compute gyromagnetic ratios for blackfolds. There are two main reasons why we would like to do this. The first reason is that blackfolds are effectively geometries that could be used in a bulk of a holographic picture. Therefore, for reasons explained above, their gyromagnetic ratio could be important in a duality with some conformal field theory on the boundary. The second is that we believe that the g-factor could be a good way to classify different types of blackfolds. The gyromagnetic ratio for blackfold will be defined in the same it was defined for black holes, meaning by the formula (3.2). The content of this section is new expected to appear in published form in the near future [18].

3.1 Magnetic dipole moment

In order to compute the gyromagnetic ratio for a blackfold, we need to understand how to compute its magnetic dipole moment first. The method for measuring the dipole transport coefficient has been analyzed by [12]. The method works as follows. The first step is to consider a multiple expansion of the energy-momentum tensor and of the current tensor, for configurations charged under a q-form gauge field:

$$\begin{aligned} T^{\mu\nu}(x) &= \int_{\mathcal{W}_{p+1}} d\sigma\sqrt{-\gamma}\left[T^{\mu\nu}_{(0)}\frac{\delta^D(x-X(\sigma))}{\sqrt{-g}} - \nabla_\rho\left(T^{\mu\nu\rho}_{(1)}\frac{\delta^D(x-X(\sigma))}{\sqrt{-g}}\right) + \dots\right] \\ J^{\mu_1\dots\mu_q}(x) &= \int_{\mathcal{W}_{p+1}} d\sigma\sqrt{-\gamma}\left[J^{\mu_1\dots\mu_q}_{(0)}\frac{\delta^D(x-X(\sigma))}{\sqrt{-g}} - \right. \\ &\qquad \left. \nabla_\rho\left(J_{(1)}\mu_1\dots\mu_q\rho\frac{\delta^D(x-X(\sigma))}{\sqrt{-g}}\right) + \dots\right] \end{aligned} \tag{3.5}$$

Where the tensors with subscript (0) represent the monopole term of the expansion, while the tensors with subscript (1) represents dipole terms that we are interested in. The tensors that represent dipole terms could be decomposed in a particular way, in order to encode degrees of freedom connected to its physical interpreation. This process is quite complex, since there are extra symmetries due to gauge symmetry and diffeomorphism invariance of the expansion (3.6), and using those symmetries some components of the dipole correction of the energy-momentum tensor and of the charge current can be gauged away. In any case, those dipole correction can be decomposed, and the physical components (meaning those that cannot be gauged away) are the following

$$T^{\mu\nu\rho}_{(1)} = u_b{}^{(\mu} j^{b\nu)\rho} + u_a{}^{\mu} u_b{}^{\nu} d^{ab\rho} \tag{3.6}$$

$$J^{\mu\nu}_{(1)} = m^{\mu\nu} + u_a{}^{\mu} p^{a\nu} \tag{3.7}$$

where $u_a{}^{\mu}$s are the projector on the worldvolume directions ($u_a{}^{\mu} = \partial_a X^{\mu}$, where $X^{\mu}(\sigma)$ are the embedding coordinates), $j^{a\mu\nu}$ is the tensor that encodes spin degrees of freedom, $d^{ab\mu}$ is the tensor that encodes the bending degrees of freedom of the brane, while $m^{\mu\nu}$ encodes the magnetic dipole moment and $p^{a\mu}$ the electric dipole moment (which will be important for the piezoeletric effect). In this discussion we restricted ourselves to particle charged branes (i.e. $q = 0$), but the method provides a way for considering also higher gauge field charge, and the paper [12] provides also the detailed analysis for string charged branes, that we are not going to consider.

The method explained in [12] for measuring those transport coefficients work as follows. We consider asymptotically flat dilationic black brane solution of a theory with the following action

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left[R - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{2(q+2)!} e^{a\phi} H^2_{q+2} \right] \tag{3.8}$$

where H_{q+2} is the field strength of the gauge potential A_q. The equations of motion of the theory are the following

$$\begin{aligned} G_{\mu\nu} &- \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2(q+1)!} e^{a\phi} \left(H_{\mu\rho_1\ldots\rho_{q+1}} H_{\nu}{}^{\rho_1\ldots\rho_{q+1}} - \frac{1}{2(q+2)} H^2 g_{\mu\nu} \right) = 8\pi G T_{\mu\nu} \\ &\nabla_\nu \left(e^{a\phi} H^{\nu\rho_1\ldots\rho_{q+1}} \right) = -16\pi G J^{\mu_1\ldots\mu_{q+1}} \ , \quad \Box\phi - \frac{a}{2(q+2)!} e^{a\phi} H^2 = 0 \end{aligned} \tag{3.9}$$

In order to measure the transport coefficient one needs to look at large r-asymptotics of the equation of motion of the theory (3.9). The large r expansion of the metric and the gauge potential can be written in the following way

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(M)}_{\mu\nu} + h_{\mu\nu}(D) + \mathcal{O}\left(\frac{1}{r^{n+2}}\right) \tag{3.10}$$

$$A_{\mu_1\ldots\mu_{q+1}} = A^{(M)}_{\mu_1\ldots\mu_{q+1}} + A^{(D)}_{\mu_1\ldots\mu_{q+1}} + \mathcal{O}\left(\frac{1}{r^{n+2}}\right) \tag{3.11}$$

The method of finding the dipole coefficients of the theory, meaning the tensors (3.6), is then to expand the equation of motion using (3.10), (3.11) and (3.6) and matching the terms with the same r dependence in both side of the equations. Expanding in particular one of the equation of motion, one obtains

$$\nabla_{\perp}^{2} A^{\mu}_{(D)} = 16\pi G \left[m^{\mu r_{\perp}} + p^{\mu r_{\perp}} \partial_{r_{\perp}} \delta^{n+2}(r) \right] \tag{3.12}$$

therefore by reading off the expansion of the gauge field one can measure the dipole magnetic moment. Moreover, by expanding one of the other equations of motion, one can also argue that

$$m^{\mu\nu} = \lambda(\sigma) u_a j^{a\mu\nu} \tag{3.13}$$

where λ is a coefficients depending on the brane one considers.

3.2 Corrections for Rotating EMD Black Holes in D dimensions

This section will start by considering the ultraspinning limit of rotating Einstein-Maxwell-Dilaton black holes in D dimensions, and showing that there exist a limit in which the horizon flattens out along one of the planes of rotation, meaning that the horizon geometry becomes locally the one of a boosted charged rotating membrane. This will be an important result since it means that we can check the blackfold description by comparing it with the ultraspinning limit of rotating Einstein-Maxwell-Dilaton black holes in D dimensions with discs obtained with the blackfold analysis.

Then in this section we will measure the coefficient $\lambda(\sigma)$ for black holes in the EMD theory, and we will compute their magnetic dipole moment $m^{\mu\nu}$.

3.2.1 Refined ultra-spinning limit

We start by considering the metric of a charged rotating solution of the EMD model, taken from [143], with two angular momenta in D dimensions

$$\begin{aligned} ds^2 = &\left(1 + \frac{m}{r^{2N-2+\epsilon}\Pi F}\sinh^2\alpha\right)^{\frac{1}{D-2}} \Bigg\{ -dt^2 + \frac{\Pi F}{\Pi - m r^{2-2N-\epsilon}} dr^2 \\ &\sum_{i=1}^{N} (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + \epsilon r^2 d\nu^2 \\ &\frac{m}{r^{2N-2+\epsilon}\Pi F + m\sinh^2\alpha}\left(\cosh\alpha dt + \sum_{i=1}^{N} a_i \mu_i^2 d\phi_i\right)^2 \Bigg\} \end{aligned} \tag{3.14}$$

where the only nonzero spins are a_1 and a_2, and where

$$\begin{aligned} \Pi &= \left(1 + \frac{a_1^2}{r^2}\right)\left(1 + \frac{a_2^2}{r^2}\right) \\ F &= 1 - \frac{a_1^2\mu_1^2}{r^2 + a_1^2} - \frac{a_2^2\mu_2^2}{r^2 + a_2^2} \end{aligned} \tag{3.15}$$

and where we defined

$$N := \left[\frac{D-1}{2}\right] \ , \quad \epsilon := \frac{1}{2}\left(1 + (-1)^D\right) \tag{3.16}$$

where $[\cdot]$ denotes the integer part. So for even D we have that $N = (D-2)/2$ and $\epsilon = 1$, whereas for odd D we have $N = (D-1)/2$ and $\epsilon = 0$. Moreover no all the μ_is are independent, we need to require the following constraint

$$\sum_{i=1}^{N} \mu_i^2 + \epsilon\nu^2 = 1 \tag{3.17}$$

We would like to choose a parametrization of the angular variables

$$\mu_1 = \sin\theta \ , \quad \mu_2 = \cos\theta \sin\psi \tag{3.18}$$

and with this parametrization we get the following metric

$$\begin{aligned} ds^2 = &\left(1 + \frac{m}{r^{D-3}\Pi F}\sinh^2\alpha\right)^{\frac{1}{D-2}} \left\{ -dt^2 + \frac{\Pi F}{\Pi - m r^{3-D}} dr^2 \right. \\ &\sum_{i=1}^{2}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + r^2\left(d\theta^2 + \cos^2\theta\left(d\psi^2 + \cos^2\psi d\Omega^2_{(n-1)}\right)\right) \\ &\left. \frac{m}{r^{D-3}\Pi F + m\sinh^2\alpha}\left(\cosh\alpha dt + \sum_{i=1}^{2} a_i\mu_i^2 d\phi_i\right)^2 \right\} \end{aligned} \tag{3.19}$$

The event horizon of this black hole is located at $r = r_+$, where r_+ is the largest positive real solution of the equation $\Pi - m r^{2(1-N)-\epsilon}$. For clarity of notation we perform the following change of labelling for the spins, by replacing a_1 and a_2 with a and b. We introduce also the coordinate ρ_1 as

$$\rho_1 = a\sin\theta \tag{3.20}$$

This coordinate can be seen as the radius of the disc in which the black hole will pancake to. Assuming that $b \ll a\cos\theta$ the angular part of the metric will reduce to

$$\sum_{i=1}^{2}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) = d\rho_1^2 + \cos^2\theta(r^2 + b^2\cos^2\psi)d\psi^2 + \ldots \tag{3.21}$$

at first order in a $a\cos\theta$ expansion. Now we can look at the local geometry at a fixed angle $\theta^\star$. The refined ultraspinning limit with respect to the spin a of this black hole solution is reached therefore by considering

$$r \ll a\cos\theta^\star \ , \quad b \ll a\cos\theta^\star \tag{3.22}$$

After performing the ultraspinning limit, it is useful to redefine some variables in the following way

$$\begin{aligned} \tilde{r} &= r\cos\theta^\star \\ \tilde{b} &= b\cos\theta^\star \\ \tilde{r}_0^{D-5} &= \frac{m(\cos\theta^\star)^{D-5}}{a^2} \end{aligned} \tag{3.23}$$

in order to make contact with the metric of a charged spinning membrane. Also other redefinitions are important

$$\Sigma = \tilde{r}^2 + \tilde{b}^2 \sin^2 \psi \ , \quad \Delta = \tilde{r}^2 + \tilde{b}^2 - \frac{\tilde{r}_0^n}{\tilde{r}^{n-2}} \tag{3.24}$$

Also, we introduce the coordinate z which will parametrize the angular direction of the disc

$$z = \rho_1^\star \phi_1 \tag{3.25}$$

Then, under this redefinitions and parametrization, in this refined ultraspinning limit, we obtain

$$ds^2 = \left\{ 1 - \left(\frac{\tilde{r}_0}{\tilde{r}}\right)^{D-5} \frac{\tilde{r}^2}{\Sigma} \frac{\sinh^2 \alpha}{\cos^2 \theta^\star} \right\}^{\frac{1}{D-2}} \left\{ -dt^2 + \frac{\Sigma}{\Delta} d\tilde{r}^2 + d\rho_1^2 + dz^2 + \Sigma d\psi^2 \right. \tag{3.26}$$

$$\left. (\tilde{r}^2 + \tilde{b}^2) \sin^2 \psi d\phi_2^2 + \frac{\left(\frac{\tilde{r}_0}{\tilde{r}}\right)^{D-5}}{\tilde{r}^2 \Sigma + \left(\frac{\tilde{r}_0}{\tilde{r}}\right)^{D-5} \frac{\sinh^2 \alpha}{\cos^2 \theta}} \left(\cosh \alpha \frac{dt}{\cos \theta^\star} + \tan \theta^\star dz + \tilde{b} \sin^2 \psi d\phi_2 \right)^2 \right\}$$

In order to compare this metric we obtained with the metric of a charged Kerr membrane, we should first compute the metric of the charged Kerr membrane itself. This kind of solution can be obtained by using a solution generating technique which consists in uplifting a vacuum (rotating) solution of pure Einstein gravity by adding an extra flat direction, consequently boosting along the (t, z) plane with a boost parameter α and later reducing the metric over the z-direction. This technique is called uplift, boost and reduce (UBR) and it leads to a charged (rotating) solution of the equations of motion derived by minimizing the action

$$S = \int \sqrt{-g} \left(R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{4} e^{-a\phi} F^2 \right) \tag{3.27}$$

where R is the Ricci curvature tensor, ϕ is the dilaton field, with a being the dilaton coupling and F is the 2-form field strength of A, which is the 1-form gauge field. The dilaton coupling is fixed and it is given by

$$a = \frac{1}{2(D-2)} \tag{3.28}$$

where D is the number of dimensions. The result of applying this technique is the following

$$\begin{aligned} ds^2 \ &= \ \left(1 + \frac{m}{r^{D-3} \Pi F}\right) \left\{ -dt^2 + \frac{a^2 \cos^2 \theta + r^2}{r^2 - \frac{m}{r^{D-3}} + a^2} dr^2 + d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 \right. \\ &\left. + (a^2 + r^2) \mu^2 d\phi^2 + \frac{\frac{m}{r^{D-3}}}{r^2 + a^2 \cos^2 \theta + \frac{m}{r^{D-3}} \sinh^2 \alpha} \left(\cosh \alpha dt - a\mu^2 d\phi \right)^2 \right\} \end{aligned} \tag{3.29}$$

and with the following definition

$$\begin{aligned} &\Delta = r^2 - \frac{m}{r^{D-3}} + a^2 \\ &\Sigma = r^2 + a^2 \cos^2 \theta \\ &\hat{r} = \rho_1 \\ &\hat{\phi} = \rho_{1\star} z \\ &\mu = \sin \psi \end{aligned} \tag{3.30}$$

becomes

$$\begin{aligned} ds^2 &= \left(1+\frac{m}{r^{D-3}\Sigma}\right)\left\{-dt^2+\frac{\Sigma}{\Delta}dr^2+d\rho_1^2+\rho_1^2dz^2+\Sigma d\theta^2\right. \\ &\left. +(a^2+r^2)\sin^2\psi d\phi^2+\frac{\frac{m}{r^{D-3}}}{\Sigma+\frac{m}{r^{D-3}}\sinh^2\alpha}\left(\cosh\alpha dt-a\sin^2\psi d\phi\right)^2\right\} \end{aligned} \tag{3.31}$$

We would like to boost the metric along the tz direction:

$$\begin{aligned} ds^2 &= \left(1+\frac{m}{r^{D-3}\Sigma}\right)\left\{-dt^2+\frac{\Sigma}{\Delta}dr^2+d\rho_1^2+\rho_1^2dz^2+\Sigma d\theta^2+(a^2+r^2)\sin^2\psi d\phi^2\right. \\ &\left. +\frac{\frac{m}{r^{D-3}}}{\Sigma+\frac{m}{r^{D-3}}\sinh^2\alpha}\left(\cosh\alpha\cosh\eta dt+\cosh\alpha\sinh\eta dz-a\sin^2\psi^2 d\phi\right)^2\right\} \end{aligned} \tag{3.32}$$

From a direct comparison with the metric we obtained by taking the ultraspinning limit of the charged rotating solution of the EMD theory, we see that they are the same with the following identifications

$$\begin{aligned} &\tilde{r}_0 = m \\ &\sinh^2\alpha_K = \frac{\sinh^2\alpha_{MP6}}{\cos^2\theta_\star} \\ &\cosh\alpha_K\cosh\eta = \frac{\cosh\alpha_{MP6}}{\cos\theta_\star} \\ &\cosh\alpha_K\sinh\eta = -\tan\theta_\star \end{aligned} \tag{3.33}$$

which means that the boost parameters are related to the $\theta^\star$ angle and to the charge of the pancake by the following relations

$$\begin{aligned} \cosh\eta &= \frac{\cosh\alpha_{EMD}}{\sqrt{\cos^2\theta_\star+\sinh^2\alpha_{EMD}}} \\ \sinh\eta &= -\frac{\sin\theta_\star}{\sqrt{\cos^2\theta_\star+\sinh^2\alpha_{EMD}}} \end{aligned} \tag{3.34}$$

This relations are the same found in [20] when you set $\alpha_{EMD}=0$.

It is also interesting to look at the refined ultraspinning limit of the Maxwell field describing the charged rotating black hole solution of the EMD model. The Maxwell field of the solution is the following

$$A_\mu dx^\mu = \frac{m}{r^{D-3}\Pi F+m\sinh^2\alpha}\sinh\alpha\left(\cosh\alpha dt-a\sin^2\theta d\phi_1-b\cos^2\theta\sin^2\psi d\phi_2\right) \tag{3.35}$$

And performing the reparametrization and the refined ultraspinning limit in the same way we explained before, we obtain

$$A_\mu dx^\mu = \frac{\left(\frac{\tilde{r}_0}{\tilde{r}}\right)^{D-5}}{\tilde{r}^2\Sigma+\left(\frac{\tilde{r}_0}{\tilde{r}}\right)^{D-5}\frac{\sinh^2\alpha}{\cos^2\theta}}\frac{\sinh\alpha}{\cos\theta_\star}\left(\frac{\cosh\alpha}{\cos\theta_\star}dt-\tan\theta_\star dz-\tilde{b}\sin^2\psi d\phi_2\right) \tag{3.36}$$

The Maxwell field of a Kerr Membrane, after the redefinitions (3.33) is the following

$$A_\mu dx^\mu = \frac{mr\sinh\alpha}{\Sigma\left(1+\frac{mr}{\Sigma}\sinh^2\alpha\right)}\left(\cosh\alpha dt - a\sin^2\psi d\phi\right) \tag{3.37}$$

We could also perform a boost:

$$A_\mu dx^\mu = \frac{mr\sinh\alpha}{\Sigma + mr\sinh^2\alpha}\left(\cosh\alpha\cosh\eta dt + \cosh\alpha\sinh\eta dz - a\sin^2\psi d\phi\right) \tag{3.38}$$

and the result matches with the identifications given before. Therefore we conclude that the refined ultraspinning limit of a charged rotating black hole solution of the EMD theory is indeed a boosted charged rotating disc. The fact that there is a limit in which the charged rotating black hole can be locally seen as a black membrane is important since it means that it is possible to describe it using the blackfold formalism. In the following we are going to analyze the setup from the blackfold prospective.

3.2.2 Magnetic dipole moment for rotating EMD black holes

The goal of this section is to find an expression for the magnetic dipole moment for rotating black holes in the EMD model. This is very important for our work because we are going to use these black holes as seed solutions in the blackfold effective theory in order to construct exotic bent black branes with intrinsic spin. The method for calculating such dipole corrections was explained in the section 3.1. We just need to expand in a large r regime the metric and the gauge potential of a boosted charged rotating kerr membrane (which will be the membrane that we are going to use as seed solution in the blackfold theory). As it can be read off from the equation (3.32), the effect of performing a boost on the metric is to generate other components in the following way

$$\begin{aligned} g_{t\phi}^{boosted} &= g_{t\phi}\cosh\eta \\ g_{tz}^{boosted} &= 2\sinh\eta\cosh\eta(g_{tt}+g_{zz}) \\ g_{z\phi}^{boosted} &= g_{t\phi}\sinh\eta \end{aligned} \tag{3.39}$$

where $g^{\mu\nu}$ is the metric of the unboosted rotating charged membrane of equation (3.31). The same can be done for the gauge potential: the boost also generates new components on the gauge potential, as can be read off from (3.38)

$$\begin{aligned} A_t^{boosted} &= A_t\cosh\eta \\ A_\phi^{boosted} &= A_\phi \\ A_z^{boosted} &= A_t\sinh\eta \end{aligned} \tag{3.40}$$

where A_μ are the components of the unboosted gauge potential of the rotating charged membrane of equation (3.37). We can look at linearized Einstein equation for those component, in particular for dipole contribution (we omit the boosted label):

$$\begin{aligned} \nabla_\perp^2\bar{h}^{(D)t\phi} &= \nabla_{r\cos\theta}(u_a^t j^{a\phi r\cos\theta}) \\ \nabla_\perp^2\bar{h}^{(D)z\phi} &= \nabla_{r\cos\theta}(u_a^z j^{a\phi r\cos\theta}) \\ \nabla_\perp^2\bar{h}^{(D)tz} &= \nabla_{r\cos\theta}(u_a^t j^{azr\cos\theta} + u_a^z j^{ar\cos\theta} + u_a^t u_b^z d^{abr\cos\theta}) \end{aligned} \tag{3.41}$$

From the first two equations we can simply read off $j^{a\phi r\cos\theta}$. We have almost the same situation for the Maxwell field

$$\begin{aligned}
\nabla_\perp^2 A^{(D)t} &= \nabla_\rho(m^{t\rho} + u_a^t p^{a\rho}) \\
\nabla_\perp^2 A^{(D)z} &= \nabla_\rho(m^{z\rho} + u_a^z p^{a\rho}) \\
\nabla_\perp^2 A^{(D)\phi} &= \nabla_\rho m^{\phi\rho}
\end{aligned} \tag{3.42}$$

Now we may use some properties of the magnetic dipole moment and of the dipole spin current which are described in [12]. The first property is that the second index of dipole spin current is orthogonal to the worldvolume of the brane, meaning that $u_\nu^a j^{b\nu\rho} = 0$. The second property we use is that the magnetic dipole moment should be transverse in both indices. Using this knowledge on the spin current and on the magnetic dipole moment, one can read off the transport coefficients from the above equations and they are the following

$$j^{t\phi r\cos\theta} = 2am\cosh\alpha\cosh\eta\cos\theta \tag{3.43}$$

$$j^{z\phi r\cos\theta} = 2am\cosh\alpha\sinh\eta\cos\theta \tag{3.44}$$

$$m^{\phi r\cos\theta} = 2am\sinh\alpha\cos\theta \tag{3.45}$$

From this equation it is possible to calculate also the $\lambda(\sigma)$ coefficients of equation (3.13) for this particular type of rotating charged membrane, which is

$$\lambda(\sigma) = 2\coth\alpha \tag{3.46}$$

We recall also that, whenever we are dealing with stationary blackfold solution, the boosted factor of the brane is connected to the $\mathbf{k}$ vector through

$$\cosh\eta = \frac{1}{\mathbf{k}} \tag{3.47}$$

3.3 Stationary blackfold solutions

In the blackfold approach one constructs new stationary black objects starting with known black brane solutions (seeds) of a supergravity theory. In the specific case we would like to study in this paper, we need to start from a seed solution which should be charged and rotating. In order to compute the potentials for a charged rotating brane, we need to apply the UBR technique. The potentials of the rotating (Kerr) brane are the following

$$\mathcal{T} = \frac{n - 2b\tilde{\omega}}{4\pi r_0} \quad , \quad \tilde{\omega} = \frac{b}{r_0^2 + b^2} \tag{3.48}$$

and after the UBR procedure they become

$$\mathcal{T} = \frac{n - 2b\tilde{\omega}}{4\pi r_0\cosh\alpha} \quad , \quad \tilde{\omega} = \frac{b}{r_0^2 + b^2}\frac{1}{\cosh^2\alpha} \quad , \quad \Phi = \tanh\alpha \tag{3.49}$$

In this paper we are interested in new solutions in stationary equilibrium since they give rise to stationary black holes. In this solutions the fluid velocities u^a must be in the same directions of a worldvolume Killing vector field $\mathbf{k}^a$ with modulus $\mathbf{k}$, and we assume its existence. Without loss of generality we can write it as

$$\mathbf{k}^a\partial_a = \partial_\tau + \sum_a \Omega_a \partial_{\phi_a} \tag{3.50}$$

with τ being the time coordinate on the submanifold, Ω_a the angular velocity on each angular coordinate ϕ_a on the worldvolume. We furthermore assume that this killing vector fields (that lives on the worldvolume of the brane) maps to a background Killing vector field ($\mathbf{k}^\mu = \mathbf{k}^a e^\mu{}_a$) and that the global transverse velocity of the solution ω is the angular velocity associated with a background transverse Killing vector field $\mathbf{k}^\mu_\perp \partial_\mu = \omega\partial_\psi$, where ϕ labels the transverse angular coordinate. In equilibrium we also have that the global temperature, the global chemical potential and the global angular momentum are determined via redshift of the local thermodynamics potentials such that $T = \mathbf{k}\mathcal{T}$, $\phi_H = \mathbf{k}\Phi$ and $\omega = \mathbf{k}\tilde{\omega}$. These equations are indeed what we usually call the intrinsic blackfold equations and they can be inverted. We find three relations between the global thermodynamics potentials and the three parameter of the solution we are seeking: r_0 (horizon size), α (charge) and b (intrinsic spin).

$$\begin{aligned}
\frac{r_0}{\mathbf{k}} &= \frac{\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}\left(2\pi n T - 2\pi T\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}} + \sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}\sqrt{n\omega^2\left(2\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}-n\right)+4\pi^2T^2}\right)}{8\pi^2T^2+2\omega^2\left(1-\frac{\phi^2}{\mathbf{k}^2}\right)} \\
\frac{b}{\mathbf{k}} &= -\frac{2\pi T\sqrt{\left(1-\frac{\phi^2}{\mathbf{k}^2}\right)\left(n\omega^2\left(2\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}-n\right)+4\pi^2T^2\right)} - n\omega^2\left(1-\frac{\phi^2}{\mathbf{k}^2}\right) - 4\pi^2T^2\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}}{2\left(\omega^3\left(1-\frac{\phi^2}{\mathbf{k}^2}\right)+4\pi^2T^2\omega\right)} \\
\alpha &= \tanh\frac{\phi}{\mathbf{k}}
\end{aligned} \tag{3.51}$$

There are several reality conditions of the square roots to take into account, depending on the values of the seeds potentials, which are a remanent of the Kerr bound of the Kerr brane and the usual conditions that one has when studying charged blackfolds.

As explained in [15] it is possible to write down an effective free energy in a derivative expansion on some parameter ϵ (that for the blackfold approach is typically related to the thickness of the blackfold as it was explained in the previous section of the thesis), and for intrinsically rotating branes it takes the following form

$$\mathcal{F} = -\int_{\mathcal{B}_p} \sqrt{|-\gamma|}P(\mathcal{T},\omega) + 2\mathcal{J}(\mathcal{T},\omega)u^a\omega_a + \dots \tag{3.52}$$

where the scalars $\mathcal{P}$ and $\mathcal{J}$ are respectively the pressure density and the density of transverse angular momentum of the blackfold and they take the following form

$$P = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(1+\frac{b^2}{r_0^2}\right) \ , \ \mathcal{J} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n b\left(1+\frac{b^2}{r_0^2}\right) \tag{3.53}$$

The result obtained in [15] can be generalized to charged blackfold by performing the UBR technique. After a direct computation one realizes that the form of the effective free energy doesn't change if you add charge, as it happened also for non intrinsically rotating objects [43]. It is impressive that, despite the presence of the charge, the effective free energy of the blackfold still depends directly just on the thickness r_0 and on the density of intrinsic rotation b.

The equation of motion of such configurations (i.e. the extrinsic blackfold equation) can be obtained by minimizing the action (3.52).

It is possible to study also configurations with boundary, that are obtained by finding the values of the parameters such that $r_0 = 0$. As in the non intrinsically rotating (but charged) configurations, the equation $r_0 = 0$ does not have as solution just $\mathbf{k} = 0$ (as it happens in the uncharged case), but it has also some other solutions. For a given fixed ϕ_H, the horizon r_0 vanishes also for $\mathbf{k} = \Phi_H$, and this is satisfied when the charge boost α diverges, therefore the brane become locally extremal.

From the free energy (3.52) it is possible to compute all the thermodynamics of the blackfold. In particular the angular momentum along the worldvolume directions and the transverse angular momentum are given by

$$J_{(l)} = -\frac{\partial \mathcal{F}}{\partial \Omega^{(l)}} \ , \quad J_\perp = -\frac{\partial \mathcal{F}}{\partial \hat{\Omega}} \tag{3.54}$$

while the charge and the entropy can be obtained by

$$S = -\frac{\partial \mathcal{F}}{\partial T} \ , \quad \mathcal{Q} = -\frac{\partial \mathcal{F}}{\partial \Phi_H} \tag{3.55}$$

The thermodynamic mass can be obtained instead by using a property of the effective free energy, which is

$$\mathcal{F} = M - TS - \sum_{(l)} \Omega^{(l)} J_{(l)} - \hat{\Omega} J_\perp - \Phi_H \mathcal{Q} \tag{3.56}$$

These thermodynamics we just wrote obey the Smarr relation

$$(D-3)(M - \Phi\mathcal{Q}) - (D-2)\left(\sum_{(l)} \Omega^{(l)} J_{(l)} + \hat{\Omega} J_\perp + TS\right) = \hat{\mathcal{T}} \tag{3.57}$$

where T is the total tension, that can be obtained, as explained in [17], by

$$\hat{\mathcal{T}} = -L\frac{\partial \mathcal{F}}{\partial L} \tag{3.58}$$

where L is the actual size of the background where the blackfold is placed. The thermodynamics also obey the 1st law

$$dM = TdS + \sum_{(l)} \Omega^{(l)} dJ_{(l)} + \hat{\Omega} dJ_\perp + \Phi_H d\mathcal{Q} \tag{3.59}$$

Moreover, as explained in [17], whenever there is an intrinsic length-scale describing the background where the blackfold is placed in, i.e. whenever there is a tension $\hat{\mathcal{T}}$, the mass M computed by (3.56) does not correspond to the physical Komar mass. The physical mass must take into account the binding energy of the blackfold with the background and it is find to be

$$\hat{M} = M - \frac{\hat{\mathcal{T}}}{D-3} \tag{3.60}$$

Some interesting limits

The first limit we are interested in is the low frequency limit, which is the limit for small ω. The parameters of the solutions in this limit take the following form

$$r_0{}^{(0)} = \frac{n}{4\pi T}\mathbf{k}\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}} \tag{3.61}$$

$$b^{(0)} = \frac{n}{4\pi T}\frac{n\omega}{4\pi T}\mathbf{k}\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}} \tag{3.62}$$

$$\alpha^{(0)} = \tanh\frac{\Phi_H}{\mathbf{k}} \tag{3.63}$$

In the following sections we are going to study solutions under this limit.

An other limit which is worth analyzing is the extremal limit, which is obtained by taking $T \to 0$. Setting $T = 0$ in (3.51) leads to

$$r_0{}^{(extr)} = \frac{\mathbf{k}\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}\sqrt{n\left(2\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}-n\right)}}{2\omega\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}} \tag{3.64}$$

$$b^{(extr)} = \frac{\mathbf{k}n}{2\omega} \tag{3.65}$$

Since $r_0{}^{(extr)}$ should be real and non-vanishing, we should require that

$$n < 2\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}} \tag{3.66}$$

We will try to analyze further also this limit in future works.

3.4 Examples of configurations with small ω

In this section we construct novel perturbative charged and slowly intrinsically spinning black holes solutions in (Anti) - de Sitter, plane wave and Lifshitz spacetimes using blackfold approach. We also compute their magnetic dipole moment, and their gyromagnetic ratio. Since the intrinsic spin is small we take as potentials the ones of equation (3.61), (3.62) and (3.63).

3.4.1 (A)dS background and low intrinsic spin

Here we would like to construct novel solutions of electrically charged intrinsically spinning blackfolds in global (A)dS background. We write the global (A)dS metric in the following form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{D-2)}\ , \quad f(r) = 1 + \frac{r^2}{L^2} \tag{3.67}$$

where L is the AdS radius.

Discs

Here we would like to construct an electrically charged intrinsically spinning rotating disc in (A)dS background. The Killing vector field on the worldvolume for this configuration takes the following form

$$\mathbf{k}^a \partial_a = \partial_\tau + \Omega \partial_\phi \tag{3.68}$$

This geometry develops a boundary when $\mathbf{k} = 0$ corresponding to the maximum radius of the disc

$$r_{max} = \frac{\sqrt{1-\Phi_H^2}}{\Omega\sqrt{\xi}} \ , \quad \xi = 1 - \frac{1}{L^2\Omega^2} \tag{3.69}$$

The thickness of the disc, the charge parameter and the intrinsic spin, at leading order in ω, are given by

$$r_0{}^{(0)} = \frac{n\sqrt{1-\xi r^2\Omega^2}\sqrt{1-\frac{\phi^2}{1-\xi r^2\Omega^2}}}{4\pi T} \tag{3.70}$$

$$b^{(0)} = \frac{n^2\omega\sqrt{1-\xi r^2\Omega^2}\sqrt{1-\frac{\phi^2}{1-\xi r^2\Omega^2}}}{16\pi^2 T^2} \tag{3.71}$$

$$\alpha^{(0)} = \tanh^{-1}\left(\frac{\phi}{\sqrt{1-\xi r^2\Omega^2}}\right) \tag{3.72}$$

We can now proceed to evaluate the free energy for these configurations from which all thermodynamic properties can be obtained. At next to leading order in ω, the free energy functional is given by

$$\mathcal{F} = \frac{\Omega_{(n+1)}}{8G}\frac{\tilde{r}_0\left(1-\Phi_H^2\right)}{\xi\Omega^2(n+2)}\left[1 - \frac{n^2(n+2)}{2(n+3)}\left(\frac{\omega}{2\pi}\right)^2\sqrt{1-\phi^2}\,{}_2F_1\left(\frac{1}{2},1;\frac{n+5}{2};1-\phi^2\right)\right] \tag{3.73}$$

where we have defined

$$\tilde{r}_0 = \left(\frac{n}{4\pi T}\right)^n\left(1-\Phi_H^2\right)^{\frac{n}{2}} \tag{3.74}$$

It is not worth to write down the thermodynamics of the configuration since the expressions are cumbersome and they do not say anything more about the configuration, but it satisfies the first law of thermodynamics and the following Smarr relation

$$(d-3)(M - J_\omega\omega - Q\phi) - (d-2)(TS + J_\Omega\Omega) - \mathcal{T} \tag{3.75}$$

where $\mathcal{T}$ is the total tension and it is computed by

$$\mathcal{T} = L\frac{\partial\mathcal{F}}{\partial L} = -\frac{\Omega_{(n+1)}}{4G}\frac{\tilde{r}_0(1-\xi)\left(1-\Phi_H^2\right)}{\xi^2\Omega^2(n+2)} \tag{3.76}$$

With the definition of magnetic moment given in (3.45), one can compute it

$$\mu = \frac{\Omega_{(n+1)}}{8G}\frac{1}{8\pi T}\frac{n^2\omega\tilde{r}_0\phi\left(1-\Phi_H^2\right)^{1/2}}{\xi\Omega^2(n+2)}\,{}_2F_1\left(\frac{1}{2},1;\frac{n+4}{2};1-\phi^2\right) \tag{3.77}$$

and one can also compute the gyromagnetic ratio for this configuration using the formula (3.2)

$$g = \frac{n\xi + (\xi - 2)\phi^2 + \xi + 2}{\xi} \tag{3.78}$$

which has two interesting limits

$$\lim_{\phi\to 1} g = 1 \qquad \text{extremal charge} \tag{3.79}$$

$$\lim_{\phi\to 0} g = \frac{(n+1)\xi + 2}{\xi} \xrightarrow{\xi\to 1} n + 3 \qquad \text{zero charge} \tag{3.80}$$

Rings

In this section we show that, using the theory of charged rotating blackfolds, we can construct novel charged black rotating rings. In the metric (3.67) we have to place a ring geometry of radius R, and we want this ring to spin in two direction, one along the worldvolume and the other orthogonal to it. In order to do so we rewrite the metric (3.67)

$$ds^2 = -\nu(r)dt^2 + \nu(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\psi^2 + \cos^2\theta d\chi^2 + d\Omega^2_{(d-5)}\right) \tag{3.81}$$

where $\nu(r)$ is the same function of (3.67), and we choose the following embedding

$$t = \tau\ , \quad r = R\ , \quad \theta = 0\ , \quad \chi = \psi \tag{3.82}$$

We set the ring to rotate with angular velocity Ω along the direction ϕ while with angular velocity $\hat{\Omega}$a long the transverse direction ψ. This setup is described by a surface Killing vector field and a transverse killing vector field

$$\mathbf{k}^a\partial_a = \partial_\tau + \Omega\partial_\phi\ , \quad \mathbf{k}^\mu_\perp\partial_\mu = \hat{\Omega}\partial_\psi \tag{3.83}$$

The first thing to notice is that the background is non-rotating. This means that $\omega^a = 0$ and that the second term in the general action of rotating blackfold (3.52) described in [15] is absent. Therefore if we set a ring rotating with angular velocity Ω along the direction ϕ and with velocity ω along the transverse direction ψ, the corresponding equilibrium condition can be easily computed. In particular, we have that the free energy functional takes the following form in terms of $\mathbf{k}$ and ϕ, at next to leading order in ω

$$F = R_0 r^p\left(\frac{n\mathbf{k}}{4\pi T}\right)^n\left(1 - \frac{\phi^2}{\mathbf{k}^2}\right)^{n/2}\left[1 + \frac{n^2\omega^2}{16\pi^2T^2}\frac{\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}} - 2 + 2\frac{\phi^2}{\mathbf{k}^2}}{\sqrt{1-\frac{\phi^2}{\mathbf{k}^2}}}\right] \tag{3.84}$$

Note that the term proportional to ω^2 is $\mathbf{k}$ independent when ϕ is set to zero. This means that whenever the charge is set to zero, the intrinsic spin does not change the equilibrium condition of the ring, as it is experienced in [15]. We are able now to compute the equilibrium condition by minimizing the free energy functional. The result we get, at next to leading order in ω is the following

$$\begin{aligned}\Omega^2R^2 \quad &= \quad (1+\mathbf{R}^2)\frac{\mathbf{R}^2\left(\cosh^2\alpha + n + p\right) + p}{\mathbf{R}^2\left(\cosh^2\alpha + n + p\right) + n\cosh^2\alpha + p}\Bigg[1 + \\ &+ \quad \frac{n^2\omega^2}{16\pi^2T^2}\frac{\sinh\alpha\tanh\alpha(p + (1+p)\mathbf{R}^2)}{\left[p + \mathbf{R}(1 + p + n\cosh^2\alpha)\right]\left[p + n\cosh^2\alpha + \mathbf{R}^2(1 + p + n\cosh^2\alpha)\right]}\Bigg]\end{aligned} \tag{3.85}$$

This result reproduces the result of [17] when ω is set to zero, and of [15] when ϕ is set to zero. Given the equilibrium condition, one can compute the thermodynamics of the ring, and in particular it is possible to compute the total tension, using the formula (3.58). We recall that, since we put the ring in a curved background, the true mass of the blackfold is not M, but one should subtract a proper factor of gravitational tension, as explained in (3.60). It is not worth to write down the thermodynamics of the configuration since the expressions are cumbersome and they do not say anything more about the configuration, but this thermodynamics obey the first law and the Smarr relation (3.57), and it reproduces the results of [17] when ω is set to zero. In this section we write down two novel thermodynamic feature of the ring, which are the angular momentum $J_{\hat{\Omega}}$ (associated to the intrinsic spin)

$$J_{\hat{\Omega}} = \frac{\Omega_{(n+1)}V_{(p)}}{8\pi G}\frac{n^2\omega}{(4\pi T)^2}\tilde{r}_0^n\sqrt{1+\mathbf{R}^2}(2-\cosh\alpha)\mathrm{sech}\alpha \tag{3.86}$$

and the magnetic dipole moment

$$\mu = \frac{\Omega_{(n+1)}V_{(p)}}{32\pi G}\frac{n\omega}{4\pi T}\tilde{r}_0^{n+1}\sinh\alpha \tag{3.87}$$

We are able now also to compute the gyromagnetic ratio for this novel configuration using formula (3.2). We write down the gyromagnetic ratio at leading order in ω

$$g = \frac{\left(1+\mathbf{R}^2\right)^{3/2}\cosh\alpha(n\cosh(2\alpha)+n+2p+2)}{\pi(2-\cosh(\alpha))\left(n\left(1+\mathbf{R}^2\right)\cosh(2\alpha)+n\left(1+\mathbf{R}^2\right)+2\left(p\mathbf{R}^2+p+\mathbf{R}^2\right)\right)} \tag{3.88}$$

This gyromagnetic ratio has an interesting zero charge limit

$$g_0 = \frac{n+p+1}{n+p}\left(1+\mathbf{R}^2\right) \tag{3.89}$$

3.4.2 Generalization of odd-spheres result to any $\nu(r)$ backgrounds

The result obtained in the last section can be easily generalized for all background with the following form of the metric

$$ds^2 = -\nu(r,L)dt^2 + \nu(r,L)^{-1}dr^2 + r^2 d\Omega^2_{D-2)} \tag{3.90}$$

keeping $\nu(r,L)$ general. Belongs to this category for instance (A)ds, which the case studied in the previous section, but also plane wave background and Schwarzshild background. Since all this backgrounds are non rotating, the term second term in the action (3.52) of [15] is zero since ω^a is zero. The free energy functional is already described in the action (3.84). The final result for the equilibrium condition is the following

$$\begin{aligned}\Omega^2R^2 &= \frac{\nu(2p\nu+r(1+n\cosh^2\alpha)\nu')}{r\nu'+2(p+n\cosh^2\alpha)\nu}\Bigg[1+ \\ &+ \frac{\omega^2n^2}{8\pi^2T}\sinh\alpha\tanh\alpha\frac{(2\nu-r\nu')(2p\nu+r\nu'))}{(2p\nu+2n\cosh^2\alpha\nu+r\nu')(2p\nu+r\nu'+nr\cosh^2\alpha\nu')}\Bigg]\end{aligned} \tag{3.91}$$

where ν is really $\nu(r,L)$ and ν' means derivative with respect to r, we wrote in this way in order to simplify the formula. This formula gives back the same equilibrium condition

(3.85) when you set $\nu(r,L)$ to be the function described in(3.67) and it reproduces know results of [17] whenever you set specific background selecting the function $\nu(r,L)$, plane wave background and Schwarzshild background and you set the intrinsic spin ω to zero.

The background function $\nu(r,L)$, in order for the configuration to obey the first law of thermodynamics and the Smarr relation (3.57), need to fullfill the following derivative relation

$$\frac{\partial}{\partial L}\nu(r,L) = -r\frac{\partial}{\partial r}\nu(r,L) \tag{3.92}$$

which is obeyed for all the background we are interested in (AdS, plane waves, and Schwarzshild).

In this section we write down two novel thermodynamic feature of the ring, which are the angular momentum $J_{\hat{\Omega}}$ (associated to the intrinsic spin)

$$J_{\hat{\Omega}} = \frac{\Omega_{(n+1)}V_{(p)}}{8\pi G}\frac{n^2\omega}{(4\pi T)^2}\tilde{r}_0^n(2\mathrm{sech}(\alpha)-1)\sqrt{\nu(r,L)} \tag{3.93}$$

and the magnetic dipole moment

$$\mu = \frac{\Omega_{(n+1)}V_{(p)}}{32\pi G}\frac{n\omega}{4\pi T}\tilde{r}_0^{n+1}\sinh\alpha \tag{3.94}$$

We are able now also to compute the gyromagnetic ratio for this novel configuration using formula (3.2). We write down the gyromagnetic ratio at leading order in ω

$$g = -\frac{\cosh(\alpha)\sqrt{\nu}(n\cosh(2\alpha)+n+2p+2)\,(2(n+p)\nu+r\nu')}{2\pi(\cosh(\alpha)-2)(n+p)\,(\nu(n\cosh(2\alpha)+n+2p)+r\nu')} \tag{3.95}$$

This gyromagnetic ratio has two interesting limits, one is zero charge

$$g_0 = \frac{n+p+1}{n+p}\nu \tag{3.96}$$

and one is extremal charge

$$g^{\mathrm{extr}} = \frac{2(n+p)\nu(r,\kappa)+r\nu^{(1,0)}(r,\kappa)}{2\pi(n+p)\sqrt{\nu(r,\kappa)}} \tag{3.97}$$

3.4.3 Odd-spheres in Lifshitz spacetime

An other interesting setup one could study is the odd-sphere in Lifshitz backgrounds. The metric that describes the background is the following

$$ds^2 - \frac{r^{2z}}{L^{2z}}\nu(r)dt^2 + \frac{L^2}{r^2\nu(r)}dr^2 + r^2 d\Omega^2_{(d-2)}\ , \quad \nu(r) = 1+\beta\frac{L^2}{r^2} \tag{3.98}$$

where we have defined the constant β as

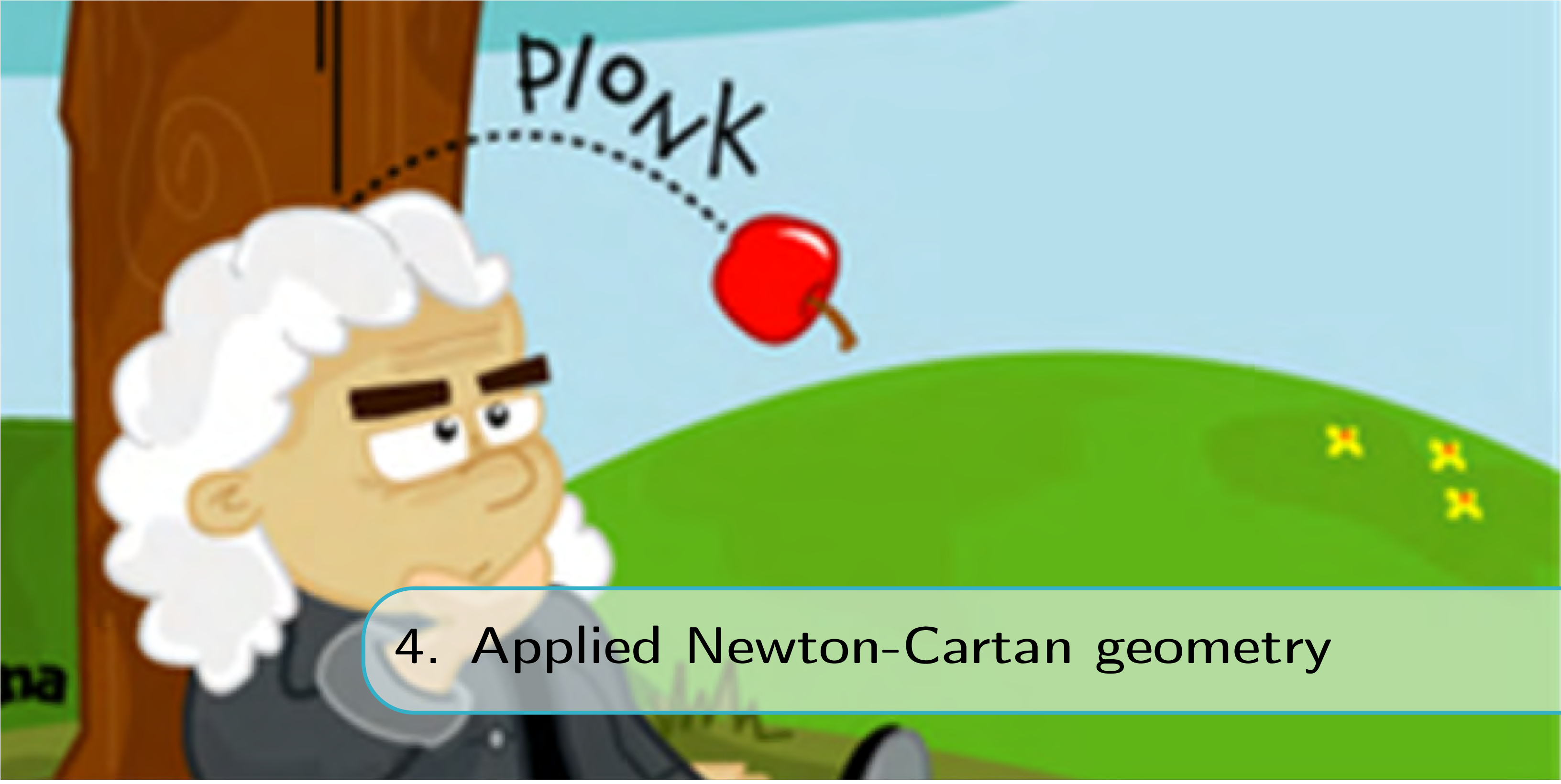
$$\beta = \left(\frac{d-3}{d+z-4}\right)^2 \tag{3.99}$$

This metric does not belong to the previous family because of the parameter z the g_{tt} and g_{rr} are not one the inverse of the other. The same consideration done in the previous sections hold for this configuration. The equilibrium condition is then

$$\Omega^2 r^2 z L^{-2z} = \frac{r^2\nu\left(2p\nu + 2z(1+n\cosh^2\alpha)\nu + r(1+n\cosh^2\alpha)\nu'\right)}{2z\nu + 2(p+n\cosh^2\alpha)\nu + r\nu'}\Bigg[1+ \tag{3.100}$$

$$+\frac{n^2\omega^2}{8\pi^2T^2}\frac{\sinh\alpha\tanh\alpha(2(z-1)\nu + r\nu')(2p\nu + 2z\nu + r\nu')}{\left(2z\nu + 2(p+n\cosh^2\alpha)\nu + r\nu'\right)\left(2p\nu + 2z(1+n\cosh^2\alpha)\nu + r(1+n\cosh^2\alpha)\nu'\right)}\Bigg]$$

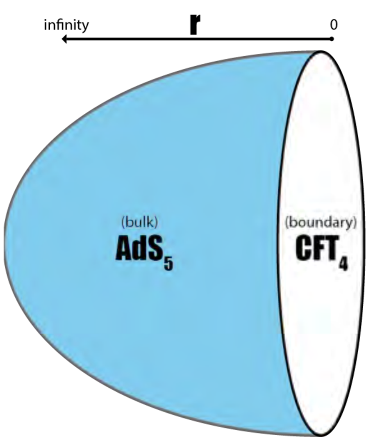
In order for the Smarr and the first law to be satisfied we need to require the same condition (3.92) and the function in (3.98) satisfies that condition. The computation of the magnetic dipole moment and of the gyromagnetic ratio for this configuration is left for the future.



4. Applied Newton-Cartan geometry

Newton-Cartan geometry can be used to geometrize Newton's theory of gravity. This geometry describes spacetimes that exhibit non relativistic symmetries. Over the recent years the interest in such geometries has grown because it has been understood that this geometrical framework for non-relativistic physics opened up many areas of applications in different directions. In fact many branches of physics are not controlled by Lorentz symmetry but by some non-relativistic symmetry or some other effective symmetries. This chapter begins with an overview of the possible application of Newton-Cartan geometry. This is not meant to be a exhaustive picture of the possible application of this novel geometrical framework, since a lot of work has been done in this research direction, but it is an overview describing some of the research developments. Three main areas of application in which the Newton-Cartan geometry plays an important role have been developed:

Non-relativistic holographic models



Holography is essentially a map between a gravity theory which lives in the bulk of a spacetime, often with negative cosmological constant, and a conformal field theory that lives on the boundary of that spacetime. This duality between field theories and gravity theories is very important because through gravity we can access some dark corners of field theory in which the particles are highly interacting, and viceversa through field theory we can access corners of gravity in which the gravitational force is very strong. In particular, concerning our interest in this thesis, this means that gravity is interesting not only in its own right, meaning describing the gravitational force that governs nature, but also as a tool for learning more about (conformal) field theories. This opened up many branches of research. In this direction it is also insightful to study holography in which the bulk space-time is not AdS. In fact if one would like to study strongly coupled conformal field theories which exhibit non-relativistic scaling, this necessitates the consideration of a bulk space-time with asymptotics which is not AdS. In the past years there has been a lot of activity in this direction studying non-AdS bulk space-times that exhibit non-relativistic symmetries, like Lifshitz or Schrödinger space-times. It has been shown in [56] that the holographic dual of Lifshitz space-time solution of the Einstein-Proca-Dilaton model is described by a boundary theory which lives on a new extension of Newton-Cartan geometry, which has been called torsional Newton-Cartan geometry.

Pushing this area of research further, one may ask if it is possible to have a bulk which is described by a dynamical Newton-Cartan geometry, or better to some non relativistic string theory. This is an interesting open area of research [102, 107] but this argument is beyond the scope of this thesis.

Field theories: Condensed matter and statistical physical systems

There are many physical systems in nature that exhibit non-relativistic symmetry. One of the first application of Newton-Cartan geometry has been to effective theories for the Fractional Quantum Hall effect [93, 163].

Moreover Newton-Cartan geometry could be used not only for studying specific non realitivistic systems, but also as a theoretical tool for writing field theories with non relativistic symmetries. In fact one can use the background field method, writing down the coupling of a non realtivistic field to the underlying (fixed) NC geometry and computing the energy-momentum tensors, Ward identities and the anomalies.

A lot of work has been done in this direction, developing the coupling of Galilean invariant field theory to curved spacetime [132] and for studying coupling of different type of matter (different spin and mass) to NC space-time using limiting or null reduction technique [33, 32].

An other important application of NC geometry is that it could be used to give a covariant formulation for non-relativistic hydrodynamics [37, 38]

Gravity

The study of non-relativistic theories of gravity has its own importance. In fact, one of the biggest difficulties in finding a meaningful theory of quantum gravity is that general relativity in non-renormalizable. Hořava-Lifshitz (HL) gravity has appeared [123, 124] and it is a tentalizing possibility of a non-Lorentz invariant and renormalizable theory of gravity, because it has been shown that it is power-counting renormalizable and it is expected to be renormalizable and unitary. Therefore, while observational constraints put severe limitations on the application of HL gravity to the studies of the universe, it still has a high theoretical interest as an effective theory of gravity. In fact it is an example of gravity with anisotropic scaling between time and space. In particular it could be very interesting to provide, through HL gravity, an alternative way for constructing gravity duals to strongly coupled system with non relativistic scaling [130]. As we are going to see later in this thesis, the dynamics of the Newton-Cartan geometry is a covariant formulation of HL gravity. Therefore, studying the dynamics of (T)NC geometry becomes really important because on one hand one can try to use it as a toy model to build theories of non-relativistic quantum gravity; on the other hand one can try to develop new holographic bulk alternatives.

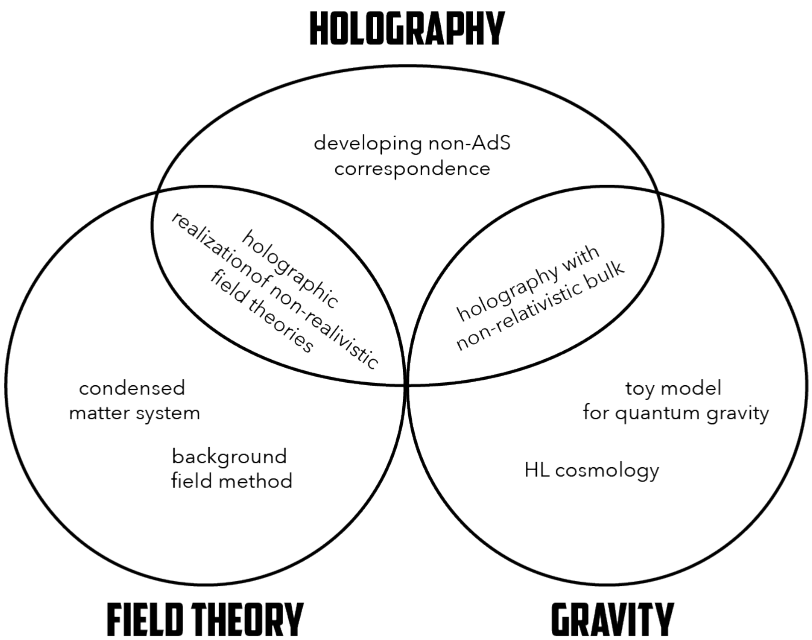


Figure 4.1: A schematic overlook to some of the applications that have been pursued over the last years in the realm of Newton-Cartan geometries

It is well known that local Lorentz symmetry give rise to (pseudo)-Riemannian geometry and therefore it is used to describe the kinematics of general relativity, which means that local symmetries of space and time are connected to the geometry of space and time, In the same way, local Galilean symmetry give rise to Newton-Cartan geometry and it can be used to describe the kinematics of non-relativistic gravity.

Examples of non relativistic symmetries are given by the Galilean or the Bargmann algebra, that are transforming space and time in the following way

$$t \to t \ , \quad \vec{x} \to \vec{x} + \vec{v}t \tag{4.1}$$

There are also ultra-relativistic symmetries: for instance the Carroll algebra, that induces on space and time the following transformations

$$t \to t + \vec{v} \cdot \vec{x} \ , \quad x \to x \tag{4.2}$$

which is a ultra-relativistic symmetry. In both these transformations, the main difference with respect to the relativistic symmetry, generated by the Poincaré algebra, is the type of boosts.

4.1 Newton Cartan Kinematics

4.1.1 A General Relativity review

In order to better understand Torsional Newton Cartan formulation of gravity, we would like to give a very brief review of Einstein gravity. In the Einstein picture, free falling frames are those without gravitational force and they are connected to Poincaré symmetry, which is described by the Poincaré algebra

$$[P_A, M_{BC}] = 2\eta_{A[B}P_{C]}\ ,\quad [M_{AB}, M_{CD}] = 4\eta_{[A[C}M_{D]B]}. \tag{4.3}$$

where P is the generator of space-time translations and M_{AB} is the generator of Lorentz rotations. All the other commutators are zero. Usually the gravitational force is described by a symmetric tensor field $g_{\mu\nu}$, and in order to describe the kinematics of gravity, one needs to introduce other geometric objects. In particular one needs a way to take covariant derivatives of tensor. In the Riemannian geometry, by requiring metric compatibility, i.e.

$$\nabla_\mu g_{\nu\rho} = 0 \tag{4.4}$$

you can introduce the Christoffel symbol and you can take covariant derivatives in the following way

$$\begin{aligned} \Gamma_\mu{}^\rho{}_\nu &= \frac{1}{2}g^{\rho\lambda}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right) && (4.5)\\ \nabla_\mu A_\nu{}^\rho &= \partial_\mu A_\nu{}^\rho + \Gamma_\mu{}^\rho{}_\lambda A_\nu{}^\lambda - \Gamma_\mu{}^\lambda{}_\nu A_\lambda{}^\rho && (4.6) \end{aligned}$$

It is possible to write down the Christoffel symbols in terms of the metric tensor by requiring metric compatibility. Moreover one can introduce the Riemann tensor, the Ricci scalar, and the Einstein tensor

$$\begin{aligned} R_{\mu\nu}{}^\rho{}_\sigma &= \partial_\nu\Gamma^\rho_{\mu\sigma} - \partial_\sigma\Gamma^\rho_{\mu\nu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\sigma\lambda} && (4.7)\\ R_{\mu\nu} &= R_{\mu\nu}{}^\rho{}_\rho && (4.8)\\ G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R && (4.9) \end{aligned}$$

and with these objects one can write down the Einstein equations, which describe how gravity couples to some matter fields through its energy-momentum tensor. Those equations can be find also from a variational principle, minimizing the Einstein-Hilbert action.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{4.10}$$

$$\mathcal{S} = \int dx\sqrt{-g}R + \mathcal{S}_M \tag{4.11}$$

It is well-known also that Einstein gravity can be written also in the vielbein formulation, just defining the vielbein field in the following way

$$g_{\mu\nu} = E_\mu{}^A E_\nu{}^B \eta_{ab} \tag{4.12}$$

where $\eta_{\mu\nu}$ is the Minkowski metric (we use mostly plus signature). In this formulation the kinematics of General Relativity will be described from the vielbeins and from their spin-connections $\Omega_\mu{}^{AB}$. The spin-connections are required to obey a constraint which is the following

$$2\partial_{[\mu}E_{\nu]}{}^A - \Omega_{[\mu}{}^{AB}E_{\nu]}{}^c\eta_{BC} = 0\ ,\quad \Omega_\mu{}^{AB} = \Omega_\nu{}^{AB}(e) \tag{4.13}$$

which means that using (4.13) one can write the spin-connections in terms of the vielbeins. The metric compatibility in the metric formulation of GR correspond to requiring that the vielbein is covariantly conserved in the vielbein formulation.

$$\nabla_\mu E_\nu{}^A = \partial_\mu E_\nu{}^A \Gamma_\nu{}^\rho{}_\mu E_\rho{}^A - \Omega_\mu^{AB} e_\nu{}^C \eta_{BC} = 0 \tag{4.14}$$

From this equation one can obtain the relation between the spin-connections and the Christoffel symbols.

General relativity by Gauging Poincare'

The vielbein formualtion of General Relativity can be viewed as the gauging procedure of the Pioncare' algebra, as it was explained in [50, 114]. The gauging procedure of Poincare', leading to GR, will be as follows. The starting point is the Poincare' algebra (4.3). It is possible to introduce a Lie algebra valued connection $\mathcal{A}_\mu$ which is a way for introducing a gauge field for each generator of the algebra

$$\mathcal{A}_\mu = P_A E_\mu{}^A + \frac{1}{2} M_{AB} \Omega_\mu{}^{AB} \tag{4.15}$$

The connection $\mathcal{A}_\mu$ will transform under the adjoint representation

$$\delta \mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu, \Lambda] \ , \quad \Lambda = P_A \xi^A + \frac{1}{2} M_{AB} \sigma^{AB} \tag{4.16}$$

For each gauge field it is possible to write down their transformations properties (under diffeomorphism and under Lorentz rotations)

$$\begin{aligned} \delta E_\mu{}^A &= \xi^\rho \partial_\rho E_\mu{}^A + E_\rho{}^A \partial_\mu \xi^\rho + \lambda^A{}_B E_\mu{}^B && (4.17) \\ \delta \Omega_\mu{}^{AB} &= \xi^\rho \partial_\rho \Omega_\mu{}^{AB} + \Omega_\mu{}^{AB} \partial_\mu \xi^\rho + \partial_\mu \lambda^{AB} + 2\lambda^{[A}{}_C \Omega_\mu{}^{CB]} && (4.18) \end{aligned}$$

and their curvatures

$$\begin{aligned} R_{\mu\nu}{}^A(P) &= 2\partial_{[\mu} E_{\nu]}{}^A - 2\Omega_{[\mu}{}^A{}_B E_{\nu]}{}^B && (4.19) \\ R_{\mu\nu}{}^{AB}(M) &= 2\partial_{[\mu]} \Omega_{\nu]}{}^{AB} - 2\Omega_{[\mu}{}^A{}_C \Omega_{\nu]}{}^{CB} && (4.20) \end{aligned}$$

In order to take contact with familiar objects in General Relativity, in is possible to relate the curvature of P to the torsion, and this is why it is called the torsion tensor, and the curvature of M to the Riemann tensor. This is done in the following way

$$\begin{aligned} R_{\mu\nu}{}^A(P) &= \Gamma_{[\mu}{}^\rho{}_{\nu]} E_\rho^A && (4.21) \\ R_{\mu\nu}{}^\sigma{}_\rho &= E_{\rho A} E_B^\sigma R_{\mu\nu}{}^{AB}(M) && (4.22) \end{aligned}$$

In general these two curvatures can be different from zero. In General Relativity usually one considers metrics which are covariantly constant (the constraint which is usually called metric compatibility). Moreover the common choice in General Relativity is to deal with theories without torsion, i.e. one usually imposes the curvature of P to zero (curvature constraint). With this choice, the spin-connection $\omega_\mu{}^{AB}$ are fully dependent connection expressible in terms of the vielbeins and their derivatives. Otherwise, without fixing the torsion, the vielbeins and the spin-connection remain independent, as it happens for instance in the Palatini formulation of Einstei gravity.

These informations are sufficient to define and understand the kinematics of General Relativity from a gauging procedure. If one is interested also in the dynamics of General Relativity, this can be obtained by putting the theory on-shell, imposing Einstein equations.

4.1.2 Newton Cartan geometry by limit of relativistic gravity

In non relativistic theories the falling frames are connected with Galilean symmetry instead of Poincare' symmetry. For reasons that will become clear at the end of this section, it is useful to work with the Bargmann Algebra instead of the Galilean algebra. The Bargmann algebra is just a central extension of the Galilean algebra, and the central extension will become important in the gauging procedure. In [31] it was argued that the Bargmann Algebra can be obtained by an Inonü-Wigner contraction of the direct sum of the Poincaré with an abelian factor with generator $\mathcal{Z}$. The abelian factor will be represetend by the gauge field M_μ and it will transform under diffeomorphism and gauge transformations in the following way

$$\delta M_\mu = \xi^\rho \partial_\rho M_\mu + M_\rho \partial_\mu \xi^\rho + \partial_\mu \Lambda \tag{4.23}$$

and its curvature will be

$$F_{\mu\nu}(M) = \partial_\mu M_\nu - \partial_\nu M_\mu \tag{4.24}$$

Starting from the Poincaré algebra (4.3), with the central extension $\mathcal{Z}$ (which is a generator that commutes with everything, as it should be since it is a central extension) it is possible to define the following generator, using a contraction parameter ω

$$P_0 \to \frac{1}{2\omega} H + \omega Z \ , \quad \mathcal{Z} \to \frac{1}{2\omega} H + Z \ , \quad M_{a0} \to \omega G_a \tag{4.25}$$

and all the other generators of the Poincaré algebra are left untouched (i.e. $P_a = P_a$ and $M_{ab} = J_{ab}$). Plugging in this ansatz (4.25) into (4.3) and taking the limit for $\omega \to \infty$ one gets the following algebra

$$\begin{aligned} [J_{ab}, J_{cd}] &= 4\delta_{[a[c} J_{d]b]} && (4.26) \\ [P_a, J_{cd}] &= 2\delta_{a[b} P_{c]} && (4.27) \\ [P_a, J_{cd}] &= 2\delta_{a[b} P_{c]} && (4.28) \\ [H, G_a] &= P_a && (4.29) \\ [P_a, G_b] &= \delta_{ab} Z && (4.30) \end{aligned}$$

where a, b, c indices run over the spatial directions and all the other commutators are zero. This is the Bargmann algebra. Now we know that General Relativity can be obtained by a gauging procedure of Poincaré, and we know that the Iononü-Winger contraction of Poincaré is Bargmann, therefore we can argue that also the Bargmann algebra can be gauged.

The gauging of the Bargmann algebra can be done, and the result of the gauging is a new geometry, that has been called (Torsional) Newton Cartan geometry (TNC).

In this few pages we would like to present yet an other point of view. In [31] it has been argued that the limiting Iononü-Winger procedure can be done not only at the level of the generators, but at the level of the gauge fields. This will not give you the most general (T)NC geometry, but a sub-case of it. We will comment later on some more general frameworks. In this setup, one can give the following redefinitions of the gauge fields

$$
\begin{aligned}
E_\mu{}^A &= \delta_0^A \left(\omega\tau_\mu + \frac{1}{2\omega} m_\mu \right) + \delta_a^A e_\mu{}^a && (4.31) \\
E^\mu{}_A &= \delta^a{}_A e^\mu{}_a + \frac{1}{\omega} \delta_a^0 v^\mu + O\left(\frac{1}{\omega^2} \right) && (4.32) \\
M_\mu &= \omega\tau_\mu - \frac{1}{2\omega} m_\mu && (4.33)
\end{aligned}
$$

From the properties of vielbeins in the relativistic picture, it is possible to derive the properties of the geometrical objects we just defined

$$
e^\mu{}_a e_\mu{}^b = \delta_a^b \ , \quad v^\mu \tau_\mu = -1 \ , \quad v^\mu e_\mu{}^a = \tau_\mu e^\mu{}_a = 0 \ , \quad e^\rho{}_a e_\mu{}^a = v^\rho \tau_\mu \qquad (4.34)
$$

Moreover, if you perform the substitution and the limit at the level of the gauge field (4.15), using the redefinitions of the gauge fields and the redefinitions of the algebra generator, you get

$$
P_A E_\mu{}^A + M_{AB} \Omega_\mu{}^{AB} = P_a e_\mu{}^a + H\tau_\mu + Z m_\mu + + M_{ab} \omega_\mu{}^{ab} - 2 G_a \omega_\mu{}^a \qquad (4.35)
$$

and the explicit form of the spin-connections $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ are given in terms of $e_\mu{}^a$, τ_μ and v^μ through the existing relativistic relations between the relativistic spin-connection $\Omega_\mu{}^{AB}$ and the relativistic vielbein $E_\mu{}^A$. Therefore we ended up by having one gauge field for each generator of the Bargmann algebra. We can push ourselves a step further, and compute the transformation properties of those gauge field (up to diffeomorphism for simplicity)

$$
\begin{aligned}
\delta\tau_\mu &= 0 && (4.36) \\
\delta e_\mu{}^a &= \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu && (4.37) \\
\delta m_\mu &= \partial_\mu \sigma + \lambda_a e_\mu{}^a && (4.38) \\
\delta \omega_\mu{}^a b &= \partial_\mu \lambda^{ab} + 2\lambda^{[a}{}_c + \omega_\mu{}^{cb]} && (4.39) \\
\delta \omega_\mu{}^a &= \partial_\mu \lambda^a + \lambda^a{}_b \omega_\mu{}^b - \omega_\mu{}^a{}_c \lambda^c && (4.40)
\end{aligned}
$$

and the expressions for the curvatures

$$
\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu} e^a_{\nu]} - 2\omega_{[\mu}{}^{ab} e^b_{\nu]} - 2\omega^a_{[\mu}\tau_{\nu]} && (4.41) \\
R_{\mu\nu}(Z) &= 2\partial_{[\mu} m_{\nu]} - 2\omega_{[\mu}{}^a e_{\nu]}{}^a && (4.42) \\
R^{ab}_{\mu\nu}(J) &= 2\partial_{[\mu} \omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^a{}_c \omega_{\nu]}{}^{cb} && (4.43) \\
R^a_{\mu\nu}(G) &= 2\partial_{[\mu} \omega^a_{\nu]} - 2\omega_{[\mu}{}^{ab} \omega^b_{\nu]} && (4.44) \\
R_{\mu\nu}(H) &= 2 e^\mu{}_a e^\nu{}_b \partial_{[\mu} \tau_{\nu]} && (4.45)
\end{aligned}
$$

The curvatures of P and Z are automatically zero if you take the expressions for $\omega_\mu{}^{ab}$ and $\omega_m u^a$ in terms of $e_\mu{}^a$, τ_μ and v^μ computed by taking the limit of $\Omega_\mu{}^{AB}$ (this resembles the imposition of the momentum constraint in the relativistic theory). Imposing the H constraint, meaning setting the curvature $R_{\mu\nu}(H)$ to zero, one has a restriction on τ, which is the following

$$
\partial_{[\mu} \tau_{\nu]} = b_{[\mu} \tau_{\nu]} \qquad (4.46)
$$

where b_μ is an arbitrary vector field. This resembles the constraint which was found in [7], where the twistless torsional Newton Cartan geometry was developed (TTNC). If you set b_μ to be the zero, you obtain a geometry without torsion where time is absolute.

This ends the kinematic of the TTNC geometry. If one is interested also in the dynamics, the exercise to do is to take the non relativistic limit of the Einstein equations. This can be done and it leads, for the specific case in which the torsion is zero, to the same equation of motion presented in [7]. One of this equations, once gauge-fixed, reproduces the first Newton-Cartan equation of motion $\Delta\Phi = 0$ (which is the well-known Poisson equation).

As in General Relativity, it si possible to write an action that can be minimized in order to obtain the NC equation of motion. This work has been done in [100].

It is also possible to introduce a spatial metric $h_{\mu\nu}$ from the NC vielbeins as follows

$$h_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{4.47}$$

Looking at the transformation properties (4.36) it is possible to derive some combinations of Newton Cartan vectors that are invariant under Galilean boosts, and these objects are going to be very important in the future works

$$\begin{aligned} \hat{v}^\mu &:= v^\mu + h^{\mu\nu} m_\nu && (4.48)\\ \bar{h}_{\mu\nu} &:= h_{\mu\nu} - \tau_{(\mu} m_{\nu)} && (4.49)\\ \tilde{\Phi} &:= -v^\mu m_\mu + \frac{1}{2} h^{\mu\nu} m_\mu m_\nu && (4.50) \end{aligned}$$

These will be the geometric invariants, together with τ_μ, $h^{\mu\nu}$ and the determinant $e = (\tau_\mu, e_\mu{}^a)$, we need to use to build tensors that transform covariantlly under local diffeomorphism transforamtion and that are invariant under the internal symmetries (boost, space and time translations, but not for the Bargamann U(1) symmetry).

The metric compatibility condition in the relativistic setup, imposes $E_\mu{}^a$ to be covariantly conserved, as we already said in the previous section. This implies that also τ_μ and $h^{\mu\nu}$ are covariantly conserved

$$\nabla_\rho h^{\mu\nu} = 0 \ , \quad \nabla_\rho \tau_\mu = 0 \tag{4.51}$$

In this non-relativistic case, however, this equations do not solve uniquely for the Christoffel symbols, as it happens in the relativistic case. One can perform the Inonü-Wigner limit of the relativistic connection, simply by substituting the redefinitions (4.31) and taking the limit for $\omega \to \infty$. With this method you will obtain a valid Christoffel connection in the non-relativistic setup. But it will not be the only connection that satisfies (4.51). In fact, in the Newton Cartan framework, we will have a family of connections, as it was argued in [114], that satisfies (4.51)

$$\Gamma^{(\alpha)\rho}_{\mu\ \ \nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left[\partial_\mu H^{(\alpha)}_{\nu\sigma} + \partial_\nu H^{(\alpha)}_{\mu\sigma} - \partial_\sigma H^{(\alpha)}_{\mu\nu} \right] \tag{4.52}$$

where

$$H^{(\alpha)}_{\mu\nu} = \bar{h}_{\mu\nu} + \alpha \tau_\mu \tau_\nu \tilde{\Phi} \tag{4.53}$$

where α is any constant. They are all equivalent, and we will use one or the other in different setup we would like to study.

4.1.3 Generic torsion and the Stückelberg Scalar

As we previously said, there is no need in setting to zero the curvatures. This happens when you construct the non-relativistic geometry using the limiting procedure explained in the previous sections. But this is not the general case. In general the curvatures can be non-zero, and this will give us the torsion feature. We will see that in order to construct a theory with generic torsion, one need to introduce an extra scalar field in order to preserve covariance.

The first thing to notice is that the antisymmetric part of the connection (4.52) is fixed because the second term of it is totally symmetric

$$\Gamma^{(\alpha)\rho}_{[\mu\ \ \nu]} = -\hat{v}^{\rho}(\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) \tag{4.54}$$

Moreover, given the compatibility conditions (4.51) and the form of the curvature of P given in (4.41), it follows that

$$R_{\mu\nu}{}^{a}(P) = e^{\mu a} m_\mu \left(\partial_\mu \tau_\nu - \partial_\nu \tau_\mu\right) \tag{4.55}$$

From this expression we see that the left hand side transforms under our internal symmetries in the same way of the right hand side only if we ignore the transformations due to the central extension N, and this is due to the presence of m_μ that transforms also under N as it is evident from equation (4.36). One way to solve the problem is to introduce and other scalar, that we call Stückelberg Scalar χ. which will transform under the N transformation as a scalar, i.e.

$$\delta\chi = \sigma \tag{4.56}$$

and it will be a shifting of m_μ in the following way

$$M_\mu = m_\mu - \partial_\mu \chi \tag{4.57}$$

In this way, M_μ will be invariant under N transformations. Therefore, in order to have a theory covariant under N, it is important to replace all the m_μ with the M_μ defined in (4.57) (note that it is not the same M_μ that it was used in the limiting procedure: that one was a field of the relativistic framework). Therefore, if we want theory which preserves this U(1) symmetry, one way is to introduce the Stückelberg Scalar χ. This new scalar is necessary if one wants the $U(1)$ invarance whenever there is torsion, i.e. whenever the right hand side of (4.55) is nonzero.

Therefore we conclude that it is possible to have a generic torsion if we add the Stückelberg Scalar χ. Depending on the features of the curvature constraint (4.54) we distinguish three cases for the torsion

- $\partial_\mu \tau_\nu - \partial_\nu \tau_m u = 0$: no torsion. This is called Newton-Cartan geometry (NC) and it has absolute time.
- $\tau_{[\mu}\partial_\nu \tau_{\rho]} = 0$, which means that τ_μ is hypersurface orthogonal (HSO). This is called Twistless Torsional Newton-Cartan geometry (TTNC). This is a geometry in which the induced geometry on the slices to which τ_μ is HSO, is described by a torsion free Riemannian geometry. In this setup is also useful to introduce a new vector

  $$a_\mu = \hat{v}^{\rho}\partial_\rho \tau_\mu - \tau_\rho \partial_\mu \hat{v}^{\rho} \tag{4.58}$$

  which can be called the torsion vector.

- No constraint on τ_μ which corresponds to the Torsional Newton-Cartan geometry (TNC). This particular new geometry is very subtle. In this case in fact, there is no foliation of time, and in this case the geometry violates causality. So what is the relevance of this novel geometry? The fact is that it still can be used as a background geometry to which a field theory can couple, and it is a more general geometry with respect to the one in which the τ_μ is closed. So as a fixed background geometry it plays an important role, because it is important to have it in order to define the most general energy-momentum tensor of a field theory coupled to a non-relativistic geometry. The fact that there is no foliation in the geometry does not imply non-causal effect on the matter field, as we are going to see in the following sections. Concerning instead its role as a proper dynamical geometry, one would like that, once you analyze the dynamical structure of the geometry generated by the Bargmann algebra, those modes are not allowed by the equation of motion.

4.1.4 Null reduction of metric

The Newton-Cartan geometry could be also obtained by null reduction of a Lorentzian spacetime with one extra dimension u, as it was studied by [68, 134, 28, 56, 55]. Consider the null reduction ansatz for the metric

$$ds^2 = \gamma_{AB}dx^A dx^B = 2\tau_\mu dx^\mu (du - m_\nu dx^\nu) + h_{\mu\nu}dx^\mu dx^\nu \,, \tag{4.59}$$
$$= 2\tau_\mu dx^\mu du + \bar{h}_{\mu\nu}dx^\mu dx^\nu \tag{4.60}$$

where $A = (u, \mu)$, in which $a = 1\,, \ldots\,, d$. The reduction ansatz is the most general metric for which $\gamma_{uu} = 0$ and such that ∂_u is a null Killing vector of γ_{AB}. The fields τ_μ and e^a_μ are the vielbeins of the $d+1$ dimensional TNC geometry. The metric (4.59) preserves the following local tangent space transformations

$$\delta\tau_\mu = 0\,, \tag{4.61}$$
$$\delta e^a_\mu = \tau_\mu\lambda^a + \lambda^a{}_b e^b_\mu\,, \tag{4.62}$$
$$\delta m_\mu = \partial_\mu\sigma + \lambda_a e^a_\mu\,. \tag{4.63}$$

The local σ transformation requires $\delta u = \sigma$. The transformations with local parameter λ^a correspond to tangent space Galilean boosts (G) and transformations with local parameter $\lambda^a{}_b$ correspond to tangent space rotations (J). The metric components $\gamma_{\mu u} = \tau_\mu$ and $\gamma_{\mu\nu} = \bar{h}_{\mu\nu}$ are invariant under these local transformations. The inverse metric is

$$\gamma^{uu} = 2\tilde{\Phi}\,, \qquad \gamma^{u\mu} = -\hat{v}^\mu\,, \qquad \gamma^{\mu\nu} = h^{\mu\nu}\,, \tag{4.64}$$

where $\tilde{\Phi}$ and $\hat{v}^\mu$ are defined in (4.48).

A TNC compatible connection can be obtained by performing the null reduction of the Levi-Civita connection of the higher dimensional spacetime. A torsionful affine connection $\Gamma^\rho_{\mu\nu}$ that is invariant under the local tangent space symmetries (G, J) and that satisfies metric compatibility, in the TNC sense, i.e.

$$\nabla_\mu\tau_\nu = 0\,, \tag{4.65}$$
$$\nabla_\mu h^{\nu\rho} = 0\,, \tag{4.66}$$

is given by

$$\bar{\Gamma}^{\rho}_{\mu\nu} = -\hat{v}^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}\bar{h}_{\nu\sigma} + \partial_{\nu}\bar{h}_{\mu\sigma} - \partial_{\sigma}\bar{h}_{\mu\nu}\right) . \tag{4.67}$$

In [109] is presented a more sophisticated way to perform the reduction. In fact, instead of speaking about null reduction, the point of view of [109] is that TNC is the geometry on the spacetime orthogonal to the null Killing vector ∂_u. This makes the reduction more general and the result is that one can access more general connections of the TNC geometry.

4.1.5 Gauging Carrol algebra: Carrollian spacetime

The Carrol algebra can be obtained as a Iononü-Wigner contraction of the Poincaré algebra, as it was explained for the Galilean algebra. The difference with the previous exercise is that instead of taking $\omega \to \infty$, one should take the opposite limit, i.e. $\omega \to 0$. This has been done by [21] and the resulting Carrol algebra is the following

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{ac}J_{bd} - \delta_{ad}J_{bc} + \delta_{bc}J_{ad} - \delta_{bd}J_{ac} && (4.68)\\
[P_c, J_{ab}] &= \delta_{bc}P_a - \delta_{ac}P_b && (4.69)\\
[C_c, J_{ab}] &= \delta_{bc}C_a - \delta_{ac}C_b && (4.70)\\
[P_a, C_b] &= \delta_{ab}H && (4.71)
\end{aligned}$$

where $a = 1, \ldots, d$. This Carrol algebra can be gauged in the same way as we did for the Bargmann algebra. This work was earlier done in [30]and more recently from an other point of view by [109]. As in the Bargmann case one could define a Lie valued algebra connection in the following way

$$\mathcal{A}_{\mu} = H\tau_{\mu} + P_A e^a_{\mu} + C_a\Omega^a_{\mu} + \frac{1}{2}J_{ab}\Omega^{ab}_{\mu} \tag{4.72}$$

and as in the previous case one can study how internal transformation acts on the gauge fields (for simplicity we don't write diffeomorphism, but they can be considered as in the Bargmann case)

$$\begin{aligned}
\delta\tau_{\mu} &= e^a_{\mu}\lambda_a && (4.73)\\
\delta e^a_{\mu} &= e^b_{\mu}\lambda^a{}_b && (4.74)\\
\delta\Omega^a_{\mu} &= \partial_{\mu}\lambda^a + \lambda^a{}_b\Omega_{\mu}{}^b - \lambda_b\Omega_{\mu}{}^{ab} && (4.75)\\
\delta\Omega^{ab}_{\mu} &= \partial_{\mu}\lambda^{ab} + \lambda^a{}_c\Omega_{\mu}{}^{cb} - \lambda^b{}_c\Omega_{\mu}{}^{ca} && (4.76)
\end{aligned}$$

and their curvatures

$$\begin{aligned}
R_{\mu\nu}(H) &= \partial_{[\mu}\tau_{\nu]} + 2e^a_{(\mu}\Omega_{\nu)a} && (4.77)\\
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu}e^a_{\nu]} - 2\Omega_{[\mu}{}^{ab}e^b_{\nu]} && (4.78)\\
R^a_{\mu\nu}(C) &= 2\partial_{[\mu}\Omega^a_{\nu]} - 2\Omega_{[\mu}{}^{ab}\Omega^b_{\nu]} && (4.79)\\
R^{ab}_{\mu\nu}(J) &= 2\partial_{[\mu}\Omega_{\nu]}{}^{ab} - 2\Omega_{[\mu}{}^a{}_c\Omega_{\nu]}{}^{cb} && (4.80)
\end{aligned}$$

The following step is to impose vielbein postulates so that we can write (4.77) in terms of the curvature and the torsion of an affine connection Γ which should be invariant under the internal symmetry transformations. This can be done and it is possible to write down a very general torsionful connection. In this framework we thus have a transformation δ, which will transform the fields as (4.73), acting on the fields τ_μ, e^a_μ, Ω^a_μ and Ω^{ab}_μ, with a connection $\Gamma^\rho_{\mu\nu}$ which is computed in such a way that is metric compatible, i.e. $\nabla_\mu \tau_\nu = 0$ and $\nabla_\mu h_{\nu\rho} = 0$. As opposed to the Bargmann case, in this Carrol framework we haven't introduced yet any vector M_μ. This is because in the algbebra (4.68) there wasn't a central extension (and one can prove that the Carrol algebra cannot be centrally extended) therefore in the connection $\mathcal{A}_\mu$ there wasn't any extra $U(1)$ gauge field. In this framework one can anyway introduce an extra M_μ field by hand. In the case of TNC geometry the gauge field M_μ was important when curvature constraints were imposed. The effect of such curvature constraints was that the connection $\Gamma^\rho_{\mu\nu}$ became a fully dependent connection. In other words the effect of adding curvature constraints was that of realizing the algebra on a smaller number of fields. In fact, before imposing the curvature constraints, one had τ_μ, e^a_μ and Ω^a_μ as field content; instead after imposing the curvature constraints one has τ_μ, e^a_μ and M_μ as field content. We are not going to add the details of this, but it is possible to do it. Once a new M^μ field is added, one can also define invariants under Carrol boost C, which are the following

$$\hat{\tau}_\mu = \tau_\mu - h_{\mu\nu} M^\nu \tag{4.81}$$

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - M^\mu v^\nu - M^\mu v^\mu \tag{4.82}$$

$$\bar{\Phi} = -M^\nu \tau_\nu + \frac{1}{2} h_{\nu\sigma} M^\nu M^\sigma \tag{4.83}$$

and it is possible to write the affine connection in terms of these invariants. Moreover it is also possible to derive the Carrollian geometry by considering it as the geometry induced on a null hyper-surface, as it was done in the section 4.1.4 for the TNC geometry. The reduction ansatz is the following

$$ds^2 = du \left(2\bar{\Phi} du - 2\hat{\tau}_\mu dx^\mu\right) + h_{\mu\nu} dx^\mu dx^\nu \tag{4.84}$$

The Carrollian spacetime can be thought in this framework of null reduction ad the geometry of the null hypersurface $u = \text{const}$ whose normal is $\partial_a u$.

Moreover, as it was suggested by [69] there is a strong duality between Newton-Cartan and Carrol spacetime, and this is evident also in the null reductions realization of such geometries. The metric-like objects in the TNC geometry are given by τ_μ and $h_{\mu\nu}$ while in the Carrol geometry are given by v^μ and $h^{\mu\nu}$. One can show that there si the following duality

$$\tau_\mu \leftrightarrow v^\mu \ , \quad h_{\mu\nu} \leftrightarrow h^{\mu\nu} \tag{4.85}$$

Moreover, also the extra $U(1)$ fields can be related in the following way

$$M_\mu \leftrightarrow M^\mu \tag{4.86}$$

and, as a consequence, also the invariants (4.48) and (4.81) can be related

$$\hat{v}^\mu \leftrightarrow \hat{\tau}_\mu \ , \quad \bar{h}_{\mu\nu} \leftrightarrow \bar{h}^{\mu\nu} \ , \quad \tilde{\Phi} \leftrightarrow \bar{\Phi} \tag{4.87}$$

Moreover, an other important thing to notice in this duality is that whenever there is no coupling to $\tilde{\Phi}$ and to $\bar{\Phi}$, also the affine connection of the two geometry coincides. Therefore, when $\bar{\Phi} = \tilde{\Phi} = 0$ the Newton-Cartan and the Carrol geometry are the same.

4.2 Field theories on TNC background

There are many ways to construct field theories in curved TNC background. One of those is of course a bottom up approach, i.e. consider a field and write down all the terms that respect the symmetries using the Newton Cartan invariants (4.48). This method was mainly developed in [110]. The first step is to write down an action that will be dependent on the Newton Cartan invariants

$$S = S\left[\hat{v}^{\mu}, h^{\mu\nu}, \Phi\right] \tag{4.88}$$

When we will perform the variations, we need to keep in mind that not all the Newton Cartan geometric objects are independent. In fact there are some orthogonality relations which are described in equation (4.34). This means that we need to choose a set of fields that we consider independent (it does not matter which field we choose at this point, what matters is just the number of degrees of freedom). We choose as independent background fields the set $\{\hat{v}^{\mu}, h^{\mu\nu}, \Phi\}$. The reason we did such a choice will become clear soon. All the variations with respect to other background field will be dependent from this set of variation. For example

$$\delta\tau_{\mu} = \tau_{\mu}\tau_{\nu}\delta\hat{v}^{\nu} - \bar{h}_{\mu\rho}\tau_{\nu}\delta h^{\nu\rho} \tag{4.89}$$

$$\delta\bar{h}_{\mu\nu} = -2\tau_{\mu}\tau_{\nu}\delta\Phi + (\tau_{\mu}\bar{h}_{\nu\rho} + \tau_{\nu}\bar{h}_{\mu\rho})\delta\hat{v}^{\rho} - \hat{h}_{\mu\rho}\hat{h}_{\nu\sigma}\delta h^{\rho\sigma} \tag{4.90}$$

Here we summarize results obtained in [56, 55, 111, 112, 33, 110, 114] regarding the most general formulation of torsional Newton–Cartan (TNC) geometry. We will focus only on those aspects that are needed for the purposes of this work. Since here we encounter TNC geometry through null reduction of the AdS_5 boundary metric we will study its properties in this context.

4.2.1 Non relativistic analogue of the energy-momentum tensor and Ward identities

To make connection with some literature in the field, and also to make connection with the holographic description that we are going to develop later, we first write the variation with respect to the background NC fields in the following way

$$\delta\mathcal{S} = \int dx\, e\,\left[-S^{0}_{\mu}\delta v^{\mu} + S^{a}_{\mu}\delta e^{\mu}_{a} + T^{0}\delta m_{0} + T^{a}\delta m_{a} + \langle O_{\chi}\rangle\,\delta\chi\right] \tag{4.91}$$

where $m_0 = -v^{\mu}m_{\mu}$ and $m_a = e^{\mu}_{a}m_{\mu}$. Now we can substitute in the Stückelberg Scalar and the field M_{μ} by $m_{\mu} = M_{\mu} + \partial_m u\chi$ and we can write the variations with respect to the boost invariant geometric NC objects. In order to do so, it is convenient to define two new tensors

$$T^{\mu} = -T^{0}v^{\mu} + T^{a}e^{\mu}_{a} \tag{4.92}$$

$$T^{\mu}{}_{\nu} = -(S^{0}_{\nu} + T^{0}\partial_{\nu}\chi) + (S^{a}_{\nu} + T^{a}\partial_{\nu}\chi)e^{\mu}_{a} \tag{4.93}$$

The variation of the action with respect to the background fields, written in terms of the invariants, with the Stückelberg Scalar turned on, and with these field redefinitions, will then become

$$\begin{aligned}\delta\mathcal{S} \;=\; & \int dx\, e\,\Big\{-\tau_{\nu}T^{\nu}{}_{\mu}\delta\hat{v}^{\mu} - \left(\hat{e}^{a}_{\nu}\hat{v}^{\mu}T^{\nu}{}_{\mu}\right)\hat{e}_{\sigma a}\tau_{\rho}\delta h^{\rho\sigma} + \frac{1}{2}\left(\hat{e}^{b}_{\nu}e^{\mu}_{a}T^{\nu}{}_{\mu}\right)\hat{e}_{\rho b}\hat{e}^{a}_{\sigma}\delta h^{\rho\sigma} \\ + \;& \tau_{\mu}T^{\mu}\delta\Phi + \left(\langle O_{\chi}\rangle - \frac{1}{e}\partial_{\mu}\left(eT^{\mu}\right)\right)\delta\chi + \left(\hat{e}^{a}_{\mu} - \tau_{\nu}e^{\mu a}T^{\nu}{}_{\mu}\right)\delta M_{a} - \frac{1}{2}\hat{e}^{[a}_{\nu}e^{b]\mu}T^{\nu}{}_{\mu}\left(\hat{e}_{\rho a}\delta e^{\rho}_{b} - \hat{e}_{\rho b}\delta e^{\rho}_{a}\right)\Big\}\end{aligned} \tag{4.94}$$

This variations of the action tells us many things. First of all that the components of the tensors defined in (4.92) can describe physical quantities since they are boost invariant. In fact this will be the case, and in particular they will represent the following physical quantites

$$\hat{e}^a_\mu T^\mu{}_\nu \quad \rightarrow \quad \text{energy density} \tag{4.95}$$
$$\hat{v}^\nu T^\mu{}_\nu \quad \rightarrow \quad \text{energy flux} \tag{4.96}$$
$$\tau_\mu T^\mu{}_\nu \quad \rightarrow \quad \text{momentum density} \tag{4.97}$$
$$e^a_\nu T^\mu{}_\nu \quad \rightarrow \quad \text{stress density} \tag{4.98}$$
$$\tau_\mu T^\mu \quad \rightarrow \quad \text{mass density} \tag{4.99}$$
$$e^a_\mu T^\mu \quad \rightarrow \quad \text{mass flux} \tag{4.100}$$

Moreover the Ward identities for the Stückelberg Scalar and for the local Galilean boost, which are associated to local shift transformations acting on M_a and χ, are

$$\frac{1}{e}\partial_\mu\left(eT^\mu\right) = \langle O_\chi\rangle \ , \quad \hat{e}^a_\mu T^\mu - \tau_\nu e^{\mu a} T^\nu{}_\mu = 0 \tag{4.101}$$

and the Ward identity associated to the local rotation symmetry is

$$\hat{e}^{[a}_\nu e^{b]\mu} T^\nu{}_\mu = 0 \tag{4.102}$$

So far we have looked at general variations of the background fields. Now we would like to look to global TNC spacetime symmetry. There is a symmetry of this type that plays an important role in the Newton-Cartan geometry and in all physics. We are talking about the diffeomorphism, which is the set of transformations that leave the background field invariant. We write the variation of the action with respect to tensors that are invariant under the internal symmetry, because it is more convenient. The variation of the action with respect to diffeomorphism acting only on the background fields is

$$\begin{aligned}\delta_{\text{bg}}S \quad = \quad & \int d^{d+1}xe\left[-\tau_\nu T^\nu{}_\mu \mathcal{L}_\xi \hat{v}^\mu - \left(\hat{h}_{\sigma\nu}\hat{v}^\mu T^\nu{}_\mu\right)\tau_\rho \mathcal{L}_\xi h^{\rho\sigma}\right. \\ & \left. +\frac{1}{2}\left(\hat{h}_{\rho\nu}\hat{h}_{\sigma\lambda}h^{\lambda\mu}T^\nu{}_\mu\right)\mathcal{L}_\xi h^{\rho\sigma} + \tau_\mu T^\mu \mathcal{L}_\xi \tilde{\Phi}\right] ,\end{aligned} \tag{4.103}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector ξ^μ. The solution to this equation is to ask that

$$\mathcal{L}_\xi \hat{v}^\mu = 0 \ , \quad \mathcal{L}_\xi h\mu\nu = 0 \ , \quad \mathcal{L}_\xi \tilde{\Phi} = 0 \tag{4.104}$$

We call K^μ the vector that solve this equations, which will be a Killing vector for TNC geometry. The variation of the action can yet be written in an other way (by explicitly writing down the Lie derivatives)

$$\delta_{\text{bg}}S \quad = \quad -\int d^{d+1}x\partial_\nu\left(e\xi^\mu T^\nu{}_\mu\right) + \int d^{d+1}xe\xi^\rho\left[e^{-1}\partial_\nu\left(eT^\nu{}_\rho\right) + \tau_\mu T^\mu \partial_\rho\tilde{\Phi}\right. \tag{4.105}$$
$$+ \quad T^\nu{}_\mu\left(\hat{v}^\mu\partial_\rho\tau_\nu - e^\mu_a\partial_\rho\hat{e}^a_\nu\right)] , \tag{4.106}$$

hence, on-shell (i.e. after throwing away boundary terms and after imposing the equation of motions for the NC fields) we have the following diffeomorphism Ward identity

$$e^{-1}\partial_\nu(eT^\nu{}_\mu) + T^\rho{}_\nu\left(\hat{v}^\nu\partial_\mu\tau_\rho - e^\nu_a\partial_\mu\hat{e}^a_\rho\right) + \tau_\nu T^\nu\partial_\mu\tilde{\Phi} = 0 \tag{4.107}$$

This Ward Identity can be rewritten also in the following way

$$\nabla_\nu T^\nu{}_\mu + \text{Torsion terms} + \tau_\nu T^\nu \partial_\mu \tilde{\Phi} = 0 \tag{4.108}$$

which tells us that the energy momentum tensor of the non relativistic matter we have coupled to the Newton-Cartan geometry is influenced in this way by the Newton potential, as we expected.

4.2.2 Local scale transformation and the dilatation connection

It is important also to study setups in which the theory under consideration could be scale invariant and coupled to a Newton-Cartan background. In order to do so, it is necessary to set appropriate weights to the fields and to the background NC field such that the scale transformation leaves the action invariant.

One way of describing such configurations is to consider the gauging of the Schrödinger algebra which is described by the Bargmann algebra plus a Dilaton generator which commutators with the generators of the Bargmann algebra are

$$[D,H] = -zH \tag{4.109}$$

$$[D,G_a] = (z-1)G_a \tag{4.110}$$

$$[D,P_a] = -P_a \tag{4.111}$$

$$[D,Z] = (z-2)N \tag{4.112}$$

Note that when D is added to the algebra, the generator Z is not central anymore, since the commutator with D is nonzero (unless when $z=2$ which is a special case). The Lie valued connection we can define for this algebra in order to gauge it will be

$$\mathcal{A}_\mu = H\tau_\mu + P_a e^a_\mu + G_a\omega^a_\mu + \frac{1}{2}J_{ab}\omega^{ab}_\mu + Z\tilde{m}_\mu + Db_\mu \tag{4.113}$$

Among the other internal transformations, there will be also a transformation associated to the dilatation with parameter Λ_D. The transformations of the gauge fields under the dilatation operator, can be read off from (4.109). Now it is important to rewrite the covariant derivatives and therefore the affine connections in such a way that they consider also the transformations under the dilatation operator through Λ_D. One way to write down expressions which are dilatation covariant is to replace the derivatives with a dilatation covariant derivative. For example one can write the following connection [33, 114, 110]

$$\tilde{\Gamma}^\rho_{\mu\nu} = \hat{v}^\rho\left(\partial_\mu - zb_\mu\right)\tau_\nu + \frac{1}{2}h^{\rho\sigma}\left[\left(\partial_\mu - 2b_\mu\right)\bar{h}_{\nu\sigma} + \left(\partial_\nu - 2b_\nu\right)\bar{h}\mu\sigma - \left(\partial_\sigma - 2b_\sigma\right)\bar{h}_{\mu\nu}\right] \tag{4.114}$$

An other question to ask is how to express the Schrödinger field $\tilde{m}_\mu$ in terms of the Bargmann field m_μ and the Stückelberg Scalar. In the definition of M_μ given in the Bargmann case we need to substitute the derivative with the dilatation covariant derivative. Therefore we define a new M_μ for Schrödinger invariant field theories in the following way

$$M_\mu := \tilde{m}_\mu - \left(\partial_\mu + (z-2)b_\mu\right)\chi \tag{4.115}$$

Therefore we conclude that

$$\tilde{m}_\mu = m_\mu + (z-2)b_\mu \tag{4.116}$$

4.2.3 Null reduction of energy-momentum tensor

In [55, 111, 112, 110] we have worked out the coupling prescriptions of non-relativistic field theories to torsional Newton–Cartan (TNC) backgrounds both directly in field theory and from Lifshitz holography. The results of course agree (see e.g. [110]). Here we briefly review these results and derive them from null reduction as done in [55].

The TNC energy-momentum tensor (EMT) is defined as the response to varying the TNC fields via

$$\begin{aligned}\delta_{\text{bg}}S &= \int d^{d+1}xe\left[-\tau_\nu T^\nu{}_\mu\delta\hat{v}^\mu - \left(\hat{h}_{\sigma\nu}\hat{v}^\mu T^\nu{}_\mu\right)\tau_\rho\delta h^{\rho\sigma}\right.\\&\left.+\frac{1}{2}\left(\hat{h}_{\rho\nu}\hat{h}_{\sigma\lambda}h^{\lambda\mu}T^\nu{}_\mu\right)\delta h^{\rho\sigma} + \tau_\mu T^\mu\delta\tilde{\Phi}\right]\,, \end{aligned}\tag{4.117}$$

where e is the determinant of the 3 by 3 matrix (τ_μ, e^a_μ) which is both boost and rotation invariant. We can alternatively define an energy-momentum tensor by varying the unhatted TNC fields via

$$\delta_{\text{bg}}S = \int d^{d+1}xe\left[-\mathcal{T}_\mu\delta v^\mu + \frac{1}{2}\mathcal{T}_{\mu\nu}\delta h^{\mu\nu} + T^\mu\delta m_\mu\right]\,. \tag{4.118}$$

The two are related via

$$h^{\nu\rho}\mathcal{T}_{\rho\mu} - v^\nu\mathcal{T}_\mu = T^\nu{}_\mu + T^\nu m_\mu\,. \tag{4.119}$$

According to the null reduction of [55] the energy momentum tensor $T^\mu{}_\nu$ and mass current T^μ are related to the higher dimensional energy-momentum tensor t^{AB} via[1]

$$t^{\mu u} = 2\tilde{\Phi}T^\mu - \hat{v}^\sigma T^\mu{}_\sigma\,, \tag{4.120}$$

$$t^{\mu\nu} = -\hat{v}^\mu T^\nu + h^{\mu\rho}T^\nu{}_\rho\,. \tag{4.121}$$

The latter relation implies due to the symmetry of $t^{\mu\nu}$

$$-\hat{v}^\mu T^\nu + h^{\mu\rho}T^\nu{}_\rho + \hat{v}^\nu T^\mu - h^{\nu\rho}T^\mu{}_\rho = 0\,, \tag{4.122}$$

from which we read off the boost and rotation Ward identities

$$0 = -\hat{h}_{\mu\nu}T^\mu + \tau_\mu h^{\rho\sigma}\hat{h}_{\nu\sigma}T^\mu{}_\rho\,, \tag{4.123}$$

$$0 = \hat{h}_{\mu\rho}\hat{h}_{\nu\lambda}h^{\lambda\sigma}T^\rho{}_\sigma - (\mu\leftrightarrow\nu)\,. \tag{4.124}$$

The definitions (4.120) and (4.121) imply

$$\begin{aligned}-\frac{1}{2}t^{AB}\delta\gamma_{AB} &= -\tau_\nu T^\nu{}_\mu\delta\hat{v}^\mu - \left(\hat{h}_{\sigma\nu}\hat{v}^\mu T^\nu{}_\mu\right)\tau_\rho\delta h^{\rho\sigma}\\&+\frac{1}{2}\left(\hat{h}_{\rho\nu}\hat{h}_{\sigma\lambda}h^{\lambda\mu}T^\nu{}_\mu\right)\delta h^{\rho\sigma} + \tau_\mu T^\mu\delta\tilde{\Phi}\,,\end{aligned}\tag{4.125}$$

in agreement with the definition of $T^\mu{}_\nu$ and T^μ as the response to varying the TNC invariants $\hat{v}^\mu$, $h^{\mu\nu}$ and $\tilde{\Phi}$ as given in (4.117). The relation between the higher and lower dimensional energy-momentum tensors holds for any reduction given that γ_{AB} admits a null Killing vector. No additional assumptions such as hypersurface orthogonality of ∂_u are needed.

[1] Since t^{AB} is the response to varying γ_{AB} there is no need for t^{uu} since $\gamma_{uu} = 0$.

The higher-dimensional energy-momentum tensor corresponds to the boundary theory of a bulk AdS_5 space-time and is thus traceless. Further by boundary diffeomorphism invariance it satisfies a Ward identity for local diffeomorphism invariance. Upon reduction these give rise to Ward identities for local scale and diffeomorphism invariance. To this end it is useful to consider[2] $t^A{}_B$, i.e.

$$
\begin{aligned}
t^u{}_u &= 2\tilde{\Phi}\tau_\mu T^\mu - \hat{v}^\nu \tau_\mu T^\mu{}_\nu \,, && (4.129)\\
t^u{}_\nu &= 2\tilde{\Phi}\tau_\mu T^\mu{}_\nu - \hat{v}^\sigma \hat{h}_{\nu\rho} T^\rho{}_\sigma + \tau_\nu \hat{v}^\rho \hat{v}^\sigma t_{\rho\sigma} \,, && (4.130)\\
t^\mu{}_u &= T^\mu \,, && (4.131)\\
t^\mu{}_\nu &= T^\mu{}_\nu \,, && (4.132)
\end{aligned}
$$

where $\hat{v}^\rho \hat{v}^\sigma t_{\rho\sigma}$, which contains t^{uu}, is unspecified in terms of lower dimensional quantities as it will drop out of the Ward identities. We can derive the following identities

$$
\begin{aligned}
\nabla_A t^A{}_u &= \partial_\mu \left(e T^\mu\right) \,, && (4.133)\\
\nabla_A t^A{}_\mu &= e^{-1}\partial_\nu \left(e T^\nu{}_\mu\right) + T^\rho{}_\nu \left(\hat{v}^\nu \partial_\mu \tau_\rho - e^\nu_a \partial_\mu \hat{e}^a_\rho\right) + \tau_\nu T^\nu \partial_\mu \tilde{\Phi} \,, && (4.134)\\
t^A{}_A &= -2\hat{v}^\nu \tau_\mu T^\mu{}_\nu + \hat{e}^a_\mu e^\nu_a T^\mu{}_\nu + 2\tilde{\Phi}\tau_\mu T^\mu \,. && (4.135)
\end{aligned}
$$

If we are dealing with a relativistic and scale invariant theory, i.e. $\nabla_A t^A{}_B = t^A{}_A = 0$ we find the diffeomorphism, $U(1)$ and the $z = 2$ version of the local dilatation Ward identities as given in [55, 111, 112, 110].

The diffeomorphism Ward identity (4.134) can also be written in a TNC covariant form using the connection (4.67) as done in [111]. Instead of the connection (4.67) we can also take the Riemann–Cartan connection of [109], that we will denote by $\check{\Gamma}^\rho_{\mu\nu}$, given by

$$
\check{\Gamma}^\lambda_{\mu\rho} = -\hat{v}^\lambda \partial_\mu \tau_\rho + \frac{1}{2} h^{\nu\lambda}\left(\partial_\mu \hat{h}_{\rho\nu} + \partial_\rho \hat{h}_{\mu\nu} - \partial_\nu \hat{h}_{\mu\rho}\right) - h^{\nu\lambda}\tau_\rho K_{\mu\nu} \,, \qquad (4.136)
$$

where $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\hat{v}} \hat{h}_{\mu\nu}$ is the extrinsic curvature. This connection obeys

$$
\check{\nabla}_\mu \tau_\nu = 0 \,, \qquad \check{\nabla}_\mu \hat{h}_{\nu\rho} = 0 \,, \qquad \check{\nabla}_\mu \hat{v}^\nu = 0 \,, \qquad \check{\nabla}_\mu h^{\nu\rho} = 0 \,, \qquad (4.137)
$$

and the relation (4.134) becomes

$$
\nabla_A t^A{}_\nu = \check{\nabla}_\mu T^\mu{}_\nu + 2\check{\Gamma}^\rho_{[\mu\rho]} T^\mu{}_\nu - 2\check{\Gamma}^\mu_{[\nu\rho]} T^\rho{}_\mu + \tau_\mu T^\mu \partial_\nu \tilde{\Phi} \,. \qquad (4.138)
$$

This is the most compact and TNC covariant way of writing the diffeomorphism Ward identity.

In [109] it was shown that TTNC geometry (but not the more general TNC geometry) can be obtained by projecting the higher dimensional metric compatibility conditions involving the Levi-Civita connection onto the surface orthogonal to ∂_u (null reduction) in the sense that

[2] Sometimes it is useful to express t_{AB} in terms of lower-dimensional quantities via

$$
\begin{aligned}
t_{\mu\nu} &= \hat{h}_{\rho\nu} T^\rho{}_\mu + \hat{h}_{\rho\mu} T^\rho{}_\nu - \hat{h}_{\mu\rho}\hat{h}_{\nu\sigma} h^{\sigma\lambda} T^\rho{}_\lambda + \tau_\mu \tau_\nu \hat{v}^\rho \hat{v}^\sigma t_{\rho\sigma} \,, && (4.126)\\
t_{\mu u} &= \tau_\rho T^\rho{}_\mu \,, && (4.127)\\
t_{uu} &= \tau_\rho T^\rho \,. && (4.128)
\end{aligned}
$$

the TNC metric compatibility conditions follow from the projection of the higher dimensional metric compatibility conditions only when ∂_u is hypersurface orthogonal. However, the condition that the TNC metric compatibility conditions follow by projection is somewhat artificial. Here we see that the diffeomorphism Ward identity takes the required form for any field theory on a TNC geometry and not just TTNC geometry. At no point in the analysis did we assume anything about τ_μ.

4.2.4 Energy-momentum tensors for Non-relativistic particles

In this section we would like just to give an example of energy-momentum tensors, taking the simplest example: the non-relativistic particle. The motion of a non-relativistic particle of mass m is governed by the following action

$$S = \int d\lambda L = \frac{m}{2}\int d\lambda \frac{\hat{h}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho} \tag{4.139}$$

where the dots denotes derivative with respect to λ. This action has a reparametrization symmetry, i.e. $\{\delta\lambda = \xi\,,\, \delta x^\mu = \xi \dot{x}^\mu\}$. This can be used to fix a gauge and in particular we choose $\tau_\rho \dot{x}^\rho$. The equation of motion of the action (4.139) is

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{4.140}$$

where $\Gamma^\mu_{\nu\rho}$ is the affine connection defined in (4.52) once α is set to zero. By varying the action(4.139) with respect to the background NC fields we obtain for -$T^\mu{}_\nu$ the following result

$$T^\mu{}_\nu = -P_\nu \dot{x}^\mu \tag{4.141}$$

where P_μ is the generalized momentum defined by

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{m}{2}\tau_\mu h_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma + m h_{\mu\nu}\dot{x}^\nu - mM_\mu = p_\mu - mM_\mu \tag{4.142}$$

and the mass current is the following

$$T^\mu = -m\dot{x}^\mu \tag{4.143}$$

4.2.5 Null reduction of a perfect relativistic fluid

Since we are interested in non-relativistic versions of the fluid/gravity correspondence we study here the null reduction of a relativistic fluid[3].

A relativistic perfect fluid is given by a conserved energy-momentum tensor t_{AB} that is of the form

$$t_{AB} = (E+P)\, U_A U_B + P\gamma_{AB}\,, \tag{4.144}$$

where U_A satisfies $U_A U^A = -1$. Consider the following parametrization of U_A,

$$\begin{aligned} U_u^2 &= \frac{\rho}{E+P}\,, & (4.145)\\ h^{\mu\nu}U_\nu &= U_u\left(\hat{v}^\mu - u^\mu\right)\,, & (4.146)\\ \hat{v}^\mu U_\mu &= \frac{1}{2}U_u\left(\hat{h}_{\mu\nu}u^\mu u^\nu + 2\tilde{\Phi} + U_u^{-2}\right)\,, & (4.147) \end{aligned}$$

[3]This has also been done in [27] but our approach differs in that we do not need to introduce what is called a null fluid in [27].

where u^μ satisfies $\tau_\mu u^\mu = -1$. It follows that

$$U_\mu = -\frac{1}{2}U_u\tau_\mu\left(\hat{h}_{\rho\sigma}u^\rho u^\sigma + 2\tilde{\Phi} + U_u^{-2}\right) - U_u\hat{h}_{\mu\nu}u^\nu \quad (4.148)$$

$$= -U_u\left[\frac{1}{2}\tau_\mu\left(h_{\rho\sigma}u^\rho u^\sigma + U_u^{-2}\right) + h_{\mu\nu}u^\nu + m_\mu\right], \quad (4.149)$$

which (except for the U_u^{-2} term) takes the form of the velocity of a point particle. The components of U^μ are given by

$$U^u = -\frac{1}{2}U_u\left(\hat{h}_{\mu\nu}u^\mu u^\nu - 2\tilde{\Phi} + U_u^{-2}\right), \quad (4.150)$$

$$U^\mu = -U_u u^\mu. \quad (4.151)$$

Further redefine the energy density E as

$$E = 2\mathcal{E} + P. \quad (4.152)$$

Using the above results it follows that $T^\mu{}_\nu$ and T^μ are given by

$$T^\mu{}_\nu = \left(\mathcal{E} + P + \rho\tilde{\Phi} + \frac{1}{2}\rho\hat{h}_{\lambda\kappa}u^\lambda u^\kappa\right)u^\mu\tau_\nu + P\delta^\mu_\nu + \rho u^\mu\hat{h}_{\nu\rho}u^\rho$$

$$= \left(\mathcal{E} + P + \frac{1}{2}\rho h_{\lambda\kappa}u^\lambda u^\kappa\right)u^\mu\tau_\nu + P\delta^\mu_\nu + \rho u^\mu h_{\nu\rho}u^\rho + \rho u^\mu m_\nu, \quad (4.153)$$

$$T^\mu = -\rho u^\mu. \quad (4.154)$$

This can be shown to agree with the notion of a Galilean perfect fluid as given in [131].

The null reduction ansatz has a local $U(1)$ symmetry which is the diffeomorphism $\delta u = -\xi^u$ and $\delta m_\mu = -\partial_\mu\xi^u$. If we act with this diffeomorphism on U^A and U_A via

$$\delta U_A = \xi^B\partial_B U_A + U_B\partial_A\xi^B, \qquad \delta U^A = \xi^B\partial_B U^A - U^B\partial_B\xi^A, \quad (4.155)$$

with $\xi^A = \delta^A_u\xi^u$ we see that U_u and U^μ are $U(1)$ invariant. It follows from (4.151) that the fluid velocity u^μ is particle number invariant.

The $z = 2$ trace Ward identity reads (for $d = 2$ spatial dimensions)

$$t^A{}_A = -2\hat{v}^\nu\tau_\mu T^\mu{}_\nu + \hat{e}^a_\mu e^\nu_a T^\mu{}_\nu + 2\tilde{\Phi}\tau_\mu T^\mu = -2\mathcal{E} + 2P = 0. \quad (4.156)$$

The null reduction only leads to theories with $z = 2$ scaling relations.

4.2.6 Schrödinger field on NC curved backgroung from limiting procedure

The NC theory can be coupled to some matter content also in an other way, using the non-relativistic limit procedure. The matter couplings can be derived by taking some relativistic matter coupling and taking the non relativistic limit of it. This method was mainly developed in [32] .For example we can consider a complex Klein-Gordon vector field coupled to relativistic gravity

$$\mathcal{S}_{rel} = \int\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}D_\mu\Phi D_\nu\Phi^\star - \frac{M^2}{2}\Phi\Phi^\star\right) \quad (4.157)$$

where we defined the following covariant derivative with respect to a $U(1)$ gauge field M_μ

$$D_\mu\Phi \quad = \quad \partial_\mu\Phi - iMM_\mu\Phi \tag{4.158}$$

This is not a coupling to the electromagnetic field, but it is really a $U(1)$ curl free field. This lagrangian is invariant under a local $U(1)$ transformation which transforms the field in the following way

$$\delta\Phi = iM\Lambda\Phi \tag{4.159}$$

As always, we can associate to this $U(1)$ a conserved current which is given by

$$j^\mu = \frac{M}{2i}\left(\Phi^\star D^\mu\Phi - \Phi D^\mu\Phi^\star\right) \tag{4.160}$$

Using the redefinitions of field described before in (4.31), and rescaling the mass parameter as $M = \omega m$, taking the limit for $\omega \to \infty$ this lead to the well defined lagrangian

$$\mathcal{S}_{rel} = \int e\left(\frac{i}{2}v^\mu\left(\Phi^\star D_\mu\Phi - \Phi D_\mu\Phi^\star\right) - \frac{1}{2m}h^{\mu\nu}D_\mu\Phi D_\nu\Phi^\star\right) \tag{4.161}$$

where

$$D_\mu\Phi = \partial_\mu\Phi + imm_\mu\Phi \tag{4.162}$$

This is the lagrangian of a Schrödinger field coupled to an arbitrary TTNC background, and it has a symmetry under the following transformation

$$\delta\Phi = \xi^\mu\partial_\mu\Phi - im\sigma\Phi \tag{4.163}$$

Also in this case it is possible to find a conserved current associated to this symmetry, and it is the following

$$j^\mu = v^\mu\Phi\Phi^\star + \frac{1}{2mi}h^{\mu\nu}\left(\Phi^\star D_\nu\Phi - \phi D_\nu\Phi^\star\right) \tag{4.164}$$

This is often called number current because it counts the number of particle, and the number of particle is conserved. In [32] the coupling to more complicated matter is derived : fermions, bosons, massless and massive matter. There are also some details on spin-3/2 field.

4.3 Dynamical Newton Cartan geometry and Hořava-Lifshitz gravity

In this section we would like to briefly analyze the work done in [114] in the context of making the geometry described in 4.1.1 dynamical. The goal is to write down an action which encodes the dynamics of the geometry described by the fields τ_μ, e^a_μ and m^a only. For simplicity we restrict ourselves to 2 spatial dimensions and to values of z between 1 and 2. This is a restriction done for simplicity, but it is straightforward to consider higher dimensions. First of all, since we demand manifest G and J symmetry, the terms that will be present in the action should involve the invariants (4.48). Moreover the action should have dilatonic weight zero. The weights of the NC fields can be read off by the Schrödinger algebra (4.109) and are given in the following table

G and J invariants	τ_μ	$\hat{h}_{\mu\nu}$	$\hat{v}^\mu$	$h^{\mu\nu}$	e
dilatation weight	$-z$	-2	z	2	$-(z+2)$

The terms that are going to appear in the action, can be build essentialy in 3 ways

- The first way is employing the torsion tensor, which is the antisymmetric part of the affine connection of the NC geometry
- The second way is taking covariant derivatives of $\hat{h}_{\mu\nu}$ and of the torsion tensor itslef
- The third way is building scalars out of curvature tensor of G and J

We are not going to classify all the terms that could be taken into account of this 3 types, it is out of the scope of this brief review. This has been done in [114] and, taking into account all the information given in this section, the action they obtain is the following

$$S = \int dtd^2xe\left[C\left(h^{\mu\rho}h^{\nu\sigma}K_{\mu\nu}K_{\rho\sigma} - \lambda(h^{\mu\nu}K_{\mu\nu})^2\right) - \mathcal{V}\right] \tag{4.165}$$

where the potential is

$$\begin{aligned} -\mathcal{V} &= 2\Lambda + c_1 h^{\mu\nu}a_\mu a_\nu + c_2\mathcal{R} + \delta_{z,2}\left[c_{10}(h^{\mu\nu}a_\mu a_\nu)^2 + c_{11}h^{\mu\rho}a_\mu a_\rho \nabla_\nu(h^{\nu\sigma}a_\sigma)\right. \\ &+ \left. c_{12}\nabla_\nu(h^{\mu\rho}a_\rho)\nabla_\mu(h^{\nu\sigma}a_\sigma) + c_{13}\mathcal{R}^2 + c_{14}\mathcal{R}\nabla_\mu(h^{\mu\nu}a_\nu) + c_{15}\mathcal{R}h^{\mu\nu}a_\mu a_\nu\right] \end{aligned} \tag{4.166}$$

and $K_{\mu\nu} = -\hat{h}_{\nu\rho}\nabla_\mu\hat{v}^\rho$, a_μ is the Lie derivative of τ_μ along $\hat{v}^\mu$ (and it was previously defined in (4.58)). What it is nice in this action is that the field m_μ is not explicitly present, but it is hidden into $\hat{v}^\mu$ and $\hat{h}_{\mu\nu}$. The explicit field content of the theory is $\{\tau_\mu, \hat{v}^\mu, \hat{h}_{\mu\nu}, h^{\mu\nu}\}$. This means that we could effectively treat $\hat{v}^\mu$ and $\hat{h}_{\mu\nu}$ as new fields and "forget" about m_μ.

As we said at the beginning of this section, this is the action which encodes the dynamics of the geometry spanned by the field τ_μ, e^a_μ and m^a only. In the work done in [114] there is a generalization. in which also the extra $U(1)$ symmetry is considered and the fields $\tilde{\Phi}$ and χ are turned on. We are not going to write details about this.

The result can be generalized to higher dimensions and in the same paper they also developed the dynamics of the geometry generated by the Schrödinger group, which should be the analogue of conformal gravity but for non-relativistic geometries.

Moreover they showed that the action (4.165) is just the action which has been considered when people studied Horaw̌a-Lifshitz gravity in 3D.

4.3.1 Horav̌a-Lifshitz cosmology

In the paper [114], which has been briefly reviewed in the last section, it was shown that the dynamics of TTNC (Twistless Torsional Newton Cartan) geometry is given by non-projectable Horav̌a-Lifshitz gravity and that the projectable case corresponds to the dynamics of NC geometries without torsion. It is worth to point out that, in the absence of a cosmological constant, the vacuum of this dynamical geometry is not Minkowski space-time, but is is flat NC space-time which has different symmetries than Minkowski space-time, as it is explained in [110]. Therefore it could be interesting to study the propagation of degrees of freedom and compare them with some known results in the studies on Horav̌a-Lifshitz cosmology. In this light is then important also to find the equations of motion for the theory and possible vacuum solutions, such as black holes or maximally symmetric spaces. This work could be important also because one can ideally use this dynamical geometry as a bulk in the holographic correspondence. Moreover, if there are some black holes solution of the theory (and they should be since black hole solutions to HL gravity possibly exist), they can be used some version of the fluid/gravity correspondence. I will report in the following some preliminary results of this work, which may become a paper in the future.

The first step in deriving the equation of motion is to choose a linear independent set of variation for NC objects. In fact, not all the NC datas are linear independent, since they obey the following orthogonality conditions

$$\hat{v}^\mu \hat{h}_{\mu\nu} = 0 \qquad \tau_\nu h^{\nu\mu} = 0 \qquad \tau_\mu \hat{v}^\mu = -1 \tag{4.167}$$

therefore also the variations $\delta\hat{v}^\mu$, $\delta\tau_\mu$, $\delta h^{\mu\nu}$, δm_μ or $\delta\tilde{\phi}$ are not independent from each other. One can choose as linear independent variations the set $\{\delta\hat{v}^\mu,\, \delta h^{\mu\nu},\, \delta\tilde{\phi}\}$, where the last variation will come into play only when the conformal symmetry is manifest, for obvious reasons. The other variations are dependent from those in a way such that (4.167) is fullfilled, i.e.

$$\delta\tau_\mu = \tau_\mu\tau_\nu\delta\hat{v}^\nu - \hat{h}_{\mu\rho}\tau_\nu\delta h^{\rho\nu} \tag{4.168}$$

$$\delta\hat{h}_{\mu\nu} = (\tau_\mu\hat{h}_{\nu\rho} + \tau_\nu\hat{h}_{\mu\rho})\delta\hat{v}^\rho - \hat{h}_{\mu\rho}\hat{h}_{\nu\sigma}\delta h^{\rho\sigma} \tag{4.169}$$

The second step is to perform the variation of the action (4.165). The first set of the equation of motion, found by varying it with respect to $\hat{v}^\mu$, for dynamical NC geometry (with torsion) is the following

$$h^{\mu\nu}\left[2C(\nabla_\mu - a_\mu)p_{\alpha\nu} - 2c_1\tau_\alpha\nabla_\nu a_\mu + 2c_2\tau_\alpha(\nabla_\mu - a_\mu)a_\nu\right] = 0 \tag{4.170}$$

This is often called in the Horavǎ-Lifshitz literature, the momentum constraint. A remarkable fact is that, ince the torsion is set to zero (i.e. $a_\mu = 0$) this equation is just the "covariant" conservation of the tensor $p_{\mu\nu}$. Having the expression of the momentum constraint, one can also try to look at linear perturbations around the flat NC background without matter, and count the number of degrees of freedom. Considering the most general linear perturbation around a flat NC background without matter, it can be decomposed in tensors as follows

$$\tau_\mu = \delta^t_\mu + t_\mu + \partial_\mu T \tag{4.171}$$

$$\hat{v}^\mu = -\delta^\mu_t + \omega^\mu + \partial^\mu B \tag{4.172}$$

$$\hat{h}_{\mu\nu} = (1+\epsilon)\delta^a_\mu\delta^b_\nu\delta_{ab} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu + \epsilon_{\mu\nu} + \partial_\mu\partial_\nu\epsilon_L \tag{4.173}$$

$$h^{\mu\nu} = (1+\tilde{\epsilon})\delta^\mu_a\delta^\nu_b\delta^{ab} + \partial^\mu\tilde{\epsilon}^\nu + \partial^\nu\tilde{\epsilon}^\mu + \tilde{\epsilon}^{\mu\nu} + 2\partial^\mu\partial^\nu\tilde{\epsilon}_L \tag{4.174}$$

where t_μ, ω^μ, ϵ_μ and $\tilde{\epsilon}^\mu$ are transverse and $\epsilon_{\mu\nu}$ and $\tilde{\epsilon}^{\mu\nu}$ are traverse traceless. Diffeomorphism invariance can be used to kill some gauge residual symmetry. In particular one can choose $\xi^t = -T$ and $\xi_a = -\epsilon_a - \partial_a\epsilon_L$. Moreover, recalling that the NC sources obey orthogonality conditions (4.167), the perturbations are further constrained. Applying orthogonality conditions and killing the residual gauge symmetry one is left with the following set of perturbations

$$\delta\hat{h}_{\mu\nu} = (\delta^a_\mu\delta^t_\nu + \delta^t_\mu\delta^a_\nu)(\omega_a + \partial_a\Omega) + \delta^a_\mu\delta^b_\nu(2\epsilon\delta_{ab} + \epsilon_{ab}) \tag{4.175}$$

$$\delta h^{\mu\nu} = -(\delta^\mu_t\delta^\nu_a + \delta^\mu_a\delta^\nu_t)t^a + \delta^\mu_a\delta^\nu_b(\tilde{\epsilon}^{ab} - 2\epsilon\delta^{ab}) \tag{4.176}$$

$$\delta\tau_\mu = \delta^t_\mu t_t + \delta^a_\mu t_a \tag{4.177}$$

$$\delta\hat{v}^\mu = \delta^\mu_t t_t + \delta^{\mu a}(\omega_a + \partial_a\Omega) \tag{4.178}$$

performing the redefinition $t_t = \chi$. Transversality on t_μ, ω^μ, ϵ_μ and $\epsilon_{\mu\nu}$ translates on t_a, ω_a, ϵ_{ab}, χ and B in the following way

$$\partial_a t^a = 0\ ,\quad \partial_a \omega^a = 0\ ,\quad \dot{\chi} = 0 \tag{4.179}$$
$$\partial_a \epsilon^{ab} = -\dot{\omega}^b\ ,\quad \partial_a \tilde{\epsilon}^{ab} = \dot{t}^b \tag{4.180}$$

Before expanding the action one should solve the momentum constraint first. In this gauge, the momentum constraint without matter is

$$C\partial_a\left[(d\lambda - 1)\dot{\epsilon} - 2(\lambda - 1)\Box B\right] + C(\partial_t^2 + \Box)\omega_a = 0 \tag{4.181}$$

where d is the space dimension. One can formally solve this equation and get the following

$$B = \frac{d\lambda - 1}{\lambda - 1}\frac{\dot{\epsilon}}{\Box}\ ,\quad \omega_a = 0 \tag{4.182}$$

which is the same result of the review [147] but for t_a that in the review is zero (projetable theories) but in general it could not zero. The difference with projectable theories is then that one has a nonzero t_a/ω_a and a nonzero t_t. Now one can expand the action (4.165) and apply the momentum constraint, getting the following result

$$\begin{aligned} S &= \int d^d x dt \left[\frac{1}{4}\epsilon_{ab}\left(-C\partial_t^2 - c_2\Box\right)\epsilon^{ab} + (d-1)\epsilon\left(-C\frac{(d\lambda - 1)}{\lambda - 1}\partial_t^2 + c_2 d\Box\right)\epsilon\right. \\ &- \left.\frac{C}{2}\omega^a\left(\frac{c_1^2}{c_2^2}\partial_t^2 + \frac{1}{2}\Box\right)\omega_a - t_t(c_1 - c_2)\Box t_t\right] \end{aligned} \tag{4.183}$$

In this preliminary work, also the Einstein tensor has been computed and it has been shown that it obeys the corresponding Bianchi identities.

4.4 Holographic renormalization of the upliftable model

The EPD model with

$$Z = e^{3\Phi}\,,\qquad W = 4\,,\qquad V(\Phi) = 2e^{-3\Phi} - 12e^{-\Phi}\,,\qquad x = 3\,. \tag{4.184}$$

can be obtained from a Scherk–Schwarz reduction of the 5-dimensional action

$$S = \frac{1}{2\kappa_5^2}\int d^5 x\sqrt{-\mathcal{G}}\left(R + 12 - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial^{\mathcal{M}}\psi\right)\,, \tag{4.185}$$

where $\kappa_5^2 = 8\pi G_5$ with G_5 the 5-dimensional Newton's constant and where $\mathcal{M} = (u, M)$. The consistency of this reduction will be shown in section 4.4.5.

In this section we will first perform the holographic renormalization in 5 dimensions for those asymptotically locally AdS space-times that have a boundary metric obeying the null reduction ansatz of section 4.1.4. We then subsequently reduce the result to obtain the counterterms and near boundary expansions in 4 dimensions for asymptotically locally $z = 2$ Lifshitz space-times.

4.4.1 Fefferman–Graham expansions and counterterms

By using the results of [108, 150, 54, 55][4] we can obtain the solution to the equations of motion of (4.185) (that are given further below in (4.201) and (4.202)) expressed as an asymptotic series in radial gauge, i.e. as a Fefferman–Graham (FG) expansion [89]. The result reads[5]

$$\begin{aligned}
\mathcal{G}_{\mathcal{M}\mathcal{N}}dx^{\mathcal{M}}dx^{\mathcal{N}} &= \frac{dr^2}{r^2}+\gamma_{AB}dx^A dx^B\,, && (4.186)\\
\gamma_{AB} &= \frac{1}{r^2}\left[\gamma_{(0)AB}+r^2\gamma_{(2)AB}+r^4\log r\gamma_{(4,1)AB}+r^4\gamma_{(4)AB}+O(r^6\log r)\right], && (4.187)\\
\psi &= \psi_{(0)}+r^2\psi_{(2)}+r^4\log r\psi_{(4,1)}+r^4\psi_{(4)}+O(r^6\log r)\,, && (4.188)
\end{aligned}$$

where the coefficients are given by

$$\begin{aligned}
\gamma_{(2)AB} &= -\frac{1}{2}\left(R_{(0)AB}-\frac{1}{2}\partial_A\psi_{(0)}\partial_B\psi_{(0)}\right)+\frac{1}{12}\gamma_{(0)AB}\left(R_{(0)}-\frac{1}{2}(\partial\psi_{(0)})^2\right), && (4.189)\\
\psi_{(2)} &= \frac{1}{4}\Box_{(0)}\psi_{(0)}\,, && (4.190)
\end{aligned}$$

at second order and by

$$\begin{aligned}
\gamma_{(4,1)AB} &= \frac{1}{4}\nabla^C_{(0)}\left(\nabla_{(0)A}\gamma_{(2)BC}+\nabla_{(0)B}\gamma_{(2)AC}-\nabla_{(0)C}\gamma_{(2)AB}\right)-\frac{1}{4}\nabla_{(0)A}\nabla_{(0)B}\gamma^C_{(2)C}\\
&\quad+\gamma_{(2)AC}\gamma^C_{(2)B}-\frac{1}{2}\partial_{(A}\psi_{(0)}\nabla_{(0)B)}\psi_{(2)}-\gamma_{(0)AB}\left(\frac{1}{4}\gamma^{CD}_{(2)}\gamma_{(2)CD}+\frac{1}{2}\psi^2_{(2)}\right), && (4.191)\\
\psi_{(4,1)} &= -\frac{1}{4}\left[\Box_{(0)}\psi_{(2)}+2\psi_{(2)}\gamma^A_{(2)A}+\frac{1}{2}\partial^A\psi_{(0)}\nabla_{(0)A}\gamma^B_{(2)B}-\gamma^{AB}_{(2)}\nabla_{(0)A}\partial_B\psi_{(0)}\right.\\
&\quad\left.-\partial^A\psi_{(0)}\nabla^B_{(0)}\gamma_{(2)AB}\right], && (4.192)
\end{aligned}$$

at order $r^4\log r$. We note that the quantity $\gamma_{(4,1)AB}$ is traceless. Indices of the expansion coefficients are raised and lowered with the AdS boundary metric $\gamma_{(0)AB}$. At order r^4 we have that $\gamma_{(4)AB}$ is constrained by

$$\begin{aligned}
\gamma^A_{(4)A} &= \frac{1}{4}\gamma_{(2)AB}\gamma^{AB}_{(2)}-\frac{1}{2}\psi^2_{(2)}\,, && (4.193)\\
\nabla^B_{(0)}\gamma_{(4)AB} &= \psi_{(4)}\partial_A\psi_{(0)}-\frac{1}{2}\psi_{(2)}\nabla_{(0)A}\psi_{(2)}-\frac{1}{4}\gamma^{BC}_{(2)}\nabla_{(0)A}\gamma_{(2)BC}\\
&\quad-\frac{1}{4}\gamma_{(2)AC}\nabla^C_{(0)}\gamma^B_{(2)B}+\frac{1}{2}\gamma^{BC}_{(2)}\nabla_{(0)B}\gamma_{(2)AC}+\frac{1}{2}\gamma^C_{(2)A}\nabla^B_{(0)}\gamma_{(2)BC}\,. && (4.194)
\end{aligned}$$

Following [108] we write the coefficient $\gamma_{(4)AB}$ as

$$\gamma_{(4)AB}=X_{AB}-\frac{1}{4}t_{AB}\,, \tag{4.195}$$

[4]We set $\hat{\chi}=0$ and redefine $\hat{\phi}=\psi$ in [55].

[5]We will denote here and further below by $a_{(n,m)}$ the coefficient at order $r^n(\log r)^m$ of the field $r^\Delta a$ where $r^{-\Delta}$ is the leading term in the expansion of a with the exception of the $a_{(n,0)}$ term which we will simply denote as $a_{(n)}$.

where t_{AB} is the boundary energy-momentum tensor defined in (4.207). The trace and divergence of t_{AB} will be given below together with the explicit form of X_{AB}. In the expansion for the scalar we have that $\psi_{(4)}$ is a fully arbitrary function of the boundary coordinates.

The complete action with Gibbons–Hawking and local counterterms (using minimal subtraction) is given by

$$S_{\text{ren}} = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} d^5x\sqrt{-\mathcal{G}}\left(R + 12 - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial^{\mathcal{M}}\psi\right) + \frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}K + S_{\text{ct}}\,, \quad (4.196)$$

where γ denotes the determinant of the metric γ_{AB} on the cut off boundary $\partial\mathcal{M}$, the extrinsic curvature K is given by

$$K = \gamma^{AB}K_{AB}\,, \qquad K_{AB} = -\frac{1}{2}\mathcal{L}_n\gamma_{AB}\,, \qquad n^{\mathcal{M}} = -r\delta_r^{\mathcal{M}}\,, \quad (4.197)$$

and where

$$S_{\text{ct}} = \frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\left(-\frac{1}{4}\left(R_{(\gamma)} + 12 - \frac{1}{2}\partial_A\psi\partial^A\psi\right) - \frac{1}{2}\mathcal{A}\log r\right)\,, \quad (4.198)$$

with

$$\begin{aligned} \mathcal{A} &= -\frac{1}{4}\left(Q^{AB}Q_{AB} - \frac{1}{3}Q^2 + \frac{1}{2}\left(\Box_{(\gamma)}\psi\right)^2\right)\,, \\ Q_{AB} &= R_{(\gamma)AB} - \frac{1}{2}\partial_A\psi\partial_B\psi\,. \end{aligned} \quad (4.199)$$

4.4.2 One-point functions

To compute one-point functions, we write the total variation of $S_{\text{ren}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{ct}}$ as

$$\begin{aligned} \delta S_{\text{ren}} &= \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} d^5x\sqrt{-\mathcal{G}}\left(\mathcal{E}_{\mathcal{MN}}\delta\mathcal{G}^{\mathcal{MN}} + \mathcal{E}_\psi\delta\psi\right) \\ &\quad + \frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\left(\frac{1}{2}T_{AB}\delta\gamma^{AB} + T_\psi\delta\psi\right)\,, \end{aligned} \quad (4.200)$$

where $\mathcal{E}_{\mathcal{MN}}$ and $\mathcal{E}_\psi$ are the equations of motion

$$\mathcal{E}_{\mathcal{MN}} = G_{\mathcal{MN}} - 6\mathcal{G}_{\mathcal{MN}} - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial_{\mathcal{N}}\psi + \frac{1}{4}\mathcal{G}_{\mathcal{MN}}(\partial\psi)^2\,, \quad (4.201)$$

$$\mathcal{E}_\psi = \Box\psi\,, \quad (4.202)$$

and where

$$T_{AB} = -2(K-3)\gamma_{AB} + 2K_{AB} - Q_{AB} + \frac{1}{2}h_{AB}Q + \log r\,T_{AB}^{(A)}\,, \quad (4.203)$$

$$T_\psi = -n^M\partial_M\psi - \frac{1}{2}\Box_{(\gamma)}\psi + \log r\,T_\psi^{(A)}\,. \quad (4.204)$$

Here we defined

$$T_{AB}^{(A)} = -\frac{2\kappa_5^2}{\sqrt{-\gamma}}\frac{\delta A}{\delta\gamma^{AB}}\,, \qquad T_\psi^{(A)} = -\frac{\kappa_5^2}{\sqrt{-\gamma}}\frac{\delta A}{\delta\psi}\,, \quad (4.205)$$

with

$$A=\frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}}d^4x\sqrt{-\gamma}\mathcal{A}\,. \tag{4.206}$$

From the expansions it follows that $\sqrt{-\gamma}=r^{-4}\sqrt{-\gamma_{(0)}}+O(r^{-2})$, $\delta\gamma^{AB}=r^2\delta\gamma^{AB}_{(0)}+O(r^4)$, $\delta\psi=\delta\psi_{(0)}+O(r^2)$, which is used to obtain the following one-point functions (we take the cut-off boundary at $r=\epsilon$)

$$\begin{aligned} t_{AB} &= \frac{4\kappa_5^2}{\sqrt{-\gamma_{(0)}}}\frac{\delta S^{\text{on-shell}}_{\text{ren}}}{\delta\gamma^{AB}_{(0)}}=\lim_{\epsilon\to0}\epsilon^{-2}T_{AB}=-4\gamma_{(4)AB}+4X_{AB}\,, && (4.207)\\ \langle\mathcal{O}_\psi\rangle &= \frac{2\kappa_5^2}{\sqrt{-\gamma_{(0)}}}\frac{\delta S^{\text{on-shell}}_{\text{ren}}}{\delta\psi_{(0)}}=\lim_{\epsilon\to0}\epsilon^{-4}T_\psi=4\psi_{(4)}+\psi_{(2)}\gamma^A_{(2)A}+3\psi_{(4,1)}\,, && (4.208)\end{aligned}$$

where

$$X_{AB}=\frac{1}{2}\gamma_{(2)AC}\gamma^C_{(2)B}-\frac{1}{4}\gamma^C_{(2)C}\gamma_{(2)AB}+\frac{1}{8}\gamma_{(0)AB}\mathcal{A}_{(0)}-\frac{3}{4}\gamma_{(4,1)AB}\,, \tag{4.209}$$

with

$$\mathcal{A}_{(0)}=\lim_{\epsilon\to0}\epsilon^{-4}\mathcal{A}=(\gamma^A_{(2)A})^2-\gamma^{AB}_{(2)}\gamma_{(2)AB}-2\psi^2_{(2)}\,. \tag{4.210}$$

Using equations (4.193) and (4.194) we can compute the trace and divergence of the boundary energy-momentum tensor and the result is

$$\begin{aligned} t^A{}_A &= \mathcal{A}_{(0)}\,, && (4.211)\\ \nabla_{(0)A}t^A{}_B &= -\langle\mathcal{O}_\psi\rangle\partial_B\psi_{(0)}\,. && (4.212)\end{aligned}$$

4.4.3 Dimensional Reduction of the action

The Scherk–Schwarz reduction leading to (5.3) with the choices (5.17) is obtained by the following reduction ansatz

$$\begin{aligned} ds_5^2 &= \mathcal{G}_{\mathcal{M}\mathcal{N}}dx^{\mathcal{M}}dx^{\mathcal{N}}=\frac{dr^2}{r^2}+\gamma_{AB}dx^Adx^B=e^{-\Phi}g_{MN}dx^Mdx^N+e^{2\Phi}\left(du+A_Mdx^M\right)^2\\ &= e^{-\Phi}\left(e^{\Phi}\frac{dr^2}{r^2}+h_{\mu\nu}dx^\mu dx^\nu\right)+e^{2\Phi}\left(du+A_\mu dx^\mu\right)^2\,, && (4.213)\\ \psi &= 2u+2\Xi\,, && (4.214)\end{aligned}$$

where the all functions are independent of the fifth coordinate u which is periodically identified $u\sim u+2\pi L$. The only exception is the term linear in ψ which means that upon going around the reduction circle ψ comes back to itself up to a constant shift. This is allowed because shifting ψ is a global symmetry of the higher dimensional theory. The consistency of the reduction is proven in section 4.4.5.

After reduction the four dimensional action is

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}\left(R-\frac{3}{2}\partial_M\Phi\partial^M\Phi-\frac{1}{4}e^{3\Phi}F_{MN}F^{MN}-2B_MB^M-V\right)\\
&\quad +2\int d^3x\sqrt{-h}K+S_{\rm ct}\,, \qquad (4.215)\\
S_{\rm ct} &= 2\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left[-\frac{1}{4}e^{\Phi/2}\left(R_{(h)}-\frac{3}{2}\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{4}e^{3\Phi}F_{\mu\nu}F^{\mu\nu}-2B_\mu B^\mu+10e^{-\Phi}\right)\right]\\
&\quad -\log r\int_{\partial\mathcal{M}} d^3x\sqrt{-h}e^{-\Phi/2}\mathcal{A}\,, \qquad (4.216)
\end{aligned}
$$

where

$$
\begin{aligned}
B_M &= A_M-\partial_M\Xi\,, \qquad (4.217)\\
F_{MN} &= \partial_MB_N-\partial_NB_M\,, \qquad (4.218)\\
V &= 2e^{-3\Phi}-12e^{-\Phi}\,, \qquad (4.219)
\end{aligned}
$$

and where we used that $\frac{2\pi L}{2\kappa_5^2}=1$.

The total variation can be written as

$$
\begin{aligned}
\delta S_{ren} &= \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\mathcal{E}_{MN}\delta g^{MN}+\mathcal{E}^N\delta B_N+\mathcal{E}_\Phi\delta\Phi\right)\\
&\quad +\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu}+\mathcal{T}^\nu\delta B_\nu+T_\Phi\delta\Phi\right)\,, \qquad (4.220)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{E}_{MN} &= G_{MN}+\frac{1}{8}e^{3\Phi}g_{MN}F_{PQ}F^{PQ}-\frac{1}{2}e^{3\Phi}F_{MP}F_N{}^P+g_{MN}B_PB^P-2B_MB_N\\
&\quad +\frac{3}{4}g_{MN}\partial_P\Phi\partial^P\Phi-\frac{3}{2}\partial_M\Phi\partial_N\Phi+\frac{1}{2}g_{MN}V\,, \qquad (4.221)\\
\mathcal{E}_\Phi &= 3\Box\Phi-\frac{3}{4}e^{3\Phi}F_{MN}F^{MN}+6e^{-3\Phi}-12e^{-\Phi}\,, \qquad (4.222)\\
\mathcal{E}^N &= \nabla_M\left(e^{3\Phi}F^{MN}\right)-4B^N\,, \qquad (4.223)
\end{aligned}
$$

and

$$
\begin{aligned}
T_{\mu\nu} &= -2Kh_{\mu\nu}+2K_{\mu\nu}-e^{\Phi/2}G_{(h)\mu\nu}+5e^{-\Phi/2}h_{\mu\nu}\\
&\quad +\frac{1}{2}e^{7\Phi/2}F_{\mu\rho}F_\nu{}^\rho-\frac{1}{8}e^{7\Phi/2}h_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}-e^{\Phi/2}h_{\mu\nu}B_\rho B^\rho+2e^{\Phi/2}B_\mu B_\nu\\
&\quad +\frac{1}{2}e^{\Phi/2}\left(\nabla^{(h)}_\mu\partial_\nu\Phi-h_{\mu\nu}\Box_{(h)}\Phi\right)+\frac{7}{4}e^{\Phi/2}\partial_\mu\Phi\partial_\nu\Phi-e^{\Phi/2}h_{\mu\nu}\partial_\rho\Phi\partial^\rho\Phi\,, \qquad (4.224)\\
T_\Phi &= -3n^M\partial_M\Phi-\frac{1}{4}e^{\Phi/2}R_{(h)}-\frac{3}{8}e^{\Phi/2}\partial_\mu\Phi\partial^\mu\Phi-\frac{3}{2}e^{\Phi/2}\Box_{(h)}\Phi\\
&\quad +\frac{7}{16}e^{7\Phi/2}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}e^{\Phi/2}B_\mu B^\mu+\frac{5}{2}e^{-\Phi/2}\,, \qquad (4.225)\\
\mathcal{T}^\nu &= -e^{3\Phi}n_MF^{M\nu}-\frac{1}{2}\nabla^{(h)}_\mu\left(e^{7\Phi/2}F^{\mu\nu}\right)+2e^{\Phi/2}B^\nu\,, \qquad (4.226)
\end{aligned}
$$

where the extrinsic curvature is

$$K = h^{\mu\nu} K_{\mu\nu}\,, \qquad K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n h_{\mu\nu}\,, \qquad n^M = -re^{-\Phi/2}\delta_r^M\,. \tag{4.227}$$

These expressions are correct up to $\log r$ terms since we did not vary those counterterms.

4.4.4 Sources and Vevs

We write the 4-dimensional metric in (4.213) as

$$ds^2 = e^\Phi \frac{dr^2}{r^2} + h_{\mu\nu}dx^\mu dx^\nu = e^\Phi \frac{dr^2}{r^2} - E^0E^0 + \delta_{ab}E^aE^b\,. \tag{4.228}$$

In order to compute the vevs we use the identity [55, 111]

$$\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} + \mathcal{T}^\nu\delta B_\nu + T_\Phi\delta\Phi = \mathcal{S}^0_\mu\delta E^\mu_0 + \mathcal{S}^a_\mu\delta E^\mu_a + \mathcal{T}_\varphi\delta\varphi + \mathcal{T}^a\delta A_a + \mathcal{T}_\Xi\delta\Xi + \mathcal{T}_\Phi\delta\Phi\,, \tag{4.229}$$

which holds up to a total derivative, where we used that $B_\nu = A_\nu - \partial_\nu\Xi$, $A_a = E^\mu_a A_\mu$ and where φ is defined by [111]

$$\varphi = E^\nu_0 A_\nu - \alpha(\Phi)\,, \tag{4.230}$$

with $\alpha = e^{-3\Phi/2}$ for the particular model studied here [55] and where

$$\begin{aligned}
\mathcal{S}^0_\mu &= -\left(T_{\mu\nu}E^\nu_0 + \mathcal{T}^\rho E^0_\rho A_\mu\right)\,, && (4.231)\\
\mathcal{S}^a_\mu &= \left(T_{\mu\nu}E^{\nu a} - \mathcal{T}^\rho E^a_\rho A_\mu\right)\,, && (4.232)\\
\mathcal{T}_\varphi &= \mathcal{T}^\nu E^0_\nu\,, && (4.233)\\
\mathcal{T}_\Phi &= T_\Phi + \mathcal{T}^\nu E^0_\nu \frac{d\alpha}{d\Phi}\,, && (4.234)\\
\mathcal{T}^a &= \mathcal{T}^\nu E^a_\nu\,, && (4.235)\\
\mathcal{T}_\Xi &= e^{-1}\partial_\mu\left(e\mathcal{T}^\mu\right)\,. && (4.236)
\end{aligned}$$

The 4-dimensional sources are defined as the leading terms in the expansions of the bulk fields appearing on the right hand side of (4.229). We find the sources $v^\mu, e^\mu_a, m_\mu, \phi, \chi$ defined via

$$\begin{aligned}
E^\mu_0 &\simeq -r^2\alpha_{(0)}^{-1/3}v^\mu\,, && (4.237)\\
E^\mu_a &\simeq r\alpha_{(0)}^{1/3}e^\mu_a\,, && (4.238)\\
A_\mu - \alpha(\Phi)E^0_\mu &\simeq -m_\mu\,, && (4.239)\\
\Phi &\simeq \phi\,, && (4.240)\\
\Xi &\simeq -\chi\,, && (4.241)\\
\varphi &\simeq r^2\alpha_{(0)}^{-1/3}v^\mu m_\mu\,, && (4.242)\\
A_a &\simeq -r\alpha_{(0)}^{1/3}e^\mu_a m_\mu\,. && (4.243)
\end{aligned}$$

Likewise the vevs are defined as the leading terms in the expansions of the objects that are the responses to the variations written in (4.229), i.e. we define the vevs $S^0_\mu, S^a_\mu, T^0, T^a, \langle O_\phi\rangle, \langle O_\chi\rangle$

$$
\begin{aligned}
\mathcal{S}^0_\mu &\simeq r^2\alpha_{(0)}^{2/3}S^0_\mu\,, &(4.244)\\
\mathcal{S}^a_\mu &\simeq r^3 S^a_\mu\,, &(4.245)\\
\mathcal{T}_\varphi &\simeq -r^2\alpha_{(0)}^{2/3}T^0\,, &(4.246)\\
\mathcal{T}^a &\simeq -r^3 T^a\,, &(4.247)\\
\mathcal{T}_\Phi &\simeq r^4\alpha_{(0)}^{1/3}\langle O_\phi\rangle\,, &(4.248)\\
\mathcal{T}_\Xi &\simeq -r^4\alpha_{(0)}^{1/3}\langle O_\chi\rangle\,, &(4.249)
\end{aligned}
$$

where

$$\alpha_{(0)} = e^{-3\phi/2}\,. \tag{4.250}$$

Using (4.228), (4.213) as well as the definitions of the 4-dimensional sources (4.237)–(4.239) we can derive the following relation between the 5-dimensional boundary metric $\gamma_{(0)AB}$ and the 4-dimensionsal sources τ_μ, m_μ and e^a_μ,

$$ds^2 = \gamma_{(0)AB}dx^A dx^B = 2\tau_\mu dx^\mu\left(du - m_\nu dx^\nu\right) + h_{\mu\nu}dx^\mu dx^\nu\,, \tag{4.251}$$

where $h_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu$ which is the form of a null reduction ansatz for a reduction along u as discussed in section 4.1.4. The fact that the boundary metric of the 5-dimensional asymptotically locally AdS space-time must have a null circle means that the source ϕ which appears in the expansion of Φ is not independent of the other sources. This can be seen by noting that $\gamma_{uu} = e^{2\Phi}$, so that the 5-dimensional FG expansion via (4.187) and (4.189) tells us that

$$e^{2\phi} = \gamma_{(2)uu} = -\frac{1}{2}R_{(0)uu} + 1 = -\frac{1}{4}\left(\epsilon^{\mu\nu\rho}\tau_\mu\partial_\nu\tau_\rho\right)^2 + 1\,, \tag{4.252}$$

where the epsilon tensor is given by $\epsilon^{\mu\nu\rho} = e^{-1}\varepsilon^{\mu\nu\rho}$ where e is the determinant of the TNC vielbein matrix $(\tau_\mu\,, e^a_\mu)$ and $\varepsilon^{\mu\nu\rho}$ is the Levi-Civita symbol. For more details we refer to [55]. The consequence of this is that the variation of the on-shell with respect to ϕ gives zero since nothing depends on ϕ.

We now relate the 5-dimensional vevs to the 4-dimensional vevs. For all solutions obeying the reduction ansatz the variation of the on-shell action can be written in both a 5-dimensional and a 4-dimensional notation. From a 5-dimensional perspective we have

$$
\begin{aligned}
\delta S^{\text{on-shell}}_{\text{ren}} &= \lim_{\epsilon\to 0}\frac{1}{2\kappa_5^2}\int_{r=\epsilon} d^4x\sqrt{-\gamma}\left(\frac{1}{2}T_{AB}\delta\gamma^{AB} + T_\psi\delta\psi\right)\\
&= \int_{\partial\mathcal{M}} d^3x e\left(\frac{1}{2}t_{AB}\delta\gamma^{AB}_{(0)} + \langle O_\psi\rangle\delta\psi_{(0)}\right)\,, \qquad (4.253)
\end{aligned}
$$

where we used the fact that $\sqrt{-\gamma_{(0)}} = e = \det\left(\tau_\mu, e^a_\mu\right)$ as follows from (4.251) and the fact that nothing depends on u so that we can perform the u integral. At the same time from a 4-dimensional perspective we also have, using (4.220), (4.229),

$$
\begin{aligned}
\delta S^{\text{on-shell}}_{\text{ren}} &= \lim_{\epsilon\to 0}\int_{r=\epsilon} d^3x\sqrt{-h}\left(\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} + \mathcal{T}^\nu\delta B_\nu + T_\Phi\delta\Phi\right) \qquad (4.254)\\
&= \int_{\partial\mathcal{M}} d^3x e\left(-S^0_\mu\delta v^\mu + S^a_\mu\delta e^\mu_a + T^0\delta m_0 + T^a\delta m_a + \langle O_\chi\rangle\delta\chi + \langle\tilde{O}_\phi\rangle\delta\phi\right)\,,
\end{aligned}
$$

with $m_0=-v^\mu m_\mu$, $m_a=e^\mu_a m_\mu$ and where we used

$$\psi_{(0)}=2u-2\chi\,,\qquad \langle O_\psi\rangle=-\tfrac{1}{2}\langle O_\chi\rangle\,,\tag{4.255}$$

so that $\delta\psi_{(0)}=-2\delta\chi$ and where furthermore $\tilde{O}_\phi$ is given by

$$\tilde{O}_\phi=O_\phi-\frac{1}{2}\left[v^\mu\left(S^0_\mu+T^0 m_\mu\right)+e^\mu_a\left(S^a_\mu+T^a m_\mu\right)\right]=0\,,\tag{4.256}$$

which must vanish because of the comment below (4.252). The extra terms added to O_ϕ come from the variation of ϕ due to the $\alpha_{(0)}(\phi)$ factors in (4.237)–(4.239). Equating (4.254) with (4.253) we obtain

$$\frac{1}{2}t_{AB}\delta\gamma^{AB}_{(0)}+\langle O_\psi\rangle\delta\psi_{(0)}=-S^0_\mu\delta v^\mu+S^a_\mu\delta e^\mu_a+T^0\delta m_0+T^a\delta m_a+\langle O_\chi\rangle\delta\chi+\langle\tilde{O}_\phi\rangle\delta\phi\,,\tag{4.257}$$

up to total derivatives. The right hand side can be rewritten as follows

$$\begin{aligned}
&-S^0_\mu\delta v^\mu+S^a_\mu\delta e^\mu_a+T^0\delta m_0+T^a\delta m_a+\langle O_\chi\rangle\delta\chi+\langle\tilde{O}_\phi\rangle\delta\phi= \\
&-\tau_\nu T_\chi{}^\nu{}_\mu\delta\hat{v}^\mu_\chi-\left(\tau_{(\mu}\hat{h}^\chi_{\nu)\rho}\hat{v}^\sigma_\chi T_\chi{}^\rho{}_\sigma\right)\delta h^{\mu\nu}+\frac{1}{2}\left(\hat{h}^\chi_{\mu\rho}\hat{h}^\chi_{\nu\lambda}h^{\lambda\sigma}T_\chi{}^\rho{}_\sigma\right)\delta h^{\mu\nu}+\tau_\mu T^\mu\delta\tilde{\Phi}_\chi \\
&+\left(\langle O_\chi\rangle-\frac{1}{e}\partial_\mu\left(eT^\mu\right)\right)\delta\chi+\left(\hat{e}_{\chi\mu}{}^a T^\mu-\tau_\nu e^{\mu a}T_\chi{}^\nu{}_\mu\right)\delta M_a-\frac{1}{2}\hat{e}_{\chi\nu}{}^{[a}e^{b]\mu}T_\chi{}^\nu{}_\mu\left(\hat{e}_{\chi\rho a}\delta e^\rho_b-\hat{e}_{\chi\rho b}\delta e^\rho_a\right)\,,
\end{aligned}\tag{4.258}$$

where $\hat{v}^\mu_\chi$, $\hat{e}_{\chi\mu}{}^a$ and $\tilde{\Phi}_\chi$ are given by (4.50), (4.48) and (4.49) but with m_μ replaced by M_μ which is

$$M_\mu=m_\mu-\partial_\mu\chi\,.\tag{4.259}$$

This does not affect their orthonormality propertier. Further we defined $M_a=e^\mu_a M_\mu$ and

$$\begin{aligned}
T_\chi{}^\mu{}_\nu &= -\left(S^0_\nu+T^0\partial_\nu\chi\right)v^\mu+\left(S^a_\nu+T^a\partial_\nu\chi\right)e^\mu_a\,, &(4.260)\\
T^\mu &= -T^0 v^\mu+T^a e^\mu_a\,. &(4.261)
\end{aligned}$$

The definitions of the 4-dimensional sources (4.251) and (4.255) imply that they transform under the local symmetries

$$\begin{aligned}
\delta e^a_\mu &= \tau_\mu\lambda^a+\lambda^a{}_b e^b_\mu\,, &(4.262)\\
\delta m_\mu &= \partial_\mu\sigma+\lambda_a e^a_\mu\,, &(4.263)\\
\delta\chi &= \sigma\,, &(4.264)
\end{aligned}$$

for the same reasons as discussed in section 4.1.4. From this we conclude that

$$\begin{aligned}
\delta e^\mu_a &= \lambda_a{}^b e^\mu_b\,, &(4.265)\\
\delta M_a &= \lambda_a+\lambda_a{}^b M_b\,, &(4.266)
\end{aligned}$$

so that we must have the off-shell Ward identities

$$\begin{aligned}
\hat{e}_{\chi\mu}{}^a T^\mu &= \tau_\nu e^{\mu a}T_\chi{}^\nu{}_\mu\,, &(4.267)\\
0 &= \hat{e}_{\chi\nu}{}^{[a}e^{b]\mu}T_\chi{}^\nu{}_\mu\,, &(4.268)\\
\langle O_\chi\rangle &= \frac{1}{e}\partial_\mu\left(eT^\mu\right)\,. &(4.269)
\end{aligned}$$

Hence we obtain the following relation between the 5- and 4-dimensional vevs

$$\begin{aligned}\frac{1}{2}t_{AB}\delta\gamma^{AB}_{(0)}+\langle O_\psi\rangle\delta\psi_{(0)} &= -\tau_\nu T_\chi{}^\nu{}_\mu\delta\hat{v}^\mu_\chi-\left(\tau_{(\mu}\hat{h}^\chi_{\nu)\rho}\hat{v}^\sigma_\chi T_\chi{}^\rho{}_\sigma\right)\delta h^{\mu\nu}\\ &\quad+\frac{1}{2}\left(\hat{h}^\chi_{\mu\rho}\hat{h}^\chi_{\nu\lambda}h^{\lambda\sigma}T_\chi{}^\rho{}_\sigma\right)\delta h^{\mu\nu}+\tau_\mu T^\mu\delta\tilde{\Phi}_\chi\,.\end{aligned} \tag{4.270}$$

Using the same reasoning as in section 4.2.3 we conclude from this that the relation between the 5- and 4-dimensional vevs can be summarized as

$$t^{\mu u} = 2\tilde{\Phi}T^\mu-\hat{v}^\nu\left(T_\chi{}^\mu{}_\nu-T^\mu\partial_\nu\chi\right)\,, \tag{4.271}$$

$$t^{\mu\nu} = -\hat{v}^\mu T^\nu+h^{\mu\rho}\left(T_\chi{}^\nu{}_\rho-T^\nu\partial_\rho\chi\right)\,. \tag{4.272}$$

Note that $T_\chi{}^\mu{}_\nu-T^\mu\partial_\nu\chi$ is independent of χ because we absorbed $T^\mu\partial_\nu\chi$ into the definition of $T_\chi{}^\mu{}_\nu$ (see also (4.260)). Put another way we can use equations (4.126)–(4.132) with

$$T^\mu{}_\nu = T_\chi{}^\mu{}_\nu-T^\mu\partial_\nu\chi\,. \tag{4.273}$$

The Ward identities are then obtained by the dimensional reduction of (4.211) and (4.212) using (4.255) and equations (4.126)–(4.132) with $T^\mu{}_\nu=T_\chi{}^\mu{}_\nu-T^\mu\partial_\nu\chi$. On a flat boundary with $\tau_\mu=\delta^t_\mu$, $h_{\mu\nu}=\delta_{ij}\delta^i_\mu\delta^j_\nu$, $m_\mu=0$ and $\chi=0$ this becomes

$$2T^t{}_t+T^i{}_i = 0\,, \tag{4.274}$$

$$\partial_\mu T^\mu{}_\nu = 0\,, \tag{4.275}$$

$$\partial_\mu T^\mu = \langle O_\chi\rangle\,. \tag{4.276}$$

4.4.5 Consistency of the reduction

In this subsection we will show that the Scherk–Schwarz reduction (4.213) and (4.214) is consistent. We performed the reduction at the level of the action in section 4.4.3. It remains to show that also the equations of motion of the 5-dimensional action reduce correctly. The 5-dimensional equations of motion (4.201)and (4.202) can be written as

$$R^{(5)}_{\mathcal{MN}} = -4\mathcal{G}_{\mathcal{MN}}+\frac{1}{2}\partial_\mathcal{M}\psi\partial_\mathcal{N}\psi\,, \tag{4.277}$$

$$0 = \partial_\mathcal{M}\left(\sqrt{-\mathcal{G}}\mathcal{G}^{\mathcal{MN}}\partial_\mathcal{N}\psi\right)\,, \tag{4.278}$$

where the superscript on the Ricci tensor is used to distinguish its MN component from the 4-dimensional Ricci tensor $R^{(4)}_{MN}$.

The Kaluza–Klein ansatz for the metric (4.213) tells us that

$$\mathcal{G}_{MN}=e^{-\Phi}g_{MN}+e^{2\Phi}A_MA_N\,,\qquad \mathcal{G}_{Mu}=e^{2\Phi}A_M\,,\qquad \mathcal{G}_{uu}=e^{2\Phi}\,, \tag{4.279}$$

$$\mathcal{G}^{MN}=e^{\Phi}g^{MN}\,,\qquad g^{Mu}=-e^{\Phi}A^M\,,\qquad \mathcal{G}^{uu}=e^{-2\Phi}+e^{\Phi}A^MA_M\,. \tag{4.280}$$

Further we have $\sqrt{-\mathcal{G}}=e^{-\Phi}\sqrt{-g}$. The reduction of the 5-dimensional Ricci tensor follows from standard results on circle reductions of gravity (see for example [149]). The components of the 5-dimensional Ricci tensor can be written as follows

$$
\begin{aligned}
R^{(5)}_{uu} &= -e^{3\Phi}\Box\Phi + \frac{1}{4}e^{6\Phi}F^2\,, && (4.281)\\
R^{(5)}_{uM} &= R^{(5)}_{uu}A_M + \frac{1}{2}\nabla^N\left(e^{3\Phi}F_{MN}\right)\,, && (4.282)\\
R^{(5)}_{MN} &= A_M R^{(5)}_{uN} + A_N R^{(5)}_{uM} - A_M A_N R^{(5)}_{uu} + R^{(4)}_{MN} - \frac{3}{2}\partial_M\Phi\partial_N\Phi \\
&\quad + \frac{1}{2}g_{MN}\Box\Phi - \frac{1}{2}e^{3\Phi}F_{MP}F_N{}^P\,. && (4.283)
\end{aligned}
$$

Using the Scherk–Schwarz reduction ansatz for ψ given in (4.214) we also have

$$
\begin{aligned}
R^{(5)}_{uu} &= -4e^{2\Phi} + 2\,, && (4.284)\\
R^{(5)}_{uM} &= -4e^{2\Phi}A_M + 2\partial_M\Xi\,, && (4.285)\\
R^{(5)}_{MN} &= -4e^{-\Phi}g_{MN} - 4e^{2\Phi}A_M A_N + 2\partial_M\Xi\partial_N\Xi\,. && (4.286)
\end{aligned}
$$

It is now straightforward to verify that combining (4.281) and (4.284) leads to the equation of motion for Φ given in (4.222). Continuing with (4.282) and (4.285) we obtain the equation of motion for $B_M = A_M - \partial_M\Xi$ given in (4.222). Finally the equations (4.283) and (4.283) lead to the trace-reversed versions of the Einstein equation given in (4.221). We also have the 5-dimensional equation of motion for ψ. This can be seen to reduce to $\partial_M\left(\sqrt{-g}B^M\right) = 0$ which is a consequence of (4.222). We have hereby shown that the reduction (4.213) and (4.214) is consistent.

4.4.6 Comparison to other approaches

The works [52, 53] and [111, 110] both study asymptotically locally Lifshitz solutions of the EPD model. The setup of [52, 53] includes in principle what we refer to as the upliftable model but does not study it explicitly. Below we will make a first attempt at a comparison between the two approaches. We will do this for the general class of EPD models for as much as possible. Some statements will however be more specific for the case of the upliftable model.

In the notation of [52, 53] the solution to the equations of motion of the EPD model near a Lifshitz boundary is written as

$$
\begin{aligned}
ds^2 &= dr^2 + \gamma_{ij}dx^i dx^j\,, && (4.287)\\
A &= A_i dx^i\,, \qquad B = A - d\omega\,. && (4.288)
\end{aligned}
$$

The $U(1)$ gauge transformations have been partially fixed by setting $A_r = 0$. In [111, 110] we make the same gauge choice only for $z = 2$. In our notation we would replace the i index by a μ index and replace r by $\log r$. Further ω here is denoted by Ξ[6] and γ_{ij} is called $h_{\mu\nu}$ here.

In [52, 53] they employ a radial gauge ($F_2 = 1$) for the metric while in [111, 110] we allow for a general function in the rr component of the metric. For example for the upliftable model it is more natural to work with a gauge in which[7] $F_2 = e^{-\Phi}$ so that the 5-dimensional

[6] The source $\omega_{(0)}$ of the Stückelberg field ω is what we call χ.

[7] Similar gauge choices for F_2 are also important for some other EPD models that do not admit an uplift (see the $z = 2$ and $\Delta = 0$ cases discussed in [111]).

uplifted asymptotically locally AdS metric is written in radial gauge. This difference is more than just a matter of choice because we have shown in [55] that one cannot transform to the $F_2 = 1$ gauge unless the leading term in the expansion of Φ, that we call ϕ, vanishes[8]. This is not always the case and when we impose this extra condition it leads via (4.252) to the condition that τ_μ is hypersurface orthogonal. Hence if we make the assumption that the asymptotically locally Lifshitz boundary conditions of [55] are compatible with radial gauge we need to put the source $\phi = 0$.

The authors of [52, 53] write the metric γ_{ij} in ADM decomposition as

$$\gamma_{ij}dx^i dx^j = -n^2 dt^2 + \sigma_{ab}\left(dx^a + n^a dt\right)\left(dx^b + n^b dt\right) , \tag{4.289}$$

where a labels the number of spatial dimensions which here is $d = 2$. In order to make contact with the way we set up the definition of the sources we write the ADM decomposition in terms of vielbeins as follows

$$\gamma_{ij}dx^i dx^j = -E^0 E^0 + \delta_{\underline{ab}} E^{\underline{a}} E^{\underline{b}} , \tag{4.290}$$

where underlined indices $\underline{a}$ refer to flat tangent space indices that take as many values as there are spatial coordinates. We keep here with the notation of [52, 53]. Since they do not use tangent space indices we introduced these underlined indices only in this section for the sake of comparison. We can take without loss of generality

$$E^0 = n dt , \qquad E^{\underline{a}} = e_a^{\underline{a}}\left(dx^a + n^a dt\right) . \tag{4.291}$$

This allows us to establish the following dictionary between the sources in [52, 53] and those defined in [111, 110]

$$\tau_\mu = \left(n_{(0)}\,, 0\right) , \qquad v^\mu = n_{(0)}^{-1}\left(-1\,, n_{(0)}^a\right) , \tag{4.292}$$

$$h_{\mu\nu}dx^\mu dx^\nu = g_{(0)ab}\left(dx^a + n_{(0)}^a dt\right)\left(dx^b + n_{(0)}^b dt\right) . \tag{4.293}$$

Hence in [52, 53] the source τ_μ is always taken to be hypersurface orthogonal.

In the work [111, 110] has been introduced the Newton–Cartan vector m_μ as a source or rather the $U(1)$ invariant combination $M_\mu = m_\mu - \partial_\mu \chi$. By fixing local tangent space transformations (with parameter λ_a) we can fix all but one component of M_μ. The remaining component is related to $\tilde{\Phi}_\chi$ which is defined by (4.50) with m_μ replaced by M_μ. The scalar $\tilde{\Phi}$ has scaling weight $2(z-1)$. Similarly there is a scalar source in the work of [52, 53] that is denoted by ψ following [160]. The main difference between the approach of [52, 53] and [111, 110] lies in the fact that the dilatation weight of ψ denoted by Δ_- does not in general agree with the dilatation weight of $\tilde{\Phi}$. The number of sources (when comparing both approaches in radial gauge and taking τ_μ to be hypersurface orthogonal) thus agrees but for one of them the scaling dimensions differ.

[8]For general EPD models we set $\Phi \simeq r^\Delta \phi$ [111, 110] where the value of Δ depends on the details of the EPD action. To the best of our knowledge this Δ parameter does not appear explicitly in [52, 53]. However there is a comment below their equation (5.25) stating that the asymptotic form of the dilaton depends on the potential which is essentially allowing for a Δ in the fall-off of the dilaton.

To see where $\tilde{\Phi}$ appears in our near boundary expansion we consider purely radial solutions, like we studied in section 5.2.2. Recall that in section 5.2.2 we set $\tilde{\Phi} = 0$ by hand. If we do not do this then we obtain, using the results of appendices 4.1.4 and 4.4.1,

$$\gamma_{(2)AB} = \delta^u_A \delta^u_B - \frac{1}{3}\tilde{\Phi}\gamma_{(0)AB}\,, \tag{4.294}$$

$$\gamma_{(4,1)AB} = \frac{4}{3}\tilde{\Phi}\delta^u_A \delta^u_B - \frac{2}{3}\tilde{\Phi}^2\gamma_{(0)AB}\,, \tag{4.295}$$

$$\gamma_{(4)AB} = -\frac{1}{2}\tilde{\Phi}\delta^u_A \delta^u_B + \frac{5}{18}\tilde{\Phi}^2\gamma_{(0)AB} - \frac{1}{4}t_{AB}\,, \tag{4.296}$$

$$\psi_{(2)} = \psi_{(4,1)} = \psi_{(4)} = 0\,. \tag{4.297}$$

Components such as $\gamma_{(4,1)AB}$ correspond to logarithmic terms in the expansion. This implies that for example the expansions of the matter fields become

$$\Phi = \frac{2}{3}\tilde{\Phi}r^2\log r - \frac{1}{8}(\rho + 2\tilde{\Phi})r^2 + \ldots\,, \tag{4.298}$$

$$A_\mu = r^{-2}\tau_\mu - \frac{4}{3}\log r\tilde{\Phi}\tau_\mu + \frac{1}{6}\tilde{\Phi}\tau_\mu + \frac{1}{4}\rho\tau_\mu + \ldots\,, \tag{4.299}$$

where the dots denote subleading terms.

It would be nice to make a direct comparison between the case studied here where V is a sum of two exponential potentials as given in (5.17). However the case where V is the sum of two exponentials is not explicitly studied in [52, 53] so this would have to be worked out first[9].

[9]There exists another model used in [55] that contains two dilatons that can be obtained by a similar Scherk–Schwarz reduction as used in the present work admitting a $z = 2$ Lifshitz solution. This model is related in [54] (see around equation (6.40)) to an EPD model with a single exponential potential, which is a case explicitly worked out in that paper. This is done by setting a linear combination of these two scalars equal to a constant, which, however, is not a consistent truncation of the model discussed in [54]. The relation (6.40) of [54] only holds asymptotically at leading order in a near boundary expansion.

5. Lifshitz Holography

Hydrodynamics is a powerful tool to describe the long wave-length physics of quantum field theories at finite temperature. Remarkably, holography provides a dual realization of such effective descriptions, typically at strong coupling in the field theory, in terms of the long-wave length dynamics of black holes. The seminal example of this has been the calculation of the viscosity to entropy ratio [154, 140] (see [162] for a review) of strongly coupled plasmas using black hole via the AdS/CFT correspondence. This deep relation between fluid dynamics and gravity has led to the fluid/gravity correspondence [36, 23] (see [157] for a review) which as sparked numerous novel insights on both sides of the duality.[1]

These developments initially focused on the dual gravitational formulation of relativistic hydrodynamics, since in the standard AdS holography the dual field theories are relativistic. Motivated by applying holography to more general settings, in particular strongly coupled non-relativistic field theories, more general bulk theories with anistropic scaling between time and space, charactererized by the dynamical exponent z, have been introduced [164, 24, 135, 170]. These include include Schrödinger and Lifshitz spacetimes, and in the latter case[2] there are different bulk realizations (e.g. Einstein-Maxwell-dilaton (EMD) and Einstein-Proca-dilaton (EPD) models) which have distinct physical features. All these bulk models serve to describe different types of non-relativistic field theories, where in the former there is Galilean boost symmetry, while in the latter there is a broken boost symmetry. For possible applications to condensed matter systems, but also to further our understanding of holography in non-AdS setups, it is thus a natural question to generalize the fluid/gravity correspondence to these different classes of non-relativistic field theories. This may also serve as a step towards a more general classification of such field theories.

For theories with Schrödinger symmetries a corresponding version of (conformal) non-relativistic fluid/gravity correspondence was developed in Refs. [121, 146, 1, 158]. Certain

[1] For asymptotically flat black branes the blackfold approach [87, 84, 46] also gives a relation between the long wave-length dynamics of black branes and fluids that live on dynamically embedded surfaces. Applied to D3-branes this has been shown to encapsulate AdS fluid/gravity [77].

[2] See Ref. [169] for a review on Lifshitz holography.

realizations of Lifshitz hydrodyamics and their holographic description have subsequently been considered in Refs. [127, 126, 125]. Another class of Lifshitz theories and their hydrodynamics was holographically studied in the context of a bulk Einstein-Maxwell-dilaton (EMD) model [138]. In these theories there is an extra bulk $U(1)$ symmetry and since the dilaton runs logarithmically close to the boundary, there is a new scaling exponent on top of the dynamical exponent z.

In this paper we will focus on yet another class of Lifshitz theories, namely those that have Lifshitz symmetry in the bulk and Schrödinger symmetry with broken particle number on the boundary. For a large class of EPD models, it was shown in [56, 55, 111] that holography in such bulk theories is dual to non-relativistic field theories of this type, coupled to a background torsional Newton-Cartan geometry that is induced on the boundary. Our aim is therefore to find a gravitational dual realization of the hydrodynamics, or rather the perfect fluid limit, of this class of non-relativistic theories by constructing appropriate Lifshitz black branes.

A classification of the different version Lifshitz hydrodynamics will be given in the companion paper [37] using a field theory perspective. The novel version of Lifshitz hydrodynamics that we find in this paper is a holographic realization of one particular class, which will be also discussed in [37] with field theory examples.[3] We will find in this paper that this version of Lifshitz hydrodynamics requires the construction of a new class of four-dimensional $z = 2$ Lifshitz black branes that have a non-zero linear momentum. While, as mentioned above, a large class of general z Lifshitz spacetimes can be constructed in the EPD model, we restrict for technical reasons to a particular EPD model with $z = 2$ solutions, that can be obtained by Scherk–Schwarz circle reduction of AdS_5 gravity coupled to a free real scalar field.

Our new $z = 2$ Lifshitz black brane solutions exhibit the following features:

- The linear momentum of the black brane cannot be obtained by a boost transformation, and hence this class of solutions is physically distinct from unboosted solutions.
- The (squared) magnitude of the boost velocity plays the role of a chemical potential dual to the mass density. Consequently, the mass density occurs asymptotically as an extra parameter on top of the energy, even when the velocity is zero.
- The black brane configurations describe a new class of Lifshitz perfect fluid that are obtained by breaking particle number symmetry in the Schrödinger perfect fluid.

In further detail, the thermodynamics for these Lifshitz black branes can be summarized by

$$\mathcal{E} + P = Ts + \frac{1}{2}\rho V^2 \,, \tag{5.1}$$

$$\delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho \,, \tag{5.2}$$

where $\mathcal{E}$ is the energy density, P the pressure with equation of state $P = \mathcal{E}$ (2 spatial dimensions with $z = 2$), T temperature, s entropy density, ρ mass density and $V^2 = V^iV^i$ with V^i the fluid velocity.

Outline

The outline of the paper is as follows. In section 5.1 we introduce the Einstein–Proca-dilaton (EPD) theory which consists of Einstein gravity coupled to a massive vector and a dilaton

[3] See also [110] for a discussion on field theories coupled to TNC with broken Schrödinger symmetries.

with arbitrary dilaton-dependent couplings between the massive vector and the dilaton. In section 5.1.2 we show that this model admits, under some mild restrictions on the dilaton-dependent coupling functions, Lifshitz solutions for any value of the dynamical exponent z. Important for the rest of this work we show in section 5.1.3 that there is one EPD model that can uplifted to a 5-dimensional AdS gravity theory coupled to a free real scalar field. This specific model will be referred to as the upliftable model and it admits $z = 2$ Lifshitz solutions.

For the higher-dimensional AdS theory it is known how to perform holographic renormalizaton and by reducing the result to one dimension lower we can obtain the relevant counterterms and near boundary asymptotic expansions. This reduction is of the Schwarz–Schwarz type meaning that the 5-dimensional scalar field is required to come back to itself up to a shift (which is a global symmetry) upon going around the compact 5th dimension. In section 4.4 we provide the details of the holographic renormalization before and after the dimensional reduction. In section 4.4.5 we give a proof that the reduction is consistent.

The reduction in the bulk is everywhere along a spacelike circle however on the boundary (due to conformal rescaling) the circle is null. Hence from a boundary perspective we are dealing with a null reduction. It is well known that null reductions of Lorentzian geometries, in this case the boundary of the asymptotically AdS_5 space-time, lead to Newton–Cartan geometries. The details of this null reduction for both the metric and energy-momentum tensor are given in chapter 4.

Section 5.2 is concerned with the construction and properties of Lifshitz black branes with linear momentum. We start with the ansatz in section 5.2.1 where we also show that the effective action, that reproduces the equations of motion of the EPD model in which our ansatz has been substituted, possesses two scale symmetries. This leads to two Noether charges or first integrals of motion that are constant along the holographic coordinate. The following sections 5.2.2–5.2.4 study the properties of the solution near the boundary and near the horizon. In the last two subsections 5.2.5 and 5.2.6 we work out the thermodynamic properties of the solution showing that the magnitude of the velocity acts like a chemical potential whose conjugate variable is the mass density. We further derive an Euler-type thermodynamic relation for Lifshitz perfect fluids using the conserved Noether charges and once more using the Killing charges associated with bulk Killing vectors. We end with a discussion of the first law of thermodynamics (summarized in (5.1)) for these Lifshitz black branes.

The final section can be read independently from sections 5.1 and 5.2. It only requires chapter 4. It takes the point of view that these Lifshitz perfect fluids can be obtained by dimensional reduction of a relativistic perfect fluid (as discussed in appendix 4.2.5) in the presence of a scalar source that is linearly dependent on the circle coordinate. It presents the Ward identities of a Lifshitz fluid and the expressions for the energy-momentum tensor and mass current at the perfect fluid level. The goal of this section is to derive the first law of our Lifshitz perfect fluids from the requirement that the Ward identities lead to the existence of a conserved entropy current.

5.1 The bulk theory

Lifshitz space-times as a solution of a theory with Einstein gravity coupled to matter fall into 2 classes. These are the Einstein–Proca-dilaton (EPD) theories of [92, 96] and the

Einstein–Maxwell dilaton (EMD) theories of [170, 51]. We are interested in Lifshitz black brane solutions of the EPD models and to ultimately use them as a starting point to set up a fluid/gravity correspondence for Lifshitz space-times. Black brane solutions of theories with massive vectors were studied for models without a dilaton in [34, 35]. However in such theories the solutions are not analytically known. Nevertheless it is possible to work out the thermodynamics of these solutions by using first integrals of motion (with respect to the holographic radial coordinate) that allows one to relate near horizon expansions to asymptotic expansions. We will follow a similar approach here.

Regarding Lifshitz black brane solutions of the EMD models, they are known analytically however they have different physical properties due to the presence of an extra bulk $U(1)$ symmetry and the fact that the dilaton is running logarithmically close to the boundary which introduces a new scaling exponent on top of the dynamical exponent z. The fluid/gravity correspondence for these black branes was studied in [138].

In this section we will introduce the EPD model and discuss its Lifshitz solutions. In the last section 5.1.3 we will show that there is a specific EPD model that can be obtained from dimensional reduction of an action in one dimension higher that admits asymptotically AdS solutions. This so-called upliftable model will be used throughout the rest of this work.

5.1.1 The EPD model

The general class of 4-dimensional bulk EPD models is described by the following family of actions

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}W(\Phi)B^2 - \frac{x}{2}(\partial\Phi)^2 - V(\Phi)\right), \tag{5.3}$$

where $F = dB$. The equations of motion are

$$\begin{aligned} G_{MN} &= \frac{x}{2}\left(\partial_M\Phi\partial_N\Phi - \frac{1}{2}(\partial\Phi)^2 g_{MN}\right) - \frac{1}{2}V(\Phi)g_{MN} \\ &+ \frac{1}{2}Z(\Phi)\left(F_{MP}F_N{}^P - \frac{1}{4}F^2 g_{MN}\right) + \frac{1}{2}W(\Phi)(B_M B_N - \frac{1}{2}B^2 g_{MN}), \qquad (5.4) \\ \frac{x}{\sqrt{-g}}&\partial_M\left(\sqrt{-g}\partial^M\Phi\right) = \frac{1}{4}\frac{dZ}{d\Phi}F^2 + \frac{1}{2}\frac{dW}{d\Phi}B^2 + \frac{dV}{d\Phi}, \qquad (5.5) \\ \frac{1}{\sqrt{-g}}&\partial_M\left(\sqrt{-g}Z(\Phi)F^{MN}\right) = W(\Phi)B^N. \qquad (5.6) \end{aligned}$$

It is convenient to keep x arbitrary. Of course one can always set it equal to one but often it is more convenient to take some other value for it. We will also write

$$B_M = A_M - \partial_M\Xi, \tag{5.7}$$

and $F = dA$. The axion Ξ has dimensions of length while all other fields are dimensionless.

5.1.2 Lifshitz solution

The equations of motion admit the following Lifshitz solutions (with $z > 1$)

$$
\begin{aligned}
ds^2 &= -\frac{1}{r^{2z}}dt^2 + \frac{1}{r^2}\left(dr^2 + dx^2 + dy^2\right) , && (5.8)\\
B &= A_0 \frac{1}{r^z} dt \, , && (5.9)\\
\Phi &= \Phi_\star \, , && (5.10)
\end{aligned}
$$

provided that

$$
\begin{aligned}
A_0^2 &= \frac{2(z-1)}{zZ_0} \, , && (5.11)\\
\frac{W_0}{Z_0} &= 2z \, , && (5.12)\\
V_0 &= -(z^2 + z + 4) \, , && (5.13)\\
V_1 &= (za + 2b)(z-1) \, . && (5.14)
\end{aligned}
$$

where

$$
a = \frac{Z_1}{Z_0} \, , \qquad b = \frac{W_1}{W_0} \, . \tag{5.15}
$$

Above we have defined the notation

$$
V_0 \equiv V(\phi_*)^V_1 \equiv \frac{dV}{d\Phi}\Big|_{\phi=\phi_*} \quad , \quad V_2 \equiv \frac{d^2V}{d\Phi^2}\Big|_{\phi=\phi_*} \tag{5.16}
$$

etc. In (5.10) we indicate that the scalar is constant in this solution. Equations (5.12) and (5.13) determine the value of ϕ_* and z in terms of the functions appearing in the action. Equation (5.11) determines A_0, and (5.14) is an extra condition that makes Lifshitz a non-generic solution of the family of actions (5.3). We note that there are also solutions of the EPD model with a running scalar whose metric is a Lifshitz space-time [96, 92]. These will not be considered here.

Without loss of generality we can always perform a constant shift of Φ and redefine the functions Z, W and V such that for the new Φ the solution has $\Phi_\star = 0$. We will from now on always assume this has been done.

In order to study the boundary fluid properties (or even only thermodynamic properties as we will do here) one needs to understand the near boundary expansion and the identification in that expansion of all the sources and vevs. Despite a lot of effort we do not feel confident that this problem has been tackled fully. We will therefore restrict our attention to a specific model for which this problem has been solved because it can be related to an AdS holographic renormalization problem in one dimension higher. The general features of Lifshitz black brane solutions of other EPD models will have to wait until we have understood fully the problem of performing holographic renormalization for asymptotically Lifshitz solutions of the general class of EPD models. The model for which we do have full control of the asymptotic expansion is called the upliftable model and will be the subject of the next subsection.

5.1.3 The upliftable model

When we make the choices

$$Z = e^{3\Phi}\,, \qquad W = 4\,, \qquad V(\Phi) = 2e^{-3\Phi} - 12e^{-\Phi}\,, \qquad x = 3\,. \tag{5.17}$$

the action (5.3) can be uplifted to

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-\mathcal{G}}\left(R + 12 - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial^{\mathcal{M}}\psi\right)\,, \tag{5.18}$$

where $\kappa_5^2 = 8\pi G_5$ with G_5 the 5-dimensional Newton's constant and where $\mathcal{M} = (u, M)$. The relation between the 5- and 4-dimensional theories is via a so-called Scherk–Schwarz reduction whereby we demand that the scalar field ψ when going around the compatification circle comes back to itself up to a shift. This is also known as a twisted reduction. This is possible because the higher dimensional theory has a shift symmetry acting on the scalar. The Scherk–Schwarz reduction leading to (5.3) with the choices (5.17) is obtained by the following ansatz

$$\begin{aligned} ds_5^2 &= \mathcal{G}_{\mathcal{M}\mathcal{N}}dx^{\mathcal{M}}dx^{\mathcal{N}} = \frac{dr^2}{r^2} + \gamma_{AB}dx^Adx^B = e^{-\Phi}g_{MN}dx^Mdx^N + e^{2\Phi}\left(du + A_Mdx^M\right)^2 \\ &= e^{-\Phi}\left(e^{\Phi}\frac{dr^2}{r^2} + h_{\mu\nu}dx^\mu dx^\nu\right) + e^{2\Phi}\left(du + A_\mu dx^\mu\right)^2\,, \qquad (5.19) \\ \psi &= 2u + 2\Xi\,, \qquad (5.20) \end{aligned}$$

where the four dimensional fields g_{MN}, A_M, Ξ and Φ are independent of the fifth coordinate u which is periodically identified $u \sim u + 2\pi L$. In our normalization the 4-dimensional Newton's constant G_4 obeys $16\pi G_4 = 1$. This means that 5-dimensional Newton's constant G_5 obeys $\frac{2\pi L}{16\pi G_5} = \frac{1}{16\pi G_4} = 1$.

What the Scherk–Schwarz reduction does is that it gauges the shift symmetry of ψ using the Kaluza–Klein vector. This results in 4 dimensions in a covariant derivative acting on Ξ. This covariant derivative can be read as a massive vector field B where B is given by (5.7). The consistency of the reduction will be proven in appendix 4.4.5. We now specialize to the case of the upliftable model (5.17) because for this theory we have full control over the asymptotic solution space.

5.2 Black branes with linear momentum

The goal of this work is to construct the gravity dual of a Lifshitz perfect fluid. The Lifshitz algebra does not contain a boost generator. We will be interested in cases where the Lifshitz algebra arises from a larger algebra that contains boosts[4] by some explicit symmetry breaking. The bulk Lifshitz space-time has no boost Killing vector and the EPD model has no additional local symmetries that could combine with a space-time transformation to give an additional global symmetry like a Galilean boost[5]. Hence in order to study perfect fluids

[4]These can only be Galilean or Carrollian boosts as these are compatible with a $z > 1$ scaling. We cannot obtain a Lifshitz algebra by breaking Lorentz boosts because these require $z = 1$.

[5]What we have in mind here is some bulk dual of the mechanism discussed in [112, 110] whereby a boundary space-time transformations combined with a certain $U(1)$ transformation leads to an additional global symmetry. For example Galilean boost symmetries of the Schrödinger equation come about by a combination of a space-time Galilean coordinate transformation and a $U(1)$ phase transformation of the wave function. The latter can be traded for a $U(1)$ transformation of a background gauge field.

with a non-zero velocity we cannot simply boost a static Lifshitz black brane and promote the boost velocity to the fluid velocity. If we do that for a static Lifshitz black brane solution of the EPD model we simply describe a static black brane in a moving coordinate system and that is not equivalent to a moving black brane in a static coordinate system because of the absence of a boost symmetry. That means that we need to construct a new class of Lifshitz black branes that have a nonzero velocity or as we shall say nonzero linear momentum. We will construct these solutions near the Lifshitz boundary and near the horizon. We will then construct first integrals of motion to relate near horizon quantities such as temperature and entropy to near boundary quantities such as energy and mass density[6].

5.2.1 The ansatz

We assume that the black brane solution admits time and space translation Killing vectors. We can perform a rotation to make sure that the linear momentum is only along the y-direction. The full non-linear solution is thus of the form

$$ds_4^2 = -F_1(r)\frac{dt^2}{r^4} + \frac{1}{F_2(r)}\frac{dr^2}{r^2} + F_3(r)\frac{dx^2}{r^2} + F_4(r)\left(\frac{dy}{r} + N(r)\frac{dt}{r^2}\right)^2 , \tag{5.21}$$

$$B = G_1(r)\frac{dt}{r^2} + G_2(r)\left(\frac{dy}{r} + N(r)\frac{dt}{r^2}\right) , \tag{5.22}$$

$$\Phi = \Phi(r) , \tag{5.23}$$

where we did not fix the r reparametrization invariance. The powers in r are chosen for convenience to match with the Lifshitz scaling of the boundary coordinates t, x and y.

We can go to Eddington–Finkelstein coordinates[7] by defining V and Y coordinates as follows

$$dt = dV + r\,(F_1F_2)^{-1/2}\,dr , \tag{5.24}$$

$$dy = dY - N\,(F_1F_2)^{-1/2}\,dr , \tag{5.25}$$

leading to

$$ds_4^2 = -F_1\frac{dV^2}{r^4} - 2\left(\frac{F_1}{F_2}\right)^{1/2}\frac{dV\,dr}{r^3} + F_3\frac{dx^2}{r^2} + F_4\left(\frac{dY}{r} + N\frac{dV}{r^2}\right)^2 , \tag{5.26}$$

$$B = G_1\frac{dV}{r^2} + \frac{G_1}{(F_1F_2)^{1/2}}\frac{dr}{r} + G_2\left(\frac{dY}{r} + N\frac{dV}{r^2}\right) . \tag{5.27}$$

Substituting the ansatz (5.21)–(5.23) into the bulk equations of motion (5.4)–(5.6) for (5.17) and integrating the equations to an action leads to the following effective Lagrangian

[6]If we assume that a Galilean boost symmetry has been broken, the velocity V^i or rather by rotational invariance, its magnitude V^2, will be a chemical potential. On dimensional grounds it follows that the dual thermodynamic variable must be a mass density denoted by ρ. We will show that Lifshitz black branes indeed contain such a quantity.

[7]Null geodesics with generalized momenta $\frac{\partial L}{\partial \dot{x}} = 0$ and $\frac{\partial L}{\partial \dot{y}} = 0$ where $L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ correspond to $V = \text{cst}$ and $Y = \text{cst}$.

for the equations of motion

$$
\begin{aligned}
L \;=\; & r^{-5}\left(F_1F_2F_3F_4\right)^{1/2}\Bigg[\frac{r^2}{2}\frac{F_1'}{F_1}\frac{F_3'}{F_3}+\frac{r^2}{2}\frac{F_1'}{F_1}\frac{F_4'}{F_4}-r\frac{F_3'}{F_3}-r\frac{F_4'}{F_4}+2r\frac{F_2'}{F_2}+\frac{r^2}{2}\frac{F_3'F_4'}{F_3F_4}-6 \\
&+\frac{1}{2}\frac{F_4(rN'-N)^2}{F_1}+\frac{1}{2}\frac{Z\left(rG_1'-2G_1\right)^2}{F_1}+2\frac{G_1^2}{F_1F_2}-\frac{3}{2}r^2\Phi'^2-\frac{V}{F_2}-2\frac{G_2^2}{F_2F_4} \\
&+\frac{1}{2}\frac{ZG_2^2\left(rN'-N\right)^2}{F_1}-2\frac{ZG_1G_2\left(rN'-N\right)}{F_1}+r\frac{ZG_2G_1'\left(rN'-N\right)}{F_1}-\frac{1}{2}\frac{Z\left(rG_2'-G_2\right)^2}{F_4}\Bigg]\,,
\end{aligned}
\tag{5.28}
$$

where the independent variables are F_1 to F_4, N, G_1, G_2, Φ and their derivatives with respect to r. This effective Lagrangian can also be obtained by substituting the ansatz (5.21)–(5.23) into the bulk action (5.3) with (5.17) and performing a few partial integrations. This ansatz is a generalization of a static black brane with zero momentum corresponding to setting $G_2 = N = 0$ and $F_3 = F_4$.

The effective Lagrangian (5.28) has the following two scaling symmetries

$$
F_1 \to \lambda^2 F_1\,, \quad F_{3,4} \to \lambda^{-1} F_{3,4}\,, \quad N \to \lambda^{3/2} N\,, \quad G_1 \to \lambda G_1\,, \quad G_2 \to \lambda^{-1/2} G_2\,, \tag{5.29}
$$

and

$$
F_3 \to \mu^2 F_3\,, \quad F_4 \to \mu^{-2} F_4\,, \quad N \to \mu N\,, \quad G_2 \to \mu^{-1} G_2\,. \tag{5.30}
$$

Both of these transformations are symmetries of the ansatz (5.21)–(5.23) provided we transform the coordinates appropriately. For the λ transformation that means that we must rescale the coordinates as

$$
t \to \lambda^{-1} t\,, \qquad x \to \lambda^{1/2} x\,, \qquad y \to \lambda^{1/2} y\,, \tag{5.31}
$$

while for the μ transformation it means that we must rescale the spatial coordinates as

$$
x \to \mu^{-1} x\,, \qquad y \to \mu y\,. \tag{5.32}
$$

Using Noether's theorem the associated charges are Q_λ and Q_μ, respectively, that are given by

$$
Q_\lambda \;=\; -2\frac{\partial L}{\partial F_1'}F_1+\frac{\partial L}{\partial F_3'}F_3+\frac{\partial L}{\partial F_4'}F_4-\frac{3}{2}\frac{\partial L}{\partial N'}N-\frac{\partial L}{\partial G_1'}G_1+\frac{1}{2}\frac{\partial L}{\partial G_2'}G_2\,, \tag{5.33}
$$

$$
Q_\mu \;=\; -2\frac{\partial L}{\partial F_3'}F_3+2\frac{\partial L}{\partial F_4'}F_4-\frac{\partial L}{\partial N'}N+\frac{\partial L}{\partial G_2'}G_2\,. \tag{5.34}
$$

Using that L is given by (5.28) these charges can be shown to be equal to

$$
\begin{aligned}
Q_\lambda \;=\; & -\frac{3}{2}r^{-1}Q_N N+r^{-4}\left(F_1F_2F_3F_4\right)^{1/2}\Bigg[-\frac{ZG_1}{F_1}\big(rG_1'-2G_1\big)-2+r\frac{F_1'}{F_1}-\frac{r}{2}\frac{F_3'}{F_3}-\frac{r}{2}\frac{F_4'}{F_4} \\
&-\frac{ZG_1G_2\left(rN'-N\right)}{F_1}-\frac{1}{2}\frac{ZG_2\left(rG_2'-G_2\right)}{F_4}\Bigg]\,,
\end{aligned}
\tag{5.35}
$$

and

$$
Q_\mu \;=\; -r^{-1}Q_N N+r^{-4}\left(F_1F_2F_3F_4\right)^{1/2}\left[r\frac{F_3'}{F_3}-r\frac{F_4'}{F_4}-\frac{ZG_2\left(rG_2'-G_2\right)}{F_4}\right]\,, \tag{5.36}
$$

where we defined the charge Q_N

$$\begin{aligned} Q_N &= \frac{\partial L}{\partial N'} = r^{-3}\left(F_1F_2F_3F_4\right)^{1/2}\left[\frac{F_4\left(rN'-N\right)}{F_1}+\frac{ZG_2^2\left(rN'-N\right)}{F_1}-2\frac{ZG_1G_2}{F_1}\right. \\ &\left.+\frac{ZG_2\left(rG_1'-G_1\right)}{F_1}\right], \end{aligned} \tag{5.37}$$

which results from the fact that L does not depend on N. The Noether charges Q_λ and Q_μ are first integrals of motion and thus independent of the radial coordinate r. This will play an important role later when we derive the thermodynamics properties. We will see that Q_λ relates to the energy and Q_μ to the linear momentum of the black brane.

The ansatz (5.21)–(5.23) has a third global scale symmetry namely

$$t \to \nu^{-1}t\,, \qquad F_1 \to \nu^2 F_1\,, \qquad N \to \nu N\,, \qquad G_1 \to \nu G_1\,. \tag{5.38}$$

However this transformation does not leave the effective Lagrangian (5.28) invariant because it is not a symmetry of the prefactor. On top of the 3 global symmetries whose parameters are λ, μ and ν the ansatz also has one local symmetry which corresponds to r-reparametrization invariance. This symmetry acts as

$$\delta F_2 = \xi^r F_2' + 2F_2\left(r^{-1}\xi^r - \partial_r\xi^r\right)\,, \qquad \delta A_I = \xi^r A_I'\,, \tag{5.39}$$

where A_I is any of the functions appearing in the ansatz that is not F_2 and ξ^r is the local parameter generating the r reparametrization. This local symmetry can be fixed by choosing a gauge. This local symmetry implies that using the F_2 equation of motion (which is first order and needs to be differentiated with respect to r) and any 6 of the other A_I equations of motion the remaining 7th A_I equation of motion can be derived.

5.2.2 The asymptotic solution

The 4-dimensional near boundary expansion follows by dimensional reduction using the reduction ansatz (5.19) and (5.20) as well as the 5-dimensional Fefferman–Graham (FG) expansion (4.187) and (4.188), the details of which are given in appendix 4.4.1.

The ansatz for the black branes with linear momentum are such that all 4-dimensional fields only depend on the radial coordinate r. From the 5-dimensional FG expansion point of view that implies that all sources and vevs must be constants. The only exception to this is of course the fact that ψ is allowed to be linear in the reduction circle coordinate u because we are performing a Scherk–Schwarz reduction. That means that our ansatz forces us to consider a FG expansion in 5D with the following sources and vevs

$$\gamma_{(0)AB} = \text{cst}\,, \qquad \text{with } \gamma_{(0)uu} = 0\,, \tag{5.40}$$

$$t_{AB} = \text{cst}\,, \tag{5.41}$$

$$\psi_{(0)} = 2u\,, \qquad \text{so that } \Xi = 0\,, \tag{5.42}$$

$$\langle O_\psi \rangle = 0\,, \tag{5.43}$$

where t_{AB} obeys the Ward identities (4.211) and (4.212). Setting $\langle O_\psi\rangle = 0$ is a consequence of the Ward identity $\nabla_{(0)A}t^A{}_B = -\langle O_\psi\rangle\partial_B\psi$ for $B = u$ and constant $t^A{}_B$. Since the field Ξ always appears differentiated it makes no difference if we set it equal to zero or equal to some constant. The choice $\gamma_{(0)uu} = 0$ is rather important and is necessary in order that the lower-dimensional theory has a $z = 2$ scaling exponent. This is explained in detail in [54, 55]. It is shown in section 2 of [55] that the reduction in the bulk is everywhere along a spacelike circle (due to $\psi_{(0)} = 2u$) but that this circle is null on the boundary[8].

It is well known that reductions along null Killing directions turn a Riemannian geometry into a torsional Newton–Cartan geometry [70, 142, 134, 55]. For details see appendix 4.4.3. In particular see the reduction ansatz for the AdS_5 boundary metric (4.251). In the language of TNC geometry the uu component of the inverse metric is called $\tilde{\Phi}$ which is defined in (4.50) where $M_\mu = m_\mu - \partial_\mu \chi$ with m_μ the Kaluza–Klein vector associated with the null reduction as given in (4.251). Since here we have $\chi = 0$ we can take $M_\mu = m_\mu$. From the inverse metric we know that

$$\gamma^{uu}_{(0)} = 2\tilde{\Phi}\,. \tag{5.44}$$

We will be interested in flat boundaries of the 4-dimensional $z = 2$ Lifshitz space-time. A flat space-time in TNC language means that there exists a coordinate system in which we have [110]

$$\tau_\mu = \delta^t_\mu\,, \qquad M_\mu = 0\,, \qquad h_{tt} = h_{ti} = 0\,, \qquad h_{ij} = \delta_{ij}\,. \tag{5.45}$$

This means in particular that $\tilde{\Phi} = 0$. Turning on $\tilde{\Phi}$ corresponds to turning on a Newtonian potential for the boundary theory [33, 110]. We will thus not consider this possibility.

The expansion of the 4-dimensional fields follows from (5.19) and (5.19) which imply that[9])

$$e^{2\Phi} = \gamma_{uu}\,, \tag{5.46}$$

$$A_\mu = \frac{\gamma_{u\mu}}{\gamma_{uu}}\,, \tag{5.47}$$

$$h_{\mu\nu} = (\gamma_{uu})^{1/2}\left(\gamma_{\mu\nu} - \frac{\gamma_{u\mu}\gamma_{u\nu}}{\gamma_{uu}}\right)\,, \tag{5.48}$$

where γ_{AB} is FG expanded using the results of appendix 4.4.1. In order to carry out this reduction we need to know how to reduce the AdS boundary energy-momentum tensor into the laguage of the energy-momentum tensor of the TNC boundary of the lower-dimensional Lifshitz space-time. The relation between a relativistic energy-momentum tensor and the TNC energy-momentum tensor related via null reduction is explained in appendix 4.1.4 where we derive the following relations

$$t_{uu} = \rho\,, \tag{5.49}$$

$$t_{u\mu} = \tau_\rho T^\rho{}_\mu\,, \tag{5.50}$$

$$t_{\mu\nu} = \hat{h}_{\mu\rho}\hat{h}_{\nu\kappa}h^{\kappa\sigma}T^\rho{}_\sigma - \left(\tau_\nu\hat{h}_{\mu\rho} + \tau_\mu\hat{h}_{\nu\rho}\right)\hat{v}^\sigma T^\rho{}_\sigma + \left(\hat{v}^\rho\hat{v}^\sigma t_{\rho\sigma}\right)\tau_\mu\tau_\nu\,, \tag{5.51}$$

where

$$\hat{v}^\rho\hat{v}^\sigma t_{\rho\sigma} = t^{uu} - 4\tilde{\Phi}^2\rho + 4\tilde{\Phi}\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma\,. \tag{5.52}$$

[8]Here we use a model that is simpler than the one used in [55] but regarding this point the properties are identical.

[9]We warn the reader that we use $h_{\mu\nu}$ both to denote the $\mu\nu$ component of the bulk metric as well as the spatial metric-like quantity (**??**) on the boundary. We hope that this will not cause any confusion.

Recall that here $\tilde{\Phi} = 0$. The TNC energy-momentum tensor is denoted by $T^\mu{}_\nu$ and the TNC mass density is denoted by ρ. We note that t^{uu} has no lower dimensional interpretation in terms of energy-momentum or mass density. As shown in appendix 4.1.4 it does not appear in any of the Ward identities involving $T^\mu{}_\nu$ and ρ. Hence we will set t^{uu} equal to zero. It would appear in 4 dimensions for the first time at order r^2 in that part of the expansion of $h_{\mu\nu}$ that is proportional to $\tau_\mu\tau_\nu$. We refer to [55] for more discussion on the role of t^{uu}.

In order to find out where the momentum flux, the spatial projection of $\tau_\rho T^\rho{}_\mu$, which is one of the quantities of interest, the spatial stress tensor etc appear upon reduction we need to know what happens with $t_{\mu\nu}$ upon reduction. Clearly in 5 bulk dimensions $t_{\mu\nu}$ appears in $\gamma_{\mu\nu}$ at order r^2. Therefore in order to see it in four bulk dimensions we need to expand A_μ and $h_{\mu\nu}$ up to order r^2. This follows from (5.47) and (5.48) and implies that we need to expand γ_{uu} to order r^6, $\gamma_{u\mu}$ to order r^4 and $\gamma_{\mu\nu}$ to order r^2.

We will now proceed to construct the 5-dimensional solution up to the required order. Since the sources and vevs only depends on r so we have that $\Box\psi = 0$ implies that

$$\sqrt{-\mathcal{G}}r^2\partial_r\psi = C\,, \tag{5.53}$$

where C is an integration constant. Using that

$$\sqrt{-\mathcal{G}} = 1 + O(r^6)\,, \tag{5.54}$$

it follows that

$$r^{-3}\partial_r\psi = C\left(1 + O(r^6)\right)\,, \tag{5.55}$$

so that the fact that $\psi_{(4)} = 0$ implies that $C = 0$. Hence $\partial_r\psi = 0$ or in other words $\psi = 2u$ to all orders. With this result the Einstein equation simplifies to

$$G_{\mathcal{MN}} = 6\mathcal{G}_{\mathcal{MN}} + 2\delta^u_{\mathcal{M}}\delta^u_{\mathcal{N}} - \mathcal{G}_{\mathcal{MN}}\gamma^{uu}\,, \tag{5.56}$$

which is equivalent to

$$R_{\mathcal{MN}} = -4\mathcal{G}_{\mathcal{MN}} + 2\delta^u_{\mathcal{M}}\delta^u_{\mathcal{N}}\,. \tag{5.57}$$

To find the solution up to order r^6 we make the following ansatz

$$\gamma_{AB} = r^{-2}\left(\gamma_{(0)AB} + r^2\delta^u_A\delta^u_B - \frac{1}{4}r^4 t_{AB} + r^6\gamma_{(6)AB} + r^8\gamma_{(8)AB} + O(r^{10})\right)\,. \tag{5.58}$$

The log terms at order $r^2\log r$ are zero and so it is expected that they are zero to all orders. This is a correct assumption as long as we do not need to put constraints on the sources and vevs coming from the nature of the expansion. The inverse metric reads

$$\gamma^{AB} = r^2\left(\gamma^{AB}_{(0)} - r^2\gamma^{Au}_{(0)}\gamma^{Bu}_{(0)} + \frac{1}{4}r^4 t^{AB} + r^6\sigma^{AB}_{(6)} + r^8\sigma^{AB}_{(8)} + O(r^{10})\right)\,, \tag{5.59}$$

where

$$\sigma^{AB}_{(6)} = -\gamma^{AB}_{(6)} - \frac{1}{4}\gamma^{Au}_{(0)}t^{uB} - \frac{1}{4}\gamma^{Bu}_{(0)}t^{uA}\,, \tag{5.60}$$

$$\sigma^{AB}_{(8)} = -\gamma^{AB}_{(8)} + \gamma^{Au}_{(0)}\gamma^{uB}_{(6)} + \gamma^{Bu}_{(0)}\gamma^{uA}_{(6)} + \frac{1}{16}t^{AC}t_C{}^B + \frac{1}{4}\gamma^{Au}_{(0)}\gamma^{Bu}_{(0)}t^{uu}\,. \tag{5.61}$$

The Christoffel symbols are

$$\Gamma^r_{rr} = -\frac{1}{r}\,, \qquad \Gamma^r_{rA} = 0\,, \qquad \Gamma^r_{AB} = -\frac{1}{2}r^2\partial_r\gamma_{AB}\,,$$
$$\Gamma^A_{rr} = 0\,, \qquad \Gamma^A_{rB} = \frac{1}{2}\gamma^{AC}\partial_r\gamma_{BC}\,, \qquad \Gamma^A_{BC} = 0\,. \tag{5.62}$$

From this we conclude that

$$R_{rr} = -4r^{-2}+r^4\left(-12\gamma^A_{(6)A} - 2t^{uu}\right)+r^6\left(-24\gamma^A_{(8)A} + 18\gamma^{uu}_{(6)} + \frac{1}{2}t^{AB}t_{AB}\right)+O(r^8)\,. \tag{5.63}$$

The rr component of (5.57) tells us that $R_{rr} = -4r^{-2}$ so that

$$\gamma^A_{(6)A} = -\frac{1}{6}t^{uu}\,, \qquad \gamma^A_{(8)A} = \frac{3}{4}\gamma^{uu}_{(6)} + \frac{1}{48}t^{AB}t_{AB}\,. \tag{5.64}$$

The rA component of (5.57) brings nothing as both sides are identically zero. Using that

$$\begin{aligned} R_{AB} \;&=\; -4r^{-2}\gamma_{(0)AB} - 2\delta^u_A\delta^u_B + r^2 t_{AB} + r^4\left(\frac{1}{4}t^{uu}\gamma_{(0)AB} - 10\gamma_{(6)AB} - \frac{1}{2}\delta^u_A t^u{}_B - \frac{1}{2}\delta^u_B t^u{}_A\right) \\ &\quad +r^6\left(-20\gamma_{(8)AB} - \gamma^{uu}_{(6)}\gamma_{(0)AB} - \frac{1}{24}t^{CD}t_{CD}\gamma_{(0)AB} + 6\delta^u_A\gamma^u_{(6)B} + 6\delta^u_B\gamma^u_{(6)A}\right. \\ &\quad \left.+\frac{1}{2}t_A{}^C t_{CB} + \frac{1}{2}\delta^u_A\delta^u_B t^{uu}\right) + O(r^8)\,, \end{aligned} \tag{5.65}$$

as well as the equation of motion $R_{AB} = -4\gamma_{AB} + 2\delta^u_A\delta^u_B$, we find that

$$\gamma_{(6)AB} \;=\; -\frac{1}{6}\delta^u_A t^u{}_B - \frac{1}{6}\delta^u_B t^u{}_A + \frac{1}{24}t^{uu}\gamma_{(0)AB}\,, \tag{5.66}$$
$$\gamma_{(8)AB} \;=\; -\frac{1}{16}t^{uu}\delta^u_A\delta^u_B - \frac{1}{384}t^{CD}t_{CD}\gamma_{(0)AB} + \frac{1}{32}t_A{}^C t_{BC}\,. \tag{5.67}$$

From the reduction (5.46)–(5.48) it follows that

$$\Phi \;=\; -\frac{1}{8}r^2\rho + r^4\left(\frac{1}{6}\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma - \frac{1}{64}\rho^2\right) + O(r^6)\,, \tag{5.68}$$
$$A_\mu \;=\; r^{-2}\tau_\mu + \frac{1}{4}\rho\tau_\mu + r^2\left(\frac{1}{12}\tau_\rho T^\rho{}_\mu + \frac{1}{16}\rho^2\tau_\mu - \frac{1}{3}\bar{h}_{\mu\rho}T^\rho\right) + O(r^4)\,, \tag{5.69}$$
$$\begin{aligned} h_{\mu\nu} \;&=\; -r^{-4}\tau_\mu\tau_\nu + r^{-2}\left(\bar{h}_{\mu\nu} - \frac{1}{8}\rho\tau_\mu\tau_\nu\right) - \frac{1}{8}\rho\bar{h}_{\mu\nu} + \frac{1}{4}\left(\tau_\mu\tau_\rho T^\rho{}_\nu + \tau_\nu\tau_\rho T^\rho{}_\mu\right) \\ &\quad -\left(\frac{3}{128}\rho^2 - \frac{1}{6}\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma\right)\tau_\mu\tau_\nu + r^2\left(-\frac{1}{4}\hat{h}_{\mu\rho}\hat{h}_{\nu\kappa}h^{\kappa\sigma}T^\rho{}_\sigma + \frac{1}{12}\left(\tau_\nu\hat{h}_{\mu\rho} + \tau_\mu\hat{h}_{\nu\rho}\right)\hat{v}^\sigma T^\rho{}_\sigma\right. \\ &\quad +\left(\frac{1}{6}\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma - \frac{1}{128}\rho^2\right)\bar{h}_{\mu\nu} - \frac{1}{16}\rho\left(\tau_\mu\tau_\rho T^\rho{}_\nu + \tau_\nu\tau_\rho T^\rho{}_\mu\right) \\ &\quad \left.+\left(\frac{3}{64}\rho\hat{v}^\sigma\tau_\rho T^\rho{}_\sigma + \frac{1}{64}T^\sigma\tau_\rho T^\rho{}_\sigma - \frac{5}{1024}\rho^3 - \frac{1}{32}t^{uu}\right)\tau_\mu\tau_\nu\right) + O(r^4)\,. \end{aligned} \tag{5.70}$$

For the interested reader we have included the term t^{uu}. But as said earlier we will set this independent quantity equal to zero. If we choose the boundary sources to correspond to a flat TNC boundary as in (5.45) then the expansions become

$$
\begin{aligned}
\Phi &= -\frac{1}{8}r^2\rho - r^4\left(\frac{1}{6}T^t{}_t + \frac{1}{64}\rho^2\right) + O(r^6)\,, && (5.71)\\
A_t &= r^{-2} + \frac{1}{4}\rho + r^2\left(\frac{1}{12}T^t{}_t + \frac{1}{16}\rho^2\right) + O(r^4)\,, && (5.72)\\
A_i &= -\frac{1}{4}r^2 T^t{}_i + O(r^4)\,, && (5.73)\\
h_{tt} &= -r^{-4} - \frac{1}{8}r^{-2}\rho + \frac{1}{3}T^t{}_t - \frac{3}{128}\rho^2 + O(r^2)\,, && (5.74)\\
h_{ti} &= \frac{1}{4}T^t{}_i + r^2\left(-\frac{1}{12}\delta_{ij}T^j{}_t + \frac{1}{32}\rho T^t{}_i\right) + O(r^4)\,, && (5.75)\\
h_{ij} &= r^{-2}\delta_{ij} - \frac{1}{8}\rho\delta_{ij} + r^2\left(\left(\frac{1}{12}T^t{}_t - \frac{1}{128}\rho^2\right)\delta_{ij} - \frac{1}{4}\delta_{ik}T^k{}_j + \frac{1}{8}T^k{}_k\delta_{ij}\right) + O(r^4)\,, && (5.76)
\end{aligned}
$$

where in the last expression we used the z-deformed trace Ward identity (equation (4.135) with zero on the right hand side)

$$
2T^t{}_t + T^k{}_k = 0\,. \tag{5.77}
$$

In this work we are interested in gravitational duals of boundary perfect fluids so without loss of generality we can assume that $T^\mu{}_\nu$ takes the form of a perfect fluid. This form is derived in appendix 4.2.5 by the null reduction of a relativistic perfect fluid. On flat TNC space-time it reads

$$
T^t{}_t = -\left(\mathcal{E} + \frac{1}{2}\rho V^2\right)\,,\qquad T^i{}_t = -\left(\mathcal{E} + P + \frac{1}{2}\rho V^2\right)V^i\,, \tag{5.78}
$$

$$
T^t{}_i = \rho V_i\,,\qquad T^j{}_i = \left(P\delta^j_i + \rho V^j V_i\right)\,, \tag{5.79}
$$

where $\mathcal{E}$ is the total energy density, P the pressure, ρ the mass density and V^i the velocity of the fluid. The z-deformed trace Ward identity tells us that the equation of state is $P = \mathcal{E}$.

It is interesting and insightful to take a closer at look at this V-dependent solution from the 5-dimensional point of view. Using the relations between the lower and higher-dimensional energy-momentum tensors (5.49)–(5.51) we see that the 5-dimensional energy-momentum is given by

$$
\begin{aligned}
& t_{uu} = \rho\,,\quad t_{ut} = -\mathcal{E} - \frac{1}{2}\rho V^2\,,\quad t_{ui} = \rho V_i\,,\\
& t_{ti} = -\left(\mathcal{E} + P + \frac{1}{2}\rho V^2\right)V_i\,,\quad t_{ij} = P\delta_{ij} + \rho V_i V_j\,,
\end{aligned} \tag{5.80}
$$

with t_{tt} being undetermined. A convenient way of writing this is in terms of $t_{AB}dx^A dx^B$ which can be seen to be equal to

$$
\begin{aligned}
t_{AB}dx^A dx^B &= \rho\left(du + V_i dx^i - \frac{1}{2}V^2 dt\right)^2 - 2\mathcal{E}dt\left(du + V_i dx^i - \frac{1}{2}V^2 dt\right) \qquad (5.81)\\
&\quad + P\delta_{ij}\left(dx^i - V^i dt\right)\left(dx^j - V^j dt\right) + \left(t_{tt} - \left(\mathcal{E} + P + \frac{1}{2}\rho V^2\right)V^2\right)dt^2\,.
\end{aligned}
$$

The rest of the solution is fully determined by the following boundary data

$$\gamma_{(0)AB}dx^A dx^B = 2dtdu + \delta_{ij}dx^i dx^j\,, \tag{5.82}$$

$$\psi_{(0)} = 2u\,, \tag{5.83}$$

$$\langle O_\psi \rangle = 0\,. \tag{5.84}$$

If we now perform the following coordinate transformation, which from a lower dimensional point of view is a Galilean boost and a $U(1)$ gauge transformation (acting on the Kaluza–Klein vector m_μ),

$$u = u' - \frac{1}{2}V^2 t' - V_i x'^i\,, \qquad t = t'\,, \qquad x^i = x'^i + V^i t'\,, \tag{5.85}$$

we obtain

$$\begin{aligned} t_{AB}dx^A dx^B &= \rho du'^2 - 2\mathcal{E}dt'du' + P\delta_{ij}dx'^i dx'^j \\ &\quad + \left(t_{tt} - \left(\mathcal{E} + P + \frac{1}{2}\rho V^2\right)V^2\right)dt'^2\,, \end{aligned} \tag{5.86}$$

$$\gamma_{(0)AB}dx^A dx^B = 2dt'du' + \delta_{ij}dx'^i dx'^j\,, \tag{5.87}$$

$$\psi_{(0)} = 2u' - V^2 t' - 2V_i x'^i\,, \tag{5.88}$$

$$\langle O_\psi \rangle = 0\,. \tag{5.89}$$

We thus see that the boundary metric $\gamma_{(0)AB}$ remained invariant and that all the V-dependence now resides in the expression for $\psi_{(0)}$. The $t't'$ component of t_{AB} is not important for the lower dimensional boundary EMT and its Ward identities. It is thus clear that due to the presence of ψ, and the Scherk–Schwarz reduction ansatz $\psi = 2u + \Xi$, solutions with different V^i are not diffeomorphic. We will later see this reflected in the fact that V^2 plays the role of a chemical potential. The ansatz in section 5.2.1 used rotations to orient the flow in the y-directions. We will see further below that indeed $V^x = 0$.

5.2.3 The near horizon solution

The near horizon expansion is entirely straightforward. Referring to the ansatz (5.26) and (5.27) in EF coordinates we can make the following observations about the behavior of the solution near the horizon.

The horizon is located at the locus where the $r = \text{cst}$ hypersurface becomes null, i.e. at $g^{rr} = 0$. That means that F_2 will have a first order zero at $r = r_h$. Regularity of the metric in EF coordinates, in particular of the component g_{Vr} then tells us that F_1 must also have a first order zero at $r = r_h$. Note that for $N \neq 0$ this is not the locus where ∂_t becomes null. In other words the stationary limit surface $g_{tt} = 0$ comes before the horizon (viewed from outside). Regularity of the massive vector at horizon forces G_1 to have a first order zero at $r = r_h$. The functions F_4 and N are both regular without any zeros at the horizon, i.e. $F_4(r_h) \neq 0$ and $N(r_h) \neq 0$. The latter quantity can be zero but as we will see in the next subsection that corresponds to a brane without any momentum so we take it to be nonzero. The remaining functions G_2 and Φ are regular at the horizon, but they do not have to be non-vanishing.

A convenient gauge choice to fix the r reparametrization invariance of the ansatz to study the near horizon solution is to take $F_3 = 1$. In this gauge we will refer to the radial coordinate as R to distinguish it from the radial coordinate r used in the previous subsection[10]. The horizon is now located at $R = R_h$.

The ansatz also has three global scale symmetries (5.31), (5.32) and (5.38) that leave the ansatz invariant. These can be viewed as rescalings of x, y and t. We have used these symmetries to set the asymptotic values of F_1, F_3 and F_4 equal to one. This fixes the asymptotic values of Φ and thus of F_2 (via the asymptotic gauge choice $F_2 = e^{-\Phi}$) as well as of G_1 via the equations of motion. That means that we cannot use these rescaling symmetries a second time to fix parameters in the near horizon solution. We thus take for the near horizon solution the following expansion

$$F_1 = f_1 \frac{R - R_h}{R_h} + \dots\,, \tag{5.90}$$

$$F_2 = h_1 \frac{R - R_h}{R_h} + \dots\,, \tag{5.91}$$

$$F_3 = 1\,, \tag{5.92}$$

$$F_4 = p_0 + p_1 \frac{R - R_h}{R_h} + \dots\,, \tag{5.93}$$

$$N = n_0 + n_1 \frac{R - R_h}{R_h} + \dots\,, \tag{5.94}$$

$$G_1 = g_1 \frac{R - R_h}{R_h} + \dots\,, \tag{5.95}$$

$$G_2 = m_0 + m_1 \frac{R - R_h}{R_h} + \dots\,, \tag{5.96}$$

$$\Phi = l_0 + l_1 \frac{R - R_h}{R_h} + \dots\,. \tag{5.97}$$

Most but not all of the coefficients appearing in the near horizon expansion will be determined by solving the equations of motion of the effective action L in an expansion around $R = R_h$. We studied the solution up to second order in $R - R_h$ and it leaves 8 parameters unfixed. These are f_1, p_0, g_1, m_0, n_0, n_1, l_0 and r_h. The parameter h_1 is fixed by the equations of motion to be[11]

$$h_1 = \frac{2f_1\left(2e^{-3l_0} - 12e^{-l_0}\right)}{4f_1 - e^{3l_0}\left(g_1 + m_0 n_1\right)^2}\,, \tag{5.98}$$

where the numerator is $2f_1$ times the potential (5.17) evaluated at $R = R_h$. We expect that most of these parameters will be determined by matching the solution onto the asymptotic region.

There are not many examples known of analytic black branes solutions of the EPD model. However in the context of Schrödinger space-times we can obtain analytic solutions by applying a sequence of duality transformations known as TsT transformations [146] to obtain black brane solutions from known AdS black branes [1, 121, 116]. The resulting

[10] We permit ourselves to also use r for the family of gauges parametrized by the ansatz (5.26) and (5.27). We hope that this will not cause any confusion.

[11] To find this result one solves the leading term of the F_1 equation of motion for p_1 and the leading term of the F_3 equation of motion for f_2. The expression then follows from the leading term in the F_4 equation of motion. A similar expression has been observed in [34].

Schrödinger black branes have a nonzero charge associated with particle number. Since in Schrödinger holography particle number is realized geometrically this means that these correspond to black branes with a linear momentum along a direction that asymptotically becomes null. If we study these black branes near the horizon in the same coordinates in which the AdS black brane has a flat boundary Minkowski metric written in Cartesian coordinates then we see the exact same near horizon boundary conditions as we imposed for our Lifshitz black brane[12].

5.2.4 Comments on the interpolating solution

We have used different radial gauges in the near horizon region ($F_3 = 1$) and in the asymptotic region ($F_2 = e^{-\Phi}$). The two coordinates are related via the coordinate transformation

$$h_{xx} = R^{-2}\,, \tag{5.99}$$

where h_{xx} is given in (5.76). In order to write both the near horizon and the near boundary expansion in the same gauge it is convenient to rewrite the expansions (5.71)–(5.76) in terms of the radial coordinate R. This can be done as follows. The expansions (5.71)–(5.76) in terms of the ansatz functions correspond to

$$F_1 = 1 + \frac{1}{8}r^2\rho + r^4\left(-\frac{1}{3}T^t{}_t + \frac{3}{128}\rho^2\right) + O(r^6)\,, \tag{5.100}$$

$$F_2 = e^{-\Phi} = 1 + \frac{1}{8}r^2\rho + r^4\left(\frac{1}{6}T^t{}_t + \frac{3}{128}\rho^2\right) + O(r^6)\,, \tag{5.101}$$

$$F_3 = 1 - \frac{1}{8}r^2\rho + r^4\left(\frac{1}{12}T^t{}_t - \frac{1}{128}\rho^2 + \frac{1}{8}\rho V^2\right) + O(r^6)\,, \tag{5.102}$$

$$F_4 = 1 - \frac{1}{8}r^2\rho + r^4\left(\frac{1}{12}T^t{}_t - \frac{1}{128}\rho^2 - \frac{1}{8}\rho V^2\right) + O(r^6)\,, \tag{5.103}$$

$$N = \frac{1}{4}r^3\rho V + O(r^5)\,, \tag{5.104}$$

$$G_1 = 1 + \frac{1}{4}r^2\rho + r^4\left(\frac{1}{12}T^t{}_t + \frac{1}{16}\rho^2\right) + O(r^6)\,, \tag{5.105}$$

$$G_2 = -\frac{1}{4}r^3\rho V + O(r^5)\,, \tag{5.106}$$

$$\Phi = -\frac{1}{8}r^2\rho - r^4\left(\frac{1}{6}T^t{}_t + \frac{1}{64}\rho^2\right) + O(r^6)\,, \tag{5.107}$$

where we remind that $V = V^y$ and $V^x = 0$. The change of gauge (5.99) implies that we define R asymptotically as

$$R^{-2} = r^{-2}\left(1 - \frac{1}{8}r^2\rho + r^4\left(\frac{1}{12}T^t{}_t - \frac{1}{128}\rho^2 + \frac{1}{8}\rho V^2\right) + O(r^6)\right)\,. \tag{5.108}$$

[12]More explicitly if we use equation (62) of [116] setting $\xi = V$ the TsT transformation (113)–(115) provides us with a $z = 2$ Schrödinger black brane solution of some EPD model. If we then perform the coordinate transformation $t = T - X$ and $2\xi = 2V = T + X$ we find that the near horizon geometry has exactly the same properties as the Lifshitz black brane solution studied here.

We can invert this order by order to obtain $r = r(R)$ up to any desired power of R. Inverting (5.108) up to order R^6 we find

$$r = R\left(1 - \frac{1}{16}R^2\rho + \frac{1}{8}R^4\left(\frac{1}{3}T^t{}_t + \frac{1}{64}\rho^2 + \frac{1}{2}\rho V^2\right) + O(R^6)\right). \tag{5.109}$$

This can be used to express (5.21)–(5.23) with the above expansions for the various functions as an asymptotic solution that is written in terms of the same radial coordinate R as the near horizon solution. If we carry out these steps we obtain the following expressions for the ansatz functions in the new gauge

$$\begin{aligned}
F_1 &= 1 + \frac{3}{8}R^2\rho + \frac{1}{2}R^4\left(-T^t{}_t + \frac{9}{64}\rho^2 - \frac{1}{2}\rho V^2\right) + O(R^6)\,, && (5.110)\\
F_2 &= 1 + \frac{3}{8}R^2\rho + R^4\left(-\frac{1}{6}T^t{}_t + \frac{11}{128}\rho^2 - \frac{1}{2}\rho V^2\right) + O(R^6)\,, && (5.111)\\
F_3 &= 1\,, && (5.112)\\
F_4 &= 1 - \frac{1}{4}R^4\rho V^2 + O(R^6)\,, && (5.113)\\
N &= \frac{1}{4}R^3\rho V + O(R^5)\,, && (5.114)\\
G_1 &= 1 + \frac{3}{8}R^2\rho + \frac{1}{2}R^4\left(\frac{9}{64}\rho^2 - \frac{1}{4}\rho V^2\right) + O(R^6)\,, && (5.115)\\
G_2 &= -\frac{1}{4}R^3\rho V + O(R^5)\,, && (5.116)\\
\Phi &= -\frac{1}{8}R^2\rho - \frac{1}{6}R^4T^t{}_t + O(R^6)\,. && (5.117)
\end{aligned}$$

In order to find an interpolating solution we thus need to solve the equations of motion of (5.28) in the $F_3 = 1$ gauge such that near the horizon the solution looks like (5.90)–(5.97) while near the boundary it looks like (5.110)–(5.117). It would be interesting to study the interpolating solution numerically. For the purposes of this work we do not need this explicit solution, but we will need to assume that it exists.

We also see from the asymptotic solution that even for $V = 0$ we still can turn on the ρ deformation. Hence static Lifshitz black branes can have a nonzero mass density. Further even though the full non-linear solution breaks rotational symmetries the near boundary solution has an asymptotic Killing vector for rotations. Hence rotations are spontaneously broken.

5.2.5 Thermodynamics

The most general Killing vector that (5.21) admits is of the form

$$K^M = (\partial_t)^M + A_1\,(\partial_x)^M + A_2\,(\partial_y)^M\,, \tag{5.118}$$

where A_1 and A_2 are constants. The norm is given by

$$||K||^2 = -\frac{F_1}{r^4} + \frac{F_4}{r^2}\left(\frac{N}{r} + A_2\right)^2 + A_1^2\frac{F_3}{r^2}\,. \tag{5.119}$$

In order to find the generator of the horizon we demand that $||K||^2$ vanishes at $R = R_h$ which will be the case if and only if

$$A_1 = 0\,, \qquad A_2 = -\frac{N(R_h)}{R_h}\,. \tag{5.120}$$

Hence the horizon generator which we will denote by X^M is given by

$$X^M = (\partial_t)^M - \frac{N(R_h)}{R_h}(\partial_y)^M\,. \tag{5.121}$$

We thus see that there is a chemical potential $-N(R_h)/R_h$ associated with the motion in the y-direction.

The metric and vector field expanded near the horizon read

$$ds_4^2 = -\tilde{\rho}^2 d\tilde{t}^2 + d\tilde{\rho}^2 + \frac{1}{R_h^2}dx^2 + \frac{p_0}{R_h^2}\left(dy + \frac{N(R_h)}{R_h}dt\right)^2\,, \tag{5.122}$$

$$B = \frac{1}{2}g_1\left(h_1 f_1^{-1}\right)^{1/2}\tilde{\rho}^2 d\tilde{t} + \frac{m_0}{R_h}\left(dy + \frac{N(R_h)}{R_h}dt\right)\,, \tag{5.123}$$

where we defined

$$\tilde{\rho} = 2\left(\frac{R - R_h}{h_1 R_h}\right)^{1/2}\,, \tag{5.124}$$

$$\tilde{t} = \frac{1}{2}(f_1 h_1)^{1/2} R_h^{-2} t\,. \tag{5.125}$$

We next ask which linear functions $f(t, x, y)$ solve the equation $X^M \partial_M f = 0$. These are x and $y + \frac{N(R_h)}{R_h}t$. The metric induced on the common intersection of the hyperplanes $x = \text{cst}$ and $y + \frac{N(R_h)}{R_h}t = \text{cst}$, after Wick rotating the time coordinate $t = -it_E$, is called the bolt and is given by

$$ds^2|_{\text{bolt}} = \frac{F_1}{R^4}dt_E^2 + \frac{dR^2}{F_2 R^2}\,. \tag{5.126}$$

We expand this metric around $R = R_h$ with a periodic t_E demanding the absence of conical singularities. Because we are on the hyperplane $y + \frac{N(R_h)}{R_h}t = \text{cst}$ this forces us to also Wick rotate $y = -iy_E$ and make it periodic as well in agreement with the interpretation of $-N(R_h)/R_h$ as a chemical potential. The inverse temperature is the periodicity of t_E. The temperature and entropy density are given by

$$T = \frac{1}{4\pi R_h^2}(f_1 h_1)^{1/2}\,, \tag{5.127}$$

$$s = 4\pi\frac{(p_0)^{1/2}}{R_h^2}\,, \tag{5.128}$$

where we used units in which $16\pi G_N = 1$.

In the Wick rotated geometry t_E and y_E are periodic. The thermal cycle parametrized by t_E is contractible while the cycle parametrized by $y_E + \frac{N(R_h)}{R_h}t_E$ is non-contractible. Hence we can compute $\oint_{R=R_h} B$ where we integrate along the cycle parametrized by $y_E + \frac{N(R_h)}{R_h} t_E$. The result is

$$\oint_{R=R_h} B = \frac{4\pi m_0 n_0}{(f_1 h_1)^{1/2}} \,. \tag{5.129}$$

In general using our ansatz the massive vector field can be written as

$$B^M = -R^2 \frac{G_1}{F_1} \left((\partial_t)^M - \frac{N}{R} (\partial_y)^M \right) + R \frac{G_2}{F_4} (\partial_y)^M \,. \tag{5.130}$$

It thus follows that for $m_0 = 0$ the massive vector field B^M is proportional to the horizon generator X^M at $R = R_h$.

It can be shown by using the near horizon solution that the charges Q_λ and Q_μ (5.35) and (5.36) are such that

$$Q_\lambda - \frac{3}{2} Q_\mu = Ts \,. \tag{5.131}$$

Using the asymptotic form of the solution (5.71)–(5.76) with (5.78) and (5.79) to compute the left hand side of (5.131) we conclude that

$$\mathcal{E} + P = Ts + \frac{1}{2} \rho V^2 \,. \tag{5.132}$$

The equations of state follows from (5.77)

$$P = \mathcal{E} \,. \tag{5.133}$$

We have thus been able to derive the thermodynamic relations without knowing the full solution analytically using the Noether charges Q_λ and Q_μ. This is similar to what has been done in [34, 35]. We will see further below that we can also derive the first law of thermodynamics without having full analytic control of the solution. All that we need to know is the near horizon expansion, the near boundary expansion and the existence of an interpolating solution. The latter we assume to be the case. It would be interesting to provide numerical evidence for the interpolating solution.

5.2.6 Charges

The goal of this subsection is to find an alternative derivation of (5.132) which can be thought of as an integral form in terms of the renormalized on-shell action and certain horizon charges. The second goal is to find additional relations between near boundary and near horizon quantities. In particular we will show that the velocity $V^y = V$ is equal to the chemical potential $-N(R_h)/R_h$.

In order to define the black brane charges we use the boundary diffeomorphism Ward identity which on a flat TNC geometry reads (4.275). Given a boundary Killing vector K^μ in the sense that

$$\mathcal{L}_K \tau_\mu = 0 \,, \qquad \mathcal{L}_K \bar{h}_{\mu\nu} = 0 \,, \qquad \mathcal{L}_K \tilde{\Phi} = 0 \,, \tag{5.134}$$

it can be shown (see [112, 110]) that we find the conserved current

$$\partial_\mu \left(K^\nu T^\mu{}_\nu \right) = 0 \,. \tag{5.135}$$

The conserved charge associated with the boundary Killing vector K^μ is thus

$$Q_K = -\int_{t=\text{cst}} dxdy K^\nu T^t{}_\nu \,. \tag{5.136}$$

For our case the integrand is independent of x and y and so it is better to consider the charge per unit boundary volume. We will often write $\int_{t=\text{cst}} dxdy$ as a formal integral that we never really perform. We can always divide the charges by it. We will assume that K^μ is the μ component of a bulk Killing vector K^M.

Using the definitions of the vevs in (4.231), (4.232) and (4.244), (4.245)[13], as well as the boundary energy-momentum tensor in (4.260) and (4.273) we find that

$$T^t{}_\nu = -\lim_{r\to 0} r^{-2}\left(T_{\mu\nu}E^\nu_0 + \mathcal{T}^\rho E^0_\rho B_\mu\right) \,. \tag{5.137}$$

Using (4.224) and (4.226) we find that for purely radial solutions (no dependence on boundary coordinates)

$$T_{\mu\nu}E^\nu_0 + \mathcal{T}^\rho E^0_\rho B_\mu = \frac{1}{\sqrt{-h}}\mathcal{L}^{\text{os}}_{\text{bdry}}E^0_\mu + 2K_{\mu\nu}E^\nu_0 + e^{3\Phi}n^M E^\nu_0 F_{M\nu}B_\mu \,, \tag{5.138}$$

where $\mathcal{L}^{\text{os}}_{\text{bdry}}$ is the on-shell value of the counterterm Lagrangian (4.216) including the Gibbons–Hawking boundary term, i.e.

$$\mathcal{L}^{\text{os}}_{\text{bdry}} = \sqrt{-h}\left(2K - 5e^{-\Phi/2} + e^{\Phi/2}B_\rho B^\rho\right) \,. \tag{5.139}$$

The extrinsic curvature $K = h^{\mu\nu}K_{\mu\nu}$ where $K_{\mu\nu}$ is the $\mu\nu$ component of $K_{MN} = -\frac{1}{2}\mathcal{L}_n h_{MN} = \nabla_M n_N - n_M n^K \nabla_K n_N$ with the unit normal vector n_M given by $n_M = -(g^{rr})^{-1/2}\delta^r_M$. Since the Killing vector K^M is a boundary Killing vector we have $K^M n_M = 0$. Further we employ a radial gauge choice such that $E^M_0 n_M = 0$. Using these results we can write

$$Q_K = \int_{t=\text{cst}} dxdy \lim_{r\to 0} r^{-2}\left(\frac{1}{\sqrt{-h}}\mathcal{L}^{\text{os}}_{\text{bdry}}K^M E^0_M + n^M E^{N0} Z_{NM}\right) \,, \tag{5.140}$$

where $Z_{NM} = -Z_{MN}$[14] is given by

$$Z_{NM} = 2\nabla_N K_M + e^{3\Phi}F_{NM}K^P B_P \,. \tag{5.141}$$

The integrand is over a $t = \text{cst}$ hypersurface. Its timelike unit normal is given by

$$u_M = U\delta^t_M \,, \qquad U = r^{-2}\left(1 + \frac{1}{16}r^2\rho + r^4\left(-\frac{1}{6}T^t{}_t + \frac{5}{2}\frac{1}{256}\rho^2\right) + O(r^6)\right) \,, \tag{5.142}$$

where we used the boundary expansions of section 5.2.2. It can be shown using these same expansions that

$$U^t = E^{t0} + O(r^8) \,, \qquad U^i = O(r^4) \,, \tag{5.143}$$

[13] The quantity $\alpha_{(0)}$ defined in (4.250) equals unity because for our solutions $\Phi = O(r^2)$ so that $\phi = 0$ as follows from (4.240).

[14] The antisymmetry follows from the fact that K_M is also assumed to be a bulk Killing vector.

where $E^{t0} = U^{-1} + O(r^8)$. We also have that $E^{i0} = O(r^4)$. Using these results together with the near boundary expansion of Z_{NM} it can be proven that we can replace E^{N0} by u^N everywhere in the integrand of Q_K, i.e. we can write

$$Q_K = \int_{t=\text{cst}} dxdy \lim_{r\to 0} r^{-2} \left(\frac{1}{\sqrt{-h}} \mathcal{L}^{\text{os}}_{\text{bdry}} K^M u_M + n^M u^N Z_{NM} \right) . \tag{5.144}$$

Let us define the projector $P^N_M = \delta^N_M + u_M u^N$ which projects onto the $t = \text{cst}$ hypersurface whose metric we will denote by H_{IJ}, i.e. using the ADM decomposition we obtain

$$ds^2 = -U^2 dt^2 + H_{IJ} \left(dx^I + u^I dt \right) \left(dx^J + u^J dt \right) . \tag{5.145}$$

Let us furthermore define $Z^M = u^N Z_{NM}$. We can derive the following identity

$$P^N_M \nabla_N Z^M = \frac{1}{\sqrt{-H}} \partial_I \left(\sqrt{H} Z^I \right) . \tag{5.146}$$

Hence it follows that

$$\int_{t=\text{cst}} d^3x \sqrt{H} P^N_M \nabla_N Z^M = \int_{t=\text{cst}} dxdy \int_\epsilon^{R_h} dR \partial_R \left(\sqrt{H} Z^R \right) , \tag{5.147}$$

where we used the radial R coordinate of section 5.2.4, i.e. the $F_3 = 1$ gauge, with a cut-off boundary at $R = \epsilon$ and the horizon at $R = R_h$. The integration measure in terms of the ansatz functions can be written as

$$\sqrt{H} = r^{-3} F_2^{-1/2} F_4^{1/2} . \tag{5.148}$$

It follows that

$$\begin{aligned} \int_{t=\text{cst}} d^3x \sqrt{H} P^N_M \nabla_N Z^M &= \int_{t=\text{cst}} dxdy R^{-2} n_M Z^M|_{R=\epsilon} \\ &+ \int_{t=\text{cst}} dxdy R^{-3} F_2^{-1/2} F_4^{1/2} Z^R|_{R=R_h} . \end{aligned} \tag{5.149}$$

We conclude that the charge Q_K can be written as

$$\begin{aligned} Q_K &= \int_{t=\text{cst}} dxdy R^{-2} \frac{1}{\sqrt{-h}} \mathcal{L}^{\text{os}}_{\text{bdry}} K^M U_M|_{R=\epsilon} + \int_{t=\text{cst}} dxdy \int_\epsilon^{R_h} dR \sqrt{H} P^N_M \nabla_N Z^M \\ &+ \int_{t=\text{cst}} dxdy R^{-2} F_4^{1/2} X^N Y^P Z_{NP}|_{R=R_h} , \end{aligned} \tag{5.150}$$

where we send ϵ to zero. In the horizon integral X^P is the horizon generator (5.121) and Y^N is given by (in the EF coordinates of (5.26) with $F_3 = 1$)

$$Y^N = R^3 \left(\frac{F_2}{F_1} \right)^{1/2} \delta^N_R . \tag{5.151}$$

The vector Y is a null vector that satisfies $X \cdot Y = -1$ at the horizon.

Using the equations of motion (5.4)–(5.6) with (5.17) as well as the fact that the Killing vector K is a symmetry of the matter fields which means that

$$\mathcal{L}_K B_M = 0\,, \qquad \mathcal{L}_K \Phi = 0\,, \tag{5.152}$$

it can be shown that

$$P_M^N \nabla_N Z^M = \frac{1}{\sqrt{-g}} \mathcal{L}_{\text{bulk}}^{\text{os}} K^M u_M\,. \tag{5.153}$$

The charge can now be written as

$$Q_K = K^t T S_E^{\text{os}} + \int_{t=\text{cst}} d^2x \sqrt{\sigma} X^N Y^P Z_{NP}|_{R=R_h}\,, \tag{5.154}$$

where we used that $K^M u_M = K^t U$ with $\sqrt{-g} = U\sqrt{H}$ and where we defined $\sqrt{\sigma} = F_4^{1/2} R^{-2}$ which is the determinant of the metric on the $t = \text{cst}$ and $R = \text{cst}$ submanifold. In this expression for the charge S_E^{os} is the Euclidean on-shell action, i.e.

$$TS_E^{\text{os}} = \lim_{\epsilon\to 0}\left[\int_\epsilon^{R_h} dR \int_{t=\text{cst}} d^2x \mathcal{L}_{\text{bulk}}^{\text{os}} + \int_{t=\text{cst}} d^2x \mathcal{L}_{\text{bdry}}^{\text{os}}|_{R=\epsilon}\right]\,. \tag{5.155}$$

Equation (5.154) is the result we were looking for. It expresses the asymptotic charge associated with the Killing vector K^M in terms of a horizon integral and the Euclidean on-shell action.

This result can be used to compute the charges associated with the Killing vectors ∂_t and ∂_y twice, once near the boundary using (5.136) and once at the horizon using (5.154). Near the boundary we find

$$\begin{aligned} Q_{\partial_t} &= -\int_{t=\text{cst}} dxdy T^t{}_t = \int_{t=\text{cst}} dxdy \left(\mathcal{E} + \frac{1}{2}\rho V^2\right)\,, & (5.156)\\ Q_{\partial_y} &= -\int_{t=\text{cst}} dxdy T^t{}_y = -\int_{t=\text{cst}} dxdy \rho V\,. & (5.157)\end{aligned}$$

Using (5.154) we can derive the following relation

$$Q_{\partial_t} - TS_E^{\text{os}} = T\int_{t=\text{cst}} dxdys - \frac{N(R_h)}{R_h} Q_{\partial_y}\,, \tag{5.158}$$

where we used (5.127) and (5.128).

The momentum Q_{∂_y} can be written in terms the Noether charge Q_μ defined in (5.36) via

$$\frac{N(R_h)}{R_h} Q_{\partial_y} = \int_{t=\text{cst}} dxdy Q_\mu\,. \tag{5.159}$$

This can be proven by computing the left and right hand side at the horizon where for the left hand side we use the integral form given in (5.154). The Noether charge Q_μ can also be computed near the boundary where it gives $Q_\mu = \rho V^2$. Hence with (5.157) we conclude that

$$\frac{N(R_h)}{R_h} = -V^y = -V\,, \tag{5.160}$$

i.e. the chemical potential is the velocity of the fluid. From this and (5.158) it follows that the Euclidean on-shell action relates to the pressure as follows

$$TS_E^{\mathrm{os}} = -\int_{t=\mathrm{cst}} dxdyP\,. \tag{5.161}$$

We believe that similar arguments allow one to derive the first law of thermodynamics for these Lifshitz holographic fluids. For example using arguments similar to those of [151] that do not require an explicit knowledge of the interpolating solution. However there are quite compelling arguments that fix the first law in a more straightforward manner so we will refrain from using a more general approach. One of these arguments uses the Ward identities of the dual holographic fluid and the existence of an entropy current. This will be discussed in the next section. The other argument results from the assumption that the pressure only depends on temperature and chemical potential. Given, say, a numerical solution this could be tested by evaluating (5.161). For us this is a rather minor assumption because it is essentially assuming that a solution with a horizon generated by X^M exists. If we assume that $P = P(T, V^2)$ we can vary it and use (5.132) to derive

$$\left(\frac{\partial P}{\partial T}\right)_{V^2} = s\,, \qquad \left(\frac{\partial P}{\partial V^2}\right)_T = \frac{1}{2}\rho\,, \qquad \delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho\,, \tag{5.162}$$

where the latter relation is the first law for our holographic Lifshitz perfect fluid. More will be said about this in the next section.

5.3 Lifshitz perfect fluids

This section is independent from holography and derives the Lifshitz perfect fluid from dimensional reduction. In appendix 4.2.5 we have discussed the null reduction of a relativistic perfect fluid. This gives rise to a Galilean perfect fluid. If furthermore the relativistic fluid is scale invariant, i.e. conformal, the lower-dimensional Galilean perfect fluid has a $z = 2$ Schrödinger invariance. The $z = 2$ Schrödinger algebra contains the $z = 2$ Lifshitz algebra as a subalgebra. Hence a Lifshitz invariant system can be obtained by starting with a Schrödinger invariant system and breaking the generators that are part of the Schrödinger algebra but not of the Lifshitz algebra. One of these symmetries is particle number N. By breaking N explicitly the $z = 2$ Schrödinger algebra reduces to the $z = 2$ Lifshitz algebra[15]. This is precisely what our holographic model for Lifshitz invariant field theories does.

We have shown that the 4-dimensional bulk theory follows from Scherk–Schwarz reduction of a 5-dimensional AdS-gravity model coupled to a scalar field. This scalar field leads to an additional source in the dual field theory and the corresponding diffeomorphism Ward identity, as derived in section 4.4.2, reads

$$\nabla_A t^A{}_B = -\langle O_\psi\rangle\partial_B\psi\,. \tag{5.163}$$

[15] In the Schrödinger algebra the commutator between Galilean boosts G_i and momenta P_i reads $[P_i\,, G_j] = \delta_{ij}N$ so by breaking N keeping P_i intact we break G_i. Further special conformal symmetries K in the Schrödinger algebra satisfy the commutation relation $[K\,, P_i] = -G_i$ so that breaking G_i leads to broken K symmetries. Hence by breaking N we loose the G_i and K generators as well and we are left with the Lifshitz algebra.

Here we will be interested in flat space only so the left hand side is simply $\nabla_A t^A{}_B = \partial_A t^A{}_B$. The Scherk–Schwarz reduction tells us that

$$\begin{aligned} \psi &= 2u - 2\chi\,, && (5.164) \\ \langle O_\psi \rangle &= -\frac{1}{2}\langle O_\chi \rangle\,, && (5.165) \end{aligned}$$

where χ and $\langle O_\chi \rangle$ are independent of u. If we now set the 4-dimensional scalar source $\chi = 0$ we obtain the 4-dimensional Ward identities (see also (4.275) and (4.276))

$$\begin{aligned} \partial_\mu T^\mu{}_\nu &= 0\,, && (5.166) \\ \partial_\mu T^\mu &= \langle O_\chi \rangle\,, && (5.167) \end{aligned}$$

where we used (4.131) and (4.132). We thus see that the mass current T^μ is not conserved due to the presence of $\langle O_\chi \rangle$. The $z = 2$ scale Ward identity follows from (4.135) with $t^A{}_A = 0$ which for the case of a flat TNC space-time (5.45) reads

$$2T^t{}_t + T^i{}_i = 0\,. \tag{5.168}$$

The null reduction also implies the identities (4.123) and (4.124) which on a flat TNC space-time read

$$T^t{}_i = T^i\,, \qquad T^i{}_j = T^j{}_i\,. \tag{5.169}$$

The null reduction in the presence of the scalar source ψ as written in (5.163)–(5.165) gives rise to a system that breaks Galilean boost symmetries and particle number. This is due to the fact the ψ in (5.164) breaks these symmetries. What we are left with is a $z = 2$ Lifshitz invariant system in one dimension lower.

We will now apply the Lifshitz Ward identities (5.166)–(5.169) to the case of a $d = z = 2$ perfect fluid where $\mathcal{E} = P$ and V^i are functions of t, x^i. It has been shown that the form of $T^\mu{}_\nu$ and T^μ for the null reduction of a relativistic perfect fluid take the form (4.153) and (4.154). We now consider the fluid equations as follows from the Ward identities and demand that there exists a conserved entropy current. The latter requirement will tell us what the thermodynamic relations for a Lifshitz perfect fluid are.

On flat TNC space-time the form of the fluid energy-momentum tensor and mass current for a perfect fluid are given by (5.78) and (5.79). The fluid equations are thus given by the Ward identities which read[16]

$$\begin{aligned} 0 &= \partial_t\left(\mathcal{E} + \frac{1}{2}\rho V^2\right) + \partial_i\left(\left(\mathcal{E} + P + \frac{1}{2}\rho V^2\right)V^i\right)\,, && (5.170) \\ 0 &= \partial_t\left(\rho V_i\right) + \partial_i\left(P\delta_i^j + \rho V^j V_i\right)\,, && (5.171) \\ \langle O_\chi \rangle &= \partial_t\rho + \partial_i\left(\rho V_i\right)\,. && (5.172) \end{aligned}$$

These equations can be used to rewrite the equation for energy conservation (5.170) as

$$\partial_t\mathcal{E} + V^i\partial_i\mathcal{E} + \left(\mathcal{E} + P - \frac{1}{2}\rho V^2\right)\partial_i V^i - \frac{1}{2}V^2\left(\partial_t\rho + V^i\partial_i\rho\right) = 0\,. \tag{5.173}$$

[16] If in the holographic setup we would make the fluid variables functions of the boundary coordinates we would have to correct the energy-momentum tensor by derivatives of the fluid variables. The Einstein equations will then lead to Ward identities for this corrected boundary energy-momentum tensor. At leading order in derivatives it will however reduce to the Ward identiites for a perfect fluid.

This gives rise to an equation for conservation of entropy,

$$\partial_t s + \partial_i \left(s V^i \right) = 0 \,, \tag{5.174}$$

provided we take

$$\begin{aligned} \mathcal{E} + P &= Ts + \frac{1}{2}\rho V^2 \,, && (5.175) \\ \delta\mathcal{E} &= T\delta s + \frac{1}{2} V^2 \delta\rho \,. && (5.176) \end{aligned}$$

These two equations together with the equation of state $P = \mathcal{E}$ (that follows from (5.168)) describe the thermodynamic properties of a Lifshitz invariant system obtained by breaking particle number symmetries. What we see here is a realization of a Lifshitz perfect fluid where the velocity or rather, due to rotational symmetries, V^2, plays the role of a chemical potential.[17] The thermodynamically conjugate variable is the mass density ρ. From the first law (5.176) it follows that

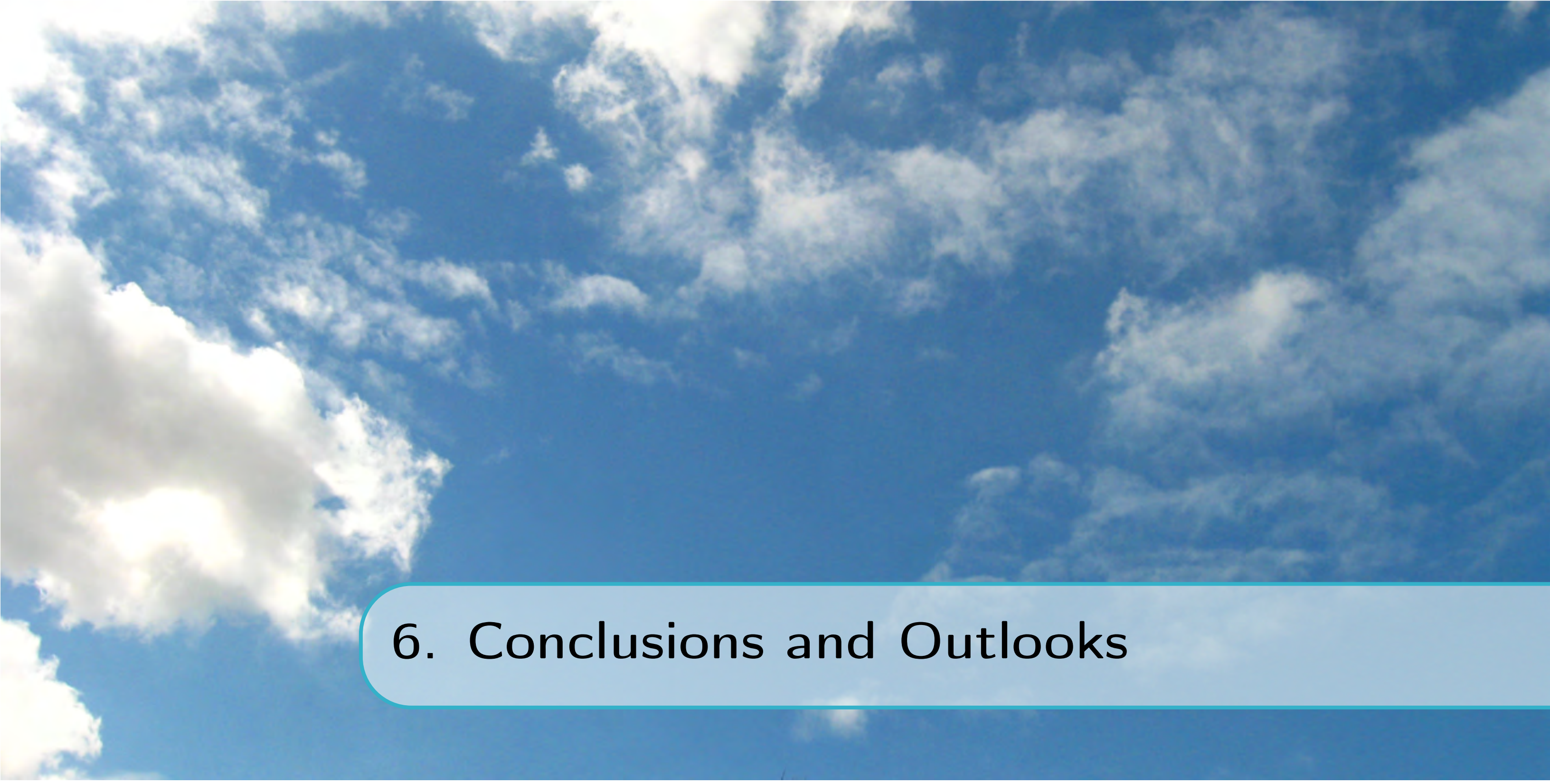
$$\delta P = s\delta T + \frac{1}{2}\rho\delta V^2 \,, \tag{5.177}$$

so that pressure is a function of T and the chemical potential V^2.

We see here that the way in which we realize Lifshitz hydrodynamics is quite different from what has been discussed in [127]. The approach in [127] is to start with a $z = 1$ relativistic perfect fluid and to break Lorentz symmetries by adding higher derivative interactions that break the symmetry of the energy-momentum tensor. One can then take a non-relativistic limit to obtain system with $z \neq 1$ that break Galilean boost symmetries. This leads to a model where Galilean boosts are broken at higher orders in a derivative expansion. On the other hand here we realize Lifshitz symmetries by breaking particle number and hence Galilean boosts already at the perfect fluid level. In [37] we will present more examples of Lifshitz hydrodynamics from a field theory perspective.

As a final comment we note that in order to solve the $d + 2$ equations (5.170)–(5.172) we need to know what $\langle O_\chi \rangle$ is in terms of the fluid variables ρ, V^i and $\mathcal{E}$. Explicit examples will be given in [37].

[17] A similar extension of the first law of thermodynamics involving a fluid with boost momentum was seen in Ref. [11] in the proposed effective theory for the dynamics of helicoidal black p-branes using the blackfold construction [87, 84].



6. Conclusions and Outlooks

In this thesis we discussed mainly two different directions in which higher dimensional gravity is important. The first was about the blackfold approach. In this framework, novel blackfold solutions were discussed. In particular chapter 2 was about finding novel charged blackfold solutions in different backgrounds, giving a physical interpretation to the gravitational tension, and showing that the definitions of pressure and volume previously given for black holes in AdS space do not have the same physical meaning in different backgrounds, and their role could be played by the gravitational tension. Chapter 3 instead was about studying novel blackfold solution with charge and intrinsic spin placed in different background, computing also their magnetic moment and their gyromagnetic ratio. The second part of the thesis was then devoted to Lifshitz Holography. In particular chapter 5 was about studying the Lifshitz/TNC correspondence, previously discovered in the literature, in a fluid/gravity regime. It has been discovered that there is a new class of Lifshitz perfect fluids in which Galilean boosts are broken at the perfect fluid level, and the holographic dual description was realized by a moving black brane solution of the EPD model.

There are still some open problems and follow up projects which would be interesting to pursue. Some fo these projects are work in progress.

6.1 Outlook for the blackfold project

In this paper we have shown that there is an infinite set of conjugate thermodynamic variables $(B_k, \mathbb{L}_k)$ that can be introduced in order to describe the modifications in the first law (2.1) due to variations of external gravitational fields. We have argued that the most natural quantity that describes these variations is the gravitational tension (or gravitational binding energy) that describes the extra energy associated to a black hole due to the presence of surrounding gravitational fields. We have furthermore argued that the popular choice of black hole volume and spacetime pressure used to describe such variations in AdS spacetimes is not the most natural one and leads to non-universal results in the flat spacetime limit.

In order to reach these conclusions we have proposed in Sec. 2.1 that modifications to

(2.1) should satisfy four different properties. We could, in principle, not demand property **(4)**, namely, the existence of a universal result when the external gravitational field is removed. Imposing it, selects the introduction of gravitational tension instead of black hole volume to describe the modifications of (2.1), namely, via (2.10). Not imposing it, would in principle render any of the choices of k in (2.4) as good as any other. However, we must also recall property **(1)**, which requires the existence of a geometrical interpretation. Since we have shown that the notion of black hole volume can be defined in spacetimes which are solutions of the vacuum Einstein equations, then the geometrical interpretation in terms of Killing potentials [137] does not hold. On the other hand, there is a well-defined prescription for evaluating the gravitational tension, following [105], that works for arbitrary black hole spacetimes, at least when there are periodic or non-compact horizon directions or whenever the black hole admits a blackfold regime. In order to complete this picture, we would need generalise the prescription of [105] to black holes which do not admit a blackfold regime. This interesting task we leave for future work.

Furthermore, when considering complete UV theories of gravity, we must in fact add an extra property, namely, **(5)** the existence of a microscopic description. As it is well known, the entropy of black holes has played a central role in developing and testing theories of quantum gravity. In particular, it has led to the celebrated holographic principle [122, 167] as embodied in the AdS/CFT correspondence [2]. The existence of a macroscopic entropy poses the challenge of a microscopic explanation, and one of the successes of string theory has been to provide this for classes of supersymmetric black holes [166]. Similarly, one may expect that such microscopic description should also exist for the quantities describing variations in the gravitational field. In the particular case of AdS, recent work [136] has given a possible CFT interpretation of black hole volume. However, we think that a similar interpretation could be given for gravitational tension and this research direction would be very interesting to pursue.

Finally, we note that we have constructed novel perturbative black hole solutions in AdS, plane wave and Lifshitz spacetimes. In particular we constructed the analogue of the higher-dimensional Kerr-Newman solution in AdS and we gave the first example of a class of black hole solutions in Lifshitz spacetimes with non-trivial horizon topology. These solutions were found using the blackfold approach and their consequences for the universality of black hole volume in the flat spacetime limit were studied. This exercise is yet another illustration of how the blackfold approach can be used in a very simple way to study interesting properties of higher-dimensional black holes.

In connection with the novel Lifshitz solutions that we have obtained, we note that since Lifshitz spacetimes play a role in holography for field theory systems with anisotropic scaling between time and space [135, 170], it would be interesting to generalise the perturbative Lifshitz solutions of this paper in order to include charge. In particular, an interesting family would be the analogue of the Kerr-Newman solutions of Secs. 2.4.2, 2.4.3 in Lifshitz spacetimes. These would be the rotating versions of the black holes constructed in [168]. Since for these solutions the Lifshitz vacuum is supported by non-trivial matter fields, this would require the use of the generalisation of the blackfold equations of motion to such backgrounds [19].

We also briefly mention a number of more general settings in which it would be interesting to examine the gravitational tension perspective proposed in this paper. These include

Taub-NUT spacetimes[1], KK bubbles [177] and the more general sequences of bubbles and holes (see e.g. [72]), other limits of the black saturn configuration, black holes in flux bagrounds, and high-derivative gravity theories.[2]

6.2 Discussion and Outlook for the Lifshitz Hydrodynamics project

We have shown that there is a new class of Lifshitz perfect fluids in which Galilean boosts are broken at the perfect fluid level. The holographic dual description is realized by a moving black brane solution of the EPD model. The motion of the black brane is not obtained by applying a boost transformation to a static black brane but follows from constructing a new class of solutions corresponding to Lifshitz black branes with linear momentum. From the dual field theory point of view the boundary fluid can be obtained by a twisted null reduction of relativistic fluid in the background of a free scalar source that depends linearly on the null circle. From the lower-dimensional point of view this corresponds to a Schrödinger fluid with broken particle number symmetry.

In this work we restricted our attention to a specific EPD model for which we obtained the counterterms and near boundary expansion by dimensional reduction from AdS holography coupled to a free real scalar. In order to consider similar solutions of other EPD models we need to be able to write down the counterterms and near boundary expansions for general EPD models. Despite a lot of effort the situation is presently still not fully understood. There are different proposals [52, 53] and [111, 110] (see [160, 22] for earlier work) that share certain similarilties but that also have some differences. A comparison between [52, 53] and [111, 110] is made in appendix 4.4.6. We believe that more work needs to be done before we can state what the near boundary expansion and counterterms are for a given EPD model in the general class that admits Lifshitz solutions. This general analysis includes asymptotically Lifshitz solutions with hyperscaling violation exponent θ and the charge hyperscaling violation exponent introduced in [92, 96].

A special subset of the EPD models are those for which $W = 0$ so that the bulk vector field becomes a Maxwell gauge potential with a $U(1)$ gauge symmetry. It has been shown in [138] that the corresponding global $U(1)$ symmetry in the boundary theory leads to mass conservation. For the EMD model we know the black brane solutions that are dual to perfect fluids analytically [170, 51]. For the solutions of the EPD models with $W \neq 0$ we only know the solution near the boundary and near the horizon but we do not know that interpolating solution. Hence we have to resort to arguments based on the existence of conserved Noether charges that are a consequence of various ansatz symmetries that allows one to relate near boundary and near horizon properties of the solution as was done in [34, 35]. Here we followed a similar approach and we added to this various integral forms of the asymptotic charges related to the existence of Killing vectors. It would be interesting to see how far one can push this kind of analysis beyond the perfect fluid level. In other words it is worth exploring if it possible to construct bulk solutions in which the fluid variables such as the temperature and velocity become slowly varying functions of the boundary coordinates in such a way that we can extract all the relevant boundary properties from the near horizon and near boundary features. Further, it would be nice to have numerical confirmation about the interpolating solution we have assumed to exist.

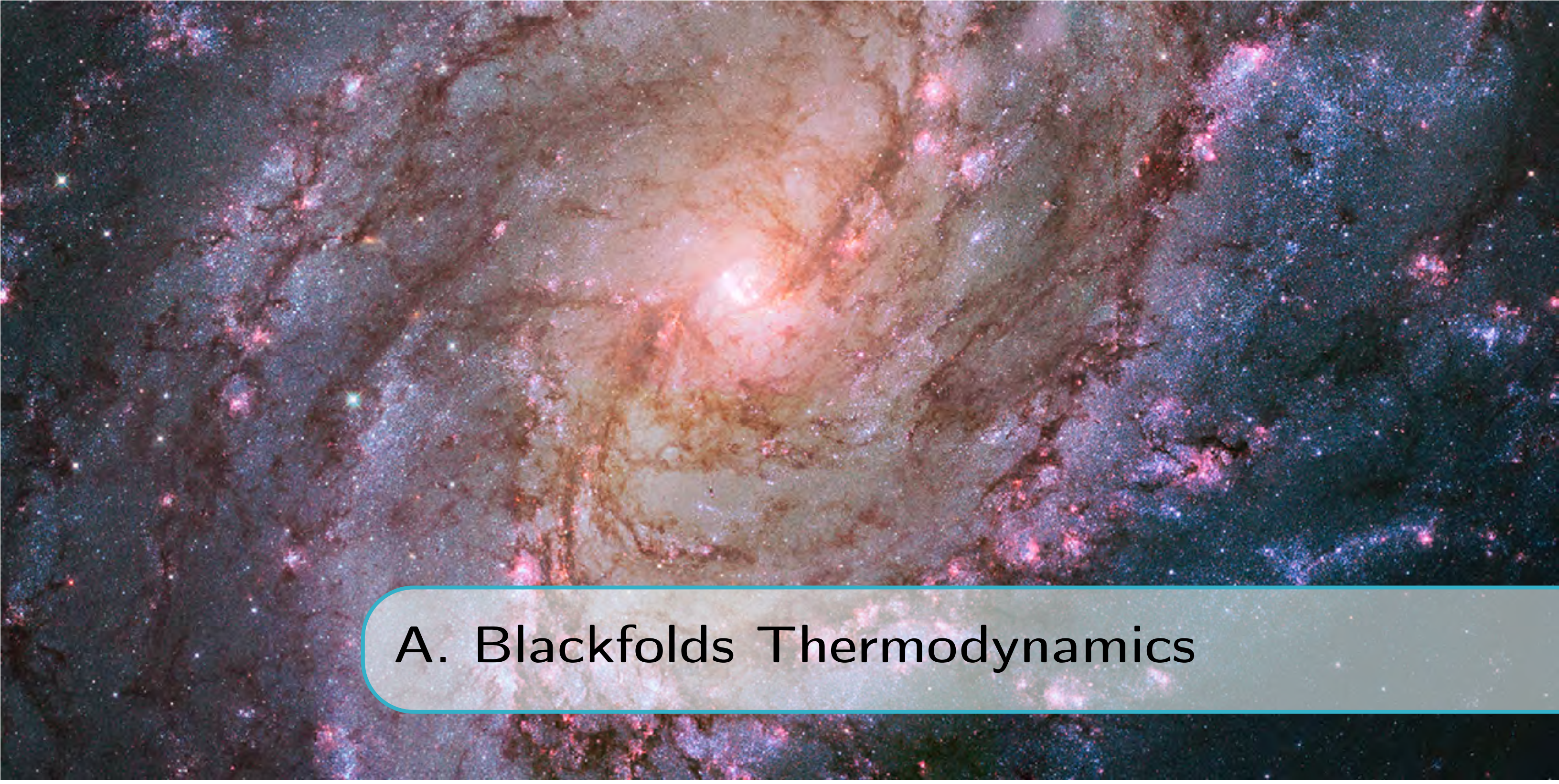
---

[1] Black hole volume and spacetime pressure have been considered for such spacetimes in e.g. [133].

[2] See e.g. [40] for RN-RN-AdS black holes in Gauss-Bonnet gravity.

We also remark that it would be interesting to study the role of charge in the boundary Lifshitz hydrodynamics by adding additional $U(1)$ vector gauge fields to the bulk description like in [168]. For this the recent results in [91] on non-relativistic electrodynamics coupled to TNC could be relevant.

Finally, another interesting direction to pursue is to use Horava-Lifshitz gravity theories as bulk theories in holography [97, 130] and examine the connection with Lifshitz hydrodynamics [71, 60]. It would be worthwhile to pursue this further in the light of the results of this paper. In particular in connection to dynamical NC geometry [114] and the finite temperature states in the 3-dimensional Chern-Simons Schrödinger gravity found in [113].



A. Blackfolds Thermodynamics

A.1 Electrically charged odd-spheres in flat background

The thermodynamic properties of the configurations found in Sec. 1.5.1 can be easily obtained from (2.42) using formulae (2.37) and (1.77), and written in terms of r_0, α. These read

$$
\begin{aligned}
M &= \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \left(nN\sinh^2\alpha + n + p + 1\right) \quad , \\
J &= \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n R \sqrt{p(nN\sinh^2\alpha + n + p)} \quad , \\
S &= \frac{\Omega_{(n+1)}V_{(p)}}{4G} r_0^{n+1} \cosh^N\alpha \sqrt{\frac{nN\sinh^2\alpha + n + p}{n\left(1 + N\sinh^2\alpha\right)}} \quad , \\
Q &= \frac{\Omega_{(n+1)}V_{(p)} n}{16\pi G} \sqrt{N}\sinh\alpha\cosh\alpha r_0^n \sqrt{\frac{nN\sinh^2\alpha + n + p}{n\left(1 + N\sinh^2\alpha\right)}} \quad ,
\end{aligned}
\tag{A.1}
$$

while the temperature T and chemical potential Φ_{H} are given in terms of r_0, R, α via the expressions

$$
\begin{aligned}
T &= \frac{n}{4\pi r_0 \cosh^N\alpha} \sqrt{\frac{n\left(N\sinh^2\alpha + 1\right)}{nN\sinh^2\alpha + n + p}} \quad , \\
\Phi_{\mathrm{H}} &= \sqrt{N}\tanh\alpha \sqrt{\frac{n\left(N\sinh^2\alpha + 1\right)}{nN\sinh^2\alpha + n + p}} \quad .
\end{aligned}
\tag{A.2}
$$

A.2 Uncharged odd-spheres in (A)ds background

The thermodynamic properties of the configurations found in Sec. 1.5.2 can be easily obtained from (2.42) using formulae (2.37) and (1.77), and written in terms of r_0, R. These read

$$\begin{aligned}
M &= \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \left(1+\mathbf{R}^2\right)^{3/2}(n+p+1) \quad , \\
J &= \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n R \frac{\sqrt{\left(\mathbf{R}^2(n+p+1)+p\right)\left((n+p)(1+\mathbf{R}^2)+\mathbf{R}^2\right)}}{n} \quad , \\
S &= \frac{\Omega_{(n+1)}V_{(p)}}{4G} r_0^{n+1}\sqrt{\frac{\mathbf{R}^2(n+p+1)+n+p}{n}} \quad ,
\end{aligned} \tag{A.3}$$

while the temperature T is given in terms of r_0, R via the expressions

$$T = \frac{n}{4\pi r_0}\sqrt{\frac{n\left(\mathbf{R}^2+1\right)}{\mathbf{R}^2(n+p+1)+n+p}} \quad , \tag{A.4}$$

Moreover using (1.43) it is possible to compute the tension of the blackfold and the result is the following

$$\boldsymbol{\mathcal{T}} = -\frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \mathbf{R}^2\sqrt{1+\mathbf{R}^2}\,(n+p+1) \quad . \tag{A.5}$$

With this tension and the previous thermodynamics the Smarr relation (1.44) is satisfied.

A.3 Black odd-spheres with string dipole in flat space

In terms of the parameters r_0, α, the thermodynamics quantities associated with the perturbative solution of Sec. A.10 are

$$\begin{aligned}
M &= \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G_N} r_0^n \left(2nN\sinh^2\alpha+n+p+1\right) \;, \\
J &= \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G_N} r_0^n R\sqrt{\frac{n+p+nN\sinh^2\alpha}{n+nN\sinh^2\alpha}} \;, \\
S &= \frac{\Omega_{(n+1)}V_{(p)}}{4G_N} r_0^{n+1}\cosh^N\alpha\sqrt{\frac{nN\sinh^2\alpha+n+p}{n}} \;, \\
Q &= \frac{\Omega_{(n+1)}V_{(p)}}{32\pi^2 RG_N} r_0^n n\sqrt{N}\sinh\alpha\cosh\alpha \quad ,
\end{aligned} \tag{A.6}$$

while the temperature and chemical potential read

$$\begin{aligned}
T &= \frac{n^{3/2}}{4\pi r_0\cosh^N\alpha}\sqrt{\frac{1}{n\left(N\sinh^2\alpha+1\right)+p}} \quad , \\
\Phi_{\mathrm{H}} &= 2\pi R\sqrt{N}\frac{n+p+nN\sinh^2\alpha}{n}\tanh\alpha \quad .
\end{aligned} \tag{A.7}$$

A.4 Black discs in flat background

For these black holes found in Sec. 1.5.4, the thermodynamic properties can be extracted from (2.53) and read

$$
\begin{aligned}
J =& \frac{\Omega_{(n+1)}}{4G}\tilde{r}_0^n \frac{{}_2F_1\left(1, \frac{1}{2}n(N-1); \frac{nN}{2}+2; 1-\frac{\Phi_{\mathrm{H}}^2}{N}\right)}{\Omega^3(2+nN)} ~, \\
S =& \frac{n\Omega_{(n+1)}}{8TG}\tilde{r}_0^n \frac{{}_2F_1\left(1, \frac{1}{2}n(N-1); \frac{nN}{2}+2; 1-\frac{\Phi_{\mathrm{H}}^2}{N}\right)}{\Omega^2(2+nN)} ~, \\
Q =& \frac{\Omega_{(n+1)}}{32G}\frac{n\Phi_{\mathrm{H}}\tilde{r}_0^n\Gamma\left(\frac{nN}{2}\right)}{\Omega^2\left(1-\frac{\Phi_{\mathrm{H}}^2}{N}\right)}\left(nN\,{}_2\tilde{F}_1\left(1, \frac{1}{2}n(N-1); \frac{nN}{2}+2; 1-\frac{\Phi_{\mathrm{H}}^2}{N}\right)\right. \\
&\left. + 2\,{}_2\tilde{F}_1\left(2, \frac{1}{2}n(N-1); \frac{nN}{2}+2; 1-\frac{\Phi_{\mathrm{H}}^2}{N}\right)\right) ~.
\end{aligned}
\tag{A.8}
$$

The mass of the black hole can be readily obtained using (2.38) together with the free energy (2.53) and the above quantities.

A.5 Electrically charged black odd-spheres in (A)dS

The thermodynamic properties of the configurations found in Sec. 2.4.2 can be easily obtained from (2.42) using formulae (2.37),(2.38) and written in terms of r_0, R, α. These read

$$
\begin{aligned}
M =& \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G}r_0^n\left(1+\mathbf{R}^2\right)^{3/2}\left(nN\sinh^2\alpha+n+p+1\right) ~, \\
J =& \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G}r_0^n R\frac{\sqrt{\left(\mathbf{R}^2(nN\sinh^2\alpha+n+p+1)+p\right)}}{n} \\
&\times\frac{\sqrt{\left((nN\sinh^2\alpha+n+p)(1+\mathbf{R}^2)+\mathbf{R}^2\right)}}{n} ~, \\
S =& \frac{\Omega_{(n+1)}V_{(p)}}{4G}r_0^{n+1}\cosh^N\alpha\sqrt{\frac{\mathbf{R}^2\left(nN\sinh^2\alpha+n+p+1\right)+nN\sinh^2\alpha+n+p}{n\left(1+N\sinh^2\alpha\right)}} ~, \\
Q =& \frac{\Omega_{(n+1)}V_{(p)}n}{16\pi G}\sqrt{N}g(\alpha)r_0^n\sqrt{\frac{\mathbf{R}^2\left(nN\sinh^2\alpha+n+p+1\right)+nN\sinh^2\alpha+n+p}{n\left(1+N\sinh^2\alpha\right)}} ~,
\end{aligned}
\tag{A.9}
$$

where $g(\alpha) = \sinh\alpha\cosh\alpha$ while the temperature T and chemical potential Φ_{H} are given in terms of r_0, R, α via the expressions

$$
\begin{aligned}
T =& \frac{n}{4\pi r_0}\sqrt{\frac{n\left(\mathbf{R}^2+1\right)\left(N\sinh^2\alpha+1\right)}{\mathbf{R}^2\left(nN\sinh^2\alpha+n+p+1\right)+nN\sinh^2\alpha+n+p}} ~, \\
\Phi_{\mathrm{H}} =& \tanh\alpha\sqrt{\frac{n\left(\mathbf{R}^2+1\right)\left(N\sinh^2\alpha+1\right)}{\mathbf{R}^2\left(nN\sinh^2\alpha+n+p+1\right)+nN\sinh^2\alpha+n+p}} ~.
\end{aligned}
\tag{A.10}
$$

A.6 Black discs in (A)dS: analogue of Kerr-Newmann black holes

For these black holes found in Sec. 2.4.2, the thermodynamic properties can be extracted from (2.53) and read

$$
\begin{aligned}
J =& \frac{\Omega_{(n+1)}}{4G} \tilde{r}_0^n \frac{{}_2F_1\left(1, \frac{1}{2}n(N-1); \frac{nN}{2}+2; 1-\frac{\Phi_{\rm H}^2}{N}\right)}{\Omega^3 \xi^2 (2+nN)} \, , \\
S =& \frac{n\Omega_{(n+1)}}{8TG} \tilde{r}_0^n \frac{{}_2F_1\left(1, \frac{1}{2}n(N-1); \frac{nN}{2}+2; 1-\frac{\Phi_{\rm H}^2}{N}\right)}{\Omega^2 \xi (2+nN)} \, , \\
Q =& \frac{\Omega_{(n+1)}}{32G} \frac{n\Phi_{\rm H} \tilde{r}_0^n \Gamma\left(\frac{nN}{2}\right)}{\Omega^2 \xi \left(1-\frac{\Phi_{\rm H}^2}{N}\right)} \left(nN \, {}_2\tilde{F}_1\left(1, \frac{1}{2}n(N-1); \frac{nN}{2}+2; 1-\frac{\Phi_{\rm H}^2}{N}\right) \right. \\
& \left. + 2 \, {}_2\tilde{F}_1\left(2, \frac{1}{2}n(N-1); \frac{nN}{2}+2; 1-\frac{\Phi_{\rm H}^2}{N}\right) \right) \, .
\end{aligned}
\tag{A.11}
$$

The mass of the black hole can be readily obtained using (2.38) together with the free energy (2.53) and the above quantities.

A.7 Electrically charged black discs in plane wave background

Here we collect the thermodynamic properties of the configurations of Sec. 2.4.3. These are given by

$$
\begin{aligned}
M =& \frac{\Omega_{(n+1)}}{8G} \tilde{r}_0^n \frac{(n+2)\left(\Omega^2 - A_1\right) - \left(\Phi_{\rm H}^2 - 1\right)\left(2A_1 + \Omega^2\right) {}_2F_1\left(-\frac{1}{2}, 1; \frac{n+4}{2}; \frac{A_1(\Phi_{\rm H}^2-1)}{\Omega^2 - A_1}\right)}{(n+2)(1-\Phi_{\rm H}^2)\left(A_1 - \Omega^2\right)^2} \, , \\
J =& \frac{\Omega_{(n+1)}}{8G} \tilde{r}_0^n \frac{\Omega \Gamma\left(\frac{n}{2}+1\right) {}_2\tilde{F}_1\left(-\frac{1}{2}, 2; \frac{n+4}{2}; \frac{A_1(\Phi_{\rm H}^2-1)}{\Omega^2 - A_1}\right)}{\left(A_1 - \Omega^2\right)^2} \, , \\
S =& \frac{n\Omega_{(n+1)}}{8TG} \tilde{r}_0^n \frac{{}_2F_1\left(-\frac{1}{2}, 1; \frac{n+4}{2}; \frac{A_1(\Phi_{\rm H}^2-1)}{\Omega^2 - A_1}\right)}{(n+2)\left(\Omega^2 - A_1\right)} \, , \\
Q =& \frac{\Omega_{(n+1)}}{16G} \tilde{r}_0^n \frac{\Phi_{\rm H} \Gamma\left(\frac{n}{2}+1\right) \left(n \, {}_2\tilde{F}_1\left(-\frac{1}{2}, 1; \frac{n+4}{2}; \frac{A_1(\Phi_{\rm H}^2-1)}{\Omega^2 - A_1}\right)\right)}{(1-\Phi_{\rm H}^2)(\Omega^2 - A_1)} \\
& + \frac{2 \, {}_2\tilde{F}_1\left(-\frac{1}{2}, 2; \frac{n+4}{2}; \frac{A_1(\Phi_{\rm H}^2-1)}{\Omega^2 - A_1}\right)}{(1-\Phi_{\rm H}^2)(\Omega^2 - A_1)} \, .
\end{aligned}
\tag{A.12}
$$

A.8 Black odd-spheres in Lifshitz background

The thermodynamic properties of the Lifshitz black holes found in Sec. 2.4.4 are given by

$$\begin{aligned}
M &= \frac{\Omega_{(n+1)}}{16\pi G} V_{(p)} r_0^n \frac{(\beta+\mathbf{R}^2)^{3/2}\mathbf{R}^{z-1}}{|\mathbf{R}^2(z-1)+\beta(z-2)|} \ , \\
J &= \frac{\Omega_{(n+1)}}{16\pi G} V_{(p)} r_0^{n-2} n\sqrt{\beta+\mathbf{R}^2}\mathbf{R}^{z-1} \\
&\quad \sqrt{\frac{(\beta+\mathbf{R}^2)\left(\mathbf{R}^2((n+2)z+p-1)+\beta(n(z-1)+p+2z-2)\right)}{\mathbf{R}^2(n+p+2z-1)+\beta(n+p+2z-2)}} \ , \\
S &= n\frac{\Omega_{(n+1)}}{16\pi TG} V_{(p)} r_0^n \sqrt{\beta+\mathbf{R}^2}\mathbf{R}^{z-1} \ ,
\end{aligned} \tag{A.13}$$

where we have assumed that (2.69) holds and where β and r_0 are given by (2.65) and (2.68) respectively.

A.9 Black odd-spheres in Schwarzschild background

In this section we collect the conserved charges for the black odd-spheres constructed in Sec. 2.5.1. These are given by

$$\begin{aligned}
J =& \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \frac{R\sqrt{\tilde{\mathbf{R}}^{n+p}(nN\sinh^2\alpha(n+p)+n(n+p+1)-p)+2p}}{2-\tilde{\mathbf{R}}^{n+p}(n+p+2)} \\
& \sqrt{2\left(nN\sinh^2\alpha+n+p\right)-\tilde{\mathbf{R}}^{n+p}\left(2nN\sinh^2\alpha+n+p\right)} \ , \\
S =& n\frac{\Omega_{(n+1)}V_{(p)}}{16\pi TG} r_0^n (1-\tilde{\mathbf{R}}^{n+p})^{1/2} \ , \\
\frac{Q}{\sqrt{nN}} =& \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n g(\alpha) \frac{\sqrt{\tilde{\mathbf{R}}^{n+p}\left(2nN\sinh^2\alpha+n+p\right)-2\left(nN\sinh^2\alpha+n+p\right)}}{\sqrt{\left(1+N\sinh^2\alpha\right)\left((n+p+2)\tilde{\mathbf{R}}^{n+p}-2\right)}} \ ,
\end{aligned} \tag{A.14}$$

while the mass was given in (2.76). Also the horizon radius and the chemical potential are given by

$$T = \frac{n}{4\pi r_0\cosh^N\alpha}\sqrt{\frac{n\left(1-\tilde{\mathbf{R}}^{n+p}\right)\left((n+p+2)\tilde{\mathbf{R}}^{n+p}-2\right)\left(N\sinh^2\alpha+1\right)}{2nN\sinh^2\alpha\left(\tilde{\mathbf{R}}^{n+p}-1\right)+(n+p)\left(\tilde{\mathbf{R}}^{n+p}-2\right)}} \ , \tag{A.15}$$

$$\Phi_{\mathrm{H}} = \sqrt{nN}\tanh(\alpha)\sqrt{\frac{n\left(1-\tilde{\mathbf{R}}^{n+p}\right)\left((n+p+2)\tilde{\mathbf{R}}^{n+p}-2\right)\left(N\sinh^2\alpha+1\right)}{2nN\sinh^2\alpha\left(\tilde{\mathbf{R}}^{n+p}-1\right)+(n+p)\left(\tilde{\mathbf{R}}^{n+p}-2\right)}} \ . \tag{A.16}$$

Since we will need to compare this with the black saturn solution in App. A.12 it is convenient to provide these thermodynamic properties in the uncharged limit. In this case we have

$$\begin{aligned}
M =& \frac{\Omega_{(n+1)}V_{(p)}}{8\pi G} r_0^n (1-\tilde{\mathbf{R}}^{n+p})^{3/2} \frac{(n+p+1)}{2-\tilde{\mathbf{R}}^{n+p}(n+p+2)} \quad , \\
J =& \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \frac{R\sqrt{\tilde{\mathbf{R}}^{n+p}(n(n+p+1)-p)+2p}}{2-\tilde{\mathbf{R}}^{n+p}(n+p+2)} \sqrt{(n+p)\left(2-\tilde{\mathbf{R}}^{n+p}\right)} \quad , \\
S =& n\frac{\Omega_{(n+1)}V_{(p)}}{16\pi T G} r_0^n (1-\tilde{\mathbf{R}}^{n+p})^{1/2} \quad , \\
T =& \frac{n}{4\pi r_0} \sqrt{\frac{n\left(1-\tilde{\mathbf{R}}^{n+p}\right)\left((n+p+2)\tilde{\mathbf{R}}^{n+p}-2\right)}{(n+p)\left(\tilde{\mathbf{R}}^{n+p}-2\right)}} \quad , \\
\Omega^2 R^2 =& \frac{\left(\tilde{\mathbf{R}}^{n+p}-1\right)\left(\tilde{\mathbf{R}}^{n+p}\left(n(n+p+1)-p\right)+2p\right)}{\left((n+p)\left(\tilde{\mathbf{R}}^{n+p}-2\right)\right)} \quad , \\
\boldsymbol{\mathcal{T}} =& -\frac{\Omega_{(n+1)}V_{(p)}r_0^n}{16\pi G} \tilde{\mathbf{R}}^{n+p}\sqrt{1-\tilde{\mathbf{R}}^{n+p}} \frac{(n+p)\,(n+p+1)}{2-\tilde{\mathbf{R}}^{n+p}(n+p+2)} \quad .
\end{aligned} \tag{A.17}$$

A.10 Black odd-spheres with string dipole

In this appendix we consider a related configuration to the one studied in Sec.2.4.2 where instead of an electric charge, the black hole has a string dipole charge. The geometry is still characterised by (2.41) and the free energy (2.42), but it must be supplemented with the polarisation vector [43],

$$v^a \partial_a = \frac{\gamma}{R}\left(\sum_{\hat{a}=1}^{[(p+1)/2]} \partial_{\phi_{\hat{a}}} + \Omega R^2 \partial_\tau\right) \quad , \quad \gamma = \frac{1}{\sqrt{1-\Omega^2 R^2}} \quad . \tag{A.18}$$

Using the free energy (2.42) where now the dependence of α on R is different than in the previous case, we obtain the equilibrium condition

$$\Omega^2 R^2 = (1+\mathbf{R}^2)\frac{\mathbf{R}^2\left(n+p+1+2nN\sinh^2\alpha\right)+p+nN\sinh^2\alpha}{\mathbf{R}^2\left(n+p+1+2nN\sinh^2\alpha\right)+n+p+nN\sinh^2\alpha} \quad . \tag{A.19}$$

Again, in the flat space limit, this reduces to the result obtained in [43]. A discussion of the different limits and static cases follows similarly to the previous section. Furthermore, we obtain the tension

$$\hat{\boldsymbol{\mathcal{T}}} = -\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \mathbf{R}^2 \sqrt{1+\mathbf{R}^2}\left(2nN\sinh^2\alpha+n+p+1\right) \quad . \tag{A.20}$$

The thermodynamic properties can be easily obtained. In terms of the parameters $r_0, \mathbf{R}, \alpha$, these read

$$
\begin{aligned}
M =& \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \left(1+\mathbf{R}^2\right)^{3/2}\left(2nN\sinh^2\alpha + n + p + 1\right) \ , \\
J =& \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n R \frac{\sqrt{\left(\mathbf{R}^2(2nN\sinh^2\alpha+n+p+1)+p\right)}}{n} \\
& \times \frac{\sqrt{\left((2nN\sinh^2\alpha+n+p+1)(1+\mathbf{R}^2)+\mathbf{R}^2\right)}}{n} \ , \\
S =& \frac{\Omega_{(n+1)}V_{(p)}}{4G} r_0^{n+1}\cosh^N\alpha \sqrt{\frac{\mathbf{R}^2\left(2nN\sinh^2\alpha+n+p+1\right)+nN\sinh^2\alpha+n+p}{n}} \ , \\
Q =& \frac{\Omega_{(n+1)}V_{(p)}}{32\pi^2 RG} r_0^n n\sqrt{N}\sinh\alpha\cosh\alpha \ ,
\end{aligned}
\tag{A.21}
$$

while the temperature and chemical potential are given by

$$
\begin{aligned}
T =& \frac{n^{3/2}}{4\pi r_0\cosh^N\alpha}\sqrt{\frac{\mathbf{R}^2+1}{n\left(N\left(2\mathbf{R}^2+1\right)\sinh^2\alpha+\mathbf{R}^2+1\right)+(p+1)\mathbf{R}^2+p}} \ , \\
\Phi_{\mathrm{H}} =& 2\pi R\sqrt{N}\sqrt{1+\mathbf{R}^2}\tanh\alpha \ .
\end{aligned}
\tag{A.22}
$$

A.11 Disc with low intrinsic spin in AdS global background

From the free energy functional we are able to compute all the thermodynamics, at leading order in ω.

$$M = \frac{\Omega_{(n+1)}}{8G}\frac{\tilde{r}_0}{\xi^2\Omega^2(n+2)}\left[2-2\Phi_H^2+\xi\left(1+n+\Phi_H^2\right)\right] \tag{A.23}$$

$$J_\Omega = \frac{\Omega_{(n+1)}}{4G}\frac{\tilde{r}_0(\xi-1)\left(1-\Phi_H^2\right)}{\xi^2\Omega^3(n+2)} \tag{A.24}$$

$$S = \frac{\Omega_{(n+1)}}{8GT}\frac{n\tilde{r}_0\left(1-\Phi_H^2\right)}{\xi\Omega^2(n+2)} \tag{A.25}$$

$$Q = \frac{\Omega_{(n+1)}}{8G}\frac{\tilde{r}_0\Phi_H}{\xi\Omega^2} \tag{A.26}$$

$$J_\omega = \frac{\Omega_{(n+1)}}{8G}\frac{n^2\omega\tilde{r}_0\left(1-\Phi_H^2\right)^{3/2}}{\xi\Omega^2(n+3)}\,{}_2F_1\left(\frac{1}{2},1;\frac{n+5}{2};1-\phi^2\right) \tag{A.27}$$

A.12 The blackfold limit of the static black saturn

In this section we take the blackfold limit of the black saturn solution of [74] for which the spherical black hole in the centre is static and compare this with the results of Sec. 2.5.1.

We are interred in the case for which the angular momentum of the black hole in the middle vanishes. Following the notation of [74], this means that $J^{BH} = 0$ which in practice amounts to set the constant $\bar{c}_2 = 0$. In this case we have the following thermodynamic properties for the black ring surrounding the black hole

$$M^{BR} = \frac{3\pi L^2}{4G} k_2 \quad , \quad J^{BR} = \frac{\pi L^3}{G} \sqrt{\frac{k_3 k_2}{2k_1}} \quad , \quad \Omega^{BR} = \frac{1}{L} \sqrt{\frac{k_1 k_3}{2k_2}} \quad , \tag{A.28}$$

$$T^{BR} = \frac{1}{2\pi L} \sqrt{\frac{k_1(1-k_3)(k_1-k_3)}{2k_2(k_2-k_3)}} \quad , \quad S^{BR} = \frac{L^3 \pi^2}{G} \sqrt{\frac{2k_2(k_2-k_3)^3}{k_1(k_1-k_3)(1-k_3)}} \quad . \tag{A.29}$$

It is also important to take a look at the mass of the black hole in the centre. This is given by

$$M^{BH} = \frac{3\pi L^2}{4G}(1-k_1) \quad . \tag{A.30}$$

The constants k_1, k_2, k_3 are related to the rod intervals of the seed structure used in [74] in order to obtain the solution. These are required to satisfy

$$0 \leq k_3 < k_2 < k_1 \leq 1 \quad , \tag{A.31}$$

but only two of them are independent. The balancing condition, in order to avoid conical singularities, fixes one of them in terms of the others . This condition reads

$$(k_1 - k_2) = \sqrt{k_1(1-k_2)(1-k_3)(k_1-k_3)} \quad , \tag{A.32}$$

and has two solutions but only one satisfies (A.31), which reads

$$k_3 = \frac{1}{2}\left(\frac{\sqrt{-k_1(k_2-1)\left(-k_1((k_1-2)k_1+9)k_2 + k_1(k_1+1)^2 + 4k_2^2\right)}}{k_1(k_2-1)} + k_1 + 1\right) \quad . \tag{A.33}$$

Limits

From the mass of the middle black hole (A.30), we see that the limit in which we recover the black ring, and hence the middle black hole disappears, is the limit for which $k_1 = 1$. It is indeed useful to consider the reparametrization $k_1 = 1 - \beta$ where now the black ring limit is achieved when $\beta = 0$. An expansion in powers of β is thus a weak background gravitational field expansion which can be thought of an expansion in powers of β/L where β is related to the middle black hole horizon and L is the radius of the black ring. Here we fix the radius L which formally can be set equal to one.

Furthermore, we wish to know the limit in which the black ring is ultraspinning and hence becomes effectively thin. Since we have fixed the radius L, this limit is achieved by sending $k_2 \to 0$ for which the black ring mass (A.28) is small and hence we have an expansion of the form k_2/L where k_2 is proportional to the black ring horizon radius.

Therefore, to first order β and to leading order in k_2 we find the following thermodynamic expressions

$$\Omega^{BR} = \frac{1}{2L}\left(1 + \frac{1}{4}\beta\right) + \mathcal{O}(\beta^2, k_2) \ , \ J^{BR} = \frac{L^3 \pi k_2}{2G}\left(1 + \frac{5}{4}\beta\right) + \mathcal{O}(\beta^2, k_2^2) \quad , \tag{A.34}$$

$$T^{BR} = \frac{1}{2\pi L k_2}\left(1 - \frac{1}{4}\beta\right) + \mathcal{O}(\beta^2, 1) \ , \ S^{BR} = \frac{L^3\pi^2 k_2^2}{2G}\left(1 - \frac{5}{4}\beta\right) + \mathcal{O}(\beta^2, k_2^4) \ . \quad \text{(A.35)}$$

Furthermore, note that M^{BR} does not get corrected with these expansions, neither does M^{BH}. To all orders in this expansion, they are given by

$$M^{BR} = \frac{3\pi L^2}{4G} k_2 \quad , \quad M^{BH} = \frac{3\pi L^2}{4G}\beta \quad . \quad \text{(A.36)}$$

If we set $\beta = 0$ in these expressions we recover the thermodynamics of 5D ultraspinning black rings. Here we have performed a weak-field expansion in order to gain intuition regarding ultraspinning regimes of the black saturn. However, this is not necessary. To all orders in β and to first order in k_2 we obtain

$$\Omega^{BR} = \frac{1}{\sqrt{2}L}\sqrt{\frac{1-\beta^2}{2-\beta}} \quad , \quad J^{BR} = \frac{L^3\pi k_2}{\sqrt{2}G}\sqrt{\frac{1+\beta}{2-3\beta+\beta^2}} \quad , \quad \text{(A.37)}$$

$$T^{BR} = \frac{1}{2L\pi k_2}\sqrt{\frac{(2-\beta)(1-\beta)^2}{2-4\beta}} \quad , \quad S^{BR} = \frac{\sqrt{2}L^3\pi^2 k_2^2}{(1-\beta)G}\sqrt{\frac{(1-2\beta)^3}{(2-\beta)^3}} \quad . \quad \text{(A.38)}$$

Comparison with the blackfold construction

We now compare this analytic solution with the construction of Sec. 2.5.1. Using the thermodynamic charges obtained in (A.17) and taking the particular values $D = 5, p = 1, n = 1$ we find

$$\Omega = \frac{1}{R}\sqrt{\frac{1-\tilde{\mathbf{R}}^4}{2-\tilde{\mathbf{R}}^2}} \quad , \quad J = \frac{\pi R^2 r_0}{2G}\frac{\sqrt{2+\tilde{\mathbf{R}}^2-\tilde{\mathbf{R}}^4}}{1-2\tilde{\mathbf{R}}^2} \quad , \quad \text{(A.39)}$$

$$T = \frac{1}{4\pi r_0}\sqrt{\frac{1-3\tilde{\mathbf{R}}^2+2\tilde{\mathbf{R}}^4}{2-\tilde{\mathbf{R}}^2}} \quad , \quad S = \frac{2\pi^2 R r_0^2}{G}\sqrt{\frac{2-\tilde{\mathbf{R}}^2}{1-2\tilde{\mathbf{R}}^2}} \quad . \quad \text{(A.40)}$$

If we now compare the angular velocity Ω given in (A.39) with the analytic solution Ω^{BR} in (A.37) we find that we must have

$$R = \sqrt{2}L \quad , \quad \tilde{\mathbf{R}}^2 = \beta \quad . \quad \text{(A.41)}$$

Furthermore, comparing the temperature T in (A.40) with T^{BR} in (A.38) we find that we must have

$$r_0 = \frac{L k_2}{\sqrt{2}}\sqrt{\frac{(1-3\tilde{\mathbf{R}}^2+2\tilde{\mathbf{R}}^4)(1-2\tilde{\mathbf{R}}^2)}{(2-\tilde{\mathbf{R}}^2)^2(1-\tilde{\mathbf{R}}^2)^2}} \quad . \quad \text{(A.42)}$$

Indeed, this identification is enough to match both the entropy and the angular momentum of the analytic solution with the blackfold approach. As for the mass, we refer to the discussion at the end of Sec. 2.5.1, in which we have noted that the mass of the blackfold construction does not match the mass of the analytic solution M^{BR}. Instead we find that, in the particular case of $D = 5$, we have an agreement by subtracting an appropriate factor of the tension, namely,

$$M^{BR} = M - \frac{1}{2}\boldsymbol{\mathcal{T}} \quad . \quad \text{(A.43)}$$

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PhD Thesis

Marco Sanchioni

Blackfolds and non-AdS Holography

Over the past 20 years we learnt that General Relativity is not "just" a theory that describes the 4 dimensional universe we live in, through the study of cosmology and black holes. It is much more. In fact we discovered that General Relativity can be used as a tool to describe also other physical settings, not just astrophysics and cosmology. As it happened with Quantum Field Theory, which was discovered in the realm of quantum electrodynamics but it was later applied in many other branches of physics, so it happened with General Relativity. After the discovery of AdS/CFT correspondence we learnt that General Relativity is the tool for describing strongly coupled system. This is one of the motivation for studying General Relativity in settings that are different with respect to astrophysics and cosmology.

The first part of the thesis is devoted to studying higher dimensional fancy black holes configurations. We don't believe that such configurations can exist, in fact we are going to study them in higher dimensions. But they can be useful for understanding more about general relativity, and they can be used, through AdS/CFT correspondence, for studying some strongly coupled Quantum Field Theory. Over the last years effective theories for black holes, together with numerical methods, have been extremely important for studying gravity in higher dimensions, which is particularly difficult to solve. Among these effective theories, there are the blackfold approach and the large D expansion. In this work we analyze the first of these two approaches.

The aim of the second part of the work is to study holographic applications which are not relativistic. Not all the system in nature are in fact relativistic. There are many system, especially in the condensed matter sector, which exhibits non-relativistic symmetries. In the past years many work has been done in order to extend the holographic principle (on which the AdS/CFT correspondence was based) also for non-relativistic physics. These works have been very useful in studying strongly coupled physical systems with non-relativistic symmetries, such as Galielan or