UNIVERSITY OF COPENHAGEN

Ph.D. Thesis

The Weyl double copy and black holes

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ABSTRACT

Using spinor formalism, we investigate the profound connection between gravity theory and gauge theory in this thesis. We concentrate on the Weyl double copy prescription. Given that higher spin massless free-field spinors can be constructed from spin-1/2 spinors (Dirac-Weyl spinors) and scalars, we introduce a map between Weyl fields and Dirac-Weyl fields and determine the corresponding Dirac-Weyl spinors in a given empty spacetime. Specifically for non-twisting vacuum Petrov type N and type D solutions, our findings elucidate a number of fundamental properties that were previously unknown. We systematically reconstruct the Weyl double copy for these solutions and demonstrate the significance of the zeroth copy in connecting gravity fields with a single copy and degenerate Maxwell fields with the Dirac-Weyl fields in curved spacetime. Moreover, we investigate the Weyl double copy relation for vacuum solutions of the Einstein equations with a cosmological constant using our new approach in which Dirac-Weyl fields are considered fundamental units. Our research demonstrates that the single and zeroth copies satisfy conformally invariant field equations in conformally flat spacetime and that the zeroth copy retains its importance in connecting gravity fields with a single copy and Dirac-Weyl fields with degenerate electromagnetic fields in curved spacetime. In addition, we find that the zeroth copy plays an important role in time-dependent radiation solutions, especially for Robinson-Trautman gravitational waves. In the limit of weak fields, the zeroth copy carries additional information indicating whether

the sources of gravitational waves are time-like, null, or space-like. Finally, we present an overview of our ongoing work and future research directions.

RESUMÉ

Ved hjælp af spinorformalisme undersøger vi den dybe forbindelse mellem gravitationsteori og gaugeteori i denne afhandling. Vi fokuserer på Weyl-dobbeltkopiforskriften. Da højere spindriftløse fri-feltspinorer kan konstrueres fra spin-1/2 spinorer (Dirac-Weyl spinorer) og skalare, introducerer vi et kort mellem Weyl-felter og Dirac-Weyl felter og bestemmer de tilsvarende Dirac-Weyl spinorer i et givet tomt rumtid. Specifikt for ikke-vridende vakuum Petrov type N og type D-løsninger belyser vores resultater en række grundlæggende egenskaber, der tidligere var ukendte. Vi genopbygger systematisk Weyl-dobbeltkopien for disse løsninger og demonstrerer betydningen af nulte kopien i at forbinde gravitationsfelter med en enkelt kopi og degenererede Maxwell-felter med Dirac-Weyl-felter i krummet rumtid. Desuden undersøger vi Weyl-dobbeltkopirelationen for vakuumløsninger af Einsteins ligninger med en kosmologisk konstant ved hjælp af vores nye tilgang, hvor Dirac-Weyl-felter betragtes som fundamentale enheder. Vores forskning viser, at den enkelte og nulte kopier opfylder konformt invariante feltligninger i konformt fladt rumtid, og at nulte kopien bevarer sin betydning ved at forbinde gravitationsfelter med en enkelt kopi og Dirac-Weyl-felter med degenererede elektromagnetiske felter i krummet rumtid. Derudover finder vi, at nulte kopien spiller en vigtig rolle i tidsafhængige strålingsløsninger, især for Robinson-Trautman gravitationsbølger. I svage feltgrænser bærer nulte kopien yderligere information, der indikerer, om kilderne til gravitationsbølger er tidslignende, nul eller rumlignende. Til sidst præsenterer vi en oversigt over vores igangværende arbejde og fremtidige forskningsretninger.

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LIST OF PUBLICATIONS

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2. Shanzhong Han. “The Weyl double copy in vacuum spacetimes with a cosmological constant”. In: JHEP 09 (2022), p. 238. DOI: 10.1007/JHEP09(2022)238. arXiv: 2205.08654 [gr-qc].

CONTENTS

Part I

INTRODUCTION

1

BACKGROUND AND CURRENT UNDERSTANDING

Is there a beautiful fundamental law, a theory of everything, governing the rule of how our nature operates? To my best knowledge, we did find two promising theories—gravity theory and the Standard Model, which have explained and predicted most phenomena in our universe. They demonstrate that there are four fundamental interactions (forces) in nature: electromagnetic interaction, weak interaction, strong interaction, and gravity.

However, they are not yet beautiful enough. The Standard Model constructed from quantum field theory unifies only three of the four fundamental forces[1–4], except for gravity which is described by Einstein's theory of General Relativity. Due to its long-range interaction of gravity and its substantial dependence on the masses of the objects involved, gravity plays a dominant role at large length scales, such as the scales of planets and all the way to the scale of the entire universe. At small length scales—the scale of subatomic particles, the other three fundamental forces play a much more important role than gravity and are described by the Standard Model. However, at an even smaller scale—the Planck scale, gravity becomes as important as the other fundamental forces. Neglecting any of them may lead to incomplete or incorrect descriptions of physical phenomena. Unfortunately, the current formulation of quantum field theory breaks down when including gravity.

For gravity theory, it states that spacetime is curved by matter following the general relativity theory proposed by Albert Einstein[5, 6]. While on the other hand, quantum field theory is formulated on a fixed background, typically the Minkowski spacetime, and is based on the principles of quantum mechanics and special relativity. In other words, quantum field theory has not fully taken into account the effects of general relativity. No self-consistent approach has been found to extend quantum field theory to a dynamic background to date. Suppose one naively applies a similar process to gravity as we do, for example, for spin-1 massless fields in quantum field theory. The results turn out to be non-renormalizable, which deviates from a physically realistic outcome. Besides, although general relativity predicts the existence of black holes, the singularity at the center of black holes leads to infinite curvature and thus the breakdown of general relativity theory.

Overall, our understanding of the natural world remains incomplete due to the limitations of current theories. This gives rise to a pivotal goal for theoretical physicists around the world: the unification of the Standard Model and gravity theory, or more ambitiously, the development of a comprehensive Theory of Everything.

Currently, a promising candidate to unify gravity theory and quantum mechanics is string theory. Instead of treating 0-dimensional particles as elementary objects, string theory regards 1-dimensional strings as elementary objects in nature. Particles, including photons, quarks, gravitons, and others, are all represented by different vibrational modes of strings. Although there is to date no direct experimental evidence to verify it (largely due to our inability to achieve super high-energy experiments), string theory does offer a framework that potentially unifies all known fundamental forces, and it generates numerous insights and developments both in physics and mathematics.

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DOUBLE COPY AND RESEARCH FINDINGS

One of the intriguing topics emerging from string theory is the double copy. It can be traced back to the discovery of the relationships between the amplitudes of closed and open strings in 1986, now known as the KLT relations[7], which shows that any closed string tree amplitude can be expressed as a sum of the products, consisting of corresponding open string tree amplitudes. As we know, in the low-energy limit, closed string theory gives rise to massless particles called gravitons, which mediate gravitational interactions; while open string theory gives rise to massless particles associated with gauge fields, which mediate the fundamental forces in Standard Model. Therefore, inspired by KLT relations, a new relationship between gravity and gauge tree amplitudes was noticed about two decades later—BCJ duality[8–10]. It demonstrates that tree-level gravity amplitudes can be obtained using double copies of gauge-theory diagram numerators. Shortly after, this finding was extended to the loop level, and at that point, the “double copy” concept was formally proposed.

Let’s take the case of tree-level as an example. The general massless m-point tree-level gauge theory amplitude can be represented by

$$\mathcal{A}_m^{(tree)} = g^{m-2} \sum_i \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{1}$$

where the sum is over all distinct diagrams with cubic vertices, the product runs over all propagators $1/p^2_{\alpha_i}$ of each diagram, and α_i refers to the α_i-th internal line in the i-th diagram. The numerators n_i are called the kinematic numerators, which are dependent on the particle momenta and polarisation vectors and can be deformed by a so-called gauge transformation. The c_i are the color factors obtained by dressing each triple vertex with structure constants. Interestingly, the BCJ duality states that as long as there exist a set of kinematic numerators n_i such that they obey the same Jacobi-like identities as the color factors c_i, one is able to write directly the tree-level gravity amplitudes by replacing the color factors with another kinematic numerator $\tilde{n}_i$ from the tree-level gauge theory amplitudes Eq.(1),

$$\mathcal{M}_m^{(tree)} = (\frac{\kappa}{2})^{m-2} \sum_i \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}, \tag{2}$$

where the coupling constant g is replaced with the gravitational counterpart κ.

In recent years, a significant amount of research has been dedicated to understanding this duality relation [11–39]. In the meantime, due to its potential implications in gravitational wave astronomy, cosmology, and particle physics, much attention has been devoted to investigating exact classical solutions related to the double copy[40–70]. Two main types of classical double copy prescriptions have been extensively studied. One is the Kerr-Schild double copy prescription, which is based on a special metric known as Kerr-Schild spacetime. This metric allows the Einstein equation to be linearized, leading to a double copy relation between solutions of linearized gravity equations and associated gauge field solutions. The other one is the Weyl double copy prescription, which demonstrates a gravity and gauge duality in a wider range of 4-dimensional spacetimes, such as non-twisting vacuum Petrov type N spacetime and Petrov type D spacetime. We will provide a more detailed introduction to these two types of classical double copy in the next part, then we will pay full attention to the research on Weyl double copy throughout this thesis.

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THESIS OUTLINE

The structure of this thesis is as follows. Chapter. 4 and Chapter. 5 review two main types of classical double copy prescriptions —the Kerr-Schild double copy and the Weyl double copy. Chapter. 6 and Chapter. 7 are two publications related to the Weyl double copy relations. In Chapter. 6, we proposed a new approach to systematically rebuild the Weyl double copy relation for Petrov type N and D solutions in 2-spinor formalism, and we also extended the discussion to Petrov type III cases. Using this new method, we examine the Weyl double copy relation for vacuum solutions of the Einstein equations with a cosmological constant in Chapter. 7. Then, Chapter. 8 introduces some of the research findings from our current work on Weyl double copy. We investigate the Weyl double copy relation for the Robinson-Trautman solution with a special metric, aiming to contribute to the field of gravitational wave astronomy. A summary and discussions are contained in Chapter. 9.

In this thesis, I adopt the convention that Greek indices $(\mu, \nu, \rho, ...)$ denote spacetime coordinates $(0, 1, 2, 3)$ and Latin indices $(i, j, k, ...)$ represent space coordinates $(1, 2, 3)$. Unless otherwise specified, we assume the use of natural units, where the speed of light (c) and the reduced Planck constant $(\hbar)$ are set to 1, i.e., $c = 1$ and $\hbar = 1$.

Part II

CLASSICAL DOUBLE COPIES

4

KERR-SCHILD DOUBLE COPY

In this part, we will review two main types of classical double copy relations— Kerr-Schild double copy and Weyl double copy. Firstly, we will focus on the Kerr-Schild double copy. The Weyl double copy will be discussed in the next chapter.

Inspired by the BCJ duality, Monteiro, O'Connell, and White examined the double copy relation in classical exact solution for the first time[41]. It is now known as Kerr-Schild double copy since it is based on the Kerr-Schild metric. This section will provide a review to introduce more details about the Kerr-Schild double copy.

The Kerr-Schild metric was first proposed by Trautman in 1962 as a simple solution to study gravitational waves[71]. The metric is of the form:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi k_\mu k_\nu, \tag{3}$$

where $\bar{g}$ is a background metric, k_μ is a null vector with respect to the background, and ϕ is a general scalar function. It is easy to observe that

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \phi k^\mu k^\nu. \tag{4}$$

Namely, both the covariant and contravariant components of the metric depend linearly on the structure-function ϕ. When one tries to show that gravitational wave is able to propagate information, this special form avoids the difficulty that the contravariant form of the metric tensor normally turns out to be complicated even though we have a simple covariant metric form. In 1965, Kerr and Schild systematically investigated this class of vacuum solutions with $\bar{g}$ being Minkowski metric $\eta_{\mu\nu}$[72]. Some important features are revealed, such as:

(a). Kerr-Schild metrics include the Schwarzchild solution and the Kerr solution.

(b). If k_μ is null with respect to $\eta_{\mu\nu}$, then it is also null with $g_{\mu\nu}$, vice versa. One thus observes that $g^{\mu\nu}k_\mu k_\nu = \eta^{\mu\nu}k_\mu k_\nu$=0.

(c). All Kerr-Schild vacuum solutions are algebraically special. In other words, the null vector k_μ is not only geodesic but also shear-free[73].

For Kerr-Schild metric $g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu = \eta_{\mu\nu} + h_{\mu\nu}$, we think of $\phi k_\mu k_\nu$ as graviton $h_{\mu\nu}$. Motivated by BCJ duality, we are prompted to put forth a daring hypothesis: can this gravity field be acquired through the double copy of a gauge field? Essentially, could there be a connection between the field k_μ and a gauge field? Monteiro, O'Connell, and White investigated this conjecture and demonstrated that a duality relation does indeed exist between gravity and Abelian gauge theory.

Given a Kerr-Schild metric, the Ricci tensor is given by

$$R^\mu_{\ \nu} = g^{\mu\rho}R_{\rho\nu} = \frac{1}{2}\left(\partial^\mu\partial_\alpha\left(\phi k^\alpha k_\nu\right) + \partial_\nu\partial^\alpha\left(\phi k_\alpha k^\mu\right) - \partial^2\left(\phi k^\mu k_\nu\right)\right), \tag{5}$$

where we have defined $\partial^\mu = \eta^{\mu\nu}\partial_\nu$. Since $k^\alpha k_\beta = g^{\alpha\gamma}k_\gamma k_\beta = \eta^{\alpha\gamma}k_\gamma k_\beta$, it is easy to see that right side of Eq.(5) can be regarded as an expression linearized in the scalar function ϕ on the Minkovski background.

Considering the stationary case where null vector k_μ and scalar function ϕ are independent of the time, and fixing a coordinate gauge condition such that $k^0 = 1$, one transfers the vacuum Einstein equations $R_{\mu\nu} = 0$ into simple formulas,

$$R^0_{\ 0} = \frac{1}{2}\partial_i\partial^i\phi, \tag{6}$$

$$R^i{}_0 = -\frac{1}{2}\partial_j \left[\partial^i \left(\phi k^j\right) - \partial^j \left(\phi k^i\right)\right], \tag{7}$$

and

$$R^i{}_j = \frac{1}{2}\partial_l \left[\partial^i \left(\phi k^l k_j\right) + \partial_j \left(\phi k^l k^i\right) - \partial^l \left(\phi k^i k_j\right)\right]. \tag{8}$$

An intriguing outcome will surface. By defining a vector field $A_\mu = \phi k_\mu$, Eq.(6) and Eq.(7) can be recast as a set of Maxwell equations on Minkowski spacetime,

$$0 = \partial_\mu F^{\mu 0} = \partial_i \partial^i \phi, \tag{9}$$

$$0 = \partial_\mu F^{\mu i} = \partial_j \left[\partial^i \left(\phi k^j\right) - \partial^j \left(\phi k^i\right)\right], \tag{10}$$

where $F^{\mu\nu}$ is nothing but an Abelian field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Furthermore, if we take k_μ away from the vector A_μ, the scalar function ϕ satisfies the equation of motion according to Eq.(6). It is believed to play an important role as the propagator in the quantum field theory[41]. The scalar function ϕ is referred to as the zeroth copy, and naturally, the gauge field A_μ is called the single copy since it is just the zeroth copy multiplied by a single copy of a factor k_μ. More interestingly, the gravity field $h_{\mu\nu}$ is obtained by multiplying two copies of the factor k_μ. This inspiring relationship between the gauge and gravity fields is now known as Kerr-Schild double copy. One can go even further. By defining

$$A^a_\mu = c^a \phi k_\mu, \tag{11}$$

one obtains a more general class of the fields that satisfy the Abelian Yang-Mills equations, where c_a are arbitrary constants. This is reminiscent of the BCJ duality, we directly get a gravity field by replacing a color factor c^a with a "kinematic" factor k_μ.

As we have demonstrated, the Kerr-Schild double copy prescription reveals a close connection between gauge theory and gravity. Then one may ask how well the two theories' classical solutions match up.

To show this, one of the interesting examples of the Kerr-Schild double copy is the case of Schwarzchild black holes. As the simplest model of black holes—a massive, non-rotating, spherically symmetric black hole, Schwarzschild metric has only a mass parameter M, originating from the energy-momentum tensor,

$$T^{\mu\nu} = Mv^{\mu}v^{\nu}\delta^{(3)}(x), \tag{12}$$

where $v^{\mu} = (1, 0, 0, 0)$ is a vector. In the Kerr-Schild coordinates system the metric is represented by

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2M}{r}k_{\mu}k_{\nu}. \tag{13}$$

The structure-function ϕ reads

$$\phi = \frac{2M}{r}, \tag{14}$$

and the null vector is given by

$$k_{\mu} = (1, \frac{\vec{x}}{r}) \tag{15}$$

where $r^2 = x_i x^i$. Following the Kerr-Schild double copy relation, the single copy should be given by

$$A_{\mu} = \frac{Q}{r}k_{\mu}, \tag{16}$$

where we let $Q = 2M$. Grasping its physical significance directly may not be straightforward. While it is not hard to verify that

$$\partial_{\mu}F^{\mu\nu} = J^{\nu}, \tag{17}$$

where the current $J^{\nu} = -Qv^{\nu}\delta^{3}(\boldsymbol{x})$, which refers to a static source. Thus, we obtain an Abelian Maxwell field. Then, without changing the electromagnetic strength, one can perform a gauge transformation

$$A_{\mu}(x) \to A'_{\mu} = A_{\mu}(x) + \partial_{\mu}\alpha(x) \tag{18}$$

with a certain scalar function α such that

$$\partial_\mu \alpha(x) = -\frac{Q}{r}(0, \frac{\vec{x}}{r}). \tag{19}$$

Under this gauge condition, one shall find that the solution we obtained is nothing but a Coulomb solution

$$A'_\mu = \frac{Q}{r}(1, 0, 0, 0). \tag{20}$$

Therefore, up to a particular gauge transformation, we illustrate a direct connection between Schwarzchild solutions and Coulomb solutions. It is easy to extend the situation to the general case A^a_μ, and the charge, in this case, will be the superposition of static color charge. This extension can even be applied to higher dimensions, with a derivation similar to that of the 4-dimensional case. For further details, one may refer to the original work [41].

Overall, Kerr-Schild double copy has been studied in wider ranges of spacetime, such as Kerr black holes, Black branes, pp-waves, Taub-NUT spacetime[42], Lifshitz spacetime[74], etc. the Kerr-Schild double copy provides an intriguing possibility to understand how gauge and gravity theories are related. Based on it, one can streamline computations and create new, precise solutions to General Relativity and other theories of gravity by using gauge theory's proven solutions. It also provides a potent tool for exploring the characteristics of gravitational systems, which may reveal hitherto undiscovered phenomena.

While the Kerr-Schild double copy marks substantial progress in uncovering the connection between gravity and gauge theories, numerous challenges remain to be addressed. Firstly, most research on Kerr-Schild double copy concentrates on stationary solutions; the time-dependent Kerr-Schild double copy has not been extensively studied. Developing a better understanding of time-dependent Kerr-Schild double copies could potentially provide new insights into the relationship between classical gauge and gravity solutions, as well as uncover new properties of the double copy correspondence in more general settings. Secondly, the research on Kerr-Schild double copy in (A)dS spacetime shows that the single and zeroth copies satisfy different equations for time-dependent and time-independent solutions[53]. The physical significance of single and zeroth copies appears to be non-intuitive in this case. In fact, this is also one of the motivations why we study the Weyl double copy in (A)dS spacetime. This problem will not arise in that framework, which will be shown in Chapter. 5. Thirdly, due to its dependence on the Kerr-Schild coordinate system, Kerr-Schild double copy only applies to a special class of spacetime, which does not even cover the whole vacuum Petrov type D spacetime. Last but not least, it is also worth extending research to general non-Abelian cases. Overall, there are still many intricate details to be discovered in the relationship between gauge theory and gravity.

5

WEYL DOUBLE COPY

Unlike the Kerr-Schild double copy, which is subject to the condition that the Kerr-Schild metric linearizes the Einstein equations, the Weyl double copy prescription draws a wider correspondence between exact solutions of gauge and gravity theories. It has been shown to hold for all vacuum Petrov type D solutions and non-twisting vacuum type N solutions[75]. The asymptotic behaviors of algebraically general spacetime and the cases in Einstein-Maxwell theory have also been discussed[62–64, 68]. Additionally, a related strategy known as the Cotton double copy has recently been put forth[69, 76], which expands the use of double copy prescriptions to 3-dimensional spacetimes for massive theories. This discovery broadens our comprehension of the relationships between gauge and gravity theories, opening the door for further research in the area. Now, let's delve into the concept of the Weyl double copy. We shall give a short review of the spinor formalism here, one may refer to Chapter. 6 for more details about the mathematical framework.

Considering an arbitrary vector $V = (V^0, V^1, V^2, V^3)$, one may write it in a 2×2 Hermitian matrix form

$$H = V^0\sigma_0 + V^i\sigma_i = \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}, \tag{21}$$

with the aid of a 2 by 2 unit matrix σ_0 and Pauli matrices σ_i. It can be further written as a product of a 2×1 complex matrix (a 2-dimensional complex vector) and its complex conjugate,

$$H = \begin{pmatrix} \xi \\ \eta \end{pmatrix} (\bar{\xi} \;\; \bar{\eta}) = \begin{pmatrix} \xi\bar{\xi} & \xi\bar{\eta} \\ \eta\bar{\xi} & \eta\bar{\eta} \end{pmatrix}. \tag{22}$$

By combining the right sides of Eq.(21) and Eq.(22), one can easily find such a type of vector. Further more, performing a non-singular complex linear transformation A,

$$\begin{pmatrix} \xi & \eta \end{pmatrix} \to A \begin{pmatrix} \xi & \eta \end{pmatrix}, \tag{23}$$

the Hermitian matrix H that represents the vector V will transform as

$$H \to \hat{H} = AHA^{\dagger}. \tag{24}$$

If we restrict that the determinate of the matrix A is equal to 1, namely det $A = 1$, one observes that det $\hat{H} = 1$ as well. That is to say, the norm of the vector V is invariant under this transformation. Since all such matrices A form the group $SL(2, \mathbb{C})$, one may realize that the group $SL(2, \mathbb{C})$ defines a restricted Lorentz transformation. Besides, since the group $SL(2, \mathbb{C})$ is isomorphic to the symplectic group $Sp(2, \mathbb{C})$, it is natural to introduce a 2-dimensional symplectic vector space (spin-space) S over $\mathbb{C}$ and its dual space S$'$.

By choosing a normalized spin basis (o, ι) for S with a skew-symmetrical structure $[,]$, so that their inner product is unity

$$[o, \iota] = 1. \tag{25}$$

Introducing Penrose's abstract index notation, the above condition can be represented by

$$[o, o] = \epsilon_{AB} o^A o^B = [\iota, \iota] = \epsilon_{AB} \iota^A \iota^B = 0, \quad [o, \iota] = \epsilon_{AB} o^A \iota^B = 1, \tag{26}$$

where ϵ_{AB} is a Levi-Civita symbol, which plays a similar role as the metric tensor in the tensor world. It is easy to verify

$$\epsilon_{AB} = o_A \iota_B - \iota_A o_B. \tag{27}$$

For an arbitrary spinor ξ_B, we have

$$\xi_B = \epsilon_{AB} \xi^A = \xi^A \epsilon_{AB}. \tag{28}$$

We do the same for the dual space, such as

$$\epsilon_{A'B'} = \bar{o}_{A'} \bar{\iota}_{B'} - \bar{\iota}_{A'} \bar{o}_{B'}, \tag{29}$$

where the bar refers to the operation of complex conjugation. By convention, we omit the bar of the $\bar{\epsilon}$ in the dual space. Then a hermitian spinor h, for which $\bar{h} = h$, can be expressed as

$$h^{AA'} = h_0 \bar{o}^{AA'} + h_1 \iota^A \bar{\iota}^{A'} + h_2 o^A \bar{\iota}^{A'} + h_3 \iota^A \bar{o}^{A'} \tag{30}$$

on bases (o, ι) and $(\bar{o}, \bar{\iota})$. It is not hard to see that the set of spinors h forms a real 4-dimensional vector space. If we regard the components of $h^{AA'}$ as h^μ, the most easiest way to connect a spinor and a tensor is relabelling $AA' \to a$. The world-vector components then are bilinear in the spin-vector components and in the complex conjugate spin-vector components. In general, the counterpart of an arbitrary tensor with abstract indices $T_{a...b}{}^{c...d}$ is conventionally denoted by $T_{AA'...BB'}{}^{CC'...DD'}$. For example, the Maxwell field strength is given by $F_{ab} = F_{ABA'B'}$. Due to its anti-symmetric property, its spinor form can be further written as[77]

$$F_{AA'BB'} = \Phi_{AB} \epsilon_{A'B'} + \epsilon_{AB} \bar{\Phi}_{A'B'}, \tag{31}$$

where Φ_{AB} is a totally symmetric spinor. Clearly, the Maxwell field in spinor formalism is represented by the spinor Φ_{AB} (and its complex conjugate). Analogously, one may show that the Weyl tensor is represented by the totally symmetric Weyl spinor Ψ_{ABCD}.

Let's turn our attention back to the Weyl double copy relation, it is formulated as

$$\Psi_{ABCD} = \frac{1}{S}\Phi_{(AB}\Phi_{CD)}, \tag{32}$$

where the Weyl spinor Ψ_{ABCD} encodes the curvature of vacuum spacetime, the single copy Φ_{AB} corresponds to the Abelian gauge electromagnetic fields in Minkowski space, and the zeroth copy S is a scalar field that satisfies the Klein–Gordon equation in Minkowski space. As shown above, the Weyl double copy effectively establishes a connection between spin-2, spin-1, and spin-0 massless free fields.

In fact, an early version of this idea originated from the study of geodesic equations of charged test particles: In 1970, the work by Walker and Penrose, and another work by the same authors with Hughston and Sommers in 1972, found a special set of test electromagnetic field solutions in vacuum type D spacetime. This relation was formulated as

$$\Phi_{AB} = \psi^{2/3} o_{(A}\iota_{B)}, \tag{33}$$

where o_A and ι_A constitute a basis in spinor space, ψ is a dyad component of the Weyl spinor $\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)}$. As we can see, this equation provides a direct connection between gravity fields and the Maxwell fields living on it.

As the method applied to validate Eq.(33) is fundamental to the whole thesis, it merits a more in-depth examination.

Let's recall the Bianchi identities in 2-spinor formalism first:

$$\nabla^{AA'}\Psi_{ABCD} = 0. \tag{34}$$

For the vacuum field equations, the curvature of spacetime is characterized by the Weyl spinor following the above equations. On the other hand, the Maxwell field is represented by the Maxwell spinor $\Phi_{AB} = \phi o_{(A}\iota_{B)}$, which satisfies the sauce free field equation

$$\nabla^{AA'}\Phi_{AB} = 0. \tag{35}$$

For Petrov type D spacetime, how do the dyad components of Bianchi identities look? Substituting $\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)}$ into Eq.(34), due to the totally symmetric property of Ψ_{ABCD} one can see that there are only four possible combinations to contract with Eq.(34): $o^Bo^Co^D$, $o^Bo^C\iota^D$, $o^B\iota^C\iota^D$, $\iota^B\iota^C\iota^D$. For instance,

$$\begin{aligned}0 =& o^Bo^Co^D\nabla^{AA'}\Psi_{ABCD}\\ =& o^Bo^Co^D\left\{o_{(A}o_B\iota_C\iota_{D)}\nabla^{AA'}\psi + 2\psi o_{(A}\iota_B\iota_C\nabla^{AA'}o_{D)} + 2\psi o_{(A}o_B\iota_C\nabla^{AA'}\iota_{D)}\right\}\\ =& \frac{1}{6}\psi o_Ao^B\nabla^{AA'}o_B.\end{aligned} \tag{36}$$

From the second line to the third line, one needs to use the fundamental identities in 2-spinor formalism

$$o_Ao^A = \iota_A\iota^A = 0, \quad o_A\iota^A = -o^A\iota_A = 1. \tag{37}$$

Notably, Petrov type D solutions admit two principal null vectors, associated with o_A and ι_A, which are geodesic and shear-free. In 2-spinor formalism, they are equivalent to the identities $o_Ao^B\nabla^{AA'}o_B = 0$ and $\iota_A\iota^B\nabla^{AA'}\iota_B = 0$. Therefore, Eq.(36) is a trivial equation. Similarly, one can show that $\iota^B\iota^C\iota^D\nabla^{AA'}\Psi_{ABCD} = 0$ is also trivial. Then, what about the case of "$o^Bo^C\iota^D$"? The result is shown below,

$$\begin{aligned}0 =& o^Bo^C\iota^D\nabla^{AA'}\Psi_{ABCD}\\ =& o^Bo^C\iota^D\nabla^{AA'}[\psi o_{(A}o_B\iota_C\iota_{D)}]\\ =& o^Bo^C\iota^D\left\{o_{(A}o_B\iota_C\iota_{D)}\nabla^{AA'}\psi + 2\psi o_{(A}\iota_B\iota_C\nabla^{AA'}o_{D)} + 2\psi o_{(A}o_B\iota_C\nabla^{AA'}\iota_{D)}\right\}\\ =& o^Bo^C\iota^D\left\{\frac{4}{24}o_Ao_B\iota_C\iota_D\nabla^{AA'}\psi + \frac{2}{24}\psi\left(2o_A\iota_B\iota_C\nabla^{AA'}o_D + 2o_D\iota_A\iota_B\nabla^{AA'}o_C\right.\right.\\ &\left.\left.+2o_D\iota_A\iota_C\nabla^{AA'}o_B + 2o_D\iota_B\iota_C\nabla^{AA'}o_A\right) + \frac{2}{24}\psi[2o_Ao_D\iota_B\nabla^{AA'}\iota_C + 2o_Ao_D\iota_c\nabla^{AA'}\iota_B]\right\}\\ =& \frac{1}{6}o_A\nabla^{AA'}\psi + \frac{1}{6}\psi o_A\iota^B\nabla^{AA'}o_B - \frac{2}{6}\psi\iota_Ao^B\nabla^{AA'}o_B + \frac{1}{6}\psi\nabla^{AA'}o_A - \frac{2}{6}\psi o_Ao^B\nabla^{AA'}\iota_B\\ =& \frac{1}{6}\left(o_A\nabla^{AA'}\psi - 2\psi\iota_Ao^B\nabla^{AA'}o_B + \psi\nabla^{AA'}o_A - \psi o_A\iota^B\nabla^{AA'}o_B\right)\\ =& \frac{1}{6}\left(o_A\nabla^{AA'}\psi - 3\psi\iota_Ao^B\nabla^{AA'}o_B\right),\end{aligned} \tag{38}$$

where one needs to use the identity $o_A\iota^B - \iota_A o^B = \epsilon_A^{\ B}$ in the last step. Analogously, one may obtain

$$\begin{aligned} 0 =& o^B\iota^C\iota^D\nabla^{AA'}\Psi_{ABCD} \\ =& \frac{1}{6}\left(\iota_A\nabla^{AA'}\psi + 3\psi o_A\iota^B\nabla^{AA'}\iota_B\right). \end{aligned} \tag{39}$$

So, two non-trivial dyad components of Bianchi identities are listed here

$$o_A\nabla^{AA'}\log\psi - 3\iota_A o^B\nabla^{AA'}o_B = 0, \tag{40}$$

$$\iota_A\nabla^{AA'}\log\psi + 3o_A\iota^B\nabla^{AA'}\iota_B = 0. \tag{41}$$

Additionally, by expanding the vacuum Maxwell equation $\nabla^{AA'}\Phi_{AB} = 0$, where the Maxwell spinor Φ_{AB} is set to be of the form $\Phi_{AB} = \phi o_{(A}\iota_{B)}$, one arrives at

$$o_A\nabla^{AA'}\log\phi - 2\iota_A o^B\nabla^{AA'}o_B = 0, \tag{42}$$

$$\iota_A\nabla^{AA'}\log\phi + 2o_A\iota^B\nabla^{AA'}\iota_B = 0. \tag{43}$$

Combining the above two sets of equations, one may find that, choosing $\phi = \psi^{2/3}$ one will get a test Maxwell field $\Phi_{AB} = \psi^{2/3}o_{(A}\iota_{B)}$ in the curved spacetime characterized by the Weyl spinor Ψ_{ABCD}. This was proposed by Walker and Penrose for the first time. However, there indeed exists a closer relationship between gravity fields and electromagnetic fields in flat spacetime that was not realized until 2019 by Luna, Monteiro, Nicholson, and O'Connell[75]. It is now known as Weyl double copy relation, and it is also the topic of this thesis. In the following chapters, I will present my findings and delve into their potential implications within this field of study.

Part III

INVESTIGATING THE WEYL DOUBLE COPY

6

WEYL DOUBE COPY AND MASSLESS FREE-FIELDS IN CURVED SPACETIMES

The work presented in this chapter is based on a paper that has been published as: 'Weyl doube copy and massless free-fields in curved spacetimes' in the journal Classical and Quantum Gravity.

ABSTRACT

In spinor formalism, since any massless free-field spinor with spin higher than $1/2$ can be constructed with spin-1/2 spinors (Dirac-Weyl spinors) and scalars, we introduce a map between Weyl fields and Dirac-Weyl fields. We determine the corresponding Dirac-Weyl spinors in a given empty spacetime. Regarding them as basic units, other higher spin massless free-field spinors are then identified. Along this way, we find some hidden fundamental features related to these fields. In particular, for non-twisting vacuum Petrov type N solutions, we show that all higher spin massless free-field spinors can be constructed with one type of Dirac-Weyl spinor and the zeroth copy. Furthermore, we systematically rebuild the Weyl double copy for non-twisting vacuum type N and vacuum type D solutions. Moreover, we

show that the zeroth copy not only connects the gravity fields with a single copy but also connects the degenerate Maxwell fields with the Dirac-Weyl fields in the curved spacetime, both for type N and type D cases. Besides, we extend the study to non-twisting vacuum type III solutions. We find a particular Dirac-Weyl scalar independent of the proposed map and whose square is proportional to the Weyl scalar. A degenerate Maxwell field and an auxiliary scalar field are then identified. Both of them play similar roles as the Weyl double copy. The result further inspires us that there is a deep connection between gravity theory and gauge theory.

6.1 INTRODUCTION

In recent years, the attempt to look for the connection between gravity theory and quantum theory has been actively investigated. As is known, Yang-Mills gauge theory is by far the most successful theory to describe the micro world. On the other hand, given the experimental confirmation of gravitational waves [78, 79], Einstein's gravity theory is further confirmed as the most promising theory to describe the macro-scale universe. Therefore, it is significant to explore the relationship between these two theories. Much work has been devoted to this study. One such attempt is the double copy. It started from the research of perturbative scattering amplitudes [7–10]; with the help of exact gravity solutions, the study was extended into the classical context [41]. There are two classes of double copies: Kerr-Schild double copy [41–43, 45, 50–53, 56, 57, 59, 80, 81] and Weyl double copy [60–64, 75, 82–84]. The latter covers a broader range of spacetimes so we will focus on the Weyl double copy in this paper.

In spinor formalism, the Weyl double copy is written as

$$\Psi_{ABCD} = \frac{\Phi_{(AB}\Phi_{CD)}}{S}, \tag{44}$$

which maps a Weyl spinor Ψ_{ABCD} (a vacuum gravity field) into a single copy Φ_{AB} (a Maxwell field which satisfies the Maxwell equation in Minkowski spacetime) and a zeroth copy S (a scalar field which satisfies the wave equation in Minkowski spacetime).

Decades ago, some works [85, 86] had already given us the prediction about the Weyl double copy. In Ref. [85], given a Weyl spinor of a vacuum type D solution on a dyad (o_A, ι_A), $\Psi_{ABCD} = \psi o_{(A} o_B \iota_C \iota_{D)}$, Walker and Penrose showed that there exists a Killing spinor of valence two $\chi_{BC} = \psi^{-1/3} o_{(B}\iota_{C)}$, which satisfies the twistor equation $\nabla^{A'}_{(A}\chi_{BC)} = 0$. Based on this, the same authors, together with Hughston and Sommers, proposed [86] that in any vacuum type D spacetime with a Weyl spinor Ψ_{ABCD}, one can construct a test electromagnetic field, such that $\Phi_{AB} = \psi^{2/3} o_{(A}\iota_{B)}$[1]. This work discovered an intriguing relation between gravity and Maxwell fields in curved spacetimes. Combined with the fact that the Maxwell field is the simplest solution of the gauge theory — the case of the group $U(1)$, the Weyl double copy relation appears to be more essential between gravity theory and gauge theory. It was proposed for the first time for vacuum type D solutions [75]. Then the Weyl double copy was proven to work also for non-twisting vacuum type N solutions [82]. For the type III case, using the twistor theory, the study only showed it holds at the linearised level [61, 83]. On the other hand, the asymptotic behaviours of the Weyl double copy have been discussed in recent works [62, 63], which state that the Weyl double copy holds asymptotically for the algebraically general solutions by using the peeling property of the Weyl scalars [87, 88]. More recently, Ref. [64] studied the Weyl double copy for general type D spacetimes with

1 It might be more enlightening if we write it in form of Eq.(44), the difference is the background that the Maxwell and scalar fields are living in is a curved spacetime instead of a Minkowski spacetime.

external sources but without a cosmological constant and introduced an extended Weyl double copy. However, Weyl double copy for a general spacetime, even for vacuum spacetime with a cosmological constant, is still unknown. More generally speaking, our understanding of the connection between gravity theory and gauge theory remains to be improved. Hopefully, there are many exciting and promising roads awaiting us. In particular, although the Weyl double copy prescription does not capture the current double copy interpretation of twisting type N solution, which might require a more general and complicated prescription, the curved double copy indeed holds in this case. This fact leads us to consider that it might be helpful to study first the map between a gravity field and a test Maxwell field in the curved spacetime. Or, it might be worth exploring first the features of spin-$n/2$ $(n = 0, 1, 2, 3)$ massless free fields that live in the curved spacetime. Then the curvature information will be reflected by these lower spin fields. By probing the features of these fields, it would be easier to look for those curvature-independent fields, such as pure Maxwell field[2]. In this paper, regarding spin-$1/2$ massless free-field spinors— Dirac-Weyl (DW) spinors, as basic units, we not only identify the DW spinors but also construct higher massless free-field spinors following the proposed map. Then, one will see that the relations between gravity fields and Maxwell fields in curved spacetime found in [82, 86] are a particular case in the present work; more fundamental properties of these fields are revealed. Especially, a natural map similar to the Weyl double copy from gravity fields to pure Maxwell fields in type III spacetime is proposed with the aid of a scalar field.

The structure of our paper is as follows. In section 6.2, we give a brief review of spinor algebra and the massless free-field spinors. Section 6.3 identifies the DW spinors in vacuum type N, type D, and type III spacetimes, respectively. Regarding them as basic units, we

2 Where "pure Maxwell fields" means that the Maxwell fields are living in Minkowski spacetime as the special solutions of gauge theory, so they are totally independent of the gravity theory.

analize the properties of different spin massless free-fields, especially the DW and Maxwell fields. Then we systematically reconstruct the Weyl double copy for non-twisting vacuum type N and vacuum type D spacetimes. A new property of the zeroth copy is discovered after that. Following this way, a degenerate electromagnetic field that lives in Minkowski spacetime and an associated scalar field are obtained from the vacuum type III solutions. The discussion and conclusions are given in section 6.4.

6.2 MASSLESS FREE-FIELDS IN SPINOR FORMALISM

Since massless free-field equations have a simple form in spinor formalism, before going on, we shall give a short introduction to spinor algebra; for more details, one may refer to Refs. [77, 89].

First, let us consider an arbitrary vector V on the basis e_i,

$$V = V^0 e_0 + V^1 e_1 + V^2 e_2 + V^3 e_3. \tag{45}$$

We transfer it to a 2×2 Hermitian matrix

$$H = V^a \sigma_a = V^0 \sigma_0 + V^j \sigma_j = \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}, \tag{46}$$

where σ_0 is a 2×2 unit matrix, and σ_j $(j = 1, 2, 3)$ are Pauli spin matrices. By introducing a pair of complex numbers (ξ, η) as follows,

$$\begin{aligned} V^0 &= \tfrac{1}{\sqrt{2}}(\xi\bar{\xi} + \eta\bar{\eta}), \quad V^1 = \tfrac{1}{\sqrt{2}}(\xi\bar{\eta} + \eta\bar{\xi}), \\ V^2 &= \tfrac{1}{i\sqrt{2}}(\xi\bar{\eta} - \eta\bar{\xi}), \quad V^3 = \tfrac{1}{\sqrt{2}}(\xi\bar{\xi} - \eta\bar{\eta}), \end{aligned} \tag{47}$$

where the bar denotes the operation of complex conjugation, Eq.(46) can be translated into a new form

$$\frac{1}{\sqrt{2}}\begin{pmatrix} V^0+V^3 & V^1+iV^2 \\ V^1-iV^2 & V^0-V^3 \end{pmatrix} = \begin{pmatrix} \xi\bar{\xi} & \xi\bar{\eta} \\ \eta\bar{\xi} & \eta\bar{\eta} \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix}. \tag{48}$$

On the other hand, there is a complex linear transformation of pair $(\xi, \eta)^T$:

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{49}$$

where the hat denotes the new quantity after the transformation. If we impose the condition $\det A = 1$, it corresponds to the spin transformation, all of such matrices form the group $SL(2, \mathbb{C})$. When a spin transformation is applied, Eq.(48) becomes

$$\begin{pmatrix} V^0+V^3 & V^1+iV^2 \\ V^1-iV^2 & V^0-V^3 \end{pmatrix} \to A \begin{pmatrix} V^0+V^3 & V^1+iV^2 \\ V^1-iV^2 & V^0-V^3 \end{pmatrix} A^\dagger = \begin{pmatrix} \hat{V}^0+\hat{V}^3 & \hat{V}^1+i\hat{V}^2 \\ \hat{V}^1-i\hat{V}^2 & \hat{V}^0-\hat{V}^3 \end{pmatrix} = \hat{H}, \tag{50}$$

where the dagger denotes the operation of conjugate transpose. It is worth noting that due to the condition $\det A = 1$, the determinant of the Hermitian matrix remains invariant,

$$\det \hat{H} = \det H = (V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2. \tag{51}$$

In other words, the norm of vector V is invariant under the transformation. Therefore every matrix element A of the group $SL(2, \mathbb{C})$ defines a restricted Lorentz transformation. As is known, $SL(2, \mathbb{C})$ is homomorphic to Lorentz group[3]. Furthermore, since $SL(2, \mathbb{C})$ is isomorphic to the symplectic group $Sp(2, \mathbb{C})$, it is natural to introduce the 2-dimensional symplectic vector space (spin-space) over $\mathbb{C}$. A tensor is defined in spinor form such that

3 Note $SO(3, 1)$ is not isomorphic to $SL(2, \mathbb{C})$, not all spinors have a tensor courterpart. In general, tensors can be regarded as special cases of spinors.

$T_{a...b}{}^{c...d} = T_{AA'...BB'}{}^{CC'...DD'}$ with abstract index notation. In practice, tensors commute with associated spinors through the Infeld-van der Waerden symbols

$$\Sigma_{\boldsymbol{a}}^{\boldsymbol{AA'}} = \frac{1}{\sqrt{2}}\sigma_{\boldsymbol{a}}^{(\boldsymbol{AA'})}, \qquad \boldsymbol{a} = 0,1,2,3, \quad \boldsymbol{A} = 0,1 \tag{52}$$

under the component transformation relation

$$T_{\boldsymbol{a...c}}{}^{\boldsymbol{d...f}} = T_{\boldsymbol{AA'...CC'}}{}^{\boldsymbol{DD'...FF'}}\Sigma_{\boldsymbol{a}}^{\boldsymbol{AA'}}...\Sigma_{\boldsymbol{c}}^{\boldsymbol{CC'}}\Sigma^{\boldsymbol{d}}_{\boldsymbol{DD'}}...\Sigma^{\boldsymbol{f}}_{\boldsymbol{FF'}}, \tag{53}$$

where matrices $\sigma_{\boldsymbol{a}}$ are conventionally chosen as Pauli matrices, small bold latin letters denote the indices of tensor components, capital bold latin letters denote the indices of spinor components and the prime marks the indices on complex conjugate space[4], e.g. $\overline{T^A} = \bar{T}^{A'}$. Note that, in general, the Infeld-van der Waerden symbols are used in the specific transformation calculation between a tensor and a spinor, they will not appear in this paper. One can refer to the last part of 3.1 of Ref. [89] for more details about this symbol.

Now, let us focus on spin-space. Any vectors ξ^A can be expanded on a spinor dyad (o, ι),

$$\xi^A = \xi^0 o^A + \xi^1 \iota^A \quad \Leftrightarrow \xi^{\boldsymbol{A}} = \begin{pmatrix} \xi^0 \\ \xi^1 \end{pmatrix}, \tag{54}$$

where o^A can be any non-zero vector, and another vector ι^A is imposed to satisfy $\{o, \iota\} = 1$. Then one may find

$$o^{\boldsymbol{A}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \iota^{\boldsymbol{A}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{55}$$

In addition, the symplectic structure implies that the inner product of two arbitrary vectors satisfies

$$\{\xi, \eta\} = \varepsilon_{AB}\xi^A\eta^B = -\{\eta, \xi\}, \tag{56}$$

where ε_{AB} plays a role analogous to the metric tensor; nevertheless, it is anti-symmetric

$$\varepsilon_{AB} = -\varepsilon_{BA}. \tag{57}$$

4 In spinor algebra, the complex conjugate space is anti-isomorphic with the spin-space.

Then normalization condition reads

$$\begin{aligned} \{o,\iota\} &= \varepsilon_{AB}o^A\iota^B = -\{\iota,o\} = -\varepsilon_{AB}\iota^A o^B = 1, \\ \{o,o\} &= \varepsilon_{AB}o^A o^B = 0, \ \ \{\iota,\iota\} = \varepsilon_{AB}\iota^A\iota^B = 0, \end{aligned} \tag{58}$$

where it is easy to see

$$\varepsilon_{\boldsymbol{AB}} = \varepsilon^{\boldsymbol{AB}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{59}$$

and

$$\varepsilon_{AB} = 2o_{[A}\iota_{B]}, \quad \varepsilon^{AB} = 2o^{[A}\iota^{B]}. \tag{60}$$

The rule of raising and lowering indices is as follows

$$\varepsilon^{AB}\xi_B = \xi^A, \quad \xi^A\varepsilon_{AB} = \xi_B. \tag{61}$$

The relations above also hold in the complex conjugate space.

In addition, the null tetrad can be written in terms of the spinor bases

$$\begin{aligned} \ell^a &= o^A\bar{o}^{A'}, \ \ n^a = \iota^A\bar{\iota}^{A'}, \ \ m^a = o^A\bar{\iota}^{A'}, \ \ \bar{m}^a = \iota^A\bar{o}^{A'}, \\ \ell_a &= o_A\bar{o}_{A'}, \ \ n_a = \iota_A\bar{\iota}_{A'}, \ \ m_a = o_A\bar{\iota}_{A'}, \ \ \bar{m}_a = \iota_A\bar{o}_{A'}, \end{aligned} \tag{62}$$

where real null vectors ℓ and n satisfy $\ell^2 = n^2 = 0$, $\ell \cdot n = 1$, complex null vectors m and $\bar{m}$ satisfy $m^2 = \bar{m}^2 = 0$, $m \cdot \bar{m} = -1$, furthermore, $\ell \cdot m = n \cdot m = \ell \cdot \bar{m} = n \cdot \bar{m} = 0$. The definitions of spin coefficients in this paper are consistent with Appendix B of Ref. [89] and Ref. [73]. They are listed as follows

$$\begin{aligned} \kappa^* &= m^a\ell^b\nabla_b\ell_a, & \pi^* &= n^a\ell^b\nabla_b\bar{m}_a, & \epsilon^* &= \frac{1}{2}(n^a\ell^b\nabla_b\ell_a + m^a\ell^b\nabla_b\bar{m}_a), \\ \tau^* &= m^a n^b\nabla_b\ell_a, & \nu^* &= n^a n^b\nabla_b\bar{m}_a, & \gamma^* &= \frac{1}{2}(n^a n^b\nabla_b\ell_a + m^a n^b\nabla_b\bar{m}_a), \\ \sigma^* &= m^a m^b\nabla_b\ell_a, & \mu^* &= n^a m^b\nabla_b\bar{m}_a, & \beta^* &= \frac{1}{2}(n^a m^b\nabla_b\ell_a + m^a m^b\nabla_b\bar{m}_a), \\ \rho^* &= m^a\bar{m}^b\nabla_b\ell_a, & \lambda^* &= n^a\bar{m}^b\nabla_b\bar{m}_a, & \alpha^* &= \frac{1}{2}(n^a\bar{m}^b\nabla_b\ell_a + m^a\bar{m}^b\nabla_b\bar{m}_a). \end{aligned} \tag{63}$$

To distinguish from other symbols, we use the star to mark the spin coefficients.

We have given a short introduction to spinor algebra. Now, let us turn to massless free-fields (source-free). We shall list the corresponding spinors without proof, one may refer to sections 5.7 of Ref. [77] for more details.

Given a symmetric spinor with n indexes $\mathcal{S}_{A_1 A_2 \ldots A_n}$, spin-$n/2$ massless free-field equations are translated into a simple form

$$\nabla^{A_1 A_1'} \mathcal{S}_{A_1 A_2 \ldots A_n} = 0. \tag{64}$$

When $n = 4$, the spinor $\mathcal{S}$ refers to a Weyl spinor Ψ_{ABCD} translated from the Weyl tensor[5]

$$C_{abcd} = C_{AA'BB'CC'DD'} = \Psi_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{CD}. \tag{65}$$

Following the vacuum Einstein's field equation, Eq.(211) in this case represents the Bianchi identity (with or without a cosmological constant)

$$\nabla^{AA'} \Psi_{ABCD} = 0. \tag{66}$$

According to Petrov classification, there are five different types of solutions:

$$\begin{aligned} I : \Psi_{ABCD} &\sim \tilde{\alpha}_{(A} \tilde{\beta}_B \tilde{\gamma}_C \tilde{\delta}_{D)}, \\ II : \Psi_{ABCD} &\sim \tilde{\alpha}_{(A} \tilde{\alpha}_B \tilde{\gamma}_C \tilde{\delta}_{D)}, \\ III : \Psi_{ABCD} &\sim \tilde{\alpha}_{(A} \tilde{\alpha}_B \tilde{\alpha}_C \tilde{\delta}_{D)}, \\ D : \Psi_{ABCD} &\sim \tilde{\alpha}_{(A} \tilde{\alpha}_B \tilde{\delta}_C \tilde{\delta}_{D)}, \\ N : \Psi_{ABCD} &\sim \tilde{\alpha}_{(A} \tilde{\alpha}_B \tilde{\alpha}_C \tilde{\alpha}_{D)}, \end{aligned} \tag{67}$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ and $\tilde{\delta}$ are four different non-proportional and non-vanishing spinors. The tilde is used to distinguish them from the spin coefficients. In addition, with the help of

5 As an exception, $\bar{\varepsilon}_{A'B'}$ is usually abbreviated as $\varepsilon_{A'B'}$, that is also applied for raised index version. In addition, one may realize that the first identity of Eq.(212) is a general tensor-spinor identity with abstract indices, we do not need to use the Infeld-van der Waerden symbols here.

Newman-Penrose formalism, the Weyl tensor is reduced to five independent complex scalars,

$$\begin{aligned}
\psi_0 &= C_{abcd}\ell^a m^b \ell^c m^d = \Psi_{ABCD}o^A o^B o^C o^D, \\
\psi_1 &= C_{abcd}\ell^a m^b \ell^c n^d = \Psi_{ABCD}o^A o^B o^C \iota^D, \\
\psi_2 &= C_{abcd}\ell^a m^b \bar{m}^c n^d = \Psi_{ABCD}o^A o^B \iota^C \iota^D, \\
\psi_3 &= C_{abcd}\ell^a n^b \bar{m}^c n^d = \Psi_{ABCD}o^A \iota^B \iota^C \iota^D, \\
\psi_4 &= C_{abcd}\bar{m}^a n^b \bar{m}^c n^d = \Psi_{ABCD}\iota^A \iota^B \iota^C \iota^D.
\end{aligned} \tag{68}$$

The second set of equalities is obtained from Eq.(215) and Eq.(212). Then the Weyl spinor can be expanded in a general form

$$\begin{aligned}
\Psi_{ABCD} = {} & \psi_0 \iota_A \iota_B \iota_C \iota_D - 4\psi_1 o_{(A}\iota_B \iota_C \iota_{D)} + 6\psi_2 o_{(A} o_B \iota_C \iota_{D)} \\
& - 4\psi_3 o_{(A} o_B o_C \iota_{D)} + \psi_4 o_A o_B o_C o_D.
\end{aligned} \tag{69}$$

When $n = 2$, the spinor $\mathcal{S}$ refers to an electromagnetic spinor Φ_{AB} translated from the Maxwell tensor

$$F_{ab} = F_{AA'BB'} = \Phi_{AB}\varepsilon_{A'B'} + \bar{\Phi}_{A'B'}\varepsilon_{AB}. \tag{70}$$

Eq.(211) in this case represents the source-free Maxwell equation

$$\nabla^{AA'}\Phi_{AB} = 0. \tag{71}$$

In analogy to the Weyl spinor, there are two different types of Maxwell spinors:

$$\begin{aligned}
I &: \Phi_{AB} \sim \tilde{\alpha}_{(A}\tilde{\delta}_{B)}, \\
N &: \Phi_{AB} \sim \tilde{\alpha}_A \tilde{\alpha}_B,
\end{aligned} \tag{72}$$

where $\tilde{\alpha}_A$ and $\tilde{\delta}_A$ are two non-proportional spinors. We also call Type N Maxwell spinor as degenerate Maxwell spinor. Because the corresponding electric fields $\boldsymbol{E}$ and magnetic fields $\boldsymbol{B}$ are of the same magnitude and they are perpendicular; namely, $|\mathbf{B}|^2 - |\mathbf{E}|^2 = 0$,

$\mathbf{B}\cdot\mathbf{E}=0$. In addition, for later convenience, we define three typical Maxwell spinors as follows,

Type I:

$$\Phi^{(1)}_{AB}=\phi_1 o_{(A}\iota_{B)}, \tag{73}$$

Type N:

$$\Phi^{(0)}_{AB}=\phi_0\iota_A\iota_B, \tag{74}$$

$$\Phi^{(2)}_{AB}=\phi_2 o_A o_B, \tag{75}$$

where the coefficients ϕ_1,ϕ_0 and ϕ_2 are called Maxwell scalars. They are expanded in three different ways in the spin space. Substituting Eq.(73) into Eq.(71), then multiplying o^B and ι^B on Eq.(71), respectively, we obtain two dyad components of the field equation,

$$o_A\nabla^{AA'}\log\phi_1-2\iota_A o^B\nabla^{AA'}o_B=0, \tag{76}$$

$$\iota_A\nabla^{AA'}\log\phi_1+2o_A\iota^B\nabla^{AA'}\iota_B=0. \tag{77}$$

Analogously, from Eq.(74) and Eq.(75) we arrive at

$$\iota_A\nabla^{AA'}\log\phi_0-2\iota_A o^B\nabla^{AA'}\iota_B+o_A\iota^B\nabla^{AA'}\iota_B=0, \tag{78}$$

$$o_A\nabla^{AA'}\log\ \phi_2+2o_A\iota^B\nabla^{AA'}o_B-\iota_A o^B\nabla^{AA'}o_B=0. \tag{79}$$

Recalling Eq.(70), the tensor forms of the above three Maxwell spinors read

$$F^{(0)}_{ab}=2\phi_0\bar{m}_{[a}n_{b]}+2\bar{\phi}_0 m_{[a}n_{b]}, \tag{80}$$

$$F^{(1)}_{ab}=2\phi_1\left(\ell_{[a}n_{b]}+\bar{m}_{[a}m_{b]}\right)+2\bar{\phi}_1\left(\ell_{[a}n_{b]}+m_{[a}\bar{m}_{b]}\right), \tag{81}$$

$$F^{(2)}_{ab}=2\phi_2\ell_{[a}m_{b]}+2\bar{\phi}_2\ell_{[a}\bar{m}_{b]}. \tag{82}$$

When $n=1$, the spinor $\mathcal{S}$ refers to a DW spinor ξ_A translated from the DW tensor

$$P_{ab}=\xi_A\xi_B\varepsilon_{A'B'}. \tag{83}$$

Eq.(211) in this case represents the DW equation

$$\nabla^{AA'}\xi_A = 0. \tag{84}$$

The tensor form is given by

$$P_{ab}\nabla_d P_c^d + P_{ad}\nabla_c P_b^d = 0. \tag{85}$$

On the spinor dyad (o, ι), clearly, there are only two types of DW spinors:

$$\xi_A = \xi o_A, \tag{86a}$$

$$\eta_A = \eta \iota_A, \tag{86b}$$

where ξ and η are called DW scalars. Substitution of the above equations into Eq.(221) yields

$$o_A \nabla^{AA'} \log\, \xi + o_A \iota^B \nabla^{AA'} o_B - \iota_A o^B \nabla^{AA'} o_B = 0, \tag{87}$$

$$\iota_A \nabla^{AA'} \log\, \eta - \iota_A o^B \nabla^{AA'} \iota_B + o_A \iota^B \nabla^{AA'} \iota_B = 0. \tag{88}$$

Throughout this paper, one will find that Eq.(76)-Eq.(79), Eq.(87)-Eq.(88), and the given Bianchi identities are the basic equations for our calculation.

It is worthwhile to mention that Dirac's equation is just a pair of coupled DW equations with a source

$$\left.\begin{aligned} \nabla^A_{A'}\xi_A &= \mu\bar{\eta}_{A'} \\ \nabla^{A'}_A\bar{\eta}_{A'} &= \mu\xi_A, \end{aligned}\right\} \tag{89}$$

where μ is a real constant related to the mass of the spinor. The tensor version is written as

$$\left.\begin{aligned} P_{ab}\nabla_d P_c^d + P_{ad}\nabla_c P_b^d &= -2\mu P_{ab} C_c, \\ Q_{ab}\nabla_d Q_c^d + Q_{ad}\nabla_c Q_b^d &= -2\mu Q_{ab} C_c, \end{aligned}\right\} \tag{90}$$

where $C_a = \xi_A\bar{\eta}_{A'}$, and the field Q_{ab} written in terms of another spin-$1/2$ spinor η reads

$$Q_{ab} = \eta_A\eta_B\varepsilon_{A'B'}. \tag{91}$$

Although we will not use Dirac's equation in this paper, the above formulas might be useful for studying the double copy in non-vacuum spacetimes in the future.

For spin-$3/2$ massless free-fields equation, the field equation is given by

$$\nabla^{AA'}\Omega_{ABC} = 0. \tag{92}$$

One may refer to section 5.8 of Ref. [77] for more details about Ω_{ABC}, and we will instead pay more attention to the other three massless free-fields in the following.

6.3 FROM GRAVITY FIELDS TO LOWER SPIN MASSLESS FREE-FIELDS

Inspired by the Weyl double copy relation Eq.(44), with the fact that any massless free-field spinor with spin higher than $\frac{1}{2}$ can be constructed by scalar fields and DW spinors, we introduce a general map between vacuum gravity fields and DW fields[6] in the curved spacetime.

$$\Psi_{ABCD} = \frac{\xi_{(A}\eta_B\zeta_C\chi_{D)}}{S_{14}}. \tag{93}$$

Notably, the above four DW spinors could be identical depending on which type of spacetime we are considering, and S_{ij} is a scalar field connecting spin-$i/2$ spinors with spin-$j/2$ spinors e.g. $i = 1, j = 4$ here. Then, with Eq.(66) and Eq.(221), it is easy to identify what kind of DW fields can exist for a specific curved spacetime. Furthermore, if we regard DW spinors as basic units, other higher spin massless free fields in the curved spacetime are able to be constructed as well. For example, we have

$$\Phi_{AB} = \frac{\xi_{(A}\eta_{B)}}{S_{12}} \tag{94}$$

6 All of the lower spin massless free-field $(i = 1, 2, 3)$ considered in this paper is assumed to be a test field, which will not curve the spacetime.

and $\Omega_{ABC} = \frac{\xi_{(A}\eta_B\zeta_{C)}}{S_{13}}$. Especially, with respect to three Maxwell spinors $\Phi^{(0)}_{AB}$, $\Phi^{(1)}_{AB}$, and $\Phi^{(2)}_{AB}$, we define the associated scalars S_{12} as follows

$$\Phi^{(0)}_{AB} = \frac{\eta_A\eta_B}{S^{(0)}_{12}}, \quad \Phi^{(1)}_{AB} = \frac{\xi_{(A}\eta_{B)}}{S^{(1)}_{12}}, \quad \Phi^{(2)}_{AB} = \frac{\xi_A\xi_B}{S^{(2)}_{12}}, \tag{95}$$

where $\xi_A = \xi o_A$, $\eta_A = \eta\iota_A$. Based on Eq.(93) and Eq.(94), it is natural to lead to a map connecting gravity fields with Maxwell fields in the curved spacetime

$$\Psi_{ABCD} = \frac{\Phi_{(AB}\Theta_{CD)}}{S_{24}}, \tag{96}$$

where Θ_{CD} is also a Maxwell spinor $\Theta_{CD} = \frac{\zeta_{(C}\chi_{D)}}{S'_{12}}$ with another scalar field S'_{12}, as long as

$$S_{24} = \frac{S_{14}}{S_{12}S'_{12}}. \tag{97}$$

Notably, Eq.(96) admits a similar form to the Weyl double copy Eq.(44). In fact, the curved double copy for type N spacetimes and the specific relation, $\Phi_{AB} = \psi^{2/3}o_{(A}\iota_{B)}$ we mentioned in Section 7.1, are just particular cases of the present work. Along the above method, one may ask about other situations. Is there a special relationship between different auxiliary scalar fields S_{ij}? What kind of spin-$i/2$ massless free-fields can exist in a specific spacetime? Can we directly map a gravity field to a DW field that is living in flat space? What about mapping to Maxwell fields for type III spacetimes? We shall answer these questions in the following.

6.3.1 *Vacuum type N solutions*

For a vacuum type N spacetime, the Weyl tensor has only one no-vanishing component ψ_4. Combining with Eq.(218), the Weyl spinor reads $\Psi_{ABCD} = \Psi_4 o_A o_B o_C o_D$ with $\Psi_4 = \psi_4$. According to Eq.(93), there is only one type of DW spinor ξ_A along the basis o in the spacetime, which follows Eq.(86a). In other words, one can always find a special DW spinor, such that

$$\Psi_{ABCD} = \frac{\xi_{(A}\xi_B\xi_C\xi_{D)}}{S_{14}}. \tag{98}$$

In this case, due to the symmetry property of Ψ_{ABCD}, the Bianchi identity Eq.(66) are expanded into two non-trivial dyad components

$$o_A \nabla^{AA'} \log\ \Psi_4 + 4 o_A \iota^B \nabla^{AA'} o_B - \iota_A o^B \nabla^{AA'} o_B = 0, \tag{99}$$

$$o_A o^B \nabla^{AA'} o_B = 0. \tag{100}$$

Based on the Goldberg-Sachs theorem [90], the congruence formed by the principal null-direction ℓ for algebraically specially spacetime (e.g. type N spacetime here) must be geodesic $\kappa^* = 0$ and shear-free $\sigma^* = 0$, the second equation should hold automatically. Thus, only Eq.(99) is left. Making use of Eq.(87), Eq.(98) and Eq.(99), it is not hard to identify the DW spinor ξ_A[7]. Before doing that, let us first pay attention to constructing the Maxwell spinor; then, the problem will be resolved automatically.

Since there is only one type of DW spinor ξ_A in the spacetime, the unique formula of the Maxwell spinor is given by

$$\Phi^{(2)}_{AB} = \frac{\xi^2}{S^{(2)}_{12}} o_A o_B = \phi_2 o_A o_B \quad \Leftrightarrow \quad \phi_2 = \frac{\xi^2}{S^{(2)}_{12}}. \tag{101}$$

According to Eq.(79), one can see that there is only one independent dyad component of the field equation. Substituting Eq.(101) into Eq.(79), one observes that the scalar field $S^{(2)}_{12}$ satisfies

$$o_A \nabla^{AA'} \log\ S^{(2)}_{12} - \iota_A o^B \nabla^{AA'} o_B = 0. \tag{102}$$

7 It is worthwhile pointing out here that Eq.(87) exactly verifies the statement of Ref. [82]—the coefficient of the middle term of the left side of the equation is the rank of the corresponding spinor.

It can be identified by solving equations

$$\ell \cdot \nabla \log\, S_{12}^{(2)} - \rho^* = 0, \quad m \cdot \nabla \log\, S_{12}^{(2)} - \tau^* = 0. \tag{103}$$

This is an interesting result, since the scalar $S_{12}^{(2)}$ shares the same equation with the scalar S_{24}[8] discovered by Ref. [82]. Further more, assuming spin-3/2 massless free-field spinor are constructed by DW spinor as follows

$$\Omega_{ABC} = \frac{\xi_A \xi_B \xi_C}{S_{13}} = \omega o_A o_B o_C. \tag{104}$$

Setting $S_{13} = (S_{12}^{(2)})^2$ (it will be soon clear why we do this), combining Eq.(221) and Eq.(92), one will get Eq.(102) again. Therefore, for vacuum type N spacetimes, the connection between Weyl spinors and other lower spin massless free-field spinors can be summarized as follows

$$\Psi_4 = \frac{\xi^4}{(S_{12}^{(2)})^3}, \quad \omega = \frac{\xi^3}{(S_{12}^{(2)})^2}, \quad \phi_2 = \frac{\xi^2}{(S_{12}^{(2)})}. \tag{105}$$

Clearly, the curved double copy is covered by the above relations. In addition, according to Eq.(83), the DW tensor on the null tetrad reads

$$P_{ab} = 2\xi^2 \ell_{[a} m_{b]} = 2S_{12}^{(2)} \sqrt{\Psi_4 S_{12}^{(2)}}\, \ell_{[a} m_{b]}. \tag{106}$$

Using Eq.(85), it is easy to verify whether the DW fields depend on the curvature or not.

Next, we shall show several specific investigations on exact vacuum type N solutions.

6.3.1.1 *Kundt solutions*

Firstly, let us focus on non-diverging solutions ($\rho^* = 0$), usually, called Kundt solutions [91]. There are two classes of Kundt solutions. One of them is plane-fronted wave with parallel propagation, called pp waves, the metric reads

$$ds^2 = 2du(dv + H du) - 2dz d\bar{z}, \tag{107}$$

8 Different from the original work, to keep consistent with the notion of this paper, here we use S_{24} to denote the harmonic scalar field of the curved double copy, instead of S.

where $H(u,z,\bar{z}) = f(u,z) + \bar{f}(u,\bar{z})$ with a general function f. Choosing a null tetrad

$$\ell = \partial_v, \quad n = \partial_\mu - H\partial_v, \quad m = \partial_z, \tag{108}$$

one can find $\tau^* = 0$, and $\Psi_4 = -\partial_{\bar{z}}^2 \bar{f}(u,\bar{z})$. Solving Eq.(226) we have $S_{12}^{(2)} = \mathcal{G}(u,\bar{z})$, which is an arbitrary function of u and $\bar{z}$. Therefore, from Eq.(105) the DW scalar is solved by

$$\xi^2 = \sqrt{-\partial_{\bar{z}}^2 \bar{f}(u,\bar{z})\mathcal{G}^3(u,\bar{z})}. \tag{109}$$

Clearly, due to the appearance of $\mathcal{G}(u,\bar{z})$, $\xi(u,\bar{z})$ can be any function of u and $\bar{z}$. Moreover, turning to tensor version, one observes

$$2\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{110}$$

the only non-vanishing components of P_{ab} are $P_{u\bar{z}} = -P_{\bar{z}u} = -\xi^2(u,\bar{z})$. Particularly, it is simple to check that this satisfies the DW equation not only in curved spacetime but also in Minkowski spacetime where we just need to set $H = f = 0$ in the metric.

The degenerate Maxwell scalar is then given by

$$\phi_2 = \sqrt{-\partial_{\bar{z}}^2 \bar{f}(u,\bar{z})\mathcal{G}(u,\bar{z})}; \tag{111}$$

this is nothing but the result of Ref. [82], which admits the Weyl double copy.

Another class is given by

$$\begin{gathered} \mathrm{d}s^2 = 2\mathrm{d}u(\mathrm{d}v + W\mathrm{d}z + \bar{W}\mathrm{d}\bar{z} + H\mathrm{d}u) - 2\mathrm{d}z\mathrm{d}\bar{z}, \\ W(v,z,\bar{z}) = \frac{-2v}{(z+\bar{z})}, \quad H(u,v,z,\bar{z}) = \left[f(u,z) + \bar{f}(u,\bar{z})\right](z+\bar{z}) - \frac{v^2}{(z+\bar{z})^2}, \end{gathered} \tag{112}$$

where $f(u,z)$ is an arbitrary function . A null tetrad is chosen as follows

$$\ell = \partial_v, \quad n = \partial_u - (H + W\bar{W})\partial_v + \bar{W}\partial_z + W\partial_{\bar{z}}, \quad m = \partial_z. \tag{113}$$

Then one obtains

$$\tau^* = -\frac{1}{z+\bar{z}}, \quad \Psi_4 = -(z+\bar{z})\partial^2_{\bar{z}}\bar{f}, \quad S_{12} = \frac{\zeta(u,\bar{z})}{z+\bar{z}}, \tag{114}$$

where $\zeta(u,\bar{z})$ is an arbitrary function. So, DW scalar is given by

$$\xi^2 = \frac{\sqrt{-\partial^2_{\bar{z}}\bar{f}(u,\bar{z})\zeta^3(u,\bar{z})}}{(z+\bar{z})}. \tag{115}$$

In this case

$$2\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{116}$$

it is easy to check that the corresponding DW equation also holds in Minkowski spacetime. As we can see from Eq.(115) and Eq.(116); the DW field is curvature-independnet. More importantly, the degenerate Maxwell spinor is given by

$$\phi_2 = \sqrt{-\zeta(u,\bar{z})\partial^2_{\bar{z}}\bar{f}}, \tag{117}$$

which is consistent with the result of Ref. [82] and will lead to the double copy.

6.3.1.2 *Robinson-Trautman solutions*

The solutions of general vacuum type N spacetimes admitting a geodesic, shear-free, non-twisting but diverging null congruence are given by Robinson and Trautman [91, 92]

$$\begin{gathered} \mathrm{d}s^2 = H\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - \frac{2r^2}{P^2}\mathrm{d}z\mathrm{d}\bar{z}, \\ H(u,r,z,\bar{z}) = k - 2r\partial_u \log P, \qquad (k = 0, \pm 1) \\ k = 2P^2\partial_z\partial_{\bar{z}} \log P(u,z,\bar{z}). \end{gathered} \tag{118}$$

Choosing a null tetrad as follows

$$\ell = \partial_r, \quad n = \partial_u - \frac{1}{2}H\partial_r, \quad m = -\frac{P}{r}\partial_z, \tag{119}$$

one obtains $\rho^* = -1/r$, $\tau^* = 0$, and

$$\Psi_4 = \frac{P^2}{r}\partial_u\left(\frac{\partial_{\bar{z}}^2 P}{P}\right), \quad S_{12} = \frac{\mathcal{R}(u,\bar{z})}{r}, \tag{120}$$

where $\mathcal{R}(u,\bar{z})$ is an arbitrary function. Since

$$2P\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r & 0 & 0 & 0 \end{pmatrix}, \tag{121}$$

which is the same as the Kundt cases; there is only one independent component. The DW spinor is represented by

$$\xi^2 = \frac{P}{r^2}\sqrt{\mathcal{R}^3(u,\bar{z})\partial_u\left(\frac{\partial_{\bar{z}}^2 P}{P}\right)}. \tag{122}$$

The information of the structure function $P(u,z,\bar{z})$ cannot be canceled by the function $\mathcal{R}(u,\bar{z})$. and it is easy to check that the DW field does not satisfy its field equation in Minkowski spacetime. While for the degenerate Maxwell scalar

$$\phi_2 = \frac{P}{r}\sqrt{\mathcal{R}(u,\bar{z})\partial_u\left(\frac{\partial_{\bar{z}}^2 P}{P}\right)}, \tag{123}$$

as expected, it is consistent with the result of Ref. [82] and leads to the double copy relation.

In summary, we rebuild the Weyl double copy with the help of DW field for non-twisting type N solutions. In addition, we find that only for the Kundt class, we can obtain a DW field such that it satisfies its field equation in Minkowski spacetime.

6.3.2 *Vacuum type D solutions*

For vacuum type D spacetimes, according to Eq.(218), the Weyl tensor has only one non-vanishing component ψ_2. In this case, the Weyl spinor is reduced to

$$\Psi_{ABCD} = \Psi_2 o_{(A} o_B \iota_C \iota_{D)}, \tag{124}$$

where we let $\Psi_2 = 6\psi_2$. The map Eq.(93) is choosen as

$$\Psi_{ABCD} = \frac{\xi_{(A}\xi_B \eta_C \eta_{D)}}{S_{14}}, \tag{125}$$

where the Weyl spinor is constructed by two mutually orthogonal DW spinors given by Eq.(86) with the condition $\xi = \eta$. Expanding the above equation on the spin bases, we obtain a scalar identity

$$\Psi_2 = \frac{\xi^4}{S_{14}}. \tag{126}$$

Following the Goldberg-Sachs theorem, the congruences are formed by two principal null-directions for type D spacetimes, namely, ℓ and n, and they should be geodesic and shear-free, i.e. $\kappa^* = \sigma^* = \nu^* = \lambda^* = 0$. So, non-trivial dyad components of the Bianchi identity Eq.(66) are given by

$$o_A \nabla^{AA'} \log(\Psi_2) - 3\iota_A o^B \nabla^{AA'} o_B = 0, \tag{127}$$

$$\iota_A \nabla^{AA'} \log(\Psi_2) + 3 o_A \iota^B \nabla^{AA'} \iota_B = 0. \tag{128}$$

In analogy to the case of type N, combining Eq.(87), Eq.(126) and Eq.(127) we have

$$o_A \nabla^{AA'} \log S_{14} + 4 o_A \iota^B \nabla^{AA'} o_A - \iota_A o^B \nabla^{AA'} o_B = 0. \tag{129}$$

Similarly, from Eq.(88), Eq.(126) and Eq.(128) we have

$$\iota_A \nabla^{AA'} \log S_{14} - 4\iota_A o^B \nabla^{AA'} \iota_B + o_A \iota^B \nabla^{AA'} \iota_B = 0. \tag{130}$$

Multiplying $\bar{o}_{A'}$ and $\bar{\iota}_{A'}$ on the above equations, respectively, S_{14} is solved by

$$\begin{aligned} \ell\cdot\nabla\log S_{14}+4\epsilon^*-\rho^*=0, \quad m\cdot\nabla\log S_{14}+4\beta^*-\tau^*=0, \\ \bar{m}\cdot\nabla\log S_{14}-4\alpha^*+\pi^*=0, \quad n\cdot\nabla\log S_{14}-4\gamma^*+\mu^*=0. \end{aligned} \tag{131}$$

This is an overdetermined system since there is only one unknown quantity, and one will soon see that S_{14} satisfies its integrability condition, so we can always find its solution. Once S_{14} is solved, the DW scalars will then be identified. Different from the case of type N, since there are two types of DW spinors for vacuum type D spacetime, we can find two types of Maxwell fields in the curved spacetime. One is degenerate, the simplest forms are

$$\Phi^{(0)}_{AB}=\frac{\xi^2}{S^{(0)}_{12}}\iota_A\iota_B=\phi_0\iota_A\iota_B \quad\Leftrightarrow\quad \phi_0=\frac{\xi^2}{S^{(0)}_{12}}, \tag{132}$$

$$\Phi^{(2)}_{AB}=\frac{\xi^2}{S^{(2)}_{12}}o_Ao_B=\phi_2 o_Ao_B \quad\Leftrightarrow\quad \phi_2=\frac{\xi^2}{S^{(2)}_{12}}. \tag{133}$$

The other one is non-degenerate, the simplest form reads

$$\Phi^{(1)}_{AB}=\frac{\xi^2}{S^{(1)}_{12}}o_{(A}\iota_{B)}=\phi_1 o_{(A}\iota_{B)} \quad\Leftrightarrow\quad \phi_1=\frac{\xi^2}{S^{(1)}_{12}}. \tag{134}$$

Correspondingly, we can build two different maps between gravity fields and Maxwell fields in the curved spacetime starting from the relation Eq.(126)

$$\Psi_2=\frac{\frac{\xi^2}{S^{(0)}_{12}}\frac{\xi^2}{S^{(2)}_{12}}}{S^{(0,2)}_{24}}=\frac{\phi_0\phi_2}{S^{(0,2)}_{24}} \quad\Leftrightarrow\quad \Psi_{ABCD}=\frac{\Phi^{(0)}_{(AB}\Phi^{(2)}_{CD)}}{S^{(0,2)}_{24}}, \tag{135}$$

$$\Psi_2=\frac{(\frac{\xi^2}{S^{(1)}_{12}})^2}{S^{(1,1)}_{24}}=\frac{(\phi_1)^2}{S^{(1,1)}_{24}} \quad\Leftrightarrow\quad \Psi_{ABCD}=\frac{\Phi^{(1)}_{(AB}\Phi^{(1)}_{CD)}}{S^{(1,1)}_{24}}, \tag{136}$$

where upper index (i,j) refers to the case of mixed Maxwell scalars $\phi_i\phi_j$. One can see that the first case involves two degenerate Maxwell spinors which might lead to mixed double copy; while for the second case, it corresponds to the classical Weyl double copy [75], for which $S^{(1,1)}_{24}=(\phi_1)^{1/2}=(\Psi_2)^{1/3}$. We will restudy this along a new way with the help of DW spinors. And we will also check whether the mixed double copy of the first case hold or not in Minkowski spacetime. The main point, in the following, is looking for source-independent Maxwell fields.

Firstly, let us focus on degenerate Maxwell spinors. Clearly, once DW spinors are identified, to obtain the Maxwell spinor, the only work left for us is to identify S_{12}. Combining Eq.(78) and Eq.(132), $S_{12}^{(0)}$ can be solved by

$$\bar{m} \cdot \nabla \log S_{12}^{(0)} + \pi^* = 0, \quad n \cdot \nabla \log S_{12}^{(0)} + \mu^* = 0. \tag{137}$$

Analogously, $S_{12}^{(2)}$ can be solved as well; the corresponding equations have been shown in Eq.(102) for the type N case. Since the equation of $S_{12}^{(2)}$ is independent of the Petrov type of spacetime. To avoid redundancy, we will not show this equation again.

For the second case, substitution of Eq.(87) and Eq.(134) into Eq.(76) yields

$$o_A \nabla \log S_{12}^{(1)} + 2 o_A \iota^B \nabla^{AA'} o_B = 0. \tag{138}$$

Similarly, substitution of Eq.(88) and Eq.(134) into Eq.(77) yields

$$\iota_A \nabla^{AA'} \log S_{12}^{(1)} - 2 \iota_A o^B \nabla^{AA'} \iota_B = 0. \tag{139}$$

Multiplying $\bar{o}_{A'}$, $\bar{\iota}_{A'}$ respectively, $S_{12}^{(1)}$ is solved by

$$\begin{aligned} \ell \cdot \nabla \log S_{12}^{(1)} + 2\epsilon^* = 0, \quad m \cdot \nabla \log S_{12}^{(1)} + 2\beta^* = 0, \\ \bar{m} \cdot \nabla \log S_{12}^{(1)} - 2\alpha^* = 0, \quad n \cdot \nabla \log S_{12}^{(1)} - 2\gamma^* = 0. \end{aligned} \tag{140}$$

This is also an overdetermined system, it is easy to check that the integrability condition holds in this case, and the solution can always be solved.

Therefore, once S_{14} is fixed, we can identify the DW fields; the Maxwell fields' property then will be revealed by S_{12}. If one is instead interested in spin-$3/2$ massless free-fields, S_{13} then will be the key point. Thus, it is a good starting point to identify first the DW fields when investigating the higher spin massless free-fields. Then, the rest information of the target fields will be encoded in the associated scalar fields. Once DW fields and the associated scalar fields are identified, the property of the target fields should be clear. Using this method we can now look for source-independent electromagnetic fields for vacuum type D solutions. More illustration on verifying the Weyl double copy relation will be given with the help of the modified Plebański–Demiański metric.

6.3.2.1 *Modified vacuum Plebański-Demiański metric*

The Plebański-Demiański metric gives a complete family of type D spacetimes [93, 94], the original line element reads

$$\mathrm{d}s^2 = \frac{1}{(1-\hat{p}\hat{r})^2}\left[\frac{\mathcal{Q}\left(\mathrm{d}\hat{\tau}-\hat{p}^2\mathrm{d}\hat{\sigma}\right)^2}{\hat{r}^2+\hat{p}^2} - \frac{\mathcal{P}\left(\mathrm{d}\hat{\tau}+\hat{r}^2\mathrm{d}\hat{\sigma}\right)^2}{\hat{r}^2+\hat{p}^2} - \frac{\hat{r}^2+\hat{p}^2}{\mathcal{P}}\mathrm{d}\hat{p}^2 - \frac{\hat{r}^2+\hat{p}^2}{\mathcal{Q}}\mathrm{d}\hat{r}^2\right], \tag{141}$$

where

$$\begin{aligned}\mathcal{P} &= k' + 2N'\hat{p} - \epsilon'\hat{p}^2 + 2M'\hat{p}^3 - \left(k' + e'^2 + g'^2 + \Lambda/3\right)\hat{p}^4 \\ \mathcal{Q} &= \left(k' + e'^2 + g'^2\right) - 2M'\hat{r} + \epsilon'\hat{r}^2 - 2N'\hat{r}^3 - (k' + \Lambda/3)\hat{r}^4.\end{aligned} \tag{142}$$

It includes seven free real parameters, $M', N', e', g', \epsilon', k'$, and Λ. Besides the cosmological constant Λ, M' is the mass parameter, N' is related to the NUT parameter, e' and g' are the electric and magnetic charges, and ϵ' and k' are related to the angular momentum per unit mass and the acceleration. Considering this metric cannot give an obvious physical interpretation; for example, it is not apparent that this line element does include the well-known Kerr metric, the NUT solution or C-metric, etc., we rescale the coordinates

$$\hat{p} = \sqrt{\alpha\omega}p, \quad \hat{r} = \sqrt{\frac{\alpha}{\omega}}r, \quad \hat{\sigma} = \sqrt{\frac{\omega}{\alpha^3}}\sigma, \quad \hat{\tau} = \sqrt{\frac{\omega}{\alpha}}\tau, \tag{143}$$

and the parameters

$$M' + iN' = \left(\frac{\alpha}{\omega}\right)^{3/2}(M + iN), \quad \epsilon' = \frac{\alpha}{\omega}\epsilon, \quad k' = \alpha^2 k. \tag{144}$$

A modified metric then is given by [94, 95]

$$\begin{aligned} \mathrm{d}s^2 =& \frac{1}{(1-\alpha pr)^2}\left[\frac{Q}{r^2+\omega^2p^2}\left(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma\right)^2\right.\\ &\left.-\frac{P}{r^2+\omega^2p^2}\left(\omega\mathrm{d}\tau+r^2d\sigma\right)^2-\frac{r^2+\omega^2p^2}{P}\mathrm{d}p^2-\frac{r^2+\omega^2p^2}{Q}\mathrm{d}r^2\right], \end{aligned} \tag{145}$$

where

$$\begin{aligned} P = P(p) &= k + 2\omega^{-1}Np - \epsilon p^2 + 2\alpha Mp^3 - \alpha^2\omega^2 kp^4,\\ Q = Q(r) &= \omega^2 k - 2Mr + \epsilon r^2 - 2\alpha\omega^{-1}Nr^3 - \alpha^2 kr^4. \end{aligned} \tag{146}$$

Since we only consider vacuum type D solutions, e', g' and Λ are set to be vanishing here. In addition, it is worthwhile to mention here that this modified metric does not include a non-singular NUT solution. In practice, to get a metric to cover all of the cases, we still need to do a coordinate transformation: $p = \frac{b}{\omega} + \frac{a}{\omega}\tilde{p}$, $\tau = t - \frac{(l+a)^2}{a}\phi$, and $\sigma = -\frac{\omega}{a}\phi$ where new parameters a and b usually correspond to a rotation parameter and a NUT parameter, respectively. However, considering the modified metric has a simple form and already covers the accelerating and rotating black hole solutions with the NUT parameter, we will use it as an example to investigate the double copy in this paper. Choosing a null tetrad

$$\begin{aligned} \ell^\mu &= \frac{(1-\alpha pr)}{\sqrt{2\left(r^2+\omega^2p^2\right)}}\left[\frac{1}{\sqrt{Q}}\left(r^2\partial_\tau-\omega\partial_\sigma\right)-\sqrt{Q}\partial_r\right],\\ n^\mu &= \frac{(1-\alpha pr)}{\sqrt{2\left(r^2+\omega^2p^2\right)}}\left[\frac{1}{\sqrt{Q}}\left(r^2\partial_\tau-\omega\partial_\sigma\right)+\sqrt{Q}\partial_r\right],\\ m^\mu &= \frac{(1-\alpha pr)}{\sqrt{2\left(r^2+\omega^2p^2\right)}}\left[-\frac{1}{\sqrt{P}}\left(\omega p^2\partial_\tau+\partial_\sigma\right)+i\sqrt{P}\partial_p\right], \end{aligned} \tag{147}$$

we have

$$\begin{aligned} \rho^* = \mu^* &= \frac{1+i\alpha\omega p^2}{\sqrt{2}(r+i\omega p)}\sqrt{\frac{Q(r)}{r^2+\omega^2p^2}},\\ \tau^* = \pi^* &= \frac{\omega - i\alpha r^2}{\sqrt{2}(r+i\omega p)}\sqrt{\frac{P(p)}{r^2+\omega^2p^2}},\\ \epsilon^* = \gamma^* &= \frac{1}{4\sqrt{2}}\left[\frac{2(1-\alpha pr)}{r+i\omega p}-2\alpha p-(1-\alpha pr)\frac{Q'}{Q}\right]\sqrt{\frac{Q(r)}{r^2+\omega^2p^2}}\\ \alpha^* = \beta^* &= \frac{1}{4\sqrt{2}}\left[\frac{2\omega(1-\alpha pr)}{r+i\omega p}+2i\alpha r+i(1-\alpha pr)\frac{P'}{P}\right]\sqrt{\frac{P(p)}{r^2+\omega^2p^2}}. \end{aligned} \tag{148}$$

The Weyl scalar is given by

$$\Psi_2 = 6\psi_2 = \frac{6(M+iN)(1-\alpha pr)^3}{(r+i\omega p)^3}, \tag{149}$$

which, as we can see, is independent of coordinates τ and σ. Plugging Eq.(147) and Eq.(148) into Eq.(131) we have

$$\begin{aligned} L_c + L_\tau \partial_\tau \log S_{14} + L_\sigma \partial_\sigma \log S_{14} + L_r \partial_r \log S_{14} = 0, \\ M_c + M_\tau \partial_\tau \log S_{14} + M_\sigma \partial_\sigma \log S_{14} + M_p \partial_p \log S_{14} = 0, \\ -M_c + M_\tau \partial_\tau \log S_{14} + M_\sigma \partial_\sigma \log S_{14} - M_p \partial_p \log S_{14} = 0, \\ -L_c + L_\tau \partial_\tau \log S_{14} + L_\sigma \partial_\sigma \log S_{14} - L_r \partial_r \log S_{14} = 0, \end{aligned} \tag{150}$$

where

$$\begin{aligned} L_c &= \frac{Q(r)[-i+\alpha p(-3\omega p + i4r)] + (\omega p - ir)(\alpha pr - 1)Q'(r)}{(\omega p - ir)\sqrt{2Q(\omega^2 p^2 + r^2)}}, \\ L_\tau &= \frac{r^2(1-\alpha pr)}{\sqrt{2Q(\omega^2 p^2 + r^2)}}, \\ L_\sigma &= -\frac{\omega(1-\alpha pr)}{\sqrt{2Q(\omega^2 p^2 + r^2)}}, \\ L_r &= -\frac{\sqrt{Q}(1-\alpha pr)}{\sqrt{2(\omega^2 p^2 + r^2)}}, \\ M_c &= \frac{P[3\alpha r^2 + i\omega(4\alpha pr - 1)] - i(\omega p - ir)(\alpha pr - 1)P'(p)}{(\omega p - ir)\sqrt{2P(\omega^2 p^2 + r^2)}}, \\ M_\tau &= -\frac{\omega p^2(1-\alpha pr)}{\sqrt{2P(\omega^2 p^2 + r^2)}}, \\ M_\sigma &= -\frac{(1-\alpha pr)}{\sqrt{2P(\omega^2 p^2 + r^2)}}, \\ M_p &= \frac{i\sqrt{P}(1-\alpha pr)}{\sqrt{2(\omega^2 p^2 + r^2)}}. \end{aligned} \tag{151}$$

From Eq.(150), one can find that S_{14} is independent of the coordinates τ and σ, which is the same as the Weyl scalar Ψ_2. The integrability condition of the above equations is then given by

$$\partial_r\left(\frac{M_c}{M_p}\right) = \partial_p\left(\frac{L_c}{L_r}\right). \tag{152}$$

One can check that this condition does hold and we arrive at

$$\log S_{14} = i \arctan \frac{\omega p}{r} - \log \frac{P(p)Q(r)}{\mathcal{C}_1 (1-\alpha pr)^3 \sqrt{r^2+\omega^2 p^2}}, \tag{153}$$

where $\mathcal{C}_1$ is an arbitrary constant of integration. Using the identity

$$\arctan(z) = -\frac{i}{2}\log(\frac{i-z}{i+z}) \qquad z \in \mathbb{C}, \tag{154}$$

we obtain

$$S_{14} = \mathcal{C}_1 \frac{(1-\alpha pr)^3 (r+i\omega p)}{P(p)Q(r)}. \tag{155}$$

The square of the coefficient of the DW spinor ξ_A reads

$$\xi^2 = \sqrt{S_{14}\Psi_2} = \sqrt{\frac{6\mathcal{C}_1(M+iN)}{P(p)Q(r)}\frac{(1-\alpha pr)^3}{(r+i\omega p)}}, \tag{156}$$

which is the coefficient of the Maxwell spinor as well as the DW tensor Eq.(106). It is simple to check that the DW equation in the curved spacetime indeed holds in this case. Moreover, one can see that the pre-factor $(M+iN)$ related to the source can be absorbed by $\mathcal{C}_1$, so we may pay attention to the rest term. Here, $P(p)Q(r)$ in the denominator is the only term related to the source, thus it is a crutial point when we look for a source-independent Maxwell field.

On the one hand, the degenerate Maxwell Spinor $\Phi^{(0)}_{AB}$ can now be identified once scalar field $S^{(0)}_{12}$ is fixed. Following Eq.(137), we arrive at

$$S^{(0)}_{12} = \mathcal{C}_2 \frac{1-\alpha pr}{r+iwp} \sim \frac{1-\alpha pr}{r+i\omega p}, \tag{157}$$

where $\mathcal{C}_2$ is an arbitrary constant of integration. Analogously, the scalar field $S^{(2)}_{12}$ is shown as

$$S^{(2)}_{12} = S^{(0)}_{12}, \tag{158}$$

up to a constant. Notably, they are independent of the source, so $P(p)Q(r)$ term of ξ^2 will stay when mapped to the Maxwell field scalar, it is simple to verify that we cannot get a source-independent degenerate Maxwell field. In addition, according to Eq.(97), the scalar $S_{24}^{(0,2)}$ is given by

$$S_{24}^{(0,2)} \sim \frac{(1-\alpha pr)(r+i\omega p)^3}{P(p)Q(r)}, \tag{159}$$

one can see that it is not equal to $S_{12}^{(0)}$.

On the other hand, for non-degenerate Maxwell spinor $\Phi_{AB}^{(1)}$, similar to the case of S_{14}, from Eq.(140) one can find that $S_{12}^{(1)}$ is independent of the coordinates τ and σ, and the integrability condition holds. With the aid of Eq.(154), we then have

$$S_{12}^{(1)} = C_3 \frac{(1-\alpha pr)(r+i\omega p)}{\sqrt{P(p)Q(r)}}, \tag{160}$$

as one can see, which depends on the source because of the appearance of the term $P(p)Q(r)$. Furthermore, one can see that the $P(p)Q(r)$ term of ξ^2 in Eq.(156) will be cancelled out by $S_{12}^{(1)}$ when mapped to ϕ_1 of Eq.(134). Therefore, we will get a source-independent non-degenerate Maxwell field[9]. Besides, $S_{24}^{(1,1)}$ is given by

$$S_{24}^{(1,1)} = \frac{S_{14}}{(S_{12}^{(1)})^2} = \frac{C_1}{(C_3)^2}\frac{1-\alpha pr}{r+i\omega p} \sim \frac{1-\alpha pr}{r+i\omega p}. \tag{161}$$

Interestingly, this scalar satisfies the wave equation even in Minkowski spacetime ($M = N = 0$). So we discover a map from a vacuum gravity field to a source-independent Maxwell field. This means that the background of the Maxwell field can be flat. One may already realize that this is nothing but the Weyl double copy.

So far, all auxiliary scalar fields connecting Weyl, Maxwell and DW fields are identified. The maps among different spin massless-free fields are summarized as follows

$$\begin{gathered}\Psi_2 = \frac{\xi^4}{(S_{12}^{(2)})^2 S_{24}^{(0,2)}} = \frac{\xi^4}{(S_{12}^{(1)})^2 S_{24}^{(1,1)}}, \\ S_{12}^{(0)} = S_{12}^{(2)} = S_{24}^{(1,1)} = (\phi_1)^{1/2} = (\Psi_2)^{1/3}.\end{gathered} \tag{162}$$

9 It is easy to check that $l_{[a}n_{b]} + \bar{m}_{[a}m_{b]}$ and $l_{[a}n_{b]} + m_{[a}\bar{m}_{b]}$ are all independent of the source, from the tensor form Eq.(81) one can see that the Maxwell scalar ϕ_1 is the only physical quantity that could be affected by the source.

Compared with the case of vacuum type N solutions Eq.(105), S_{24} is not equal to S_{12} anymore regarding the above two cases of the first line of Eq.(162). While, unexpectedly, one can see that $S_{12}^{(0)}$ and $S_{12}^{(2)}$ are equal to $S_{24}^{(1,1)}$ up to a constant. That means, the zeroth copy not only connects the vacuum gravity fields with single copy but also connects degenerate electromagnetic fields with DW fields in the curved spacetime for non-twisting vacuum type N solutions and vacuum type D solutions. The success of mapping gravity fields to the single and zeroth copies by using the DW spinors encourages us to extend the study to non-twisting vacuum type III solutions.

6.3.3 *Vacuum type III solutions*

For vacuum type III solutions, $\psi_0 = \psi_1 = \psi_2 = 0$. By making a null rotation about null vector ℓ [73],

$$\ell \to \ell, \quad n \to n + A^* m + A\bar{m} + AA^*\ell, \quad m \to m + A\ell, \quad \bar{m} \to \bar{m} + A^*\ell, \tag{163}$$

where A^* is the complex conjugate of a complex number A, then the Weyl scalars transform like

$$\begin{aligned} \psi_0 \to \psi_0, \quad \psi_1 \to \psi_1 + A^*\psi_0, \quad \psi_2 \to \psi_2 + 2A^*\psi_1 + (A^*)^2\psi_0, \\ \psi_3 \to \psi_3 + 3A^*\psi_2 + 3(A^*)^2\psi_1 + (A^*)^3\psi_0, \\ \psi_4 \to \psi_4 + 4A^*\psi_3 + 6(A^*)^2\psi_2 + 4(A^*)^3\psi_1 + (A^*)^4\psi_0. \end{aligned} \tag{164}$$

Clearly, we can let ψ_4 vanish without changing other three Weyl scalars by requiring

$$A^* = -\frac{\psi_4}{4\psi_3}, \tag{165}$$

and ψ_3 will be only non-vanishing Weyl scalar. The spinor form thus reduces to

$$\Psi_{ABCD} = \Psi_3 o_{(A} o_B o_C \iota_{D)}, \tag{166}$$

where we set $\Psi_3 = -4\psi_3$. Based on this, Eq.(93) can be written in the form

$$\Psi_{ABCD} = \frac{\xi_{(A}\xi_B\xi_C\eta_{D)}}{S_{14}}, \tag{167}$$

where $\xi_A = \xi o_A$ and $\eta_A = \eta\iota_A$, or in scalar from

$$\Psi_3 = \frac{\xi^3\eta}{S_{14}}. \tag{168}$$

According to Eq.(66), two independent dyad components of the Bianchi identity read

$$o_A\nabla^{AA'}\log\Psi_3 + 2\nabla^{AA'}o_A = 0, \tag{169}$$

$$\iota_A\nabla^{AA'}\log\Psi_3 + 4o_A\iota^B\nabla^{AA'}\iota_B + 2\iota_A o^B\nabla^{AA'}\iota_B = 0. \tag{170}$$

Since

$$\begin{aligned} o_A\iota^B\nabla^{AA'}o_B - \iota_A o^B\nabla^{AA'}o_B &= (o_A\iota^B - \iota_A o^B)\nabla^{AA'}o_B \\ &= \epsilon_A^{\ B}\nabla^{AA'}o_B \\ &= \nabla^{AA'}(\epsilon_A^{\ B}o_B) \\ &= \nabla^{AA'}o_A, \end{aligned} \tag{171}$$

Eq.(87) can be rewritten as

$$o_A\nabla^{AA'}\log\xi + \nabla^{AA'}o_A = 0. \tag{172}$$

Combining Eq.(169) and Eq.(172), we have

$$\Psi_3 = \mathcal{C}\xi^2, \tag{173}$$

where $\mathcal{C}$ is a non-vanishing constant of integration. This result is even independent of our assumption Eq.(167). In order to keep the total spin invariant for the above equation[10], the constant $\mathcal{C}$ here should correspond to a field with a total spin of 1. However, we need to point

10 We thank Ricardo Monteiro for bringing this up.

out that it is not necessary to require it to be a source-free Maxwell scalar. This point can also be verified from Eq.(167) or Eq.(168), then one observes

$$\mathcal{C} = \frac{\xi\eta}{S_{14}}; \tag{174}$$

this is nothing but a constraint equation about spinor ξ_A and spinor η_A. Furthermore, one can see that $\mathcal{C}$ indeed corresponds to a field with a total spin of 1 in view of the right side of Eq.(174). Yet there is no reason to require $S_{14} = S_{12}$, $\mathcal{C}$ does not have to be a source-free Maxwell scalar. We will come back to talk more about this in the discussions. Regarding spinor ξ_A, according to Eq.(95), we can construct a degenerate Maxwell spinor $\Phi^{(2)}_{AB}$ with it. Repeating the same calculation as the case of type N, $S^{(2)}_{12}$ can be solved, and then the Maxwell field will be identified.

As for another DW spinor $\eta_A = \eta\iota_A$, the equation of motion is given by

$$\iota_A \nabla^{AA'} \log(\frac{\Psi_3}{\eta}) + 3o_A\iota^B\nabla^{AA'}\iota_B + 3\iota_A o^B \nabla^{AA'}\iota_B = 0 \tag{175}$$

following Eq.(88) and Eq.(170). The tensor version then reads

$$\bar{m}\cdot\nabla\log\left(\frac{\Psi_3}{\eta}\right) + 3\pi^* + 3\alpha^* = 0, \quad n\cdot\nabla\log\left(\frac{\Psi_3}{\eta}\right) + 3\mu^* + 3\gamma^* = 0. \tag{176}$$

Recalling the type N and type D cases, the DW scalars mapped from the gravity fields all depend on the same coordinates as the Weyl scalars, we expect η also behaves like that and we are in fact only interested in this case in the present work. However, one will see that its solution is also related to the other coordinates unless we impose an extra condition. Therefore, generally, there is no trivial relationship between the Weyl scalar Ψ_3 and the DW scalar η, we thus shall pay more attention to the DW tensor ξ_A in this section.

Further investigation on exact non-twisting vacuum type III solutions is given in the following.

6.3.3.1 *Kundt solutions*

There are two kinds of Kundt solutions for the type III case, the metric in general is given by [91]

$$\mathrm{d}s^2 = 2\mathrm{d}u(H\mathrm{d}u + \mathrm{d}v + W\mathrm{d}z + \bar{W}\mathrm{d}\bar{z}) - 2\mathrm{d}z\mathrm{d}\bar{z}, \tag{177}$$

with a real function H and a complex function W.

For the case of $W_{,v} = 0$,

$$\begin{aligned} W = W(u,\bar{z}), \quad H = \frac{1}{2}\left(W_{,\bar{z}} + \bar{W}_{,z}\right)v + H^0, \\ H^0_{,z\bar{z}} - \mathfrak{Re}\left[W^2_{,\bar{z}} + WW_{,z\bar{z}} + W_{,u\bar{z}}\right] = 0. \end{aligned} \tag{178}$$

We choose a null tetrad

$$\ell = \partial_v, \quad n = \partial_u - (H + W\bar{W})\partial_v + \bar{W}\partial_z + W\partial_{\bar{z}}, \quad m = \partial_{\bar{z}}. \tag{179}$$

The Weyl scalars in this case are given by

$$\psi_3' = -\frac{1}{2}\partial^2_{\bar{z}}W(u,\bar{z}), \quad \psi_4' = -\bar{W}(u,z)\partial^2_{\bar{z}}W(u,\bar{z}) - \frac{1}{2}v\partial^3_{\bar{z}}W(u,\bar{z}) - \partial^2_{\bar{z}}H^0(u,z,\bar{z}). \tag{180}$$

Note $\partial^2_{\bar{z}}W(u,\bar{z}) \neq 0$ here, otherwise the metic reduces to type N solution. By making a null rotation with the help of Eq.(165), the only non-vanishing Weyl scalar left is [96]

$$\Psi_3 = -4\psi_3 = 2\partial^2_{\bar{z}}W(u,\bar{z}). \tag{181}$$

From Eq.(173), one of the corresponding Dirac-Weyl fields is given by

$$\xi^2 = \frac{2}{C}\partial^2_{\bar{z}}W(u,\bar{z}). \tag{182}$$

In addition, some spin coefficients are given by

$$\begin{aligned}\rho^* &= \tau^* = \alpha^* = 0, \\ \mu^* &= \frac{[\partial_{\bar z}^3 W(2W\partial_z^2\bar W + v\partial_z^3\bar W + 2\partial_{\bar z}^2 H^0) + 8\partial_z^2\bar W(\partial_{\bar z}W\partial_{\bar z}^2 W - \partial_z\partial_{\bar z}^2 H^0)]}{16\partial_{\bar z}^2 W\partial_z^2\bar W}, \\ \pi^* &= -\frac{\partial_{\bar z}^3 W}{4\partial_{\bar z}^2 W}, \\ \gamma^* &= \frac{1}{2}\partial_{\bar z}W.\end{aligned} \tag{183}$$

solving Eq.(226) we have $\partial_v S_{12}^{(2)} = \partial_z S_{12}^{(2)} = 0$. $S_{12}^{(2)}$ is independent of v and z, namely, it can be an arbitrary function of u and $\bar z$,

$$S_{12}^{(2)} = S_{12}^{(2)}(u, \bar z). \tag{184}$$

And, it is easy to check that $S_{12}^{(2)}$ satisfies the wave equation even in the flat spacetime. Then the degenerate Maxwell scalar is given by

$$\phi_2 = \frac{\xi^2}{S_{12}^{(2)}} = \frac{2}{\mathcal{C}}\frac{\partial_{\bar z}^2 W(u,\bar z)}{S_{12}^{(2)}(u,\bar z)}. \tag{185}$$

Combining the fact

$$2\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\ell_{[a}\bar m_{b]} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{186}$$

one can find that this degenerate Maxwell field also satisfies the field equation in Minkowski spacetime, for which we may let $W = H^0 = 0$ in the metric. Recalling the relationship Eq.(173), the Weyl scalar can be written as

$$\Psi_3 = \mathcal{C}S_{12}^{(2)}\phi_2. \tag{187}$$

Besides, we also probe another DW spinor's form. Keeping consistent with the Weyl scalar Ψ_3, we are only interested in the solution that η is independent of the coordinates v and z. In this case, by solving Eq.(176) we obtain

$$\partial_u \log(\frac{\Psi_3}{\eta}) = \mathcal{M}, \quad \partial_{\bar{z}} \log(\frac{\Psi_3}{\eta}) = \mathcal{N}, \tag{188}$$

where

$$\begin{aligned} \mathcal{M} &= -\frac{3(4\partial_{\bar{z}}W\partial_{\bar{z}}^2 W + W\partial_{\bar{z}}^3 W - 2\partial_z\partial_{\bar{z}}^2 H^0)}{4\partial_{\bar{z}}^2 W}, \\ \mathcal{N} &= \frac{3\partial_{\bar{z}}^3 W}{4\partial_{\bar{z}}^2 W}. \end{aligned} \tag{189}$$

The integrability condition is given by $\partial_{\bar{z}}\mathcal{M} = \partial_u \mathcal{N}$. Clearly, to have a solution we have to impose one more condition, $\partial_z^2\partial_{\bar{z}}^2 H^0 = 0$. In general, however, there is no solution which depends on the same coordinates as the Weyl scalar, and there is no trivial relation between DW scalar η_A and Weyl scalar Ψ_3. In the following, we will focus on the DW tensor ξ_A.

For the case of $W_{,v} \neq 0$,

$$\begin{aligned} W =& W^0(u,z) - \frac{2v}{z+\bar{z}}, \quad H = H^0 + v\frac{W^0 + \bar{W}^0}{z+\bar{z}} - \frac{v^2}{(z+\bar{z})^2}, \\ &\left(\frac{H^0 + W^0\bar{W}^0}{z+\bar{z}}\right)_{,z\bar{z}} = \frac{W^0_{,z}\bar{W}^0_{,\bar{z}}}{z+\bar{z}}. \end{aligned} \tag{190}$$

We choose a null tetrad

$$\ell = \partial_v, \quad n = \partial_u - (H + W\bar{W})\partial_v + \bar{W}\partial_z + W\partial_{\bar{z}}, \quad m = \partial_{\bar{z}}. \tag{191}$$

By doing a null rotation with Eq.(165), the Weyl scalar is given by [96]

$$\Psi_3 = -4\psi_3 = 4\frac{\partial_{\bar{z}}\bar{W}^0(u,\bar{z})}{z+\bar{z}}. \tag{192}$$

Correspondingly, we arrive at

$$\xi^2 = \frac{1}{\mathcal{C}}\Psi_3 = \frac{4}{\mathcal{C}}\frac{\partial_{\bar{z}}\bar{W}^0(u,\bar{z})}{z+\bar{z}}. \tag{193}$$

The spin coefficients ρ^* and τ^* are given by

$$\rho^* = 0, \quad \tau^* = -\frac{1}{z+\bar{z}}. \tag{194}$$

Following Eq.(226), the auxiliary scalar field is solved by

$$S_{12}^{(2)} = \frac{\mathcal{V}(u,\bar{z})}{z+\bar{z}}, \tag{195}$$

where function $\mathcal{V}(u,\bar{z})$ is arbitrary. One can check that $S_{12}^{(2)}$ satisfies the wave equation even in the flat spaceitme. Moreover, we have

$$\phi_2 = \frac{4}{\mathcal{C}}\frac{\partial_{\bar{z}}\bar{W}^0(u,\bar{z})}{\mathcal{V}(u,\bar{z})}. \tag{196}$$

Since

$$2\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad 2\ell_{[a}\bar{m}_{b]} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{197}$$

similar to the case of $W_{,v} = 0$, one can show that the Maxwell field also satisfies its field equation in Minkowski space. The Weyl scalar is written as

$$\Psi_3 = \mathcal{C}S_{12}^{(2)}\phi_2. \tag{198}$$

6.3.3.2 *Robinson-Trautman solutions*

The vacuum solution of diverging non-twisting type III case is given by [91, 92],

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}u(H\mathrm{d}u + 2\mathrm{d}r) - \frac{2r^2}{P^2(u,z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z}, \\ \Delta \log P &= \mathcal{K} = -3\left[f(u,z) + \bar{f}(u,\bar{z})\right], \quad f_{,z} \neq 0, \\ H &= \Delta \log P - 2r\partial_u \log P, \quad \Delta \equiv 2P^2\partial_z\partial_{\bar{z}}. \end{aligned} \tag{199}$$

where the structure function f(u,z) is complex.

Choosing a null tetrad

$$\ell = \partial_r, \quad n = \partial_u - \frac{H}{2}\partial_r, \quad m = -\frac{P}{r}\partial_z, \tag{200}$$

the non-vanishing Weyl scalars read

$$\psi_3' = \frac{3P\partial_{\bar{z}}\bar{f}}{2r^2}, \qquad \psi_4' = \frac{3P^2\partial_{\bar{z}}^2\bar{f} - 2r\partial_{\bar{z}}^2 P\partial_u P + 2P(3\partial_{\bar{z}}P\partial_{\bar{z}}\bar{f} + r\partial_u\partial_{\bar{z}}^2 P)}{2r^2}. \tag{201}$$

Same to the case of Kundt class of the last section, by doing a null rotation with Eq.(165), the only non-vanishing Weyl scalar reads

$$\Psi_3 = -4\psi_3 = -4\psi_3' = -\frac{6P\partial_{\bar{z}}\bar{f}}{r^2} \tag{202}$$

In the new null tetrad, according to Eq.(173), one of the DW fields mapping from the gravity side is given by

$$\xi^2 = -\frac{6}{\mathcal{C}}\frac{P\partial_{\bar{z}}\bar{f}}{r^2}. \tag{203}$$

The spin coefficients ρ^* and τ^* are solved by

$$\rho^* = -\frac{1}{r}, \qquad \tau^* = \frac{3P^2\partial_z^2 f - 2r\partial_u P\partial_{\bar{z}}^2 P + 2P(3\partial_z f\partial_{\bar{z}}P + r\partial_u\partial_{\bar{z}}^2 P)}{12Pr\partial_z f}. \tag{204}$$

Making use of Eq.(226), we find $S_{12}^{(2)}$ has to satisfy

$$\frac{1}{r} + \frac{\partial_r S_{12}^{(2)}}{S_{12}^{(2)}} = 0, \quad \partial_z S_{12}^{(2)} = 0. \tag{205}$$

Therefore, we arrive at a general solution

$$S_{12}^{(2)} = \frac{\mathcal{X}(u,\bar{z})}{r}, \tag{206}$$

where $\mathcal{X}(u,\bar{z})$ is an arbitrary function. From Eq.(133) the degenerate Maxwell scalar reads

$$\phi_2 = \frac{\xi^2}{S_{12}^{(2)}} = -\frac{6}{\mathcal{C}}\frac{P\partial_{\bar{z}}\bar{f}(u,\bar{z})}{r\mathcal{X}(u,\bar{z})}. \tag{207}$$

Going to the tensor version Eq.(82), we have

$$2\frac{P}{r}\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad 2\frac{P}{r}\ell_{[a}\bar{m}_{b]} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{208}$$

Clearly, only the $[uz]$ and $[u\bar{z}]$ components are non-vanishing. Similar to the case of type N, it is easy to show that this field satisfies the field equation in Minkowski spacetime. In addition, combining Eq.(173) and Eq.(207), we have

$$\Psi_3 = \mathcal{C} S_{12}^{(2)} \phi_2, \tag{209}$$

where the scalar field $S_{12}^{(2)}$ satisfies the wave equation not only in this curved spacetime but also in Minkowski spacetime.

Therefore, with the help of the DW spinors, we have successfully proved that there indeed exists a natural map between pure Maxwell fields and gravity fields for non-twisting vacuum type III spacetimes. Moreover, we found that the auxiliary scalar field, connecting the DW field with the degenerate electromagnetic field in the curved spacetime, plays a similar role to the zeroth copy.

6.4 DISCUSSION AND CONCLUSIONS

In this paper, based on the fact that any massless free-field spinors with spin higher than $1/2$ can be constructed with DW spinors (spin-$1/2$) and scalar fields, we introduced a map between vacuum gravity fields and DW fields in spin-space. The form of associated DW spinors are identified. Regarding these DW spinors as basic units, we investigated the other higher spin massless-free fields, especially the Maxwell fields, and showed some hidden fundamental features among these fields.

In particular, for Petrov type N solutions, inspired by the work [82], we found that only one type of DW spinor exists in the curved spacetime; combining with the zeroth copy, the DW spinor can construct any other higher spin massless free-fields. Following this, we studied the Petrov type D solutions. In this situation, there are two types of DW spinors in the curved spacetime. Unlike the case of type N, we found that S_{24} (a scalar field connecting a Maxwell field with a gravity field) is not equal to S_{12} (a scalar field connecting a DW field with a Maxwell field) anymore for each case. While, there remains an interesting relation, $S_{12}^{(0)} = S_{12}^{(2)} = S_{24}^{(1,1)}$, the scalar fields connecting the DW fields with the degenerate electromagnetic fields are equal to the zeroth copy up to a constant. In general, by using the DW spinors and the auxiliary scalar fields, we systematically rebuilt the Weyl double copy for non-twisting vacuum type N and vacuum type D solutions in this paper. Our results are consistent with previous work [75, 82]. Moreover, we showed that the zeroth copy not only connects the gravity fields with the single copy but also connects DW fields with those degenerate electromagnetic fields living in the curved spacetime.

We also investigated the case of non-twisting vacuum type III solutions. Independent of the proposed map, we found that the square of a DW scalar is just proportional to the Weyl scalar Ψ_3. Such an interesting result produces a natural relationship between the gravity fields and the Maxwell fields in the flat spacetime, which is summarized as $\Psi_3 = \mathcal{C} S_{12}^{(2)} \phi_2$, where $S_{12}^{(2)}$ and ϕ_2 correspond to a scalar field and a degenerate Maxwell field, respectively. Interestingly, both of them not only satisfy their field equation in curved spacetime but also in Minkowski spacetime. As an auxiliary scalar field associated with the degenerate electromagnetic field, it is not surprising that $S_{12}^{(2)}$ plays a role similar to the zeroth copy considering our discovery in the cases of type N and type D solutions. However, why this scalar can play such an important role in connecting gravity theory with gauge theory is still unclear. On the whole, with the help of the chosen DW spinors, we systematically show that there indeed exists a deep connection between gravity theory and gauge theory by investigating non-twisting vacuum type N, III and vacuum type D solutions. The Weyl double copy proposed before is covered in the present work.

Next, it would be fascinating to study the case in non-vacuum spacetime using Dirac equation Eq.(89) instead of Dirac-Weyl equation Eq.(221). The situation could be viewed as turning from a DW equation to DW equations with a source. In addition, so far, all of the works related to the Weyl double copy only focus on classical gravity solutions without a cosmological constant. Along the road of this work, it would be interesting to show a specific situation about the Weyl double copy for asymptotically (anti-)de Sitter spacetimes. In fact, we found that the Weyl double copy, in general, satisfies conformally invariant field equations even in conformally flat spacetimes, which is consistent with the result of twistorial version of Weyl double copy[61]. Progress on this has been shown in another work [97].

In the end, we have to point out that although we have shown a natural map for type III cases between gravity fields and the Maxwell fields living in Minkowski spacetime, we did not prove if type III spacetime admits the classical Weyl double copy prescription. In terms of Kundt class with $W_{,v} = 0$, we only shown that the DW scalar η does not depends on the same coordinates as the Weyl scalar, unless we impose one more condition — $H^0_{,zz\bar{z}\bar{z}} = 0$. If the Weyl double copy prescription does exist for vacuum type III solutions, the possible way to show it may start from regarding $S^{(2)}_{12}$ as the zeroth copy. Then, it would be interesting to probe the physical meaning of the constant $\mathcal{C}$, since it corresponds to a field with a total spin of 1. Alternatively, we may need to extend the Weyl double copy to a more general form to cover even the twisting case. All in all, to get full knowledge about the relation between gravity theory and gauge theory, there is still a long way to go. We hope this paper provides new insights for a better understanding of double copy and the connection between gravity theory and gauge theory.

7

THE WEYL DOUBLE COPY IN VACUUM SPACETIMES WITH A COSMOLOGICAL CONSTANT

The work presented in this chapter is based on a paper that has been published as: 'The Weyl double copy in vacuum spacetimes with a cosmological constant' in the Journal of High Energy Physics.

ABSTRACT

We examine the Weyl double copy relation for vacuum solutions of the Einstein equations with a cosmological constant using the approach we previously described, in which the spin-$1/2$ massless free-field spinors (Dirac-Weyl fields) are regarded as basic units. Based on the exact non-twisting vacuum type N and vacuum type D solutions, the finding explicitly shows that the single and zeroth copies fulfill conformally invariant field equations in conformally flat spacetime. In addition, irrespective of the presence of a cosmological constant, we demonstrate that the zeroth copy connects Dirac-Weyl fields with the degenerate electromagnetic fields in the curved spacetime in addition to connecting gravity fields with the single copy in conformally flat spacetime. Moreover, the study also demonstrates the critical

significance the zeroth copy plays in time-dependent radiation solutions. In particular, for Robinson-Trautman (Λ) gravitational waves, unlike the single copy, we find that the zeroth copy carries additional information to specify whether the sources of associated gravitational waves are time-like, null, or space-like, at least in the weak field limit.

7.1 INTRODUCTION

The double copy originates from the study of perturbative scattering amplitudes[8–10], which brings forth a fascinating connection between gauge amplitudes and gravity amplitudes. Moreover, this idea has been extended to the classical context. In Kerr-Schild coordinate system, a map between gravity theory and gauge theory was proposed, called Kerr-Schild double copy [41]. A wide array of such classes of spacetimes has been studied []. Inspired by this, a new type of double copy relation called Weyl double copy is drawing more attention [60–64, 75, 82–84]. This prescription is represented by

$$\Psi_{ABCD} = \frac{\Phi_{(AB}\Phi_{CD)}}{S}, \tag{210}$$

where Ψ_{ABCD} is a Weyl spinor describing vacuum gravity fields, Φ_{AB} is an electromagnetic spinor referring to a Maxwell field in Minkowski spacetime—the simplest solution of the gauge theory, and S is an auxiliary scalar field satisfying the wave equation in Minkowski spacetime. The last two fields are called single copy and zeroth copy, respectively. Starting from the gravity fields, the Weyl double copy relation leads to a gauge field that is completely independent of the gravity theory. As a result, it is thought that, the Weyl double copy relation could serve as a link between gravity theory and gauge theory.

Luna et al. proposed for the first time the Weyl double copy relation for the case of vacuum type D solutions [75]. Then, in spinor language, this relation was extended to non-twisting vacuum type N solutions by Godazgar et al. [82]. Making use of the peeling property [87, 88] of the Weyl tensor, they further showed that the Weyl double copy relation also holds asymptotically for algebraically general solutions [63]. In addition, at the linearised lever, the Weyl double copy relation was shown to hold for arbitrary Petrov type solutions using the twistor formalism [61, 83]. An extended Weyl double copy prescription was also proposed recently for non-vacuum solutions, whose Weyl spinor is decomposed into a sum of source terms [64]. Very recently, regarding the Dirac-Weyl (DW) spinors (spin-$1/2$ massless free-field spinors) as the basic units of other higher spin massless free-field spinors, we systematically revisited the Weyl double copy relation for non-twisting vacuum type N and vacuum type D solutions [98]. We further found a map similar to the Weyl double copy prescription for non-twisting vacuum type III spacetimes.

However, the Weyl double copy relation for the exact vacuum solutions with a cosmological constant has not yet been investigated. This is the primary objective of the current effort. In fact, since 1998, by the observations of supernovae of Ia type [99, 100], studies have shown that the expansion of our universe is accelerating, which strongly supports the condition that the cosmological constant Λ is nonzero and positive. On the other hand, although Anti-de Sitter (AdS) spacetime does not appear to have direct cosmological applications, it plays a crucial role in AdS/CFT correspondence. Therefore, investigating the Weyl double copy relation in the presence of a cosmological constant would be of interest. Currently, there are two possible research directions: one is to interpret the cosmological constant as a source of the single and zeroth copies in the flat spacetime; the other is to consider the (A)dS spacetime to be the background of the single and zeroth copies. The former idea was proposed for the first time in Kerr-Schild double copy in Taub-NUT spacetime [42] and it would be natural in the direct investigation of the relationship between gravity theory and gauge theory. On the other hand, the latter can be viewed as a precursor to the former. Moreover, it is also advantageous for extending the remit of the Weyl double copy, including cosmological applications and perturbation theory. This has been done in Ref. [53] for Kerr-Schild(Λ) double copy, which shows that the single and zeroth copies satisfy different equations for time-dependent and time-independent solutions. These outcomes encourage us to study whether or not the Weyl double copy relation shares this property. In this paper, we shall give an explicit demonstration to show that, different from the Kerr-Schild(Λ) double copy, the single and zeroth copies in the Weyl double copy prescription all satisfy conformal invariant field equations in conformally flat spacetime, both for time-independent solutions and time-dependent solutions. Our finding coincides with the statement of Ref. [61] in the twistorial version. Some interesting relations between the zeroth copy and gravitational waves will also be discussed.

The structure of this paper is as follows. In Sec. 7.2, we will briefly review how to construct electromagnetic spinors in vacuum type N and type D spacetimes by regarding DW spinors as the basic units . Then, we will study the Weyl double copy for exact vacuum solutions with a cosmological constant in Sec. 7.3. The interpretations of the single copy and the zeroth copy will also be included. Discussion and conclusions are given in Sec. 7.4. The notation of this paper follows the conventions of Ref. [98].

7.2 MASSLESS FREE-FIELDS IN SPINOR FORMALISM

In this section, we will briefly review how to construct electromagnetic spinors in order to verify the Weyl double copy relation using the methodology of the previous work [98].

In spinor formalism, spin-$k/2$ massless free-field equations have a simple form [77]

$$\nabla^{A_1 A_1'} \mathcal{S}_{A_1 A_2 \ldots A_k} = 0, \tag{211}$$

where the spinor $\mathcal{S}_{A_1 A_2 \ldots A_k}$ is totally symmetric.

For spin-2 massless free-fields, the spinor $\mathcal{S}$ refers to the Weyl spinor Ψ_{ABCD} translated from the Weyl tensor C_{abcd}

$$C_{abcd} = C_{AA'BB'CC'DD'} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}. \tag{212}$$

It is easy to find that the Weyl spinor Ψ_{ABCD} plays the same role as the Weyl tensor C_{abcd}. For a vacuum spacetime (with or without a cosmological constant Λ), the Einstein field equation is absorbed into the Bianchi identity, which reads

$$\nabla^{AA'}\Psi_{ABCD} = 0. \tag{213}$$

This is nothing but a spin-2 massless free-field equation. Notably, the fact that this field equation remains the same regardless of the presence or absence of a cosmological constant motivates us to generalize the original Weyl double copy to the case with a cosmological constant. As is well known, ten independent real components of the Weyl tensor can be reduced to 5 independent complex scalars with the aid of a null tetrad, as defined in Ref.[101]. Using the totally symmetric property of the Weyl spinor, we can define them as follows,

$$\begin{aligned}
\psi_0 &= \Psi_{ABCD} o^A o^B o^C o^D = C_{abcd} \ell^a m^b \ell^c m^d, \\
\psi_1 &= \Psi_{ABCD} o^A o^B o^C \iota^D = C_{abcd} \ell^a m^b \ell^c n^d, \\
\psi_2 &= \Psi_{ABCD} o^A o^B \iota^C \iota^D = C_{abcd} \ell^a m^b \bar{m}^c n^d, \\
\psi_3 &= \Psi_{ABCD} o^A \iota^B \iota^C \iota^D = C_{abcd} \ell^a n^b \bar{m}^c n^d, \\
\psi_4 &= \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D = C_{abcd} \bar{m}^a n^b \bar{m}^c n^d,
\end{aligned} \tag{214}$$

where the second equations hold based on the definition of the null tetrad in the spinor bases

$$\begin{aligned} \ell^a &= o^A \bar{o}^{A'}, \quad n^a = \iota^A \bar{\iota}^{A'}, \quad m^a = o^A \bar{\iota}^{A'}, \quad \bar{m}^a = \iota^A \bar{o}^{A'}, \\ \ell_a &= o_A \bar{o}_{A'}, \quad n_a = \iota_A \bar{\iota}_{A'}, \quad m_a = o_A \bar{\iota}_{A'}, \quad \bar{m}_a = \iota^A \bar{o}^{A'}. \end{aligned} \tag{215}$$

It is easy to check that the above correspondence indeed defines a null tetrad such that

$$\begin{aligned} \ell^2 = n^2 = m^2 = \bar{m}^2 = 0, \\ \ell \cdot n = 1, \quad m \cdot \bar{m} = -1, \quad \ell \cdot m = n \cdot m = \ell \cdot \bar{m} = n \cdot \bar{m} = 0. \end{aligned} \tag{216}$$

The spin coefficients are defined the same as in the preceding work [98],

$$\begin{aligned}
\kappa^* &= m^a \ell^b \nabla_b \ell_a, & \pi^* &= n^a \ell^b \nabla_b \bar{m}_a, & \epsilon^* &= \frac{1}{2}(n^a \ell^b \nabla_b \ell_a + m^a \ell^b \nabla_b \bar{m}_a), \\
\tau^* &= m^a n^b \nabla_b \ell_a, & \nu^* &= n^a n^b \nabla_b \bar{m}_a, & \gamma^* &= \frac{1}{2}(n^a n^b \nabla_b \ell_a + m^a n^b \nabla_b \bar{m}_a), \\
\sigma^* &= m^a m^b \nabla_b \ell_a, & \mu^* &= n^a m^b \nabla_b \bar{m}_a, & \beta^* &= \frac{1}{2}(n^a m^b \nabla_b \ell_a + m^a m^b \nabla_b \bar{m}_a), \\
\rho^* &= m^a \bar{m}^b \nabla_b \ell_a, & \lambda^* &= n^a \bar{m}^b \nabla_b \bar{m}_a, & \alpha^* &= \frac{1}{2}(n^a \bar{m}^b \nabla_b \ell_a + m^a \bar{m}^b \nabla_b \bar{m}_a).
\end{aligned} \tag{217}$$

For more details, one may refer to Ref. [101, 102]. To distinguish from other symbols in this paper, we use $*$ to mark these spin coefficients in the following, such as κ^*, α^*, β^*, etc. Expanding out the Weyl spinor, the general form reads

$$\begin{aligned} \Psi_{ABCD} = {} & \psi_0 \iota_A \iota_B \iota_C \iota_D - 4\psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + 6\psi_2 o_{(A} o_B \iota_C \iota_{D)} \\ & - 4\psi_3 o_{(A} o_B o_C \iota_{D)} + \psi_4 o_A o_B o_C o_D. \end{aligned} \tag{218}$$

For vacuum type N and type D solutions, the Weyl spinors are reduced to

$$\text{type N}: \quad \Psi_{ABCD} = \psi_4 o_A o_B o_C o_D, \tag{219}$$

$$\text{type D}: \quad \Psi_{ABCD} = 6\psi_2 o_{(A} o_B \iota_C \iota_{D)}. \tag{220}$$

For spin-1/2 massless free-fields, the spinor $\mathcal{S}$ refers to a DW spinor ξ_A. Eq.(211) in this case represents the DW field equation

$$\nabla^{AA'} \xi_A = 0. \tag{221}$$

With the map proposed in the preceding work [98]

$$\Psi_{ABCD} = \frac{\xi_{(A}\eta_B\zeta_C\chi_{D)}}{S_{14}}, \tag{222}$$

where the four DW spinors on the right side can be chosen to be the same (depending on which type of spacetime we are focusing on), we are now able to derive the DW spinors in a certain vacuum spacetime with a cosmological constant. Correspondingly, the electromagnetic spinors in curved spacetime will be formulated.

Specifically, for vacuum type N solutions, the map Eq.(222) reduces to

$$\Psi_{ABCD} = \frac{\xi_{(A}\xi_B\xi_C\xi_{D)}}{S_{14}} = \frac{\xi_{(A}\xi_B\xi_C\xi_{D)}}{(S_{12})^3}. \tag{223}$$

From Eq.(219) one can see that $\xi_A = \xi o_A$. According to Ref. [98], we know that $S_{12} = S_{24} = S_{14}^{1/3}$, where S_{ij} is an auxiliary scalar connecting a spin-$i/2$ massless free-field spinor with a spin-$j/2$ massless free-field spinor. The independent dyad components of the Weyl field equation Eq.(213) then read

$$o_A \nabla^{AA'} \log\ \Psi_4 + 4 o_A \iota^B \nabla^{AA'} o_B - \iota_A o^B \nabla^{AA'} o_B = 0, \tag{224}$$

where we define the Weyl scalar $\Psi_4 = \psi_4$. On the other hand, the dyad component of the DW field equation Eq.(221) is given by

$$o_A \nabla^{AA'} \log\ \xi + o_A \iota^B \nabla^{AA'} o_A - \iota_A o^B \nabla^{AA'} o_B = 0. \tag{225}$$

Combining Eq.(223), Eq.(224) and Eq.(225), the auxiliary scalar S_{12} and the DW scalar ξ will be identified by solving

$$\ell \cdot \nabla \log\ S_{12} - \rho^* = 0, \quad m \cdot \nabla \log\ S_{12} - \tau^* = 0. \tag{226}$$

Since there is only one type of DW spinor $\xi_A = \xi o_A$, correspondingly, only one type of electromagnetic spinor can exist—the degenerate electromagnetic spinor

$$\Phi_{AB} = \frac{\xi_A \xi_B}{S_{12}} = \frac{\xi^2}{S_{12}} o_A o_B = \phi_2 o_A o_B. \tag{227}$$

Furthermore, the electromagnetic tensor $F_{ab} = F_{AA'BB'} = \Phi_{AB}\varepsilon_{A'B'} + \bar{\Phi}_{A'B'}\varepsilon_{AB}$, where $\varepsilon_{AB} = 2o_{[A}\iota_{B]}$, in the null tetrad we have

$$F_{ab} = 2\phi_2 \ell_{[a} m_{b]} + 2\bar{\phi}_2 \ell_{[a} \bar{m}_{b]}. \tag{228}$$

For vacuum type D solutions, many of spacetimes that we are familiar with belong to this class, such as Kerr (A)dS black holes, Reissner–Nordström (A)dS black holes, NUT solutions, C-metric, etc. In this case, the map Eq.(222) reduces to

$$\Psi_{ABCD} = \frac{\xi_{(A}\xi_B\eta_C\eta_{D)}}{S_{14}}, \tag{229}$$

where we choose two DW spinors with the same coefficient, in other words,

$$\xi_A = \xi o_A, \qquad \eta_A = \xi \iota_A. \tag{230}$$

The dyad components of gravity field equation Eq.(213) are then given by

$$o_A \nabla^{AA'} \log(\Psi_2) - 3\iota_A o^B \nabla^{AA'} o_B = 0, \tag{231}$$

$$\iota_A \nabla^{AA'} \log(\Psi_2) + 3o_A \iota^B \nabla^{AA'} \iota_B = 0, \tag{232}$$

where we let the Weyl scalar $\Psi_2 = 6\psi_2$. Two dyad components of the DW field equations read

$$o_A \nabla^{AA'} \log\, \xi - \iota_A o^B \nabla^{AA'} o_B + o_A \iota^B \nabla^{AA'} o_B = 0, \tag{233}$$

$$\iota_A \nabla^{AA'} \log\, \xi + o_A \iota^B \nabla^{AA'} \iota_B - \iota_A o^B \nabla^{AA'} \iota_B = 0. \tag{234}$$

By making use of the map Eq.(229), the auxiliary scalar S_{14} and the DW scalar will be identified by solving

$$\begin{aligned} &\ell \cdot \nabla \log S_{14} + 4\epsilon^* - \rho^* = 0, \quad m \cdot \nabla \log S_{14} + 4\beta^* - \tau^* = 0, \\ &\bar{m} \cdot \nabla \log S_{14} - 4\alpha^* + \pi^* = 0, \quad n \cdot \nabla \log S_{14} - 4\gamma^* + \mu^* = 0. \end{aligned} \tag{235}$$

Different from the type N case, since there are two different types of DW spinors, we thereby have two different types of electromagnetic spinors. Apart from the degenerate electromagnetic spinor we discussed above, the other type is a non-degenerate electromagnetic spinor,

$$\Phi^{(1)}_{AB} = \phi_1 o_{(A}\iota_{B)} = \frac{\xi^2 o_{(A}\iota_{B)}}{S^{(1)}_{12}}. \tag{236}$$

In order to distinguish two different types of electromagnetic spinors, we use a upper index (1) to refer to non-degenerate ones and (0), (2) to refer to degenerate ones $\Phi^{(0)}_{AB} = \phi_0 \iota_A \iota_B$ and $\Phi^{(2)}_{AB} = \phi_2 o_A o_B$, repectively[1]. The dyad components of the non-degenerate electromagnetic field equation are given by

$$o_A \nabla^{AA'} \log \phi_1 - 2\iota_A o^B \nabla^{AA'} o_B = 0, \tag{237}$$

$$\iota_A \nabla^{AA'} \log \phi_1 + 2 o_A \iota^B \nabla^{AA'} \iota_B = 0. \tag{238}$$

Substitution of the map Eq.(236) into the above equations and multiplying $\bar{o}_{A'}$ and $\bar{\iota}_{A'}$ respectively yield

$$\begin{aligned} \ell \cdot \nabla \log S^{(1)}_{12} + 2\epsilon^* = 0, \quad m \cdot \nabla \log S^{(1)}_{12} + 2\beta^* = 0, \\ \bar{m} \cdot \nabla \log S^{(1)}_{12} - 2\alpha^* = 0, \quad n \cdot \nabla \log S^{(1)}_{12} - 2\gamma^* = 0. \end{aligned} \tag{239}$$

Solving the above equations, we are able to obtain the auxiliary scalar field $S^{(1)}_{12}$. The electromagnetic scalar ϕ_1 will then be determined from Eq.(236). In analogy to Eq.(228), the non-degenerate electromagnetic tensor in the null tetrad reads

$$F_{ab} = 2\phi_1 \left(\ell_{[a} n_{b]} + \bar{m}_{[a} m_{b]} \right) + 2\bar{\phi}_1 \left(\ell_{[a} n_{b]} + m_{[a} \bar{m}_{b]} \right). \tag{240}$$

1 The complete expression of the degenerate electromagnetic spinors in Eq.(227) should be $\Phi^{(2)}_{AB}$, since there is only one type of the field in the type N case, for simplicity, we omit the superscript (2) there.

Correspondingly, the auxiliary scalar field connecting the Weyl field and the non-degenerate electromagnetic field is denoted by $S_{24}^{(1,1)}$, which satisfies

$$\Psi_2 = \frac{(\phi_1)^2}{S_{24}^{(1,1)}}, \tag{241}$$

where superscript $(1,1)$ corresponds to the product of two electromagnetic scalars ϕ_1.

In general, both for type N solutions and type D solutions, once DW spinors are identified, all electromagnetic fields (or other higher spin massless free-fields) in the curved spacetime principally can be formulated with the aid of an auxiliary scalar field. To verify the Weyl double copy relation, it is only necessary to locate a specific set of electromagnetic fields, which are independent of the source parameters or structure functions that determine how the spacetime deviates from the (A)dS background. If such electromagnetic fields do exist, they will also satisfy the field equation in (A)dS spacetime. These fields are nothing but the single copy, and the associated auxiliary scalar fields are the zeroth copy.

7.3 THE WEYL DOUBLE COPY IN CURVED SPACETIMES

Regarding DW spinors as basic units, electromagnetic spinors living in a certain curved spacetime are constructed according to the maps Eq.(227) and Eq.(236). Then, they are converted to tensor form and expanded in terms of the products of the null tetrad bases, such as Eq.(228) and Eq.(240). As we will see later, except for the electromagnetic scalars, the products of the null tetrad bases of the degenerate electromagnetic tensors in non-twisting vacuum type N spacetimes are independent of the structure functions and source parameters. The same is true for non-degenerate electromagnetic tensors in vacuum type D spacetimes. For the sake of brevity, structure functions and source parameters will be referred to as deviation-information in the following. Once we have demonstrated that the electromagnetic scalars are independent of deviation-information, the same will be true of electromagnetic fields. Therefore, they should satisfy the field equations in conformally flat spacetime. Surprisingly, one will find that the associated scalar fields automatically satisfy their conformally invariant field equations in conformally flat spacetime. An explicit demonstration of the Weyl double copy for non-twisting vacuum type N and vacuum type D solutions is given in the following. The signature of the spacetime metric is chosen as $(+,-,-,-)$ in this work.

7.3.1 *The case of non-twisting vacuum type N solutions*

As the solutions of gravitational waves, non-twisting vacuum type N solutions (Λ) are composed of two classes [103, 104], one is the non-expanding Kundt(Λ) class, and the other is the expanding Robinson-Trautman(Λ) class.

7.3.1.1 *The Kundt(Λ) class*

The metric in this case reads

$$\mathrm{d}s^2 = -F\mathrm{d}u^2 + 2\frac{q^2}{p^2}\mathrm{d}u\mathrm{d}v - 2\frac{1}{p^2}\mathrm{d}z\mathrm{d}\bar{z}, \tag{242}$$

with

$$\begin{aligned} p &= 1 + \frac{\Lambda}{6}z\bar{z}, \qquad q = \left(1 - \frac{\Lambda}{6}z\bar{z}\right)\alpha + \bar{\beta}z + \beta\bar{z}, \\ F &= \kappa\frac{q^2}{p^2}v^2 - \frac{(q^2)_{,u}}{p^2}v - \frac{q}{p}H, \qquad \kappa = \frac{\Lambda}{3}\alpha^2 + 2\beta\bar{\beta}, \\ H &= H(u,z,\bar{z}) = \left(f_{,z} + \bar{f}_{,\bar{z}}\right) - \frac{\Lambda}{3p}\left(\bar{z}f + z\bar{f}\right). \end{aligned} \tag{243}$$

where f is an arbitrary complex function of u and z, analytic in z. Further more, α and β are two arbitrary real and complex functions of u, respectively. In fact, according to Ref. [103], one can see that the parameter κ is sign invariant. For the case $\Lambda = 0$, there are two classes of solutions—generalised pp-waves ($\kappa = 0$) and generalised Kundt waves ($\kappa > 0$). If our universe admits a positive cosmological constant, namely $\Lambda > 0$, there is no limit on α and β, and there is only one kind of solution—generalised Kundt waves. For the case $\Lambda < 0$, the values of parameters α and β classify the metric into three types of solutions—generalised Kundt waves ($\kappa > 0$), generalised Siklos waves ($\kappa = 0$), and generalised pp-waves ($\kappa < 0$). We will soon see that the zeroth copy inherits this property to classify the gravity solutions.

Choosing the null tetrad

$$\ell = \mathrm{d}u, \quad n = -\frac{F}{2}\mathrm{d}u + \frac{q^2}{p^2}\mathrm{d}v, \quad m = \frac{1}{p}\mathrm{d}\bar{z}, \tag{244}$$

we have

$$\rho^* = 0, \quad \tau^* = -\frac{2\Lambda\bar{z}\alpha + \Lambda\bar{z}^2\beta - 6\bar{\beta}}{(6-\Lambda z\bar{z})\alpha + 6(z\bar{\beta} + \bar{z}\beta)}, \tag{245}$$

$$\Psi_4 = \frac{1}{72}\left(\Lambda z\bar{z} + 6\right)\left[(\Lambda z\bar{z} - 6)\alpha - 6(\bar{z}\beta + z\bar{\beta})\right]\partial_{\bar{z}}^3 \bar{f}. \tag{246}$$

Recalling Eq.(228), it is easy to check

$$2\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\ell_{[a}\bar{m}_{b]} = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{247}$$

where $I = \frac{6}{6+\Lambda z\bar{z}}$. Both matrices do not depend on the deviation-information, so the electromagnetic scalar will decide whether this kind of electromagnetic field is dependent of the deviation-information or not. From Eq.(226), the auxiliary scalar S_{12} is solved by

$$S_{12} = \mathcal{C}(u,\bar{z})\frac{\Lambda z\bar{z} + 6}{\left(\Lambda z\bar{z} - 6\right)\alpha - 6\left(z\bar{\beta} + \bar{z}\beta\right)}, \tag{248}$$

where $\mathcal{C}(u,\bar{z})$ is an arbitrary function of u and $\bar{z}$. Clearly S_{12} itself is independent of the deviation-information. According to Eq.(223) and Eq.(227), the DW scalar ξ and the electromagnetic scalar ϕ_2 are solved by

$$\xi^4 = \frac{(6+\Lambda z\bar{z})^4 \mathcal{C}(u,\bar{z})^3 \partial_{\bar{z}}^3 \bar{f}}{72\left[(\Lambda z\bar{z}-6)\alpha - 6(\bar{z}\beta + z\bar{\beta})\right]^2}, \tag{249}$$

$$\phi_2 = \frac{(6+\Lambda z\bar{z})}{6\sqrt{2}}\sqrt{\mathcal{C}(u,\bar{z})\partial_{\bar{z}}^3 \bar{f}}. \tag{250}$$

The structure function $\bar{f}$, which measures the value of Ψ_4, is absorbed by an arbitrary function $\mathcal{C}(u,\bar{z})$. The electromagnetic scalar thus does not depend on $\partial_{\bar{z}}^3 \bar{f}(u,\bar{z})$. So we obtain a particular degenerate electromagnetic field which is independent of the deviation-information. It is easy to check that this type of electromagnetic field satisfies its field equation even for the case $\partial_{\bar{z}}^3 \bar{f} = 0$. Namely,

$$\tilde{\nabla}_a F^{ab} = 0, \tag{251}$$

where the symbol tilde denotes that the background is (A)dS spacetimes—conformally flat spacetimes—where we just need to let $f = 1$ in the original metric. In fact, there is a freedom to choose a polynomial function $f = c_0(u) + c_1(u)\bar{z} + c_2(u)\bar{z}^2$ as long as $\partial_{\bar{z}}^3 \bar{f}(u,\bar{z}) = 0$, where $c_i(u)$ are expanding parameters of $\bar{z}$. Furthermore, with the fact that the Ricci scalar $R = -4\Lambda$, it is easy to verify that the auxiliary scalar field $S_{24} (= S_{12})$ satisfies the conformally invariant scalar field equation not only in the curved spacetime but also in conformally flat spacetime. So we have

$$\tilde{\nabla}^a \tilde{\nabla}_a S_{24} - \frac{1}{6}\tilde{R} S_{24} = 0. \tag{252}$$

When $\Lambda \to 0$, the result reduces to the Kundt ($\Lambda = 0$) class, the single copy and the zeroth copy satisfy their field equations in Minkowski spacetime.

More interestingly, one can find that the single copy only confines the structure function. For example, it does not depend on the parameters α, β, and Λ; the function f in Maxwell scalar only needs to be a function of coordinates u and z, and there are no other restrictions. In contrast, the zeroth copy is closely associated with α, β, and Λ. With a negative cosmological constant and in conjunction with the introduction in the first paragraph of this section, one can see that for different κ, it is the zeroth copy that specifies the sort of curved spacetimes they map.

7.3.1.2 *The Robinson-Trautman(Λ) class*

One of the familiar form of the metric for Robinson-Trautman (Λ) solutions is given by García Díaz and Plebański [103, 105]

$$\begin{aligned} \mathrm{d}s^2 &= -2(A\bar{A}+\psi B)\mathrm{d}u^2 - 2\psi \mathrm{d}u\mathrm{d}v - 2v\bar{A}\mathrm{d}u\mathrm{d}z - 2vA\mathrm{d}u\mathrm{d}\bar{z} - 2v^2\mathrm{d}z\mathrm{d}\bar{z}, \\ A &= \epsilon z - vf, \quad B = -\epsilon + \frac{v}{2}(f_{,z}+\bar{f}_{,\bar{z}}) + \frac{\Lambda}{6}v^2\psi, \quad \psi = 1+\epsilon z\bar{z}, \end{aligned} \tag{253}$$

where $\epsilon = +1, 0, -1$ corresponds to the source of the transverse gravitational waves being time-like, null, or space-like, respectively, at least in the weak field limit. This is consistent with the case that $\Lambda = 0$ [106]. One can also refer to Ref. [107] for more details on the interpretation of the Robinson-Trautman solutions. In addition, this metric only depends linearly on an arbitrary structure function $f(u,z)$, which will help facilitate the following discussions.

Choosing the null tetrad

$$\ell = du, \quad n = -(A\bar{A}+\psi B)du - \psi dv - \bar{A}vdz - vAd\bar{z}, \quad m = vd\bar{z}, \tag{254}$$

we have

$$\rho^* = \frac{1}{v(1+\epsilon z\bar{z})}, \quad \tau^* = \frac{\bar{f}}{1+\epsilon z\bar{z}}, \tag{255}$$

$$\Psi_4 = \frac{(1+\epsilon z\bar{z})\partial_{\bar{z}}^3\bar{f}}{2v}. \tag{256}$$

In this case, the Weyl scalar does not even depend on the cosmological constant. Recalling Eq.(228), one observes

$$2\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -v & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\ell_{[a}\bar{m}_{b]} = \begin{pmatrix} 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{257}$$

both terms are independent of the structure function $f(u,\zeta)$. Solving Eq.(226), the auxiliary scalar field S_{12} is given by

$$S_{12} = \frac{C(u,\bar{z})}{v(1+\epsilon z\bar{z})}, \tag{258}$$

where the function $C(u,\bar{z})$ is arbitrary. Following Eq.(223) and Eq.(227), the DW scalar and the electromagnetic scalar are solved by

$$\xi^4 = \frac{C(u,\bar{z})^3 \partial_{\bar{z}}^3 \bar{f}}{2v^4(1+\epsilon z\bar{z})^2}, \tag{259}$$

$$\phi_2 = \sqrt{\frac{C(u,\bar{z})\partial_{\bar{z}}^3 \bar{f}}{2}}\frac{1}{v}. \tag{260}$$

Clearly, the function $C(u,\bar{z})$ lets ϕ_2 be independent of the structure function f. Thus, the electromagnetic field also satisfies the field equation in conformally flat spacetime. We can further check that the auxiliary scalar filed $S_{24}(= S_{12})$ satisfies Eq.(252) both in the curved spacetime and in conformally flat spacetime.

It is worth noting that the single copy does not depend on the parameter ϵ. It is the zero copy that decides what kind of sources of gravitational waves they are mapping, at least in the weak field limit. For example, given the same electromagnetic field in conformally flat spacetime, following the map Eq.(210) the scalar field S_{12} with $\epsilon = 1$ will lead to a class of transverse gravitational waves whose source is time-like. On the other hand, a scalar field S_{12} with $\epsilon = 0$ will lead to another class of transverse gravitational waves whose source is null.

So far, we only consider the time-dependent vacuum solutions. Next, we will investigate time-independent vacuum solutions by focusing on type D spacetimes. More interpretations about the single copy and the zeroth copy will be discussed later.

7.3.2 *The case of vacuum type D solutions*

7.3.2.1 *Kerr-(A)dS black holes*

As we know, rotating black holes are believed to be the most typical astrophysical black holes in the universe. It is necessary to take the case of Kerr-(A)dS black holes as a specific example to study the double copy relation before going to the most general vacuum type D solutions.

The metric of Kerr-(A)dS black holes in the Boyer-Lindquist coordinates reads [108–110]

$$\mathrm{d}s^2 = \frac{\mathcal{R}}{\rho^2}(\mathrm{d}t - \frac{a}{\Sigma}\sin^2\theta \mathrm{d}\phi)^2 - \frac{\rho^2}{\mathcal{R}}\mathrm{d}r^2 - \frac{\rho^2}{\Theta}\mathrm{d}\theta^2 \tag{261}$$

$$- \frac{\Theta}{\rho^2}\sin^2\theta(a\mathrm{d}t - \frac{r^2+a^2}{\Sigma}\mathrm{d}\phi)^2, \tag{262}$$

where

$$\mathcal{R} = (r^2+a^2)(1+\frac{r^2}{l^2}) - 2Mr, \quad \Theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \tag{263}$$

$$\Sigma = 1 - \frac{a^2}{l^2}, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad l^2 = -\frac{3}{\Lambda}, \tag{264}$$

with mass M/Σ^2 and angular momentum $J = aM/\Sigma^2$. Clearly, M and a can be regarded as mass parameter and angular momentum parameter, respectively.

Since the metric has already been written in the orthogonal tetrad $\{e^i\}$ $(i = 1, 2, 3, 4)$ such that $ds^2 = (e^1)^2 - (e^2)^2 - (e^3)^2 - (e^4)^2$, the null tetrad $\{e'^i\}$ then is easily given under the transformation

$$\begin{aligned} e'^1 =& \frac{1}{\sqrt{2}}(e^1 + e^2), \qquad e'^2 = \frac{1}{\sqrt{2}}(e^1 - e^2), \\ e'^3 =& \frac{1}{\sqrt{2}}(e^3 + ie^4), \qquad e'^4 = \frac{1}{\sqrt{2}}(e^3 - ie^4). \end{aligned} \tag{265}$$

Thus we have

$$e'^1 = \ell = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\mathcal{R}}{\rho^2}}\mathrm{d}t + \sqrt{\frac{\rho^2}{\mathcal{R}}}\mathrm{d}r - \sqrt{\frac{\mathcal{R}}{\rho^2}}\frac{a}{\Sigma}\mathrm{d}\phi\right), \tag{266}$$

$$e'^2 = n = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\mathcal{R}}{\rho^2}}\mathrm{d}t + \sqrt{\frac{\rho^2}{\mathcal{R}}}\mathrm{d}r - \sqrt{\frac{\mathcal{R}}{\rho^2}}\frac{a}{\Sigma}\mathrm{d}\phi\right), \tag{267}$$

$$e'^3 = m = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\Theta}{\rho^2}}a\sin\theta\mathrm{d}t + i\sqrt{\frac{\rho^2}{\Theta}}\mathrm{d}\theta - \sqrt{\frac{\Theta}{\rho^2}}\frac{(r^2+a^2)}{\Sigma}\sin\theta\mathrm{d}\phi\right), \tag{268}$$

$$e'^4 = \bar{m} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\Theta}{\rho^2}}a\sin\theta\mathrm{d}t - i\sqrt{\frac{\rho^2}{\Theta}}\mathrm{d}\theta - \sqrt{\frac{\Theta}{\rho^2}}\frac{(r^2+a^2)}{\Sigma}\sin\theta\mathrm{d}\phi\right). \tag{269}$$

We obtain the Weyl scalar

$$\Psi_2 = 6\psi_2 = \frac{6M}{(r + ia\cos\theta)^3}, \tag{270}$$

and the spin coefficients

$$\begin{aligned} \rho^* = \mu^* =& -\frac{i}{\sqrt{2}(a\cos\theta - ir)}\sqrt{\frac{\mathcal{R}}{\rho^2}}, \\ \tau^* = \pi^* =& -\frac{ia\sin\theta}{\sqrt{2}(a\cos\theta - ir)}\sqrt{\frac{\Theta}{\rho^2}}, \\ \epsilon^* = \gamma^* =& -\frac{a\cos\theta[l^2(r-M) + r(a^2+2r^2)] + i(a^2l^2 - r^4 - l^2Mr)}{2\sqrt{2}l^2(a\cos\theta - ir)}\frac{1}{\sqrt{\rho^2\mathcal{R}}}, \\ \alpha^* = \beta^* =& \frac{r\cos\theta(a^2\cos 2\theta - l^2) + ia(a^2\cos^4\theta - l^2)}{2\sqrt{2}l^2\sin\theta(a\cos\theta - ir)}\frac{1}{\sqrt{\rho^2\Theta}}. \end{aligned} \tag{271}$$

According to Eq.(229), to identify the DW scalar we need to solve the auxiliary scalar field S_{14}. Using Eq.(235) and the identity

$$\arctan(z) = -\frac{i}{2}\log\left(\frac{i-z}{i+z}\right) \qquad z \in \mathbb{C}, \tag{272}$$

it is not hard to obtain that

$$S_{14} = \mathcal{K}_1 \frac{\csc^2\theta(r + ia\cos\theta)}{l^4 \mathcal{R}\Theta}, \tag{273}$$

where all of the constant coefficients have been absorbed by a constant of integration $\mathcal{K}_1$. The DW scalar can then be solved by

$$\xi^2 = \sqrt{\Psi_2 S_{14}} = \frac{\sqrt{6\mathcal{K}_1 M}\csc\theta}{(r + ia\cos\theta) l^2 \sqrt{\mathcal{R}\Theta}}. \tag{274}$$

Note this is also the coefficient of DW tensor in the null tetrad [98]. We are not going to talk about DW tensor in more detail in this paper, the only reason we show this is to construct an electromagnetic scalar. With the help of the auxiliary scalar $S_{12}^{(1)}$, which is solved from Eq.(239)

$$S_{12}^{(1)} = \mathcal{K}_2 \frac{\csc\theta(r + ia\cos\theta)}{l^2\sqrt{\mathcal{R}\Theta}}, \tag{275}$$

we obtain

$$\phi_1 = \frac{\xi^2}{S_{12}^{(1)}} = \frac{\sqrt{6\mathcal{K}_1 M}}{\mathcal{K}_2} \frac{1}{(r + ia\cos\theta)^2} \sim \frac{1}{(r + ia\cos\theta)^2}, \tag{276}$$

where $\mathcal{K}_2$, similar to $\mathcal{K}_1$, is an arbitrary constant of integration. One can observe that the mass parameter M is absorbed by a constant of integration. So ϕ_1 actually does not depend on the source. In addition, it is easy to verify that $(l_{[a}n_{b]} + \bar{m}_{[a}m_{b]})$ and $(l_{[a}n_{b]} + m_{[a}\bar{m}_{b]})$ are all independent of the mass parameter M. Hence, from Eq.(240), we conclude that the non-degenerate electromagnetic field we construct is independent of the source. It even satisfies the conformally invariant field equation in conformally flat spacetime, where we just need to let $M = 0$ in the metric. What about the auxiliary scalar field $S_{24}^{(1,1)}$ associated to this electromagnetic field? Using the formula Eq.(210), one observes that

$$S_{24}^{(1,1)} = \frac{(\phi_1)^2}{\Psi_2} = \frac{\mathcal{K}_1}{(\mathcal{K}_2)^2} \frac{1}{(r + ia\cos\theta)} \sim \frac{1}{r + ia\cos\theta}. \tag{277}$$

As we expect, this satisfies the conformally invariant scalar field equation

$$\tilde{\nabla}^a \tilde{\nabla}_a S_{24}^{(1,1)} - \frac{1}{6}\tilde{R} S_{24}^{(1,1)} = 0. \tag{278}$$

When $\Lambda \to 0$, it describes the wave equation on Minkowski background. Thus, we have shown that the single copy and the zeroth copy of Kerr-AdS spacetimes satisfy their conformally invariant field equations in conformally flat spacetime.

Moreover, in analogy to Eq.(226) of the type N case, the auxiliary scalars associated with the degenerate electromagnetic fields are given by

$$S_{12}^{(2)} = S_{12}^{(0)} \sim \frac{1}{r + ia\cos\theta}. \tag{279}$$

Combining with Eq.(277), one can see that $S_{12}^{(2)}$ and $S_{12}^{(0)}$ are equivalent to $S_{24}^{(1,1)}$ up to a constant. Therefore, the zeroth copy connects not only gravity fields with the single copy living in the conformally flat spacetime, but also DW fields with those degenerate electromagnetic fields residing in the curved spacetime. This property is consistent with the discovery of the preceding work [98] in the absence of the cosmological constant Λ.

7.3.2.2 *The most general vacuum type D solutions*

Now, we shall investigate the Weyl double copy relation for the most general vacuum type D solutions with a cosmological constant. The metric has been given by Plebanski and Demianski[2] [111],

$$ds^2 = \frac{1}{(p+q)^2}\left(-\frac{1+(pq)^2}{\mathcal{P}}\mathrm{d}p^2 - \frac{\mathcal{P}}{1+(pq)^2}(\mathrm{d}\sigma + q^2\mathrm{d}\tau)^2 - \right. \tag{280}$$

$$\left. \frac{1+(pq)^2}{\mathcal{L}}\mathrm{d}q^2 + \frac{\mathcal{L}}{1+(pq)^2}(-p^2\mathrm{d}\sigma + \mathrm{d}\tau)^2\right), \tag{281}$$

where the structure functions read

$$\mathcal{P} = (-\frac{\Lambda}{6} + \gamma) + 2np - \epsilon p^2 + 2mp^3 + (-\frac{\Lambda}{6} - \gamma)p^4, \tag{282}$$

$$\mathcal{L} = (-\frac{\Lambda}{6} - \gamma) + 2nq + \epsilon q^2 + 2mq^3 + (-\frac{\Lambda}{6} + \gamma)q^4, \tag{283}$$

2 By doing a coordinate transformation $q \to -1/q$ and some rescalings following in eq. (3) of Ref. [94], we will go back to the modified form of the metric applied in the preceding work [98].

m and n are dynamical parameters measuring the curvature, ϵ and γ are called kinematical parameters which will affect the properties of the solutions.

By choosing null tetrad

$$\ell = \frac{1}{\sqrt{2}(p+q)}\left(\sqrt{\frac{1+(pq)^2}{\mathcal{L}}}dq - p^2\sqrt{\frac{\mathcal{L}}{1+(pq)^2}}d\sigma + \sqrt{\frac{\mathcal{L}}{1+(pq)^2}}d\tau\right), \quad (284)$$

$$n = \frac{1}{\sqrt{2}(p+q)}\left(-\sqrt{\frac{1+(pq)^2}{\mathcal{L}}}dq - p^2\sqrt{\frac{\mathcal{L}}{1+(pq)^2}}d\sigma + \sqrt{\frac{\mathcal{L}}{1+(pq)^2}}d\tau\right), \quad (285)$$

$$m = \frac{1}{\sqrt{2}(p+q)}\left(\sqrt{\frac{1+(pq)^2}{\mathcal{P}}}dp + i\sqrt{\frac{\mathcal{P}}{1+(pq)^2}}d\sigma + iq^2\sqrt{\frac{\mathcal{P}}{1+(pq)^2}}d\tau\right), \quad (286)$$

$$\bar{m} = \frac{1}{\sqrt{2}(p+q)}\left(\sqrt{\frac{1+(pq)^2}{\mathcal{P}}}dp - i\sqrt{\frac{\mathcal{P}}{1+(pq)^2}}d\sigma - iq^2\sqrt{\frac{\mathcal{P}}{1+(pq)^2}}d\tau\right), \quad (287)$$

we obtain

$$\Psi_2 = 6\psi_2 = 6(m+in)\left(\frac{p+q}{1-ipq}\right)^3. \quad (288)$$

Obviously, the cosmological constant does not affect the Weyl scalar according to the above result. Some spin coefficients are given by

$$\begin{aligned}
\rho^* = \mu^* &= \frac{(p^2-i)\sqrt{\mathcal{L}(q)}}{\sqrt{2}(pq+i)\sqrt{1+p^2q^2}},\\
\tau^* = -\pi^* &= \frac{(q^2-i)\sqrt{\mathcal{P}(p)}}{\sqrt{2}(pq+i)\sqrt{1+p^2q^2}},\\
\epsilon^* = \gamma^* &= \frac{2(p^2+2pq+i)\mathcal{L}(q) - (p+q)(pq+i)\mathcal{L}'(q)}{4\sqrt{2}(pq+i)\sqrt{1+p^2q^2}\sqrt{\mathcal{L}(q)}},\\
\beta^* = -\alpha^* &= \frac{2(q^2+2pq+i)\mathcal{P}(p) - (p+q)(pq+i)\mathcal{P}'(q)}{4\sqrt{2}(pq+i)\sqrt{(1+p^2q^2)\mathcal{P}(p)}}.
\end{aligned} \quad (289)$$

Solving Eq.(235) with the help of Eq.(272), the auxiliary scalar field S_{14} is given by

$$S_{14} = \mathcal{D}_1\frac{(p+q)^3(1-ipq)}{\mathcal{P}(p)\mathcal{L}(q)}, \quad (290)$$

where $\mathcal{D}_1$ is a constant of integration. Then from Eq.(229) we have

$$\xi^2 = \frac{\sqrt{6\mathcal{D}_1(m+in)}}{\sqrt{\mathcal{P}(p)\mathcal{L}(q)}}\frac{(p+q)^3}{1-ipq}. \quad (291)$$

Recalling Eq.(240), one observes that

$$2\left(\ell_{[a}n_{b]}+\bar{m}_{[a}m_{b]}\right)=\begin{pmatrix}0 & 0 & iA & ip^2A\\ 0 & 0 & q^2A & -A\\ -iA & q^2A & 0 & 0\\ -ip^2A & A & 0 & 0\end{pmatrix}, \tag{292}$$

and

$$2\left(\ell_{[a}n_{b]}+\bar{m}_{[a}m_{b]}\right)=\begin{pmatrix}0 & 0 & -iA & -ip^2A\\ 0 & 0 & q^2A & -A\\ iA & q^2A & 0 & 0\\ ip^2A & A & 0 & 0\end{pmatrix}, \tag{293}$$

where $A=\frac{(p+q)^2}{1+p^2q^2}$. Clearly, they are independent of the dynamical parameters. Consequently, if the electromagnetic scalar is independent of the deviation-information, the electromagnetic field we construct will be independent of the deviation-information as well and the field equation will hold even in conformally flat spacetime. From Eq.(239), one obtains

$$S^{(1)}_{12}=\mathcal{D}_2\frac{(p+q)(1-ipq)}{\sqrt{\mathcal{P}(p)\mathcal{L}(q)}}, \tag{294}$$

where $\mathcal{D}_2$ is a constant of integration. Following Eq.(236), the non-degenerate electromagnetic scalar ϕ_1 is given by

$$\phi_1=\sqrt{\frac{6\mathcal{D}_1(m+in)}{\mathcal{D}_2}}\frac{(p+q)^2}{(1-ipq)^2}\sim\frac{(p+q)^2}{(1-ipq)^2}. \tag{295}$$

One can see that the dynamical parameters which measure the curvature are absorbed by the constants of integration, so ϕ_1 is independent of the dynamical parameters. Thus, we have discovered a particular non-degenerate electromagnetic field which is independent of the deviation-information. Correspondingly, the auxiliary scalar field $S^{(1,1)}_{24}$ is given by

$$S^{(1,1)}_{24}=\frac{(\phi_1)^2}{\Psi_2}=\frac{\mathcal{D}_1}{(\mathcal{D}_2)^2}\frac{p+q}{1-ipq}\sim\frac{p+q}{1-ipq}. \tag{296}$$

It is easy to check that $S_{24}^{(1,1)}$ satisfies the conformal invariant field equation Eq.(252) in conformally flat spacetime, where we just need to set $m = n = 0$.

Therefore, for vacuum type D solutions with a cosmological constant, the single copy and the zeroth copies satisfy their conformal invariant field equations in conformally flat spacetime. When $\Lambda \to 0$, the background goes back to Minkowski spacetime and the situation is consistent with the previous result [75].

In addition, similar to the case of Kerr-AdS spacetime, for the general vacuum type D solutions with or without a cosmological constant, we find that

$$S_{12}^{(0)} = S_{12}^{(2)} \sim \frac{p+q}{1-ipq} \sim S_{24}^{(1,1)}. \tag{297}$$

Thus, not only does the zeroth copy connect gravity fields to the single copy, but it also links DW fields to degenerate electromagnetic fields living in curved spacetime. Recalling the previous section, it is evident that this property also applies to non-twisting type N solutions. While, distinct from the type N cases, the zeroth copy now does not possess any extra information about the source. This is mirrored clearly by the double copy scalar relation $(\Psi_2)^{1/3} = (\phi_2)^{1/2} = S_{24}^{(1,1)}$. Therefore, we find that only for the time-dependent solutions, the zeroth copy carries extra information about the source. This provides support for constructing other exact time-dependent radiation solutions in future work.

7.4 DISCUSSION AND CONCLUSIONS

In this paper, using DW spinors (massless spin-$1/2$ spinors) as basic units, we constructed a particular set of electromagnetic fields in 4-dimensional non-twisting vacuum type N and vacuum type D spacetimes in the presence of a cosmological constant Λ. These electromagnetic fields are independent of the deviation-information for a given curved metric. Thus they also satisfy the field equation in conformally flat spacetime. Regarding these electromagnetic fields as the single copies in the curved (A)dS spacetime, we verified the Weyl double copy prescription. We found that the single and zeroth copies satisfy the conformally invariant field equations in conformally flat spacetime both for the time-dependent solutions (type N cases) and time-independent solutions (type D cases). When $\Lambda \to 0$, the result reduces to the original case. Namely, they satisfy the field equations in Minkowski spacetime. This is an intriguing outcome. For Kerr-Schild (Λ) double copy prescription [53], the single and zeroth copies satisfy different equations for time-independent solutions and time-dependent solutions. Specifically, in time-independent cases, the zeroth and single copies satisfy the conformally invariant field equations in conformally flat spacetime; whereas, in time-dependent cases, the zeroth copy satisfies the wave equation and does not admit good conformal transformation properties anymore. Moreover, the single copy does not satisfy the conformally invariant Maxwell's field equation because of an extra term proportional to the Ricci scalar appearing in the equation of motion. Therefore, from this point of view, the Weyl double copy prescription appears as a more fundamental map between gravity theory and gauge theory. This is also consistent with the fact that the Kerr-Schild double copy prescription is linear, whereas the Weyl double copy prescription is essentially more general.

Apart from the above results, we found that the preceding finding [98] also holds in the presence of Λ. Not only does the zeroth copy connect gravity fields with the single copy in the conformally flat spacetime, but it also connects DW fields with degenerate electromagnetic fields in the curved spacetime, both for non-twisting vacuum type N solutions and vacuum type D solutions. More interestingly, we found that the zeroth copy plays a more important role than expected for time-dependent radiation solutions (type N cases). Unlike the single copy, which only restricts the form of the structure function, the zeroth copy carries additional information characterizing the curved spacetimes it is mapping. Specifially for the Robinson-Trautman (Λ) class, we discovered that it is the zeroth copy that determines whether the sources of associated gravitational waves are time-like, null, or space-like, at least in the weak field limit. This result is reminiscent of previous research on the fluid/gravity duality [60], which showed that all of the information about the fluid is encoded in the zeroth copy for the type N case. Their work further supports our result that the zeroth copy can indeed carry additional information compared with the single copy. We hope this discovery will contribute to constructing other exact time-dependent radiation solutions.

All in all, we have shown explicitly that the single copy and the zeroth copy satisfy conformally invariant equations in conformally flat spacetime by concentrating on non-twisting vacuum type N and vacuum type D solutions. Several novel interpretations of the Weyl double copy prescription are provided, particularly with regard to the zeroth copy. Next, it would be intriguing to check whether the generalized Weyl double copy holds asymptotically for the algebraically general case with a cosmological constant. It is also significant to investigate the applications of the Weyl double copy on astrophysical observations, such as the specific correspondence between the source of gravitational waves and the Weyl double copy. In addition, a natural progression of this work is to analyse the classical Weyl double copy in the flat spacetime instead of the (A)dS background. That would be essential for establishing a bridge between gravity theory and gauge theory. The cosmological constant, in this case, should be considered as the source of the single and zeroth copies. Further studies which treat the Minkovski spacetime as the background of the Weyl double copy prescription in the presence of a cosmological constant will need to be undertaken in the future. Since all of the discussion in this paper is limited to 4-dimensional spacetimes, it would also be worthwhile to extend the study to high-dimensional spacetimes.

8

ROBINSON-TRAUTMAN SOLUTIONS AND THE WEYL DOUBLE COPY

In this chapter, I will introduce my ongoing research that builds upon the findings presented in the Chapters. 6 and 7, highlighting some promising preliminary results and exploring the ramifications of those discoveries.

8.1 INTRODUCTION

In the last chapter, the results show that the zeroth copy carries additional information characterizing the associated curved spacetime. In particular, for the Robinson-Trautman (RT) (Λ) class, I found that it is the zeroth copy that determines whether the sources of associated gravitational waves are time-like, null, or space-like, at least in the weak field limit. This work makes it possible to develop exact gravitational wave solutions in the future. In fact, over the years, research on gravitational waves based on RT spacetime background has been ongoing, such as Ref.[112–114]. Inspired by this, I restudy the RT solution in an unfamiliar metric, which is linear in a function q, in the hope of finding a new interpretation of the Weyl double copy and potential applications on gravitational wave astronomy.

Conventionally, most researchers prefer focusing on the RT metric with a vanishing parameter q, which is made zero by coordinate transformation. However, as discussed by Robinson and Trautman in section (v) of their work Ref. [92], the curvature tensor and metric will be linear in parameter q if we keep it non-vanishing. This provides a new possible way to extend the Weyl double copy prescription to perturbative theory and gravitational wave astronomy research.

8.2 REVIEW OF ROBINSON TRAUTMAN SOLUTIONS

8.2.1 *General Robinson Trautman spacetime*

Robinson Trautman solutions describe a class of spacetime admitting a shear-free, diverging and hypersurface-orthogonal null vector field σ_k, which satisfies $R^{ik}\sigma_i\sigma_k = 0$. The metric is given by

$$\mathrm{d}s^2 = -\frac{\rho^2}{p^2}[(\xi - a\mathrm{d}\sigma)^2 + (\mathrm{d}\eta - b\mathrm{d}\sigma)^2] + 2d\rho\mathrm{d}\sigma + c\mathrm{d}\sigma^2, \quad \partial p/\partial\rho = 0. \tag{298}$$

One can rewrite it as

$$\mathrm{d}s^2 = 2\mathrm{d}\sigma\left[\mathrm{d}\rho + \left(c - \frac{\rho^2}{p^2}(a^2+b^2)\right)\mathrm{d}\sigma + \frac{a\rho^2}{p^2}\xi + \frac{b\rho^2}{p^2}\mathrm{d}\eta\right] - \frac{\rho^2}{p^2}(\xi^2 + \mathrm{d}\eta^2) \tag{299}$$

Introducing a complex variable $\sqrt{2}\zeta = (\xi + i\eta)$, or $\xi = \frac{\sqrt{2}}{2}(\zeta + \bar{\zeta})$ and $\eta = -\frac{\sqrt{2}i}{2}(\zeta - \bar{\zeta})$, the metric is transferred to

$$\begin{aligned}\mathrm{d}s^2 =& 2\mathrm{d}\rho\mathrm{d}\sigma + \left(c - \frac{\rho^2}{p^2}(a^2+b^2)\right)\mathrm{d}\sigma^2 + \frac{\sqrt{2}\rho^2}{p^2}\left((a-ib)\mathrm{d}\sigma\mathrm{d}\zeta + (a+ib)\mathrm{d}\sigma\mathrm{d}\bar{\zeta}\right) \\ &- \frac{2\rho^2}{p^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}.\end{aligned} \tag{300}$$

We list some useful identities in the following

$$\partial_\xi = \frac{1}{\sqrt{2}}(\partial_\zeta + \partial_{\bar{\zeta}}), \quad \partial_\eta = \frac{i}{\sqrt{2}}(\partial_\zeta - \partial_{\bar{\zeta}}), \tag{301}$$

$$a + ib = i\sqrt{2}\partial_\zeta q, \quad a - ib = -i\sqrt{2}\partial_{\bar{\zeta}} q, \tag{302}$$

$$a^2 + b^2 = 2\partial_\zeta q \partial_{\bar{\zeta}} q. \tag{303}$$

By doing a co-ordinate transformation,

$$\zeta = \psi(\tilde{\zeta}, \tilde{\sigma}), \quad \rho = \frac{\tilde{\rho}}{\gamma'(\tilde{\sigma})}, \quad \sigma = \gamma(\tilde{\sigma}), \tag{304}$$

the metric keeps the same form as Eq.(299), as shown below

$$\begin{aligned} ds^2 =& 2d\tilde{\rho}d\tilde{\sigma} + \Bigg[c\gamma'^2 - \frac{2\tilde{\rho}\gamma''}{\gamma'} - \frac{\tilde{\rho}^2}{p^2\gamma'^2}\Big((a^2+b^2)\gamma'^2 + 2\partial_{\tilde{\sigma}}\bar{\psi}\partial_{\tilde{\sigma}}\psi - \sqrt{2}\gamma'((a+ib)\partial_{\tilde{\sigma}}\bar{\psi} \\ &+(a-ib)\partial_{\tilde{\sigma}}\psi)\Big)\Bigg] d\tilde{\sigma}^2 + \frac{\sqrt{2}\tilde{\rho}^2\partial_{\tilde{\zeta}}\psi}{p^2\gamma'^2}\left((a-ib)\gamma' - \sqrt{2}\partial_{\tilde{\sigma}}\bar{\psi}\right) d\tilde{\sigma}d\tilde{\zeta} \\ &+ \frac{\sqrt{2}\rho^2\partial_{\bar{\tilde{\zeta}}}\bar{\psi}}{p^2\gamma'^2}\left((a+ib)\gamma' - \sqrt{2}\partial_\sigma\psi\right) d\sigma d\bar{\zeta} - \frac{2\tilde{\rho}^2\partial_{\tilde{\zeta}}\psi\partial_{\bar{\tilde{\zeta}}}\bar{\psi}}{p^2\gamma'^2} d\tilde{\zeta}\bar{\tilde{\zeta}}. \end{aligned} \tag{305}$$

Comparing Eq.(300) with Eq.(305), one arrives at

$$\begin{aligned} p'^2 &= \frac{p^2\gamma'^2}{\partial_{\tilde{\zeta}}\psi\partial_{\bar{\tilde{\zeta}}}\bar{\psi}}, \\ a' + ib' &= \frac{(a+ib)\gamma' - \sqrt{2}\partial_{\tilde{\sigma}}\psi}{\partial_{\tilde{\zeta}}\psi}, \\ c' &= c\gamma'^2 - \frac{2\tilde{\rho}\gamma''}{\gamma'}. \end{aligned} \tag{306}$$

Choosing a null tetrad,

$$\begin{aligned} \ell &= d\sigma, \\ n &= d\rho + \frac{1}{2}\left(c - \frac{\rho^2}{p^2}(a^2+b^2)\right) d\sigma + \frac{\sqrt{2}\rho^2}{2p^2}(a-ib)d\zeta + \frac{\sqrt{2}\rho^2}{2p^2}(a+ib)d\bar{\zeta}, \\ m &= \frac{\rho}{p}d\zeta, \\ \bar{m} &= \frac{\rho}{p}d\bar{\zeta}, \end{aligned} \tag{307}$$

the equation $\ell^a m^b R_{ab} = 0$ reduces to

$$\partial_\rho (a + ib) = 0. \tag{308}$$

The equation $m^a m^b R_{ab} = 0$ reduces to

$$\partial_{\bar{\zeta}} (a + ib) = 0. \tag{309}$$

Therefore, $(a + ib)$ is a function independent of ρ and $\bar{\zeta}$. Based on the above two equations, it is easy to find that we can eliminate a and b by transformation Eq.(304). However, one should bear in mind that the form of p might be intricate in this case. We could get the formula of p simpler if we keep a and b non-vanishing. Additionally, from the above two equations, one can see that there exists an analytic function q of σ, ξ, and η, such that,

$$a = \frac{\partial q}{\partial \eta}, \quad b = \frac{\partial q}{\partial \xi}, \quad \text{and} \quad \Delta q = p^2 (\partial_\xi^2 + \partial_\eta^2) q = 0. \tag{310}$$

Note, in $\zeta, \bar{\zeta}$ coordinates system,

$$(\partial_\xi^2 + \partial_\eta^2) = 2\partial_\zeta \partial_{\bar{\zeta}} \tag{311}$$

Substituting the formulas of a and b into the metric Eq.(298) and choosing a null tetrad with two complex null bases,

$$\begin{aligned} l &= \mathrm{d}\sigma, \\ n &= \mathrm{d}\rho + \frac{1}{2}\left(c - \frac{\rho^2}{p^2}\left((\partial_\xi q)^2 + (\partial_\eta q)^2\right)\right)\mathrm{d}\sigma + \frac{\rho^2 \partial_\eta q}{p^2}\mathrm{d}\xi + \frac{\rho^2 \partial_\xi q}{p^2}\mathrm{d}\eta, \\ m &= \frac{\rho}{\sqrt{2}p}\mathrm{d}\xi + \frac{i\rho}{\sqrt{2}p}\mathrm{d}\eta, \end{aligned} \tag{312}$$

one obtains 7 independent Einstein field equations. $l^a l^b R_{ab} = 0$, $l^a m^b R_{ab} = 0$, and $m^a m^b R_{ab} = 0$ are already satisfied. $m^a \bar{m}^b R_{ab} = 0$ leads us to

$$c = -\frac{2M}{\rho} + K - 2H\rho, \tag{313}$$

where

$$H(\sigma,\xi,\eta) = p^{-1}\frac{\partial p}{\partial \sigma} - \frac{\partial^2 q}{\partial \xi \partial \eta} + p^{-1}\frac{\partial p}{\partial \xi}\frac{\partial q}{\partial \eta} + p^{-1}\frac{\partial p}{\partial \eta}\frac{\partial q}{\partial \xi}, \tag{314}$$

$$K(\sigma,\xi,\eta) = \Delta \ln p, \tag{315}$$

and M is an integral function independent of the variable ρ. The above formulas are consistent with the convention of the original work by Robinson and Trautman. Notably, we correct a typo of H in eq. (17) of their work; instead of a plus sign, there should be a minus sign in the second term of H. Then $l^a n^b R_{ab} = 0$ is satisfied automatically. Furthermore, $n^a m^b R_{ab} = 0$ leads us to

$$\partial_\xi M = 0, \quad \partial_\eta M = 0. \tag{316}$$

Namely, M is a function of σ alone. Based on the previous six equations, the last one $\rho^2 n^a n^b R_{ab} = 0$ turns out to be

$$\frac{1}{2}\Delta K + 2(3H - \frac{\partial}{\partial_\sigma})M = 0. \tag{317}$$

Therefore, the Einstein field equations are reduced to two equations: Eq.(310) and Eq.(317). We list them together shown below

$$(\partial_\xi^2 + \partial_\eta^2)q = 0, \quad \frac{1}{2}\Delta K + 2(3H - \frac{\partial}{\partial_\sigma})M = 0. \tag{318}$$

So the spacetime information is characterized by three functions

$$q(\sigma,\xi,\eta), \quad p(\sigma,\xi,\eta), \quad M(\sigma). \tag{319}$$

8.2.2 *Robinson Trautman vacuum type N spacetime*

To get a vacuum type N solution in terms of a function q, we need to find an appropriate coordinate transformation. Firstly, we need to know how variables change with the transformation. We list some important results,

$$p'^2 = \frac{p^2\gamma'^2}{\partial_\zeta\psi\partial_{\bar{\zeta}}\bar{\bar{\psi}}}, \tag{320}$$

$$a' + ib' = \frac{(a+ib)\gamma' - \sqrt{2}\partial_{\tilde{\sigma}}\psi}{\partial_{\bar{\zeta}}\psi}, \tag{321}$$

$$c' = c\gamma'^2 - \frac{2\tilde{\rho}\gamma''}{\gamma'}, \tag{322}$$

$$M' = \gamma'^2 M, \tag{323}$$

$$K' = \gamma'^2 K, \tag{324}$$

$$H' = \gamma' H + \gamma''/\gamma'. \tag{325}$$

According to RT's work, under a specific coordinate transformation by which q is vanishing, vacuum type N solutions correspond to

$$M = 0, \quad \frac{\partial K}{\partial \zeta} = 0, \quad \frac{1}{\rho^2}\frac{\partial}{\partial \zeta}\left(p^2\frac{\partial H}{\partial \zeta}\right) \neq 0. \tag{326}$$

Instead of eliminating function q by the coordinate transformation, we are going to do another transformation so that the function p is reduced to

$$p = 1 + \frac{1}{2}K\zeta\bar{\zeta} = 1 + \frac{1}{4}K(\xi^2 + \eta^2). \tag{327}$$

There is no way to ensure that q (or, a and b) is vanishing at the same time under the coordinate transformation. Therefore, the field equation left is

$$\Delta q = 0. \tag{328}$$

Since K in this case only depends on σ, it is easy to reduce it to $-1, 0$ or 1 by doing one more simple coordinate transformation, such that

$$\gamma'^2 = \frac{1}{|K|}. \tag{329}$$

In the end, one will be able to obtain the vacuum RT type N solutions by solving Laplace's Equation Eq.(328).

8.3 WEYL DOUBLE COPY RELATION

Based on the method proposed in the previous two chapters, we calculate the Weyl spinor in the coordinates system Eq.(300). The Weyl scalar is given by

$$\psi_4 = -i\frac{(1+\frac{1}{2}K\zeta\bar{\zeta})^2}{2\rho}\partial_\zeta^4 q(\sigma,\zeta,\bar{\zeta}). \tag{330}$$

Solving Eq.(328), the function q is divided into two distinct parts,

$$q(\sigma,\zeta,\bar{\zeta}) = \mathcal{L}(\sigma,\zeta) + \mathcal{R}(\sigma,\bar{\zeta}), \tag{331}$$

where $\mathcal{L}$ and $\mathcal{R}$ are two arbitrary functions. Then, we further reduce the Weyl scale to the form

$$\psi_4 = -i\frac{(1+\frac{1}{2}K\zeta\bar{\zeta})^2}{2\rho}\partial_\zeta^4 \mathcal{L}(\sigma,\zeta). \tag{332}$$

The zeroth copy and Maxwell scale are identified, respectively,

$$S = \frac{W(\sigma,\zeta)}{\rho}, \tag{333}$$

$$\phi_2 = \frac{(1+\frac{1}{2}K\zeta\bar{\zeta})}{2\rho}X(\sigma,\zeta), \tag{334}$$

where W and X are two arbitrary functions of σ and ζ. It's easy to verify that both the zeroth copy and the single copy satisfy respective equations of motion on flat spacetime. It is worth pointing out that the single copy here carries the factor K, instead of the zeroth copy. This is different from the result of the last chapter.

8.4 DISCUSSIONS AND CONCLUSIONS

This chapter concludes by presenting ongoing research and preliminary findings that build upon the work presented in the Chapters. 6 and 7. These preliminary findings provide promising insights into the Weyl double copy and have the potential to substantially advance our understanding of the topic. For example, we show that the single copy carries the factor K, which characterizes the curved spacetimes. This differs from the case discussed in the last chapter, in which we found that the zeroth copy carries the factor to determine whether the sources of associated gravitational waves are time-like, null, or space-like in the weak field limit. From this point of view, the zeroth copy and single copy may both be able to inherit information from the gravitational field. On the other hand, it also reflects that the Weyl double copy does not provide a one-to-one correspondence between gravity solutions and gauge solutions, so we obtain different types of zeroth and single copies on the different coordinate systems. Regarding this non-uniqueness property, further research is still needed.

It is worth pointing out that, in this chapter, we didn't discuss whether the case $p = 1 + 1/2K\zeta\bar{\zeta}$ can represent all the solutions of the Robinson Trautman vacuum type N solutions. We temporarily assume that we can always find a coordinated transformation to make it. Based on the associated research, such as Ref. [112] and Ref. [115], we will proceed to further examine this matter in order to ascertain its validity. While some restrictions and difficulties still exist, resolving them will open the door for more development in this area. The results of this research and those from the published publications will be combined in the following chapter, together with their broader implications, limits, and suggested future research trajectories.

Part IV

DISCUSSIONS AND CONCLUSIONS

9

DISCUSSIONS AND CONCLUSIONS

Based on the property of spinors that any massless free-field spinors with spin higher than 1/2 can be constructed with DW spinors (spin-1/2) and scalar fields (spin-0), I proposed a new approach to study the Weyl double copy prescription. Specifically, by investigating non-twisting vacuum type N and vacuum type D solutions, this thesis has systematically explored the Weyl double copy relation between gravity theory and gauge theory. The case of vacuum type III solutions has also been discussed. In addition, I discovered that the zeroth copy plays a crucial role in connecting not only the gravity fields to the single copy but also the DW fields to degenerate electromagnetic fields in curved spacetime.

Additionally, in the presence of the cosmological constant, I constructed a particular set of electromagnetic fields and scalar fields that satisfy the conformally invariant field equation in conformally flat spacetime for non-twisting vacuum type N and vacuum type D spacetimes. In curved (A)dS spacetime, these electromagnetic fields serve as single copies, scalar fields serve as zeroth copies, and we validated the Weyl double copy prescription. Compared to the Kerr-Schild double copy prescription, I have shown that the Weyl double copy prescription appears to be a more fundamental map between gravity theory and gauge theory. In addition, the function of the zeroth copy, which carries additional information describing the curved spacetimes it is mapping, has been emphasized further, specifically for the Robinson-Trautman gravitational wave solutions.

In the last chapter of this thesis, I introduced the ongoing research about the Weyl double on a linearized Robinson-Trautman gravitational wave solution. The result shows that not only the zeroth copy, but the single copy may also be able to inherit spacetime information from the gravity field. This point may reflect the fact that the Weyl double copy does not have a one-to-one correspondence between gravity solutions and gauge solutions. To gain a deeper comprehension of the complexities of these theories and their interrelationships, additional study is still ongoing.

Overall, our research has shed light on the significance of the spin-1/2 massless free-fields and scalar fields in building other high spin massless free fields and provided a comprehensive analysis of the connections between gravity theory and gauge theory. We hope that these discoveries will lead to the development of additional precise time-dependent radiation solutions and inspire additional research into the compelling relationships between these fundamental theories.

Next, it would be fascinating to determine whether the generalized Weyl double copy holds asymptotically for the algebraically general case containing a cosmological constant. It is also important to investigate the implications of the Weyl double copy to astrophysical observations, such as the specific correspondence between the gravitational wave source and the Weyl double copy. In addition, it is a natural progression of this work to analyze the classical Weyl double copy in the flat spacetime background as opposed to the (A)dS background. This would be crucial for bridging the gap between gravity theory and gauge theory. In this instance, the cosmological constant should be considered the origin of the single and zeroth copies. Future research will be required to consider the Minkowski spacetime as the background of the Weyl double copy prescription in the presence of a cosmological constant. In addition, since all of the discussion in this paper is restricted to four-dimensional spacetimes, it would be beneficial to extend the research to higher-dimensional spacetimes. Lastly, as a final note, further investigation of RT spacetime could potentially uncover connections between cosmic strings and the Weyl double copy, paving the way for new insights in this field.

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